THE MATHEMATICS STUDENT

EDITED BY

R. P. BAMBAH and P. L. BHATNAGAR

WITH THE ASSISTANCE OF

G. BANDYOPADHYAY P. T. BATEMAN S. D. CHOPRA
H. GUPTA V. S. KRISHNAN D. J. LEWIS
V. V. NARLIKAR T. PATI U. N. SINGH
T. P. SRINIVASAN M. VENKATARAMAN

Vol. XXVII

PUBLISHED BY
THE INDIAN MATHEMATICAL SOCIETY
Annual Subscription Rs. 8

PRINTED IN INDIA
AT THE COMMERCIAL PRINTING PRESS LTD.
BOMBAY

CONTENTS

PAPERS

MATHEMATICAL NOTES

CLASSROOM NOTES

BOOK REVIEWS

ERRATUM

THE MATHEMATICS STUDENT,

Vol. XXIV, 1956

On page 84, lines 3-4 from the bottom *read* 'enchainment' *for* enchantment'.

ON TWO DIMENSIONAL SUPERPOSABLE FLOWS

By C. D. GHILDYAL

INTRODUCTION. The present paper aims at discussing two dimensional fluid motions mutually superposable. The case of finding a rotational flow superposable on a given irrotational flow is capable of being solved easily. This problem has already been discussed by Ballabh (1943 ; 1952). In this paper we have derived results which are more general than Ballabh's and have given simpler proofs.

1. General derivation. The equations of motion of a viscous homogeneous incompressible fluid in two dimensions can be written as

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - v\zeta &= -\frac{\partial \chi'}{\partial x} - \nu\frac{\partial \zeta}{\partial y}, \\ \text{and} \qquad \frac{\partial v}{\partial t} + u\zeta &= -\frac{\partial \chi'}{\partial y} + \nu\frac{\partial \zeta}{\partial x}, \end{aligned}\right\} \tag{1.1}$$

where the symbols have their usual meanings.

To these we add the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1.2}$$

Let (u_1, v_1, ζ_1) and (u_2, v_2, ζ_2) be the two solutions of (1.1). Using the definition of superposability as given by Ballabh (1940), after some calculation we get

$$\frac{\partial}{\partial x}(u_1\zeta_2 + u_2\zeta_1) + \frac{\partial}{\partial y}(v_1\zeta_2 + v_2\zeta_1) = 0, \tag{1.3}$$

as the condition of superposability. Let ψ_1 and ψ_2 be the stream functions for these two flows. The above equation can then be written as

$$\frac{\partial(\psi_1, \zeta_2)}{\partial(x, y)} + \frac{\partial(\psi_2, \zeta_1)}{\partial(x, y)} = 0, \tag{1.4}$$

since $u_r = -\dfrac{\partial \psi_r}{\partial y}$; $v_r = \dfrac{\partial \psi_r}{\partial x}$, $r = 1, 2$.

Two flows (ψ_1, ζ_1) and (ψ_2, ζ_2) are therefore superposable if and only if the sum of the Jacobians of (ψ_1, ζ_2) and (ψ_2, ζ_1) vanishes.

2. Let the flow whose stream function is ψ_1, be irrotational. For convenience we denote ψ_1 by ψ and ψ_2 by $\bar{\psi}$. We have to find the rotational flow superposable on the given irrotational flows. From equation (1.4), we get

$$\frac{\partial(\psi, \zeta)}{\partial(x, y)} = 0, \tag{2.1}$$

since $\zeta_1 = 0$ and $\zeta_2 = \zeta$. This gives

$$\zeta = f(\psi). \tag{2.2}$$

A rotational flow is therefore superposable on a given irrotational flow if and only if the vorticity of the former is constant along the stream lines of the latter.

The condition of integrability for steady flow is

$$\frac{\partial}{\partial x}(\bar{u}\zeta) + \frac{\partial}{\partial y}(\bar{v}\zeta) = \nu \nabla^2 \zeta, \tag{2.3}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Substituting the value of ζ from (2.2) and using the equation of continuity, we get,

$$\frac{\partial \bar{\psi}}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \bar{\psi}}{\partial y}\frac{\partial \psi}{\partial x} = \nu \left\{ \left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 \right\} F(\psi), \tag{2.4}$$

since

$$\nabla^2 \psi = 0; \; \bar{u} = -\frac{\partial \bar{\psi}}{\partial y}, \; \bar{v} = \frac{\partial \bar{\psi}}{\partial x}$$

and

$$F(\psi) = \frac{d}{d\psi} \log \frac{df}{d\psi}.$$

Here we are assuming that $\dfrac{df}{d\psi} \neq 0$.

Following Ballabh (1952), we get

$$\bar{\psi} = \nu\phi F(\psi) + G(\psi), \tag{2.5}$$

where ϕ is the velocity potential of the irrotational flow.

This value of $\bar{\psi}$ obtained from (2.5) has to satisfy the equation

$$\zeta \equiv \nabla^2 \bar{\psi} = f(\psi),$$

i.e.

$$f(\psi) = \left\{ \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 \right\} \left\{ \nu\phi F''(\psi) + G''(\psi) \right\},$$

where dashes denote differentiation with respect to ψ.

Now the above equation can be written as

$$\frac{1}{q^2} = \frac{1}{\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2} = \nu\phi \frac{F''(\psi)}{f(\psi)} + \frac{G''(\psi)}{f(\psi)}, \tag{2.6}$$

where $\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 \neq 0$, i.e. we assume that the velocity of the irrotational flow does not vanish anywhere.

Assuming $F''(\psi) = 0$, Ballabh obtained a general expression for $\bar{\psi}$ and ζ. In this way the velocity of the irrotational flow becomes a function of ψ alone. In one of his earlier papers he (1943) has discussed this very problem for non-viscous homogeneous incompressible fluids and finds the same results for the stream function of the irrotational flow in the case of viscous fluid also. He, however, gives no reason for assuming $F''(\psi) = 0$. We shall see that $F''(\psi)$ must necessarily vanish. We shall further see that steady irrotational flows on which a steady rotational flow is superposable consist either of uniform streaming or of motion outside a rectilinear vortex filament.

3. Using a well-known property of conjugate functions equation (2.6) can be written as

$$\left|\frac{d(x+\iota y)}{d(\phi+\iota\psi)}\right|^2 = \left|\frac{d(x+\iota y)}{d(-\psi+\iota\phi)}\right|^2 = \nu\phi\frac{F''(\psi)}{f(\psi)} + \frac{G''(\psi)}{f(\psi)}, \tag{3.1}$$

where $\iota = \sqrt{(-1)}$. Also, we have

$$\frac{d(x+\iota y)}{d(-\psi+\iota\phi)} = \frac{1}{\dfrac{d(-\psi+\iota\phi)}{d(x+\iota y)}} = -\frac{1}{v+\iota u}. \tag{3.2}$$

Also, $(x+\iota y)$ and $\dfrac{d(x+\iota y)}{d(-\psi+\iota\phi)}$ are analytic functions of $(-\psi+\iota\phi)$.

Therefore we can write

$$-\frac{1}{v+\iota u} = \sum_{n=0}^{\infty} a_n(-\psi+\iota\phi)^n.$$

The right hand side is equivalent to

$$a_0 + (-\psi+\iota\phi)\,a_1 + \dots + (-\psi+\iota\phi)^n\,a_n + \dots .$$

Putting $H = \sum\limits_{n=0}^{\infty} (-)^n\,a_n\,\psi^n$, we can write above expansion as

$$H + \sum_{n=1}^{\infty} \frac{(-)^n(\iota\phi)^n}{n!}\,\frac{d^n H}{d\psi^n},$$

or

$$H + \sum_{n=1}^{\infty} \frac{(-\phi^2)^n}{(2n)!}\,\frac{d^{2n}H}{d\psi^{2n}} + i\phi\sum_{n=1}^{\infty}\frac{(-)^n\,\phi^{2n-2}}{(2n-1)!}\,\frac{d^{2n-1}H}{d\psi^{2n-1}}.$$

Taking the square of the modulus, we get

$$\frac{1}{q^2} = \left\{H + \sum_{n=1}^{\infty}\frac{(-\phi^2)^n}{(2n)!}\,\frac{d^{2n}H}{d\psi^{2n}}\right\}^2 + \phi^2\left\{\sum_{n=1}^{\infty}\frac{(-1)^n\,\phi^{2n-2}}{(2n-1)!}\,\frac{d^{2n-1}H}{d\psi^{2n-1}}\right\}^2$$

$$\frac{1}{q^2} = H^2 - \frac{\phi^2}{2!}\left\{2H\frac{d^2H}{d\psi^2} - \frac{2!}{(1!)^2}\left(\frac{dH}{d\psi}\right)^2\right\} +$$

$$+\frac{\phi^4}{4!}\left\{2\left(H\frac{d^4H}{d\psi^4} - \frac{4!}{1!\,3!}\,\frac{dH}{d\psi}\,\frac{d^3H}{d\psi^3}\right) + \frac{4!}{(2!)^2}\left(\frac{d^2H}{d\psi^2}\right)^2\right\} -$$

$$-\frac{\phi^6}{6!}\left\{2\left(H\frac{d^6H}{d\psi^6} - \frac{6!}{1!\,5!}\,\frac{dH}{d\psi}\,\frac{d^5H}{d\psi^5} + \frac{6!}{2!\,4!}\,\frac{d^2H}{d\psi^2}\,\frac{d^4H}{d\psi^4}\right) -\right.$$

$$\left.-\frac{6!}{(3!)^2}\left(\frac{d^3H}{d\psi^3}\right)^2\right\} + \dots +$$

$$+(-)^n \frac{\phi^{2n}}{(2n)!}\left\{2\sum_{p=1}^{n-1}(-)^p \frac{(2n)!}{p!(2n-p)!}\frac{d^pH}{d\psi^p}\frac{d^{2n-p}H}{d\psi^{2n-p}}+\right.$$
$$\left.+(-)^n\frac{(2n)!}{(n!)^2}\left(\frac{d^nH}{d\psi^n}\right)^2\right\}+\ldots,$$

where n is a positive integer.

But from (2.6), we have

$$\frac{1}{q^2}=\nu\phi\frac{F''(\psi)}{f(\psi)}+\frac{G''(\psi)}{f(\psi)}.$$

From these two equations we easily get

$$\frac{d^2F}{d\psi^2}=0;\ \frac{1}{f(\psi)}\frac{d^2G}{d\psi^2}=H^2;\ H\frac{d^2H}{d\psi^2}-\left(\frac{dH}{d\psi}\right)^2=0, \text{ etc.}$$

From the first equation, we have

$$F(\psi)=\frac{d}{d\psi}\log\frac{df}{d\psi}=A\psi+B,$$

giving

$$f=K\int e^{(A\psi+B)^2/2A}\,d\psi+K_1,$$

where A, B, K_1 and K are constants. This determines the vorticity of the rotational flow. From the equation

$$H\frac{d^2H}{d\psi^2}-\left(\frac{dH}{d\psi}\right)^2=0,$$

we have

$$H=a\,e^{\lambda\psi},$$

where a and λ are arbitrary constants.

It is easy to see that the coefficients of $\phi^4, \phi^6, \ldots, \phi^{2n}, \ldots,$ also vanish identically for this value of H.

Also the value of $G(\psi)$ is

$$G(\psi)=Ka^2\int\left[\int\left\{e^{2\lambda\psi}\int e^{(A\psi+B)^2/2A}\,d\psi+\frac{K_1}{K}\right\}d\psi\right]d\psi+$$
$$+K_2\psi+K_3.$$

4. Now we wish to determine the possible form of streamlines of the irrotational on which a rotational flow is superposable. We have already seen that the velocity of the irrotational flow is constant along its streamlines. Thus equation (2.6) merely reduces to

$$|\nabla\psi|^2 = \frac{f(\psi)}{G''(\psi)} = \frac{1}{H^2}, \tag{4.1}$$

where

$$|\nabla\psi|^2 = q^2 = \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2.$$

The radius of curvature ρ of any curve $\psi(x, y) =$ constant, at any point (x, y) of the curve, is given by the relation

$$\nabla\cdot\left(\frac{\nabla\psi}{|\nabla\psi|}\right) = \frac{1}{\rho},$$

where $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}$; i and j are the unit vectors in x and y directions respectively.

Expanding the above equation, we get

$$\frac{1}{\rho} = \frac{\nabla\cdot(\nabla\psi)}{|\nabla\psi|} - \frac{\nabla\psi\cdot\nabla(|\nabla\psi|)}{|\nabla\psi|^2}$$

Using the relation (4.1) and condition of irrotationality $\nabla^2\psi = 0$, we get

$$\frac{1}{\rho} = \frac{1}{H^2}\frac{dH}{d\psi},$$

where $H = 1/q \neq 0$.

Two cases arise

(i) $\dfrac{dH}{d\psi} = 0$;

(ii) $\dfrac{dH}{d\psi}$ is a function of ψ, different from zero.

In the first case the curvature of the streamlines of the irrotational flow is zero which are therefore straight lines with constant velocity everywhere. In this case stream function is given by

$$\psi = \alpha_1 x + \beta_1 y + \gamma_1,$$

where α_1, β_1 and γ_1 are constants.

In the second case the curves $\psi =$ constant, represent the curves of constant curvature, and are therefore circles. The velocity is constant along the streamlines. In this case stream function is given by

$$\psi = A_1 \log r + B_1 .$$

Jeffery (1915) studied the steady two dimensional flow of a viscous homogeneous liquid by using the orthogonal curvilinear coordinates without assuming the motion to be slow. The coordinate system is defined by conjugate functions α, β of x and y. He further assumed that either the streamlines or the lines of constant vorticity are identical with one family of the coordinate curves. He obtained some exact solutions of the equations of motion of a viscous homogeneous incompressible fluid. His results are similar to those obtained above by using the principle of superposability.

Jeffery's procedure seems to be more involved than what we have given above.

My thanks are due to Dr. Ram Ballabh for guidance and to the Scientific Research Committee, U.P. for financial assistance.

REFERENCES

1. R. Ballabh : On two dimensional superposable flows, *J. Indian Math. Soc.* (1952).

2. R. Ballabh : Superposable fluid motions, *Proc. Benaras Math. Soc.* (1940).

3. R. Ballabh : Steady uniplanar superposable fluid motion, *J. Indian Math. Soc.* (1943).

4. G. B. Jeffery : Two dimensional steady motion of a viscous fluid, *Phil. Mag.* 29 (1915), 455.

Lucknow University
Lucknow

NOTE ON AN ENTIRE FUNCTION OF INFINITE ORDER

By S. M. SHAH

1. *Let $\theta(x)$ satisfy the following conditions :*

(i) *$\theta(x)$ is positive and non-decreasing for $x \geqslant x_0$, and tends to infinity with x,*

(ii) $I(x) = \int_{x_0}^{x} \frac{dt}{t\,\theta(t)}$ *tends to infinity with x,*

(iii) $x\,\theta'(x)/\theta(x) \leqslant c < 1$ *for $x > x_0$.*

Let N be an integer such that $I(N) > 1$, and

$$f(z) = \sum_{N}^{\infty} \left(\frac{z}{I(n)} \right)^n. \tag{1}$$

It is known [1] ***that $f(z)$ is an entire function of infinite order such that***

$$\lim_{r\to\infty} \frac{\log \mu(r, f)\, \theta(\log \mu(r, f))}{\nu(r, f)} = 0. \tag{2}$$

Further if $\theta(x)$ also satisfies the condition

(iv) $I(n^p) - I(n) > \dfrac{2}{\theta(n+1)}$,

where p is some (fixed) integer, for all large n, then

$$\lim_{r\to\infty} \frac{\log M(r, f)\, \theta(\log M(r, f))}{\nu(r, f)} = 0. \tag{3}$$

The purpose of this note is to show that a modified form of (iv) (see lemma below) follows from (i)-(iii) and this modified form of (iv) is used to deduce (3). Thus we show that condition (iv) is superfluous. Clunie [2, pp 180-2] deduced (3) from (i)-(iii) and also his theorem [2, p. 175].

2. Lemma. *If $\theta(x)$ satisfies* (i)-(iii) *then for any (fixed) $p > 1$,*

$$\liminf_{n\to\infty} \left[\theta(n+1) \{ I(n^p) - I(n+1) \} \right] \geqslant \frac{1}{c}; \tag{4}$$

$$\liminf_{n\to\infty} [\theta(n+1)\{I(pn)-I(n+1)\}] \geqslant \frac{1-p^{-c}}{c}. \tag{5}$$

The following proof of the lemma is due to W. K. Hayman. We have from (iii)

$$\log \frac{\theta(x_2)}{\theta(x_1)} \leqslant \log \left(\frac{x_2}{x_1}\right)^c, \quad x_1 \leqslant x_2.$$

Hence

$$\theta(t) \leqslant \theta(n+1) \left(\frac{t}{n+1}\right)^c, \quad t \geqslant n+1;$$

$$I(n^p) - I(n+1) = \int_{n+1}^{n^p} \frac{dt}{t\,\theta(t)}$$

$$> \frac{1}{\theta(n+1)} \int_{n+1}^{n^p} \frac{(n+1)^c}{t^{1+c}}\, dt$$

$$= \frac{1}{c\,\theta(n+1)} \left\{1 - \frac{(n+1)^c}{n^{pc}}\right\},$$

$$\theta(n+1)\{I(n^p)-I(n+1)\} > \frac{1}{c} + O\left(\frac{1}{n^{pc-c}}\right) \tag{6}$$

and (4) follows. Similarly

$$\theta(n+1)\{I(pn) - I(n+1)\} \geqslant \frac{1}{c}\left(1 - \frac{1}{p^c}\right) + O\left(\frac{1}{n}\right) \tag{7}$$

and (5) follows.

3. We have in the notation of [1, p. 84]

$$\log M(r) \leqslant (1 + o(1)) \log \mu(r) + 2 \log \nu\left(r + \frac{1}{r\,\nu^2(r)}\right).$$

Now $\nu(r) = n$ for $R_n \leqslant r < R_{n+1}$, where

$$R_n = I(n) \exp\left\{\frac{(n-1)}{\xi\,\theta(\xi)\,I(\xi)}\right\}, \quad n-1 < \xi < n.$$

Further, for x in the range (0, 1), $e^x < 1 + x + (e-2)x^2$. Hence

$$R_n < I(n) + \frac{1}{\theta(n)} + O\left(\frac{1}{\theta^2(n)\,I(n)}\right)$$

and so from (6)

$$R_{np} - R_{n+1} > I(n^p) - I(n+1) - \frac{1}{\theta(n+1)} + O\left(\frac{1}{\theta^2(n)\, I(n)}\right)$$

$$> \left(\frac{1}{c} - 1\right)\frac{1}{\theta(n+1)} + O\left(\frac{1}{\theta(n)\, n^{(p-1)c}}\right) +$$

$$+ O\left(\frac{1}{\theta^2(n)\, I(n)}\right).$$

Hence for all large n, $R_{np} - R_{n+1} > 1/n^2$, and so

$$R_{n+1} + \frac{1}{n^2 R_n} < R_{np} - \frac{1}{n^2}\left(1 - \frac{1}{R_n}\right) < R_{np},$$

$$\nu\left(R_{n+1} + \frac{1}{n^2 R_n}\right) \leqslant \nu(R_{np}) \leqslant n^p.$$

Hence

$$\log M(r) \leqslant (1 + o(1)) \log \mu(r) + 2 \log \nu\left(R_{n+1} + \frac{1}{n^2 R_n}\right)$$

$$\leqslant (1 + o(1)) \log \mu(r) + 2\, p \log n$$

and so we get [1, p. 85], $\log M(r) = (1 + o(1)) \log \mu(r)$, and (3) follows.

4. To show that when $\frac{1}{2} < c < 1$, (iv) need not follow from (i)-(iii), we construct $\theta(x)$ which satisfies (i), (ii) and (iii) except at an enumerable set of points where the right and left derivatives of $\theta(x)$ exist and satisfy (iii). We define $\theta(x)$ as follows.

Let $a_1 = 10!$, $b_k = e^{a_k}$, $a_{k+1} = \exp\left\{\left(\frac{b_k}{a_k}\right)^c \log a_k\right\}$, where $\frac{1}{2} < c < 1$, and $k = 1, 2, 3, \ldots$ Let

$$\theta(x) = \frac{\log a_k}{(a_k)^c}\, x^c, \qquad a_k \leqslant x \leqslant b_k,$$

$$= \frac{\log a_k}{(a_k)^c}\, b_k^c, \qquad b_k < x < a_{k+1}, \; k = 1, 2, 3, \ldots$$

Then $\theta(x)$ is positive, non-decreasing for $x \geqslant a_1$, and tends to infinity with x. Further

$$\limsup_{x \to \infty} \frac{\log x}{\theta(x)} = 1,$$

and so $I(x) = \int_{a_1}^{x} \frac{dt}{t\,\theta(t)}$ tends to infinity with x. Also

$$\frac{x\,\theta'(x)}{\theta(x)} = 0, \quad \text{when } b_k < x < a_{k+1},$$

$$= c, \quad \text{when } a_k < x < b_k,$$

and when $x = a_k (\text{or } b_k)$, $\frac{x\,\theta'(x \pm 0)}{\theta(x)} = c \text{ or } 0.$

Now given an integer $p > 1$, choose K so large that $a_k^p < e^{a_k}$ for all $k \geqslant K$.

Let $[a_k] = n$. Then when $k > K$,

$$I(n^p) - I(n+1) = \int_{n+1}^{n^p} \frac{dt}{t\,\theta(t)} = \int_{n+1}^{n^p} \frac{(a_k)^c\,dt}{(\log a_k)\,t^{1+c}}.$$

Hence

$$\theta(n+1)\,\{I(n^p) - I(n+1)\} = \frac{1}{c}\left\{1 - \frac{(n+1)^c}{n^{pc}}\right\}$$

$$\sim \frac{1}{c}$$

as k (and so n) tends to infinity.

Hence if $\frac{1}{2} < c < 1$, $\{I(n^p) - I(n+1)\}\,\theta(n+1)$ is not greater than 2, for all large n.

REFERENCES

1. S. M. SHAH and S. K. SINGH : The maximum term of an entire series, *Proc. Royal Soc.*, Edinburgh, 64 (A), (1954), 80-89.

2. J. CLUNIE : The behaviour of integral functions determined from their Taylor series, *Quart. J. Math.* (Oxford), 7 (1956), 175-182.

Muslim University
Aligarh

SOME CONGRUENCES INVOLVING RAMANUJAN'S FUNCTION $\tau(n)$

By P. J. McCARTHY

RAMANUJAN'S function $\tau(n)$ is defined by

$$\sum_{1}^{\infty} \tau(n)\, x^n = x \prod_{1}^{\infty} (1 - x^n)^{24} \quad (|x| < 1).$$

Our purpose is to obtain certain congruences involving $\tau(n)$. Such congruences have been obtained by several authors, and in particular, by Lahiri [1]. The congruences which he obtained for modulus 11 and modulus 13 were not true congruences for $\tau(n)$, for the coefficient of $\tau(n)$ contains, in the respective cases, the factor 11 or 13. In this note we obtain congruences for $\tau(n)$ for the moduli 11 and 13, and in addition obtain a congruence for $\tau(n)$ for the modulus 17.

Our method is standard. We make use of the expressions

$$P = 1 - 24 \sum_{1}^{\infty} \sigma(n)\, x^n,$$

$$Q = 1 + 240 \sum_{1}^{\infty} \sigma_3(n)\, x^n,$$

$$R = 1 - 504 \sum_{1}^{\infty} \sigma_5(n)\, x^n,$$

which were introduced by Ramanujan [3, p. 140]. Here $\sigma_k(n)$ is the sum of the kth powers of all the positive divisor of n : $\sigma(n) = \sigma_1(n)$. Ramanujan proved the identity [3, p. 144]

$$Q^3 - R^2 = 1728 \sum_{1}^{\infty} \tau(n)\, x^n.$$

From relations 3 and 5 of Table II on page 142 of [3] we have

$$5.1008 \sum_1^\infty n\, \sigma_5(n) x^n.\ Q = 5Q^3 - 5PQR$$

$$= 2(Q^3 - R^2) + 1584 \sum_1^\infty n\, \sigma_9(n)\, x^n.$$

Hence,

$$2.1728 \sum_1^\infty \tau(n)\, x^n = 5.1008 \sum_1^\infty n\, \sigma_5(n) x^n.\ Q - 1584 \sum_1^\infty n\, \sigma_9(n) x^n. \quad (*)$$

If we now use the fact that $2.1728.6 \equiv 1 \pmod{11}$, and reduce all coefficients modulo 11, we have

$$\sum_1^\infty \tau(n) x^n \equiv \sum_1^\infty n\, \sigma_5(n)\, x^n \left(1 + 9 \sum_1^\infty \sigma_3(n)\, x^n \right) \pmod{11}.$$

If we now compare the coefficients of x^n we obtain

$$\tau(n) \equiv n\, \sigma_5(n) + 9\, S_{5,3}(n) \pmod{11},$$

where

$$S_{r,s}(n) = \sum_1^{n-1} k\, \sigma_r(n)\, \sigma_s(n-k),\ n > 1\,;\ S_{r,s}(1) = 0.$$

To obtain such a congruence for the modulus 13, we use the relation

$$6(R^2 - I) = -3 \sum_1^\infty \tau(n)\, x^n + 13\, I,$$

where I is a power series in x with integral coefficients. This relation may be found on page 886 of [2]. Now,

$$R^2 \equiv 1 - 7 \sum_1^\infty \sigma_5(n)\, x^n + 9 \sum_1^\infty T_5(n)\, x^n \pmod{13},$$

where

$$T_r(n) = \sum_1^{n-1} \sigma_r(n)\, \sigma_r(n-k),\ n > 1,\ T_r(1) = 0.$$

Using the fact that $-3.4 \equiv 1 \pmod{13}$, we have

$$\sum_1^\infty \tau(n) x^n \equiv 24 \left(-7 \sum_1^\infty \sigma_5(n)\, x^n + 9 \sum_1^\infty T_5(n)\, x^n \right) \pmod{13}.$$

Then, upon comparing coefficients of x^n, and reducing these modulo 13, we obtain

$$\tau(n) \equiv \sigma_5(n) + 8T_5(n) \pmod{13}.$$

From the sixth relation in Table III on page 888 of [2] we have

$$8(Q^3 - R^2) + 14R^2 + 3 \equiv \sum_1^\infty \sigma_{11}(n)\, x^n \pmod{17}.$$

Hence, using the fact that $8.15 = 1 \pmod{17}$,

$$\sum_1^\infty \tau(n)\, x^n \equiv 11R^2 + 6 + 15 \sum_1^\infty \sigma_{11}(n)\, x^n \pmod{17},$$

Since

$$R^2 \equiv 1 - 5 \sum_1^\infty \sigma_5(n)\, x^n + 2 \sum_1^\infty T_5(n)\, x^n \pmod{17},$$

we have

$$\sum_1^\infty \tau(n)\, x^n \equiv 13 \sum_1^\infty \sigma_5(n)\, x^n + 5 \sum_1^\infty T_5(n)\, x^n + 15 \sum_1^\infty \sigma_{11}(n)\, x^n \pmod{17}.$$

Comparison of coefficients of x^n gives

$$\tau(n) \equiv 13\, \sigma_5(n) + 15\, \sigma_{11}(n) + 5T_5(n) \pmod{17}.$$

Another interesting congruence can be obtained from the relation (*). If we use the fact that $5.1008.240 = 2^8.3^3.5^2.7$, we obtain

$$24\, \tau(n) \equiv 35n\, \sigma_5(n) - 11n\, \sigma_9(n) \pmod{2^4.3.5^2.7}.$$

REFERENCES

1. D. B. Lahiri : On Ramanujan's function $\tau(n)$ and the divisor function $\sigma_k(n)$. II, *Bull. Calcutta Math. Soc.* 39, (1947), 33-52.

2. W. H. Simons : Congruences involving the partition function $p(n)$, *Bull. American Math. Soc.* 50, (1944), 883-892.

3. S. Ramanujan : *Collected Papers*, Cambridge, (1927).

Florida State University
Tallahasse, Fa.

ODD PERFECT NUMBERS

By M. SATYANARAYANA

1. Introduction. It is well known that an even number is perfect if and only if it is of the form

$$2^{n-1}(2^n - 1),$$

where $2^n - 1$ is a prime. It is not yet known if odd perfect numbers exist. Jacques Touchard [2] has shown that if odd perfect numbers exist, they must be found among numbers of the form $12n + 1$ and $36n + 9$.

In this note, I show that if any odd perfect number exists it must be of the form

$$p^{4k+1} N^2,$$

where p is a prime of the form $4j + 1$ and $(N, p) = 1$.

We deduce Touchard's result from this with the help of a congruence property of $\sigma(n)$ due to Ramanathan [1].

2. In what follows p's denote odd primes. Let

$$p_1^{\beta_1} p_2^{\beta_2} \dots p_r^{\beta_r}$$

be the canonical form for an odd perfect number M. Then

$$\sigma(M) = \sigma(p_1^{\beta_1}).\, \sigma_2^{\beta_2} \dots \sigma(p_r^{\beta_r}) = 2M.$$

Since $\sigma(M) \equiv 0 (\text{mod } 2)$ but $\not\equiv 0 (\text{mod } 4)$ all the σ's on the right except one, say $\sigma(p_1^{\beta_1})$, must be odd.

Now

$$\sigma(p^\beta) = 1 + p + p^2 + \dots + p^\beta \equiv \beta + 1 \cdot (\text{mod } 2).$$

Therefore $\beta_2, \dots, \beta_r$ must be even.

Since $\sigma(p_1^{\beta_1}) \equiv 0 (\text{mod } 2)$ but $\not\equiv 0 (\text{mod } 4)$, β_1 is odd. Also because

$$\frac{p^{\beta_1 + 1} - 1}{(p_1 - 1)} = (p_1 + 1)(1 + p_1^2 + p_1^4 + \dots p_1^{\beta_1 - 1}),$$

we must have $p_1 + 1 \equiv 0 (\text{mod } 2)$ but $\not\equiv 0 (\text{mod } 4)$ and

$$1 + p_1^2 + \ldots + p_1^{\beta_1 - 1}$$

must be odd.

Hence p_1 must be of the form $4j + 1$ and $\frac{\beta_1 - 1}{2} \equiv 0 \pmod 2$, i. e. $\beta_1 = 1 \pmod 4$.

This proves the statement in § 1.

3. Evidently M in § 2 is of the form $4t + 1$.

The only admissible forms for M are, therefore,

$$12n + 1,\ 12n + 5,\ 12n + 9.$$

Ramanathan has shown that

$$\sigma(3m - 1) \equiv 0 \pmod 3 \text{ for } m > 1.$$

No perfect number can thus be of the form $3m - 1$, for if it were so then we would have

$$\sigma(3\,m - 1) = 2(3\,m - 1) \equiv 1 \pmod 3.$$

Hence numbers of the form $12n + 5$ are ruled out.

Moreover in the canonical decomposition of M, primes of the form $4j + 3$ occur only in even powers, therefore if

$$M = 0 \pmod 3), \text{ it must be} \equiv 0 \pmod 9).$$

Hence M must be of the form $12n + 1$ or $36n + 9$. Since odd powers of numbers of the form $(12j \pm 5)$ are also of the form $12j \pm 5$, while even powers of all numbers prime to 12 are of the form $12j + 1$, in the canonical decomposition of M, as in § 2,

$$p_1 \equiv 1 \pmod{12}, \text{ when } M \equiv 1 \pmod{12}.$$

REFERENCES

1. K. G. Ramanathan : Congruence properties of $\sigma(N)$, *Math. Student*, 11 (1943), 33-35.

2. J. Touchard : On prime numbers and perfect numbers, *Scripta Mathematica*, 19 (1953), 35-39.

S. V. University
Tirupati

SOME PROPERTIES OF FIBONACCI NUMBERS, II

By K. SUBBA RAO

IN this paper, I follow the notation of my recent papers ([2], [3]). It is known [1] that if u_n is the nth Fibonacci number, then

$$V_p = u_{n+p} + (-1)^p u_{n-p} \equiv 0 \pmod{u_n}, = v_p u_n, \text{ say.} \tag{1}$$

Putting $p = 1, 2, 3, \ldots$ we see that $V_1, V_2, V_3, V_4, \ldots$ are respectively equal to $u_n, 3u_n, 4u_n, 7u_n, \ldots$, which is the series of Lucas. That this is a recurring series of the Fibonacci type can be easily proved by using (1) and by induction on p. Further, it is easily seen that the pth term of the Lucas series 1, 3, 4, 7, ... is equal to

$$\left(\frac{1+\sqrt{5}}{2}\right)^p + \left(\frac{1-\sqrt{5}}{2}\right)^p$$

More generally, considering the Fibonacci series of the type $a, b, a+b, a+2b, 2a+3b, \ldots$, where a and b are arbitrary positive integers and denoting the nth term of this series by U_n, we can show that

$$U_{n+p} + (-1)^p U_{n-p} = v_p u_n.$$

Many results analogous to those proved in ([2], [3]) hold good in respect of the Lucas series.

I now prove some theorems of a general nature concerning Lucas numbers.

THEOREM 1. *If v_n be the n-th Lucas number, then for $k > 1$ and $m > m_0$,*

(i)
$$v_{2km} < v_{2m}^k < v_{2km+1} \cdots, \tag{2}$$

$$v_{2km+k-1} < v_{2m+1}^k < v_{2km+k} \cdots; \tag{3}$$

(ii) *there lie exactly $(k-1)$ Lucas numbers between v_{2m}^k and v_{2m+1}^k and $(k+1)$ Lucas numbers between v_{2m+1}^k and v_{2m+2}^k.*

PROOF OF (i). We have $v_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n = a^n + b^n$, say; where a is therefore positive, b is negative and $|b| < a$. Therefore

$$v_{2m}^k = (a^{2m} + b^{2m})^k = a^{2km} + b^{2km} + \binom{k}{1} a^{2(k-1)m} b^{2m} + \\ + \dots + \binom{k}{k-1} a^{2m} b^{2(k-1)m},$$

$$v_{2km} = a^{2km} + b^{2km}, \quad v_{2km+1} = a^{2km+1} + b^{2km+1}.$$

Thus $v_{2m}^k - v_{2km} > 0$, and

$$\frac{v_{2km+1}}{v_{2m}^k} = \frac{a^{2km+1} + b^{2km+1}}{a^{2km} + \binom{k}{1} a^{2(k-1)m} b^{2m} + \dots + b^{2km}}$$

$$= \frac{a\left\{1 + \left(\frac{b}{a}\right)^{2km+1}\right\}}{1 + \binom{k}{1}\left(\frac{b}{a}\right)^{2m} + \dots + \left(\frac{b}{a}\right)^{2km}}$$

$$\to a > 1, \text{ as } m \to \infty.$$

Therefore $\frac{v_{2km+1}}{v_{2m}^k} > 1$ for $m > m_1$. Hence (2) is proved. Similarly (3) can be proved for $m > m_2$. Thus for $m > m_0 = \max(m_1, m_2)$, (2) and (3) are true.

PROOF OF (ii). We have, by (i), for $m > m_0$, that (2) and (3) hold. Replacing m by $m + 1$ in (2) we get

$$v_{2km+2k} < v_{2m+2}^k < v_{2km+2k+1} \cdots \tag{4}$$

From (2), (3) and (4), (ii) follows.

COROLLARY. *Between v_m^k and v_{m+2}^k, $(m > m_0)$, there lie exactly $2k$ Lucas numbers and, more generally, between v_m^k and v_{m+2p}^k, $(m > m_0)$, there lie $2kp$ numbers.*

THEOREM 2. *Given a prime p, there are infinitely many Lucas numbers each of which $\equiv 1 \pmod p$.*

PROOF. From the identity

$$a^n + b^n = n\left[\frac{(a+b)^n}{n} - \binom{n-1}{1}\frac{(a+b)^{n-2}}{(n-1)} ab + \\ + \binom{n-2}{2}\frac{(a+b)^{n-4}}{(n-2)}(ab)^2 - \dots\right],$$

where n is a positive integer, putting

$$a = \frac{1+\sqrt{5}}{2},\ b = \frac{1-\sqrt{5}}{2} \text{ and therefore } a+b=1,\ ab=-1$$

we have

$$v_n = n\left[\frac{1}{n} + \frac{\binom{n-1}{1}}{(n-1)} + \frac{\binom{n-2}{2}}{(n-2)} + \frac{\binom{n-3}{3}}{(n-3)} + \ldots\right]$$

$$= 1 + n\left[\frac{\binom{n-1}{1}}{(n-1)} + \frac{\binom{n-2}{2}}{(n-2)} + \ldots\right].$$

Therefore

$$v_{p^\alpha} = 1 + p^\alpha\left[\frac{\binom{p^\alpha-1}{1}}{(p^\alpha-1)} + \frac{\binom{p^\alpha-2}{2}}{(p^\alpha-2)} + \ldots\right],$$

where p is the given prime and α is an arbitrary positive integer. Since v_{p^α} is an integer,

$$p^\alpha\left[\frac{\binom{p^\alpha-1}{1}}{(p^\alpha-1)} + \frac{\binom{p^\alpha-2}{2}}{(p^\alpha-2)} + \ldots\right]$$

should be an integer.

Since the highest power of p which can divide the denominators of the several terms in the square brackets is $p^{\alpha-1}$, it follows that

$$p^\alpha\left[\frac{\binom{p^\alpha-1}{1}}{(p^\alpha-1)} + \ldots\right]$$

is divisible by p. Hence $v_{p^\alpha} \equiv 1 \pmod{p}$ and the theorem is proved.

COROLLARY. $v_{p^\alpha} \equiv v_{p^\beta} \pmod{p}$ *for arbitrary positive integers* α *and* β.

To find the last digit of a Lucas number, we use the fact that the residues of the Lucas numbers modulo 10 recur periodically, in the order :

1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2.

The length of the period is thus 12 and the last digit of any Lucas number can be obtained from the following table :

Rank of Lucas Number	$(12n+1)$ and $(12n+5)$	$12n$	$(12n+2)$ and $(12n+10)$	$(12n+3)$	$(12n+9)$	$(12n+4)$ and $(12n+8)$	$(12n+6)$	$(12n+7)$ and $(12n+11)$
Last digit of Lucas Number	1	2	3	4	6	7	8	9

It is curious to note that no Lucas number is divisible by 5.

I give now a new proof of the following well-known

THEOREM 3. *The number of primes is infinite.*

In proving this I require the following

LEMMA. *No two Lucas numbers of rank 2^n have a common factor greater than* 1.

PROOF. The Lucas numbers $v_2, v_4, \dots, v_{2^n}$, are all odd. Also, on account of the identity

$$v_{2m} + 2(-1)^m = v_m^2,$$

we have, if $v_{2m} \equiv 0 \pmod p$, where p is any odd prime,

$$v_{2^{m+1}} \equiv -2 \pmod p,\ v_{2^{m+k}} \equiv 2 \pmod p,$$

where k is a positive integer > 1. It follows that any common odd prime factor of v_{2m} and v_{2m+1} should divide 2 and also that of v_{2m} and v_{2m+1} should divide 2. Since $v_2, v_4, \dots$ are odd ; it follows that no two numbers of the form v_{2^n} have a common divisor other than 1.

PROOF OF THOREM 3. By the above lemma, each of the numbers $v_2, v_4, \dots, v_{2^n}$, is divisible by an odd prime which does not divide any of the others. Therefore there are at least n odd primes not exceeding v_{2^n}. This proves the theorem.

REFERENCES

1. L. E. Dickson : *History of the theory of numbers,* Vol. I. Ch. XV.

2. K. Subba Rao : Some properties of Fibonacci numbers, *American Math. Monthly,* 60 (1953), 680-684.

3. K. Subba Rao : Some properties of Fibonacci numbers—I, *Bull. Calcutta Math. Soc.* 46 (1954), 253-257.

Maharaja's College
Vizianagaram

ON AN ENTIRE FUNCTION DEFINED BY A GAP DIRICHLET SERIES

By K. N. SRIVASTAVA

INTRODUCTION. The sequence $\{\lambda_n\}$ will have, throughout the present paper, the following properties :

(a) $\{\lambda_n\}_1^\infty \uparrow \infty, \lambda_1 > 0.$

(b) $\liminf_{n \to \infty} \lambda_{n+1} - \lambda_n = h > 0.$

(c) Let $\nu(x)$ be the greatest of n such that $\lambda_n < x$. Then $\nu(x)$ will be called the distribution function of the sequence $\{\lambda_n\}$. It is such that $\nu(x) = 0$, for $x \leqslant \lambda_1$. The quantity

$$\limsup_{x \to \infty} \frac{\nu(x)}{x} = D$$

is called the upper density of the sequence $\{\lambda_n\}$. We suppose that D is finite.

Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(\lambda_n s)$, where $s = \sigma + it$, be a Dirichlet series convergent in the whole plane. It is interesting to note that, since $D < 1/h$, the series which represents $f(s)$ will be absolutely convergent in the whole plane, since, according to a well-known result, a Dirichlet series whose exponents form a sequence of finite upper density has its abscissa of convergence equal to its abscissa of absolute convergence [1]. Therefore, $f(s)$ is an entire function. For this class of functions, for any given $\mathrm{Re}(s) = \sigma$, $\limsup_{-\infty < t < \infty} |f(\sigma + it)|$ has a finite value $M(\sigma, f)$. After Ritt [3], we define the order of $f(s)$ in the following way:

$$\rho = \limsup_{\sigma \to \infty} \frac{\log \log M(\sigma, f)}{\sigma},$$

and the lower order of $f(s)$ is defined as

$$\lambda = \limsup_{\sigma \to \infty} \frac{\log \log M(\sigma, f)}{\sigma}.$$

Since the series $f(s)$ is absolutely convergent in the whole plane, for any given $\text{Re}(s) = \sigma$, there is at least one term of the series whose modulus is greater than that of all the other terms. We denote this term by $\mu(\sigma, f)$. When more than one term of the series are in modulus equal to $\mu(\sigma, f)$, we shall agree to regard the term with the greatest value of λ_n amongst them as the maximum term ; with this convention $\lambda_{N(\sigma,f)} = \lambda_N$ will be called indicative index, as it denotes the index $N(\sigma, f)$ of the maximum term.

1. An entire Dirichlet series of order infinity.

1.1. If $\rho = \infty$, the function $f(s)$ is of infinite order.

Following a procedure similar to Hoing [2], it may be proved that in this case there always exists a function $W(\sigma)$ with the following properties :

$$W\left\{\sigma - \frac{1}{\log W(\sigma)}\right\} < [W(\sigma)]^{1+\epsilon(\sigma)},$$

where $\epsilon(\sigma)$ tends to zero as σ tends to infinity, and

$$\limsup_{\sigma\to\infty} \frac{\log\log M(\sigma, f)}{\log W(\sigma)} = 1,$$

and finally $\log W(\sigma)$ is a convex function of σ. Any function with the above properties is called an order of $f(s)$.

In another note [4] I have proved

$$\liminf_{\sigma\to\infty} \frac{\log \mu(\sigma, f)}{\lambda_{N(\sigma,f)}} \leqslant \frac{1}{\rho} \leqslant \frac{1}{\lambda} \leqslant \limsup_{\sigma\to\infty} \frac{\log \mu(\sigma, f)}{\lambda_{N(\sigma,f)}}$$

Hence for a function of infinite order we have

$$\liminf_{\sigma\to\infty} \frac{\log \mu(\sigma, f)}{\lambda_{N(\sigma,f)}} = 0.$$

Moreover, according to Sugimura ([5], Theorem 5), as D is finite,

$$\log \mu(\sigma, f) = (1 - \epsilon(\sigma)) \log M(\sigma, f),$$

and consequently we shall have

$$\liminf_{\sigma\to\infty} \frac{\log M(\sigma, f)}{\lambda_{N(\sigma,f)}} = 0.$$

It is possible to refine this result and to prove

THEOREM 1. *If* $M(\sigma, f^p) = \limsup_{-\infty<t<+\infty} |f^p(\sigma + it)|$, *where* $f^p(s)$ *denotes the p-th derivative of* $f(s)$ *and* p *is any function of*

$$\lambda_{N(\sigma,f)} = \lambda_{N(\sigma)}, \quad p(\lambda_N) = o\,(\lambda_{N/\log}\,\lambda_N),$$

then

$$\liminf_{\sigma\to\infty} \frac{\log M(\sigma, f^p)}{\lambda_{N(\sigma,f)}} = 0. \tag{1.1.1}$$

Before coming to the proof, we shall establish a result that we require for its proof.

1.2. THE ORDINARY INTERVALS. The argument of this section is similar to that of Valiron ([5], pp. 93-95).

The absolute convergence of $f(s)$ in the whole plane requires

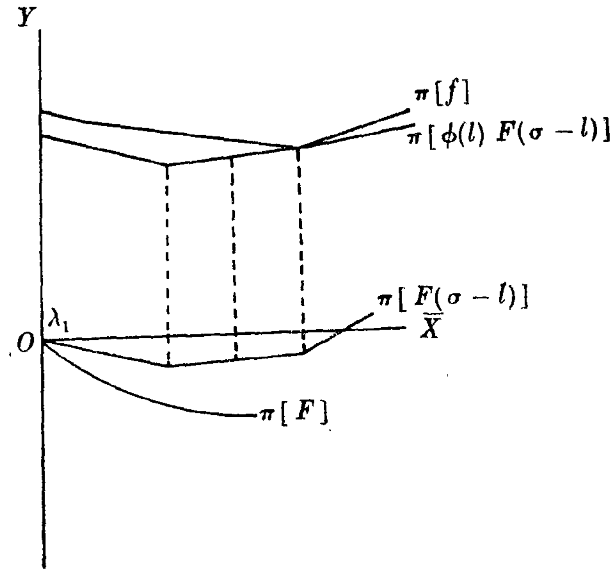
$$\limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n} = -\infty.$$

As pointed out by Yung [7], we can construct a Newton's polygon with the help of the coefficients. We shall compare it with a polygon corresponding to a function of simple growth. This amounts to the comparing of the coefficients of the two functions.

Let $F(u)$ be a Dirichlet series in the real variable u with positive unbounded coefficients,

$$F(u) = \sum_{n=1}^{\infty} \exp\,[H(\lambda_n) + \lambda_n u]. \tag{1.2.1}$$

This series is convergent for $u < 0$, where the numbers $H(\lambda_n)$ tend to infinity and $H(x)/x$ tends to zero as x tends to infinity. It is clear that the points A_n, $x_n = \lambda_n$, $y_n = -H(\lambda_n)$ are the vertices of a polygon concave in the positive direction of y-axis. Let this polygon be denoted by $\pi(F)$; for every $u < 0$, $F(u)$ has a maximum term which can be determined by finding the tangent to $\pi(F)$ of slope u.



Now let l be any number and σ be a quantity less than l, such that $\sigma - l < 0$. To the series $F(\sigma - l)$ there corresponds a polygon obtained by adding $\lambda_n l$ to the ordinates of A_n. The slope of the sides of $\pi[F(\sigma - l)]$ is an increasing function tending to l. Since the slope of the sides of $\pi(f)$ tends to infinity, the polygon $\pi[F(\sigma - l)]$ lies below $\pi(f)$, hence as is evident from the figure, a translation of $\pi[F(\sigma - l)]$ parallel to OY can be effected, so that, in this new position no vertex of $\pi[F(\sigma - l)]$ lies above the corresponding vertex of $\pi(f)$, while the two polygons have at least one common vertex. If we denote this translation by $[-\log\phi(l)]$, the polygon in its new position corresponds to the coefficients of $[\phi(l)\,F(\sigma - l)]$ regarded as a function of σ, it will be denoted by $\pi[\phi(l)\,F(\sigma - l)]$.

The polygons $\pi(f)$, $\pi[\phi(l)\,F(\sigma - l)]$ have one or more common vertices. Let $\lambda_{n(l,F)}$ be the greatest of the abscissa of these vertices. Now at the common vertex every tangent to $\pi[\phi(l)\,F(\sigma - l)]$ is also a tangent to $\pi(f)$. In particular the line of slope given by the equation

$$\sigma(l) = l - H'[\lambda_{n(l,F)}],$$

where

$$H'(x) = \frac{d}{dx}[H(x)],$$

will be a common tangent to $\pi[\phi(l) F(\sigma - l)]$ and $\pi(f)$ at the point of abscissa $\lambda_{n(l,F)}$. For this value $\sigma(l)$ of σ the maximum terms of the two functions $f(s)$ and $[\phi(l)F(\sigma - l)]$ are equal and of the same order of magnitude and have the same indicative index $\lambda_{n(l,F)}$ while the second function dominates the first.

If l increases steadily, $\lambda_{n(l,F)}$ is non-decreasing, for the slope of sides of $[\phi(l) F(\sigma - l)]$ increases indefinitely with l. Thus $\lambda_{n(l,F)}$ cannot be bounded. Similarly $\phi(l)$ cannot be bounded, hence it is an unbounded increasing function. $H'(\lambda_{n(l,F)})$ is thus a decreasing discontinuous function of l which tends to zero as l tends to infinity. Hence it follows that $\phi(l)$ is an unbounded non-decreasing function of l and its only discontinuities are those of $H'(l)$. These occur where l is such that $\lambda_{n(l,F)}$ is discontinuous and correspond to those values of l for which the ploygons $\pi(f)$ and $\pi[\phi(l) F(\sigma - l)]$ have several common vertices. The total number of such discontinuities between 0 and $\sigma(l)$ cannot exceed the total variation of $H'(x)$ and this is finite. Hence we have

THEOREM 2. *Given an entire Dirichlet series $f(s)$ and a series $F(u)$, we can, in general find two numbers l and $\phi(l)$ corresponding to a given value of σ, such that for this value of σ, the maximum terms of the two functions $f(s)$ and $[\phi(l)\ F(\sigma - l)]$ are equal and have the same indicative index, and the first function dominates the second. The values of σ in the segment $(0, \sigma)$ for which this property does not hold good, constitute a set of not more than $N(\sigma)$ intervals, where $N(\sigma)$ corresponds to $\lambda_{N(\sigma,f)} = \lambda_{N(\sigma)}$ and the measure of these intervals is finite.*

Those values of σ, for which this property holds, shall be called ordinary values. It is of course understood that they are ordinary with respect to a given function $F(u)$. Values which are not ordinary are called exceptional.

1.3. We are now in a position to prove Theorem 1. We require the following lemmas :

LEMMA 1. *If $\mu(\sigma, f)$ is the maximum term of $f(s)$ for* $\mathrm{Re}(s) = \sigma$, *then*

$$\liminf_{\sigma\to\infty} \frac{\log \mu(\sigma, f)}{\lambda_{N(\sigma,f)}} = 0. \tag{1.3.1}$$

LEMMA 2. *If $f(s) = \sum_{n=1}^{\infty} a_n \exp(\lambda_n s)$ then $\phi(s) = \sum_{n=1}^{\infty} [a_n/\lambda_n \exp(\lambda_n s)]$, is also an integral function, whose central indicative indices are a sub-sequence of those of $f(s)$. If $p(\lambda_n) = o(\lambda_n/\log \lambda_n)$, then for some large ρ*

$$M(\rho, f^p) < K[\lambda_N^{2p}\, \mu(\rho, f)], \tag{1.3.2}$$

where $\lambda_N(\rho, f) = \lambda_N$ is also the central indicative index of $\phi(s)$.

PROOF. It is clear that $\phi(s)$ is also an integral function, so that, we have for $\lambda_N = \lambda_N(\sigma, \phi)$,

$$|a_n| \exp(\lambda_n \sigma)/\lambda_n \leqslant a_N \exp(\lambda_N \sigma)/\lambda_N,$$

or

$$\frac{|a_n| \exp(\lambda_n \sigma)}{|a_N| \exp(\lambda_N \sigma)} \leqslant \frac{\lambda_n}{\lambda_N}.$$

Choose $R < 0$, such that

$$\lambda_n \exp(\lambda_n R) \leqslant \lambda_N \exp(\lambda_N R)$$

which is possible for all λ_n. Hence we have

$$\frac{|a_n| \exp[\lambda_n(\sigma + R)]}{|a_N| \exp[\lambda_n(\sigma + R)]} \leqslant \frac{\lambda_n \exp(\lambda_n R)}{\lambda_N \exp(\lambda_N R)}.$$

So that if

$$F(s_1) = \sum_{n=1}^{\infty} \lambda_n \exp(\lambda_n s_1), \text{ where } s_1 = R + it,\ R < 0$$

we have

$$\mu(\sigma + R, f) = |a_N| \exp[\lambda_N(\sigma + R)] = \mu(\sigma, \phi).\ \mu(R, F);$$

hence it follows that the central indices of $\phi(s)$ form a sub-sequence of those of $f(s)$. Hence

$$\frac{\lambda_n^p\,|a_n|\exp\left[\lambda_n(\sigma+R)\right]}{\lambda_N^p\,|a_N|\exp\left[\lambda_N(\sigma+R)\right]}\leqslant\frac{\lambda_n^{p+1}\exp(\lambda_n R)}{\lambda_N^{p+1}\exp(\lambda_N R)}$$

and the lemma is proved, if we show that

$$\sum_{n=1}^{\infty}\lambda_n^{p+1}\exp(\lambda_n R)<K[\lambda_N^{2p}\;\mu(R.F)]. \tag{1.3.3}$$

To prove this, we see that since $p(\lambda_n)=o(\lambda_n/\log\lambda_n)$ the series (1.3.3) is convergent for $R<0$. We express this as a definite integral in the following way :

$$\sum_{n=1}^{\infty}(\lambda_n)^{p+1}\;\;e^{(\lambda_n R)}=\int_0^{\infty}x^{p+1}\,e^{xR}\,d\nu(x).$$

Let $R=-Y$, $Y>0$, then

$$\sum_{n=1}^{\infty}(\lambda_n)^{p+1}\,e^{(\lambda_n R)}=\int_0^{\infty}x^{p+1}\,e^{-xY}\,d\nu(x).$$

If ξ is the value of x for which $[-xY+\log x]$ is a maximum, so that $|\xi-\lambda_n|<0$, then with $p(\lambda_n)=o(\lambda_n/\log\lambda_n)$, we have

$$\int_0^{\infty}x^{p+1}\;\exp(-xY)\;d\nu(x)=\int_0^{\infty}\frac{\nu(x)}{x}\;[x^{p+2}\;Y-(p+1)\,x^{p+1}]\;\;e^{-xY}.\,dx$$

$$<D\int_0^{\infty}[x^{p+2}\;Y-(p+1)\,x^{p+1}]\;e^{-xY}\,dx$$

$$=\frac{D.\,(p+1)!}{Y^{p+2}}=D.\,(p+1)!\;\xi^{p+2}$$

$$<K\;[\lambda_N^{2p}\,\mu\,(R.\,F)]$$

since $Y=1/\xi$, $\limsup\limits_{x\to\infty}\dfrac{\nu(x)}{x}=D$ (finite). The lemma is thus proved.

LEMMA 3. *If* $\lambda_{N(\sigma,\phi)}=\lambda_{N(R,F)}=\lambda_{N(\sigma+R,f)}$, *then*

$$\liminf_{\sigma\to\infty}\frac{\log\mu\,(\sigma,\phi)}{\lambda_{N(\sigma,\phi)}}=0.$$

The result now follows immediately, for choose $\mathrm{Re}(s)=\sigma$ to which Lemma 3 applies, then we have

$$\liminf_{\sigma\to\infty} \frac{\log M(\sigma, f)}{\lambda_{N(\sigma,f)}} \leqslant \liminf_{N\to\infty} \frac{2p \log \lambda_N}{\lambda_N} + \liminf_{\sigma\to\infty} \frac{\log \mu(\sigma, f)}{\lambda_{N(\sigma,f)}}.$$

This is zero by the choice of p and Lemma 3.

1.4. THEOREM 3. *Given any increasing function $\phi(x)$ tending to infinity (however rapidly) with x then there are entire Dirichlet series $f(s)$ and $F(s)$ both of infinite order, such that*

$$\liminf_{\sigma\to\infty} \frac{\phi(\sigma) \log M(\sigma, f)}{\lambda_{N(\sigma,f)}} = 0, \tag{1.4.1}$$

$$\limsup_{\sigma\to\infty} \frac{\log M(\sigma, f)}{\lambda_{N(\sigma,f)}} = 1, \tag{1.4.2}$$

$$\limsup_{\sigma\to\infty} \frac{\log M(\sigma, F)}{\phi[\sigma \lambda_{N(\sigma,F)}]} = \infty. \tag{1.4.3}$$

PROOF. We may suppose that $\phi(x) \geqslant 2$ for $x \geqslant 1$ and

$$\frac{\log \phi(x)}{x} \to \infty \text{ as } x \to \infty.$$

Let $\mu_1 = 1 + e^e$, $\mu_{n+1} = \exp \exp(\mu_n)$ and let β_n be a rapidly increasing sequence of positive numbers such that $\beta_0 = 1$, and

$$\beta_n \geqslant 2 \frac{\log \phi(\mu_n)}{\mu_n} + \beta_{n-1}, C_n = \exp[\beta_n \log \mu_n], n = 1, 2, \ldots$$

and

$$f(s) = \sum_{n=1}^{\infty} \left[\frac{e^s}{\mu_n}\right]^{C_n}$$

It is easily seen that $f(s)$ is an entire Dirichlet series of order infinity. Let

$$\sigma_n = \log \left[\mu_n \left\{1 + \frac{1}{\phi(\mu_n)\, ll\, \mu_n}\right\}\right],$$

then for $n > n_0$, $\lambda_N = \lambda_{N(\sigma_n, f)} = C_n$,

$$\left[\frac{e^{\sigma_n}}{\mu^n}\right]^{C_n} \leqslant M(\sigma_n, f) < n(e^{\sigma_n})^{C_{n-2}} + \left(\frac{e^{\sigma_n}}{\mu_{n-1}}\right)^{C_{n-1}} + \left(\frac{e^{\sigma_n}}{\mu_n}\right)^{C_n}.$$

Hence $\log M(\sigma_n, f) \sim C_n/\phi(2\mu_n)\, ll\, \mu_n$, hence

$$\frac{\phi(\sigma_n)\log M(\sigma_n, f)}{\lambda_{N(\sigma_n,f)}} \sim \frac{\phi(2\mu_n)}{\phi(2\mu_n)\, ll\, \mu_n}, \to 0 \text{ as } n \to \infty,$$

where $ll\,x = \log\log x$.

Further it can be shown that $\log M(\sigma, f) \sim \log \mu(\sigma, f)$ and for $\sigma = \mu_{n+1}$,

$$\frac{\log M(\sigma, f)}{\lambda_{N(\sigma,f)}} \sim \frac{C_n(\mu_{n+1} - \log\mu_n)}{\mu_{n+1}\,.\,C_n} \to 1 \text{ as } n \to \infty.$$

To prove 1.43, let $\xi_n = n^n$, $\theta_1 = 2$, $k_n = \exp\,[n\phi\{(\theta_n \log \xi_{n+1})/\sqrt{n}\}]$,

$$\theta_n = \exp\,(k_n + \theta_n + 1),\ n = 1, 2, \ldots,$$

$$\alpha_{n,m} = \frac{1}{(n+m)(m+\theta_n)},\ m = 1, 2, \ldots, k_n,$$

$$a_{n,m} = \frac{\xi_{n+1}}{\sqrt{n}}\cdot\frac{1}{1-\alpha_{n,m}}$$

and

$$F(s) = \sum_{n=1}^{\infty}\left[\left(\frac{e^s}{\xi_n}\right)^{\theta_n} + \left(\frac{e^s}{a_{n,1}}\right)^{1+\theta_n} + \ldots + \left(\frac{e^s}{a_{n,k_n}}\right)^{k_n+\theta_n}\right].$$

The function $F(s)$ is an entire Dirichlet series of order infinity. Further for $\sigma = \log(\xi_{n+1}/\sqrt{n})$, we have

$$M(\sigma, F) \geqslant \left[\left(1 - \frac{1}{(n+1)(1+\theta_n)}\right)^{1+\theta_n} + \ldots + \right.$$

$$\left. + \left(1 - \frac{1}{(k_n+n)(k_n+\theta_n)}\right)^{k_n+\theta_n}\right]$$

$$\geqslant e^{-1/n} + \ldots + e^{-1/(k_n+n+1)} > k_n\, e^{-1/n},$$

hence

$$\frac{\log M(\sigma, F)}{\phi(\sigma\,.\,\lambda_{N(\sigma,F)})} = \frac{\log k_n + o(1)}{\phi[\theta_n\cdot\log(\xi_{n+1}/\sqrt{n})]} \to \infty \text{ as } n \to \infty.$$

2. On the growth of an entire Dirichlet series.

2.1. In this part we prove two theorems on the growth of an entire function defined by a gap Dirichlet series.

THEOREM 4. *Let* $f(s) = \sum_{n=1}^{\infty} a_n \exp(\lambda_n s)$ *be an entire Dirichlet series and* α *be any positive number. If* $[\lambda_{n+1}/\lambda_n]^{\alpha}. |a_n/a_{n+1}|$ *is ultimately an increasing function of* n, *then*

$$M(\sigma, f) < D(1 + o(1))\, \alpha^{-\alpha-1}\, e^{\alpha}\, \mu(\sigma, f)\, \lambda_{N(\sigma, f)}. \qquad (2.1.1)$$

THEOREM 5. *If* $f(s)$ *satisfies the conditions of Theorem* 1, *then*

$$\limsup_{\sigma\to\infty} \frac{\log M(\sigma, f)}{\log \mu(\sigma, f)} \leqslant 1 + \alpha^{-1} \qquad (2.1.2)$$

2.2. These theorems follow easily from Theorem 2. If $f(s)$ satisfies the conditions of Theorem 1, then

$$\phi_{\alpha}(s) = \sum_{n=1}^{\infty} \frac{a_n}{(\lambda_n)^{\alpha}} \exp(\lambda_n s)$$

is also an entire Dirichlet series, for which all λ_n beyond some definite λ_n are central indicative indices.

Let $\lambda_N = \lambda_{N(\sigma, \phi_{\alpha})}$, then

$$\frac{|a_n|\, e^{\lambda_n \sigma}}{\lambda_n^{\alpha}} \leqslant \frac{|a_N|\, e^{\lambda_N \sigma}}{\lambda_N^{\alpha}}$$

or

$$\frac{|a_n|\, e^{\lambda_n \sigma}}{|a_N|\, e^{\lambda_N \sigma}} \leqslant \frac{\lambda_n^{\alpha}}{\lambda_N^{\alpha}}.$$

For any λ_N, an $R < 0$ can be chosen, such that for all λ_n the inequality $(\lambda_n)^{\alpha}. e^{(\lambda_n R)} \leqslant (\lambda_N)^{\alpha}\, e^{(\lambda_N R)}$ is true.

Hence

$$\frac{|a_n| \exp \lambda_n (\sigma + R)}{|a_N| \exp \lambda_N (\sigma + R)} \leqslant \frac{(\lambda_n)^{\alpha}. \exp(\lambda_n R)}{(\lambda_N)^{\alpha}. \exp(\lambda_N R)}, \qquad (2.2.1)$$

which gives

$$\lambda_N(\sigma, \phi_{\alpha}) = \lambda_N(\sigma + R, f) = \lambda_N(R, F_{\alpha}), \qquad (2.2.2)$$

where $F_{\alpha}(R) = \sum_{n=1}^{\infty} (\lambda_n)^{\alpha}\, e^{\lambda_n R}$.

Further

$$\mu(\sigma + R, f) = |a_N| \exp [\lambda_n (\sigma + R)]$$
$$= \frac{|a_N|}{(\lambda_N)^\alpha} e^{\lambda_N \sigma} . \lambda_N^\alpha e^{\lambda_N R}$$
$$= \mu(\sigma, \phi_\alpha) \mu(R, F_\alpha). \tag{2.2.3}$$

Since all λ_n in turn become central indicative indices of both $\phi_\alpha(s)$ and $F_\alpha(R)$, it follows that the values $\sigma + R$ include all the numbers exceeding some definite bound. From (2.2.1) we have

$$\frac{\sum_{n=1}^{\infty} |a_n| e^{\{\lambda_n(\sigma+R)\}}}{|c_N| e^{\lambda_N(\sigma+R)}} \leqslant \frac{\sum_{n=1}^{\infty} (\lambda_n)^\alpha . e^{\lambda_n R}}{(\lambda_N)^\alpha e^{\lambda_N R}}.$$

Now let $R = -Y$, $Y > 0$, then

$$\sum_{n=1}^{\infty} (\lambda_n)^\alpha e^{(\lambda_n R)} = \sum_{n=1}^{\infty} (\lambda_n)^\alpha e^{-\lambda_n Y}$$
$$= \int_0^\infty x^\alpha e^{-xY} d\nu(x)$$
$$= \int_0^\infty \frac{\nu(x)}{x} (Yx^{\alpha+1} - x^\alpha) e^{-xY} dx$$
$$\leqslant D (\alpha + 1)!/Y^{\alpha+1}.$$

Hence

$$\frac{\sum_{n=1}^{\infty} |a_n| \exp \{\lambda_n (\sigma + R)\}}{|a_n| \exp \{\lambda_N (\sigma + R)\}} \leqslant \frac{D(1+\alpha)!}{Y^{\alpha+1} e^{-\lambda_N Y} \lambda_N^\alpha}. \tag{2.2.4}$$

Now $d(\alpha \log x + R x)/dx = \alpha/x + R$. Hence $x^\alpha \exp (xR)$ increases steadily till $xR = -\alpha$ and $\xi = x = -\alpha/R$ and then decreases steadily. Hence, if $\lambda_N = \lambda_N(R, F)$, then $|\lambda_N - \xi| \leqslant 0$. Therefore

$$\frac{\sum_{n=1}^{\infty} |a_n| \exp\{ \lambda_n(\sigma + R)\}}{|a_N| \exp \{\lambda_N(\sigma + R)\}} \leqslant D(1+\alpha)! \, e^\alpha \alpha^{-\alpha-1} \lambda_N. \tag{2.2.5}$$

Since, as has been observed, all large values are assumed by $\sigma + R$, from (2.2.2), (2.2.3) and (2.2.5) we get the result.

Also from (2.1.1)

$$\limsup_{\sigma\to\infty} \frac{\log M(\sigma, f)}{\log\ \mu(\sigma, f)} \leqslant 1 + \limsup_{\sigma\to\infty} \frac{\log\{\lambda_N(\sigma, f)\}}{\log\{\ \mu(\sigma, f)\}}.$$

From (2.2.2) and (2.2.3)

$$\frac{\log\{\lambda_N(\sigma + R, f)\}}{\log\{\ \mu(\sigma + R, f)\}} = \frac{\log\{\lambda_{N(R, F\alpha)}\}}{\log\{\ \mu(R,\ F_\alpha)\} + \log\{\mu(\sigma,\ \phi_\alpha)\}}$$

$$\leqslant \frac{\log\ \lambda_N(R,\ F_\alpha)}{\log \mu(R,\ F_\alpha)}.$$

But

$$\limsup_{R\to -0} \frac{\log\ \lambda_N(R,\ F_\alpha)}{\log\ \ \mu(R,\ F_\alpha)} \leqslant \limsup_{\lambda_N\to\infty} \frac{\lambda_N}{\alpha \log \lambda_N + \alpha} = 1/\alpha.$$

Hence

$$\limsup_{\sigma\to\infty} \frac{\log M(\sigma, f)}{\log\ \mu(\sigma, f)} \leqslant 1 + \alpha^{-1}.$$

I am thankful to Prof. P. L. Srivastava for his helpful criticism.

REFERENCES

1. V. Bernstein : *Lecons sur les progress recents de la theorie des series de Dirichlet*, Paris, (1933).
2. K. L. Hoing : Sur les fonctions entieres et les fonctions meromorphes d'ordre infini, *Journal de Mathematiques* (9) 14 (1935), 233.
3. J. M. Ritt : On certain points in the theory of Dirichlet series, *American J. Math.* 50 (1928), 73-86.
4. K. N. Srivastava : On maximum term of an entire Dirichlet series, *Proc. Nat. Acad. Sci. India.* 27 (1958).
5. K. Sugimura : Übertragung einiger Sätze and der Theorie der ganzen Funktionen auf Dirichletschen Reihen, *Math. Z.* 29 (1929), 264.
6. G. Valiron : *Lectures on integral functions*, New York, (1949).
7. Y. C. Yung : Sur les droites de Borel de certaines fonctions entieres, *Ann. Ecole Norm. Sup.*, 68 (1951), 65.

University of Allahabad
Allahabad

ON THE SEQUENCE $\{V_n\}$, $V_n = \sum_{1 \leqslant i \leqslant k} \alpha_i V_{n-i}$, $k \geqslant 2$

By U. V. SATYANARAYANA

LET $\{V_n\}$, $n \geqslant 1$, be a sequence defined by the recurrence relation $V_n = \sum_{1 \leqslant i \leqslant k} \alpha_i V_{n-i}$ where α_i, V_i, $1 \leqslant i \leqslant k$ are given real numbers with $\alpha_k \neq 0$.

For the case, where $k = 2$, $V_1 = 1$, $V_2 = 2$, $\alpha_1 = \alpha_2 = 1$, K. Subba Rao [3] proved that if μ be an integer > 1, there exist integers L and $N_0(\mu)$ such that

(a) $$V_{\left(\sum_{1 \leqslant i \leqslant \mu} n_i\right) - L} \lesseqgtr \prod_{1 \leqslant i \leqslant \mu} V_{n_i} \lesseqgtr V_{\left(\sum_{1 \leqslant i \leqslant \mu} n_i\right) - L + 1}$$

according as $r \gtrless 1$ for $n_i > N_0(\mu)$, $1 \leqslant i \leqslant \mu$; and

(b) m being any integer $\geqslant 1$, there are exactly $m\mu$ members of the sequence lying between V_n^μ and V_{n+m}^μ for $n > N_0(\mu)$. Theorem B below presents an extension of the results for $k \geqslant 2$ and a simultaneous widening of the context to sequences $\{V_n\}$ satisfying (1), (2), (3) and (4) of Theorem A below:

THEOREM A. *Suppose that numbers r, a_1, a_2, ..., a_{k-1} exist such that*

(1) $0 < r \neq 1$;

(2) $x^k - \alpha_1 x^{k-1} - \ldots - \alpha_k \equiv (x - r) f(x)$, *where*

$$f(x) \equiv x^{k-1} + a_1 x^{k-2} + \ldots + a_{k-1};$$

(3) *r is greater than the modulus of every root of $f(x) = 0$;*

and

(4) $\delta = V_k + \sum_{1 \leqslant i \leqslant k-1} a_i V_{k-i} > 0$.

Then the sequence $\{V_n\}$ is ultimately positive and increases strictly to $+\infty$ or decreases strictly to 0 according as $r > 1$ or $r < 1$

THEOREM B. *Suppose further that*

(5) *μ is an integer > 1 and is such that* $\frac{(1-\mu)\log A}{\log r}$, *where* $A = \frac{\delta}{r\, f(r)}$ *is not an integer or zero, and let* $L(\mu)$ *stand for* $\left[\frac{(1-\mu)\log A}{\log r}\right] + 1$.

Then integers L and $N_0(\mu)$ exist such that

(I) (a) $$V_{\left(\sum_{1\leqslant i\leqslant \mu} n_i\right)-L} \lessgtr \prod_{1\leqslant i\leqslant \mu} V_{n_i} \lessgtr V_{\left(\sum_{1\leqslant i\leqslant \mu} n_i\right)-L+1}$$

according as $r \gtrless 1$ *for* $n_i > N_0(\mu)$, $1 \leqslant i \leqslant \mu$;

(b) *L is unique and is equal to* $L(\mu)$;

and

(II) *m being any integer* $\geqslant 1$, *there are exactly* $m\mu$ *members of the sequence lying between* V_n^{μ} *and* V_{n+m}^{μ} *for* $n > N_0(\mu)$.

It may be noted that Subba Rao's proof of results I(a) and II depends implicity on the conditions (1) to (5) above.

M. Perisastry [2], attempting to generalize the results of Subba Rao, gave a proof in [2] of I(a) and II for the case $k = 2$, $\alpha_1 > 0$, $\alpha_2 > 0$, $V_1 > 0$, $V_2 > 0$ subject to conditions (1) to (4). His disregard of condition (5) vitiates the left part of the inequality in I(a) (and consequently the truth of II) as can be seen by the following example.

Take $V_n = 4(2)^n + (-1)^n$, for $n \geqslant 1$ and let μ be even. Then it can be easily verified that $\frac{V_n^{\mu}}{V_{n\mu-L}} < 1$ if n is odd and > 1 if n is even, for $n > N_0(\mu)$.

We shall first prove a lemma.

LEMMA. *Let* $r_1, r_2, \ldots, r_t$ *be the distinct roots of* $f(x) = 0$ *with multiplicities* $d_1, d_2, \ldots, d_t$ *respectively. Also let* $\Delta(x)$ *denote the determinant of order k whose first row is*

$$(x^{k-1}, x^{k-2}, \ldots, x, 1)$$

and whose i-th row, where $i = 1 + d_0 + d_1 + \dots + d_s + \sigma$, $0 \leqslant s \leqslant t-1$, $1 \leqslant \sigma \leqslant d_{s+1}$ *with* $d_0 = 0$, *is*

$$(k^{\sigma-1} r_{s+1}^{k-1}, (k-1)^{\sigma-1} r_{s+1}^{k-2}, \dots, j^{\sigma-1} r_{s+1}^{j-1}, \dots, 2^{\sigma-1} r_{s+1}, 1).$$

Then

$$\Delta(x) \equiv v(x-r_1)^{d_1}(x-r_2)^{d_2} \dots (x-r_t)^{d_t} \equiv v f(x),$$

where

$$v = \prod_{1 \leqslant i \leqslant t} \left[(-r_i)^{\frac{1}{2}(d_i-1)d_i} \left\{ \prod_{j=i+1}^{t} (r_i - r_j)^{d_j} \right\}^{d_i} \left\{ \prod_{m=1}^{(d_i-1)} m! \right\} \right] \neq 0.$$

PROOF. For any integer $k \geqslant 2$ let $G(x, \epsilon_1, \dots, \epsilon_{k-1})$ denote the determinant of order k whose first row is

$$(x^{k-1}, x^{k-2}, \dots, x, 1)$$

and whose ith row, $2 \leqslant i \leqslant k$ is

$$(\epsilon_{i-1}^{k-1}, \epsilon_{i-1}^{k-2}, \dots, \epsilon_{i-1}, 1),$$

where the ϵ's are all distinct. Now on subtracting the first row from each of the remaining rows, it follows that

$$G(x, \epsilon_1, \dots, \epsilon_{k-1}) \equiv G(\epsilon_1, \dots, \epsilon_{k-1}) \prod_{1 \leqslant r \leqslant k-1} (x - \epsilon_r).$$

Hence it follows that

$$G(x, \epsilon_1, \dots, \epsilon_{k-1}) \equiv \prod_{1 \leqslant r < s \leqslant k-1} (\epsilon_r - \epsilon_s) \prod_{r=1}^{k-1} (x - \epsilon_r).$$

Now carrying out the operation

$$\left(\frac{\partial}{\partial \epsilon_{d_1}} \epsilon_{d_1}\right)^{d_1-1} \dots \left(\frac{\partial}{\partial \epsilon_i} \epsilon_i\right)^{i-1} \dots \left(\frac{\partial}{\partial \epsilon_2} \epsilon_2\right)$$

on $G(x, \epsilon_1, \dots, \epsilon_{k-1})$ and setting $\epsilon_1 = \epsilon_2 = \dots = \epsilon_{d_1} = r_1$, we obtain the determinant $G(x, \epsilon_1, \dots, \epsilon_{k-1})$ with the ith row, $2 \leqslant i \leqslant d_1 + 1$ replaced by that of $\Delta(x)$. Continuing in a similar way with the corresponding operators $(t-1)$ times more, we obtain the identity under consideration.

PROOF OF THEOREM A. Since none of the roots of $f(x)$ is zero, it follows from the above lemma and the usual methods of solving recurrence equations that there exist constants (real or complex) $B, C_{ij}, 1 \leqslant i \leqslant t, 1 \leqslant j \leqslant d_i$, such that

$$V_n = Br^n + \sum_{i=1}^{t} r_i^n \left(\sum_{j=1}^{d_i} C_{ij} \, n^{j-1} \right).$$

By the above lemma, we can verify that $B = \dfrac{\delta}{r f(r)} = A$ of (5) which shows that $B > 0$ by (1), (3) and (4) of the hypothesis.

Now, since

$$\lim_{n \to \infty} \frac{V_n}{r^n} = A > 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{V_n}{V_{n-1}} = r \neq 1,$$

there exists a positive integer N_1 such that $\{V_n\}$ is positive and strictly monotonic (increasing to $+\infty$ if $r > 1$ and decreasing to zero if $r < 1$) for $n > N_1$.

We now prove Theorem B.

Proof of (I) (a). Taking L to be equal to $L(\mu)$, we have

$$\frac{(1-\mu)\log A}{\log r} < L < \frac{(1-\mu)\log A}{\log r} + 1 \tag{i}$$

by virtue of (5) of the hypothesis.

Now it can be easily verified that

$$\lim_{\substack{n_i \to \infty \\ 1 \leqslant i \leqslant \mu}} \frac{\prod_{i=1}^{\mu} V_{n_i}}{V_{\left(\sum_{1 \leqslant i \leqslant \mu} n_i\right) - L + 1}} = A^{\mu-1} \, r^{L-1},$$

which is $\lesseqgtr 1$ according as $r \gtrless 1$, by virtue of (i).

Hence there exists a positive integer $N_2(\mu)$ such that

$$\prod_{1 \leqslant i \leqslant \mu} V_{n_i} \lesseqgtr V_{\left(\sum_{1 \leqslant i \leqslant \mu} n_i\right) - L + 1}$$

according as $r \gtrless 1$, for $n_i > N_2(\mu)$, $1 \leqslant i \leqslant \mu$.

Similar argument shows that there exists an integer $N_3(\mu)$ such that

$$\prod_{1 \leqslant i \leqslant \mu} V_{n_i} \gtrless V_{\left(\sum_{1 \leqslant i \leqslant \mu} n_i\right)-L}$$

according as $r \gtrless 1$, for $n_i > N_3(\mu)$, $1 \leqslant i \leqslant \mu$.

Now, taking $N_0(\mu) = \max(N_1, N_2(\mu), N_3(\mu))$, we have

$$V_{\left(\sum_{1 \leqslant i \leqslant \mu} n_i\right)-L} \lessgtr \prod_{1 \leqslant i \leqslant \mu} V_{n_i} \lessgtr V_{\left(\sum_{1 \leqslant i \leqslant \mu} n_i\right)-L+1} \tag{ii}$$

according as $r \gtrless 1$ for $n_i > N_0(\mu)$, $1 \leqslant i \leqslant \mu$.

Proof of (I) (b). Suppose an integer L satisfies the inequalities (ii). For definiteness, let $r > 1$. Then

$$A^{\mu-1} r^{L-1} = \limsup_{\substack{n_i \to \infty \\ 1 \leqslant i \leqslant \mu}} \frac{\prod_{1 \leqslant i \leqslant \mu} V_{n_i}}{V_{\left(\sum_{1 \leqslant i \leqslant \mu} n_i\right)-L+1}} \leqslant 1, \tag{iii}$$

and

$$A^{\mu-1} r^{L} = \liminf_{\substack{n_i \to \infty \\ 1 \leqslant i \leqslant \mu}} \frac{\prod_{1 \leqslant i \leqslant \mu} V_{n_i}}{V_{\left(\sum_{1 \leqslant i \leqslant \mu} n_i\right)-L}} \geqslant 1. \tag{iv}$$

Inequalities (iii) and (iv) show that

$$\frac{(1-\mu)\log A}{\log r} \leqslant L \leqslant \frac{(1-\mu)\log A}{\log r} + 1$$

and by virtue of (5) of the hypothesis, we have $L = L(\mu)$.

This completes the proof.

Proof of (II). For this we need only take $n_1 = n_2 = \ldots = n_\mu = n$ in (I)(a) and apply Theorem A.

Corollary. *If V_i, α_i, $1 \leqslant i \leqslant k$ are all positive and condition* (5) *of Theorem* B *is satisfied, then both the conclusions* (I) *and* (II) *hold for the sequence $\{V_n\}$.*

Proof. Obviously conditions (1), (2) and (4) of Theorem A are satisfied; and by Kakeya's theorem [1], condition (3) is also satisfied.

I am thankful to Dr. V. Ramaswami for his valuable suggestions and criticisms.

REFERENCES

1. F. W. Levi : *Algebra,* Vol. I, 210, Calcutta (1942).

2. M. Perisastry : On the sequence $\{V_n\}$, $V_n = \alpha V_{n-1} + \beta V_{n-2}$, *Math. Student,* 25 (1957), 162.

3. K. Subbarao : Some properties of Fibonacci numbers, *American Math. Monthly,* 60 (1953), 682.

Andhra University
Waltair

ON THE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

By OMAR ALI SIDDIQI

1. Generalized method of parameters in the solution of non-linear partial differential equations. In the method of parameters given by Srinivasiengar [1] the substitutions $p = \dfrac{f_1(x, a)}{\phi(z, a)}$, $q = \dfrac{f_1(y, a)}{\phi(z, a)}$ make the partial differential equation of the 1st order $f(p, q, x, y, z) = 0$ an indentity and the solution is determined on integrating $dz = pdx + qdy$. This method applies to a large number of equations of all the standard forms and some general linear and non-linear partial differential equations of the 1st order in two variables. The scope of the applications can however be increased if the equation becomes an identity on substituting $p = \phi(x, y, z, a)$ and $q = \psi(y, y, z, a)$. Now $dz = \phi\, dx + \psi\, dy$ is an integrable total differential equation if $\dfrac{\partial \phi}{\partial y} = \dfrac{\partial \psi}{\partial x}$. The solution will be of the form $F(x, y, z, a, b) = 0$ which is a complete integral containing two necessary arbitrary constants a and b. Srinivasiengar's method of parameters is a particular case of this. Many non-linear partial differential equations of the 1st order which are not standard forms and to which Charpit's method does not apply are integrable by this generalized method of parameters.

EXAMPLES. (1) *Solve* $(p + y)^m = l(q + x)^n$, *where* l, m, n *are constants.*

This reduces to an indentity if

$$p = -y + a \text{ and } q = -x + (a^m/l)^{1/n}.$$

Now $dz = pdx + qdy = (-y + a)\, dx + \{-x + (a^m/l)^{1/n}\}\, dy$

giving $z = -xy + ax + (a^m/l)^{1/n} y + b$ as the complete integral.

(2) *Solve* $xp + xq = -(z + y^2)$.

We put $p = (-z/x + a/x)$, $q = (-y^2/x - a/x)$.

Now from $dz = pdx + qdy$, $xdz = (-z + a)\, dx - (y^2 + a)\, dy$, we get $xz + \frac{1}{3} y^3 = ax - ay + b$ as the complete integral.

(3) *Solve* $p/x^2 + q^3 = -z/x^3$.

We put $p = \dfrac{-a^3 - z}{x}$, $q = a/x$.

Now $dz = \dfrac{-a^3 - z}{x}\, dx + \dfrac{a}{x}\, dy$, or $(zdx + xdz) = -a^3 dx + ady$ giving $xz = -a^3 x + ay + b$ as the complete integral.

2. The generalized method of parameters can be extended to partial differential equations of the second order as well.

Let $f(x, y, z, p, q, r, s, t) = 0$ be a partial differential equation of the second order and let

$$r = f_1(x, y, z, p, q, a),$$

$$s = f_2(x, y, z, p, q, a),$$

and

$$t = f_3(x, y, z, p, q, a)$$

satisfy it. If $dp = rdx + sdy$ and $dq = sdx + tdy$ are integrable then the integral of $dz = pdx + qdy$ is a complete integral of the differential equation.

EXAMPLES. (1) *Solve* $r - t\, y^2 = \cos x$.

Put $r = a + \cos x$ and $t = a/y^2$. The equation reduces to an identity. Further

$$dp = (a + \cos x)\, dx, \quad dq = (a/y^2)\, dy,$$

so

$$p = ax + \sin x + b, \quad q = -a/y + c.$$

Now $dz = pdx + qdy$ and hence $dz = (ax + \sin x + b)\, dx + (-a/y + c)\, dy$, so $z = \frac{1}{2} ax^2 - \cos x + bx - a \log y + cy + d$ is the solution.

(2) *Solve* $xr + 2p = 0$.

Here $r = -2p/x$ satisfies the equation and $\partial p/\partial x = -2p/x$, hence, $\log p = -2 \log x + \phi(y)$ or $p = f(y)/x^2$ or $z = -f(y)/x + F(y)$.

(3) *Solve* $pt - qs = q^3$.

Here $s = -q^2$, $t = 0$ satisfy the equation.

Now $dq = sdx + tdy$, so $dq/(-q^2) = dx$ or $1/q = x + a = x + \psi(z)$ or

$$\frac{\partial y}{\partial z} = x + \psi(z) \text{ or } y = xz + f(z) + F(x).$$

This equation is otherwise worked out by Monge's method.

3. Example (1) of § 1 is a particular case of the partial differential equation $\phi(p, y) = \psi(q, x)$.

Let $p = f_1(y, a)$, $q = f_2(x, a)$ be the values of p and q which satisfy the equation. Now from $dz = pdx + qdy$, $\dfrac{\partial p}{\partial y} = \dfrac{\partial q}{\partial x}$, so f_1 and f_2 must be linear in y and x respectively, and so $p = ky + l$ and $q = kx + l'$, or the equation is of the form

$$\phi(p + ky) = \psi(q + kx).$$

Hence $\phi(p, y) = \psi(q, x)$ occurs as an integrable partial differential equation only in the form $\phi(p + ky) - \psi(g + kx)$.

4. **The form** $f_1(p, z, x) = f_2(q, z, y)$. $f_1(p, x) = f_2(q, y)$ is one of the standard forms; we find that $f_1(p, z, x) = f_2(q, z, y)$ can as well be treated as a standard form under certain conditions.

Putting $f_1(p, z, x) = f_2(q, z, y) = a$, an arbitrary constant, we get

$$p = \phi(x, z, a) \text{ and } q = \psi(y, z, a).$$

Now the condition $\dfrac{\partial p}{\partial y} = \dfrac{\partial q}{\partial x}$ is satisfied and so $dz = pdx + qdy$ gives the general solution.

EXAMPLES. (1) *Solve* $z^2 = pqxy$.

We write this as $z/px = qy/z = a$, so $p = z/ax$ and $q = az/y$.

Hence $dz = (z/ax)\,dx + (az/y)\,dy$, and $dz/z = dx/ax + ady/y$ giving $z = bx^{1/a}\,y^a$ as the general solution.

(2) *Solve* $2x\,(z^2 q^2 + 1) = pz$.

Here $\dfrac{pz}{x} = 2\,(z^2q^2 + 1) = a$ giving $p = \dfrac{ax}{z}$ and $q = \pm\,\dfrac{1}{z}\left(\dfrac{a-2}{2}\right)^{1/2}$

Hence $z\,dz = ax\,dx \pm \left(\frac{a-2}{2}\right)^{1/2} dy$ and $z^2 = ax^2 \pm \sqrt{(2(a-2))}\,y + b$

or $z^2 = 2(a^2+1)\,x^2 + 2\,ay + b$ is the general solution.

(3) *Solve* $p^2 x + q^2 y = z$.

Writing it as $\dfrac{p^2 x}{z - p^2 x} = \dfrac{z - q^2 y}{q^2 y} = a$, we get

$$p = \pm \left(\frac{az}{(1+a)x}\right)^{1/2} \text{ and } q = \left(\frac{z}{(1+a)y}\right)^{1/2}.$$

$$\text{Hence} \quad dz = \pm \left(\frac{az}{(1+a)x}\right)^{1/2} dx \pm \left(\frac{z}{(1+a)y}\right)^{1/2} dy$$

and the solution is

$$\sqrt{((1+a)z)} = \sqrt{(ax)} + \sqrt{y} + b.$$

These are otherwise solved by Charpit's method or by that of Srinivasiengar.

I thank Dr. S. M. Shah for checking the results.

REFERENCES

1. C. N. Srinivasiengar : *Mathematical Gazettee*, 14 (1929), 423.

2. Piaggio : *Differential Equations*, Art. 154.

Mulsim University
Aligarh

ON THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF THE PERTURBED DIFFERENTIAL EQUATIONS

By PAVMAN MURTHY

1. Introduction. N. Levinson [2] and Hermann Weyl [4] considered the asymptotic behaviour of the solutions of the perturbed linear systems where the perturbations could be majorised by linear functions. Viswanatham [3] considered the case where the perturbations could be majorised by functions $w(z, t)$ which have a monotonic character. The aim of this note is to consider, in a way, a more general case where the majorising functions need not have any monotonic character.

Viswanatham [3] assumed every solution of the original unperturbed equations to be bounded as $t \to \infty$ and considered the behaviour of the solutions of the perturbed equations as $t \to \infty$. We shall assume that every solution of the unperturbed equations is bounded as $t \to \infty$ and $t \to -\infty$, and consider the behaviour of the solutions of the perturbed equations as $t \to \infty$.

2. As in [3] let the perturbed equation be

$$z'_i = \sum_{j=1}^{n} a_{ij} z_j + f_i(z_1, \ldots, z_n, t) \quad (i = 1, \ldots, n), \tag{A}$$

where a_{ij} are all constants.

We can write these equations symbolically as $z' = Az + f(z, t)$, where $A = (a_{ij})$ is the $n \times n$ constant matrix and $f(z, t)$, z are column vectors with components $f_1, \cdots, f_n$ and $z_1, \ldots z_n$ respectively. We shall say that a variable matrix $P(t) = (p_{ij})$ is bounded if its norm $\| P \| = \sum_{i,j=1}^{n} | p_{ij} |$ is bounded.

Any solution of the equations (A) with $z(0) = y_0$ is a solution of the integral equation

$$z = \exp(tA)\, y_0 + \int_0^t \exp((t-s)A)\, f(z, s)\, ds. \tag{B}$$

Suppose now that the following conditions are satisfied :

(i) $\|\exp(tA)\| \leqslant c$ *for every value of* t, *i.e. every solution of* $z' = Az$ *is bounded.*

(ii) $\|f(\exp(tA)z, t)\| \leqslant w(\|\exp(tA)\|.\|z\|, t)$ *where* $w(x, t)$ *is continuous, non-negative in the region* R *defined by* $-\infty < t < +\infty$ *and* $x \geqslant 0$.

(iii) *The maximal solution* $b(t)$ *of* $x' = cw(\|\exp(tA)\|x, t)$ *through* $(0, \|y_0\|)$ *is bounded as* $t \to \infty$.

Then every solution of (A) *is bounded as* $t \to \infty$.

PROOF. Define $y(t) = \exp(-tA)z$, then from (B) it follows that

$$y(t) = y_0 + \int_0^t \exp(-sA) f(\exp(sA)\, y(s), s)\, ds. \qquad \text{(C)}$$

Suppose $b(t, \epsilon)$ is the solution of $x' = cw(\|\exp(tA)\|x, t) + \epsilon$ through $(0, \|y_0\|)$, where ϵ is a small positive quantity. The maximal solution of $x' = cw(\|\exp(tA)\|x, t)$ through $(0, \|y_0\|)$ is given by $\lim_{\epsilon \to 0} b(t, \epsilon) = b(t)$ [1].

We shall first show that

$$\|y(t)\| \leqslant b(t, \epsilon).$$

Now this inequality is satisfied at $(0, \|y_0\|)$.

Suppose at a point $t > 0$, the inequality is not satisfied. Then on account of the continuity of the functions involved, there is a greatest interval in which the inequality is not satisfied. Let this interval be $0 \leqslant a < t < e$.

At a this relation reduces to equality. In other words

$$\|y(a)\| = b(a, \epsilon) \text{ and } \|y(t)\| > b(t, \epsilon) \text{ for } a < t < e.$$

Taking the right hand derivative at 'a' we get

$$\|y'(a)\| \geqslant \|y(a)\|' \geqslant b'(a, \epsilon),$$

i.e.

$$\|\exp(-aA)f(\exp(aA)\,y(a), a)\| \geqslant cw(\|\exp(aA)\|.\,\|y(a)\|, a) + \epsilon$$

i.e.

$$cw\,(\|\exp(aA)\|\,.\,\|y(a)\|,\,a) \geqslant cw(\|\exp(aA)\|\,.\,\|y(a)\|,\,a) + \epsilon$$

(since (ii) is satisfied). This is obviously a contradiction.

Therefore $\|y(t)\| \leqslant b(t, \epsilon)$ for every value of t.

Therefore $\|z(t)\| = \|\exp(tA)\, y(t)\| \leqslant c\,\|y(t)\| \leqslant cb(t, \epsilon)$.

Making $\epsilon \rightarrow 0$, we get $\|z(t)\| \leqslant cb(t)$, i.e. $\|z(t)\|$ is bounded as $t \rightarrow \infty$.

NOTE. Instead of condition (ii), Viswanatham [3] imposed the condition $\|f(z, t)\| \leqslant w(\|z\|, t)$, where $w(x, t)$ is continuous, non-negative and non-decreasing in x. Condition (ii) is obviously a less restrictive condition and our result is therefore in a way a generalization of the corresponding theorem in [3].

In the end, I thank Dr. B. Viswanatham for his encouragement and interest shown in this work.

REFERENCES

1. E. KAMKE : *Differential Gleichungen reeller Functionen*, p. 83.

2. NORMAN LEVINSON : The asymptotic behaviour of a system of linear differential equations, *American J. Math.* 68 (1946), 1-6.

3. B. VISWANATHAM : On the asymptotic behaviour of solutions of non-linear differential equations, *Proc. Indian Acad. Sci.* (Sec. *A*) 36 (1952), 340-342.

4. HERMANN WEYL : Comment on the preceding paper [2], *American J. Math.* 68 (1946), 7-12.

Osmania University
Hyderabad

FLOW OF A COMPRESSIBLE VISCOUS FLUID ROUND A CORNER

By J. N. KAPUR

1. Introduction. In a recent paper [4], Ray has studied Prandtl's problem of expansion of a uniform supersonic stream of gas flowing round a corner. He takes both conductivity and viscosity into account, but in spite of starting with non-adiabatic conditions, he finds that the equations lead to an adiabatic flow and that the transverse component of the velocity is equal to the local velocity of sound. In spite of this similarity, he finds that the expressions for the components of velocity and for pressure and density are analytically different from those obtained by the other methods.

In the present paper, we have examined the reason for this difference and we find that it is due to the additional assumption about μ, as being proportional to some power of the enthalpy i, that Ray has introduced. We find that this assumption is inconsistent with the rest of his equations and assumptions, and in a fluid which follows this law, p, ρ, i, u, v cannot be functions of θ alone. The inconsistency is easily seen by verifying that the final expressions for u, v, p, ρ which Ray obtains do not satisfy all his equations. We have found the correct law of variation of μ which makes the equations consistent and find that for this law the flow is the same as for inviscid fluids.

2. The basic equations and their solution. As deduced by Ray [4], these are :

$$\frac{du}{d\theta} - v = 0 \tag{1}$$

$$\rho u + \frac{d}{d\theta}(\rho v) = 0 \tag{2}$$

$$\rho v \left(u + \frac{dv}{d\theta} \right) = - \frac{dp}{d\theta} \tag{3}$$

$$\rho \frac{di}{d\theta} = \frac{dp}{d\theta} \tag{4}$$

$$i\,\rho = \frac{\gamma}{\gamma - 1}\, p \tag{5}$$

$$\frac{d}{d\theta}\left(\frac{\mu}{\sigma}\frac{di}{d\theta}\right) = 0 \tag{6}$$

$$\lambda + 2\mu = 0. \tag{7}$$

From (4) and (5), eliminating i and integrating, we get

$$\frac{p}{p_s} = \left(\frac{\rho}{\rho_s}\right)^{\gamma}. \tag{8}$$

For (2) and (3) to be consistent, either

(i) $$u + \frac{dv}{d\theta} = 0, \quad \frac{dp}{d\theta} = 0, \quad \frac{d\rho}{d\theta} = 0$$

which together with (1) give a stream with uniform velocity, pressure and density. This may be the original stream or the stream that may be reached after the expansion is over; or

(ii) $$v^2 = \frac{dp}{d\rho} = a^2. \tag{9}$$

From (1), (3) and (8), we get

$$\frac{1}{2}(u^2 + v^2) + \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = \frac{1}{2}q^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = \frac{1}{2}q^2_{\max}, \tag{10}$$

which is Bernoulli's equation for steady adiabatic flow.

Equations (8), (9) and (10) have been obtained by Ray by making use of his assumption

$$\mu / i^n = \text{const.} \tag{11}$$

but it is obvious that they can be obtained independently of this assumption.

In fact from (1), (2), (3), (4) and (5), we can solve for u, v, p, ρ and i as functions of θ giving : (Howarth [2])

$$u = q_{\max} \sin(\lambda\theta), \quad v = \lambda q_{\max} \cos(\lambda\theta) \tag{12}$$

$$\frac{\rho}{\rho_s} = \cos^{2\beta}(\lambda\theta), \quad \frac{p}{p_s} = \cos^{2\beta+2}(\lambda\theta). \tag{13}$$

$$\frac{a}{a_s} = \cos^2(\lambda\theta), \quad i = \frac{\gamma}{\gamma - 1}\frac{p}{p_s}\cos^2(\lambda\theta), \tag{14}$$

where

$$\lambda^2 = \frac{\gamma - 1}{\gamma + 1}, \quad \beta = \frac{1}{\gamma - 1} \tag{15}$$

and the suffix s refers to the sonic conditions.

Now from (6)

$$\frac{\mu}{\sigma} \frac{di}{d\theta} = \text{const.} = -A \tag{16}$$

say, so that assuming Prandtl number σ to be constant and using (5) and (14),

$$\mu = -\frac{A\sigma}{\dfrac{di}{d\theta}} = \frac{A\sigma}{2\dfrac{\gamma}{\gamma-1}\dfrac{p_s}{\rho_s}\lambda\cos\lambda\theta\sin\lambda\theta}$$

$$= \frac{A\sigma\rho_s}{p_s(2\beta+2)} \frac{1}{\sin\lambda\theta\cos\lambda\theta}. \tag{17}$$

From (14) and (17)

$$\mu \propto \frac{1}{\sqrt{(i(1-i))}}. \tag{18}$$

This is different from the form (11) assumed by Ray. If Ray's assumption is replaced by (18), it is obvious and can be easily verified that we get Meyer's [3] results (12) to (15).

Knowing μ, (7) determines λ.

3. Components of stress and strain. With the assumption of u, v, ρ, p, i, being functions of θ alone, we have

$$e_{rr} = 2\frac{\partial u}{\partial r} = 0 \tag{19a}$$

$$e_{\theta\theta} = \frac{2}{r}\frac{\partial v}{\partial \theta} + \frac{2u}{r} = \frac{2}{r}\left[u + \frac{dv}{d\theta}\right] = \frac{2}{r} q_{\max}(1 - \lambda^2) \sin\lambda\theta \tag{19b}$$

$$e_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = \frac{1}{r}\left[\frac{du}{d\theta} - v\right] = 0. \tag{19c}$$

Also divergence is

$$\Delta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\left[u + \frac{dv}{d\theta}\right] = \frac{1}{r} q_{\max}(1 - \lambda^2) \sin\lambda\theta. \tag{20}$$

Therefore stress components are :

$$p_{rr} = -p + \lambda\,\Delta + \mu\, e_{rr}$$

$$= -p_s \cos^{2\beta+2} \lambda\,\theta - \frac{A\,\sigma\,\rho_s}{p_s(\beta+1)}\,\frac{1-\lambda^2}{r}\,\frac{q_{\max}}{\cos\lambda\,\theta}, \qquad (21a)$$

$$p_{\theta\theta} = -p + \lambda\,\Delta + \mu\, e_{\theta\theta} = -p + \lambda\,\Delta + 2\mu\,\Delta = -p_s \cos^{2\beta+2} \lambda\,\theta, \quad (21b)$$

$$p_{r\theta} = \mu\, e_{r\theta} = 0. \qquad (21c)$$

4. Conclusion. We find in the light of the above discussion of Ray's treatment that—

(i) if the equations of motion, continuity, energy and state hold and

(ii) u, v, ρ, p, i are functions of θ alone, then it follows that

(a) the adiabatic conditions hold, the dissipation function is zero and the radii vectors are the Mach lines;

(b) either the viscosity is absent or it follows the law (18);

(c) if $\lambda + 2\mu \neq 0$, then the only flow which is possible with the above assumptions is that of a stream with uniform pressure, density and velocity.

Further for the flow of a Fermi-Dirac gas [1], the above treatment holds with $\gamma = 5/3$.

REFERENCES

1. P. L. Bhatnagar and Pyare Lal : Shock relations in a Fermi-Dirac gas, *Proc. Nat. Inst. Sci.* 23A(1957), 9-14.

2. L. Howarth : *Modern Developments in Fluid Dynamics. High Speed Flow* Vol. I. (1953), 164-170, Oxford University Press.

3. T. Meyer : Ph. D. Thesis (1908), Göttingen.

4. M. Ray : Flow of a compressible fluid round a corner. *Proc. Nat. Inst. Sci. Ind.*, 21 (1955), 155-160.

Hindu College
Delhi University

MATHEMATICAL NOTE

Triangular numbers which are also squares

By M. N. Khatri, *University of Baroda*

A *triangular number* is of the form $n(n+1)/2$ and a square number is of the form m^2. We consider the solutions of the diophantine equation

$$n(n+1) = 2m^2, \text{ i.e. } (2n+1)^2 - 2(2m)^2 = 1.$$

Evidently the solutions are provided by the convergents of $\sqrt{2}$, viz.

$$\frac{1}{0}, \frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots .$$

In fact, if we denote these convergents by N_k/D_k, $k \geqslant 0$; $m_k = N_k D_k$ and $n_k = 2D_k^2$ or N_k^2 according as k is even or odd. Thus we have

$n_0 =$	$0 =$	2.0^2	$m_0 =$	$0 =$	0.0
$n_1 =$	$1 =$	1^2	$m_1 =$	$1 =$	1.1
$n_2 =$	$8 =$	2.2^2	$m_2 =$	$6 =$	2.3
$n_3 =$	$49 =$	7^2	$m_3 =$	$35 =$	5.7
$n_4 =$	$288 =$	2.12^2	$m_4 =$	$204 =$	12.17
$n_5 =$	$1681 =$	41^2	$m_5 =$	$1189 =$	29.41
$n_6 =$	$9800 =$	2.70^2	$m_6 =$	$6930 =$	70.99
$n_7 =$	$57121 =$	239^2	$m_7 =$	$40391 =$	169.239
$n_8 =$	$332928 =$	2.408^2	$m_8 =$	$235416 =$	408.577
$n_9 =$	$1940449 =$	1393^2	$m_9 =$	$1372105 =$	985.1393

The following properties are noteworthy:

$$n_k - n_{k-1} = m_k + m_{k-1}. \tag{1}$$

If $n_k - m_k = t$, then $m_{k-1}\, m_k = (t+1)\, t = 2\Delta t.$ (2)

$$n_k\, n_{k-1} = 2\,(n_{k-1} + m_{k-1})^2. \tag{3}$$

$$\sum_{k=1}^{t} m_{2k-1} = (m_t)^2. \tag{4}$$

$$\sum_{k=1}^{t} n_k\, m_k = \left(\frac{n_t + m_t}{2}\right)^2. \tag{5}$$

$$n_k\, m_{k-1} - m_k\, n_{k-1} = m_{k-1} + n_{k-1}. \tag{6}$$

$$(2m_k + n_k)^2 + (2m_k + n_k + 1)^2 = (m_{k+1} - m_k)^2. \tag{7}$$

If $\sum_{k=1}^{t} (2n_k + 1) = S$, then $S^2 + (S+1)^2 = (n_t + S + 1)^2$. (8)

If $n_t + m_t = q$, $n_t + 2m_t = s$,

$$2\,\Delta_q = \Delta_s, \text{ where } \Delta_q = \frac{q(q+1)}{2} \text{ as usual.} \tag{9}$$

$$\sum_{k=1}^{t} m_{(2k)} = 2\,\Delta(n_k + m_k) = \Delta\{2(m_k) + n_k\}. \tag{10}$$

CLASSROOM NOTES

An interesting property of numbers

By K. SUBBA RAO, *Maharajah's College, Vizianagram*

D. R. Kaprekar has enquired if numbers

$$(a_n, a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0) \equiv 10^n a_n + 10^{n-1} a_{n-1} + \ldots + 10 a_1 + a_0,$$

where

$$a_0 \neq 0,\ a_n \neq 0, \text{ and } 0 \leqslant a_j \leqslant 9,\ j \neq 0 \text{ or } n,$$

exist, such that

$$\frac{(a_0, \ldots, a_{n-2}, a_{n-1}, a_n)}{(a_n, \ldots, a_2, a_1, a_0)} = k, \text{ an integer.}$$

Evidently $1 \leqslant k \leqslant 9$.

If such numbers exist, we must have

$$ka_0 - a_n \equiv 0 \pmod{10}$$

and

$$a_0 = ka_n + r,\ 0 \leqslant r \leqslant 9. \tag{1}$$

Hence

$$(k^2 - 1)\, a_n + kr \equiv 0 \pmod{10}. \tag{2}$$

Since

$$0 < a_0 \leqslant 9,\ a_n \leqslant \left[\frac{a_0}{k}\right] \leqslant \left[\frac{9}{k}\right], \tag{3}$$

these three relations can be shown to hold only in the following three cases :

$$k = 1,\ a_n = 1, 2, 3, \ldots, 9;$$

$$k = 4,\ a_n = 2;$$

$$k = 9,\ a_n = 1.$$

For $k = 1$, we must have $a_{n-j} = a_j,\ 0 \leqslant j \leqslant n$.

For $k = 4$, $a_n = 2$, $a_{n-1} = 1$, $a_0 = 8$, $a_1 = 7$ and $a_j = 9$ for j other than 0, 1, $n - 1$ or n.

For $k = 9$, $a_n = 1$, $a_{n-1} = 0$, $a_0 = 9$, $a_1 = 8$ and $a_j = 9$ for j other than 0, 1, $n - 1$ or n.

Thus there are an infinity of numbers satisfying the given condition. It is noteworthy that a number of type 2 is double the corresponding number of type 3, thus

21978 is double of 10989.

In conclusion I am indebted to the referee for his help in drafting the note in its present form.

Functional analysis and semi-groups. By E. Hille and R. S. Phillips (American Math. Soc. Colloq. Publ. Vol. 31, revised edition) American Mathematical Society, Providence R. I. (1957), xii + 808 pp. $ 13.80.

THIS revised edition of a very popular book on Functional analysis brings to the reader a wealth of information and presents the same in a lucid manner. Much of the older material has been recast, developed more extensively and modified in the light of later work.

An idea of the extensive coverage of the book may be obtained from the following listing of the chapter headings :—

Part One : (*Functional analysis*) : Abstract spaces, Linear transformations, Vector-valued functions, Banach algebras, Analysis in a Banach algebra, Laplace integrals and Binomial series.

Part Two : (*Basic properties of semi-groups*) : Subadditive functions, Semi-modules, Addition theorems in a Banach algebra, Semi-Groups in the strong topology, Generator and resolvent, Generation of semi-groups.

Part Three ; (*Advanced analytical theory of semi-groups*) : Perturbation theory, Adjoint theory, Operational Calculus, Holomorphic semi-groups, Applications to ergodic theory.

Part Four : (*Special semi-groups and applications*) : Translations and powers, Trigonometric semi-groups, Semi-groups in $L_p(-\infty, \infty)$. Semi-groups in Hilbert space, Miscellaneous applications.

Part Five : (*Extensions of the theory*) : Notes on Banach algebras, Lie semi-groups, Functions vectors to vectors.

A welcome feature of the book is the summary given at the beginning of each part, and the orientation prefixed to each chapter. These provide the reader with a good idea of the coming section of the chapter. An extensive bibliography and a general index add to the facilities of reference and study. In short, it is a valuable book on the subject, and is likely to be the basic work of reference for many years to come.

V. S. KRISHNAN

Tables of Partitions. By H. Gupta, C. E. Gwyther and J. C. P. Miller, (Royal Society Mathematical Tables No. 4) Cambridge: At the University Press, (1958), xxxix + 132 pp. 63 sh.

THESE tables are a welcome addition to the literature of tables on Arithmetical functions. They were initiated by H. Gupta and were calculated independently by H. Gupta at the Panjab University, and Gwyther, Miller and their associates at the Cambridge mathematical laboratory.

The tables consist of four parts :—Tables I give the values of $p(n, m)$, the number of partitions of n into at most m parts for the following values of n and m :—

$n =$	$1 - 200$	$201 - 400$
	$0 < m \leqslant n$	$50.$

These are fundamental for combinatorial analysis because various types of partitions can be expressed in terms of $p(n, m)$.

Define $p_s(n, m)$, $p_s(n)$ by the relations :

$$\sum_{n=0}^{\infty} p_s(n, m)\, t^n = \prod_{r=1}^{m} \frac{1}{(1-t^r)^s} \prod_{r=m+1}^{\infty} \frac{1}{(1-t^r)^{s-1}},$$

$$\sum_{n=0}^{\infty} p_s(n)\, t^n = \prod_{r=1}^{\infty} \frac{1}{(1-t^r)^s}.$$

($p_{s+1}(n, 0) = p_s(n)$, $p_2(n, 0) = p_1(n) = p(n)$, the number of unrestricted partitions of n). Tables II give $p_2(n, m)$ for

$n =$	$1 - 50$	$51 - 100$	$101 - 150$	$151 - 200$	$201 - 250$
$0 \leqslant m \leqslant$	n	23	20	12	11

$n =$	$251-300$	$301-350$	$351-400$	$401-450$	$451-500$
$0 \leqslant m \leqslant$	7	6	5	4	3
$n =$	$501-550$		$551-1000$		
$0 \leqslant m \leqslant$	1		0		

These tables contain the value of $p(n)$ upto 1,000.

In Tables III $p_3(n, m)$ is given for

$n =$	$1-50$	$51-100$	$101-150$	$151-200$
$0 \leqslant m \leqslant$	n	19	6	0

while Tables IV give $p_3(n) = p_4(n, 0)$ for $n \leqslant 200$.

In addition to the tables there is a useful introduction by Gupta and Miller. Starting with historical remarks on the tables of partition functions in § 1, the authors define various types of partitions in § 2. They show how a simple graphical argument gives various inter-relationships between these functions. In particular, they show how various partition functions can be expressed in terms of $p(n, m)$. In § 3, entitled generating functions, they start with the generating functions for $p(n)$, $p(n, m)$ and introduce the fundamental function

$$\Phi(a, t) = \prod_{r=0}^{\infty} \frac{1}{(1 - at^r)} = \sum_{m=0}^{\infty} a^m \left(\prod_{r=1}^{m} \frac{1}{(1 - t^r)} \right)$$
$$= \sum_{n,m=0}^{\infty} p(n, m)\, a^m t^n.$$

Various formulae can be derived by taking a and t of special forms. In this section the authors also introduce identities for $\dfrac{1}{\Phi(t^{m+1}, t)}$ $\Phi(t^m, t)$ and mention various other identities including those of Roger-Ramanujan. § 4 is introductory to the next three sections. In § 5, starting with

$$\prod_{r=1}^{m} \frac{1}{(1 - t^r)} = \sum_{r=0}^{\infty} (-1)^r t^{r(m+\frac{1}{2}r+\frac{1}{2})} \Phi(t, t) \left(\prod_{s=1}^{r} \frac{1}{(1 - t^s)} \right),$$

they deduce

$$p(n, m) = p_2(n, 0) - p_2(n - m - 1, 1) + p_2(n - 2m - 3, 2) - \ldots .$$

This formula provides a valuable check on Tables I in terms of entries of Tables II, and can also be used to extend the function for many values outside the range of Tables I. A similar formula

$$p_2(n, m) = p_3(n, 0) - p_3(n - m - 1, 1) + \ldots$$

is useful for Tables II. In § 6 the authors describe various methods of deriving formulas for $p(n, m)$ for fixed small m and general n in terms of circulators and binomial coefficients due to Cayley, Glaisher, the authors and others. In § 7 some asymptotic results for large n are mentioned. § 8 describes the tables, while in § 9 they give the formulae used for actual calculations, and various checks applied by them. In § 10 they describe the thorough reading of proofs carried out at two places independently. The introduction ends with an exhaustive bibliography.

R. P. BAMBAH

NEWS AND NOTICES

THE following have been admitted to the life-membership in the Society : N. A. Khan, and S. Rajan.

The following persons have been admitted to membership in the Society.

G. L. Bakshi, V. B. Buch, B. N. Deo, V. M. Deshpande, W. Hahn, K. Krishna, M. Leela, N. K. Mehta, S. Mukerji, Muthulakshmi Iyer, M. Parthasarathy, D. M. Patel, M. S. Rajajee, B. N. Sahaney, A. C. Shamihoke, V. Srinivasa Upadhyaya, S. K. Srinivasan, M. L. Srivastava, B. N. Tagore and B. S. Yadav.

The council of the Society has been reconstituted as follows for the Session 1959-1961. Prof. B. S. Madhava Rao (President), Prof. S. Mahadevan (Secretary), Prof. R. P. Bambah (Editor), Prof. P. L. Bhatnagar (Treasurer), Prof. C. T. Rajagopal (Librarian). Other members are: Prof. G. L. Chandratreya, Prof. V. Ganapathy Iyer, Prof. V. S. Krishnan, Prof. S. Minakshisundaram, Prof. N. S. Nagendranath, Prof. V. V. Narlikar, Prof. B. N. Prasad, Prof. Ram Behari, Prof. N. R. Sen and Prof. N. G. Shabde.

Dr. S. M. Shah is extending his stay in America and is now visiting professor, North Western University, Evanston for the Session 1959-60.

Dr. S. M. Shah has resigned the editorship and the Council has appointed Prof. R. P. Bambah (Chief Editor) and Prof. P. L. Bhatnagar as editors.

Members of the Society intending to present papers for the 25th Conference of the Society to be held in Allahabad from December 25-27, 1959, are requested to send their papers in full along with two copies of abstracts (in special forms obtainable from the Secretary) to Prof. R. P. Bambah, Panjab University, Chandigarh-3 to reach him on or before 1st November 1959.

Sri M. V. Jambunathan has been awarded the Ph.D. degree by the University of Mysore for his thesis on 'Some studies in Statitical Sampling and Sample Survey.'

Miss Abha Mitra who was in London for further studies has been awarded the Ph.D. degree by the London University for her thesis on 'Generalized Nilpotent Groups.'

Sri Sahib Ram has been awarded the Ph.D. degree by the Indian Institute of Technology, Kharagpur, for his thesis on 'Triangles and Tetrahedra, Circles and Spheres, Conics and Quadrics.'

Dr. S. Swaminathan has been appointed lecturer in Mathematics, Madras University.

Dr. J. N. Kapur has been appointed Reader in Mathematics, Delhi University.

Dr. M. S. Ramanujan and Dr. S. C. Saxena are joining the Universities of Michigan and Atlanta respectively.

The next International Congress of Mathematicians will be held in Stockholm in 1962.

The Fifth Congress on Theoretical and Applied Mechanics will be held in the University of Roorkee under the presidentship of its Vice-chancellor Dr. A. N. Khosla from December 23-26, 1959. This will be preceded by a symposium on 'Non-linear Physical Problems' under the joint sponsorship of the Society and the UNESCO on December 21 and 22, 1959.

The second Summer School of Mathematics under the auspices of the Mathematics Seminar and the University of Delhi was held in Ramjas College, Delhi, for a month from the 11th May, 1959.

THE INDIAN MATHEMATICAL SOCIETY

Statement of Account for the
year ending 31st March 1959

Receipts and Payments Account for the

RECEIPTS	Rs. nP.	Rs. nP.
To **Subscriptions and Memberships**		8,783 64
,, **Life Memberships**		935 50
,, **Grant-in-Aids :**		
Bombay University	200 00	
Osmania University ; Hyderabad (Dn.) ...	100 00	
Madras University :	150 00	
Others	100 00	
National Institute of Sciences of India, New Delhi	2,000 00	2,550 00
,, **Golden Jubilee Publication :**		
Grants from various Universities (kept in Deposit as Special Grant for the expenses of the publication)		7,350 00
,, **Sale of Publications**		861 91
,, **Interest on Bank Balances**		8 68
,, **Advances Received Back**		2,092 31
,, **Associate Society Memberships**		278 65
,, **Suspense**		80 00
,, **Opening Balance :**		
Cash on hand	10 54	
In Current A/c with Indian Bank, Mylapore ..	579 89	
In Saving Bank A/c with Indian Bank, Mylapore	947 68	
In Current A/c with Sangli Bank Ltd., Wellingdon College Branch Sangli	371 44	
		1,909 55
Total Rs...		24,850 24

Bombay,

Dated : 22nd August, 1959,

MATHEMATICAL SOCIETY

year ending 31-3-1959.

PAYMENTS	Rs. nP.		Rs. nP.	
By **A. Narsing Rao Gold Medal Expenses**			202	00
, **Printing and Stationery**			923	15
,, **Office Expenses**			32	50
,, **Audit Fees**			50	00
,, **Postage and Railway Freight for Journals** ...			1,304	32
,, **Office Postage**			938	91
,, **Library Purchases**			3,302	44
,, **Advances Made**			1,877	82
,, **Outstanding Bills Paid**			217	99
,, **Travelling and Conveyance**			141	42
,, **Book Binding Charges**			943	94
,, **Honorariums and Remunerations**			445	00
,, **Bank Commission**			24	21
,, **Printing of Journals**			9,400	65
,, **Closing Balance :**				
Cash on hand	12	33		
In Current A/c with Indian Bank Ltd., Mylapore, Madras	175	47		
In Saving Bank Account with Indian Bank Ltd., Mylapore, Madras	47	68		
In Current A/c with Sangli Bank Ltd., Wellingdon College Branch, Sangli ...	4,810	41	5,045	89
Total Rs....			24,850	24

Examined and found correct.

P. G. BHAGWAT

Chartered Accountants.

REPORT OF
THE TWENTY-FOURTH CONFERENCE
AND
THE GOLDEN JUBILEE CELEBRATIONS
OF
THE INDIAN MATHEMATICAL SOCIETY

SUCCESSION LIST OF OFFICE-BEARERS

PRESIDENTS :

B. HANUMANTHA RAO (1907-12), R. N. APTE (1912-15), E. W. MIDDLEMAST (1915) R. RAMACHANDRA RAO (1915-17), A. C. L. WILKINSON (1917-21), H. BALAKRAM (1921-26), V. RAMASWAMI IYER (1926-30), M. T. NARANIENGAR (1930-32), P. V. SESHU IYER (1932-34), H. G. GHARPURY (1934-36), R. P. PARANJPYE (1936-40), R. VAIDYANATHASWAMY (1940-42), F. W. LEVI (1942-47), M. R. SIDDIQI (1947-49), A. NARASINGA RAO (1949-51), T. VIJAYARAGHAVAN (1951-53), RAM BEHARI (1953-57), V. GANAPATHI IYER (1957-59), B. S. MADHAVA RAO (1959-).

SECRETARIES :

V. RAMASWAMI IYER (1907-10), D. D. KAPADIA (1910-22), P. V. SESHU IYER (1922-26), N. M. SHAH (1926-28), G. S. CHOWLA (1928-29), MUKUNDA LAL (1929-32), S. B. BELEKAR (1932-36), RAM BEHARI (1936-43), M. R. SIDDIQI (1943-47), T. VIJAYARAGHAVAN (1947-51), S. MAHADEVAN (1951-).

TREASURERS :

K. J. SANJANA (1907-10), C. POLLARD (1910-14), S. NARAYANA IYER (1914-28), S. R. RANGANATHAN (1928-33), L. N. SUBRAMANIAN (1933-44), A. NARASINGA RAO (1944-47), RAM BEHARI (1947-53), C. N. SRINIVASIENGAR (1953-55), G. L. CHANDRATREYA (1955-).

EDITORS : JOURNAL

M. T. NARANIENGAR (1907-27), R. VAIDYANATHASWAMY (1927-50), K. CHANDRASEKHARAN (1950-58), S. M. SHAH (1958-).

EDITORS : STUDENT

A. NARASINGA RAO (1933-50), C. N. SRINIVASIENGAR (1950-53), K. CHANDRASEKHARAN (1953-58), S. M. SHAH (1958-).

LIBRARIANS :

R. P. PARANJPYE (1907-22), V. B. NAIK (1922-36), R. P. SHINTRE (1936-44), D. D. KOSAMBI (1944-50), T. VIJAYARAGHAVAN (1950-55), C. T. RAJAGOPAL (1955-).

MESSAGES

Messages wishing the Golden Jubilee Session success were received from :

Vice-President of India, Governor of Bombay, Minister of Education, Bombay ; Minister of Public Health, Bombay ; Vice-Chancellors of the Universities of Baroda, Bombay, Gujarat, Karnatak, Jammu and Kashmir, Jadhavpur, Marathwada, Nagpur, Peshawar, Panjab, Rangoon, Venkateswara ; Registrar, Womens' University ; Principal, Chathrapati, Shivaji College ; Dr. C. D. Deshmukh of University Grants Commission ; Prof. P. C. Mahalanobis ; Prof. M. S. Thacker ; Dr. P. V. Shukatame ; Dr. M. R. Jayakar ; Dr. C. P. Ramaswami Iyer ; Dr. A. Narasinga Rao ; Dr. B. R. Seth ; Dr. S. R. Ranganathan ; Dr. M. R. Siddiqi ; Professor D. D. Kosambi.

Messages of congratulations were received from the Cambridge Philosophical Society (a fascimile copy of the message appears in the opposite page), American Mathematical Society, London Mathematical Society, Moscow Mathematical Society, 10th British Mathematical Colloquium, National Research Council of Japan, Calcutta Mathematical Society and National Institute of Sciences of India.

The Cambridge Philosophical Society

learns with great pleasure that

The Indian Mathematical Society

is this year celebrating the Golden Jubilee of its foundation.

We like to remember at this time the long connection that India has with Cambridge mathematics, through the many students who have come here from your country for study and research, and in particular we recall with pleasure that from its inception the Indian Mathematical Society has been in continuous communication with the Cambridge Philosophical Society through the interchange of our respective publications.

We hope that the contact between our two Societies will grow and strengthen in the years to come, and we wish the Indian Mathematical Society all prosperity on this auspicious occasion.

President F. P. Bowden

Mathematical Secretary P Swinnerton-Dyer

December 1958

THE INDIAN MATHEMATICAL SOCIETY

XXIV CONFERENCE (GOLDEN JUBILEE), POONA, 1958

Sitting : V. S. KRISHNAN, Y. MUSIELAK, MRS. LEHMER, B. N. PRASAD, V. V. NARLIKAR, S. MINAKSHISUNDARAM, RAM BEHARI, D. H. LEHMER, S. MAHADEVAN (*Secretary*), S. V. RAMAMURTY, V. GANAPATHEY IYER (*President*), R. P. PARANJPYE (*Vice-Chancellor*), M. V. BHIDE, H. G. GHARPURE, D. D. KAPADIA, K. S. KRISHNAN, V. S. HUZURBAZAR (*Local Secretary*), N. G. SHABDE, MUKUND LAL, G. L. CHANDRATREYA, K. R. GUNJIKAR, P. L. BHATNAGAR, HANSRAJ GUPTA, C. N. SRINIVASIENGAR.

Standing 1st Row : V. N. SINGH, N. SANKARAN, B. D. AGRAWAL, S. RAMAKRISHNAN, P. TIWARI, M. M. LAL, D. R. KAPREKAR, G. C. NIWAS, P. N. VIJAYVERGIA, S. L. MALURKAR, K. M. SHAH, M. N. KHATRI, C. B. L. VERMA, S. S. SUBRAMANIAM, B. R. BHONSLE, J. DUTTA, D. V. RAJALAKSHMAN, T. VENKATARAYUDU, M. N. BHAT, V. LAKSHMI KANT, P. S. V. NAIDU, M. VENKATESWARA RAO, J. RAMAKANTH, SAHIB RAM, S. PARAMESWARA IYER, P. D. S. VARMA.

Standing 2nd Row : M. N. VARTAK, S. P. BANDYOPADHYAY, A. C. CHOUDHARI, NARAYAN SINGH, S. L. GUPTA, D. L. SHARMA, O. P. GUPTA, P. S. VENKATESAN, K. MARKANDESWARA RAO, P. S. RAU, P. C. JAIN, B. B. MEHRA, S. C. SAXENA, B. S. K. R. SOMAYAJULU, M. RAGHAVACHARYALU, M. L. CHANDRATREYA, K. K. DESHPANDE, P. K. SRINIVASAN, J. N. PANDA, A. KAMESWARA RAO, T. R. SAWHONEY, D. K. GADRE, M. S. LUTHAR, SHANTI NARAYAN, C. S. VENKATARAMAN, R. VENKATARAMAN.

Standing 3rd Row : M. V. SUBBA RAO, J. D. GUPTA, J. N. PATNAIK, REV. GONSALVES, M. RANGANATHAN, J. A. SIDDIQI, M. BHASKARAN, A. AHMED, S. C. MALIK, D. N. VERMA, R. MANOHAR, J. M. GANDHI, L. N. KAUL, L. V. SUBRAMANIAN, V. K. BALACHANDRAN, W. F. KIBBLE, P. S. SUBRAMANIAN, S. S. CHEEMA, H. G. S. SHARMA, T. J. BALWANI, P. D. GUPTA, K. M. GARDE, R. K. JAGGI, R. S. MISHRA.

Standing 4th Row : K. V. SHANKARANARAYANAN, P. V. RANGANATHAN, Y. R. PHATAK, M. K. AGRAWALA, A. V. RANGARAJAN, P. R. SREENATH, S. VISWANATHAN, K. S. REDDY, K. KISHAN RAO, B. VISWANATHAN, A. G. LELE, V. VENUGOPAL RAO, M. RAJAGOPALAN, R. BOJANIC, A. K. DESHMUK, B. N. SREENIVASA RAO, C. JAGANNATHA CHARI, V. D. GOPALAKRISHNAN, S. SWETHARANYAM, M. S. RAMANUJAN, S. SWAMINATHAN, S. R. SINHA, S. K. HINDI, A. SHAMIHOKE, J. N. KAPUR, G. BANDYOPADHYAY, M. K. SINGAL.

Sitting on ground : MRS. MARAKATHA KRISHNAN, MISS PADMAVALLI, MRS. KAMALAMMA, MRS. N. PRAKASH, MRS. JYOTI LUTHAR, MISS GIRIJA KHANNA, MISS K. SAVITRI, MRS. VYDEHI VENKATARAMAAN.

PROGRAMME

Wednesday, December 31, 1958

10-00 A.M.	Inaugural Function
	Prayer Song
	Reading of Messages
	Welcome Speech by Dr. Sir R. P. Paranjpye, President, Reception Committee, and Vice-Chancellor, University of Poona.
	Inauguration of the Conference and opening of the Mathematical Exhibition by Shri Y. B. Chavan, Chief Minister, Bombay State.
	Report by Prof. S. Mahadevan, Hon. Secretary.
	Presentation of Medals.
	Jubilee Address by Dr. K. S. Krishnan, F.R.S., Director, National Physical Laboratory, New Delhi.
	Presidential Address by Prof. V. Ganapathy Iyer, Speech by Prof. Ram Behari—past President.
12-00 NOON	Photo (Delegates only).
12-10 P.M.	Visit to the Mathematical Exhibition.
1-00 P.M.	Lunch to delegates.
2-15 P.M.	Business Meeting of the Council of the Society.
2-45 P.M.	Annual General Meeting of the Society.

3-15 P.M. Symposium on *Ordered Structures* led by Dr. V. S. Krishnan. Other participants: Dr. V. K. Balachandran, Dr. S. Swaminathan, Shri R. Venkatraman, Shri N. Sankaran, Miss Iqbal Unnisa.

4-45 P.M. Tea.

5-30 P.M. Popular lecture on *Parity in Nature* by Dr. B. S. Madhava Rao.

8-00 P.M. Dinner to delegates.

9-00 P.M. Entertainment.

Thursday, January 1, 1959

9-00 A.M. Reading of Papers.

11-00 A.M. Invited Address by Prof. D. H. Lehmer on '*Some functions of Ramanujan.*'

12-00 NOON Invited Address by Dr. J. Musielak on '*Some remarks on modular spaces.*'

12-30 P.M. Invited Address by Prof. Ram Behari on '*New ideas in mathematical education in Europe and the United States.*'

1-00 P.M. Lunch to delegates.

2-15 P.M. Visit to the Mathematical Exhibition.

3-15 P.M. Symposium on *Boolean Algebra.* Opening remarks by Mr. C. H. Smith. Other participants: Dr. V. S. Krishnan, Dr. B. S. Ramakirshnan, Dr. B. S. Madhava Rao.

5-30 P.M. Popular lecture on *Sputniks* by Prof. V. V. Narlikar.

6-45 P.M. 'Glimpses of the Poona University Campus' (Film Show)

8-00 P.M. Jubilee Dinner (By invitation)

Friday, January 2, 1959

9-00 A.M. Reading of Papers.

10-00 A.M. Invited Address by Dr. R. Bojanic on '*Slowly oscillating functions and their applications*'.

11-00 A.M. Invited Address by Dr. V. S. Huzurbazar on '*Remarks on Induction*'.

11-30 A.M. Symposium on '*Research in Statistics*', led by Dr. V. S. Huzurbazar. Other participants: Dr. A. R. Kamat, Shri D. S. Rangarao, Shri G. M. Panchang, Dr. (Mrs.) Vatsala Mukherjee, Shri S. R. Adke, Shri B. Raja Rao, Dr. D. V. Rajalakshman.

1-00 P.M. Lunch to delegates.

2-30 P.M. Excursion to the National Chemical Laboratory and the National Defence Academy, Kharakwasla.

8-00 P.M. Dinner to delegates.

9-00 P.M. Entertainment.

Saturday, January 3, 1959

9-00 A.M. Reading of Papers.

11-00 A.M. Invited Address by Dr. V. Venugopal Rao, on '*Lattice-point problems and quadratic forms.*'

11-30 A.M.	Symposium on '*Magneto-hydrodynamics*' led by P. L. Bhatnagar. Other participants : J. De, J. D. Gupta R. K. Jaggi, J. N. Kapur, P. C. Jain, K. S. Raja Rao, and B. S. Madhava Rao.
1-00 P.M.	Lunch to delegates.
2-15 P.M.	Address by Prof. Mukund Lal, on '*New approach to fundamentals of arithmetic*'.
2-45 P.M.	Address by Sir S. V. Ramamurty, on '*Science, Religion and Mathematics*'.
3-15 P.M.	Reading of papers.
4-15 P.M.	Tea.
8-00 P.M.	Dinner to Delegates.

REPORT OF THE GOLDEN JUBILEE SESSION

Inauguration

The twenty-fourth Conference and the Golden Jubilee celebrations of the Indian Mathematical Society were held in Poona from the 28th December, 1958 to January 3, 1959, on the invitation of the University of Poona. More than one hundred and fifty delegates were present.

The Conference was held in the spacious quadrangle attached to the Department of Mathematics, where a special pandal was put up and tastefully decorated for the occasion. The inaugural function began at 10 A.M. with a prayer and a sloka from *Ganita-sara-Sangraha* of Mahaviracharya. The Secretary read messages wishing the Session success, from prominent persons and mathematical societies in India and abroad. Dr. R. P. Paranjpye, the Chairman of the Reception Committee and Vice-Chancellor of the University of Poona in welcoming the delegates expressed his pleasure that the Jubilee session was held in Poona which had been the headquarters of the Society for a number of years and where the Library of the Society was located of which he was the Librarian. He recalled his connection with the Society and its growth and observed 'research in mathematics is getting more and more difficult on account of the extensive front on which progress is being made......Facilities in the form of extensive mathematical libraries have to be amply provided by the Universities and Governments. I hope that they will not be backward in providing sufficient funds for these purposes'.

The Conference was formally inaugurated by Shri Y. B. Chavan, the Chief minister of Bombay. He paid a warm tribute to the great progress made by the Society during the last 50 years and wished a brighter future in the coming years. He also declared the mathematical exhibition open.

SECRETARY'S ANNUAL REPORT

Professor S. Mahadevan, Secretary of the Society then presented the report of the Society for the year 1958. He conveyed the thanks of the Society to the University of Poona for its kind invitation to hold the Conference and for the excellent arrangements the University had made for the same and for celebrating the Golden Jubilee also. He welcomed all the delegates and was grateful to Dr. J. Musielak of the University of Poznan and Dr. R. Bojanic of the University of Belgrade who were members of the Tata Institute and to Prof. D. H. Lehmer and Mrs. Lehmer of the University of California for their active participation in this Conference.

He referred to the loss sustained by the Society by the death of Sir V. Ramesam, a retired judge of the High court of Madras, one of the oldest members of the Society and an ardent devotee of mathematics. He conveyed the society's condolences to the members of the bereaved family.

He traced the history of the Society for the last fifty-one years ever since its foundation in 1907 by V. Ramaswami Iyer. It was in the fitness of things, he stated, that the Golden Jubilee was celebrated in Poona which had been the headquarters for the last forty years. He was glad that two of the foundation members Dr. R. P. Paranjpye and Prof. D. D. Kapadia were able to participate actively in the celebrations.

Recounting the activities of the Society, the Secretary said that from 1907, progress reports were published which contained among other things mathematical notes and questions. The first mathematical note was by Principal Paranjpye 'On the cardioide' in 1908 and the first Question to be published was from Balak Ram. Continuing, the Secretary said "Encouraged by contributions, the Society started the *Journal* in February 1909 under the editorship of M. T. Naraniengar of Bangalore with the collaboration of Principal Paranjpye and A. C. L. Wilkinson of Bombay. It is a pleasure to recall that the early contributions of Ramanujan appeared in the

Journal from 1911 and his first paper on "Bernoulli's numbers" attracted great attention.

Proceeding, the Secretary mentioned that after the Silver Jubilee in 1932 it was decided to start a new periodical *Mathematics Student* containing short papers, notes and questions and book-reviews, etc. He observed further, 'M. T. Naraniengar continued to be the editor till 1927 when Dr. R. Vaidyanathaswami took up the work till he retired in 1950. Prof. A. Narasinga Rao was the editor of the *Student* from the beginning till 1950. Both these periodicals were afterwards under the able guidance of Prof. K. Chandrasekharan who was helped by a well-chosen team of referees and workers and also by the Commercial Printing Press which spared no pains to improve the get up of the periodicals. Owing to pressure of work Prof. Chandrasekharan had to resign early this year and Prof. S. M. Shah of Aligarh has been appointed as Editor. I wish to thank Prof. Vaidyanathaswami and Prof. Chandrasekharan for their unselfish and hard work. I wish to thank Prof. A. Narasinga Rao for the able manner in which he guided the *Student* for the last 18 years".

Tracing the history of the Conference, the Secretary stated that the first Conference was held in Madras in 1917 and since then Conferences were held in University centres every two years. From 1950, the Secretary said, the Conference was held annually. He recalled that the 4th Conference was held in Poona in 1924 when Balak Ram presided and Principal Paranjpye welcomed the delegates.

Coming to the work of the Library, the Secretary stated that much valuable work had been done by the first Librarian Principal Paranjpye and his associates. He said that the Library was transferred from Poona to the Ramanujan Institute in Madras in 1951, to help the growing institute. He also stated that there were about 1000 books and 4000 bound volumes of periodicals. He pleaded for a liberal grant by the Government of India for the Library. The Secretary thanked the various librarians for the care they bestowed in maintaining the Library in good condition.

He thanked the Universities of Madras, Bombay and Osmania, the Tata Institute of Fundamental Research, the National Institute of Sciences of India and the Government of India for their annual grants towards the publications. He pleaded for a more generous grant from the Government to meet the heavy cost of paper and printing.

Referring to mathematical research the Secretary welcomed the recognition by the Government of India of the Tata Institute of Fundamental Research as a national centre for research and pleaded for the Government strengthening the Ramanujan Institute. He also pressed on the Government to open at least two more institutes one in Calcutta and another in Delhi.

Concluding he observed: " We have every reason to be proud of our work for the last 50 years in creating the necessary climate for research at various centres of learning and encouraging research by publishing these in our periodicals which have attained international reputation. We are conducting our conferences on a par with international congresses. We will be publishing soon a sumptuous Jubilee Volume covering various aspects of mathematical research. If we have reason to be proud of these achievements, they are due to the unselfish work of the various presidents and secretaries and to the diligent care bestowed by the editors and members of the council all these years, and above all to the enthusiastic support given to us all along by the members of the Society."

Award of the Narasinga Rao Medal

Since no medal was awarded last year, two medals were presented this time to two persons. The one is to Dr. V. Venugopal Rao for his paper on 'The lattice point problem' and the other is to Dr. C. S. Seshadri for his paper on ' Multiplicative meromorphic functions'. Both these appeared in the *Journal of the Indian Mathematical Society.*

Jubilee Address

After the award of the medals Dr. K. S. Krishnan, Director, National Physical Laboratory delivered the Jubilee address.

President's Address

Professor V. Ganapathy Iyer then delivered the address which is printed separately.

Address by Past Presidents

The President requested Prof. Ram Behari a past president to address a few words. His speech appears elsewhere. Then Professor D. D. Kapadia an oldest foundation member also addressed a few words.

Vote of Thanks

Dr. V. S. Huzurbazar the local secretary proposed a vote of thanks, bringing the proceedings of the inaugural session to a close.

Meeting of the Society

The Council of the society met in the afternoon of December 31. At the meeting of the General Body which followed, a condolence resolution touching the death of Sir V. Ramesam was adopted, all members standing. Prof. K. R. Gunjikar suggested that the Society should give its views on the new Government Calendar. After some discussion it was agreed that the council should appoint a small committee to examine the calendar. The Secretary announced that the next Conference would be held in Allahabad and the 26th Conference in December 1960 in Nagpur at the invitation of both these Universities.

Proceedings of the Conference

There was a crowded mathematical programme which consisted of presentation of papers, invited addresses and symposia. Four sessions were devoted to the reading of papers and abstracts of these appear elsewhere. Among the invited addresses were one by Prof. D. H. Lehmer on ' Some functions of Ramanujan ', one by Dr. J. Musielak on ' Some remarks on Modular spaces ', a third by Dr. R. Bojanic on ' Slowly oscillating functions and their applications ', a fourth by Dr. V. S. Huzurbazar on ' Remarks on induction ', and a fifth by Dr. V. Venugopal Rao on ' Lattice point problems and quadratic forms'. The last address was given by Prof. Ram Behari on " New ideas in mathematical education in Europe and America". There was a symposium on ' Ordered Structures' in which Dr. V. S. Krishnan, Dr. V. K. Balachandran, Dr. S. Swaminathan, among others participated. There was another on ' Boolean Algebra' in which besides others Mr. C. H. Smith, Prof. B. S. Madhava Rao and Dr. V. S. Krishnan took part. There was a third one on ' Research in Statistics ' in which Dr. V. S. Huzurbazar and others took part. In the symposium on ' Magneto Hydrodynamics ' Prof. P. L. Bhatnagar and others took active part. Proceedings of these are printed elsewhere.

There were two popular lectures one by Prof. B. S. Madhava Rao on ' Parity in nature ' and another by Prof. V. V. Narlikar on ' Sputniks'. In addition to these, Sir S. V. Ramamurti gave a talk on ' Science, Religion and Mathematics '. This was an illuminating and interesting account of his ideas regarding the relation of ' spirit ' to the usual concepts in Science and Mathematics. Prof. Mukunda Lal of Punjab gave a talk on ' New approach to the fundamentals of arithmetic '. Here he demonstrated vividly quick and one line multiplication and division of two big numbers. As an adjunct to the Jubilee Session a mathematical exhibition was got up. The exhibits included among other interesting items, charts, drawings of geometrical patterns, portraits of mathematicians, interesting models and rare books on mathematics. Of special interest was the first issue of the *Journal* of the Society (1907), the foundation volume of the Cambridge Philosophical Society (1822) and ' Primum Mobile ' published in 1658.

Social Programme

The delegates were entertained on a lavish scale. The Reception Committee organized a grand Jubilee dinner on the 1st January in which the elite of the town took part. There was an excursion to National Chemical Laboratory and to the National Defence Academy, Kharakwasla. There was a variety entertainment consisting of music recital, folk dance and a drama. There was also a film show in which the activities of the Science department of the University were depicted.

Thanks of the Secretary

On the final day the secretary thanked the authorities of the University, the participants in the symposia and those who gave invited addresses and popular lectures. He also thanked the local secretary and volunteers for the excellent arrangements and for their unstinted service to the delegates.

PRESIDENTIAL ADDRESS

By V. GANAPATHY IYER

FELLOW MATHEMATICIAN, LADIES AND GENTLEMEN:

I have the honour and the privilege of addressing this Conference of the Indian Mathematical Society when it is celebrating its Golden Jubilee. To mark this occasion, the Conference is held for four days instead of the usual three days. The Society is bringing out a Jubilee Volume consisting of invited articles on mathematics by distinguished mathematicians from India and abroad. So far about 25 contributions have been received and the Jubilee Volume is expected to be ready in April, 1959.

As I indicated in my last year's address, India is very backward in mathematical development. In pursuance of a resolution adopted at the Conference held last year, the Council of the Indian Mathematical Society has appealed to the Government of India to open Institutes devoted to mathematical research to accelerate the pace of mathematical development in the country. The Government is already giving substantial help to the School of Mathematics in the Tata Institute of Fundamental Research. The Ramanujan Institute now taken over by the Government of India and managed by the Madras University is awaiting development. The Council has appealed to the Government to hasten the development of the Ramanujan Institute and to open two more Institutes, one at Delhi and another at Calcutta where active research work in mathematics is carried on already at the Universities. In these days when the total body of mathematical knowledge is very vast, no single individual, however eminent, can claim expert knowledge in all its branches and a large number of scholars in the same locality interested in the same or allied branches of the subject and holding frequent discussions will enable flow of ideas and consequent development in mathematics. Team work in mathematics as in several other sciences is becoming the order of the day. These research institutes besides making original contributions will also train up young men with aptitude who will man the Research Departments in the Universities and Colleges and thus accelerate the pace of mathematical development in the country. It is hoped that the Central Government will be taking concrete steps in this direction in the immediate future.

Before proceeding to the mathematical part of the address, I desire to draw attention to one of the disturbing features of the present-day trends in education in this country. It is true that no one facet of a nation's activity can remain isolated or uninfluenced by the trends in other fields. But it is certainly questionable whether the predominant influence which the political leaders and those endowed with the task of running the State are able to exert in shaping the pattern of education in this country is a healthy feature in the growth of education in this country. The experienced teacher has very little voice in this matter where, if progress is to be natural and healthy, he should have the final voice. At best the teacher is allowed to play the role of the approver for the policies adumbrated by the political leaders and other non-academic persons. Even before the dawn of political independence, the State had a complete control over secondary education through their education department. So it was easy for the politicians to make their voice felt in the field of secondary education. A glance at the successive changes that have been introduced in the field of secondary education, during the past decade, ostensibly on the advice of educational experts, will show that each change merely reflected the pet ideas of the person or persons in power at that time. The influence of the political leaders is slowly extending to University education as well. As one instance, I refer to the splitting up of the present Intermediate course into a Pre-University course with a conglomoration of subjects to be studied under the impression that it is liberal education and the introduction of the three year degree course. From the press reports it is evident that the State is actively supporting this reform. Every colleague in my profession whom I had the opportunity to consult feels that this is not a healthy step in the progress of higher education in the country and yet he has to support the scheme because it has been directed from persons in authority. It is always possible to produce plausible arguments for introducing any change which a person endowed with power desires. But on that account it does not become a healthy change. Anyway, this phenomenon in the educational atmosphere of this country is not one about which the nation should be proud of. The politicians on the one hand and teachers and educationists on the other hand should find a remedy for this situation before it is too late.

I have chosen for the mathematical part of the address a brief review of the theory of Topological Vector spaces. A knowledge of what is meant by a topological space and by a vector space is presumed in the following exposition though I am going over the relevant definitions rapidly.

Topological Space.—Let X be any set. A distinguished family Γ of subsets of X closed for finite intersections and arbitrary unions and containing the empty set and the whole set X is said to define a topology on X. The pair (X, Γ) is called a *topological space.* The elements of X are called points of the space and the elements of Γ are called the open sets of the topological. On the same set X several topologies can be defined. Starting with a collection M of subsets of X there is a unique topology having the smallest family of open sets and containing the sets of M among its open sets. This may be called the topology generated by the family of sets M. Let f be a map of a topological space X into another topological space Y. The map f is said to be continuous if the inverse images of open sets in Y are open in X. Let X and Y be topological spaces. The cartesian product of the sets X and Y, denoted by $X \times Y$, is the pair (x, y) of elements, where x and y vary over X and Y respectively. The maps $(x, y) \rightarrow x$ and $(x, y) \rightarrow y$ are called the projections of $X \times Y$ onto X and Y respectively. If X and Y are topological spaces and $X \times Y$ is endowed with the topology generated by the inverse images of open sets in X and Y by their respective projections, we call the resulting topological space *the topological product of X and Y.* A topology on a set is said to be *separated or a Hausdorff topology*, if any two distinct points of the set are contained in disjoint open sets of the topology.

Vector Spaces.—We consider vector spaces over the complex number field C (which is supposed to be endowed with the usual topology when it is to be considered as a topological space). A set V of elements closed with respect to an operation denoted by $+$ with respect to which it forms an abelian group and closed with respect to the operation of multiplication by numbers of C is said to be Vector space over C if (1) $a(x+y) = ax + ay$, $(a+b)\,x = ax + bx$, (2) $a(bx) = b(ax) = (ab)\,x$ and (3) $1 \cdot x = x$, $0 \cdot x =$ the zero element of the space V (the identity of the abelian group) which we denote by ϕ, where x and y are elements of V and a and b are numbers of C. If E and F are subsets of a vector space V, the set $E + F$ is the set of all elements of the form $x + y$, $x \in E$ and $y \in F$. Similarly, aE (where a is a complex number) is the set of elements of the form ax, $x \in E$. If x and y are elements of V, the set of elements $tx + (1-t)y$, $0 \leqslant t \leqslant 1$, is called a segment in V joining x and y. A subset E of V is called *convex* if the segment joining any two points of E lie in E. A subset E is said to be a *disc* if $aE \subset E$ for every complex number a with $|a| \leqslant 1$. Now given any set E in V, there is always a smallest convex set containing E called *the convex hull* of E and a smallest disc containig E which we call the disc generated by E. A subset E of V is said to *absorb* the subset F if there exists a $t > 0$ such that $aE \supset F$ for $|a| \geqslant t$. A set is said to be *absorbing* if it absorbs all sets consisting of single points. A *subspace* of V is a vector space over the field C contained in V. A map f of one vector space V into another W is called *linear* if $f(ax + by) = af(x) + bf(y)$ for $x, y \in E$ and $a, b, \in C$. If W is the space C itself, f is called a linear functional. The set of all linear functionals on a vector space V is called the *Algebraic Dual* of V and is itself a vector space over C with the usual definition of addition and multiplication by complex numbers. In a vector space a finite set x_i, $i = 1, 2, \ldots, k$, of elements is said to be linearly independent if $\Sigma a_i x_i = \phi$ (where a_i are complex numbers) implies that $a_i = 0$ for $i = 1, 2, \ldots, k$. A family of elements of V is said to be inearly independent if every finite subset of the family is linearly independent. Now it is known that every vector space over C contains a linearly independent family such that every element of V is a finite linear combination of elements selected from the family. Such a family is called a Hamel basis in V. Two Hamel bases can be proved to have the same cardinal number. This number is called the linear dimension of V. If this dimension is finite, V is called a finite dimensional vector space. The finite dimensional Euclidean spaces are the typical finite dimensional vector spaces. The set of all complex valued continuous functions on a closed interval on the straight line or on a bounded region in the complex plane is a typical example of an infinite dimensional vector space.

Semi-norms and norms. A non-negative function $p(x)$ defined on a vector space V is called a semi-norm if (1) $p(\phi) = 0$; (2) $p(ax) = |a| \, p(x)$ for any complex number a and (3) $p(x+y) \leqslant p(x) + p(y)$. If in addition, $p(x) = 0$ implies that $x = \phi$ then $p(x)$ is called a norm on V

Topological Vector Spaces: Let V be a vector space over C. Suppose a topology T is given on V. We say that T is compatible with the structure of the vector space if the two maps $(x, y) \to x + y$ of $X \times X$ onto X and $(a, x) \to ax$ of $C \times X$ onto X are continuous where $X \times X$ and $C \times X$ denote the topological products of X and X and C and X respectively. In this case the system (V, T) is called a topological vector space. It is evident that the same vector space V can be converted into different topological vector spaces by different choices of the compatible topology T. But it can be shown that if V is finite dimensional there is essentially only one topology compatible with the vector space, that is, if V_1 and V_2 are two topological vector spaces of the same finite dimensions, then there is a one-to-one bi-continuous linear map of V_1 onto V_2 so that regarded as topological vector spaces the two are indistinguishable. But this is not true for infinite dimensional vector spaces.

A neighbourhood of a point x in a topological space X is defined as any set containing an open set containing x. It is possible to specify the topology on a space by specifying the neighbourhoods of each point. Now one of the important properties of a topological vector space is that its topology can be completely specified by specifying the neighbourhoods of the element ϕ the neighbourhood of any other element x being of the form $x + N$, where N is neighbourhood of ϕ. At any point in a topological space, a fundamental system of neighbourhoods is one with the property that any other neighbourhood of the point contains a neighbourhood of the system. Now a topological vector space is completely specified if a fundamental system of neighbourhoods of ϕ is specified. It can be shown that such a system Δ can be chosen with the additional properties ; (1) each $N \in \Delta$ is an absorbing disc, (2) if $N \in \Delta$ then so does aN for any complex $a \neq c$, (3) if $N \in \Delta$ then there is an $N \in \Delta$ such that $N_1 + N_1 \subset N$ and (4) if $N_1, N_2 \in \Delta$, there is an $N_3 \in \Delta$ such that $N_3 \subset N_1 \cap N_2$. Conversely any system of subsets of V with properties (1) to (4) can be taken as a fundamental system of neighbourhoods of ϕ defining uniquely a compatible topology on V thus making it a topological vector space.

Topologies defined by norms and semi-norms. — Let **S** be a family of seminorms defined on a vector space V over C. If $d > 0$ and $p \in$ **S**, let $N(p\,;d)$ denote the set of elements $x \in V$ such that $p(x) < d$. Let Δ denote the class of all sets obtained as finite intersections of sets $N(p\,;d)$ as p varies over **S** and d varies over the positive real numbers. Then Δ satisfies the conditions (1)-(4) of the last paragraph and defines a topology compatible with the vector space which we call the topology defined by the family of semi-norms on V.

Now suppose **S** consists of a single norm. The corresponding topological vector space obtained is called a normed vector space. It can be shown that the topology defined coincides with that defined by the metric $p(x - y)$ on V. If V is complete with respect to this metric, the space is called a Banach Space. Banach spaces are among the most widely studied class of topological vector spaces and I shall be referring to a few important properties of such spaces a little later. It may be noted in passing that the topology defined by the family of semi-norms **S** is separated or a Hausdorff space if and only if the relation $p(x) = 0$ for all $p \in$ **S** implies that $x = \phi$.

Locally convex topological vector spaces.—The simplest among topological vector spaces are the normed spaces and Banach spaces mentioned in the last paragraph. All finite dimensional topological vector spaces can be specified in terms of suitable norms and are Banach spaces. After these come the space of all bounded complex valued functions on any arbitrary set. If the set is a compact Hausdorff space, the class of all continuous functions on the set form a subspace of the space of all bounded functions, the norm in both cases being the l.u.b. of the modulus of the function as the variable runs over the set in question. The space of all bounded complex sequences with the l.u.b. of the moduli of the terms as the norm and the subspaces of convergent sequences and null sequences are other examples of Banach spaces.

The next in importance are the locally convex topological vector spaces. A topological vector space V over C is said to be locally convex if there exists a fundamental system of neighbourhoods of ϕ consisting of convex sets. It can be shown that in this case there exists a fundamental system of neighbourhoods each set of which is a closed convex absorbing disc. Each such disc determines uniquely a semi-norm on V, the disc consisting of those elements for which this semi-norm does not exceed one. Now the family of semi-norms determined by the sets of a fundamental system of neighbourhoods defines a topology on V which is precisely the given locally convex topology. Conversely the topology defined by any family of semi-norms is locally convex. In case the locally convex Hausdorff topology possesses an enumerable fundamental system of neighbourhoods at ϕ, the space is metrisable. All normed vector spaces and Banach spaces are locally convex.

F-spaces.—There is another generalization of a Banach space. A topological vector space whose topology can be specified by a metric $d(x, y)$ with respect to which it is complete is called an F-space. In this case an equivalent metric $d_1(x, y)$ can be introduced for which $d_1(x, y) = d(x - y, \phi)$. Such a metric is called an invariant metric. Every metrisable complete locally convex topological space is an F-space but every F-space need not be locally convex. The space of integral functions which I have been investigating is an example of a locally convex F-space which is not normable.

The topological dual of a topological vector space.—Let V be a topological vector space. The class of all continuous linear functionals on V is called the topological dual of V and denoted by V^*. With the usual definition of addition and scalar multiplication V^* is a vector space over C and is a subspace of the algebraic dual. If on V we introduce the topology generated by the inverse images of open sets in the complex plane by the various functionals of V^*, the corresponding topology on V is called the weak topology induced by V^*. It can be shown that the topological dual of V endowed with the weak topology is also the set V^*.

Properties of Banach spaces.—The following are some of the important properties of Banach spaces :

1. Let V_i, $i = 1$, 2 be two Banach spaces with norms p_i, $i = 1$, 2 respectively. Let T be a linear map of V_1 into V_2. A necessary and sufficient condition that T is continuous is that there exists a fixed positive number M with property that $p_2(T(x)) \leqslant Mp_1(x)$ for all $x \in V_1$. The number l.u.b. $p_2(T(x))$ when $p_1(x) = 1$ defines a norm on the class of all continuous linear maps $L(V_1, V_2)$ of V_1 into V_2. In the special case when V_2 is the space C we get the topological dual V^*_1 of V_1 and endowed with the norm described above V^*_1 becomes a Banach space. This statement is true in the general case of $L(V_1, V_2)$ with the norm mentioned above.

2. Let V be a normed space and E be a subspace. Let f be a continuous linear functional defined on the subspace E. Then there exists a continuous linear extension of f to the whole space V, the extension having the same norm over the whole space V (as defined in (1) above) as f over the sub-space V. This result is known as the Hahn-Banach theorem on the extension of continuous functionals. A consequence of this result is that if x is at a positive distance from E, then there is a functional f of V vanishing on E with $f(x) = 1$.

3. Let L_1 be a subset of the space $L(V_1, V_2)$ where V_i, $i = 1, 2$ are Banach Spaces. If for each $x \in V_1$, the set of numbers $p_2(T(x))$ is bounded as T varies over L_1, then the set of norms of the map in L_1 is bounded. This is known as the Banach Stienhaus theorem. When V_2 is C, this reduces to the uniform boundedness of the set of functionals in V^*_1 when the values of the functionals form bounded sets for each element in V_1. Another consequence of this theorem is that if E is subset of V_1 and the set of numbers $f(x)$ as x varies over E is bounded for each $f \in V^*_1$, the norms of the elements in E form a bounded set.

4. If a sequence T_n of elements of $L(V_1, V_2)$ be such that $T_n(x) \to T(x)$ for each $x \in V_1$ then $T \in L(V_1, V_2)$. In other words, the pointwise limit of a sequence of continuous linear transformation of one Banach space into another is also one such transformation.

5. Let V be a Banach space and V^* its topological dual endowed with the norm topology mentioned in (1) above, making it a Banach space. Consider the unit sphere S in V^*, that is elements of V whose norms do not exceed one. Each element $x \in V$ determines a functional defined by $x(f) = f(x)$, $f \in V^*$, which is an element of the topological dual of V^*. The family of such functionals as x varies over V can be regarded as an isometric subset of the second dual V^*_1 of V. This subset can be used to define on V^* a weak topology similar to that indicated earlier. It is an important theorem that the set S is bi-compact in this weak topology on V^*, that is, every open covering of S in this topology contains a finite covering.

6. Let V_1 and V_2 be two Banach spaces and T a linear transformation of V_1 onto V_2. A necessary and sufficient condition that T is continuous is that the graph $(X, T(x))$ of points in $V_1 \times V_2$ is closed in the topological product $V_1 \times V_2$. This is known as the closed graph theorem.

7. Suppose T is a continuous one-one transformation of a Banach space V_1 onto V_2. Then the inverse transformation which exists by hypothesis is automatically continuous.

The above list constitutes some of the most important properties of Banach spaces. Many of these have been generalized to locally convex topological vector spaces and to F-spaces. For instance, the Hahn-Banach extension theorem is valid in any locally convex topological vector space. Again the closed graph theorem and the property of bi-continuity of one-way continuous one-to-one transformations remain valid for F-spaces. Some of the other properties can be extended to more general situations but a detailed examination of these will make this address too long. Those who are interested may look up the references given at the end. I shall conclude this talk after touching upon a few more important landmarks in the general theory.

Topologies on $L(V_1, V_2)$.—Let V_1 and V_2 be two topological vector spaces over C and let $L(V_1, V_2)$ denote the set of all continuous linear transformations of V_1 into V_2. It is a vector space over C with the usual definitions of addition and scalar multiplication. A good deal of work has been done in investigating the properties of this vector space endowed with different topologies. Let Δ be a family of subsets of V_1. For a set $E \in \Delta$ and a neighbourhood N of ϕ in V_2, let $T(E, N)$ denote all the elements of $L(V_1, V_2)$ which transforms the set E into a subset of N. Under very general restrictions on Δ, V_1 and V_2 the sets $T(E, N)$ as E runs over Δ and N over a fundamental system of neighbourhoods of ϕ in V_2 constitute a system of neighbourhoods of ϕ in $L(V_1, V_2)$. By specialising the set Δ we get different topologies on L. We mention a few such.

A set in a topological vector space is said to be bounded if it is absorbed by every neighbourhood of ϕ in that space. Let V_1 and V_2 be locally convex topological vector spaces. If we take for Δ the family of all bounded closed convex discs, in V_1 we get the *topology of bounded convergence* on L. If the V_i are normed spaces we get the usual norm topology on L (as mentioned in (1) above) under the properties of Banach spaces. When Δ is the set of all finite subsets of V_1, we get the topology of pointwise convergence. If Δ is the set of all bi-compact subsets of V_1, we get the topology of convergence on bi-compact sets.

Vector spaces in duality and weak topologies.—Two vector spaces V_1 and V_2 are said to be in duality if there is a bilinear functional (usually called a bilinear form) $B(x, y)$, $x \in V_1$, $y \in V_2$ [that is, for each $x \in V_1$, $B(x, y)$ is an element of the algebraic dual of V_2 and for each $y \in V_2$, $B(x, y)$ belongs to the algebraic dual of V_1] if $B(x, y)$ is not the identically zero functional on V_1 for a $y \neq \phi$ and on V_2 for an $x \neq \phi$. This implies that each V_1 can be identified with a subspace of the algebraic dual of the other. Now let $\sigma(V_1, V_2)$ denote the topology on V_1 generated by the inverse images of open sets in the complex plane by the functionals of V_2. Then $\sigma(V_1, V_2)$ is called the weak topology induced on V_1 by the functionals of V_2. The topological dual of V_1 with respect of this weak topology is precisely the functionals in V_2 and similarly the functionals of V_1 constitute the topological dual of V_2 with respect to the $\sigma(V_2, V_1)$. When V_1 is a Banach space and $V_2 = V^*_1$ we get the weak topology mentioned in the para on the topological dual of a vector space. When V_1 is the dual of a Banach space V and V_2 is the subspace of V_1 determined by the elements of V as stated in the property (5) of Banach spaces listed above, $\sigma(V_1, V_2)$ becomes the weak topology mentioned there with respect to which the unit sphere of V_1 is bi-compact. It is to be noted that $\sigma(V_1, V_2)$ is always a locally convex topological vector space. We say that a topology T on V_1 is compatible with the duality between V_1 and V_2 if V_2 is the topological dual of V_1 with respect to the topology T on V_1. The weak topology $\sigma(V_1, V_2)$ is compatible with the duality as already mentioned above. There are in general, several such topologies. Some of the recent developments in the theory of topological vector spaces have been in connection with the inter-relation between such topologies. Details will be found in the references given at the end. I shall conclude by mentioning one more result generalizing the Banach-Stienhaus theorem. In a locally convex topological vector space V a barrel is defined as a closed convex absorbing disc. V is said to be a t-space if every barrel is a neighbourhood of ϕ and its topology is separated. All normed spaces are t-spaces. Now let V be a t-space. Let V' be a locally convex separated space. A subset H of $L(V, V')$ is said to be bounded if for every $x \in V$ the set $u(x)$, where u varies over H is bounded in V'. Then the set H is equi-continuous, that is, given a neighbourhood N_1 of ϕ in V' there is a neighbourhood N of ϕ in V such that $u(N) \subset V'$ for every $u \in H$. This reduces to the Banach Stienhaus theorem when V and V' are Banach spaces.

I have attempted in the previous paragraphs to give a brief sketch of the fundamental notions and results in the theory of topological vector spaces. I have not made the account exhaustive. For instance, I have not referred to such notions and results as inductive and projective limits, reflexivity, tensor products, vector lattices, fixed point theorems and normed rings or Banach Algebras. The theory of topological spaces has applications in several branches of mathematics pure and applied. For instance, classical closure theorems and the theory of best approximations are consequences of the Hahn-Banach theorem. Continuous linear transformations and fixed point theorems have applications in the theory of Integral equations and boundary value problems in partial differential equations. Generalizations of the notion of Harmonic Analysis typified by the classical theory of Fourier series are best expressed in the language of the theory of Banach algebras.

I close my address with an appeal to all interested in the sound progress of higher education in this country to see that changes in the structure of education are not introduced merely for the sake of change, to the politicians and other non-academic persons to curb their desire to use the power with which they happen to be endowed to push through their ideas of what education should be and to the teachers and educationists to try to be honest to themselves and not give approval to proposals for changes in the educational structure in the country merely because they come from persons in authority and to express boldly their views on educational matters.

REFERENCES

1. N. BOURBAKI : *Espaces Vectoriels Topologiques*, Chap. I to V and Fascicule de Resultats Actualitiès Scientifiques et Industrièlles, Nos. 1189, 1229 and 1230 (Gàuthier Villars, Paris.)

2. S. BANACH : *Thèorie des Operations Linèaires* Monographs on Mathematics, Warasaw, Tome I.

3. J. A. DIEUDONNE : Recent developments in the theory of locally convex vector spaces, *Bull. American Math. Soc.* 59 (1953), 495-512.

4. E. HILLE : Functional Analysis and Semi-groups, *American Math. Soc.* Colloq. Publications.

ADDRESS

By Dr. RAM BEHARI (*Past President*)

MR. PRESIDENT, LADIES AND GENTLEMEN,

My connection with the Society dates back to the year 1921 when I attended the 3rd Conference of the Society at Lahore and was an active worker of the Reception Committee. It is a great joy to me to have attended the Silver Jubilee Celebrations of the Society at Bombay in 1932 and to be present at the Golden Jubilee Celebrations today. I met the founder of the Society, the late V. Ramaswamy Iyer in 1932 at Bombay. He expressed his keen desire to extend the activities of the Society far and wide and to hold conferences of the Society in various parts of the country. It was under his inspiring influence that I have been closely connected with the Society in various capacities as Secretary, Treasurer and President. In fulfilment of his wish, conferences of the Society have since been held in all parts of the country. After 1951, on account of the greater output of research work and the larger number of invitations from various universities to hold conferences under their auspices, it was decided to have conferences every year instead of biannually and it is gratifying that the number of papers contributed to each yearly conference has been not less than fifty. The achievements and reputation of our Society have not been confined to this country only but have extended to other parts of the world also. This year when I visited the States and also attended the International Congress of Mathematicians at Edinburgh, I had an opportunity of meeting several foreign mathematicians, and I was very glad to hear praise of the work done by Indian Mathematicians, almost all of whom are members of our Society.

Our Society and its journals are held in high esteem abroad and the work that has been done by our members starting from the late Srinivasa Ramanujan, whose first paper 'On some properties of

Bernoulli's numbers' was published in the Journal of our Society in 1911, is spoken of in very high terms. I offer my greetings to the Society on its successful completion of 50 years of excellent and useful service in the cause of Advanced Mathematics and Mathematical Research in our country.

Mathematics today is a different subject than it was a generation ago, having been transformed in many respects by the activities of research mathematicians. School and College instruction has, however, not adequately reflected these changes with the result that, there is today a big gap between the research literature on one hand and the curriculum for text book writers on the other.

The applications of mathematics have been greatly extended in recent years, particularly in the social sciences. Previously mathematical methods could generally be applied only to phenomena amenable to a deterministic description, now methods have been developed for dealing with phenomena in which *chance* plays a role. The modern approach to mathematics is as "*the study of all possible patterns*" (Sawyer). The needs of mathematics itself, of physical science, biological science, social science, technology, engineering, and industry as these needs exist in the second half of the twentieth century, should determine the orientation and content of the school and college curriculum in mathematics.

On account of the dramatic successes of mathematical theory in nuclear physics and the emergence of the digital computer technology new reforms are being introduced in mathematics teaching in the scientifically advanced countries. These reforms consist mainly in two ways, viz. introducing "modern" ideas and throwing out the dead wood, and in modifying the teaching of mathematics in the light of computation technology, i.e. to teach programming, switching theory, etc.

In mathematics we aim to teach, in essence, two kinds of skills. Skill in carrying out the calculations and demonstrations needed in treating a given mathematical problem and skill in making and understanding mathematical abstractions so that new situations may be mastered. To acquire these skills our students need drill, practice, and the experience of struggling with hard problems and difficult concepts. They need discipline—first the discipline imposed by a good teacher ; ultimately the self-discipline learnt under his guidance. It is still true that there is no royal road to the mastery of mathematics. The schools have to produce this discipline. School students should learn to think or to speak with precision, to write clearly and in good order, to finish a task down to the last details, or to persist in the face of real intellectual difficulties. The school mathematics of tomorrow will be very different from that of today. Teachers will be better grounded in the fundamentals of the mathematics they are to teach, and in order to increase their effectivenes, they will make use of various kinds of audio-visual aids like films, film-strips, television, didactic machines, etc.

Every teacher of mathematics should try to become a creator and should try to discover and arouse creativeness in his pupils. We are living in a scientific and technological age. In the 2nd five year plan of our country, stress has been laid on technical and industrial development. We need therefore to train more engineers and scientists. Scientific knowledge and its application to human affairs have expanded. If knowledge expands, then the content of education must be altered accordingly, and if knowledge is to be applied extensively, then education must include training those who are to make the applications. The gulf between mathematics and its applications can be bridged in two ways : *firstly*, by inducing mathematicians to bring forward their results in a form easily understandable by 'appliers' with *scanty* mathematical training and *secondly*, by teaching such appliers the special mathematical results and techniques they require.

Our great task is and will continue to be, the task of kindling intellectual curiosity in our students besides elaborating new curriculums, inventing new methods of pedagogy, revising the virtues of intellectual discipline with all the zeal and wisdom of which we are capable.

Our Society will perform a highly constructive service if it plays an important role in the task of national reconstruction by contributing to the solution of some national problems like that of the prevention of floods, and thus renders still more useful service to Mathematics and the progress of our country.

SOME FUNCTIONS OF RAMANUJAN*

By D. H. LEHMER

THE two basic operations, addition and multiplication give rise to a simple branching of the theory of numbers into additive and multiplicative number theory. Often however, a result or a problem is a Combination of the two features. In the case of Goldbach's problem one may say that this attempt to combine multiplicative primes by addition is unnatural and so unfortunate. There are many happier instances of this type of marriage however especially those connecting the divisors of a given number or set of numbers. Results initiated by Euler and Jacobi, have been added to by Glaisher and greatly extended by Ramanujan and broadened by many recent mathematicians, especially members of the I.M.S. It is about this class of results that I propose to talk.

Perhaps the most conspicuous class of numerical functions consists of the so-called *multipticative* functions, namely those for which the property

$$f(m)\,f(n) = f(m\,n), \quad (m,n) = 1 \tag{1}$$

holds for every pair of coprime integers m and n. The complete solution of this functional equation is obtained by assigning arbitrary values to $f(p^\alpha)$ for each prime p and each exponent α. The remaining values of f are then determined uniquely and consistently by (1).

A function g is called *purely multiplicative* in case

$$g(m)\,g(n) = g(m\,n) \tag{2}$$

holds for every pair of integers m and n, coprime or not. This small subclass of multiplicative functions is generated by assigning arbitrary values to $g(p)$ for each prime p. All composite values are then determined by (2). In particular

*Invited address delivered at the Golden Jubilee Session of the Indian Mathematical Society, December 1958 in Poona.

$$g(p^\alpha) = g(p)\, g(p^{\alpha-1}) = \{g(p)\}^\alpha.$$

A larger subclass of multiplicative functions, that includes the purely multiplicative ones, may (for want of a more descriptive name) be called *specially multiplicative*. For these functions

$$f(m)\, f(n) = \sum_{\delta/(m,n)} f(mn/\delta^2)\, g(\delta), \tag{3}$$

the sum, as indicated extending over all divisors δ of the greatest common divisor of m and n. The function g is supposed to be purely multiplicative. The most general solution of (3) may be found by assigning arbitrary values to $f(p)$ for each prime p and determining $f(p^\alpha)$ recursively by

$$f(p^{\alpha+1}) = f(p)\, f(p^\alpha) - g(p)\, f(p^{\alpha-1})$$

which is, after all, an instance of (3). Such functions appear quite naturally in several branches of number theory. Perhaps the simplest functions, that are not purely multiplicative but are specially so, are the number of divisors of n and the sum

$$\sigma(n) = \sum_{\delta/n} \delta$$

of all the divisors of n. More generally

$$\sigma_k(n) = \sum_{\delta/n} \delta^k$$

and, still more generally, $n^\lambda \sigma_k(n)$ are specially multiplicative functions. For the latter function, which we call a *basic divisor function*, $f(p) = p^r + p^{r+k}$ and $g(n) = n^{2r+k}$.

The simple fact that $\sigma(n)$ is multiplicative is a comparatively trivial example of a combined additive and multiplicative property. There are many more elaborate results. One class of such results we proceed to consider. The first member of this class was discovered by Glaisher [1] in 1884 when he observed that

$$12 \sum_{k=1}^{n-1} \sigma(k)\, \sigma(n-k) = 5\, \sigma_3(n) - 6\, n\, \sigma(n) + \sigma(n). \tag{4}$$

This connects the divisors of n, on the right, with those of the numbers $< n$ on the left. This is the simplest example of the composition of basic divisor functions expressed as a linear combination of a fixed number of such functions. Another Glaisher result is

$$12 \sum_{k=1}^{n-1} k\,\sigma(k)\,(n-k)\,\sigma(n-k) = n^2\,\sigma_3(n) - n^3\,\sigma(n).$$

He also evaluated

$$\sum_{k=1}^{n-1} \sigma_i(k)\,\sigma_j(u-k) \tag{4'}$$

for $(i, j) = (3, 3), (3, 5), (3, 9)$.

These results were rediscovered and extended by Ramanujan [2] in 1918 who discovered a basic theory of such sums pervading other parts of number theory and arising from the Weierstrassian theory of elliptic functions. In particular he gave the nine possible examples of sums of the type (4′) that are expressible in terms of basic divisor functions, namely those corresponding to

$(i, j) = (1, 1), (1, 3), (1, 5), (1, 7), (1, 11), (3, 3), (3, 5), (3, 9), (5, 7).$

We note in passing that in all cases these indices are odd. There is no theory for $\sigma_{2k}(n)$ and no observed phenomena.

S. Chowla [3] in 1947 gave a result equivalent to

$$192 \sum_{k_1+k_2+k_3=n} \sigma(k_1)\,\sigma(k_2)\,\sigma(k_3) = 7\,\sigma_5(n) - 10(3n-1)\,\sigma_3(n) + (24n^2 - 12n + 1)\,\sigma(n)$$

and stated that similar formulas hold for such sums of multiplicities 4 and 5 but apparently not for 6.

All these sums are instances of the general m-fold composition of m basic divisor functions

$$S_n(r_1, r_2, \dots, r_m \mid s_1, s_2, \dots, s_m) = \sum k_1^{r_1}\,\sigma_{s_1}(k_1) \dots k_m^{r_m}\,\sigma_{s_m}(k_m) \tag{5}$$

the sum extending over all positive integral solutions $(k_1, \dots, k_m)$ of

$$k_1 + k_2 + \dots + k_m = n.$$

It is natural to ask if such a sum is a linear combination of basic divisor functions $n^r \sigma_3(n)$. The answer is no, except for 37 cases. These exceptional cases can be read from a table of Lahiri [4]. Sums S_n may be classified by their " weight " which we take to be the even number

$$w = m + 2(r_1 + \dots + r_m) + s_1 + s_2 + \dots + s_m.$$

Of the 37 formulas mentioned above one is of weight 4 and is Glaisher's (4). Three are of weight 6, nine of weight 8, nineteen of weight 10, one of weight 12, and four of weight 14. For example the single sum of weight 12 is $S_n(1, 0, 0, 0, 0, \mid 1, 1, 1, 1, 1,)$. In fact,

$$1658880 \sum k_1\, \sigma(k_1)\, \sigma(k_2) \dots \sigma(k_5)$$

$$= 11\, n\, \sigma_9(n) - 50 n(3n-2)\, \sigma_7(n) + 30\, n(24 n^2 - 28\, n + 7)\, \sigma_5(n) -$$

$$- 20\, n\,(72\, n^3 - 108\, n^2 + 45\, n - 5)\, \sigma_3(n) +$$

$$+ n\,(864\, n^4 - 1440\, n^3 + 720\, n^2 - 120\, n + 5)\, \sigma(n).$$

Incidentally

$$5\, S_n(1, 0, 0, 0, 0 \mid 1, 1, 1, 1, 1) = n\, S_n(0, 0, 0, 0, 0 \mid 1, 1, 1, 1, 1).$$

Except for these 37 cases the sum (5) involves new and more or less imperfectly understood numerical functions of n the simplest of which is the celebrated τ-function of Ramanujan discussed later on.

To avoid such functions one may form certain linear combination of two or more sums (5) of equal weight and obtain again basic divisor functions. Thus for example

$$\sum_{k=1}^{n-1} k^4\, \sigma(k)\, \sigma(n-k) - \sum_{k=1}^{n-1} k^2\, \sigma(k)\, (n-k)^2\, \sigma(n-k)$$

$$= n^7\, [\, \sigma_3(n) - (2n-1)\, \sigma(n)\,]/24$$

or, again,

$$22 \sum_{k=1}^{n-1} \sigma_7(k)\, \sigma_9(n-k) - \sum_{k=1}^{n-1} \sigma_3(k)\, \sigma_{13}(n-k)$$

$$= [\, \sigma_{13}(n) - 11\, \sigma_9(n) + 20\, \sigma_7(n) - 10\, \sigma_3(n)\,]/240.$$

By comparing orders of magnitude in this last expression one sees that the second sum is very nearly 22 times as large as the first. This is a rather unexpected fact about the powers of the divisors of consecutive integers. Such formulas are sources of congruence properties of $\sigma_k(n)$ since the right members must be integers.

The number of such formulas is unlimited. However if one restricts the number of sums being combined, only a finite number of these combinations result in basic divisor functions alone. The basic theory here is that of Eisenstein's sum

$$G_k = G_k(w_1, w_2) = \sum_{m_1, m_2 = -\infty}^{\infty} (m_1 w_1 + m_2 w_2)^{-k}$$

which vanishes identically for odd integers k but for $k = 2n \geqslant 4$ has the Fourier development

$$\tfrac{1}{2}(2n-1)!\,\{w_1/2\pi i\}^{2n}\, G_{2n} = -\frac{B_{2n}}{4n} + \sum_{m=1}^{\infty} \sigma_{2n-1}(m)\, e^{2\pi i \tau m},$$

where $\tau = w_2/w_1$ lies in the upper half plane, $I(\tau) > 0$, and where B_{2n} is the Bernoulli number in the even suffix notation of Lucas. ($B_4 = -1/30$). Thus G_{2n} and its successive derivatives with respect to τ generate the basic divisor functions. On the other hand the G's themselves are generated by Weierstrass' function

$$\wp(Z) = Z^{-2} + \sum_{n=1}^{\infty} (2n+1)\, G_{2n+2}\, Z^{2n}.$$

Finally

$$\wp'''(Z) = 12\,\wp(Z)\,\wp'(Z).$$

This differential equation implies relations between the G's and hence other relations between basic divisor functions.

Ramanujan preferred to normalize G_4 and G_6 by dividing them by their unwieldy constant terms. Introducing a function P, which is essentially the η-function of Weierstrass, he took the following three functions as a basis

$$P = 1 - 24 \sum_{n=1}^{\infty} \sigma(n)\, \chi^n = 1 - 24\, \Phi_{0,1}$$

$$Q = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, \chi^n = 1 + 240\, \Phi_{03}$$

$$R = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)\, \chi^n = 1 - 504\, \Phi_{05},$$

where $\chi = e^{2\pi i \tau}$ and in general

$$\Phi_{r,s+r} = \sum_{n=1}^{\infty} n^r\, \sigma_3(n)\, \chi^n$$

is the power series generator of the basic divisor function. We are concerned with the case in which S is odd and we define the weight w of $\Phi_{r,s}$ to be the even number

$$w = r + s + 1.$$

To the monomial $P^a Q^b R^c$ we give the weight

$$w = 2a + 4b + 6c.$$

Every $\Phi_{r,w-r-1}$ of weight w can be expressed as a polynomial in P, Q, R whose terms are monomials of weight w (except for a constant term in case $r = 0$). In case $w < 12$ there can be no term in which the Q occurs to a power greater than two since the weight of Q^3 is 12. In these cases explicit formulas for $\Phi_{r,s}$ can be given. There are two cases according as $s = r + 1$ or not.

$$24\, \Phi_{r,r+1} = \left[\frac{1}{r+1}\right] - \frac{r!}{12^r} \sum_{a=0}^{r+1} (-1)^{r-a} (r-a) \binom{r+1}{a} P^a Q^b R^c \quad (6)$$

$$\Phi_{r,w-r-1} = \frac{(-1)^r (w-r-1)!\, B_{w-2r}}{2(w-2r)!\, 12^r} \left\{ \left[\frac{1}{r+1}\right] - \right.$$

$$\left. - \sum_{a=0}^{r} (-1)^a \binom{r}{a} P^a Q^b R^c \right\}, \quad (7)$$

where $r < w - r - 2$. The exponents b and c are functions of a. In fact in both formulas,

$$b = w - 2a - 3\left[\frac{w-2a}{3}\right]$$

$$c = a + 2\left[\frac{w-2a}{3}\right] - \tfrac{1}{2}w.$$

Both formulas may be inverted to give

$$P^a Q^b R^c = 1 - 2 \sum_{0 \leqslant r \leqslant \frac{1}{2}w} \binom{a}{r} \frac{12^r (w-2r)!}{(w-r-1)!\, B_{w-2r}} \Phi_{r,w-r-1}, \quad (8)$$

Provided $w = 2a + 3b + 4c < 12$. Hence any product of Φ's whose combined weight does not exceed 10 can be expressed as a linear combination of other Φ's. The corresponding sum (5) is then a linear combination of basic divisor functions.

When the weight $w \geqslant 12$ formulas (6), (7) and (8) become congruences modulo Δ where

$$\Delta = Q^3 - R^2 = 12^3 \chi \prod (1 - \chi^n)^{24} = 12^3 \sum_{n=1}^{\infty} \tau(n)\, \chi^n.$$

This introduces the Ramanujan function $\tau(n)$. We see that by successively replacing Q^3 by $R^2 + \Delta$ any Φ or product of Φ's has an expansion of the type

$$F(P, Q, R) + \Delta F_1(P, Q, R) + \Delta^2 F_2(P, Q, R) + \cdots$$

in which F_i is a polynomial not involving Q^b for $b > 2$.

Examples of the actual occurrence of $\tau(n)$ in sums (5) of weight 12 are

$$174132 \sum_{k=1}^{n-1} \sigma_5(k)\, \sigma_5(n-k) = 65\, \sigma_{11}(n) - 691\, \sigma_5(n) - 756\, \tau(n) \quad (9)$$

$$840 \sum_{k=1}^{n-1} k^2\, \sigma(k)\, (n-k)^2\, \sigma(n-k) = 15\, n^4\, \sigma_3(n) - 14\, n^5\, \sigma(n) - \tau(n). \quad (10)$$

Both formulas have been used to calculate $\tau(n)$ for isolated values of n. The first was used with a punched card table of $\sigma_5(n)$ and a comparatively slow multiplier. The second has been used with a fully automatic highspeed computer. Values of $\tau(n)$ for n about 16000 are obtained in a few minutes.

A large number of other sums (5) of weight 12 have been given by Lahiri [4].

Perhaps the most startling fact about $\tau(n)$ is that it behaves like a basic divisor function in being also specially multiplicative with $g(n) = n^{11}$, that is

$$\tau(p^{\alpha+1}) = \tau(p)\,\tau(p^{\alpha}) - p^{11}\,\tau(p^{\alpha-1})$$

for p a prime. However $\tau(p)$ is certainly not a polynomial in p. Our knowledge about the order of magnitude of $\tau(n)$, though not complete, is sufficient to show this.

Nevertheless $\tau(p)$ behaves like a polynomial (mod m) for many small moduli m. For example it follows at once from (10) that

$$\tau(n) \equiv 14\,n^5\,\sigma(n) - 15\,n^4\,\sigma_3(n) \pmod{840}$$

and from (9) that

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691} \tag{11}$$

a remarkable fact discovered by Ramanujan. Contributors to the study of the congruence properties of $\tau(n)$ include nearly every number theoretic member of the I.M.S. and quite a few non-members. Thus far $\tau(p)$ is congruent to a polynomial in p with respect to the moduli 2^{11}, 3^7, 5^3, 7, 691. There appear to be no other primes for which this is true, certainly none less than 1250. There are however certain known properties modulo 49 and 23. The moduli 3^7 or 5^3 cannot be replaced by 3^8 or 5^9.

The simple question of the possible vanishing of $\tau(n)$ remains unsolved. However it can be shown that $\tau(n) = 0$ implies $n \geqslant 113740236287999$.

The fact that $\tau(p)$ is specially multiplicative was discovered by Ramanujan empirically and proved by Mordell [5] in 1917. This property follows from the fact that

$$\Delta = 18662400\,[20\,G_4^3 - 49\,G_6^2]$$

is a modular form of dimension -12 when considered as a function of w_2 and w_1. That is

$$\Delta(w_1, w_2) = \Delta(\alpha w_1 + \beta w_2, \gamma w_1 + \delta w_2)$$
$$= \lambda^{12} \Delta(\lambda w_1, \lambda w_2)$$

whenever $\alpha\delta - \beta\gamma = 1$, facts which follow immediately from the definition of G_{2n}. Hecke [6] developed an extensive theory of such functions whose Fourier coefficients are specially multiplicative. There are in fact six functions $\tau_1(n), \ldots, \tau_6(n)$ which we may call the functions of Ramanujan since he made very brief mention of their generators

$$\Delta, \; Q\Delta, \; R\Delta, \; Q^2\Delta, \; RQ\Delta, \; Q^2R\Delta$$

which are functions of weights 12, 16, 18, 20, 22, and 26 respectively. These weights we denote in general by d_k $(k = 1, 2, \ldots, 6)$. For example

$$Q^2\Delta = 12^3 \sum{}' \tau_4(n) \chi^n = 12^3 (\chi + 456 \chi^2 + 50652 \chi^3 + \ldots)$$

and

$$132.174611 \sum \sigma_9(k) \sigma_9(n-k) = 25 \sigma_{19}(n) + 174611 \sigma_9(n) - \\ - 174636 \tau_4(n). \quad (12)$$

All the six functions are specially multiplicative with $g(n) = n^{d_k - 1}$. There are many congruence properties of the $\tau_k(n)$. Thus if one takes the formula (12) modulo 174611 one obtains at once as a counterpart of (11)

$$\tau_4(n) \equiv \sigma_{19}(n) \pmod{174611}.$$

In general

$$\tau_k(n) \equiv \sigma_{d_k - 1}(n) \pmod{N_{d_k}},$$

where N_{d_k} is the numerator of the Bernoulli number of suffix d_k. Other examples are

$$\tau_2(n) \equiv n^2 \tau(n) \pmod{13}$$

$$\tau_5(n) \equiv \tau(n) \pmod{11}.$$

There are other properties like those of the Riemann zeta-function. If we let

$$Z_k(S) = \Gamma(S)(2\pi)^{-s} \sum_{n=1}^{\infty} \tau_k(n) n^{-s},$$

then there is the functional equation

$$Z_k(S) = i^{d_k} \; Z_k(d_k - S)$$

and the hypothesis that the complex zeros of $Z_k(S)$ lie on the critical line $Re\ (S) = \frac{1}{2}\, d_k$.

A more elaborate identity involving the Bessel functions

$$K_\nu(2t) = t^\nu \int_0^\infty y^{-1-\nu} \exp\{-y - t^2 y^{-1}\}\, dy$$

is

$$2 \sum_{n=1}^{\infty} \tau_k(n)\, K_\nu(4\pi s \sqrt{n})\, n^{-\nu/2}$$

$$= i^{d_k}(2\pi)^{\nu - d_k} s^{-\nu}\, \Gamma(d_k - \nu) \sum_{n=1}^{\infty} \frac{\tau_k(n)}{(n + s^2)^{d_k - \nu}}\,.$$

For example if $k = 4$, $S = 1$ and $\nu = -\frac{1}{2}$ we have

$$\sum_{n=1}^{\infty} \tau_4(n)\, e^{-4\pi\sqrt{n}} = \sqrt{2}(2\pi)^{-41/2}\, \Gamma\left(\frac{41}{2}\right) \sum_{n=1}^{\infty} \tau_4(n)\, (n+1)^{-41/2}.$$

There are simpler results like

$$\sum_{n=1}^{\infty} \tau_4(n)\, e^{-2\pi n} = \begin{cases} T^6 & \text{if } k = 1 \\ T^8/12 & \text{if } k = 2 \\ T^{10}/12^2 & \text{if } k = 4 \\ 0 & \text{otherwise,} \end{cases}$$

where $T = \frac{1}{2}\,\Gamma(\frac{1}{4})^4\,(2\pi)^{-3}$.

If we admit answers in terms of these six functions we can give many more formulas for sums of the type (5), even including $\sum_{k=1}^{n-1} \sigma_{11}(k)\,\sigma_{13}(n-k)$ which, involves only $\tau_6(n)$.

However, when we ask about the sum $\Sigma\ \sigma_{11}(k)\ \ \sigma_{11}(n-k)$, that goes with Δ^2, the corresponding function fails to be multiplicative even in the weak sense. The road seems to stop. There is nevertheless a narrow pathway still open if one is searching for new specially multiplicative functions. For this, one must blend two linearly independent generators like $R^2\Delta$ and Δ^2 of weight 24. The appropriate blend in this case is

$$12^{-3} R^2\Delta + B\Delta^2 = \sum_{n=1}^{\infty} \tau_7(n)\, \chi^n \text{ with } B = 12^{-5}\{131 + (\sqrt{144169})\}.$$

Thus the values of $\tau_7(n)$ are integers in the quadratic field $K\sqrt{(144169)}$ as noted by Hecke. Thus

$\tau_7(1) = 1$, $\tau_7(2) = 12(45 + \sqrt{D})$, $\tau_7(3) = 36(4715 - 16\sqrt{D}), \ldots$ where $D = 144169$. Still $\tau_7(n)$ is specially multiplicative with $g(n) = n^{a^3}$. There are altogether six functions $\tau_k(n)$ $(k = 7(1)\,12)$ with values in the field $K(\sqrt{D})$ for the following values of D

144169, 131.139, 51349, 18295849, 479.4919, 181.349.1009

which are specially multiplicative with $g(n) = n^{d_k - 1}$ for the following values of d_k

$$d_k = 24, 28, 30, 32, 34, 38.$$

Next we come to the generator Δ^3 which we must now blend with two other functions ΔG_{24}, $\Delta^2 G_2$ to obtain a new specially multiplicative function with $g(n) = n^{35}$. The values however lie in a cubic field in fact the field determined by the cubic equation

$$y^3 - 1376111721422y^2 - 5742145719432261916155855y$$
$$- 140057.1281422085609163714549762600000$$

the discriminant of the field being

$$D = 2^{12}\, 3^4\, 5^2\, 7^2\, 23.1259\, (236364091)^6.269461929553.$$

There are, in all, six such functions corresponding to

$$d_k = 36, 40, 42, 44, 46, 50$$

whose properties are almost completely unknown.

Without following this trail any further we should say in conclusion that the theory of Ramanujan's $\tau_k(n)$ is really much more elaborate than this one-dimensional picture I have attempted. Thus nothing has been said about the quadratic forms in many variables which go with these functions, nor of the analytic theory for studying their asymptotic behaviour. The beautiful theory of Hecke and its generalizations have only been alluded to. Such developments would require much more time on my part and patience on your part.

REFERENCES

1. J. W. L. Glaisher : On the square of the series in which the coefficients are the sums of the divisors of the exponents, *Messenger of Math.* 14 (1884-5) 156-163.

2. S. Ramanujan : On certain arithmetical functions, *Trans. Camb. Phil. Soc.*, 22, (1916), 158-184 ; and *Collected Papers*, 136-162.

3. S. Chowla : Note on a certain arithmetical sum, *Proc. Nat. Insti. Sci. India*, 13, No. 5, (1947).

4. D. B. Lahiri : On Ramanujan's function $\tau(n)$ and the divisor function $\sigma(n)$ I, II, *Bull. Calcutta Math. Soc.* 38 (1946), 193-206, and 39 (1947), 33-52.

5. L. J. Mordell : On Mr. Ramanujan's empirical expansions of modular functions, *Proc. Camb. Phil. Soc.* 19 (1917), 117-124.

6. E. Hecke : "Ober Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwiklung", I, II *Math. Annalen*, 114 (1936), 1-28 ; 316-351.

SLOWLY OSCILLATING FUNCTIONS AND THEIR APPLICATIONS*

R. BOJANIC

1. A real-valued function $L(x)$ defined for all $x \geqslant 0$ belongs to the class of *slowly oscillating* functions at infinity if

I_1 : $L(x)$ is positive and continuous in $0 \leqslant x < \infty$;

I_2 : $\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1$ for every fixed $\lambda > 0$.

A slowly oscillating function has the following representation :

$$L(x) = c(x) \exp \left(\int_1^x \frac{\epsilon(t)}{t} \, dt \right), \tag{1.1}$$

where $c(x)$ is a positive, continuous function which tends to a positive limit as $x \to \infty$, and $\epsilon(x)$ is a continuous function which tends to 0 as $x \to \infty$.

From the representation (1.1) many properties of slowly oscillating functions can be obtained. We shall mention here some of these properties.

(i) The asymptotic relation

$$\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1$$

holds uniformly on every closed interval $a \leqslant \lambda \leqslant C$, $(0 < a < C < \infty)$.

(ii) For every $\alpha > 0$

$$x^{\alpha} L(x) \to \infty, x^{-\alpha} L(x) \to 0, x \to \infty.$$

(iii) If $\alpha \geqslant 0$ and

$$L_1(x) = x^{-\alpha} \max_{0 \leqslant t \leqslant x} \{t^{\alpha} L(t)\}, L_2(x) = x^{\alpha} \max_{x \leqslant t < \infty} \{t^{-\alpha} L(t)\},$$

* The invited address delivered at the Golden Jubilee Session of the Indian Mathematical Society, December 1958 in Poona. This forms part of the work done when the author was the visiting member of the Tata Institute of Fundamental Research, Bombay. The author wishes to thank Prof. K. Chandrasekharan for valuable discussions and suggestions during the preparation of this paper.

then $L_k(x) \simeq L(x)$, $x \to \infty$, $k = 1, 2$, and $L_1(x)$, $L_2(x)$ are slowly oscillating functions.(**)

(iv) If $f(x)$ is such that both integrals

$$\int_0^1 t^{-x} \, |f(t)| \, dt, \int_1^\infty t^x \, |f(t)| \, dt \tag{1.2}$$

exist for some $x > 0$, then

$$\int_0^\infty f(t) L(xt) \, dt \simeq L(x), \int_0^\infty f(t) \, dt, x \to \infty. \tag{1.3}$$

The definition of slowly oscillating functions given here, as well as their representation (1.1), is due to J. Karamata [1, 2]. J. Korevaar, T. van Aardenne-Ehrenfest and N. G. de Bruijn [3] and H. Delange [4] have deduced the uniform convergence property (i) of slowly oscillating functions directly from the definition. The other properties, and in particular, Karamata's representation theorem, follow easily.

Finally, S. Aljančić, R. Bojnaić, and M. Tomic [5] have proved property (iv) of slowly oscillating functions.

A slightly more general class of functions of " regular behaviour " at infinity is defined as follows :

A real-valued function $\phi(x)$ defined for all $x \geqslant 0$ belongs to the class of functions of " regular behaviour " at infinity if

Π_1 $\phi(x)$ is positive and continuous in $0 \leqslant x < \infty$

Π_2 $\lim\limits_{x \to \infty} \dfrac{\phi(\lambda x)}{\phi(x)} = h(\lambda)$ exists for every fixed $\lambda > 0$.

J. Karamata (2) has shown that Π_1 and Π_2 imply that $h(\lambda) = \lambda^\alpha$, where $-\infty < \alpha < \infty$. The number α is called the exponent of $\phi(x)$. It follows that a function of regular behaviour with the exponent α has the representation

**$f(x) \simeq g(x)$, $x \to \infty$ means that $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = 1$.

$$\phi(x) = x^{\alpha} L(x), \tag{1.4}$$

where $L(x)$ is a slowly oscillating function.

The well-known examples of slowly oscillating functions are for instance

$$lg^{\sigma}(x^{+2}),\ -\infty < \sigma < \infty,\ \left(2 + \frac{x_n x}{x}\right) lg(2 + x),$$

then all finite iterations of these functions, every continuous function which oscillates between

$$lg(x^{+2}) \text{ and } lg(x^{+2}) + lg^{\delta}(x^{+2}),\ 0 \leqslant \delta < 1,$$

every positive function $f(x)$ which tends to a positive limit as $x \to \infty$, etc.

The slowly oscillating functions, and more generally, the functions of regular behaviour appear naturally in problems connected with asymptotic evaluations of certain integrals and sums. In section 2 we shall consider a problem of this type from the theory of multiple Fourier series. Section 3 contains a remark on a class of averaging functions used by S. Bochner and K. Chandrasekharan [6] in connection with some extensions of Wiener's general Tauberian theorem.

2. Let $f(x, y)$ be a real-valued, L-integrable function, periodic with period 2π in each variable. For a fixed point (x, y), the circular mean of $f(x, y)$ is defined by

$$M_{xy}(t) = \frac{1}{2\pi} \int_0^{2\pi} f(x + t \cos \theta, y + t \sin \theta)\, d\theta.$$

If

$$C_{pq} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\xi, \eta)\, e^{-i(p\xi + q\eta)}\, d\xi\, d\eta$$

and

$$A_n(x, y) = \sum_{p^2+q^2=n} c_{pq}\, e^{i(px+qy)},$$

then the Riesz mean of circular partial sums of $f(x, y)$ of order δ is defined by

$$S_R^\delta(x, y) = \sum_{\nu=0}^{n} \left(1 - \frac{\nu}{R^2}\right)^\delta A_\nu(x, y), \quad n \leqslant R^2 < n + 1.$$

The following theorems which connect the asymptotic behaviour of $S_R^\delta(x, y)$ as $R \to \infty$, with the properties of the circular mean. $M_{xy}(t)$ as $t \to 0$, are well known [8].

If $M_{xy}(t) \to l$ as $t \to 0$, for a fixed (x, y), then

$$\lim_{R \to \infty} S_R^\delta(x, y) = l \text{ if } \delta > \tfrac{1}{2}.$$

If at a point (x, y), $M_{xy}(t) = O(t^\theta)$, $\theta > 0$, as $t \to 0$, then for $\delta > \theta + \frac{1}{2}$

$$S_R^\delta(x, y) = O(R^{-\theta}), \quad R \to \infty.$$

We shall prove here a slightly more general theorem of this type.

THEOREM 1. *Assume that $\delta > \frac{1}{2}$ and $\frac{1}{2} - \delta < \alpha < 2$. If at a fixed point (x, y)*

$$M_{xy}(t) \simeq t^{-\alpha} L(1/t), \; t \to 0, \tag{2.1}$$

where $L(x)$ is a slowly oscillating function at infinity, then

$$S_R^\delta(x, y) \simeq 2^{-\alpha} \frac{\Gamma(\delta_{+1})\, \Gamma(1 - \alpha/2)}{\Gamma(\delta + \alpha/2 + 1)} R^\alpha L(R), \; R \to \infty.$$

A function $f(x, y)$ which satisfies the condition (2.1) at the point (0,0) is for instance, any function of the form

$$f(x, y) = (x^2 + y^2)^{-\frac{1}{2}}\, lg\,\{2 + (x^2 + y^2)^{-\frac{1}{2}}\}\, g(x, y),$$

where $g(x, y) \to A > 0$ as $(x, y) \to (0, 0)$. In this case,

$$M_{00}(t) = \frac{1}{2\pi} t^{-1}\, lg\left(2 + \frac{1}{t}\right) \int_0^{2\pi} g(t\cos\theta, t\sin\theta)\, d\theta \simeq A\, t^{-1}\, lg\, \frac{1}{t}, \, t \to 0.$$

Therefore,

$$S_R^\delta(0, 0) \simeq \tfrac{1}{2} A \sqrt{\pi^{\frac{1}{2}}} \frac{\Gamma(\delta + 1)}{\Gamma(\delta + 3/2)} R\, lg\, R, \; R \to \infty.$$

The proof of the theorem depends on the well-known formula of Bochner which expresses the Riesz mean $S_R^\delta(x, y)$ in terms of the circular mean $M_{xy}(t)$ of $f(x, y)$, for $\delta > \frac{1}{2}$.

$$S_R^\delta(x, y) = 2^\delta \Gamma(\delta + 1) R^{1-\delta} \int_0^\infty M_{xy}(t) t^{-\delta} J_{\delta+1}(Rt)\, dt, \qquad (2.2)$$

where $J_\mu(x)$ is the Bessel function of the first kind and of order μ

$$J_\mu(x) = \left(\frac{x}{2}\right)^\mu \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!\, \Gamma(\mu + \nu + 1)} \left(\frac{x}{2}\right)^{2\nu}$$

PROOF. We choose first η so that

$$\left| M_{xy}(t) - t^{-\alpha} L\left(\frac{1}{t}\right)\right| \leqslant \epsilon\, t^{-\alpha} L\left(\frac{1}{t}\right) \text{ for } 0 < t \leqslant \eta. \qquad (2.3)$$

Then we split the integral (2.2) in the following way :

$$S_R^\delta(x, y) = c_\delta R^{1-\delta} \int_0^\eta L\left(\frac{1}{t}\right) t^{-\alpha-\delta} J_{\delta+1}(Rt)\, dt +$$

$$+ c_\delta R^{1-\delta} \int_0^\eta \left\{ M_{xy}(t) - t^{-\alpha} L\left(\frac{1}{t}\right)\right\} t^{-\delta} J_{\delta+1}(Rt)\, dt$$

$$+ c_\delta R^{1-\delta} \int_\eta^\infty M_{xy}(t)\, t^{-\delta} J_{\delta+1}(Rt)\, dt$$

$$= I_1 + I_2 + I_3,$$

where $c_\delta = 2^\delta \Gamma(\delta + 1)$.

The evaluation of I_3 follows directly from the inequality [7, p. 117]

$$|I_3| = c_\delta R^{1-\delta} \left| \int_\eta^\infty M_{xy}(t)\, t^{-\delta} J_{\delta+1}(Rt)\, dt \right| \leqslant \frac{M}{R^{\delta-\frac{1}{2}}}$$

$$\leqslant \frac{M_1}{R^{\alpha+\delta-\frac{1}{2}} L(R)} R^\alpha L(R).$$

Since $\alpha + \delta - \frac{1}{2} > 0$, we have by the property (ii) of slowly oscillating

$$R^{\alpha+\delta-\frac{1}{2}} L(R) \to \infty,\ R \to \infty$$

and so

$$I_3 = o\{R^\alpha L(R)\}.$$

Now, from (2.3) it follows that

$$|I_2| \leqslant \epsilon\, c_\delta\, R^{1-\delta} \int_0^{\eta} L\left(\frac{1}{t}\right) t^{-\alpha-\delta}\, |J_{\delta+1}(Rt)|\, dt$$

$$\leqslant \epsilon\, c_\delta\, R^{\alpha} \int_{1/R\eta}^{\infty} L(Rt)\, t^{\alpha+\delta-2} \left|J_{\delta+1}\left(\frac{1}{t}\right)\right| dt$$

$$\leqslant \epsilon\, c_\delta\, R^{\alpha} \int_0^{\infty} L(Rt)\, t^{\alpha+\delta-2} \left|J_{\delta+1}\left(\frac{1}{t}\right)\right| dt.$$

From the inequalities

$$|J_{\delta+1}(x)| \leqslant M_2 x^{\delta+1},\ 0 \leqslant x \leqslant 1,\ |J_{\delta+1}(x)| \leqslant M_3/\sqrt{x},\ x \geqslant 1,$$

and $\frac{1}{2} - \delta < \alpha < 2$ follows that the function

$$f(t) = t^{\alpha+\delta-2}\, |J_{\delta+1}(1/t)|$$

satisfies conditions (1.2) if we choose $x > 0$ such that $x < 2 - \alpha$ and $x < \alpha + \delta - \frac{1}{2}$.

Then, as $R \to \infty$, by (1.3).

$$\int_0^{\infty} L(Rt)\, t^{\alpha+\delta-2} \left|J_{\delta+1}\left(\frac{1}{t}\right) dt \simeq L(R) \int_0^{\infty} t^{\alpha+\delta-2} \left|J_{\delta+1}\left(\frac{1}{t}\right)\right| dt,\right.$$

Hence

$$I_2 = o\,\{R^{\alpha} L(R)\},\ R \to \infty.$$

Finally,

$$I_1 = c_\delta\, R^{1-\delta} \int_0^{\eta} L\left(\frac{1}{t}\right) t^{-\alpha-\delta}\, J_{\delta+1}(Rt)\, dt$$

$$= c_\delta\, R^{\alpha} \int_{1/R\eta}^{\infty} L(Rt)\, t^{\alpha+\delta-2} J_{\delta+1}\left(\frac{1}{t}\right) dt$$

$$= c_\delta\, R^{\alpha} \left(\int_0^{\infty} - \int_0^{1/R\eta}\right) L(Rt)\, t^{\alpha+\delta-2} J_{\delta+1}\left(\frac{1}{t}\right) dt$$

$$= I_1' + I_1''.$$

Since the function

$$f(t) = t^{\alpha+\delta-2} J_{\delta+1}\left(\frac{1}{t}\right)$$

satisfies also conditions (1.2), it follows from (1.3) that

$$\int_0^\infty L(Rt)\, t^{\alpha+\delta-2} J_{\delta+1}\left(\frac{1}{t}\right) dt \simeq L(R) \int_0^\infty t^{\alpha+\delta-2} J_{\delta+1}\left(\frac{1}{t}\right) dt,\ R \to \infty.$$

Hence

$$I'_1 \simeq c_\delta R^\alpha L(R) \int_0^\infty t^{\alpha+\delta-2} J_{\delta+1}\left(\frac{1}{t}\right) dt,\ R \to \infty.$$

But, for $\frac{1}{2} - \delta < \alpha < 2$ we have

$$\int_0^\infty t^{\alpha+\delta-2} J_{\delta+1}\left(\frac{1}{t}\right) dt = \int_0^\infty t^{-\alpha-\delta} J_{\delta+1}(t)\, dt = \frac{\Gamma(1-\alpha/2)}{2^{\delta+\alpha}\,\Gamma(\delta+\alpha/2+1)}.$$

Therefore

$$I_1' \simeq 2^{-\alpha} \frac{\Gamma(\delta+1)\,\Gamma(1-\alpha/2)}{\Gamma(\delta+\alpha/2+1)} R^\alpha L(R),\ R \to \infty.$$

On the other hand,

$$\begin{aligned}
|I''_1| &\leqslant c_\delta R^\alpha \int_0^{1/R\eta} L(Rt)\, t^{\alpha+\delta-2} \left|J_{\delta+1}\left(\frac{1}{t}\right)\right| dt \\
&\leqslant M_4 R^\alpha \int_0^{1/R\eta} L(Rt)\, t^{\alpha+\delta-3/2}\, dt \\
&\leqslant \frac{M_4}{R^{\delta-\frac{1}{2}}} \int_0^{1/\eta} L(t)\, t^{\alpha+\delta-3/2}\, dt \\
&\leqslant \frac{M_5}{R^{\alpha+\delta-\frac{1}{2}} L(R)} R^\alpha L(R),
\end{aligned}$$

or

$$I''_1 = o\{R^\alpha L(R)\},\ R \to \infty.$$

Hence

$$I_1 = 2^{-\alpha} \frac{\Gamma(\delta+1)\,\Gamma(-\alpha/2)}{\Gamma(\delta+\alpha/2+1)} R^\alpha L(R) + o\{R^\alpha L(R)\},\ R \to \infty,$$

and the theorem is proved.

3. In connection with some extensions of Wiener's general Tauberian theorem, S. Bochner and K. Chandrasekharan [6] have defined a class of averaging functions of "slow growth" in the following way:

A function $\psi(x)$ in $-\infty < x < \infty$ belongs to the class Ψ_p, $p = 1, 2, \ldots$

1° $\psi(x)$ is positive continuous and has the value 1 in $-\infty < x < 0$;

2° for any finite real numbers a and c, $a < c$,

$$\lim_{x \to \infty} \frac{\psi(x + \lambda)}{\psi(x)} = 1$$

uniformly in $a \leqslant \lambda \leqslant c$;

3° there exist numbers m and M, depending on a and c such that

$$0 < m \leqslant \frac{\psi(x+\lambda)}{\psi(x)} \leqslant M < \infty$$

for $a \leqslant \lambda\ c$;

4° there is a constant C such that

$$\int_{-\infty}^{\infty} \frac{\psi(t)}{1 + |x - t|^{p+1}}\, dt \leqslant C\, \psi(x), \quad -\infty < x < \infty.$$

We wish to indicate here a closely related class of functions which have the properties 2° — 4°.

If we denote by R_α the class of functions of regular behaviour at infinity, with the exponent α, it is easy to see that for a function $\phi(x) \in R_\alpha$ the properties 2° and 3° hold for every α. The property 4° holds if $-1 < \alpha < p$ in a slightly more precise and general form.

THEOREM 2. *Suppose that the function $f(x)$ is integrable on very finite interval and that*

$$f(x) = o(x^{-p-1}),\ x \to \infty, \tag{3.1}$$

where $p > 0$. If $\phi(x) \in R_\alpha$, $-1 < \alpha < p$ then

$$\int_0^{\infty} f(|x - t|)\, \phi(t)\, dt \simeq 2\, \phi(x) \int_0^{\infty} f(t)\, dt,\ x \to \infty.$$

If we take, in particular, $f(x) = 1/(1 + x^{p+1})$, *we obtain that for every* $\phi(x) \in R_\alpha$, $-1 < \alpha < p$ *the following asymptotic relation holds:*

$$\int_0^\infty \frac{\phi(t)}{1 + |x - t|^{p+1}}\, dt \simeq \frac{2\pi}{p+1} \operatorname{cosec}\left(\frac{\pi}{p+1}\right) \phi(x),\; x \to \infty.$$

PROOF. The existence of the integral

$$I = \int_0^\infty f(|x - t|)\, \phi(t)\, dt$$

for $-1 < \alpha < p$ follows immediately from (3.1) and the properties of functions of regular behaviour mentioned in §1. In order to prove the theorem we split the integral I into three parts

$$I = \left(\int_0^{\frac{1}{2}x} + \int_{\frac{1}{2}x}^{(3/2)x} + \int_{(3/2)x}^{\infty}\right) f(|x - t|)\, \phi(t)\, dt = I_1 + I_2 + I_3.$$

The first of these integrals can be evaluated as follows: If $0 < \alpha + 1$, we can find η so that $0 < \eta < \alpha + 1$. Then, using (1.4) we have

$$\begin{aligned} |I_1| &= \left|\int_0^{\frac{1}{2}x} f(|x - t|)\, \phi(t)\, dt\right| \\ &\leqslant \int_0^{\frac{1}{2}x} |f(x - t)|\, t^{\alpha-\eta} \max_{0 \leqslant u \leqslant t} \{u^\eta L(u)\}\, dt \\ &\leqslant \max_{0 \leqslant u \leqslant x} \{u^\eta L(u)\} \int_0^{\frac{1}{2}x} |f(x - t)|\, t^{\alpha-\eta}\, dt \\ &\leqslant x^\eta L_1(x) \int_{\frac{1}{2}x}^{x} |f(t)|\, (x - t)^{\alpha-\eta}\, dt. \end{aligned}$$

Now, using (3.1) we see that

$$\int_{(3/2)x}^{x} |f(t)|\, (x - t)^{\alpha-\eta}\, dt \leqslant M_1 \int_{\frac{1}{2}x}^{x} t^{-p-1}(x - t)^{\alpha-\eta}\, dt$$

$$= M_1\, x^{\alpha-\eta-p} \int_{\frac{1}{2}}^{1} t^{-p-1}(1 - t)^{\alpha-\eta}\, dt.$$

The last integral exists since $\alpha - \eta > -1$. Therefore

$$|I_1| \leqslant M_2 x^{\alpha-p} L_1(x)' = M_2 x^{-p} \frac{L_1(x)}{L(x)} \phi(x),$$

or

$$I_1 = 0 \{\phi(x)\}, x \to \infty,$$

by the property (iii) of slowly oscillating functions.

The integral I_3, can be evaluated similarly. Since $0 < p - \alpha$ we can determine δ so that $0 < \delta < p - \alpha$. Then using again (1.4) we have

$$\begin{aligned} |I_3| &= \left| \int_{(3/2)x}^{\infty} f(|x-t|) \phi(t)\, dt \right| \\ &\leqslant \int_{(3/2)x}^{\infty} |f(t-x)|\, t^{\alpha+\delta} \max_{t \leqslant u < \infty} \{u^{-\delta} L(u)\}\, dt \\ &\leqslant \max_{x \leqslant u < \infty} \{u^{-\delta} L(u)\} \int_{(3/2)x}^{\infty} |f(t-x)|\, t^{\alpha+\delta}\, dt \\ &\leqslant x^{-\delta} L_2(x) \int_{\frac{1}{2}x}^{\infty} |f(t)|\, (t+x)^{\alpha+\delta}\, dt. \end{aligned}$$

Next, we have

$$\begin{aligned} \int_{\frac{1}{2}x}^{\infty} |f(t)|\, (t+x)^{\alpha+\delta}\, dt &\leqslant M_3 \int_{\frac{1}{2}x}^{\infty} t^{-p-1}(t+x)^{\alpha+\delta}\, dt \\ &= M_3\, x^{\alpha+\delta-p} \int_{\frac{1}{2}}^{\infty} t^{-p-1}(1+t)^{\alpha+\delta}\, dt \end{aligned}$$

and the last integral is finite since $p - \alpha - \delta + 1 > 1$. Consequently

$$|I_3| \leqslant M_4 x^{\alpha-p} L_2(x) = M_4 x^{-p} \frac{L_2(x)}{L_1(x)} \phi(x)$$

or

$$I_3 = o \{\phi(x)\}, x \to \infty,$$

by the property (iii) of slowly oscillating functions.

Hence we have to evaluate the integral

$$I_2 = \int_{\frac{1}{2}x}^{3x/2} f(|x-t|)\,\phi(t)\,dt = x\int_{\frac{1}{2}}^{3/2} f(|1-t|x)\,\phi(tx)\,dt.$$

Using the representation (1.4) and the property (i) of slowly oscillating functions, we can choose x_ϵ so large that

$$\left|\frac{\phi(xt)}{\phi(x)} - t^\alpha\right| < \epsilon \text{ for all } t \in [1/2, 3/2] \text{ and } x \geqslant X_\epsilon.$$

Then we have

$$I_2 = x\,\phi(x)\int_{\frac{1}{2}}^{3/2} f(|1-t|x)\,t^\alpha\,dt + x\phi(x)\int_{\frac{1}{2}}^{3/2}\left\{\frac{\phi(xt)}{\phi(x)} - t^\alpha\right\} f(|1-t|x)\,dt$$

$$= \phi(x)\,(I'_2 + I''_2).$$

First we have

$$I'_2 = x\int_{\frac{1}{2}}^{1} f((1-t)x)\,t^\alpha\,dt + x\int_{1}^{3/2} f((t-1)x)\,t^\alpha\,dt$$

$$= x\int_0^{\frac{1}{2}x} f(t)\left\{\left(1-\frac{t}{x}\right)^\alpha + \left(1+\frac{t}{x}\right)^\alpha\right\}dt$$

$$= 2\int_0^{\frac{1}{2}x} f(t)\,dt + \int_0^{\frac{1}{2}x} f(t)\left\{\left(1-\frac{t}{x}\right)^\alpha + \left(1+\frac{t}{x}\right)^\alpha - 2\right\}dt.$$

Now, by the mean-value theorem we have

$$(1-t)^\alpha + (1+t)^\alpha = 2 + \alpha t\,\{(1-\theta)^{\alpha-1} + (1+\theta)^{\alpha-1}\},\ 0 \leqslant \theta \leqslant t.$$

Hence, if $0 \leqslant t \leqslant \frac{1}{2}$, we have

$$|(1-t)^\alpha + (1+t)^\alpha - 2| \leqslant C_\alpha\,t,$$

where

$$c_\alpha = \begin{cases} (3/2)^{\alpha-1}\,|\alpha|, \text{ if } \alpha \geqslant 1 \\ (2^{1-\alpha}+1)\,|\alpha|, \text{ if } -1 < \alpha < 1. \end{cases}$$

Therefore

$$\left|\int_0^{\frac{1}{2}x} f(t)\left\{\left(1-\frac{t}{x}\right)^\alpha + \left(1+\frac{t}{x}\right)^\alpha - 2\right\}dt\right|$$

$$\leqslant M_5.\,\frac{1}{x}\int_0^x t\,|f(t)|\,dt \to 0,\ x \to \infty$$

because of (3.1). Hence

$$I'_2 \to 2 \int_0^\infty f(t)\, dt, x \to \infty.$$

Similarly,

$$|I''_2| \leqslant x \int_{\frac{1}{2}}^{3/2} \left| \frac{\phi(xt)}{\phi(x)} - t^\alpha \right| f(|1-t|x) |\, dt$$

$$\leqslant \epsilon x \int_{\frac{1}{2}}^{3/2} |f(|1-t|x)\, dt$$

and so

$$\leqslant 2\epsilon \int_0^{\frac{1}{2}x} |f(t)|\, dt,$$

$$I''_2 \to 0, x \to \infty.$$

Collecting all these evaluations, we find that

$$\lim_{x\to\infty} \frac{1}{\phi(x)} \int_0^\infty f(|x-t|)\, \phi(t)\, dt = 2 \int_0^\infty f(t)\, dt,$$

and the theorem is proved.

REFERENCES

1. J. KARAMATA : Sur un mode de croissance reguliere des fonctions *Mathematica,* (Cluj) 4 (1930), 38-53.

2. J. KARAMATA : Sur un mode de croissance reguliere, *Bull. Soc. Math.* (France) 61 (1933), 55-62.

3. J. KORE VAAR, T. VAN AARDENNE-EHRENFEST, N. G. DE BRUIJN : A note on slowly oscillating functions, *Nieuw Archief voor Wiskunde,* 23 (1940), 77-86.

4. H. DELANCE : Sur un théorème de Karamata, *Bull. Sci. Math.* II, 79 (1955), 9-12.

5. S. ALJANCIC, R. BOJANIC, M. TOMIC : Sur la valeur asymptotigue d-une classe des integrales definies, *Publ. Inst. Math. Acad. Serbe Sci.* 7 (1954), 81-94.

6. S. BOCHNEAR, K. CHANDRASEKHARAN : *Fourier Transforms,* Princeton (1949).

7. K. CHANDRASEKHARAN, S. MINAKSHISUNDARAM : *Typical Means,* Bombay, (1952).

ON SOME MODULAR SPACES CONNECTED WITH STRONG SUMMABILITY†

By J. MUSIELAK

1. In a paper by W. Orlicz and the author "On modular spaces", *Studia Mathematica*, 18, are considered modular spaces of some general type. I shall refer to this paper as (*). Here examples of such modular spaces will be given, connected with strong summability. First, I shall outline some auxiliary definitions and results from (*).

2. Given a linear space X, a functional $\rho(x)$ defined for all $x \in X$ with values in $(-\infty, +\infty)$ is called modular, if the following conditions are satisfied :

A.1. $\rho(x) = 0$ if and only if $x = 0$; A.2. $\rho(-x) = \rho(x)$;

A.3. $\rho(\alpha x + \beta y) \leqslant \rho(x) + \rho(y)$, $x, y \in X$, $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$.

Evidently, $\rho(x) \geqslant 0$. The following conditions will also be used :

B.1. if $\alpha_n \to 0$, then $\rho(\alpha_n x) \to 0$; B.2. if $\rho(x_n) \to 0$, then $\rho(\alpha x_n) \to 0$ for any α.

The following sets are of importance :

$X_\rho = \{x \in X : \rho(x) < +\infty\}$,

$X_\rho^* = \{x \in X : \rho(kx) < +\infty \text{ for some } k > 0\}$,

$\bar{X}_\rho^* = \{x \in X : x \text{ satisfies B.1}\}$.

Obviously, $\bar{X}_\rho^* \subset X_\rho^* \subset X$ and X_ρ^* and $\bar{X}_\rho^*$ are linear spaces. The notions of convergence, completeness and separability are the following :

†The invited address delivered at the Golden Jubilee session of the Indian Mathematical Society, December 1958 in Poona. This forms part of the work done when the author was the visiting member of the Tata Institute of Fundamental Research, Bombay.

*An F-norm is a non-negative functional $\|x\|$ defined for all $x \in X_\rho^*$ such that (a) $\|x\| = 0$ if and only if $x = 0$, (b) $\|x+y\| \leqslant \|x\| + \|y\|$, (c) $\|-x\| = \|x\|$, (d) $\vartheta_n \to \vartheta$ and $\|x_n - x\| \to 0$ implies $\|\vartheta_n x_n - \vartheta x\| \to 0$. A linear space with an F-norm complete with respect to this norm is called an F-space.

(α) A sequence $\{x_n\} \subset X$ is *modular convergent* to $x \in X$, if $\rho[k(x_n - x)] \to 0$ as $n \to \infty$ for a number $k > 0$, dependent on $\{x_n\}$.

(β) A set $X_1 \subset X^*_\rho$ is called *strongly modular complete*, if there exists a constant $k > 0$ such that for any sequence $\{x_n\} \subset X_1$ the condition $\rho(x_n - x_m) \to 0$ as $m_1 n \to \infty$ implies $\rho[k(x_n - x)] \to 0$, as $n \to \infty$, where $x \in X_1$.

(γ) A set $X_1 \subset X^*_\rho$ will be called *strongly modular separable* if there exist a sequence $\{w_n\} \subset X_1$ and a number $k > 0$ such that for any $x \in X_1$ there exists a subsequence $\{w_{n_\nu}\} \subset \{w_n\}$ such that $\rho[k(w_{n_\nu} - x)] \to 0$ as $\nu \to \infty$; if the number k depends on x, X_1 will be merely called *modular separable*. Of course, the limit operation is 'unique,' additive and homogeneous. For any $x \in \overline{X}^*_\rho$ we write $\|x\| = \inf\{\epsilon > 0 : \rho(x/\epsilon) \leqslant \epsilon\}$. Then $\|x\|$ is an F-norm*. Moreover, norm-convergence implies modular-convergence to the same limit; both convergences are equivalent if and only if B.2 holds for all sequences of elements of $\overline{X}^*_\rho$. Strong modular completeness implies norm-completeness; and separability in norm implies strong modular separability and thus, obviously, modular separability, too.

3. I shall further mention that $\|x\|$ is not a B-norm**. Assuming the convexity instead of A.3, i.e. $\rho(\alpha x + \beta y) \leqslant \alpha\,\rho(x) + \beta\,\rho(y)$, $x_1 y \in X_1$, $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$, a B-norm equivalent to the F-norm $\|x\|$ may be defined by the formula $\|x\|_* = \inf\ \{\epsilon > 0 : \rho(x/\epsilon) \leqslant 1\}$. Indeed, for $\rho(x)$ convex the following inequalities hold :

$$\text{if } \|x\| = 1 \text{ or } \|x\|_* = 1, \text{ then } \|x\| = \|x\|_* = 1;$$

$$\text{if } \|x\| < 1 \text{ or } \|x\|_* < 1, \text{ then } \|x\|_* \leqslant \|x\| \leqslant \sqrt{\|x\|_*} < 1;$$

$$\text{if } \|x\| > 1 \text{ or } \|x\|_* > 1, \text{ then } 1 < \sqrt{\|x\|_*} \leqslant \|x\| \leqslant \|x\|_*.$$

4. Now, some concrete modular spaces connected with strong summability will be considered. Let $(\alpha_{n\nu})$ be an infinite matrix of non-negative numbers satisfying the following two conditions :

** An F-norm is called B-norm, if it is positive-homogeneous, i.e. if (c′) $\|\alpha x\| = |\alpha|\ \|x\|$; obviously, (c′) and (b) imply (c) and (d).

1° $$0 < \limsup_{n\to\infty} \sum_{\nu=1}^{\infty} \alpha_{n\nu} < +\infty,$$

2° For any positive integer ν there exists n such that $\alpha_{n\nu} \neq 0$. For instance, 1° is satisfied by all non-negative Toeplitz-matrices. For further use we introduce the notation

$$K = \sup_n \sum_{\nu=1}^{\infty} \alpha_{n\nu}, \quad \sigma = \limsup_{n\to\infty} \sum_{\nu=1}^{\infty} \alpha_{n\nu}.$$

Denote by X the space of all numerical sequences $x = \{a_\nu\}$. Moreover, let $M(u)$ be a continuous, even function, non-decreasing for $u \geqslant 0$, $M(0) = 0$, $M(u) > 0$ for $u \neq 0$. Then

$$\rho(x) = \sup_n \sum_{\nu=1}^{\infty} \alpha_{n\nu} M(a_\nu)$$

is modular in X. Some special cases of modular functionals of this type were considered in (*), namely the two following :

(i) $\alpha_{n\nu} = 1$ for $\nu = n$, $\alpha_{n\nu} = 0$ for $\nu \neq n$,

(ii) $\alpha_{n\nu} = 1/n$ for $\nu \leqslant n$, $\alpha_{n\nu} = 0$ for $\nu > n$.

We denote by

$X_m =$ the set of all $x = \{a_\nu\}$ such that there exists a number a with the property

$$\lim_{n\to\infty} \sum_{\nu=1}^{\infty} \alpha_{n\nu} M[k(a_\nu - a)] = 0 \text{ for any real } k;$$

$X_m^0 =$ the subset of all $x \in X_m$ such that $a = 0$.

Let us note that for any $x \in X_m$ there exists only one number a with the above property ; indeed,

$$M(a-b) \sum_{\nu=1}^{\infty} \alpha_{n\nu} \leqslant \sum_{\nu=1}^{\infty} \alpha_{n\nu} M[2(a_\nu - a)] + \sum_{\nu=1}^{\infty} \alpha_{n\nu} M[2(a_\nu - b)].$$

Assuming that the right side of this inequality tends to zero as $n \to \infty$, we obtain $M(a-b) = 0$, i.e. $a = b$.

5. X_ρ^* and $\overline{X}_\rho^*$ are *strongly modular complete,* X_m *aud* X_m^0 *are linear spaces contained in* $\overline{X}_\rho^*$, *complete with respect to the norm.*

For proving the strong modular completeness of X^*_ρ, let us take $x_n = \{a^n_\nu\} \in X^*_\rho$ such that $\rho(x_p - x_q) \to 0$ as $p, q \to \infty$. By 2° it follows $M(a^p_\nu - a^q_\nu) \to 0$ as $p, q \to \infty$; hence there exists $x = \{a_\nu\}$ such that $a^n_\nu \to a_\nu$ for $\nu = 1, 2, \ldots$. It follows

$$\sum_{\nu=1}^{\infty} \alpha_{n\nu} M(a^p_\nu - a_\nu) \leqslant \liminf_{q\to\infty} \sum_{\nu=1}^{\infty} a_{n\nu} M(a^p_\nu - a^q_\nu) \leqslant \epsilon$$

for p sufficiently large, uniformly in n. Thus, $\rho(x_p - x) \to 0$ as $p \to \infty$. The strong modular completeness of $\overline{X}^*_\rho$ follows from that of X^*_ρ.

Now, we shall prove that $X_m \subset \overline{X}^*_\rho$. Taking $x \in X_m$ and $0 \leqslant \alpha \leqslant 1/2$ we have

$$\sum_{\nu=1}^{\infty} \alpha_{n\nu} M(\alpha a_\nu) \leqslant \sum_{\nu=1}^{\infty} \alpha_{n\nu} M[2\alpha(a_\nu - a)] + \sum_{\nu=1}^{\infty} \alpha_{n\nu} M(2\alpha a)$$
$$\leqslant \sum_{\nu=1}^{\infty} \alpha_{n\nu} M(a_\nu - a) + K\, M(2\alpha a).$$

Given an $\epsilon < 0$ we choose a positive number $\alpha_0 < 1/2$ such that $KM(2\alpha a) < \epsilon/2$ for $0 \leqslant \alpha \leqslant \alpha_0$. Now, since $\sum\limits_1^\infty \alpha_{n\nu} M(a_\nu - a) \to 0$ as $n \to \infty$, we can find N such that $\sum\limits_1^\infty \alpha_{n\nu} M(a_\nu - a) < \epsilon/2$ for $n > N$. Then $\sum\limits_1^\infty \alpha_{n\nu} M(\alpha a_\nu) < \epsilon$ for $n > N$ and $0 \leqslant \alpha \leqslant \alpha_0$. Now, it is easily seen that $\rho(x) < +\infty$. Therefore there exists ν_0 such that

$$\sum_{\nu=\nu_0+1}^{\infty} \alpha_{n\nu} M(a_\nu) < \frac{\epsilon}{2}$$

for $n \leqslant N$. Thus we obtain

$$\sum_{\nu=1}^{\infty} \alpha_{n\nu} M(\alpha a_\nu) \leqslant \sum_{\nu=1}^{\nu_0} \alpha_{n\nu} M(\alpha a_\nu) + \frac{\epsilon}{2} \quad \text{for } n \leqslant N.$$

Finally, take $0 < \alpha' \leqslant \alpha_0$ such that

$$\sum_{\nu=1}^{\nu_0} \alpha_{n\nu} M(\alpha a_\nu) < \frac{\epsilon}{2} \text{ for } 0 \leqslant \alpha \leqslant \alpha' \quad \text{and } n \leqslant N.$$

Then $\sum_{1}^{\infty} \alpha_{n\nu} M(\alpha a_\nu) < \epsilon$ for $n \leqslant N$. This yields $\rho(\alpha x) \leqslant \epsilon$ for $0 \leqslant \alpha \leqslant \alpha'$, i.e. $x \in \overline{X}_\rho^*$.

The linearity of X_m and X_m^0 is obvious. Since the proof of the completeness of X_m and X_m^0 by application of 1^0 is similar to that in the case of the first arithmetic means, given in (*), it will be omitted here.

Let us further remark that from the above theorem it follows that $\overline{X}_\rho^*$, X_m and X_m^0 are F-spaces. In the case (i) the following isomorphisms hold :

$$\overline{X}_\rho^* \sim m,\ X_m \sim \lambda,\ X_m^0 \sim \lambda_0.$$

6. *Assuming $\alpha_{n\nu} \to 0$ as $n \to \infty$ for all ν, $\overline{X}_\rho^*$ is not modular separable ; hence it is non-separable in norm, too. The spaces X_m and X_m^0 are separable in norm.*

To prove the modular non-separability of $\overline{X}_\rho^*$, let us first define two increasing sequences of indices $\{k_m\}$ and $\{n_m\}$ as follows. We choose k_1 and n_1 so that $\sum_{1}^{k_1} \alpha_{n_1\nu} > \sigma/2$; that is possible, since we can find by 1^0, n_1 such that $\sum_{1}^{\infty} \alpha_{n_1\nu} > 3\sigma/4$ and then k_1 such that $\sum_{k_1+1}^{\infty} \alpha_{n_1\nu} < \sigma/4$. Now, let us assume that the numbers $K_1, K_2, \dots, K_{m-1}$ and $n_1, n_2, \dots, n_{m-1}$ are already chosen. We take $k_m > k_{m-1}$ and $n_m > n_{m-1}$ so that

$$\sum_{\nu=k_{m-1}+1}^{k_m} \alpha_{n_m\nu} > \frac{\sigma}{2}.$$

Such numbers k_m and n_m exist ; it is sufficient to take any $n_m > n_{m-1}$ so large that

$$\sum_{\nu=1}^{\infty} \alpha_{n_m\nu} > \frac{3}{4}\sigma \text{ and } \sum_{\nu=1}^{k_{m-1}} \alpha_{n_m\nu} < \frac{\sigma}{8},$$

applying 1^0 and the assumption of our theorem and then to choose $k_m > k_{m-1}$ in such way that

$$\sum_{\nu=k_m+1}^{\infty} \alpha_{n_m\nu} < \frac{\sigma}{8}.$$

Now, we take all sequences $y = \{a_\nu\}$ of the form $a_\nu = b_\mu$ for $k_{\mu-1} < \nu \leqslant k_\mu$, $k_0 = 0$ and $b_\mu = 0, 1$ (i.e. $\{b_\mu\}$ are zero-one sequences). Take two sequences $y' = \{a'_\nu\}$ and $y'' = \{a''_\nu\}$ of this form and let $\{b'_\mu\}$ and $\{b''_\mu\}$ be the two corresponding zero-one sequences respectively. If $b'_\mu = b''_\mu$ for $\mu = 1, 2, \ldots, m-1$ and $b'_m \neq b''_m$ and if k is an arbitrary positive constant, we obtain

$$\rho[k(y'-y'')] = \sup_n \sum_{\nu=1}^{\infty} \alpha_{n\nu} M[k(a'_\nu - a''_\nu)]$$
$$\geqslant \sum_{\nu=k_{m-1}+1}^{k_m} \alpha_{n_m\nu} M(k) > \frac{\sigma}{2} M(k).$$

Moreover, the set of all sequences $y = \{a_\nu\}$ of the above form is non-countable. Now, the proof of modular non-separability may be finished in the usual way.

It is easily seen that, for proving the separability in norm of X_m and X_m^0, it is sufficient to show that for any sequence $x = \{a_\nu\} \in X_m$ with the corresponding number a, the sequence

$$x_n = \{a_1, a_2, \ldots, a_n, a, a, \ldots\} \text{ tends to } x \text{ in norm.}$$

Taking any $\epsilon > 0$ we can find n_0 such that

$$\sum_{\nu=1}^{\infty} \alpha_{n\nu} M\left(\frac{a_\nu - a}{\epsilon}\right) < \frac{\epsilon}{3} \text{ for } n > n_0,$$

and k_0 such that

$$\sum_{\nu=k_0}^{\infty} \alpha_{n\nu} M\left(\frac{a_\nu - a}{\epsilon}\right) < \frac{\epsilon}{2} \text{ for } n \leqslant n_0.$$

Hence

$$\rho\left(\frac{x_k - x}{\epsilon}\right) = \sup_n \sum_{\nu=k}^{\infty} \alpha_{n\nu} M\left(\frac{a_\nu - a}{\epsilon}\right) \leqslant$$
$$\leqslant \sup_{n>n_0} \sum_{\nu=1}^{\infty} \alpha_{n\nu} M\left(\frac{a_\nu - a}{\epsilon}\right) + \sup_{n\leqslant n_0} \sum_{\nu=k}^{\infty} \alpha_{n\nu} M\left(\frac{a_\nu - a}{\epsilon}\right) < \epsilon$$

for $k \geqslant k_0$. Consequently, $\| X_k - X \| \to 0$ as $k \to \infty$.

7. *If the following condition is satisfied : for any* $\epsilon > 0$ *there exist numbers* $A_\epsilon > 0$ *and* $\alpha_\epsilon > 0$ *such that* $M(\alpha u) < \epsilon\, M(u)$ *for every* $0 \leqslant \alpha \leqslant \alpha_\epsilon$ *and for every* $u \geqslant A_\epsilon$, *then* $X_\rho^* = \overline{X}_\rho^*$.

The proof of this theorem is easy and follows from the inequality $\rho(k\,x) \leqslant \epsilon\, \rho\, (k\, x) + K\, M(\alpha\, A_\epsilon/k)$, valid for $0 \leqslant \alpha \leqslant k\, \alpha_\epsilon$, where $\rho(k\, x) < +\infty$. Let us note that in the case of the first arithmetic means (ii) the above condition is also necessary for the equality $X_\rho^* = \overline{X}_\rho^*$, as proved in (*), while in the general case the necessity does not hold ; as a counter example we may take the matrix $(\alpha_{n\nu})$ as in (i) and $M(u) = \log(1 + |u|)$.

8. *If the following condition holds* :

(Δ_2). *there exist* $\eta_0 > 0$ *and* $x > 0$ *such that for any* $u \geqslant \eta_0$, $M(2u) \leqslant x\, M(u)$, *then* B. 2 *is satisfied in* X_ρ^* (whence the modular convergence and the norm-convergence are in $\overline{X}_\rho^*$ equivalent).

As is well known, from (Δ_2) it follows that for every $\eta > 0$ there exists a $x_\eta > 0$ such that $M\,(2u) \leqslant x_\eta\, M\,(u)$ for all $u \geqslant \eta$. Now, take $x = \{a_\nu\} \in X_\rho$ and put for an arbitrary $\eta > 0$, $A = \{\nu : |a_\nu| < \eta\}$, $A' = \{\nu : |a\nu| \geqslant \eta\}$.

Then

$$\sum_{\nu=1}^{\infty} \alpha_{n\nu}\, M(2a_\nu) = \sum_{A} \alpha_{n\nu}\, M(2a_\nu) + \sum_{A'} \alpha_{n\nu}\, M(2a_\nu) \leqslant KM(2\eta) + x_\eta\, \rho(x),$$

whence $\rho(2x) \leqslant KM(2\eta) + x_\eta\, \rho(x)$ and the theorem follows easily.

It is again easily seen that in general B.2 does not imply (Δ_2) ; for instance, if $(\alpha_{n\nu})$ is defined as in (i), B. 2 holds always. But on the other hand we shall see that in the case of the first arithmetic means, B. 2 in X_ρ^* implies (Δ_2).

9. *If* $(\alpha_{n\nu})$ *is defined as in* (ii), *then* B. 2 *holds in* X_ρ^* *if and only if* $M(u)$ *satisfies the condition* (Δ_2).

Let us assume (Δ_2) is not satisfied. Then $M(u) \to \infty$ as $u \to \infty$. We define a sequence $\{u_n\}$ by induction. First we take $u_1 > 0$ such that

$M(u_1) > 1$ and $M(2u_1) > 2\,M(u_1)$. If $u_1, u_2, \ldots, u_{n-1}$ are already defined, we take u_n such that $M(u_n) > 2\,M(u_{n-1})$ and $M(2u_n) > 2^n\,M(u_n)$. Then we have $M(u_\nu) < 2^{-n+\nu}\,M(u_n)$ for any $\nu < n$ and

$$M(u_{n+s}) \leqslant \sum_{\nu=n}^{n+s} M(u_\nu) \leqslant M(u_{n+s}) \times$$

$$\times \sum_{\nu=n}^{n+s} 2^{-n-s+\nu} < 2M(u_{n+s}) \text{ for } s = 0, 1, 2, \ldots .$$

Now, we shall define a sequence $x_n = \{a_\nu^n\} \in X_\rho$ as follows. Let $p_r = [2^r M_r]$, where $[\quad]$ denotes the integral part and $M_r = M(u_r)$. We put

$$a_\nu^n = \begin{cases} u_\mu \text{ for } \nu = p_\mu,\ \mu \geqslant n, \\ 0 \text{ for } \nu < p_\mu \text{ and for } p_\mu < \nu < p_{\mu+1},\ \mu \geqslant n. \end{cases}$$

Fixing n, we have for $p_\mu \leqslant m < p_{\mu+1},\ \mu \geqslant n,\ s = \mu - n,$

$$\frac{1}{m}\sum_{\nu=1}^{m} M(a_\nu^n) \leqslant \frac{1}{p_\mu}\sum_{\nu=1}^{p_\mu} M(a_\nu^n) = \frac{1}{p_{n+s}}\sum_{\nu=n}^{n+s} M(u_\nu)$$

$$< \frac{2M_{n+s}}{p_{n+s}} \leqslant \frac{2M_\mu}{2^\mu M_\mu - 1} \leqslant \frac{2}{2^\mu - 1} \leqslant \frac{2}{2^n - 1}$$

Hence

$$\rho(x_n) = \sup_{m \geqslant p_n} \frac{1}{m}\sum_{\nu=1}^{m} M(a_\nu^n) \leqslant \frac{2}{2^n - 1} \to 0 \text{ as } n \to \infty.$$

On the other hand,

$$\rho(2x_n) = \sup_{m \geqslant p_n} \frac{1}{m}\sum_{\nu=1}^{m} M(2a_\nu^n) = \sup_{\mu \geqslant n} \frac{1}{p_\mu}\sum_{\nu=n}^{\mu} M(2u_\nu)$$

$$\geqslant \frac{1}{p_n} M(2u_n) > \frac{1}{p_n} 2^n M(u_n) \geqslant 1$$

and B. 2 is not satisfied.

LATTICE POINT PROBLEMS AND QUADRATIC FORMS

By V. VENUGOPAL RAO

THE classical lattice point problem associated with the circle is concerned with the study of the function $P(x)$ defined by

$$R(x) = \sum_{0 \leqslant n \leqslant x} r(n) = \pi x + P(x),$$

$r(n)$ denoting the number of integral representations of the integer n as the sum of squares of two other integers. Two problems regarding $P(x)$ have been studied intensively; firstly regarding the growth of the function $P(x)$ as $x \to \infty$ and secondly an exact formula for $P(x)$ as an infinite series of analytic functions similar to the Riemann prime number formula. We will be concerned with the second problem. It has been conjectured by Voronoi [24] that

$$R(x) - \delta_x\, r(x) = \pi x + x^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{r(n)\, J_1(2\pi \sqrt{(nx)})}{n^{\frac{1}{2}}}, \tag{1}$$

$J_\nu(x)$ denoting the Bessel function of the first kind and δ_x being $\frac{1}{2}$ or 0 according as x is integral or not. This result was proved for the first time in 1915 by Hardy [9] and his proof of (1) appeals to methods of complex analysis, especially to difficult theorems on the singularities of Dirichlet series of the type $\sum\limits_{n=1}^{\infty} a_n e^{-\sqrt{(n)s}}$. Hardy further showed that the series on the right of (1) is uniformly convergent in any closed interval, contained in the positive real axis, which is free from integers n such that $r(n) \neq 0$ and that the series is boundedly convergent in every closed interval (a, b), $a > 0$. This fact led Hardy [10] to the proof of the formula

$$\sum_{0 < n \leqslant x} r(n)\, (x-n)^\alpha = \frac{\pi x^\alpha}{1+\alpha} - x^\alpha + \pi^{-\alpha}\, \Gamma(\alpha+1)\, x^{\frac{1}{2}+\frac{1}{2}\alpha} \times$$
$$\times \sum_{n=1}^{\infty} \frac{r(n)\, J_{1+\alpha}(2\pi \sqrt{(nx)})}{n^{(1+\alpha)/2}}, \tag{2}$$

α being any positive real number. The non-uniformity of convergence of the series on the right of (1) disappears in (2) and the series on the right of (2) is uniformly convergent in any closed interval (a, b) with $a > 0$. Further the series on the right of (2) is absolutely convergent if $\alpha > \frac{1}{2}$. It may be remarked that the formula (2) has been utilized by Hardy [10] in showing that

$$\frac{1}{x}\int_{1}^{x} |P(t)|\, dt = O(x^{\frac{1}{4}+\epsilon})$$

as $x \to \infty$, for every $\epsilon > 0$. In 1920 Landau [15] gave a proof of (1), using the so called Pfeiffer method, which appeals to methods of real analysis. It was felt desirable to give an elementary proof of (1) and this was done in 1924 jointly by Hardy and Landau [13] who gave two proofs one of which makes use of complex analysis and the other of real analysis.

The formulae (1) and (2) can be generalized in several ways. Let us consider the number of integral representations of n, not necessarily as the sum of two squares but, as the sum of m squares ($m \geqslant 2$) or more generally the number of integral representations of a positive real number by a real, symmetric, positive definite quadratic form of rank m. Moreover every representation need not necessarily be integral but real and congruent to a fixed set of m real numbers and every such solution may be counted with a " weight". More precisely let S be a m rowed, real, symmetric, positive definite matrix and let $\mathbf{A}$, $\mathbf{H}$ be two real column vectors with m rows. Let X denote a real column vector with m rows, X' the transpose of X and $S[X] = X'\,S\,X$. Let

$$A(s, \mathbf{A}, \mathbf{H}, t) = \sum_{\substack{S[\mathbf{X}+\mathbf{A}]=t,\\ \mathbf{X}\, integral}} e^{2\pi i \mathbf{X}'\mathbf{H}} \tag{3}$$

As S is positive definite the number of summands on the right of (3) is finite. If $S = E_m$ the unit matrix of order m, $\mathbf{A} = \mathbf{H} = \mathbf{O}$, $\mathbf{O}$ being the column with all elements zero, then $A\,(s, \mathbf{A}, \mathbf{H}, t)$ represents the number of integral representations of t as the sum of m squares. The analogue of (1) and (2) is then given by

$$\sum_{0<\lambda_l\leqslant x}' A(S, \mathbf{A}, \mathbf{H}, \lambda_l)\,(x-\lambda_l)^\alpha =$$

$$\frac{\pi^{m/2}\,|S|^{-\frac{1}{2}}\,\Gamma(\alpha+1)}{\Gamma(\alpha+1+m/2)}\,x^{\alpha+m/2} - x^\alpha +$$

$$+\,|S|^{-\frac{1}{2}}\,e^{-2\pi i\mathbf{A}'\mathbf{H}}\,\pi^{-\alpha}\,\Gamma(\alpha+1)\,x^{\alpha/2+m/4}\times$$

$$\times\sum_{l=1}^{\infty}\frac{A(S^{-1},\mathbf{H},-\mathbf{A},\mu_l)\,J_{\alpha+m/2}(2\pi\sqrt{(\mu_l x)})}{\mu_l^{\alpha/2+m/4}} \qquad (4)$$

where $\{\lambda_l\}$ and $\{\mu_l\}$ represent the sequences of positive values of $S[\mathbf{x}+\mathbf{A}]$ and $S^{-1}[\mathbf{x}+\mathbf{H}]$ when $\mathbf{x}$ runs through all integral column vectors, the dash on the left of (4) indicating that for $\alpha = 0$ and $x = \lambda_l$ (for some l) the last term in the summation on the left of (4) is to be replaced by half its value. The formula (4) is valid for $\alpha > (m-1)/2$ with the series occurring on the right converging absolutely. For $\alpha \leqslant (m-1)/2$, the series is either conditionally convergent or divergent. In the case when the series diverges it has been shown in special cases that the series can be summed by Riesz typical means of type μ and of appropriate order with the formula (4) remaining valid. The best result in this direction is due to Walfisz [26, 27] who proved that (4) is valid for $\alpha \geqslant 0$, in the case of S, $\mathbf{A}$ rational and $\mathbf{H} = \mathbf{O}$, with the series on the right of (4) summable $(R, \mu, (m-3)/2-\alpha)$. He further proved that this order of summability is the best. The same order of summability was proved by Oppenheim [18] in the case $S = E_m$, $\mathbf{A} = \mathbf{H} = \mathbf{O}$. Walfisz and Oppenheim further proved that the series obtained by deriving the series on the right of (4), with respect to x is summable $\left(R;\, \mu, \frac{m-1}{2} - \alpha\right)$ in case $x \neq \mu_l$ and not summable $(R;\, \mu, \alpha')$ for any real α', if $x = \mu_l$. The considerations of Walfisz are limited to the case S rational, $\mathbf{A} = \mathbf{H} = \mathbf{O}$ and those of Oppenheim in addition, $S = E_m$. The proofs of the results of Walfisz are similar to those of Hardy [9] and those of Oppenheim similar to the first proof given in the joint paper of Hardy and Landau [13]. Thus we are led to the study of the convergence and summability of the series

$$\sum_{n=1}^{\infty} A(S^{-1}, \mathbf{H}, -\mathbf{A}, \mu_l)\, J_\mu(2\pi\sqrt{(\mu_l x)})\, \mu_l^{\alpha}, \tag{5}$$

α being real. This has been done by various authors in particular cases. Some of the results of Wilton [29, 30] concerning the series (5) have been improved by Dixon and Ferrar [7] in the case $S = E_2$, $\mathbf{A} = \mathbf{H} = \mathbf{O}$. Further the results of Dixon and Ferrar are limited to the case $x > 0$ and $r(x) = 0$.

The proofs of all the results mentioned so far make use either of the transformation formula,

$$\vartheta(S, \mathbf{A}, \mathbf{H}, s) = s^{-m/2}\,|S|^{-\frac{1}{2}}\, e^{2\pi i \mathbf{A}'\mathbf{H}}\, \vartheta(S^{-1}, \mathbf{H}, -\mathbf{A}, s^{-1}), \tag{6}$$

of the theta function

$$\vartheta(S, \mathbf{A}, \mathbf{H}, s) = \sum_{X} e^{-\pi s S[\mathbf{X}+\mathbf{A}] + 2\pi i \mathbf{X}'\mathbf{H}} \tag{7}$$

the summation on the right of (7) being over all integral column vectors X with m rows and s a complex number with positive real part or one of its "equivalents". In view of the definition (3) one can rewrite (7) as

$$\vartheta(S, \mathbf{A}, \mathbf{H}, s) = \delta_{\mathbf{A}}\, e^{-2\pi i \mathbf{A}'\mathbf{H}} + \sum_{n=1}^{\infty} A(S, \mathbf{A}, \mathbf{H}, \lambda_n)\, e^{-\pi \lambda_n s}, \tag{7*}$$

where $\delta_{\mathbf{A}} = 1$ or 0 according as $\mathbf{A}$ is integral or not. If S, $\mathbf{A}$, $\mathbf{H}$ are rational all the λ_n's are rational numbers with bounded denominators so that at least for special cases of S, $\mathbf{A}$ and $\mathbf{H}$, $\vartheta(S, \mathbf{A}, \mathbf{H}, s)$ considered as a function of $z = is$ is an automorphic form defined in the upper half plane $y > 0$ ($z = x + iy$, x and y real). Recently, S. Bochner and K. Chandrasekharan [3, 4, 5] obtained some general results concerning the convergence and Riesz summability of the series (5) in the case $\mathbf{H} = \mathbf{O}$ and their results include many of the earlier results as special cases. Their considerations are limited to those positive real values of x for which $A(S^{-1}, 0, -\mathbf{A}, x) = 0$. One of their best results states that, for S and $\mathbf{A}$ rational, $\mathbf{H} = \mathbf{O}$ and $A(S^{-1}, 0, -\mathbf{A}, x) = 0$, the series (5) is summable $(R\,;\, n, \eta)$ for $\eta \geqslant 0$ and $\alpha < (3/4) - (m/2) + (\eta/2)$, as long as $\mu > -1$. The only disadvantage of their method is that it depends on some properties which are peculiar to positive definite quadratic forms and as such cannot be applied to similar problems which arise when one replaces $A(S^{-1}, \mathbf{H}, -\mathbf{A}, t)$ by other arithmetical functions. Before we proceed further we mention that the convergence and Riesz summability of the series (5) for those x for which $A(S^{-1}, \mathbf{O}, -\mathbf{A}, x) \neq 0$ has been considered by Avadhani [1] in the case $S = E_m$.

In 1951 Bochner completely generalized the problems considered earlier by replacing $A(S, \mathbf{A}, \mathbf{H}, t)$ by arithmetical functions of certain type and obtained far reaching results. In particular his results include those of Bochner and Chandrasekharan as special cases. Bochner considers Riesz summability of series of the type

$$\sum_{n=1}^{\infty} a_n J_\mu(2\pi\sqrt{(\lambda_n x)})\lambda_n^{\alpha}, \quad (\mu > -1, \alpha \text{ real}) \tag{8}$$

where the a_n's are coefficients of a Dirichlet series

$$f(s) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s}$$

which is convergent in a half plane $\sigma > 0$ ($s = \sigma + it$, σ and t real) and satisfying

$$\sum_{n=0}^{\infty} a_n e^{-\lambda_n s} = s^{-\delta} \sum_{n=0}^{\infty} b_n e^{-\mu_n/s}, \quad (\delta > 0), \tag{9}$$

$$g(s) = \sum_{n=0}^{\infty} b_n e^{-\mu_n s},$$

being some other Dirichlet series converging for $\sigma > 0$. One can, for instance, take for a_n the Ramanujan function $\tau(n)$; for this special case, Wilton [31] and Hardy [12] have obtained a series of Bessel functions for $\sum'_{0<n\leqslant x} \tau(n)(x-n)^{\alpha}$. Another example is obtained by setting $a_n = (1/n) \sum_{d|n} d$. An exact formula, in this case, for $\sum'_{0<n\leqslant x} a_n(x-n)^{\alpha}$ and the summability of the corresponding series have been considered by Wigert [28] and Oppenheim [18].

We now proceed to consider the case in which the matrix S is indefinite. In this case $A(S, \mathbf{A}, \mathbf{H}, t)$ is not finite in general. A special case in which $A(S, \mathbf{A}, \mathbf{H}, t)$ is finite is that in which S is the matrix of a binary decomposable, rational, quadratic form. One may, without loss of generality, assume that $2S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and then $A(S, \mathbf{O}, \mathbf{O}, n)$ is $d(n)$, the number of divisors of n. Voronoi [25] proved, in 1904, that

$$\sum_{n \leqslant x}{}' d(n) = x \log x + (2C - 1)x + \tfrac{1}{4} -$$

$$- x^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{d(n)}{n^{\frac{1}{2}}} \left\{ Y_1(4\pi\sqrt{(nx)}) + \frac{2}{\pi} K_1(4\pi\sqrt{(nx)}) \right\}, \quad (10)$$

the series on the right being convergent, and the dash on the left of (10) indicating that the last term on the left of (10) is to be halved if x is an integer, and C denoting the Euler constant. The Y and K functions on the right of (10) are the usual Bessel functions which are so denoted in the notation of Watson [32]. The series on the right of (10) is boundedly convergent in any closed interval (a, b), $a > 0$ and uniformly convergent if (a, b) is free from integral values. Thus one can integrate (8) with respect to x and in this way Hardy [10] obtains for positive *integral* values of α,

$$\sum_{n \leqslant x} d(n)(x-n)^{\alpha} = \Gamma(\alpha+1)\phi_{\alpha}(x) - \frac{X^{\frac{1}{2}+\frac{1}{2}\alpha}}{(2\pi)^{\alpha}} \Gamma(\alpha+1) \sum_{n=1}^{\infty} \frac{d(n)}{n^{\frac{1}{2}+\frac{1}{2}\alpha}} \times$$

$$\times \left\{ Y_{1+\alpha}(4\pi\sqrt{(nx)}) + \frac{2}{\pi} \cos \pi\alpha \, K_{1+\alpha}(4\pi\sqrt{(nx)}) \right\}, \quad (10^*)$$

$\phi_{\alpha}(x)$ denoting the sum of the residues of $\dfrac{\zeta^2(s)\, x^{s+\alpha}}{s(s+1)\dots(s+\alpha)}$ at its poles, $\zeta(s)$ being the Riemann zeta function.

For α positive and non-integral, (10*) has to be replaced by a general formula which includes (10*) as a particular case. This is in complete contrast to all the previous considerations that we have mentioned so far and was pointed out, perhaps for the first time, by Dixon and Ferrar [6]. The analogue of (4) to indefinite quadratic forms, as will be mentioned in the next paragraph, involves an infinite series of analytic functions which in addition to the usual Bessel functions include a function studied by Lommel [32, pp. 345-352] and a function "similar" to Lommel's. The formula of Dixon and Ferrar is

$$\sum_{n\leqslant x} d(n)\,(x-n)^{\alpha}=\frac{x^{\alpha}}{4}+\frac{x^{\alpha+1}}{\alpha+1}\left\{C+\log x-\psi(\alpha+2)\right\}+$$
$$+\,2\pi x^{\alpha+1}\,\Gamma(\alpha+1)\sum_{n=1}^{\infty} d(n)\,\lambda_{\alpha+1}(4\pi\sqrt{(nx)}), \quad (11)$$

where

$$\lambda_{\alpha}(Z)=-\frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(Z/2)^{4m}}{\Gamma(2m+1)\,\Gamma(2m+\alpha+1)}\{2\log Z/2-$$
$$-\psi(2m+1)-\psi(2m+\alpha+1)\},$$

and

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}.$$

Formula (11) is valid for all real values of $\alpha>0$ with the series on the right converging. For positive integral values of α, (11) coincides with (10*) and to see this fact explicitly it will be advantageous to express the function $\lambda_{\alpha}(Z)$ in terms of the Bessel functions Y, K and a "residuary" function. It turns out that with $\lambda_{\alpha}(Z)$ so expressed, the part of the infinite series on the right of (11) arising out of the K function and the residuary function are always convergent, thus revealing that the part of the series involving the Y function have to be considered separately. Thus one may sum up by saying that the "critical" part of the series on the right of (11) is the series involving the Y function. At the moment we shall not proceed to express $\lambda_{\alpha}(x)$ in terms of Bessel functions as, in the next paragraph, we will do it in the more general situation when we consider general, indefinite, quadratic forms.

Now let S be an m-rowed real, symmetric, indefinite matrix with signature n, $m-n$ and $\mathbf{A}$, $\mathbf{H}$ two real columns with m rows. As S is indefinite, A (S, $\mathbf{A}$, $\mathbf{H}$, t) is, in general, infinite and one seeks an analogue of this arithmetical function. Siegel [20], in his researches on the analytical theory of quadratic forms, defined a function $\mu(S, \mathbf{O}, \mathbf{O}, t)$ which is defined for all *rational* S and is finite in all cases except the following : $m = 3$, with $-t|S|$ the square of a rational number, and $m = 4$, $t = 0$, $-|S|$ the square of a rational number. In these exceptional cases $\mu(S, \mathbf{O}, \mathbf{O}, t)$ is infinite. Later Siegel [23] gave an analogue of $A(S, \mathbf{A}, \mathbf{O}, t)$ for $\mathbf{A}$ rational. We consider another rational column vector $\mathbf{H}$ with m rows and following the ideas of Siegel define μ $(S, \mathbf{A}, \mathbf{H}, t)$ which is a generalization of A $(S, \mathbf{A}, \mathbf{H}, t)$. $\mu(S, \mathbf{A}, \mathbf{H}, t)$ is finite in all cases except in the cases mentioned above. Further for $\mathbf{A} = \mathbf{H} = \mathbf{O}$, μ $(S, \mathbf{A}, \mathbf{H}, t)$ coincides with the function defined by Siegel in [20]. The precise definition of $\mu(S, \mathbf{A}, \mathbf{H}, t)$ is given in [19]. The case $m = 3$, $-t|S|$ square of a rational number occurs if and only if $S[\mathbf{X} + \mathbf{A}]$ represents zero non-trivially ; in this case we shall refer to S as the matrix of a ternary zero form. The case $m = 4$, $t = 0$, $|S|$ the square of a rational number will be referred to as S being the matrix of a quaternary zero form. Hereafter we shall assume that S is an m rowed, symmetric, rational, indefinite matrix with signature $n, m - n$ and $\mathbf{A}, \mathbf{H}$ two rational column vectors with m rows. We shall *exclude* those cases in which S is the matrix of either a ternary zero form or a quaternary zero form. We shall also exclude the case in which S is the matrix of a binary decomposable form as in this case μ $(S, \mathbf{O}, \mathbf{O}, t)$ is "essentially" $d(t)$. Let $\{\lambda_l\}$, $\{\mu_l\}$, $\{\nu_l\}$ denote the sequences of positive values of $S[\mathbf{x} + \mathbf{A}]$, S^{-1} $[\mathbf{x} + \mathbf{H}]$, $-S^{-1}$ $[\mathbf{x} + \mathbf{H}]$ respectively, arranged in increasing order of magnitude, when $\mathbf{x}$ runs through all integral column vectors with m rows. Then the analogue of (4), for indefinite forms, is given by (*i*) for $|S| > 0$,

$$\text{(i)} \sum_{0<\lambda_l \leqslant x} \mu(S, \mathbf{A}, \mathbf{H}, \lambda_l)\,(x - \lambda_l)^{\alpha} =$$

$$= \frac{\rho_2\, x^{\alpha + m/2}\, \Gamma(m/2)\, \Gamma(\alpha + 1)}{\Gamma(\alpha + 1 + m/2)} + \zeta(S, \mathbf{A}, \mathbf{H}, 0)\, x^{\alpha} +$$

$$+ (-1)^{(m-n)/2}\, |s|^{-\frac{1}{2}}\, e^{-2\pi i \mathbf{A}'\mathbf{H}}\, \pi^{-\alpha}\, \Gamma(\alpha + 1)\, x^{\alpha/2 + m/4} \times$$

$$\times \sum_{l=1}^{\infty} \mu(S^{-1}, \mathbf{H}, -\mathbf{A}, \mu_l). \quad \frac{J_{\alpha + m/2}(2\pi\sqrt{(\mu_l x)})}{\mu_l^{\alpha/2 + m/4}}, \qquad (12)$$

and (ii) for $|S| < 0$,

$$\sum_{0<\lambda l\leqslant x} \mu(S, \mathbf{A}, \mathbf{H}, \lambda_l)\,(x-\lambda_l)^\alpha =$$

$$= \frac{\rho_1 x^{\alpha+1}}{\alpha+1} + \frac{\rho_2 \Gamma(m/2)\,\Gamma(\alpha+1)}{\Gamma(\alpha+1+m/2)}\, x^{\alpha+m/2} + \zeta(S, \mathbf{A}, \mathbf{H}, 0)\, x^\alpha +$$

$$+ (-1)^{(m-n+1)/2}\, \| S \|^{-\frac{1}{2}}\, e^{-2\pi i \mathbf{A}'\mathbf{H}}\, \pi^{-\alpha}\, \Gamma(\alpha+1)\, x^{\alpha/2+m/4} \times$$

$$\times \Bigg[\sum_{l=1}^{\infty} \frac{\mu(S^{-1}, \mathbf{H}, -\mathbf{A}, \mu_l)\, Y_{\alpha+n/2}\,(2\pi\sqrt{(\mu_l x)})}{\mu_l{}^{\alpha/2+m/4}} +$$

$$+ \frac{2}{\pi} \cos \pi\alpha \sum_{l=1}^{\infty} \frac{\mu(-S^{-1}, \mathbf{H}, \mathbf{A}, \nu_l)\, K_{\alpha+m/2}(2\pi\sqrt{\nu_l x})}{\nu_l{}^{\alpha/2+m/4}} +$$

$$+ \frac{1}{\pi} \frac{\Gamma(m/2+1)}{\Gamma(\alpha)} \frac{1}{2^{\alpha-m/2-1}} \times$$

$$\times \Bigg\{ \sum_{l=1}^{\infty} \frac{\mu(S^{-1}, \mathbf{H}, -\mathbf{A}, \mu_l)\, S_{\alpha-m/2-1,\alpha+m/2}(2\pi\sqrt{(\mu_l x)})}{\mu_l^{\alpha/2+m/4}} -$$

$$- \sum_{l=1}^{\infty} \frac{\mu(-S^{-1}, \mathbf{H}, -\mathbf{A}, \nu_l)\, G_{\alpha-m/2-1,\alpha+m/2}(2\pi\sqrt{(\nu_l x)})}{\nu_l^{\alpha/2+m/4}} \Bigg\} \Bigg]. \qquad (13)$$

Some terms in (12) and (13) need explanation. $S_{\mu,\nu}(x)$ is the Lommel function, its definition being dependent on $\mu \pm \nu$ being a negative odd integer or not, is a solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\nu^2 - x^2) = x^{\mu+1}.$$

Its precise definition is found in Watson [32]. $G_{\mu,\nu}(x)$ is a function similar to the Lommel function and is a solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (\nu^2 + x^2) = x^{\mu+1}.$$

For $\mu \pm \nu$ not a negative odd integer, $G_{\mu,\nu}(x)$ is defined by

$$G_{\mu,\nu}(x) = g_{\mu,\nu}(x) - 2^{\mu-1}\,\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right) \times$$

$$\times \left\{ I_\nu(x) + \cos\pi\left(\frac{\mu-\nu}{2}\right) K_\nu(x) \right\}$$

with

$$g_{\mu,\nu}(x) = x^{\mu-1} \sum_{r=0}^{\infty} \frac{\Gamma((\mu-\nu+1)/2)\ \Gamma((\mu+\nu+1)/2)}{\Gamma((\mu-\nu+3)/2+r)\,\Gamma((\mu+\nu+3)/2+r)} \left(\frac{x}{2}\right)^{2r+2};$$

if $\mu \pm \nu$ is a negative odd integer $-(2\,p+1)$, by definition

$$G_{\nu-2\rho-1,\nu}(x) = \frac{G_{\nu-1,\nu}(x)}{2^{2\rho}\rho!\,(1-\nu)_p} + \sum_{r=0}^{\rho-1} \frac{x^{\nu-2\rho+2r}}{2^{2r+2}\,(-\rho)_{r-1}\,(\nu-\rho)_{r+1}},$$

with

$$G_{\nu-1,\nu}(x) = \tfrac{1}{4}\,x^{\nu}\,\Gamma(\nu) \sum_{r=0}^{\infty} \frac{(x/2)^{2r}}{r!\,\Gamma(\nu+1+r)} \times$$

$$\{2 \log x/2 - \Psi(r+1) - \Psi(\nu+r+1)\} + 2^{\nu-1}\,\Gamma(\nu)\cos(\pi\nu)\ K_{\nu}(x),$$

and $(B)_r = \prod_{K=0}^{r-1} (B+K)$.

$\zeta(S, \mathbf{A}, \mathbf{H}, 0)$ is the value of the zeta function $\zeta(S, \mathbf{A}, \mathbf{H}, s)$ at $s = 0$. It is defined in the half plane $\sigma > m/2$, by the absolutely convergent Dirichlet series,

$$\zeta(S, \mathbf{A}, \mathbf{H}, s) = \sum_{t>0} \frac{\mu(S, \mathbf{A}, \mathbf{H}, t)}{t^s}, \tag{14}$$

the summation on the right being all positive rational t, and over the rest of the complex s plane by analytic continuation. The zeta function $\zeta(S, \mathbf{A}, \mathbf{H}, s)$ is regular for all s except possibly for $s = 1$ and $s = m/2$ where it has simple poles and ρ_1, ρ_2 are respectively the residues at these points. If either $|\,s\,| > 0$ or $S[\mathbf{x}+\mathbf{A}]$ is not a zero form, $s = 1$ is a point of regularity, $s = m/2$ is a pole if and only if $\mathbf{H}$ is integral. Further $\zeta(S, \mathbf{A}, \mathbf{H}, s)$ satisfies the functional equation

$$\phi(S, \mathbf{A}, \mathbf{H}, s) = (-1)^{(m-n)/2}\,|\,s\,|^{-\frac{1}{2}}\,e^{-2\pi i \mathbf{A}'\mathbf{H}}\,\phi(s^{-1}, \mathbf{H}, -\mathbf{A}, m/2 - s),$$
$$\text{if } |\,s\,| > 0; \tag{15}$$

and

$$\sin(\pi s)\ \phi(S, \mathbf{A}, \mathbf{H}, s) = e^{-2\pi i \mathbf{A}'\mathbf{H}}\,\|\,s\,\|^{-\frac{1}{2}}\,(-1)^{m-n-1/2}, \tag{16}$$
$$\{\cos(\pi s)\ \phi(S^{-1}, \mathbf{H}, -\mathbf{A}, m/2 - s) - \phi(-S^{-1}, \mathbf{H}, -\mathbf{A}, m/2 - s)\}$$
$$\text{for } |\,s\,| < 0,$$

where

$$\phi(S, \mathbf{A}, \mathbf{H}, s) = \pi^{-s}\,\Gamma(s)\,\zeta(s, \mathbf{A}, \mathbf{H}, s).$$

The zeta function $\zeta(S, \mathbf{A}, \mathbf{H}, s)$ has been studied by Siegel [21] in the case $\mathbf{A} = \mathbf{H} = \mathbf{O}$. The proofs of the formulae (12) and 13, which are both valid [19] for $\alpha > (m-1)/2$ with all the occurring series absolutely convergent, are based on the functional equations (15) and (16). The difference in the nature of (15) and (16) is reflected in the formulae (12) and (13). Let us examine the situation a little more closely. For a moment we go back to the case of S being positive definite. We had remarked that the proof of (4) is based on the transformation formula (6) or one of its equivalents. It has been proved by Epstein [8], on the basis of (6), that the Dirichlet series

$$\zeta^*(S, \mathbf{A}, \mathbf{H}, s) = \sum_{t>0} \frac{A\,(S, \mathbf{A}, \mathbf{H}, t)}{t^s}, \tag{17}$$

which converges absolutely in the half plane $\sigma > m/2$, can be continued analytically into the entire complex s plane and that the resulting function is everywhere regular with the possible exception of a simple pole at $s = m/2$. $m/2$ is a point of regularity if and only if $\mathbf{H}$ is not integral. Further (6) implies [8] for $\zeta^*(S, \mathbf{A}, \mathbf{H}, s)$ the functional equation

$$\zeta(S, \mathbf{A}, \mathbf{H}, s) = e^{-2\pi i\, \mathbf{A}'\mathbf{H}}\,|S|^{-\frac{1}{2}}\,\zeta(S^{-1}, \mathbf{H}, -\mathbf{A}, m/2 - s), \tag{18}$$

where

$$\zeta(S, A, H, s) = \pi^{-s}\,\Gamma(s)\,\zeta^*(S, \mathbf{A}, \mathbf{H}, s).$$

Hecke [14] more generally proved that if the Dirichlet series

$$f(s) = \sum_0^\infty a_n\, e^{-\lambda_n s}, \quad g(s) = \sum_0^\infty b_n\, e^{-\mu_n s}$$

satisfies, in addition to (9), $f(\sigma + it) = O(t^{-c_1})$, $g(\sigma + it) = O\,(t^{-c_2})$, uniformly in σ for suitable constants c_1 and c_2, then the Dirichlet series

$$\Phi(s) = \sum_1^\infty \frac{a_n}{\lambda_n^s}, \quad \Psi(s) = \sum_1^\infty \frac{b_n}{\mu_n^s} \tag{19}$$

converge absolutely in certain half planes, admit analytic continuation into the entire complex s plane and satisfy the functional equation

$$\Gamma(s)\,\Phi(s) = \Gamma(\delta - s)\,\Psi(\delta - s); \tag{20}$$

and conversely if the Dirichlet series (17) which converge absolutely in two half planes satisfy (18) and certain regularity conditions, then for the Dirichlet series $f(s)$ and $g(s)$, (9) holds. In view of this theorem of Hecke one can regard (9) and (20) as equivalents. This theorem of Hecke applied to the functional equation (15) implies for the Dirichlet series

$$\mu_0 + \sum_{l=1}^{\infty} \mu(S, \mathbf{A}, \mathbf{H}, \lambda_l)\, e^{-\lambda_l s}, \tag{21}$$

in the case $|S| > 0$, a transformation formula of the type (9), μ_0 being a suitable constant. Now one, if so desires, may invoke a theorem of Bochner [2, Theorem 10] and immediately arrive at the proof of (12). This method fails for the proof of (13) as the functional equation (16) is not of the type (20). Hence we obtain the proof of (13) directly [19] without resorting to any transformation formula similar to (9). The same method can also be applied to the proof of (12) and we thus arrive at an alternative proof of (12). Before we proceed further we remark that, for $|S| > 0$, in the light of a transformation formula for (21) of the type (9), a general theorem of Bochner [2, Theorem 13] allows us to conclude that the series

$$\sum_{l=1}^{\infty} \mu(S^{-1}, \quad \mathbf{H}, -\mathbf{A}, \mu_l)\, J_\mu(2\pi\sqrt{(\mu_l x)})\,\mu_l^{\alpha}$$

is summable $(R;\, \mu, \eta)$ for all $\eta \geqslant 0$ such that $\alpha < \frac{3}{4} - m/2 + \eta/2$, provided that $x \neq \mu_l$ and $\mu > -1$.

There arises the question of seeking an analogue of the theta function (5) for indefinite quadratic forms. This analogue is implicit in the work of Seigel [21, II]. Following Hecke one asks for the construction of this function from the zeta function (14). Maass [16] showed that these functions are not analytic functions as in (7)* but, non-analytic functions defined in the upper half plane of $Z = is$ which are solutions of a partial differential equation of second order of the elliptic type and possessing the properties of an automorphic form under a discontinuous group of mappings acting in the upper half-plane. More precisely let $g_1(x, y)$, $g_2(x, y)$ be a pair of complex-valued functions which are twice differentiable in the upper-half plane $y > 0$. Further let $g_1(x, y) = f_1(z, \bar{z})$, $g_2(x, y) = f_2(z, \bar{z})$, α and β a pair of real numbers and Q_1, Q_2 positive constants. Then one requires $g_1(x, y)$ and $g_2(x, y)$ to satisfy the following conditions ;

$$\left.\begin{array}{ll}
\text{(i)} & g_1(x, y) \text{ and } g_2(x, y) \text{ satisfy} \\
& y^2\left(\dfrac{\partial^2 g}{\partial x^2} + \dfrac{\partial^2 g}{\partial y^2}\right) - (\alpha - \beta) iy \dfrac{\partial g}{\partial x} + (\alpha + \beta) y \dfrac{\partial g}{\partial y} = 0. \\
\text{(ii)} & g_j(x + Q_j, y) = e^{2\pi i \nu_j} g_j(x, y) \qquad (0 \leqslant \nu_j < 1, j = 1, 2) \\
\text{(iii)} & g_i(x, y) = O(y^{\lambda i}) \text{ as } y \to \infty, \\
& \quad = O(y^{-\mu i}) \text{ as } y \to 0, \\
& \text{uniformly in } x, \lambda_i, \mu_i \ (i = 1, 2) \text{ being suitable constants.} \\
\text{(iv)} & f_1\left(-\dfrac{1}{z}, -\dfrac{1}{\bar{z}}\right) = \gamma(-iz)^{\alpha} (iz)^{\beta} f_2(z, \bar{z}), \\
& \gamma \text{ being another constant.}
\end{array}\right\} \quad (22)$$

It has been proved by Maass [16] that by Mellin's inversion one can associate with $\zeta(S, \mathbf{A}, \mathbf{H}, s)$ and $\zeta(S^{-1}, \mathbf{H}, -\mathbf{A}, s)$ two functions of $g_1(x, y)$ and $g_2(x, y)$ satisfying (22). In this case $\alpha = n/2$ and $\beta = (m - n)/2$. Thus $g_1(x, y)$ is the analogue of the theta function for $\zeta(S, \mathbf{A}, \mathbf{H}, s)$ and the functional equation for the zeta function appears for $g_1(x, y)$ and $g_2(x, y)$ under (22) (iv). Maass [16] has shown that conditions (22) (i), (ii), (iii) imply for $g_1(x, y)$ the Fourier expansion

$$g_1(x, y) = a_0 u(y, \alpha + \beta) + b_0 + \sum_{n + \nu_1 \neq} a_{n+\nu_1} \times$$

$$\times W\left(\frac{2\pi |n + \nu_1|}{Q_1} y ; \alpha, \beta, \operatorname{sgn}(n + \nu)\right) e^{2\pi i (n + \nu_1) x / Q_1}, \quad (23)$$

the series on the right of (23) being absolutely convergent for $y > 0$ on the right of (23),

$$u(y, \theta) = \frac{y^{1-\theta} - 1}{1 - r} \text{ and } W(y ; \alpha, \beta, \epsilon) = y^{-(\alpha+\beta)/2} W_{(\alpha-\beta)\epsilon/2, (\alpha+\beta-1)/2} \quad (24)$$

where W_l, m is the Whittaker solution of the confluent hypergeometric differential equation in the reduced form [17, Chapter 6].

In the case of quadratic forms the Fourier coefficients $a_{n+\nu}$ upto a multiplicative constant are precisely the $\mu(S, \mathbf{A}, \mathbf{H}, t)$. In the case of those non-analytic automorphic forms which arise from indefinite quadratic forms, the " simple " nature of the functional equation (15) of $\zeta(S, \mathbf{A}, \mathbf{H}, s)$ for $|S| > 0$, is reflected in the following elegant property for $g_i(x, y)$, $(i = 1, 2)$. For $|S| > 0$, $m - n$ is even and thus $(m - n)/2$ is an integer. Siegel [22] considers the $(m - n)/2$ the iterate of the differential operator $n/2 + |z - \bar{z}| \dfrac{\partial}{\partial z}$ which may be denoted by Θ and proves that $\Theta(g_1(x, y))$ is a constant multiple of (21) when one sets $is = z$ and the property (22), (iv) implies for (21) a transformation formula of the type (9). This property has been generalized by Maass [16] for general non-analytic automorphic forms for which β is a non-zero integer. Thus we are led to the fact that the functional equation characterizes the " indefinite " nature of quadratic forms. Now we are in a position to formulate a generalization of (13). One starts with a pair of functions $g_i(x, y)$, $(i = 1, 2)$ satisfying the conditions (22) and then seeks to obtain a formula expressing

$$\sum_{0 < n+\nu \leqslant x} a_{n+\nu}(x - \overline{n + \nu})^{\alpha}$$

as a series of analytic functions. The formulae (11) and (13) enable us to guess the nature of these expansions.

REFERENCES

1. T. V. Avadhani : On summations over lattice points, *J. Indian Math. Soc.* 16 (1952), 103-125.

2. S. Bochner : Some properties of modular relations, *Annals of Math.* 53 (1951), 332-363.

3. S. Bochner and K. Chandrasekharan : Summations over lattice points in K-space, *Quarterly J. Math.*, Oxford, 19 (1948), 232-248.

4. S. Bochner and K. Chandrasekharan : A supplementary note, *ibid*, (1950), 80.

5. S. Bochner and K. Chandrasekharan : Lattice points and Fourier expansions, *Acta Szged*, 12 (1950), 1-15.

6. A. L. Dixon and W. L. Ferrar : Lattice point summation formulae, *Quarterly J. Math.*, Oxford, 2 (1931), 31-54.

7. A. L. Dixon and W. L. Ferrar : Some summations over the lattice points of a circle I, II, *Quarterly Jour. Math.*, Oxford, 5 (1934), 48-63, 172-185.

8. P. Epstein : Zur theorie der allgemener Zeta-funktionen I, II, *Math. Annalen*, 56 (1903), 615-644, 62 (1906), 205-246.

9. G. H. Hardy : On the expression of a number as the sum of two squares, *Quarterly J. Math.* 46 (1915), 263-283.

10. G. H. Hardy : The average order of the arithmetical functions $P(x)$ and $\Delta(x)$, *Proc. London Math. Soc.* (2), 15 (1916), 192-213.

11. G. H. Hardy : On Dirichlet's divisor problem, *Proc. London Math. Soc.* 2, 15 (1916), 1-25.

12. G. H. Hardy : A further note on Ramanujan's arithmetical function $\tau(n)$, *Proc. Cambridge Philos. Soc.* 34 (1938), 309-315.

13. G. H. Hardy and E. Landau : The lattice points of a circle *Proc. Royal Soc.* cv. (1924) 244-258.

14. E. Hecke : Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung, *Math. Annalen* 112 (1936), 664-669.

15. E. Landau : Über die Gitterpunkte in einem Kreise (3), *Göttinger Nachrichten* (1920), 109-134.

16. H. Maass : Die Differentialaleichungen in der Theorie der. elliptischen Modulfunktionen, *Math. Annalen*, 125(1953), 235-263.

17. O. Magnus and F. Oberhettinger : *Formeln und Sätze* für *die spezieller Funktionen der mathematischen physik*, Springer-verlag 1948.

18. A. Oppenheim : Some identities in the theory of numbers, *Proc. London Math. Soc.* (2), 26 (1927) 295-350.

19. V. V. Rao : The lattice point problem for indefinite quadratic forms with rational coefficients. *J. Indian Math. Soc.* 21 (1957), 1-40.

20. C. L. Siegel : Über die analytische Theorie der quadratische Formen II, *Annals of Math.* 37 (1936), 230-263.

21. C. L. Siegel : Über die Zetafunktionen indefiniter quadratischer Formen I, II, *Math. Zeit.* 43 (1938), 682-708, 44 (1939), 398-426.

22. C. L. Siegel : Indefinite quadratische Formen und Modulfunktionen, *Courant Anniv.* Vol. 1948, 395-406.

23. C. L. Siegel : Indefinite quadratische Formen und Funktionentheorie I, *Math. Annalen*, 124 (1951), 17-54.

24. G. Voronoï : Sur le developpement, ā l' aide des fonctions Cylindriques, der sommes doubles Σ $f(pm^2 + 2qmn + rn^2)$ or $pm^2 + 2qmn + rn^2$ est une forme positive ā Coefficients entiers, verhandlungen der driuen internationalen Math-Kongresses in Heidelberg (1904), Leipzig, B.G. Tenbner 1905, 241-245.

25. G. Voronoï : Sur une fonction transcendante et ses application a la sommations de quelque series, *Ann. Ecole Norm.* (3) 21 (1904), 207-267, 459-533.

26. A. Walfisz : Über die Summatorischen Funktionen liniger Dirichletscher Reihen, *Inaugural dissertation*, Göttingen, 1922.

27 A. Walfisz : Über das piltzsche Teilerproblem in algebraischen Zahlkörperm, *Math. Zeit.* 22 (1925), 153-188.

28. S. Wigert : Sur quelques fonctions arithmetiques, *Acta Math.* 37 (1914), 113-140.

29. J. R. Wilton : The lattice points of an n dimensional ellipsoid, *Jour. London Math. Soc.* 2 (1927), 227-33.

30. J. R. Wilton : A series of Bessel functions connected with the theory of lattice points, *Proc. London Math. Soc.* (2) 29 (1929), 168-88.

31. J. R. Wilton : A note on Ramanujan's arithmetical function $\tau(n)$, *Proc. Cambridge Philos. Soc.* 25 (1929), 121-29.

32. G. N. Watson : *Theory of Bessel functions,* Cambridge, 1944.

Faculty of Science
M. S. University of Baroda

ON ORDERED STRUCTURES

By V. S. KRISHNAN

1. Introduction. In studying the characteristic properties of a (axiomatically defined) mathematical system, it is convenient to consider the algebraic, order-based, and topological characteristics separately first and then study their inter-relations. Thus the real number system is a commutative field which is infinite, totally ordered, and which has a metric topology under which it is complete; further the positive elements (elements greater than or equal to zero, under the total order) form an integrity domain which has the original field as its field of quotients, the topology is also determined by order convergence of sequences and the totally ordered set is conditionally complete as a lattice. It is this richness of properties of the real number system under the basic structural features that makes this system so fundamental for mathematics. While the role of algebraic structure and topological structure have been recognised for some time, the place of order in the structural analysis of mathematical systems has come to be studied only in recent years. This symposium is intended to present some of the order based features in different branches of the subject.

The simplest of well-ordered sets, the sequence, and its cardinal have played a dominant role in all classical analysis and early topology. This is partly due to the place of the enumerability of the integral domain of positive integers from which in stages the real numbers are built up, by immersing in a group, then in a field and then completing it. That a somewhat similar procedure gives rise to an ordered field when one starts from an integral domain which is well ordered and has the order type of a regular initial ordinal, has been worked out by R. Venkataraman, who will be presenting here some special features of his generalized system.

In the rational field there is a partial order different from the usual total order; namely the relation of divisibility : we set x/y (in words x divides y) if there is an integer n such that $y=nx$. With respect to this order the subsystem of all integers is a lattice, with the g.c.d. and the l.c.m. of two integers forming their lattice product and lattice sum. Ideal theory and valuation theory of rings deal with this type of partial orderings in rings. N. Sankaran will be giving a report on this type of order and on the use of valuation theory in rings. Birkhoff, Ore and others have observed the essential lattice-theoretic form of many structure theorems relating to algebraic systems. The results on normal series and composition series, are all extensible to corresponding results on congruence relations on an Algebra. This relation of lattice theory to abstract algebra (or metamathematics), and the relation between the general projective spaces and certain complemented modular lattices will be treated by Miss Iqbal Unnisa, who also gives the basic material required for this from lattice theory proper.

How the lattice formulation leads on naturally from the finite projective geometries to the infinite dimensional 'atomic' or 'continuous' geometries will be explained by V. K. Balachandran. Here the projective geometry itself means a special type of modular lattice.

Finally the place of order in the study of vector spaces, and the deduction of a type of extended Hahn-Banach theorem for topological vector spaces from a similar result proved for ordered vector spaces will be discussed by S. Swaminathan.

After this introduction to the nature of the symposium and the scope of the talks by the succeeding participants, I shall take up an example to show how order properties are involved in very general structural questions, by discussing the nature of immersion problems and treating in some detail one such problem.

The Immersion problem

Two typical immersion problems will illustrate our further discussion. The first is the immersion of the additive semigroup of positive integers (and zero) in the group of all integers; the second is the immersion of the topological (additive) group of rational numbers in the complete topological group of reals. In each case the original system is immersed by an isomorphism (which is also a homomorphism in the second case) as a subsystem of a larger system of similar nature but having some further properties that the original system lacks (in the first case, the existence of inverses is the extra property, in the second the existence of limits for Cauchy filters). Thus, we can formulate our immersion problem in the following manner :

P denotes a certain type of mathematical structure and P^* denotes a restricted type of P-structure, that is, any P^*-structure is a P-structure satisfying some further properties. There is also a notion of isomorphism or structural identity for P-structures; isomorphic structures have identical properties in terms of the basic structural concepts defining P-structures. The questions that can then be asked are :

(1) Given a P-structure A are there P^*-structures A^* containing substructures isomorphic to A? When such a P^*-structure exists we call it a P^*-extension of A, and the isomorph of A contained n A^* is called the image of A in A^*. By replacing the image of A by A itself, it is clear that A^* can be treated as a P^*-structure containing A itself as a substructure; it is for this we call the isomorphism of A in A^* an isomorphism immersing A in A^*.

(2) Given two P^*-extensions A^*_1 and A^*_2 of A, with f, g as the isomorphisms mapping A in A^*_1 and A^*_2, if the isomorphism $g.f^{-1}$ of the image of A in A^*_1 on the image of A in A^*_2 can be extended to an isomorphism of A^*_1 in A^*_2, then we say that the extension A^*_1 is smaller than A^*_2. The second question is then: if there are P^*-extensions of A, are there minimal P^*-extensions (such that there is no non-isomorphic smaller P^*-extension)? Is there a minimal smaller than each given P^*-extension? Can there be non-isomorphic minimal P^*-extensions?

(3) Finally is there a P^* extension smaller than all P^*-extensions? Such an extension would be unique upto isomorphisms.

The study of extensions of a lattice or partially ordered set relative to closure under various order-based operations gives examples where the above questions have sometimes positive and sometimes negative answers (see [3, 4 and 5]). In the two examples mentioned at the beginning the extensions are smallest in the sense explained above. We shall examine here another example which arose in connection with the study of quality between uniform semi-groups.

By a demigroup we shall mean a set closed for a binary, associative operation, denoted by $+$. A zero for the demigroup is a unit under the operation. A demigroup is a half group if no element other than the unit has an inverse. As P structure we take the commutative P group with zero. As P^*-structure we take a demigroup isomorphic to a subdemigroup of a direct sum ΣR_i of replicas of the additive demigroup of real numbers. (In the direct sum each element is a finite sum of elements from the R_i, or it is the subset of the Cartesian product in which only a finite number of components is nonzero.) So the question is to find under what conditions a commutative half group with zero can be immersed in a direct sum of replicas of R. The answer requires the formulation of some further concepts.

A demigroup D is said to be torsionless if $nx = ny$, for a positive integer n and elements x, y of D, implies that $x = y$ (where nx is the sum of n x's). D is said to be divisible if for any element x of D and any positive integer n there exists a y in D such that $ny = x$. An element x of D is said to be less than another y (in symbols $x < y$) relative to a subdemigroup D' if there is an element z of D' such that $x+z=y$. When D' is the demigroup D itself the associated relation is called the natural ordering relation in D. Given an ordering relation $<$, an element x of D is said to be infinitesimal relative to y under $<$ if for each positive integer n, $nx < y$. If the ordering relation is the natural ordering relation we omit to mention 'under $<$'. Evidently 0 is infinitesimal relative to any element x of D. If 0 is the only element infinitesimal relative to x in D, then x is said to be regular. If all elements of D are regular, D is called regular.

We can now state the main result regarding the immersion problem :

THEOREM. *A commutative halfgroup with zero can be immersed in a direct sum of replicas of the group of reals if and only if, it is torsionless, divisible, regular, and every enumerable set of its elements has a lattice product (relative to the natural ordering). When this immersion is possible, there is a smallest extension of the sort considered upto isomorphisms.*

The proof of this is derivable from the results (Theorems 2 and 3) proved in another paper [7]. While the conditions are seen to be necessary for such an immersion to be possible, it can be shown that, under the conditions given, the halfgroup admits, the halfring R of positive reals (and zero) as a operator halfring. Being torsionless and divisible, for any x of D, $1/(nx)$ and so $m/(nx)$ can be uniquely defined (for positive m, n). Then by taking rx to be the lattice product of $r_i\, x$, where r_i is a decreasing sequence of rationals converging to a positive real r, the real operators are defined. If then, using Zorn's principle, we find a maximal direct sum D^* of replicas of R' contained (isomorphically immersible) in D, its enveloping group G (or group of differences) is a direct sum of replicas of R (the group of reals) and this is also the enveloping group of D. Finally any direct sum G of replicas of R containing D would contain also D^*, and so also G (upto isomorphism).

REFERENCES

1. G. BIRKHOFF: 'Lattice Theory,' *American. Math. Coll.* Publ. 2nd edn.

2. P. DUBREIL: *'Algebre'*, Cahiers Scient.

3. V. S. KRISHNAN: Extensions of partially ordered sets I, II *Jour. Indian Math. Soc.* 11, (1947), 49-58. ,

4. V. S. KRISHNAN: Extensions of partially ordered sets I, II, *Jour. Indian. Math. Soc.* 12, (1948), 99-106.

5. V. S. Krishnan : Extensions of multiplicative systems and modular lattices, *J. Ind. Math. Soc.* 13 (1949), 49-59.

6. V. S. Krishnan : Theory of demigroup structures, I, *Jour. Madras Univ.* 26, (1957), 305-15.

7. V. S. Krishnan : Theory of demigroup Structures, II, to appear in the Jubilee number of the Indian Mathematical Society.

Madras University
Madras 5

GENERALIZATION OF REAL NUMBERS

By R. VENKATARAMAN

1. Introduction. One of the basic structures in all branches of mathematics is the real number system. It has quite a richness of properties in that it is a field which is totally ordered and also topological; totally ordered in the sense that the field operations are monotone with respect to order; topological in the sense that the field operations are continuous under the topology introduced through sequential convergence. The order and the algebra of the real number system are so related that it could be characterized up to isomorphism as a complete ordered field. The relationship between the topology and algebra of the real number system is brought out in the result that a locally compact, connected topological field is isomorphic with one of the three topological fields, viz. the field of real numbers, the field of complex numbers or the field of quaternions (cf. Pontrajagin [10]).

The topic of the generalization of real numbers has two natural divisions: (1) a generalization with reference to its order properties and (2) a generalization with reference to some order and algebraic properties it has. In the following sections 2 and 3, I summarize the work carried out under the divisions (1) and (2) respectively.

2. Following are some properties of the real number system as an ordered structure: If θ denotes the ordertype of all real numbers in the closed interval [0, 1], it is well known that θ is a complete ordertype. Also it is imbeddable in every one of its non-null intervals. Several interesting order properties of θ arise out of its relationship with the ordertype η of all rational numbers. η is dense in θ; again, η is similar to an isolated subset of θ (i.e. η similar to a set of disjoint intervals of θ). Further every countable order type can be realized in the ordertype η and consequently, in the ordertype θ.

A generalization of the system of real numbers as an ordertype possessing order properties similar to those of θ was initiated by

Webber [14] and Cuesta Dutari Norberto [4] and was carried out in detail by K. Padmavally [8] and [9].

Starting from the definition of a complete power (Hausdorff [5]), some relations between θ and η are generalized for complete powers of certain ordertypes.

The complete power $C(\alpha)$ with basis C, for any ordertype C, and argument α, α any ordinal number, is defined as the aggregate of all (finite, infinite, transfinite) sequences each of α terms $\{x_\beta\}_{\beta<\alpha}$, β an ordinal $<$ the given ordinal α, $x_\beta \in C$, ordered lexicographically, viz. $\{x_\beta\}_{\beta<\alpha} < \{y_\beta\}_{\beta<\alpha}$ if and only if, $x_\beta = y_\beta$ whenever $\beta < \gamma$ and $x_\gamma < y_\gamma$ for some $\gamma < \alpha$.

It could be shown that for any ordertype C, $C(\alpha)$ is its own completion if C is its own completion. Further, if α is an indecomposible ordinal (i.e. an ordinal number α such that whenever β and γ are ordinal numbers with $\beta+\gamma=\alpha$ then $\gamma=\alpha$), then $C(\alpha)$ can be imbedded in every one of its non-null intervals.

For every ordertype C and every limiting ordinal α, $\overline{C(\alpha)}$, i.e. the completion of the ordertype $C(\alpha)$, has a dense subset similar to an isolated subset of itself. If further, C has a highest or lowest element, $C(\alpha)$ has a dense subset similar to an isolated subset of itself.

Bearing in mind the Cantor-Bendixon theorem that every ordertype imbeddable as an isolated subset of θ is countable, the property of η, that every countable ordertype can be realized in it, can be stated in the following equivalent form: The union of a countable family of ordertypes imbeddable as an isolated subset of θ is itself imbeddable as an isolated subset of θ.

The generalization of the above result can be given for complete powers of argument, a regular initial ordinal number. (An ordinal number ω_μ is said to be regular if every cofinal subset of it is of ordertype of the ordinal ω_μ itself. An ordinal number is said to be initial, if it is the least ameng equi-potent ordinals. A regular ordinal is initial, but the converse, in general, is not true.) If ω_μ is a regular initial ordinal number and $\aleph_\mu$ the power of ω_μ, the union of an aggregate of power $C\ \aleph_\mu$ of ordertypes imbeddable as an isolated subset of $\overline{C(\omega_\mu)}$ is imbeddable as an isolated subset of $\overline{C(\omega_\mu)}$.

3. In this section we shall consider the question of generalization of the system of real numbers as an ordered field. I shall enumerate certain properties of ordered fields in general. The cofinal character and the coinitial character of an ordered field are equal and infinite. Every element of an ordered field has equal and symmetric character, which is the character of the field itself. Also, no ordered field of character $\omega_\mu > \omega$, ω_μ a regular initial ordinal number, can be order-complete, for all ordered fields which are order-complete are isomorphic to the ordered field R^* of all real numbers which is of character ω. Hence every ordered field of character ω_μ, $\omega_\mu > \omega$ has necessarily gaps.

Let $[A_1, A_2]$ denote a decomposition of an ordered set A. By the characters of the decomposition $[A_1, A_2]$ we mean the cofinal character of $(A_1 - a_1)$ or A_1 according as A_1 has a last element a_1 or not and the coinitial character of $(A_2 - a_2)$ or A_2 according as A_2 has a first element a_2 or not. Then it is easy to see that a necessary condition for a decomposition of an ordered field to be a cut is that it is of character ω_μ, where ω_μ is the character of the ordered field. Again, if we define that an ordered set is ω_ν-complete if every decomposition of character $[\omega_\nu, \omega_\nu]$ is a cut, then no ordered field of character ω_μ can be ω_ν-complete for $\omega_\nu < \omega_\mu$. So we shall define an ordered field F of character ω_μ to be complete if every decomposition of character $[\omega_\mu, \omega_\mu]$ is a cut.

Bearing in mind that the real number system is an ordered field of character ω which is also complete, we can define its generalization as an ordered field of character a given initial ordinal number which is complete. The question arises, whether for every regular initial ordinal number ω_μ, there exists a complete ordered field of character ω_μ. This question has been considered by Roman Sikorski [12] and also by myself [13] and has been answered in the affirmative.

I shall present a summary of the construction of a complete ordered field of character ω_μ, where ω_μ is any given regular initial ordinal, through a systematic generalization of the ordered domain of integers.

I shall first define a class of ordered sets, called the symmetrically ordered sets.

An ideal I (coideal U) of an ordered set P is said to have extremal symmetry if there exists an ultimate segment (initial segment) of I/U, anti-isomorphic with an initial segment (ultimate segment) of the coideal (ideal) constituted by set complementation of I/U in P. An ordered set in which every proper ideal (coideal) has extremal symmetry is said to be symmetrically ordered. The order-types ω, ω^*, $\omega^* + \omega$, are examples of symmetrically ordered sets.

Let J denote the ordered domain of all integers and α some ordinal. By a symmetric power $J(\alpha)$ of index α, we mean the aggregate of all integer-valued functions defined on the set of all ordinals $< \alpha$, such that each function has at most a finite number of non-zero values, and ordered by last differences. (If f and g be distinct elements of $J(\alpha)$, then as f and g have at most a finite number of non-zero values there can be at most a finite number of places where f and g can differ.) If at the last place (say) $\beta(< \alpha)$, where f and g differ, $f(\beta) < g(\beta)$, we say $f < g$. It is easy to see that this ordering relation $<$ is a total-ordering on $J(\alpha)$.

It could be proved that an ordered set is symmetrically ordered if and only if it can be imbedded as a segment of the symmetric power of index a suitable ordinal. (By a segment of an ordered set we mean a non-null subset which with every pair of its elements contains all intermediate elements of the ordered set.)

In the symmetric power $J(\alpha)$, α any ordinal, the binary operation '$+$' of point-wise sum as functions defined on the set of all ordinals $< \alpha$, could be seen to be a group operation and under this operation $J(\alpha)$ is seen to be an ordered Abelian group. $J(\alpha)$ is the least ordered Abelian group (up to isomorphism) containing the system of all ordinals $< \omega^\alpha$ under the Hessenberg natural sum (cf. P. W. Carruth [3]).

The element g_β in $J(\alpha)$ such that g_β has the value zero for all places except at β where it has the value 1, is called the βth generator of $J(\alpha)$. The set of all generators of $J(\alpha)$ constitutes a basis over the domain of integers for it. Define a binary operation $\times$ for the generators of the ordered group $J(\alpha)$ thus : $g_\beta \times g_\gamma = g_\delta$ where g_β, g_γ, g_δ are βth, γth, δth generators respectively and $\delta = \sigma(\beta, \gamma)$, viz. the Hessenberg natural sum of β and γ. This binary operation defined for the basis elements of $J(\alpha)$ could be extended as a binary operation over $J(\alpha)$ itself. The necessary and sufficient condition that $J(\alpha)$ may be closed for this operation '$\times$' is that α is an indecompossible ordinal. If α is an indecompossible ordinal, $J(\alpha)$ is an ordered integral domain, under the binary operations $+$ and $\times$ defined above.

The following characterization theorem is true. If ω_μ is any regular initial ordinal then every ordered domain of character ω_μ which is also symmetrically ordered is isomorphic to the ordered domain $J(\omega_\mu)$. This is the generalization of the following theorem which characterizes the ordered domain of integers : Every ordered domain, the set of whose positive elements is well ordered, is isomorphic to the ordered domain of all integers. So we shall refer to the ordered domain $J(\omega_\mu)$ as the ordered domain of all ω_μ-integers.

It is well known that there exists a unique (up to isomorphism) minimal extension of a given ordered domain into an ordered field. Such an extension for the ordered domain of ω_μ-integers is called the ordered field of ω_μ-rationals.

If ω_μ is any regular initial ordinal, $\omega_\mu > \omega$, then Roman Sikorski [12] has proved that the ordered field of all ω_μ-rationals is complete.

It must be noted that the above result is not true in the particular case when $\omega_\mu = 1$.

It is of interest to note that there exists one and (up to isomorphism) only one complete ordered field of character ω, while for $\omega_\mu > \omega$, there exist many non-isomorphic ordered fields of character ω_μ and complete (cf. Roman Sikorski [12]).

Further, while the complete ordered field cf character ω is of power $2^{\aleph_0} > \aleph_0$, the ordered field of character ω_μ we have constructed is of power $\aleph_0$. For $\omega_\mu > \omega$ there exist ordered fields of power $2^{\aleph_0}$.

REFERENCES

1. G. Birkhoff : Lattice theory, New York, (1948).
2. Birkhoff and Maclane : A Survey of Modern Algebra, (1948).
3. P. W. Carruth : *Bull. American. Math. Soc.* (1942).
4. Guesta Dutari Norberto : Generalized real numbers, *Revista Math. Hisp. Amer.* (4) 2 (1942), 5-12, 62-66, 104-109, 218-225.
5. Hausdorff : Theorie der Geordneten Mengen, *Math. Ann.* 65, 439-505.
6. Hausdorff : *Grundzüge der Mengenlehre*, Leipzig, (1914).
7. Hausdorff : *Mengenlehre*, Berlin, (1927).
8. K. Padmavalli : Generalization of rational numbers. *Revist. Math. Hisp. Amer.* (4) 12 (1952), 3-19.
9. K. Padmavalli : Generalization of the ordertype of rational numbers, *Revist. Math. Hisp. Amer.* (4) 14 (1954), 1-24.
10. Pontrajagin : *Topologische Grouppen*, Leipzig, (1957).
11. W. Sierpinski : *Lecons sur les Nombres Transfinis*, Paris, (1928).
12. Sikorski Roman : On an ordered Algebraic field, *Soc. Sci. Lett., Varsoive, C.R.C1.* III *Sci. Math. Phys*, 41 (1946), 69-96
13. R. Venkataraman : On a class of ordered structures and related Topological spaces, Thesis (Unpublished) 1960.
14. Webber : Ordinal characterisation of linear sets, *Trans. Roy. Soc.* (Canada), 25 (1931), 65-74.

Madras University
Madras 5.

ORDER STRUCTURE IN RINGS AND FIELDS

By N. SANKARAN

1. Introduction. Divisibility, as an ordering relation induces an order structure in a ring and in the collection of ideals of it. This is one way of introducing an order in a ring. Another is by means of valuation. We study in the following pages the nature of a valuation ring, the relation between the divisibility order of the ring and order induced by the valuation, and the conditions for a topological field to have valuations compatible with its topology. We mention briefly the characterization of valuation rings in terms of its various ideal systems and indicate the application of valuation theory to algebraic geometry.

2. Ordered Rings & Fields. We call a ring A an *ordered ring* if an order structure can be introduced in it which is compatible with the ring operations. That is, the following are true.

$$x \geqslant 0, y \geqslant 0 \Rightarrow x . y \geqslant 0$$

$$\text{for each } z \in A,\ x \leqslant y \Rightarrow x + z \leqslant y + z.$$

Now the positive part of the ring (denoted by P) determines the order structure and the order structure determines the positive part. The conditions are: (i) $P + P \subset P$, (ii) $P . P \subset P$, (iii) $P \cap (-P) = (0)$. For a total ordering of A we further demand that $P \cup (-P) = A$. From the fact that $nx = 0$ implies $x = 0$ for a non-zero natural integer n, we deduce that any totally ordered ring is of characteristic zero. If we have an ordered integral domain with a unit then we have one and only one order structure on the field of quotients which preserves the order of the ring. Now the problem is that if E is an extension of the ordered field K, can we introduce an order in E which will preserve the ordering of K?

A necessary and sufficient condition for this to happen is that the relation $\Sigma p_i x_i^2 = 0 \Rightarrow p_i x_i = 0$ for all i, where $x_i \in E$ and $p_i > 0$, $p_i \in K$. As a corollary we get the theorem due to Artin and Schreier which states that for the existence of an order structure on a commutative field E it is necessary and sufficient that the relation $\sum_{1}^{n} x_i^2 = 0 \Rightarrow x_i = 0$ for $i = 1, 2, 3, \ldots, n$.

We agree to call an ordered field *maximal* if it coincides with all its extensions. We can prove that every ordered field has a maximal ordered extension field.

3. Normed Rings. The collection R is called a *normed ring* when (i) R is a linear, normed complete space in the sense of Banach, (ii) in R the operation of multiplication of elements is defined, which satisfies the algebraic properties

$$x(\lambda y + \mu z) = \lambda xy + \mu xz, \quad x(yz) = (xy)z, \quad \| x.y \| \leqslant \| x \| \cdot \| y \|,$$

and

$$\| x + y \| \leqslant \| x \| + \| y \|, \quad \| e \| = 1 \text{ and } \| 0 \| = 0.$$

For a detailed discussion one can refer to Gelfand [5].

4. Valuation Rings. The theory of valuations can be viewed as a sort of generalization of the normed ring in that the triangular inequality with respect to multiplication becomes an equality and the valuation is not necessarily a positive valued function. It can also be looked upon as a method of constructing fields with the properties of absolute value. Before coming to the general theory of valuations we will give certain particular definitions. A field K is said to have a valuation v if a function $v(a)$ is defined for every $a \in K$ such that

(i) $v(a)$ is an element of the ordered field P,

(ii) $v(a) > 0$ for $a \neq 0$, $v(0) = 0$,

(iii) $v(ab) = v(a).\, v(b)$,

(iv) $v(a + b) \leqslant v(a) + v(b)$.

These conditions are fulfilled for any ordered field K if $v(a) = |a|$. Each field has a trivial valuation: $v(a) = 1$, for a non-zero a, and $v(0) = 0$. For the field of rational numbers Γ we can define another type of valuation the p-adic valuation (for every prime p) because if p is any prime then any rational number a can be represented as $a = b/c \cdot p^n$ where $(b, c) = 1$, and prime to p. Put now $V_p(a) = p^{-n}$, $V_p(0) = 0$. This satisfies the first three conditions and instead of the fourth we get the stronger inequality

$$V_p(a+b) \leqslant \max\,(V_p(a),\ V_p(b)).$$

If now we define an Archimedian ordered field as one in which for any two non-zero elements α, β we can find a natural number n such that $n\alpha > \beta$ we see that the absolute value gives an Archimedian order while the p-adic valuation introduces the non-Archimedian order. The necessary and sufficient condition for the valuation v of the field K to be non-Archimedian ordered is that the stronger inequality $v(a + b) \leqslant \max\,(v(a),\ v(b))$ is satisfied. This condition shows that for fields with non-Archimedian valuations it is needless to consider the field of values as we use only one operation.

Now Ostrowski [13] has shown that any field with an Archimedian valuation is topologically isomorphic to a subfield of the complex numbers with absolute value as its valuation. So for deeper results in valuation theory we consider only non-Archimedian valuations.

In general we take the valuations v to satisfy the following postulates. v is a mapping of the field K onto a simply ordered Abelian group Γ such that

(i) for every $a \neq 0$ in K, there exists an α in Γ such that $v(a) = \alpha$; (ii) $v(a.\,b) = v(a) + v(b)$; (iii) $v(a+b) \geqslant \min\,(v(a),\ v(b))$.

The elements of the skew field K for which the valuation is non-negative form a ring R called the valuation ring. In this valuation ring the elements with zero valuation form a two-sided ideal. In fact we could prove that every ideal in the valuation ring is a two-sided ideal. If now we define an upper class in the collection of the positive elements of the valuation domain Γ as the set which with any α contains all $\beta > \alpha$, then we could show that the upper classes form a simply ordered set and that it is isomorphic to the two-sided ideals of the valuation ring.

If now, we are given an integrity domain I with a unit, then the problem is that under what conditions will it be a valuation ring of its field of quotients K ? Krull [8] has given the following : The necessary and sufficient condition for I to be a valuation ring of its quotient field K is that (i) all non-units of I form an ideal ; (ii) any over-ring R which contains I and is contained in K contains an inverse of a non-unit of I.

Further if I is integrally closed in K then there exists at least one valuation ring $I_v \supset I$ and that I is the intersection of all such valuation rings.

5. Order and Valuation. Now the valuation induces an order on the field. We say that $a < b$ where $a, b \epsilon K$ if $v(a) < v(b)$ in the value group Γ. In the integrity domain I with a unit there is an intrinsic order that of divisibility order ($a < b$ if a divides b). Now the question is what the relation is between the divisibility order and the order induced by the containing valuation rings. This has been investigated by Lorenzen [9, 10] who gives the following results : If B_t denotes the valuation over-ring containing the integrity domain I then the divisibility order is the conjunction order (lattice product order) of all the induced orders. That is to say, $a < b$ implies $a < b_t$ for each t.

Each principal ideal ring can be represented as the intersection of valuation rings and the multiplicative group of its quotient field is a lattice group with respect to divisibility.

6. Topological fields and valuations. Let K be a commutative field where addition and multiplication are continuous operations and Γ be a linearly ordered Abelian group. The valuation introduces in K the order topology of Γ in the following manner: The neighbourhoods of 0 in K are given by

$$U(\gamma) = \{x \epsilon K \mid v(x) < \gamma, \gamma \epsilon \Gamma\}.$$

If this topology is compatible with the topology of the topological field K we say that the valuation preserves the topology. Now the question is which of the topological fields have valuations preserving the topology ? Kaplansky [6] has given the necessary and sufficient conditions for a topological field to have Archimedian valuation preserving the topology based on a conjecture of Shaferevitch [18] and Zelinsky [21] has given in the non-Archimedian case.

For non-commutative fields with Archimedian valuations the conditions are: (i) the set $\{a\}$ of nilpotent elements ($a^n \to 0$) forms a right bounded set ; (ii) if a is nilpotent and b is either nilpotent or neutral ($b^n \not\to, 0,\ b^{-n} \not\to 0$) then ba is nilpotent ; (iii) the commutator subgroup of the multiplicative group of non-zero elements is right bounded.

For a commutative field the conditions are rephased as (i) the set of nilptent elements form an open set ; (ii) if $A \subset K$ is bounded away from zero (A is disjoint from the neighbourhood of 0) then A^{-1} is bounded.

For non-Archimedian valuations the conditions read as (i) some neighbourhood of zero generates an additive group which is bounded; (ii) if $A \subset K$ is bounded away from zero then A^{-1} is bounded.

7. Generalizations. Schilling [17] has considered the non-commutative valuations by taking a non-commutative group. As an example we consider the following : Let Γ be a lexicographically ordered group of all motions in the plane where the law of combination is defined as $(\alpha, \beta) + (\gamma, \delta) = (\alpha + \gamma, e^{\gamma}. \beta + \delta)$. The set Γ^{+} consists of all couples for which either $\alpha > 0$ or $\alpha = 0$ and $\beta > 0$. This Γ will be the value group of the formal power series $D = \{\sum_{(\alpha,\beta)} a_{\alpha,\beta}\ t^{(\alpha,\beta)}\}$, where $a_{\alpha,\beta} \in F$ a field and t is a transcendental over F. The valuation is defined as

$$t^{(0,0)} = 1\ ;\ t^{(\alpha,\beta)}.\ \ t^{(\gamma,\delta)} = t^{\xi} \text{ where } \xi = (\alpha, \beta) + (\gamma, \delta).$$

Schilling generalizes the theorems on general valuations to valuations with value group non-commutative. Now Zelinsky [22] considers non-associative valuations by taking an ordered loop L as a valuation domain. Recently Fuchs has generalized the valuation theory by considering a partially ordered group instead of a linearly ordered group for the valuation domain. By taking the following law: $v(a) \geqslant v(c)$ and $v(b) \geqslant v(c)$ implies $v(a-b) \geqslant v(c)$ instead of the triangular inequality, he shows that every integral domain with unit can be exhibited as a valuation ring of its quotient field and that the value group of an integrally closed ring is a subdirect sum of linearly ordered groups.

8. Complete fields. For every field K with a valuation we can construct an extension field in such a way that the arithmetic properties of the original field with respect to the given valuation are preserved and the algebraic structure of the extended field is considerably simplified by the adjunction of the new elements. We can complete the field either by taking fundamental sequences or by considering a system of ideals and an infinite system of congruences, that is to say, that if $\{\mathfrak{A}_n\}$ is a collection of ideals of the valuation ring R subject to (i) $\mathfrak{A}_{n+1} \subset \mathfrak{A}_n$, (ii) $\bigcap_n \mathfrak{A}_n = (0)$ and the sequences are such that $a_m \equiv a_n \pmod{\mathfrak{A}_n}$ for $m \geqslant n$. In this completion the elements consist of all solutions of all systems of congruences. But fields having more than one complete extension fields which are not analytically isomorphic are known to exist. For further information about complete fields one can refer to Schilling [16], Ostrowski [14] and Kaplansky [7].

9. Ideal theory and Valuations. Aubert [1] has given the following characterization of the valuation ring and the various system of ideals of it. A total system of r-ideals in a quasiordered directed Abelian group G is defined as follows: To every bounded set $\mathfrak{A}$ of G is associated a subset $\mathfrak{A}_r$ of G such that

(i) $\mathfrak{A} \subseteq \mathfrak{A}_r$; (ii) $\mathfrak{A} \subseteq \mathfrak{E}_r \rightarrow \mathfrak{A}_r \subseteq \mathfrak{E}_r$;

(iii) $a \in G \rightarrow \{a\}r = (a)$; (iv) $a.\ \mathfrak{A}r = (a.\ \mathfrak{A})_r$.

For the ring case we have for $a, b \in \mathfrak{A}_r$, $a+b \in \mathfrak{A}_r$. For a detailed discussion of such abstract ideal systems one can refer to Prufer's paper [15]. The different r-systems form a partially ordered set with respect to $<$: $r_1 < r_2$ if each r_1 ideal is also an r_2 ideal. The v-ideal system is the greatest element in this partially ordered set, i.e. $\mathfrak{U}_v = \bigcap_{\mathfrak{U} \subseteq (a)} (a)$ and the s-ideal system is the least element, i.e.

$\mathfrak{U}_s = \bigcup_{a \in \mathfrak{U}} (a)$. We further define $\mathfrak{U}_{r_v} = \bigcap_{\mathfrak{U} \subseteq \mathfrak{P}_r} \mathfrak{B}_r$ and $\mathfrak{U}_{r_s} = \bigcup_{\mathfrak{P} \subseteq \mathfrak{U}} \mathfrak{B}_s$, where $\mathfrak{B}$ denotes a finite subset of G. Then the following statements are equivalent.

(i) I is an integral domain with a unit valuation ring.
(ii) Every s-ideal in I is a d-ideal (the usual Dedekind ideal).
(iii) Every s_v-ideal in I is a d-ideal.
(iv) Every s-ideal in I is a v_s-ideal.
(v) Every s_v-ideal in I is a v_s-ideal.
(vi) Every s_v-ideal in I is a v-ideal.

If $r = s$ we get the usual valuation due to Krull and $r = v$ gives the non-associative valuation due to Zelinsky [21].

For the value group of an integral domain to be linearly ordered it is necessary and sufficient that the ideals of R are not reducible and the value group to be Archimedian ordered a necessary and sufficient condition is that every ideal in R is primary.

10. Application to algebraic geometry. The theory of valuations and the theory of ideals in algebraic function fields enable us to prove the arithmetic proof of the theorem on the reduction of singularities with great ease and rigour. Corresponding to the notion of a branch of an algebraic curve we have the zero dimensional valuation of the field of rational functions. Zariski [22] proves the following fundamental lemma : Given any zero dimensional valuation of Σ there exists a projective model F of Σ, on which the centre of the valuation is a simple point. Schilling and Maclane [11] give a general survey of all possible value groups for valuations on n-dimensional algebraic varieties.

REFERENCES

1. Aubert : Some Characterizations of Valuation Rings, *Duke Math. J.* (1954), 517-525.
2. N. Bourbaki : *Algebre,* Vol. 14. No. 1179.
3. L. Fuchs : Generalizations of Valuation Rings, *Duke Math. J.* (1951), 17-23.
4. O. Frink : Ideals in partially ordered sets, *American. Math. Monthly,* 61, (1954), 223-234.

5. I. GELFAND : Normierte Ringe, *Math. Sbornik*, (1941), 3-24.

6. I. KAPLANSKY : Topological methods in valuation theory, *Duke Math. J.* (1947), 527-541.

7. I. KAPLANSKY : Maximal fields with valuations, *Duke Math. J.* 9 (1942), 303-321.

8. W. KRULL : Allgemeine Bewertungstheorie, *Journal für Mathematik*, 168 (1932), 163-196.

9. P. LORENZON : Über halbgeordnete Gruppen *Math. Zeit.* (1949), 483-526.

10. P. LORENZON : Über halbgeordnete Gruppen *Archiv für Math.* 1949-50.

11. S. MACLANE and O. F. G. SCHILLING : O-Dimensional branches of rank 1 on algebraic varieties, *Annals of Math.* (1939).

12. B. H. NEUMANN : Ordered division rings, *Trans. Amer. Math. Soc.* 66 (1949), 202-252.

13. O. OSTROWSKI : Über einige Losungen der Funktionalgleichung $q(x) \cdot q(y) = q(x.\ y)$, *Acta. Math.* 41 (1918), 271-284.

14. A. OSTROWSKI : Unter Suchungen Zur A Arithmetischen Theorie der Korper, *Math. Zeit.* 39 (1935), 269-404.

15. H. PRUFER : Untersuchungen uber Teilbarkeitseigenschaften in Korpern, *Journal für Mathematik*, 168 (1932), 1-36.

16. O. F. G. SCHILLING : *The Theory of Valuations*, Mathematical Surveys, (A.M.S.) Vol. 4 (1950).

17. O. F. G. SCHILLING : On non-commutative Valuations, *Bull. Amer. Math. Soc.* (1945), 297-304.

18. SHAFARAVITCH : On the normalisability of topological fields, *C.R. Acad. Sci.* U.S.S.R. (1940), 83-84.

19. B. L. VAN DER WAERDAN : *Modern Algebra*, Vol. 1. New York, (1953).

20. R. J. T. WALKER : *Introduction to algebraic curves*, Princeton Series, Vol. 13 (1950).

21. D. ZELINSKY : Topological characteristics of fields with valuations, *Bull. Amer. Math. Soc.* (1948), 1145-1150.

22. D. ZELINSKY : On non-associative Valuations, *Bull. Amer. Math. Soc.* (1948), 175-184.

23. O. ZARISKY : Reduction of singularities of algebraic surface, *Annals of Mathematics*, (1939), 639-689.

Madras University
Madras 5

PARTIALLY ORDERED LINEAR TOPOLOGICAL SPACES

By S. SWAMINATHAN

1. The study of linear topological spaces presents new features when an order structure is also introduced relative to which the algebraic operations are monotone. The interest in this began with the study of vector lattices in functional analysis about three decades ago. Partial order in vector spaces has been the subject of recent study by M. G. Krein, M. A. Rutman, F. F. Bonsall, I. Namioka and others. I. Namioka has made a systematic investigation of partially ordered linear topological spaces in general in [5]. Referring to some of his results, we shall briefly deal with some aspects of the relationship between order structure and topological structure in linear spaces over the scalar field of real numbers.

2. Partially ordered linear spaces. *A partially ordered linear space* is a real linear space $(E, +, .)$ with a partial ordering $\geqslant$ on E which is monotonic with respect to addition and non-negative scalar multiplication, i.e. for x, y in E such that $x \geqslant y$, we have (i) $x + z \geqslant y + z$ for each z in E, and (ii) $ax \geqslant ay$ for $a \geqslant 0$. Such a partial ordering is called a *vector ordering* on E.

A vector ordering on E can be associated with a geometric object in E called the cone. *A cone* is defined as a linear subset C of E such that $C + C \subset C$ and $aC \subset C$ for all $a \geqslant 0$. A partial ordering $\geqslant$ can be defined for E with respect to a cone C by prescribing that for x, y in E, $x \geqslant y$ if and only if $x - y$ is in E. This partial ordering is a vector ordering $\geqslant$ on E and is said to correspond to the cone C. Conversely, given a vector ordering $\geqslant$ on E, the elements x such that $x \geqslant 0$ form a cone, which is called the positive cone of E. The vector ordering $\geqslant$ corresponds to the cone C. Thus we see that a vector ordering on E determines and is determined by the positive cone C of E. Hence we many denote the partially ordered linear space by (E, C).

If the vector ordering on (E, C) be anti-symmetric (or strict) the necessary and sufficient condition for it can be expressed in terms of the cone C as $C \cap \{-C\} = \{0\}$. The cone is then called a *proper* cone. The vector ordering is a directing relation if and only if $C - C = E$. In such a case the cone is called a *generating* cone. When (E, C) is a lattice also, it is called a *vector lattice*.

There are many examples of partially ordered linear spaces. To mention one, let X be a locally compact Hausdorff space. The space $C(X)$ of all real valued continuous functions can be ordered by defining that f, g in $C(X)$, $f \geqslant g$ if and only if $f(x) \geqslant g(x)$ for all x in X. This ordering is a vector ordering and the positive cone in $C(X)$ is both proper and generating. The case in which the space $C(X)$ consists of all real valued continuous functions with compact supports is important in the theory of integration in locally compact spaces.

3. Extension of positive linear functionals. One of the most important problems concerning linear spaces is about the extensions of linear functions from a linear subspace to the whole space. In partially ordered linear spaces we consider the extensions of a *positive linear functional* which is defined as a linear functional which is non-negative on the positive cone of E. Theorems concerning such extensions have been given by Krein and Rutman [4], Bonsall [2,3] and Namioka [5]. We shall now prove an extension theorem of the Hahn-Banach type for partially ordered linear spaces due to Namioka. We use the concept 'radial at x' which is defined as follows: A subset U of (E, C) is *radial at a point* $x \in E$ if there exists a real number s such that for any z in E, $z \in t(A - x)$ for $t \geqslant s$, t real.

THEOREM I *Let F be a linear subspace of a partially ordered linear space (E, C), and let f be a linear functional on F. Then the following statements are equivalent:*

(i) *f can be extended to a positive linear functional on E.*

(ii) *There is a convex set U, radial at 0, such that $f(x) \leqslant 1$ whenever $x \in F$ and $x \leqslant y$ for some y in U.*

Furthermore, when the statement (ii) *is satisfied, an extension* $\bar{f}$ *can be chosen so that* $\bar{f}(x) \leqslant 1$ *whenever* $x \leqslant y$ *for some* y *in* U.

PROOF. (i) implies (ii). Suppose that a positive linear functional $\bar{f}$ is an extension of f. Then the set $U = \{x : \bar{f}(x) \leqslant 1\}$ is convex and radial at 0. If x is an element in F such that $x \leqslant y$ for some y in U, then $f(x) = \bar{f}(x) \leqslant \bar{f}(y) \leqslant 1$.

(ii) implies (i). (ii) is equivalent to saying that there exists a convex set U, radial at 0, such that f is bounded from above on $F \cap (U - C)$, i.e. $f(x) \leqslant 1$ whenever $x \in F \cap (U - C)$. Consider the Minkowski functional p of the convex set $U - C$, which is defined as follows :

$$p(x) = \inf \{t : t > 0, \quad x \in t \ (U - C)\} \text{ for all } x \text{ in } E.$$

p is subadditive, i.e. $p(x + y) \leqslant p(x) + p(y)$, and non-negatively homogeneous, i.e. $p(ax) = ap(x)$ for $a \geqslant 0$.

Now $f(x) \leqslant p(x)$ for all x in F. We thus have the hypothesis of the classical Hahn-Banach theorem (Banach, [1]) and so, applying it, there is a linear functional $\bar{f}$ on E, which is an extension of f, such that $\bar{f}(x) \leqslant 1$ for all x in $U - C$. The functional f is necessarily positive, for take x in C; then, for all positive numbers t — tx is in $U - C$. Hence $\bar{f}(-tx) = -t\bar{f}(x) \leqslant 1$ for all positive numbers t, whence it follows that $\bar{f}(x) \geqslant 0$. Hence the theorem.

COROLLARY I. *Let* F *be a subspace of a partially ordered linear space* (E, C) *such that for each positive element* x *in* E *there is an element* y *in* F *such that* $y \geqslant x$. *Then each positive linear functional on* F *can be extended to a positive linear functional on* E.

COROLLARY II. *Let a linear subspace* F *of* (E, C) *contain a point at which the positive cone is radial. Then each positive linear functional on* F *can be extended to a positive linear functional on* E.

A stronger form of Corollary I was proved by Dixmier, while a weaker form of Corollary II was proved by Krein.

4. Partially ordered linear topological spaces. We next introduce a topology in a partially ordered linear space (E, C). We take the usual vector topology T of the linear space E, which is determined completely by its local base of neighbourhoods of 0. Thus (E, C, T) is a *partially ordered linear topological space.* When the vector topology T is locally convex, (E, C, T) is called a partially *ordered locally convex space.*

Consider the extension of linear functionals for the space (E, C, T). We have the following theorem for partially ordered locally convex spaces.

THEOREM II. *Let F be a linear subspace of a partially ordered locally convex space (E, C, T) and let f be a linear functional on F. Then the following statements are equivalent.*

(i) *f can be extended to a T-continuous linear functional on (E, C, T).*

(ii) *There is a T-neighbourhood U of 0 such that $f(x) \leqslant 1$ whenever $x \in F$ and $x \leqslant y$ for some y in U.*

This theorem easily follows from Theorem I. In fact, the condition (ii) above is essentially the same as that of Theorem I, and also implies that f is a T-continuous positive functional on the subspace F of E. This being so, condition (i) above follows from that of Theorem I, since T-continuity of f can be deduced from its positivity using the equivalent form of the condition (ii) given in the proof of Theorem I.

We emphasise that we have been able to derive a topological theorem from a corresponding theorem with only the order structure. It should be observed that Theorem II, though stated only for partially ordered locally convex spaces, can be modified for the general case.

5. Locally full topologies. By using the order structure, we can construct a new vector topology out of the old one, which has some interesting consequences. We shall now show how this can be done, though we do not propose to go into the details of the consequences. We start with the notion of an order interval in a partially ordered linear space (E, C).

A subset of E of the form $\{z : x \leqslant z \leqslant y\}$ is called an *order interval* and is denoted by $[x, y]$. For x, y, z and u in E, we have $x + [y, z] = [x + y, x + z]$ and $[x, y] + [z, u] \subset [x + z, y + u]$. It is not necessarily true that $[x, y] + [z, u] = [x + z, y + u]$. The spaces for which this is true form a very special class of partially ordered linear spaces, which includes the vector lattices.

Let (E, C, T) be a partially ordered linear topological space and let $\mathscr{U}$ be the family of all T-neighbourhoods of 0. Consider the family of sets $\mathscr{V} = \{(U + C) \cap (U - C) : U \in u\}$. This family forms a local base and determines a unique topology which we shall denote by $F(T)$. The following properties of $F(T)$ can be easily verified: (i) $F(T) \subset T$; (ii) $F(F(T) = F(T)$; (iii) if $T_1 \subset T_2$, then $F(T_1) \subset F(T_2)$; (iv) the closure of C relative to T is identical with the closure of C relative to $F(T)$; (v) if T is pseudo-metrizable, pseudo-normable, or locally convex, then $F(T)$ is pseudo-metrizable, pseudo-normable, or locally convex.

A subset F of a partially ordered linear space is called *full* if $x, y \in F$ implies that $[x, y] \subset F$. A partially ordered linear topological space (E, C, T) is called *locally full* if full T-neighbourhoods of 0 form a local base for T. Now, for any subset F of E, $(F + C) \cap (F - C)$ is full. Therefore, the space (E, C, T) is locally full if and only if $F(T) = T$.

Let the adjoint of (E, T) be $(E, T)^*$, i.e. the space of all linear continuous functionals on (E, T). Let $(E, C, T)^{\Delta}$ denote the space of all linear functionals on E which can be exhibited as the difference of two T-continuous positive linear functionals. Then $(E, T)^*$ is identical with $(E, C, T)^{\Delta}$ for a partially ordered linear space with a locally convex and locally full topology T. For proof we refer to the memoir of Namioka [5] which contains a host of other interesting results.

REFERENCES

1. S. Banach : Theorie des operations lineaires, *Monografie Matematyczne*, 1, Warsaw, (1932).

2. F. F. Bonsall : Sublinear functionals and ideals in partially ordered vector spaces, *Proc. Lond. Math. Soc.* (3) 4 (1954), 402-18.

3. F. F. Bonsall : Regular ideals in partially ordered vector spaces, *Proc. Lond. Math. Soc.* (3) 6 (1956), 626-640.

4. Krein and Rutman : Linear operators leaving invariant a cone in a Banach Space (Russian), *Uspehi Math. Nauk.* (N. S.) 3, No. 11, (23) (1948), 3-95.

5. I. Namioka : Partially ordered linear topological spaces, *Memoirs of the American Math. Soc.* No. 24, (1957).

Madras University
Madras 5

INFINITE DIMENSIONAL PROJECTIVE GEOMETRIES

By V. K. BALACHANDRAN

1. Introduction. We shall be concerned with two types of infinite dimensional extensions of finite dimensional projective geometries. In one type of extension the notion of 'point' continues to play (as in the finite dimensional case) the dominant role, while in the other this concept is completely banished, thereby leading to the 'pointless' or 'continuous' geometries of Von Neumann, wherein the dimension function assumes the fundamental role. We conclude the discussion with some remarks on ring-coordinatisation of infinite dimensional projective geometries.

2. Atomic Projective Geometry. In order to motivate the first type of extension, let us recall the connection between a projective space Γ and the associated projective geometry $L(\Gamma)$, which is the lattice of all flats in Γ, and $L(\Gamma)$ is not only complemented and modular but also upper-continuous and atomic. Conversely, starting with any upper-continuous, atomic, complemented modular lattice L_1 we have the associated projective space $\Gamma(L_1)$ with atoms in L_1 as its points and elements covering atoms as lines. Further $L(\Gamma(L_1)) = L_1$. Therefore it is appropriate to call an upper-continuous atomic, complemented modular lattice L_1 a *projective geometry* or more precisely an *atomic projective geometry*—to distinguish it from the continuous geometry considered later. We wish to point out that the two properties 'upper continuity' and 'atomicity' which are simple consequences of 'modularity' and 'complementedness' when the lattice is finite-dimensional are no longer implied by these when the lattice is infinite-dimensional.

Given a division ring F and a cardinal d, the lattice $PG(F; d)$ of all subspaces of the d-dimensional vector space $V(F; d)$ formed by taking all d-vectors over F having only a finite number of non-zero coordinates, is an atomic projective geometry. $PG(F; d)$ is always *irreducible.* We shall call an atomic projective geometry coordinatisable if it is isomorphic to some $PG(F; d)$. This definition clearly includes the usual coordinatisable, classical (finite dimensional) projective geometry when the space has homogeneous coordinates from the field.

Frink has proved the following theorems regarding atomic projective geometries (see [1, pp.130-131]) :

THEOREM 1. *Any irreducible atomic projective geometry (apart from certain finite dimensional projective lines and non-Desarguesian plane projective geometries) is coordinatisable.*

THEOREM 2. *An atomic projective geometry is a sublattice of a direct union of irreducible atomic projective geometries.*

Theorem 2 generalizes partially the result of Birkhoff that a finite dimensional complemented modular lattice is a direct union of projective geometries [1, p. 120, Theorem 6].

3. Continuous Geometries. The motivation for this type of extension is obtained from the observation that an n-dimensional projective geometry (= an irreducible complemented modular lattice of dimension n in the lattice sense) can be viewed as a complemented modular lattice over which a 'normalized' dimension function $D = D(a)$ can be defined, whose range $R(D)$ is a subset of the unit interval I and which has the following properties:

(1) $D(0) = 0$, $D(1) = 1$; (2) $D(a \vee b) + D(a \wedge b) = D(a) + D(b)$;

(3) $a \sim b \langle = \rangle D(a) = D(b)$, $a < b \langle = \rangle D(a) < D(b)$, ($a \sim b$ means that a, b are perspective, that is, have a common complement, and $a < b$ means $a < a_1$ for some $a_1 < b$);

(4) $R(D) = S_n = (0, 1/n, 2/n, \ldots, 1)$.

This observation raises the following question. Given an infinite dimensional lattice L, under suitable conditions, is it possible to introduce in L a dimension function D satisfying the properties (1) — (3) and (4'): $R(D) = S_\infty = I$? Von Neumann gave a positive answer to this question by proving the following remarkable result [4, p. 101, Satz 2.1] :

THEOREM 3. *In any irreducible continuous complemented modular lattice L^* it is possible to introduce a (unique) normalized dimension function D satisfying properties* (1)—(3) *and* (4*): $R(D) = S_n$ *for some* n, *or* S_∞.

(A lattice is called continuous if it is both upper-continuous and lower-continuous, that is, dually upper-continuous.)

$R(D) = S_n$ occurs precisely when L^* has dimension n, and corresponds to a projective space Γ of dimension $n-1$; if L^* has dimension n, then $D = d/n$.

When $R(D) = S_\infty$, we call L^* a continuous geometry (in the proper sense); note that in this case, since there are in L^* elements a of arbitrary small positive dimension $D(a)$, the notion of 'point' cannot come in at all, as a point is considered as an element of minimum positive dimension.

THEOREM 4. *Associated with any division ring F, there is a continuous geometry $CG(F)$.*

$CG(F)$ is obtained from the finite dimensional projective geometries $PG(k) = PG(F; k)$ by a sort of limiting process (see, [4, p.121, Anm. 2.4] or [1, p.125]). $PG(k)$, the lattice of subspaces of the k-dimensional vector space over F, can be imbedded isomorphically in $PG(2k)$ so as to preserve the normalised dimension D. Repeating this we get a sequence of extensions :

$$PG(2) \subseteq PG(4) \subseteq \dots \subseteq PG(2^n) \subseteq \dots .$$

Each $PG(2^n)$ is a metric lattice the metric being induced by the valuation D. The union Σ of these metric lattices is again a metric lattice. D is defined over Σ and takes as values all rationals of the form $k/2^n (k = 0, \dots, 2^n)$. The metric completion $\bar{\Sigma}$ of Σ is the continuous geometry $CG(F)$. The function D can be extended in a natural way to $\bar{D}$ over $\bar{\Sigma}$, and $\bar{D}$ is the normalized dimension function of $CG(F)$; if x in Σ is the limit of a sequence x_n of Σ, then $D(x_n)$ will converge to a limit, which is then taken to be $\bar{D}(x)$.

Curiously we have the

THEOREM 5. *The continuous geometry $CG(R)$ associated to the real field is isomorphic to $CG(Q)$ associated with the quaternion field, but not to $CG(C)$ associated with the complex field.* (See [6]).

If in the definition of continuous geometry we replace the condition 'irreducible' by 'reducible', then we shall call the corresponding lattice a reducible continuous geometry. Regarding this we have the following representation theorem of Iwamura [4, p.128, Sz. 3.2]

THEOREM 6. *A reducible continuous geometry L_1 is a subdirect union of (irreducible) continuous geometries.*

A concept of dimension can be introduced in L_1: the dimension of an element in L_1 is no longer a number but a function, in fact, a certain continuous function $D_a(p)$ defined over the Boolean space $S = S(Z)$ associated with the Boolean algebra $Z = (z)$ of central elements z of L_1. $D_a(p)$ satisfies: (i) for all p, $0 \leqslant D_a(p) \leqslant 1$; (ii) for a central element z, $D_z(p)$ is 0 or 1 according as z is or is not in p; (iii) $D_{a\vee b}(p) + D_{a\wedge b}(p) = D_a(p) + D_b(p)$ (see [4, p. 129, Sz. 3.3]).

4. Ring Coordinatisation. The classical field coordinatisation theorem of Von Staudt asserts that an irreducible projective geometry L of finite dimension $n \geqslant 4$ is isomorphic with the lattice $PG(F; n)$ of all subspaces of the n-dimensional vector space $V(F; n)$ over a suitable division ring F. Since it can be shown that $PG(F; n)$ is isomorphic with the lattice of all (equivalently, all principal right) ideals of the semi-simple $n \times n$ matrix ring R over F, the above result can be reformulated as: L is isomorphic with the lattice of principal right ideals of a suitable semi-simple ring R. This was generalized by Von Neumann into

THEOREM 7. *A complemented modular lattice with a basis of $n(\geqslant 4.)$ pairwise perspective elements is isomorphic with the lattice of all principal right ideals of a suitable regular ring.* (See [4, p.225, Sz. 3.2]).

A ring with unit element in which to each element a there is a 'relative inverse' x such that $axa = a$ is called a 'regular ring'. It may be noted that the regular rings with finite basis are precisely the semi-simple rings. Furthermore, the concept of regularity for a ring is precisely that required to make the lattice of its principal right ideals a complemented modular lattice. (For an account of regular rings, see [4, ch. 6]).

REFERENCES

1. G. BIRKHOFF : Lattice Theory, *Amer. Math. Colloq.* Pub. 25, 2nd edn. (1948).

2. G. BIRKHOFF : Von Neumann and Lattice theory, *Bull. Amer. Math. Soc.* 64 (1958), (Supplement).

3. O. FRINK : Complemented modular lattices and projective spaces of infinite dimension, *Trans. Amer. Math. Soc.* 60 (1946), 462-67.

4. F. MAEDA : *Kontinuierliche Geometrien, Grund. Math. Wiss.* (Springer) (1958).

5. J. VON NEUMANN : Continuous geometries and examples of continuous geometries, *Proc. Nat. Acad. Sc.* 22 (1936), 92-108.

6. J. VON NEUMANN : *Lectures on Continuous Geometries*, I, II, III (Princeton) 1936-37.

Madras University
Madras 5

ENGINEERING APPLICATIONS OF BOOLEAN ALGEBRA

By C. H. SMITH

THERE is, I believe, an essential difference in outlook between the mathematician and the engineer, which needs to be understood if maximum benefit is to be obtained from any cooperative effort. I postulate this difference knowing fully well that there have been mathematicians with an interest in science and scientists who are competent mathematicians. I will enumerate some of the differences. The mathematician is primarily interested in the consistency of the relation between the premises and the conclusions. If the premises bear any resemblance to a set of existing circumstances then the scientist is welcome to use them. The engineer is interested in the degree of approximation of the premises to a set of events in the physical world and, since the fit can never be exact, the conclusions also can never be an exact statement about the real world. Newtonian mechanics, for example, is impeccable logic to the mathematician but to the physicist a very poor approximation to some physical events and a very close one to others.

To the mathematician, the relation between premises and conclusion is all important. To the engineer, if the premises are accepted and the conclusions are to be useful, then the intermediate logic must also be of physical significance. For example, although the mathematician is happy with $Z = \int Ydx$, to the engineer $\int Vdt$ is meaningless. He has no name for such a quantity and he always thinks in terms of $\frac{1}{T}\int Vdt$ which is dimensionally acceptable.

A third difference of outlook arises from the natural tendency of the mathematician to proceed from the general to the particular. The experimentalist must proceed from the particular to the general. He builds his theories by induction from a limited number of observations, and he is always interested in the simplest concept which is consistent with his observations. His premises then are always of doubtful validity and he is willing to discard them as experience dictates.

How do these remarks apply to Boolean algebra ? On the one hand the logic is capable of demonstrable proof and this makes it appeal to the engineer. He can never prove by example the identity $(x+y)^2 \equiv x^2+2xy+y^2$. He can illustrate it by example but can never exhaust all possible values. However in Boolean algebra proofs are readily demonstrable. All possible cases of the identity $(x+y)^2 = x+y$ are demonstrated in the truth table :–

x	y	$x+y=(x+y)^2$
0	0	0
0	1	1
1	0	1
1	1	1

On the other hand Boolean algebra offends in that the premises are not consistent with experience. To ask an engineer to accept a logic built on the premise $1+1=1$ and his immediate reaction is " If the premise is contradictory to experience, how can the result be useful ".

This difficulty is best bridged by the concepts of the point set theory. If, instead of $+$ and $\times$, we use "union" and "intersection" then we get physically real concepts :—(1) $x.u.y.$ is the total included region of two overlapping regions x and y, and (2) $x \cap y$ is the area common to two overlapping regions.

The importance of Boolean algebra to the engineer arises in two main fields : in relay circuitry and in digital computing. The application of Boolean algebra to relay circuits, i.e. to automatic telephone exchanges was systematised by Shannon in 1938. The application to digital computors is really a logical extension of Shannon's work. Although relays are now little used in computors because of their slow speed the diode gates are logically no more than simple quick acting relays.

An allied field is that of automation where the principles of digital computation apply but the objective is different. It may be the limited computation necessary for a particular set of industrial processes rather than a versatile all purpose computor.

For example, in servomechanism design a linear or quasi-linear analysis is usually used. However the most economical design is one which uses the smallest acceptable driving motor which is always driven to saturation. There is then no region of linear behaviour and the equation of motion becomes $\ddot{\theta} = \pm K$. The design problem is then to determine the conditions at which the torque shall be reversed and this is amenable to logical analysis on Boolean lines.

The essential process in any computor is addition. We wish the machine to carry out the process $1 + 1 = 2$. To do this we set up a circuit to add in modulo 2 ; i.e. to give $1 + 1 = 0$ and to design this circuit we apply Boolean algebra and say $1 + 1 = 1$. Can anyone deny that $+$ is an overworked symbol?

In the field of automation there are a number of interesting coding problems to which I feel that the application of Boolean algebra might give some useful results. It is usual to work to a radix 2 and, in general purpose computors use the ordinary binary code :—

0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

The laws of addition for this code are well established ; they are for the half adder :—

a_n	b_n	S	C
0	0	0	0
1	0	1	0
0	1	1	0
1	1	0	1

i.e. $S = a\bar{b} + a\bar{b}$

$C = ab.$

However in some machines a zero is stored as a negative of a one. This occurs, for example, in a ferrate core storage unit or a magnetic drum.

It is usual to include circuits to cancel the negative signal so that ordinary binary arithmetic will apply. It would be possible however to make an adder using + and − signals. The truth table for a half adder would be :—

a_n	b_n	S_n	C_n
−1	−1	−1	−1
1	−1	1	−1
−1	1	1	−1
1	1	−1	1

whence $S = -\ ab$

$$C = \frac{ab + a + b - 1}{2}.$$

Incidentally this concept of + = true ; − = false would appear to be a more logical basis on which to build a logical algebra than Boolean concept of one and zero.

In engineering processes the information is often in radix 10 whereas computors normally work in radix 2. To convert from radix 10 to radix 2 will require 4 binary digits, not all of which will be used. We can therefore choose a variety of transformation codes.

For example we might define a decimal digit by its values modulo 5 and modulo 2.

	Mod. 5	Mod. 2
0	000	0
1	001	1
2	010	0
3	011	1
4	100	0
5	000	1
6	001	0
7	010	1
8	011	0
9	100	1

and then construct a code box

	00	01	10	11
00	0	2	4	
01	5	7	9	
10	6	8		
11	1	3		

In engineering processes the data is often a shaft position and processing to be applied to the data involves addition.

The binary digital code (1) is not very convenient because of the simultaneous change of several digits which

	(1)	
0	000	000
1	001	001
2	010	011
3	011	010
4	100	110
5	101	111
6	110	101
7	111	100

may lead to gross errors. This is avoided by the use of the C.P. Code (2) in which only one digit changes at a time. The reading error then cannot exceed one unit.

The digits of the C.P. code are obtained from the binary code by the relation :—

$$(A_r)_{cp} = (A_{r+1} + A_r)_b \text{ (mod 2).}$$

Another code which is convenient for shaft rotation is the chain code. It is formed by a sequence of n brushes on the periphery of a coded disc. Thus at each step the most significant digit is discarded and a new least significant digit added.

0000
0001
0011
0111
1111
1110
1101
1010
0100
1000

0000
0001
0011
0111
1111
1110
1101
1011
0110
1100
1001
0010
0101
1010
0100
1000

This sequence is constructed by writing a 1 in the l.s.p. if permissible, otherwise write 0. An interesting point about this sequence is that it can be terminated prematurely to make a scale of 10.

This code would be convenient for conversion from, say a cash register to a digital computor.

There is one common unsolved problem in all these codes :– " What are the rules of addition ? " Boolean algebra might provide the answer and I would welcome your help.

	00	01	10	11
00	0	8	9	
01	1			6
10			7	5
11	2	3		4

Institute of Armament Studies,
Poona 3

BOOLEAN ALGEBRA

By B. S. MADHAVA RAO

Boolean algebra or the algebra of classes, of which lattice theory is a natural generalization, has recently been used as an important tool of application in several branches of mathematics, and of applied science. I shall indicate briefly a few of these important applications.

One such example of the first category is the subject of mathematical logic wherein developments relating to the so-called symbolic or algebraic or Boolean logic have not only been of great interest in themselves, but have also proved vital for many applications. Another example is the subject of topology in which basic results have been obtained regarding the topological structure of several types of Boolean algebras, thus leading to the notion of topological Boolean algebras. As a typical illustration of this, I might mention a recent theorem that a complete Boolean algebra B has a compact Ti-topology if and only if B is atomic, i.e. isomorphic to the lattice of all subsets of a fixed set. A third example is provided by the recent attempts made to build up an axiomatic foundation of probability theory based on the observation that the objects to which probabilities are ascribed always form a Boolean algebra, and the consequent development of the notion of measures on such algebras.

As regards other applications, striking examples can be found in the field of electronics. Behind the remarkable development of digital computers in the last decade lie the applications of Boolean logic to devise suitable algorithms. The theory of switching (both binary and non-binary) leading to the design of multivalued and sequential circuits makes use of generalized types of Boolean algebras. With the notion of entropy clarified on the basis of measures on Boolean algebras, one finds applications to information theory also. Analogous to computer techniques are recent attempts made towards machine translation of languages by digital data processing, words to words and sentences to sentences, structure being important for the latter. Treated as a binary algebra of classes, Boolean algebra has been employed in actuarial science to check classification of numerical data. Treated as an algebra of propositions isomorphic to the above, it has been used to simplify sets of complicated propositions thus finding application in the field of the social sciences. Finally mention may be made of application in the field of quantum statistics, and the logic of quantum mechanics.

University of Poona
Poona 7

SYMPOSIUM OF MAGNETO-FLUID-DYNAMICS

Chairman : Prof. V. GANAPATHY IYER

PROF. P. L. Bhatnagar will lead the symposium by introducing the fundamental equations of magneto-gas-dynamics and discussing the production of discontinuities in subsonic flows if the impressed magnetic field is of sufficient magnitude, a situation which can never arise in ordinary-gas-dynamics. He will be followed by Shri J. De who will present his work on the possibility of the existence of steady self-excited fluid dynamics. It may be mentioned that the Dynamo theories have been proposed to explain the existence of cosmical magnetic field. Shri J. D. Gupta will discuss how the Rankine-Hugonoit equations of ordinary gas-dynamics have to be modified in case of hydromagnetic shocks while Shri R. K. Jaggi will talk on the hydromagnetic stability of a constricted gas discharge. Then Dr. J. N. Kapur will review the progress in hydromagnetic turbulence, while Shri P. C. Jain will report on his work on the gravitational instability of turbulent medium in the presence of a magnetic field. Shri K. S. Raja Rao will discuss applications of the Magneto-hydrodynamic theory to Ionospheric problems and the symposium will be concluded with some remarks from Prof. B. S. Madhava Rao. Dr. S. L. Malurkar's paper on "Exceptionally large Solar and Geophysical events" will be taken as read.

MAGNETO-GAS-DYNAMICS AND LINES OF DISCONTINUITY IN STEADY TWO-DIMENSIONAL FLOW

By P. L. BHATNAGAR

1. Under Magneto-gas-dynamics we study the motion of an electrically conducting fluid in the presence of magnetic field. The fluid motion induces currents, which experience mechanical force, called Lorentz's force, due to the presence of magnetic field. This force tends to modify the initial state of motion. On the other hand, the electric currents are associated with magnetic field which is added on to the parent magnetic field. This interlocking will be clear from the equations governing the Magneto-gas-dynamical flows given below. Magneto-gas-dynamics finds application in a large number of cosmical phenomena. Some of them are : (i) Variability of magnetic stars, (ii) shape and temperature of Corona, (iii) inhibition of convection in sunspots by magnetic field, (iv) stability of Quiescent prominences, etc. It will be exaggerating the case of Magneto-gas-dynamics to regard it a master-key to unfold all the mysteries of the universe. Being a union of two well-developed disciplines —Electro-magnetic theory and Gas-dynamics, it has wider applicability than the parent disciplines and the researches of past decade and half have amply proved this statement.

The equations governing the Magneto-gas-dynamical flows, in the usual notation, are :

(1) *Maxwell's Equations* :

$$\text{div}\ \mathbf{B} = 0, \quad \text{div}\ \mathbf{D} = 4\pi q, \qquad (1.1,\ 1.2)$$

$$\text{curl}\ \ \mathbf{H} = \frac{4\pi}{c}\,\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}, \quad \text{curl}\,\mathbf{E} = \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad (1.3,\ 1.4)$$

$$\mathbf{B} = \mu\,\mathbf{H}, \quad \mathbf{D} = \epsilon\,\mathbf{E}. \qquad (1.5,\ 1.6)$$

(2) *Current Equation* :

$$\mathbf{j} = q\,\mathbf{V} + \mathbf{J}, \quad \mathbf{J} = \sigma\left[\mathbf{E} + \frac{1}{c}(\mathbf{V}\times\mathbf{B})\right]. \qquad (1.7,\ 1.8)$$

The first term in (1.7) is the convection current. The first term in (1.8) is the conduction current, while the second term is the induced current.

(3) *Equation of Continuity* :

$$\frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \operatorname{grad} \rho + \rho \operatorname{div} \mathbf{V} = 0. \tag{1.9}$$

(4) *Momentum Equation* :

$$\rho \left[\frac{\partial \mathbf{V}}{\partial t} - \mathbf{V} . \times \mathbf{w} + \tfrac{1}{2} \operatorname{grad} \mathbf{V}^2 \right] = \rho \mathbf{F} - \operatorname{grad} p + \tfrac{1}{3} \eta \operatorname{grad} \operatorname{div} \mathbf{V} + \eta \Delta \mathbf{V} + \mathbf{f}, \tag{1.10}$$

where $\mathbf{F}$ is the external force per unit mass, η the viscosity coefficient, and $\mathbf{f}$ the Lorentz force :

$$\mathbf{f} = q \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B}. \tag{1.11}$$

(5) *Equation of State* :

$$p = R \rho T. \tag{1.12}$$

(6) *Energy Equation* :

$$C_v \frac{DT}{Dt} = - p \operatorname{div} V + \operatorname{div} (k \operatorname{grad} T) + \rho Q + \frac{\mathbf{J} . \mathbf{J}}{\sigma} + \phi, \tag{1.13}$$

where Q is the rate of addition of heat from the sources which are not taken into account in the equation and ϕ is the rate of dissipation due to viscous forces.

We may point out that in writing the above equations we have used the unrationalized gaussian units, neglected the terms of the order of $\left(\frac{V}{C}\right)^2$ and taken μ, ϵ, η as constants.

There are in all thirteen unknown scalar quantities $\mathbf{V}(u, v, w)$, $\mathbf{H}$ (H_x, H_y, H_z), $\mathbf{E}$ (E_x, E_y, E_z), q, ρ, p, T and fourteen equations. From (1.4) we have

$$\frac{\partial}{\partial t} (\operatorname{div} \mathbf{B}) = 0.$$

Hence if initially div $\mathbf{B} = 0$, it will be so throughout. In this manner, the equation (1.1) can be replaced by the initial condition and we have just the right number of equations.

The interlocking between the electromagnetic phenomenon and the fluid motion enters through the terms marked below by an asterisk, namely convection current, induced current, and Lorentz force.

We may also note that leaving the Maxwell's equations all the equations are non-linear and hence a general study of a magneto-gas-dynamical phenomenon is bound to be extremely involved mathematically.

2. In this section we shall discuss briefly the existence of lines of discontinuity in a two-dimensional steady subsonic flow provided we put some restriction on the magnitude of the magnetic field. We may point out that such a situation cannot occur in ordinary gas-dynamics.

The equations of the problem: Neglecting the viscosity and displacement current and taking the electrical conductivity to be infinite, the relevant equations are :

Maxwell's Equations :

$$\operatorname{div} \mathbf{H} = 0, \qquad \operatorname{curl} \mathbf{H} = \frac{4\pi}{c} j, \tag{2.1, 2.2}$$

$$\operatorname{div} \mathbf{E} = 0, \qquad \operatorname{curl} \mathbf{E} = 0, \tag{2.3, 2.4}$$

and

$$\mathbf{E} = -\frac{\mu}{c} \mathbf{v} \times \mathbf{H}. \tag{2.5}$$

Equation of Continuity :

$$\operatorname{div} (\rho \mathbf{v}) = 0. \tag{2.6}$$

Equation of Momentum :

$$(\mathbf{v} . \nabla) \mathbf{v} = -\frac{1}{\rho} \operatorname{grad} p + \frac{\mu}{4\pi \rho} (\operatorname{curl} \mathbf{H}) \times \mathbf{H}. \tag{2.7}$$

From (2.3) — (2.5), we have

$$\operatorname{div} (\mathbf{v} \times \mathbf{H}) = 0, \qquad \operatorname{curl} (\mathbf{v} \times \mathbf{H}) = 0. \tag{2.8, 2.9}$$

In a two-dimensional motion in the (x, y) plane,

$$\mathbf{v} = (u, v, 0), \ \frac{\partial}{\partial z} \equiv 0 \tag{2.10}$$

and if the magnetic field lies in the plane of motion

$$\mathbf{H} = (H_1, H_2, 0) \tag{2.11}$$

the condition (2.8) is automatically satisfied, while (2.9) gives us

$$\mathbf{H} = \Phi(x, y)\,\mathbf{v} + \alpha \frac{\mathbf{k} \times \mathbf{v}}{\mathbf{v}^2}, \tag{2.12}$$

where **k** is unit vector perpendicular to the plane of motion, α is an arbitrary constant and $\Phi(x, y)$ is an arbitrary function. The inclusion of the second term on the right hand side of (2.12) makes the treatment mathematically cumbersome and hence we consider the particular case of (2.12) obtained by taking $\alpha = 0$, i.e.

$$\mathbf{H} = \Phi(x, y)\,\mathbf{v}. \tag{2.13}$$

Recently Taniuti has discussed the particular case of (2.13) where he takes $\Phi(x, y) \propto \rho$. In both of these cases the direction of magnetic lines of force and flow lines coincide.

Following the usual procedure we find that the characteristic directions are given by

$$(udy - vdx)^2 [(vdx + udy)^2 - \{a^2(1 - k) + kV^2\}(dx^2 + dy^2)] = 0, \tag{2.14}$$

where a is the local speed of sound, $V = (u^2 + v^2)^{1/2}$, (2.15)

and

$$k = \frac{\mu \Phi^2}{4\pi\rho} = \frac{\mu H^2}{4\pi\rho V^2} = \frac{\dfrac{\mu H^2}{8\pi}}{\frac{1}{2}\rho V^2} = \frac{\dfrac{\mu H^2}{4\pi\rho}}{V^2} \tag{2.16}$$

From (2.16) k is the ratio of magnetic energy density to kinetic energy density or the ratio of the squares of the Alfvén wave velocity and the fluid velocity.

The first factor in (2.14) gives us the flow lines

$$\frac{dy}{dx} = \frac{v}{u}, \tag{2.17}$$

while the directions given by the second factor are real if

$$(1 - k)(M^2 - 1)[kM^2 + (1 - k)] > 0, \tag{2.18}$$

where M is the local Mach number.

Case (A) : In the supersonic flow, i.e. when $M > 1$, the condition (2.8) is satisfied if $k < 1$, i.e. if at each point

$$\frac{\text{Alfvén wave velocity}}{\text{fluid velocity}} < 1. \tag{2.19}$$

Case (B) : In the subsonic case, i.e. when $M < 1$, the condition (2.18) is satisfied if

$$1 < \frac{\text{Alfvén wave velcoity}}{\text{fluid velocity}} < \frac{1}{(1-M^2)^{1/2}} \tag{2.20}$$

Thus the two cases arising in the paper under reference arise here too. It is clear that in Case (A) no lines of discontinuity can form if the magnetic field is so large that the Alfvén wave velocity is greater than the fluid velocity. In Case (B) by suitable choice of the magnitude of the magnetic field the lines of discontinuity can always be produced.

We can easily show that along *a flow line*

$$\tfrac{1}{2}v^2 + i = \text{constant}, \tag{2.21}$$

where i is the specific enthalpy

and

$$\frac{\Phi}{\rho} = \frac{H}{\sqrt{\rho}} = \text{constant}. \tag{2.22}$$

In (2.22) the constant may vary from one stream line to another, while in Taniuti's case this constant is the same throughout the flow field.

In Case (A) the characteristic directions are given by

$$\left(\frac{dy}{dx}\right)_{\text{I, II}} = \frac{-\dfrac{uv}{a^2} \pm [(1-k)(M^2-1)\{kM^2+(1-k)\}]^{1/2}}{\dfrac{u^2}{a^2} - (kM^2 + \overline{1-k})} = \zeta_{I,II}, \tag{2.23}$$

while in Case (B) they are given by

$$\left(\frac{dy}{dx}\right)_{\mathrm{I,\,II}} = \frac{\dfrac{-uv}{a^2} \pm [(k-1)(1-M^2)\{1-k(1-M^2)\}]^{1/2}}{\dfrac{u^2}{a^2} - \{1-k(1-M^2)\}} \equiv \zeta_{I,II}. \tag{2.24}$$

We can easily show that along the Mach lines

$$(1-k)(u\zeta_{I,II}-v)^2\left(dv+v\frac{d\Phi}{\Phi}\right)+(u\,\zeta_{I,II}-v)(u+v\,\zeta_{I,II})\left\{(k-1)x\right.$$

$$\left. x\,du+ku\frac{d\Phi}{\Phi}\right\}+(V\,dV+\frac{a^2}{\rho}d\rho)\{k(u+v\,\zeta_{I,II})+(1-k)(u\zeta_{I,II}-v)\}x$$

$$x\,\zeta_{I,II}+a^2\left(\frac{d\rho}{\rho}-\frac{d\Phi}{\Phi}\right)(u+v\,\zeta_{I,II})\,\zeta_{I,II}=0, \tag{2.25}$$

where $\zeta_{I,II}$ have to be substituted from (2.23) in Case (A) and from (2.24) in Case (B).

REFERENCE

1. TOSIYA TANIUTI: *Prog. Theo. Phys.* 19 (1958), 749-750.

Indian Inst. of Science
Bangalore 12

ON STEADY SELF-EXCITED DYNAMOS

By J. DE

The possibility of the existence of steady self-excited fluid dynamos is investigated. It is found that the existence of a steady dynamo solution is intimately connected with geometry of the magnetic lines and the boundary condition satisfied by the magnetic fields. For some general types of fields, a steady dynamo action may be possible, while for some other types, Cowling's case included, dynamo maintenance is found to be impossible.

The particular solutions constructed show an interesting feature. It is found that in every case the fluid motion becomes infinite or discontinuous over certain regions. Drawing an analogy from the mechanism of technical dynamos it is conjectured that such discontinuities will perhaps occur in every possible solution.

Science College
Calcutta

JUMP CONDITIONS FOR MAGNETO-GAS-DYNAMIC SHOCKS

By J. D. GUPTA

A SHOCK wave is a surface of discontinuity in flow variables, when it is propagated. The possibility of the existence of such discontinuities is a distinguishing feature of supersonic flows. The two-fold problem of shock waves is (i) to derive relations between the values of the flow variables on the two sides of the shock, and (ii) to study the shock structure, i.e. the process taking place in the narrow width of the shock that brings about the discontinuity. In ordinary gas-dynamics, the interest in the problem may be traced back to early 19th century, with Stokes, Earnshaw, Rankine and Hugonoit contributing tc its development.

When the medium is a conducting fluid in the presence of a magnetic field, there is interaction between the magnetic field and the flow. Alfvén, 1942, was the first to study the propagation of waves of *infinitesimal amplitude in a conducting incompressible* fluid, in the presence of a magnetic field. Van-de-Hulst, independently extended the theory to compressible fluids and showed that in general, five different modes of motion were possible, each characterised by a different velocity.

Hoffman and Teller, 1950, considered the propagation of shock disturbances in a conducting fluid. Making a relativistic approach, they obtained the general form of jump conditions and deduced therefrom the conditions in the non-relativistic case. Now, in physical phenomena of interest in this connection, the velocities are much less than that of light. So it will be of interest to derive these conditions in the non-relativistic case directly. This will be described in brief before referring to another paper on the subject.

The fundamental laws to draw upon are the Maxwell's equations, the conservation of mass, the conservation of momentum, the conservation of energy, and increase or conservation of entropy.

We shall consider a plane shock in a homogenous isotropic medium of infinite conductivity. As the material velocities are assumed small compared to that of light, there will be no free charges and the displacement current will be neglected. The energy of the electric field will be neglected compared to that of the magnetic field. The mechanical effect of the magnetic field is the same as that of a hydrostatic pressure $\frac{\mu H^2}{8\pi}$ and a tension $\frac{\mu H^2}{4\pi}$ along the lines of force. Since the conductivity is infinite we have

$$\bar{E} = -\bar{v} \times \bar{H}, \quad \frac{\partial \bar{H}}{\partial t} = \text{curl}\,(\bar{v} \times \bar{H}). \tag{1}$$

Thus the field is rigidly attached to the material. Three cases of interest arise (i) when $\bar{v}$, $\vec{H}$ are parallel and normal to the shock plane, (ii) $\bar{v}$ is normal to the shock plane but $\bar{H}$ lies in the plane and (iii) $\bar{v}$ and $\bar{H}$ are parallel but oblique to the plane. We shall take them up in turn. We shall use suffixes 1, 2, to denote the variables on the front and back side of the relatively stationary shock. Let ρ, p, U, $\bar{E}$, denote the density, the pressure, the internal energy per unit mass and the electric field.

(1) *Longitudinal shock* (A): $\bar{v}$ and $\bar{H}$ are parallel and normal to the shock front. Let the normal to the plane be taken as x-axis. Here

$$\bar{v} \times \bar{H} = 0, \qquad \therefore \frac{\partial H}{\partial t} = 0 \tag{2}$$

the field $\bar{H}$ is constant in direction and magnitude on the front side. Also,

$$H_{1x} \neq 0, \quad H_{1y} = 0 \quad H_{1z} = 0. \tag{3}$$

Then Maxwell's equations give at the surface of separation,

$$H_{2x} = H_{1x}$$

and also since $\bar{E} = 0$, $H_{2y} = 0$, $H_{2z} = 0$. (4)

Now the principle of conservation, applied to mass, momentum and energy gives the familiar equations

$$\rho_1 v_1 = \rho_2 v_2 \tag{5}$$

$$\rho_1 v_1^2 + p_1 = \rho_2 v_2^2 + p_2 \tag{6}$$

$$\tfrac{1}{2} \rho_1 v_1^3 + \rho_1 v_1 U_1 + p_1 v_1 = \tfrac{1}{2} \rho_2 v_2^3 + \rho_2 v_2 U_2 + p_2 v_2. \tag{7}$$

For a perfect gas after some simplifications we get

$$\tfrac{1}{2} q_1^2 + \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} = \tfrac{1}{2} q_2^2 + \frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2}. \tag{8}$$

All these are independent of $\bar{H}$ which also remains invariant through the shock. Hence the shock takes place as if the field were absent.

(2) *Longitudinal shock* (B): The field perpendicular to the direction of flow, which is normal to the plane.

Here the lines of force move with the fluid, and cross the shock front. Thus $H \propto \rho$,

hence
$$\frac{H_{1y}}{\rho_1} = \frac{H_{2y}}{\rho_2}. \tag{9}$$

Conservation of mass, momentum and energy give as before, equations similar to (5), (6), (7), where we replace p and U by

$$p^* = p + \frac{\mu H^2}{8\pi}, U^* = U + \frac{\mu H^2}{8\pi\rho}. \tag{10}$$

Equation (8) is replaced by

$$\tfrac{1}{2} v_{1x}^2 + \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} + \frac{\mu H_{1y}^2}{4\pi\rho_1} = \tfrac{1}{2} v_{2x}^2 + \frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2} + \frac{\mu H_{2y}^2}{4\pi\rho_2}. \tag{11}$$

Combining (5) and (9), we have $H_{1y} v_{1x} = H_{2y} v_{2x}$. (12)

The equations become determinate if the strength of the shock and the strength of the magnetic field be given.

(3). *Oblique shock* : Field parallel to the direction of flow which is oblique to the shock front.

Let us take x-axis to be along the normal to the shock plane, y-axis in the plane containing the normal and the direction of flow.

Under the assumptions we have made it can be easily shown that the field will be parallel to the direction of flow, behind the shock front also, that is, both are equally refracted. Thus we get the equations

$$\frac{H_{1x}}{H_{1y}} = \frac{v_{1x}}{v_{1y}}, \quad \frac{H_{2x}}{H_{2y}} = \frac{v_{2x}}{v_{2y}}. \tag{13}$$

The boundary conditions at the surface of separation give

$$H_{1x} = H_{2x}. \tag{14}$$

The conservation of mass gives

$$\rho_1 v_{1x} = \rho_2 v_{2x}. \tag{15}$$

The principle of conservation of momentum gives two equations

$$\rho_1 v_{1x}^2 + p_1 + \frac{H_{1y}^2}{8\pi} = \rho_2 v_{2x}^2 + p_2 + \frac{H_{2y}^2}{8\pi} \tag{16}$$

$$\rho_1 v_{1x} v_{1y} - \frac{H_{1x} H_{1y}}{4\pi} = \rho_2 v_{2x} v_{2y} - \frac{H_{2x} H_{2y}}{4\pi}. \tag{17}$$

The principle of conservation of energy gives

$$\tfrac{1}{2} \rho_1 v_{1x}(v_{1x}^2 + v_{1y}^2) + \rho_1 v_{1x} U_1 + v_{1x} p_1 = \tfrac{1}{2} \rho_2 v_{2x}(v_{2x}^2 + v_{2y}^2) + $$
$$+ \rho_2 v_{2x} U_2 + p_2 v_{2x}. \tag{18}$$

Combining (18) and (15) we get the same equation as (8). It is to be noted that this equation is independent of the field. A complete solution of these equations can be found if the shock-strength, the magnetic field strength and the obliquity of the field are all known.

When the field is small, these lead to the conventional hydrodynamic equations.

Helfer, 1953, has discussed these in detail, by a choice of suitable parameters. He has shown that small magnetic fields (of strength such that $H^2/8\pi << p$) are magnified by the passage of the shock disturbance. If the field is large, then there is no magnification. The presence of the field always causes a decrease in the compression. He has applied this theory to discuss the presence of magnetic field in the interstellar clouds and some aspects of the internal motions of prominences.

REFERENCES

1. R. COURANT and K. O. FRIEDRICHS: Supersonic flow and Shock waves, Interscience Publishers, INC., New York, (1948).

2. D. DE. HOPPMAN and E. TELLER.: Magneto-Hydrodynamic Shocks, *Physical Review*, 80 (1950), 672.

3. HELFER, H. LAWRENCE: Magneto-Hydrodynamic Shock Waves, *Astrophysical Journal*, 117, (1953), 177.

Kirorimal College
Delhi

ON HYDROMAGNETIC STABILITY OF A CONSTRICTED GAS DISCHARGE

By R. K. JAGGI

THIS problem is one of the application of Magneto-hydro-dynamics to physics, more particularly to 'Controlled thermonuclear fusion'. I will therefore say a few words about the fusion problem and how Magneto-hydro-dynamics is of help to physicists in this important problem. It may be stated that the fusion reactions are the only hope of world's future energy sources.

Prof. Narlikar remarked the other day, "that it is a very healthy thing that at the conferences of mathematicians there should be simpler exposition of some current problems although the so called popular lectures may be solids, liquids, gases or even plasmas." We shall here be dealing with the fourth state of matter, viz 'The Plasma' (completely ionized gas) during the following talk.

WHAT ARE FUSION REACTIONS

We know that atoms of an element consist of electrons revolving round their nuclei. The combinations or fusion of nuclei of light elements to form heavier nuclei (or even sometimes lighter than its own constituent [1, 2]) with the release of energy is what is called a fusion reaction. We are already familiar with the hydrogen bomb in which fusion reactions are used for destructive purposes. This particular problem 'The controlled fusion research' is an example of the peaceful uses of nuclear energy.

Fusion reactions of this type are in fact responsible for the enormous amount of energy generated in the Sun and other stars.

It is important to recognize that in a fusion reaction each of the interacting nuclei is positively charged. As a result they strongly repel one another. This repulsion is called the potential barrier. In order that the two nuclei may fuse together, they must be made to collide with high enough relative velocity so as to overcome the potential barrier which tends to keep them apart. We will return to this later.

CHIEF MERITS OF THE PROBLEM

These are : (i) Unlimited energy sources: Deuterium which is used for fusion reaction can be got from the oceans. As an order of magnitude of the energy content we may note that the amount of deuterium in one gallon of sea water is 1/8 of a gram and the cost of extraction would be less than 20 np. However its energy content if it were burnt as a fuel in a fusion reactor would be equivalent to 100 gallons of kerosene oil.

(ii) Inherent safety : The amount of fuel in a fusion reactor would be so small that there is no possibility of explosion. The products of the reaction are non-radioactive, so that there is no danger of radiation hazards.

(iii) The best point about the problem is that the device can offer direct generation of electrical power, so that the costly and inefficient stages of converting heat into mechanical work are eliminated.

Having studied the good points of the problem let us look at the dark side of the picture, i.e. our inability to control thermonuclear reactions.

Basic acquirements of achieving controlled fusion are (i) high temperatures, (ii) low density and (iii) adequate confinement means, the third requirement being the consequence of the first.

We have already seen that to make fusion possible the energy and therefore the temperature of the constituents must be very high, of the order of 10^8K°. High temperatures can only be obtained if we are able to reduce the conduction of heat of the ionized gas by its contact with the material walls of its containers. No material can stand the temperatures of the order of 10^8K°. We are therefore led to the problem of confining the hot gas so that it is not in contact with the solid container.

CONFINEMENT MEANS

These can be three: The application of (i) the electric field, (ii) the gravitational field and (iii) the magnetic field.

The first possibility can be ruled out at once, because the electric field which would confine one kind of particles would drift the other kind of particles in the opposite direction.

An example of the second confinement means is the Sun itself which is supplying energy to us at a tremendous rate. Such a complete confinement means is not possible in the laboratory.

The application of a strong magnetic field seems to be the only promise for laboratory work. Various geometries for a fusion reactor have been studied, but we will here confine our attention to one that is a right cylinder of circular cross-section. As an ideal case we will assume the length of the cylinder to be infinite.

We have noticed that to reach reacting conditions it is necessary to keep the hot gas away from its container. It may be observed that there is a self-magnetic field of the reactor. The potential difference applied across the cylindrical tube produces a current j along the length of the tube and a consequent toroidal magnetic field of the form $B_0 r_0/r$, where r_0 is the radius of the current channel. The magnetic field produces the pressure $B_0^2/2\mu_0$ on the ionized gas. By increasing B_0 we can reduce the radius of the current channel.

It is important to note that the apparatus is rendered useless due to the instabilities.

Most of the theoretical and experimental work on the problem will be found in the references [3, 4].

We will now consider one theoretical aspect of the problem. The self-magnetic field acts upon the plasma and drives it towards the axis of the discharge. This phenomenon is called the pinch effect. The macroscopic equations of compressible flow cannot be applied to a fast pinch during the whole of its motion. Because at the high temperatures, the mean free path of the particle will be long compared to the dimensions of the apparatus. It may however give a fair representation of the process when the whole gas is practically concentrated at the axis of the tube. At this point the large increase in particle density will substantially reduce the mean free path.

Kruskal and Schwarzschild (1954) imagined the cylindrical pinched fluid to attain static equilibrium when the magnetic pressure balances the plasma pressure. In static equilibrium the plasma is held in by the pressure of a toroidal magnetic field which exists in the vacuum surrounding the plasma. No magnetic field exists inside the plasma. The separation between the plasma and vacuum is supposed to be a thin boundary, so that the following continuity conditions can be applied at the boundary :

(i) normal component of the magnetic field,

(ii) tangential component of the electric field,

(iii) sums of the plasma and magnetic pressures are all continuous on the boundary.

To solve the stability problem we have to linearize the magneto-hydro-dynamic equations about the above mentioned equilibrium solution [3]. To solve these it is assumed that the fluctuation of a physical entity from its equilibrium value is proportional to $e^{im\theta+ikz+\omega t}$, exponential dependence on these quantities being justified since these linear equations do not contain θ, z, t explicitly. The vacuum equations are similarly linearized and solved. The boundary conditions are then made use of to obtain the dispersion equation, viz. a relation between ω, k, m and the equilibrium quantities.

For the case described above it can be shown [3] that for the $m = 0$ mode the cylindrical plasma is unstable. Let us now assume a uniform axial magnetic field B (produced externally) to exist inside as well outside the current channel and the material tube to be (a conductor of electricity) of radius R_0. Now if $a = B/B_0$ and $\Lambda = R_0/r_0$, the critical value of $x_0 = |k|\, r_0$ at which the instability sets in is given by

$$a^2 = \frac{1}{x_0} \frac{I'_m(x_0)\, K'_m(\Lambda\, x_0) - I'_m(\Lambda\, x_0)\, K'_m(x_0)}{I'_m(\Lambda\, x_0)\, K_m(x_0) - I_m(x_0)\, K'_m(\Lambda\, x_0)},$$

where I, K are Bessel's functions. In the absence of the conductor a is given by

$$a^2 = -\frac{1}{x_0}\frac{K'_m(x_0)}{K_m(x_0)}.$$

The following table gives the value of a against the values of $\lambda/r_0 = \frac{2\pi}{x_0}$ for $m = 0$ at which the instability sets in.

TABLE

$\lambda/r_0 = 2\pi/x_0$	a (No conductor)	a (Conductor present)
62.832	6.372	1.222
31.416	3.691	1.213
12.566	1.893	1.159
6.283	1.196	1.018
3.141	0.784	0.768
2.094	0.621	0.619

The table clearly shows that the axial magnetic field stabilizes the plasma, the conductor has a stabilizing effect and that it has greater effect on large wave length disturbances than on small ones.

REFERENCES

1. (a) GEORGE WAR-FIELD : *RCA Review*, (1958), p. 137.
   (b) E. W. HEROLD : *ibid.* p. 162.
2. AMASA S. BISHOP : U. S. program on Thermonuclear Fusion, Project Sherwood, 1958.
3. KRUSKAL and SCHWARZSCHILD : *Proc. Roy. Soc.* 223 A (1954), 348.
4. R. J. TAYLER, *Proc. Phys. Soc.* B 70.1, 1958; *Nature*, January 25, 1958.

University of Delhi
Delhi-6

HYDROMAGNETIC TURBULENCE

By J. N. KAPUR

In view of the large dimensions of cosmical masses—of interstellar gases and of stellar envelopes—turbulence is basic in many applications in cosmic physics. In most of these applications the electrical conductivity is high enough for electro-magnetic forces to play an essential role and for turbulence to be governed by the laws of magneto-hydrodynamics. The main problems of hydromagnetic turbulence are :

I. Partition of energy into energy of the velocity field and energy of the magnetic field. The problem includes the study of (a) the distribution of magnetic and kinetic energies over the various scales of motion, and (b) the mechanism of interchange of energy between the velocity and the energy field.

II. Decay of magneto–turbulence in the presence of a magnetic field and Coriolis force. The various attempts at solving the first problem have been the following :

(a) Fermi (1949), Elassasser (1950), Biermann and Schlüter (1950) have given qualitative arguments for supporting the hypothesis of equipartition of energy, i.e. for the hypothesis

$$\frac{1}{8\pi}\langle H^2 \rangle \sim \tfrac{1}{2}\rho \langle u^2 \rangle,$$

where ρ denotes the density and $\langle u^2 \rangle$ and $\langle H^2 \rangle$ are mean square velocity and magnetic intensity respectively.

(b) Batchelor (1950) has used the analogy between the equations for the vorticity vector

$$\nabla \cdot \vec{w} = 0, \quad \frac{\partial \vec{w}}{\partial t} - \nabla \times (\vec{v} \times \vec{w}) = \nu \nabla^2(\vec{w})$$

in the absence of a magnetic field and the equations for the magnetic field vector

$$\nabla \cdot \vec{H} = 0, \quad \frac{\partial \vec{H}}{\partial t} - \nabla \times (\vec{v} \times \vec{H}) = \lambda \nabla^2 \vec{H}, \left(\lambda = \frac{1}{4\pi \mu \sigma} \right),$$

and some results about mean square vorticity in ordinary turbulence to establish the hypothesis that the ultimate balance between the magnetic and the hydrodynamic systems is such that the *large wave-number* components contain comparable amounts of kinetic and magnetic energy. He also uses this analogy together with Kolmogoroff's (1941) universal theory to deduce that a small random magnetic field introduced into a conducting liquid in homogeneous turbulent motion will be amplified if $4\pi \mu \sigma \nu > 1$ and that the initial rate of growth of the magnetic energy will be exponential with a doubling time of the order $(\nu/\epsilon)^{\frac{1}{2}}$, where ν is the kinematic viscosity and ϵ is the rate of energy dissipation per unit mass. He also suggests that ultimately the magnetic field reaches a statistically steady state with energy of order $\rho(\epsilon \nu)^{\frac{1}{2}}$ per unit volume of the fluid and that this magnetic energy will have a spectral distribution which is concentrated in the neighbourhood of wave numbers of order $(\epsilon/\nu^3)^{\frac{1}{4}}$. Batchelor's arguments have been critically examined by Lundquist (1952), Chandrasekhar (1955 *a*) and Cowling (1957).

(c) Chandrasekhar (1950) has extended the treatment of Von Karman and Howarth (1938) for ordinary turbulence to confirm Batchelor's result that in a stationary state, the magnetic energy is contained principally in the small eddies. He also defines various double and triple correlations involving the components of velocity and the magnetic field intensity and obtains three equations governing the scalars defining these tensors. He has also obtained equations exhibiting the exchange of energy between the velocity and the magnetic fields. In another paper, Chandrasekhar (1951) has extended this paper to include the correlations including the total pressure $P(= p + \frac{1}{2}\rho \,|\, h \,|^2)$. Later Chandrasekhar (1955 *a*) has extended his new theory of turbulence (1955 *b*) in which the concept of correlation between velocity components at two different points and at two different times has been introduced to obtain a 'deductive' theory of hydromagnetic turbulence.

(d) An ‘ elementary ’ theory of hydromagnetic turbulence based on Heisenberg’s (1948) theory for ordinary turbulence has given by Chandrasekhar (1955c) The basic physical idea used is to conceive the transformation of the kinetic energy at a particular wave number into kinetic and magnetic energies at higher wave numbers and similarly the transformation of the magnetic energy at a given wave number into kinetic energy and magnetic energy of higher wave numbers, as a cascade process which can be visualised in terms of suitably defined coefficients of eddyviscosity and eddy-resistivity.

(e) Chandrasekhar (1957) has proved that in the inertial subrange, i.e. in the range of eddy sizes which are small compared to the largest energy-containing eddies but still large enough for the non-linear exchange of energy between them to be a dominant factor—the energy in the magnetic field is 1.6265 times the energy in the velocity field. This result is based on Chandrasekhar’s new theory (1955 *b*) which is itself based on the statistical hypothesis that the fourth order moments are related to the second—order moments as in a normal distribution—a hypothesis which has been severely criticized by Kraichnan (1957). Moreover Chandrasekhar has tried to develop a universal theory on the lines of Kolmogroff —only he has to consider three parameters ν, ϵ and $1/4\pi\mu\sigma$ — and this theory cannot answer the question of the *actual* ratio of energies in the two fields without going into the structure of energy-containing eddies, i.e. outside the framework of a universal theory. In fact the ratio in the inertial range is not of much interest from the point of view of answering the real physical question.

Though the above studies have led to a great deal of understanding about the energy–partition problem, the situation is not very satisfactory, as in the words of Cowling (1957)“ Nearly every argument for or against equipartition between magnetic and turbulent kinetic energy field can be strongly criticised. A more fundamental approach to the subject seems necessary ”. Cole (1956) in his review article on magneto-hydrodynamics omitted discussion of hydromagnetic turbulence as an area in which least general agreement exists.

Two extensions, viz. to compressible fluids and to non-metal type conductors and fully ionized gases seem desirable. The first has been attempted by Krzywoblocki (1952 *a*, *b*). The electromagnetic phenomena caused by turbulent motion of an ionized gas of low density have been discussed by Biermann and Schlüter (1950).

The decay of turbulence for large wave-numbers has been discussed by Lundquist (1952). If $G(k)$ and $F(k)$ devote the spectral functions for the sum and difference of kinetic and magnetic energies, he proves

$$\frac{G}{F} = \frac{C\,e^{-4k^2\,Bt} + 1}{C\,e^{-4k^2\,Bt} - 1}\,, \quad B = \tfrac{1}{2}\,(\nu - \lambda)\,.$$

It is seen that for $\nu \geqslant \lambda$, only magnetic energy will remain after a certain time and for $\nu < \lambda$ only kinetic energy. The distribution of energy will be such as to make the dissipation as small as possible in that part of the spectrum where the energy is consumed. This result is not inconsistent with Batchelor's. There is a fundamental difference in the assumptions, however. In Batchelor's case only kinetic energy was present in the large eddies. In Lundquist's case both types may exist and hence the feeding mechanism to the smaller eddies may be different.

The decay of magneto–turbulence in the presence of an external homogeneous magnetic field has been considered by Lehnert (1955). He considers only small amplitudes and neglects triple correlations which, however, represent the basic mechanism of turbulent interaction. Lehnert however considers only the final period of decay in which the triple correlations are not important. He finds that turbulence developes pronounced axi-symmetric properties which however are destroyed by the introduction of an angular velocity.

Finally we may mention the paper by Sweet (1950) in which he studies the kinematic effects of turbulence in a magnetic field and finds that the apparent conductivity of the medium should be decreased by the turbulent motion. In the pure hydro-dynamic case it is found that the apparent viscosity is increased by turbulence and the result of Sweet appears to be consistent with this result.

REFERENCES

G. K. Batchelor : (1950) *Proc. Roy. Soc. 201 A*, 405.

L. Biermann and A. Schlüter : (1950) *Zs. f. Natur forschung 5a*, 237.

S. Chandrasekhar : (1950) *Proc. Roy. Soc. 204A*, 435.

S. Chandrasekhar : (1951) *Proc. Roy. Soc. 207A*, 305.

S. Chandrasekhar : (1955*a*) *Proc. Roy. Soc. 233A*, 322.

S. Chandrasekhar : (1955*b*) *Proc. Roy. Soc. 229A*, 1.

S. Chandrasekhar : (1955*c*) *Proc. Roy. Soc. 233A*, 330.

S. Chandrasekhar : (1957) *Ann. Phy. 2*, 615.

G. H. A. Cole : (1956) *Adv. Phy. 3*, 452.

T. G. Cowling : (1956) *Magneto Hydro-dynamics*—Inter Science, New York.

W. M. Elssasser : (1950) *Phy. Rev. 79*, 183.

E. Fermi : (1949) *Phy. Rev. 75*, 1169.

W. Heisenberg : (1948) *Zeit Phy. 124*, 628.

A. N. Kolmogroff : (1941) *C. R. Acad. Sci.* U. S. S. R *30*, 301, *32*, 16.

B. Lehnert : (1955) *Quart. App. Math. 12*, 321.

R. H. Kraichnan : (1957) *Phy. Rev. 107*, 1485.

M. Z. Krzywoblocki : (1952) *Act. Phy.* Austr. *6*, 157, 250.

M. Z. Krzywoblocki : (1952) *J. Phy. Soc.* Japan *7*, 299, 511.

S. Lundquist : (1952) *Ark. Fysik. 5*, 338.

P. A. Sweet : (1950) *Month. Not. R. Ast. Soc. 110*, 69.

Von Kármán and L. Howarth : (1938) *Proc. Roy. Soc. 164A*. 192.

Hindu College
Delhi 8

GRAVITATIONAL INSTABILITY OF AN INFINITE HOMOGENEOUS TURBULENT MEDIUM IN THE PRESENCE OF A MAGNETIC FIELD

By P. C. JAIN

THE compressible fluid is taken to be turbulent in the presence of a magnetic field of intensity $\vec{H}$. The equations to be satisfied are:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\mu}{4\pi}\left(\frac{\partial H_i}{\partial x_m} H_m - \frac{\partial H_n}{\partial x_i} H_n\right) -$$

$$- \frac{\partial p}{\partial x} + \mu \nabla^2 u_i + \rho X^i + \frac{1}{3}\mu \frac{\partial^2 u_j}{\partial x_i \partial x_j} \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 . \tag{2}$$

First, we take the fluid to be of infinite electrical conductivity so that

$$\frac{\partial H_k}{\partial t} - \text{curl}\,(\epsilon_{ijk}\, u_i\, H_j) = 0 . \tag{3}$$

If the external force is derived from a gravitational potential, then

$$X_i = \frac{\partial V}{\partial X_i} . \tag{4}$$

Using these equations., we have

$$\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial^2}{\partial x_i\, \partial x_j}\left(\rho u_i u_j - \frac{\mu}{4\pi} H_i H_j\right)$$

$$= \nabla^2\left[p + \frac{\mu}{8\pi} H_n H_n\right] - \frac{4}{3}\mu \nabla^2 \frac{\partial u_i}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\rho \frac{\partial V}{\partial x_i}\right) . \tag{5}$$

In order to discuss the gravitational instability of the turbulent medium, we introduce

$$p = \bar{p} + \delta p, \quad \rho = \bar{\rho} + \delta \rho, \quad V = \bar{V} + \delta V . \tag{6}$$

With the help of (5) and (6) and following the usual method of taking the averages we get

$$\frac{\partial^2 \bar{w}}{\partial t^2} = \left(c^2 + \frac{1}{3}\bar{u}^2\right)\nabla^2 \bar{w} + 4\pi G \bar{\rho}\, \bar{w} - \frac{\mu}{4\pi}\frac{\partial^2}{\partial \xi_i \partial \xi_j}\overline{(\rho' H_i H_j)} + $$

$$+ \frac{\mu}{8\pi}\nabla^2\, \overline{\rho' \, |H|^2}. \tag{7}$$

If

$$V_{ij} = \overline{\rho' H_i H_j} = V_1 \xi_i \xi_j + V_2 \delta_{ij}, \tag{8}$$

where, V_1 and V_2 are the defining scalars of V_{ij}, we get

$$\left[\frac{\partial^2}{\partial \tau^2} - (c^2 + \tfrac{1}{3}\bar{u}^2)\nabla^2 - 4\pi G \bar{\rho}\right]\bar{w} +$$

$$+ \frac{\mu}{8\pi}\left(r^2 \frac{\partial^2}{\partial r^2} + 10\, r \frac{\partial}{\partial r} + 18\right) V_1 - \frac{\mu}{8\pi}\left(\frac{2}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right) V_2 = 0 \tag{9}$$

which contains three unknowns $\bar{\bar{w}}$, V^1 and V^2.

Then, we introduce the quasi-Gaussian hypothesis for the fourth order correlation in terms of the second order correlations and following the usual method of taking the averages, we have from (3)

$$\frac{\partial V_{ij}}{\partial \tau} = 0. \tag{10}$$

Hence the final result is

$$\left[\frac{\partial^2}{\partial \tau^2} - (c^2 + \tfrac{1}{3}\bar{u}^2)\nabla^2 - 4\pi G \bar{\rho}\right]\frac{\partial \bar{w}}{\partial \tau} = 0. \tag{11}$$

The criterion of stability is

$$k^2 < \frac{4\pi G \bar{\rho}}{c^2 + \frac{1}{3}\bar{u}^2} \tag{12}$$

which is the same as that obtained by the present author (*Proc. Nat. Inst. Sc. Ind. Vol.* 24, 1958 *b*) in the absence of the magnetic field.

Next, the fluid is taken to be of finite electrical conductivity and it is found that there are two modes of wave propagation, which become unstable if

$$\frac{4}{r^2} < k^2 < \frac{4\pi G \bar{\rho}}{c^2 + \frac{1}{3}\bar{u}_2}. \tag{13}$$

Hence the presence of the magnetic field does not affect the stability criterion of Chandrasekhar (1951) provided the fluid is of infinite electrical conductivity; but it will excite another mode of wave propagation for finite electrical conductivity of the fluid. In the latter case the criterion for instability is given by (13). The results so obtained are in agreement with those obtained by Chandrasekhar and Fermi for non-turbulent homogeneous medium.

Hindu College
Delhi 8

SOME APPLICATIONS OF MAGNETO HYDRO-DYNAMIC THEORY TO IONOSPHERIC PROBLEMS

By K. S. RAJA RAO

THE study of the dynamics of the ionosphere has been mostly based on the kinetic theory of gases. It was Alfvén who first suggested that in the ionospheric regions the electromagnetic effect is of far greater importance than the effect resulting from considerations of kinetic theory. Alfvén's ideas are based on the effect of earth's permanent magnetic field. It can be easily seen from the following figures how the electromagnetic effects dominate.

$$P \text{ kinetic} = n K T = 10^3 \times 1.37 \times 10^{-16} \times 1.5 \times 10^{+3}$$
$$\approx 2 \times 10^{-5} \times \text{ dynes/cm}^2;$$

$$P \text{ magnetic} = \frac{H^2}{8\pi} = \frac{(.3\Gamma)^2}{8\pi} = \approx 3.6 \times 10^{-3} \text{ dynes/cm}^2.$$

Starting with the fundamental equations of electro-dynamics

$$\left.\begin{aligned} \text{Curl } H &= 4\pi j \\ \text{Curl } E &= -\mu \frac{\partial H}{\partial t} \\ \text{and} \qquad j &= \sigma(E + v \times \mu H), \end{aligned}\right\} \tag{1}$$

and the Navier Stokes equation of hydro-dynamics one can deduce the induction equations

$$\frac{\partial H}{\partial t} = \text{Curl } (v \times H\mu) \tag{2}$$

and

$$\frac{\partial H}{\partial t} = \eta \nabla^2 H, \tag{3}$$

where

$$\eta = (4\pi \mu \sigma)^{-1}.$$

If the gravitational and coriolis forces are neglected the equation of motion becomes

$$\rho \frac{dv}{dt} = j \times H = \frac{1}{4\pi} (H \times \text{Curl } H), \tag{4}$$

where $\mu = 1$ and $\sigma = \infty$

Treating the perturbation due to small oscillations, we write $H = H_0 + h$, where $H_0 =$ earth's permanent magnetic field, and h is the perturbing field. Remembering that Curl $H_0 = 0$, the equation of motion can be written as

$$4\pi\rho\frac{\partial v}{\partial t} = -H_0 \operatorname{Curl} h. \tag{5}$$

Differentiating with respect to time and making use of equation (1), we have oscillations in a non-uniform magnetic field, the relation

$$4\pi\rho\frac{\partial^2 E}{\partial t^2} = -H_0 \times (\operatorname{Curl}\operatorname{Curl} E) \times H_0. \tag{6}$$

Consider the simple case of axial symmetry with the magnetic dipole placed at the origin, viz. the centre of the earth parallel to the axis of symmetry. Thus H_0 lies in the meridional plane.

Following Dungey, the equation (5) can be written as

$$\left(4\pi\rho H_0^{-2}\frac{\partial^2}{\partial t^2} - r^{-2}\sin\theta\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial r^2}\right)(r\sin\theta\, E_\phi) =$$

$$\sin\theta\left(H_r\frac{\partial}{\partial\theta} - H_\theta r\frac{\partial}{\partial r}\right)\left\{(r\sin\theta)^{-1}\frac{\partial u_\phi}{\partial\phi}\right\}. \tag{7}$$

In this equation the two modes of oscillations are expressed in a coupled state. To simplify matters, we put $\partial/\partial\phi = 0$, i.e. the disturbance occurs in phase over the whole earth; there the coupling between the two modes disappears. These two modes are called the poloidal and toroidal fields. The lines of force are dragged along the circles of latitude and thus give rise to the toroidal field. The poloidal field corresponds to the field coil of the technical dynamo, and the toroidal field to the armature. When $\partial/\partial\phi = 0$ there is no coupling between the two modes. The equation (7) can be transformed into

$$\left(4\pi\rho H_0^{-2}\frac{\partial^2}{\partial t^2} - r^{-2}\sin\theta\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial r^2}\right) r\sin\theta\, E_\phi = 0. \tag{8}$$

Put $\frac{4\pi\rho\omega^2}{H_0} = \eta$ assuming harmonic oscillation with $e^{i\omega t}$.

Then $\eta = \frac{V_A}{\omega}$ where $V_A = \frac{H_0}{\sqrt{4\pi\rho}}$, the Alfvén wave velocity $T = \frac{2\pi\eta}{V_A}$ and $\omega = \frac{2\pi}{T}$, the frequency. Therefore it is necessary to determine the Alfvén wave velocity.

To a first approximation, let us assume that all atoms are singly ionized in the ionosphere, although actually there is a mixture of ions and neutral particles. This gives a value of 200 kms/sec for the Alfvén wave velocity.

It is possible that the mechanism of generation of these oscillations is of solar origin. The particles emanating from the M regions in the sun may produce these oscillations in the ionosphere. With Allen, Keipenhauer and others, one may conceive of the M region being present in the solar corona itself. Such ionospheric oscillation may in turn produce minor disturbances in the earth's magnetic field.

The Damping Effect

When the medium has a finite conductivity and viscosity, it gives rise to dissipation of energy. The most general treatment is due to Van de Hulst. According to him, the damping time is given by

$$\tau_1 = \frac{1}{q\omega},$$

where $q = (a' + b')\, X_m$, $a' = \frac{\omega}{4\pi\sigma}$ and $b' = \frac{\mu\omega}{\rho c^2}$, $X_m = \frac{4\pi\rho}{H^2} = c^2$, ($c$ = velocity of light, μ = viscosity coefficient) a' gives the Joule heat according to Batchelor and b' the dissipation of energy due to viscosity. The relative importance of resistivity and viscosity to energy dissipation is given by the ratio $\frac{b'}{a'}$, where

$$\frac{b'}{a'} = \frac{4\pi\sigma\mu}{c^3\rho} \approx 1.5 \times 10^9.$$

Poona.

SUMMARY OF TALK

By B. S. MADHAVA RAO

The subject of Magneto-hydrodynamics, including Magneto-gas-dynamics has been vigorously pursued in recent years, and literature pertaining to the same has therefore been expanding very rapidly. I wish to present briefly some of these developments by dividing the subject into three categories specified respectively by the macroscopic, normal, and microscopic scales of the phenomena concerned.

It need hardly be said that much of the advance in all the three regions has been due to recent intensive study, to a great extent initiated by S. Chandrasekhar, of the solutions of the fundamental equations themselves. For example, the systematic discussion of the hydromagnetic equations for axi-symmetric fields in case of fluids of finite electrical conductivity, based on the decomposition of the magnetic and velocity fields into their poloidal and toroidal components has yielded valuable results. Further generalization by dropping the assumption of a static field, and using instead the condition that the undisturbed $\vec{v}$ and $\vec{H}$ fields are stationary, and that the $\vec{v}$ field is everywhere proportional to the $\vec{H}$ field, a great freedom can be obtained in the choice of the undisturbed field. By such a suitable choice, expressions can be derived for the periods of small oscillations about the equilibrium configuration, and lower bounds deduced for the periods of the lowest modes of vibration. Other types of work consist in the discussion of wave motion in compressible and incompressible media, thermal instability in the presence of a magnetic field, and the propagation and structure of magneto-hydrodynamic shock waves. Consideration of the stability of the simplest solutions of the fundamental equations has also given significant results.

At the macroscopic level, lie the results of an astrophysical significance. It has been possible to discuss from the point of view of the hydromagnetic equations, the question of generation and maintenance of cosmic magnetic fields. Some recent Russian work in the field of relativistic magneto-gas-dynamics also falls in this category, but it is still in the introductory stage of setting up the equations of motion in the Minskowski 4-space, there being, however no derivation so far of results of astrophysical interest. Attempts at obtaining the possible equilibrium configurations of magnetic stars, and the nature of dissipation of magnetic energy in interstellar space are other examples.

In the second category, the most striking result recently obtained is that due to S. Chandrasekhar about the decay of the Earth's magnetic field. Considering the effect of internal motions on the decay of a magnetic field in a fluid conductor, he has applied the general theory relating to axi-symmetric fields to this particular problem, and shown by numerical analysis that reasonable motions, if they occur appropriately, can lengthen the decay time of the Earth's magnetic field from 17,000 years—the value it would have in the absence of internal motions—to 500,000 years, a value suggested by recent experimental results. Theories have also been set up to serve as a basis for explaining the origin of the Earth's magnetic field. One such assumes a non-uniform rotation which generates a toroidal magnetic field from an initial poloidal one, next a succession of " cyclones " creating out of this toroidal field loops of flux in the meridian planes, and finally these coalescing and regenerating a poloidal field. Mention must also be made of recent work done at the Japanese Earthquake Research Institute about the Earth's magnetic field based on the model of a self-exciting dynamo, and investigating the stability of the Earth's dynamo. This institute has also done work on the explanation of the origin of earthquakes based on magneto-hydrodynamic theory.

On the microscopic level, the outstanding recent contribution is the development of the stellarator which is an experimental device employing a twisted magnetic field to contain an ionized gas, and electric fields to heat it, and which is perhaps the first step towards the deriving of useful power from thermo-nuclear reactions, a suggestion first made by H. J. Bhabha. When two nuclei of heavy hydrogen collide at high energies, they interact forming a new nucleus, and liberating either a proton or neutron of high energy. To obtain a useful power yield from this reaction, a gas of deuterium (H_2) or of mixed deuterium and tritium (H_3) must be brought to an enormously high temperature—equal to about 10^8 degrees absolute. The problem then is to confine this gas within a limited region away from contact with any solid matter, as otherwise it will cool and the solid material will evaporate. Magnetic forces seem to offer the only way to contain a thermo-nuclear reaction using the *pinch effect*, viz. the flow of a heavy electric current through the hot gas, thereby generating a strong magnetic field which at once compresses the gas and brings it up to a high temperature. In the stellarator, the gas is originally contained in a magnetic field produced by the above process thereby reaching a temperature of about 10^6 degrees, and then the ultra-high temperatures are reached through an effect called *magnetic pumping* induced by a second extremely rapidly pulsating magnetic field. Besides the stellarator, recent work done which belongs to this third category relates to the oscillations of an electron plasma in a magnetic field, and plasma losses by fast electrons in thin films.

Magneto-hydrodynamics is a very young and rapidly expanding subject, and by its very nature a meeting ground of several disciplines of theoretical physics. A consequence of this is that at each step in the development of the subject, unexpected problems arise which need to be solved, and exact solutions are not, in general, possible because of the mathematical complexity of the fundamental equations. When, however, solutions are obtained under reasonable assumptions and approximations, the interpretation of these solutions leads to results as unexpected as the problems that arise. Surely there are exciting possibilities for research in magneto-hydrodynamics.

University of Poona
Poona 7

EXCEPTIONALLY LARGE SOLAR AND GEOPHYSICAL EVENTS.

By S. L. MALURKAR

In the laboratory, attempts have been made to verify several results of Magneto-hydrodynamics. Of late the subject has been applied for designing the elaborate experiments of thermonuclear processes, under controlled conditions. In the sun and in the stars, the processes involved are determined by magneto-hydrodynamics due to the order of various quantities. The extra terrestrial observations would still be very useful. The chromospheric eruptions (solar flares) of solar active regions affect almost contemporaneously (taking account of the velocity of light) on the day light side of the earth's hemisphere within an area of about 70° of the sub-solar point (McNish cone) and give rise to a temporary short time change in the geomagnetic field (crochet) and to ionospheric disturbances in D and E layers leading to a radio fade-out. After 24 to 48 hours, often a world-wide geomagnetic storm may follow. The incoming cosmic rays, the solar noise and details of spectroscopic phenomena may also contemporaneously change while the cosmic rays may decrease and closely follow the changes in the one to two day later world wide geomagnetic storm. Among the large number of solar events which happen, the sorting out requires considerable preparation. The exceptionally large events assume great importance as they naturally classify themselves out. In the last twenty years, five very large events with contemporary cosmic ray bursts occurred (Feb. 28, 1942; Mar. 7, 1942; July 25, 1946; Nov. 19, 1949; Feb. 23, 1956). While a day or so after the eruptions of Feb. 28, 1942 and July, 25, 1946, world wide geomagnetic storms with simultaneous cosmic ray changes were recorded, no such combined or inter-related subsequent geomagnetic and cosmic ray disturbances were recorded in the other three instances. To these five, two more events when geomagnetic disturbances (two biggest ones recorded at Alibag) with no corresponding cosmic ray changes were added and all the seven were studied with the evolutionary history of the connected solar active regions. It was found:

1. The cosmic ray bursts happened with solar active regions which were very active and had a history of great activity > 6 days.
2. When the chromospheric eruptions of such a solar active region happened near the C. M. of the sun, there was an increase of cosmic ray intensity (a burst) and after about a day, a world wide geomagnetic storm with corresponding cosmic ray changes followed. (Feb. 28, 1942 ; July 25, 1946).
3. When the chromospheric eruption of the same type of very active region took place near the western limb of the sun there was a great burst of cosmic rays with short time geomagnetic changes (crochets) in the McNish cone. No inter-related geomagnetic cosmic ray disturbance followed in the next 24 to 48 hours. (Mar. 7, 1942 ; Nov. 19, 1949 ; Feb. 23, 1956).
4. No such event has yet been recorded at the eastern limb of the sun with very active regions.
5. On Nov. 19, 1949 and Feb. 23, 1956, the solar flares caused an increase in electron density in the F_2 which according Minnis and Bazzard is rare unlike the usual solar flare effects which give " a simultaneous increase in the electron density in the *D* and *E* layers." The geomagnetic crochet at Alibag on Mar. 7, 1942 is very marked and similar to those recorded by Kodaikanal on Feb. 23, 1956 that it is likely that F_2 layer was also affected on Mar. 7, 1942.
6. The purely geomagnetic storms were related to very active solar regions with more ephemeral history soon after their C. M. passage (Mar. 1, 1941 and Mar. 28, 1946).

Most of these along with the tabulated data and immediately deducible conclusions about the nature of particles (positively charged particles from the middle of the periodic table) have been published (*Acta Physica Hungarica*, 8.235. 1958). These show that the surface of the sun, even those belts where the solar active regions form cannot be considered as uniform in the causation of the big events, which seem to be confined to two small longitudinal sectors (vide Table, p. 236).

The next problem that was tackled was the nature of solar active regions that give rise to big events. With the data available, it was found that on days of big events, the details are very limited or are absent. Even for the one day that has been used July 25, 1946, it had to be got by interpolation. Hence an attempt was made to study eight second order type events along with that for July 25, 1946 —a first order event included earlier. It showed that the events with cosmic ray enhancements were determined by characteristic or distinguishable differences along with positional value stated earlier (*Acta Phys*. Hung. 9,353, 1959.)

The geophysical control or distribution of these events can easily be obtained from records of observations. Some items of solar control have been studied above. If in addition, the conditions between the sun and the earth are reasonably assumed, the self consistent picture would add to our theoretical knowledge.

Colaba Observatory
Bombay 5

TABLE

		SOLAR ACTIVE REGION					SOLAR FLARE			DATE OF		INTERVAL BETWEEN		REMARKS	
Year	*No.*	*Coordinates* ϕ	*L.*	*C. M. Passage Date*	*Age*	*No. of Distinct Observ-ed Flares*	*Date*	*C. M. Dist.*	*Age*	*Cosmic Ray Burst*	*Magnetic Storm*	*Magnetic Storm and Flare*	*Magnetic Storm and C.M. Passage*	*Cosmic ray affected Mag./Stm*	*Electron density increase F_2 layer*
1941	7	15N	354	Feb.27·5	2	7					Mar.1·15	...	1·85	No	
1942	12	7N	197	Feb.28·8	+26?	17	Feb.28·5	4E	+26?	Feb.28·5	Mar.1·31	0·81	0·51	Yes	
*			,,				Mar.7·18	91W	+32?	Mar. 7·18	...	...	...	No	
1946	15	23N	10	Mar.26·9	>6	3	Mar.27·2	4W	>6	...	Mar.28·1	1·1	1·41	No	
1946	51	21N	198	July26·8	>6	37	July25·67	16E	>5	July25·67	July26·8	1·13	0·0	Yes	
1949/4	23	2S	116	Nov.14·1	3	17	Nov.19·40	75W	8	Nov.19·44	...	...	...	No	Yes
1956/1	17	22N	176	Feb.17·8	>6	32	Feb.23·13	80W	11	Feb.23·13	...	...	...	No	Yes

Note :—

(*a*) C. M. distances and age refer to those of the solar active region numbered in each row.

(*b*) C. M. distances approximate to within a few degrees only due to inherent observational approximations.

(*c*) For No. 23 (of the fourth quarter of 1949) active region, the age at C. M. Passage is given as—1 in Q. *Bull of I.A.U.*, Zurich. It had been observed at Mitaka on Nov. 11, at Kanzelhohe on 13th and again at Mitaka on 14th. The age has been changed accordingly. It is also mentioned that this active region was a fresh formation on the return of active region No. 13 of 1949/4.

(*d*) No data of F_2 region behaviour available for Feb. 28, 1942, March 7, 1942 and July 25, 1946.

SUMMARY OF POPULAR LECTURE ON 'PARITY IN NATURE'

By B. S. MADHAVA RAO

By parity is meant the principle that all phenomena in nature could be explained on the basis of a few simple and fundamental laws. The whole progress of the physical and biological sciences is a striking confirmation of the fact that man has struggled for ages to perceive such parity in nature.

But one can immediately notice striking differences between biologists and physicists in their approach towards finding such fundamental laws of nature. While mighty revolutions are taking place in the concepts of the physical sciences, the biologist appears to be looking on these with detachment and exercising great restraint about stating new laws in his own subject. This attitude could perhaps be appreciated if we note that biology is still a young subject, and a most complex one because of the hypothesis of evolution by natural selection, containing the laws of adaptations, competition and survival, and implying that there need be no theory of the origin of life at all. Another kind of complexity arises because of the validity in biology of the view that the design of an existing product is relative to its way of life. This is something like the principle of indeterminacy in physics, and leads to the result that the technique of recognising that not only general statements but their opposites also are meaningful, appears valid in biology. A third complexity is the prevalence of the notion of a general type of complementarity which states that one understands the laws of nature only when considers all the three questions each independent of the others, viz. the question of mechanism, the question of adaptation, and the twin questions of embryogeny and evolution. The most complex nature of biology, however, is that it is a science of life without defining life, but only recognising it. It is because of such complexities that biologists do not want to theorise like physicists, but a time is certain to come when the biological sciences, having grown old enough, will indulge in abstract theories which will put into shade the extravagant theories of modern physics.

Coming to the physical sciences themselves, one could trace an interesting evolution in the attempts of physicists to perceive parity in nature. Based on Newton's laws of motion, and generalized by deep and beautiful mathematical analysis developed by Euler, Lagrange, Laplace, Jacobi and Hamilton, classical physics held sway for nearly two centuries until the discovery of electromagnetism by Faraday and Maxwell. It was then that Einstein formulated relativistic mechanics, and quantum mechanics was created by Bohr, de Broglie, Schrödinger, Dirac, Heisenberg and Pauli, based on Rutherford's epoch making experiments. Further attempts at the search for uniform laws are best illustrated by considering advances in the region of elementary particle physics. A fundamental development was the relativistic quantum theory of Dirac leading to the discovery of new anti-particles and the processes of annihilation and creation of elementary particles. Further experimental discovery of other elementary particles like the several types of mesons and hyperons, and a new process of spontaneous decay of several particles into other types greatly complicated the position. The interaction between several types of elementary particles holds the centre of interest in the subject today, and new theories have been developed to re-examine Dirac's work keeping the quantum principles intact, but trying to consider the invariance of these interactions under discontinuous relativistic transformations like space reflexion denoted by P, time-reversal denoted by T, and also under the new type of transformation of particles into antiparticles denoted by C. In this category falls the famous CPT-theorem of Pauli that if one of the operators is not conserved, at least one other also must not be conserved, leading to five possibilities. Recent examination of this by Lee and Yang for the particular interaction of β-decay, viz. $n^0 \rightarrow p^+ + e^- + \nu^0$($\nu^0$ being the neutrino) showed that P and C are not conserved in this interaction, and this conclusion was verified by the experiment of Wu on Cobalt-60, thus leading to the result that ν^0 is a right-handed screw, and $\bar{\nu}^0$ (the anti-neutrino) is a left handed screw, and mirror images are nothing! A consequence of Pauli's theorem in this case is that T may or may not be conserved, and experiments have not yet decided this. If the latter be true, it leads to difficulties in statistical mechanics and thermodynamics about reactions and reversible reactions, and also to speculations, that at some previous history of the universe when conditions of extreme densities and extreme temperatures existed, ordinary thermodynamics did not hold. At least if CP invariance is valid, and hence also T invariance, then the world we see in a mirror would obey different physical laws, but would obey the same physical laws as in an anti-world and this difference may be due to an accident in the history of our part of the universe.

The rapid advance of experimental work in this field, which has left theory far behind, has yielded many more new types of particles some of which by their very nature have to be classified as *strange* particles and have necessitated ascribing to them a new quantum number called the *strangeness* S. A further study of the large number of interactions between the several types of elementary particles, which number 32 at present, has led to the classification of these interactions as strong, electromagnetic and weak with relative strengths $1 : 10^{-2} : 10^{-14}$ with β decay mentioned above as a typical example of the last type, and besides a fourth type called a *strange* interaction, i.e. decay of strange particles has also been added. Complex experiments are being performed to decide the question of conservation of C, P and T in the several types of interactions, and the physicist of today appears to be back in the position of the chemist in the pre-Rutherfordian era working with 92 particles and amassing vast information regarding interactions between them, but failing to perceive a parity in nature.

Among the elementary particles, the photon and neutrino appear the most romantic, and the mysterious ν^0, apart from having helped in proving the violation of P in weak interactions, appears destined to play an important role in future developments attempting to absorb Einstein's general relativistic theory of gravitation into the main stream of quantum mechanics. Gravitational attraction as compared with other interactions is the weakest, being of the order of 10^{-34} and if gravitational theory be quantised, and we can talk of 'gravitions' as elementary particles, it would be most interesting to find how they are related to the neutrinos. For this the great open questions of cosmology dealing with the 'boundary conditions' satisfied by the universe at its beginning have to be answered by observation. Important innovations in this direction may arise in the next few decades out of astronomical observations relating to cosmological questions made possible by elaborate equipment assembled in sputniks in interplanetary space. But a clear understanding of the problems of elementary particle physics appears possible only when a still deeper mathematical analysis is made of existing quantum field theories of the nature that Hamilton's theory did to classical physics.

University of Poona
Poona 7.

SOME DYNAMICAL PROBLEMS OF ASTRONAUTICS

By PROFESSOR V. V. NARLIKAR

"*Like one that on a lonesome road*
Doth walk in fear and dread,
And having once turned round walk on,
And turns no more his head;
Because he knows, a frightful fiend
Doth close behind him tread"

The Rime of the Ancient Mariner—Coleridge

AT a recent meeting in New York, when W. Pauli gave an exposition of Heisenberg's unified theory of fundamental particles, Neils Bohr is reported to have remarked that the theory was crazy but not crazy enough to be acceptable. In solving their problems the mathematicians have frequently to resort to plausible reasoning and fruitful new ideas in science often appear crazy until we become familiar with them. It is therefore a very healthy thing that at the conferences of mathematicians there should be simpler expositions of some current problems although the so-called popular lectures may be solid, liquid, gaseous or even plasma. Attempts at enriching common understanding are necessary when every branch of the tree of knowledge is accessible only to a privileged few.

By astronautics is meant the science of space flight. From the point of view of a cosmic spectator we are all participating in a space flight : for the earth moves about the sun with an average velocity of about 18.5 miles per second and the sun moves, with a velocity of about 12 miles per second, in a stellar cluster and the stellar cluster rotates about the centre of the Central Galaxy so as to cover about 140 miles in a second. He would call us sputniks or fellow-travellers of the earth.

But the point of view of man is different.

According to Eddington, "Man, in his search for knowledge of the universe, was like a potato bug in a potato in the hold of a ship trying to fathom from the ship's motions the nature of the vast sea." Eddington was contrasting here man's helpless confinement to the earth against his insatiable thirst for knowledge of the vast outer universe. At the bottom of a vast atmospheric ocean, which may be stretching up to the moon or even beyond, man—a prisoner on earth—has to be content with the messages he can get through the radio window and through the optical window.

It is therefore a great thing to be able to travel from the earth to the moon or Mars, or even to a space-station beyond the ionosphere, in a space-ship under a short-lived, limited motive force, mainly with the help of the gravitational field of the solar system. The flight would be mostly ballistic, a gravitational 'fall'. We will be considering a few dynamical problems of such space flights.

To begin with we may use the terminology of space-time mapping which is popular with relativists and which Professor J. L. Synge* has recently used to present diagrammatically 'an elopement situation' and 'what every father should know' under the circumstances. With sputniks and 'explorers' travelling in the outer space the cosmic era has dawned and we should look upon ourselves as citizens of the cosmos, consider our events as world events and treat life careers of individuals as world lines.

The mere existence of a particle here and now defines an event describable as a set of four ordered numbers and the totality of all possible events forms a four-dimensional continuum. The sequence of events associated with the existence of a particle constitutes the world line which is the history of the particle. Each world line progresses until it merges in another, or indefinitely, as time passes. In the four-dimensional space-time representation Newton's first law states that the world line of a free particle is a straight line. If two persons are moving towards each other on a straight line we can show (by treating the persons as particles) that their world lines

*"An Introduction to Space-Time", *The New Scientist*, May 1, 1958.

would intersect, the point of intersection giving the place and time of their encounter. The world line of a pulse of light is also a straight line. A simple calculation shows that the world line of the Newtonian apple, falling from the tree, is a parabola.

The main problem of space travel can now be thus described. There is the world line S of the starting station and there is also the world line D of the destination. Initially, S is the world line of the passenger and at the end of the journey it is D. We have to find a world line for the passenger which branches off from some point on S and proceeds, with a few breaks in direction if necessary, so that it finally touches D somewhere and merges in it. There is generally a world line corresponding to the least time of travel and there may also be the least expensive track, the total rocket energy needed being minimum.

For this we must know thoroughly the law governing the propagation of world lines and we have to determine how rockets can be advantageously used for effecting the necessary change of direction at the right stage.

The simple problem that we may consider first is that of Professor Singer's MOUSE project. MOUSE is an abbreviation for 'Minimum Orbital Unmanned Satellite of the Earth'. Let us assume the earth to be spherically symmetrical, the radius being a and the gravitational acceleration at the surface being g. If an object is projected at a height $(a + b)$ from the centre, with the velocity $\sqrt{(ga^2)/(a + b)}$ parallel to the surface of the earth underneath it continues to move with that speed in a circle of radius $(a + b)$ about the centre of the earth. If there is no resistance such as that of the atmosphere the motion is non-stop.

With all the works done from Newton to Einstein the mystery of this motion is not yet fully understood. If we adopt the Newtonian method several logical difficulties are encountered and certain subtle relativistic observed effects remain unaccounted for. Einstein's method, on the other hand, makes the calculation of the world lines too complicated for known mathematical techniques. Besides, there are mathematical difficulties in representing rapidly changing fields and gravitational radiation. In launching a satellite into its orbit the scientists find the Newtonian method of calculation, with its approximations and limitations, more convenient.

Incidentally, in the case $b = 0$, that is, for a circular motion just close to the surface of the earth, the velocity of propagation is $\sqrt{(ga)}$. It is roughly 18000 miles per hour and is known as the primary cosmic velocity for the space travellers in a satellite who would keep very close to the terrestrial coast. The velocity $\sqrt{(2ga)}$ is roughly 25000 miles per hour and is often referred to as the secondary cosmic velocity. It is the velocity of escape from the earth. The world line of an object flung vertically from a point O on the earth, with this velocity, becomes ultimately parallel to the time line which is the world line of O. In other words, the object never returns to the earth.

In practice it is not prudent for space travellers to keep very close to the terrestrial coast, because of the resistance of the atmosphere. One has to go up a certain safe distance b if the circular motion is to continue undisturbed for a fairly long time. It will be some time before man succeeds in guarding against the hazards of space travel. At the present stage the unmanned satellite can be used as a space observatory for observation and transmission of scientific data. The necessary data are so important that costly rocket experiments have been performed, even at the risk of the utter destruction of the registering and measuring instruments, which could be used for observation for not more than five or six minutes. A space observatory, housed in a satellite, with a life of several weeks and months would be definitely an advantage for collecting some of the scientific data. It must be understood, however, that some of the experiments and observations at lower levels in the atmosphere can be carried out by short-range rockets which cannot be replaced by satellites for these purposes.

We may take $a=4000$ miles and $b = 22{,}300$ miles so that the period of circular motion $2\pi(a+b)^{3/2}/\sqrt{(ga^2)}$ is just one day. If the motion is in the plane of the equator and in the sense of diurnal rotation the satellite would be practically fixed relative to the terrestrial observer underneath. But the distance would be too much for the satellite being used for the observation of the atmospheric phenomena and for broadcasting the observed data to the earth. Some time back von Braun considered $b=1075$ miles as most suitable, the period of orbital motion being 2 hours. In this case the effort required for the satellite to be set moving in its orbit is considered as beyond the then resources of science and technology before October 1957. The smaller the value of b, the greater is the danger to the orbit through atmospheric resistance but the smaller is the rocket power needed. For the minimum orbit for the unmanned satellite under the MOUSE project b was taken as 600 miles.

From the data available to us of the first two sputniks we know that their periods diminished to the dangerous figures of 89-90 minutes from 96.2m. and 103.7m. in 92 and 161 days respectively. The daily loss in the periodic time first increases slowly and is rather fast after some time. The average daily loss in period was found to be about four to five seconds in the case of Sputniks I and II. This effect is a measure of the resistance of the atmosphere.

In the programme of launching a satellite into its orbit there is a rocket thrust to start with, producing vertical motion. Another rocket thrust is imparted subsequently to change the direction of motion. The satellite then coasts along under gravity, until it reaches the orbit, when the rocket is fired again to give the satellite the necessary direction and velocity. If the rocket becomes too fast too soon considerable energy is wasted in overcoming the air resistance. Here are two problems. What is the optimal trajectory for launching the satellite ? How should the limited thrust of the rocket motors be most economically employed ? These are urgent problems the solution of which is complicated by the uncertain parts played by the air resistance in the process. Oberth*, who called the optimal trajectory, the "synergy curve," was probably the first to suggest a tentative solution.

*"Rockets, Missiles and Space Travel" by Willy Ley, *Chapman and Hall*, p. 335 (1957).

Equally urgent and still more difficult is the problem of the safe return journey of the satellite or space-station to the earth. There is to be a rapid fall in velocity from about 18,000 miles per hour to zero. Under the resisting action of the atmosphere there is a danger of kinetic heating. According to Professor W. J. Duncan† "A close estimate of the temperature increase is obtained from the equation that the rise in degrees centigrade at the nose of the body is equal to the square of the speed measured in hundreds of miles per hour." Thus at a speed of 5000 miles per hour the temperature rise would be 2500°C. The problem is how to prevent a heat death and a crash and provide for a safe return of the satellite.

A smooth landing implies that the world line of the passenger should touch ultimately the world line of the destination. Under gravitational forces alone as in the Newtonian theory the world lines of two objects cannot touch. Even in general relativity this is not possible on the geodesic principle. An extraneous non-gravitational controlling force is needed to effect a smooth merging of the two world lines into each other. This is supposed to be done by a suitable rocket action.

Great precision is demanded in the direction and magnitude of the velocity imparted in the last rocket operation in launching a satellite into its circular orbit. The error makes the orbit elliptic and however distant the apogee may be the perigee is on or within the circle and it may be dangerously close to the earth, in the denser layers of the atmosphere, affecting the life of the satellite.

In applied mathematics all numbers which arise out of physical measurement and which have physical dimensions associated with them are known to have some fringe. We cannot assert that the breadth of the fringe is 2ϵ and that the positive number ϵ can be made as small as one likes. If in determining the world line of the destination or in controlling by rocket action the world line of the passenger the necessary degree of precision is not achieved, the two world lines may not meet according to the plan and the passenger

‡ "High Speed Flight", *Engineering*, 186, 278, (1958).

may be left permanently in the cosmic lurch. Let us imagine a passenger starting for the moon from a distance of 350 miles from the earth. At the precise moment of starting the initial velocity in the direction of motion is expected to be in the neighbourhood of 6.06 miles per second. From Hans A. Lieske's† calculations the following conclusions follow.

If the moon is to be hit at all the error in the velocity of projection should not exceed 40 ft. per sec. and the error in the direction of projection should not exceed a fourth of a degree. If the passenger is to go to the moon orbit, and land back on the earth, with an uncertainty of 1000 miles in landing, these errors should not exceed 1.12ft. per sec. and 0.01 of a degree respectively. This gives us an idea of the exacting demands on technology in implementing a programme of space travel.

The large fluctuations in distance in the case of Sputniks II and III, if they were unintentional, mean that the error in the velocity vector in the last rocket stage was unexpectedly high. It was not clear at first why the Russians had selected an inclination of 65° to the equator for the orbit of Sputnik I. The MOUSE was to have a polar orbit so that receiving stations all over the world and especially at the poles would be regularly getting the scientific data transmitted from the satellite and every part of the entire surface of the earth would be periodically under observation.

We have already remarked that if there is a small error of projection at the last rocket stage the orbit of the satellite becomes an ellipse. The oblateness of the figure of the earth produces certain systematic changes in the ellipse one* of which is that it rotates in its own plane with an angular velocity proportional to $(5 \cos^2\alpha - 1)$, where α is the inclination of the orbit to the equator. For $\alpha = 63.4°$ this effect is zero. In Sputnik I, the Russians succeeded in reducing the rotational motion of the ellipse to 0.4 degree per day by making the inclination sufficiently close to 63.4°.

†" Practical Aspects of Earth Satellites ", *Engineering*, 184, 484, (1957).

*" Perturbations of the orbit of a satellite near to the earth " by D. G. King. Hele, *Proc. Roy. Soc.*, Series A, 248, 55-62, (1958).

Our experience shows that when the last rocket is fired in launching a satellite there may be two or more objects almost in the same orbit. This suggests a simple mathematical problem in which two satellites are supposed to be moving in the same sense and in the same orbit. If the velocity of either is changed it starts moving in a different orbit. Assuming that we want the two satellites to be brought together and describe the same circular orbit let us see how it may be done.

Let us suppose that the satellite which is moving ahead is fitted with two rockets. The problem may be simplified further by assuming that the rocket action is fast enough to be treated as an impulse. The impulse may be so directed that the satellite moving ahead makes a detour on an elliptic path reaching the original circle again at a point when the other satellite has just arrived there. Here another impulse will be necessary so that the elliptic path is knocked back into the original circle. The questions now arise: What is the least time in which the two satellites can thus be brought together? What is the most economical manoeuvre for making the two satellites trace a common world line? The mathematical treatment of these questions is within the scope of an honours student of the B.A.—B.Sc. mathematics class. But the calculations involved are so frightful that the trailing satellite would remind the student of Coleridge's famous lines quoted in the beginning.

In a single stage of propulsion, when the propellant is all burnt out, the velocity v of the rocket is given by

$$v = c \log_e \lambda - v_1 - v_2,$$

where c is the exhaust velocity of the outgoing gas which roughly varies directly as the square root of the combustion temperature and inversely as the square root of the mean molecular weight of gas, v_1 is the velocity loss due to the gravitational opposition to the climb and v_2 is the velocity loss due to the atmospheric resistance. v_1 and v_2 depend upon the path and the burning out time which may be anything like 100 seconds or more. λ is the ratio of the total weight of the rocket to what is left when the propellant is all burnt out. It may be as high as 6. It is possible to build up a multi-stage rocket. In such a rocket, the original weight w is reduced to w/λ_1, after the first burnout. In the second stage the weight is reduced from w/λ_1 to $w/\lambda_1\lambda_2$. If there are n stages $\lambda = \lambda_1 \lambda_2 \ldots \lambda_n$. The fluorine-hydrogen combination** for the propellant is known to give for c as high a value as 7 miles per sec. The other combinations which can be more safely used give for c a value less than 2.5 miles per sec. The engineers often speak of specific impulse τ which has the dimension of time and which expresses both the exhaust velocity and the thrust t through the relations,

$$\tau = c/g = t/p,$$

where p is the time-rate of propellant consumption. For the creation of the modern multi-stage rocket, which has brought space travel within the realms of practicability, credit must be given to the recent research done in many branches of science and technology. But the most exciting adventure has been, as always, in the realm of thought, in tackling the mathematical problems arising at each step in research.

Even in very simple problems of space travel we are confronted with a situation which makes the standard* mathematical procedure of calculus of variations of little avail. At $t=0$ a rocket is at one place in space with a certain velocity. At $t = t_0$ the rocket is at another place with a given velocity. These two are essential requirements for a space trip which is not to end in a disaster. The problem is to find the optimal path for the rocket. The expected solution is one for which the velocity cannot be continuous. The ballistic flight in the gravitational field is to be so organized that at strategic moments in the course of the journey the rocket is used for changing the course and speed. The mathematical problem of finding the trajectory is quite complicated as can be seen from some of the idealized simple cases that have received attention.

**"Science News 48", *Penguin Books*, p. 30 (1958).

*"Mathematical Problems of Astronautics" by D. F. Lawden, *Math. Ga.*, XLI, 172, (1957).

As I remarked at the outset we are all spuntniks or fellow-travellers of the earth tracing with it a world line in the midst of stars and galaxies. As we are falling freely under cosmic gravitation we experience only the gravitational force of the earth. Mr. Pickwick travelling in a space-ship would experience the gravitation only of the ship which would be negligible. For all practical purposes everything in the company of Mr. Pickwick (including his body) would be weightless. If "he opened the mouth of a bottle and applied it to his own" there would be no flow and no gratifying results. The adventures of a dry Pickwick, in a space-ship bound for the moon, would be a challenging theme for a scientific fiction.

Already the satellite programme has supplied us valuable information about the figure of the earth, the density of the atmosphere, the electromagnetic fields in the ionosphere and among other things about the intense belts of 40 million-volt protons, a thousand kilometers or so above the earth in particular latitudes. We are learning every day of the new hazards of space travel through the atmosphere and a new subject called "Space Medicine" has come into existence. The resources of mechanical calculating prodigies are being devised for the guidance of rockets. Condon tells us of a visitor to the U.S. who enquired of the meaning of the legend, "What is past is prologue", inscribed on a building in Washington. The motor driver gave his interpretation in the words, "You ain't seen nothing' yet." The wonders of Nature and of human achievement that we have seen will be as nothing compared to the wonders that we are going to see—if we survive the third world war.

Banaras Hindu University
Banaras 5

REMARKS ON INDUCTION†

By V. S. HUZURBAZAR

The fundamental problem of scientific progress, and a fundamental one of everyday life, is that of learning from experience. Knowledge obtained in this way is partly mere description of what we have already observed, but part consists of making inferences from past experience to predict future experience. This part may be called generalization or induction. Induction is the backbone of all empirical sciences. Inductive inferences are probable, and cannot be absolutely certain.

Induction by simple enumeration is the following principle: " Given a number of n of α's which have been found to be β's, and no α which has been found to be not a β, then the two statements: (a) 'the next α will be a β', (b) 'all α's are β' s', both have a probability which increases as n increases, and approaches certainty as n approaches infinity ". Russell [*Human Knowledge*, (London 1948)] calls (a) ' particular induction ' and (b) ' general induction ' Thus (a) will argue from our knowledge of the past mortality of human beings that probably Mr. So-and-so will die, whereas (b) will argue that probably all men are mortal.

From the time of Laplace onward, various attempts have been made to show that the probable truth of an inductive inference follows from the mathematical theory of probability. It is now generally agreed that these attempts were all unsuccessful. Some extra postulate is needed to justify induction. Laplace's famous ' Law of succession ' was supposed, for a long time, to justify induction. As pointed out by Broad [*Mind*, 27 (1918), 389-404] it justifies only particular induction but not general induction.

In his logic of scientific inference, Jeffreys (*Theory of Probability* Oxford, 1939) lays down the ' simplicity postulate ': ' Any clearly

† A summary of the half-hour address delivered at the Golden Jubilee Session of the Indian Mathematical Society, Poona, December 1958.

stated hypothesis has a non-zero prior probability unless there is definite evidence against it.' As a consequence of this postulate, he shows that repeated verifications of the consequences of a scientific hypothesis will make the *next* consequence almost certain, when the number of verified consequences approaches infinity. This again justifies particular induction only.

Huzurbazar [*Proc. Camb. Phil. Soc.* 51 (1955, 761-762)] has shown that the simplicity postulate leads to the justification of general induction also: 'repeated verifications of the consequences of a scientific hypothesis will make *any number* of further consequences almost certain, when the number of verified consequences approaches infinity'

An interesting point about the propositions proved by Jeffrey and Huzurbazar is that the propositions hold good irrespective of whether the scientific hypothesis is true or not. Now in science, one of our difficulties is that the alternatives available for consideration are not always an exhaustive set. An unconsidered one may escape attention for centuries. The propositions of Jeffreys and Huzurbazar show that this is of minor importance. The unconsidered alternative hypothesis, if it had been thought of, would either lead to the same consequences as the considered hypothesis, or to different consequences at some stage. In the latter case, the data would have been enough to dispose of it, and the fact that it was not thought of has done no harm. In the former case, the considered and the unconsidered alternatives would have the same consequences, and will presumably continue to have the same consequences. The unconsidered alternative becomes important only when it is explicitly stated and a type of observation can be found where it would lead to different predictions from the old one.

The rise into importance of the theory of general relativity is a case in point. Even though we now know that the systems of Euclid and Newton need modification, it was still legitimate to base inferences on them until we know what particular modification way needed. As Jeffreys remarks, 'the theory of probability makes it possible to respect the great men on whose shoulders we stand '. What is more remarkable, laws which ultimately turn out to be inexact, are often far more exact than the data on which they are based. As Jeffreys points out, when Einstein's modification to Newton's law of gravitation was adopted, the agreement of observations was three hundred times as good as Newton ever knew!

University of Poona
Poona 7

ABSTRACTS

ALGEBRA AND THEORY OF NUMBERS

M. BHASKARAN Annamalainagar. *Factorization in Cyclotomic Field*

It is proved in this paper that any algebraic integer n prime to a rational prime p can be expressed as the sum of two factors of p in the algebraic number field $R\{[(\pm p)^{1+r/p-1} - n^p]^{1/p}\, n,\ \zeta\}$ where R is the rational number field, $1 < r \leqslant p$, and ζ is a primitive pth root of unity. By making use of this result, the following theorem is proved:

THEOREM. *Let $q > 3$ and $lq + 1 = p$ be rational primes such that $(2l, q-1) = 2$. Then q spilts in the cyclotomic field of p-th roots of unity.*

J. M. GANDHI, Patiala. *On logarithmic numbers.*

This paper is the result of the study of the coefficients $G_r^{(n)}(t)$ defined by the polynomial

$$\log(1 - x^n)\, e^{-xt} = -\sum_{r=1}^{\infty} G_r^{(n)}(t)\, x^r/r\,!,\ G_0^{(n)}(t) = 0.$$

Some results connecting these coefficients (which are called logarithmic numbers) with the Möbius function $\mu(n)$, Euler's function $\phi(n)$ and the function $r(n)$ defined as the number of representations of n as the sum of two squares, are established early in the paper. The rest of the paper deals with the logarithmic numbers $G_r^{(n)} = G_r^{(n)}$ (1), and $L_r^{(n)} = G_r^{(n)}(-1)$, arising out of the special cases when $t = 1$ and $t = -1$ respectively. Tables of values of $G_r^{(n)}$ and other allied functions defined in the paper, for small values of r and n, have been constructed and some of the interesting properties have been discussed therefrom. Some results involving the numbers $G_r^{(n)}$ and $L_r^{(n)}$ and the functions $d(n)$ and $\sigma(n)$ and another set of results connecting these numbers with the well-known numbers like Bernoulli's and Euler's have been obtained. There are also a few results on these numbers in connection with the partition function

$p(n)$. The logarithmic numbers independently by themselves, satisfy certain congruences which have also been worked out.

D. R. KAPREKAR, Devlali. *Circulating constants from five digited numbers.*

Let $N_1 = abcde$ represent a number of five digits where all the digits a to e are in descending order of magnitude. The number got by reversing the digits in N_1 will be called R_1.

Let $N_1 - R_1 = S_1$. Now arrange the digits in S_1 to form a new number in descending order. It will be called N_2. The reversed number of N_2 will be R_2. Let $N_2 - R_2 = S_2$. Arrange the digits of S_2 in descending order to get our next starting number N_3. The process can be continued for any number of times ; however it will be seen after a few steps that we come to only one of the three following circulating number series A, B or C.

A	—	62964,	71973,	83952,	74943
B	—	75933,	63954,	61974,	82962
C	—	59994,	53955		

The process can be similarly applied to numbers of 6 digits or more and we get circulating constants of six or more digits. If however the process is applied to any number of 4 digits we get only the constant 6174 within 8 steps.

INDAR SINGH LUTHAR, Chandigarh. *Uniqueness of the invariant mean on an Abelian semigroup.*

Let **S** be a semigroup and let **m**(**S**) be the space of bounded real-valued functions x on **S** with $\|x\| = \text{Sup}_s |x(s)|$. An invariant mean on **S** is a positive linear functional of norm one on **m**(**S**) which is invariant under all the left and right translation operators. It is known that every abelian semigroup has an invariant mean. It is proved here that an abelian semigroup **S** has a unique invariant mean if and only if **S** has a finite ideal in it. If the abelian semigroup **S** is finitely generated then each of the above two conditions is equivalent to the third condition that there are no non-trivial homomorphisms of **S** into the additive semigroup of integers. Further if **S** has many invariant means, then the diameter of the set of invariant means is two.

M. B. PANT, Poona. *Divisibility of any number by any other number.*

In this paper is given a general method for test of divisibility by any number.

M. RAJAGOPALAN, Banaras. *H-Algebras.*

An *H-algebra* is an algebra A over the field C of complex numbers whose underlying vector space is also a Hilbert space. Moreover an involution operator denoted by * is defined in A such that

$$(x\, y, z) = (y, x^* z) = (x, z\, y^*) \text{ for all } x, y, z, \in A.$$

The first result is that $A = I_1 \dot{+} I_2$ where I_1 and I_2 are two closed ideals of A. $I_1^2 = 0$ and I_2 has no non-zero annihilators. I_2 is called a proper algebra. The next result is that $I_2 = \dot{+}_{\alpha \in J} I_\alpha$ where I_α is an ideal of I_2 and is isomorphic to a full matrix algebra led by the trace topology and J is some index set. To prove this result, the first important step taken is to show the existence of self-adjoint idempotents. First of all, sequence of polynomials $P_n(x^2)$ in x^2 and without the constant term and decreasing to the step function $\phi_\partial(x)$ is chosen where $\phi_\partial = 0$ for $0 \leqslant x \leqslant \partial$ and 1 for $\partial < x \leqslant 1$. Then it is shown that $\lim_{n\to\infty} P_n(a^2)$ exists for a self-adjoint element $a \in A$ and such that the uniform norm $|||a|||$ of $a \leqslant 1$. This element $e = \lim_{n\to\infty} P_n(a^2)$ is a self-adjoint idempotent. Then the existence of minimal idempotents is proved and the structure of each I_α is proved along the same lines as for Wedderburn's structure theorem.

K. SAVITHRI, Baroda. *Quadratic forms on the Rational Function Fields.*

In this paper the generalization of the main theorem of Siegel on the representation theory of quadratic forms to the rational function field in one variable over a finite field, is given. The proofs for definite and indefinite formsup to the evaluation of a certain multiplicative constant are summarized. If **T** is the symmetric matrix that is representable by the symmetric matrix **S**, we prove that this constant depends only on **S**. The proofs of Hasse's theorem, the theorem on the semi-equivalence of quadratic forms as done by Siegel for the rational number field, the generalization of the Generalized Formula of Gauss and Eisenstein for definite and indefinite forms, the modification of the same for indefinite forms and the method of induction following the results of Artin for binary forms are indicated in detail. We also have the result that the units of a given indefinite symmetric matrix are finitely generated. This is done by means of the theory of discontinuous groups of Siegel. We make use of the discreteness of the valuation on the rational function fields in question in almost all the proofs. Finally other possible generalizations are indicated.

S. V. Sirdesai, Poona. *On the possibility of expressing general equations of degree $n \geqslant 4$ in the form $A(x+\lambda)^n + B(x+\mu)^n = 0$.*

The following theorem is proved for the general equation of the nth degree.

Theorem: *The necessary and sufficient conditions that the equation $a_0x^n + na_1x^{n-1} + \frac{n(n-1)}{1.2}a_2x^{n-2} + \dots + a_n = 0$ may be expressed in the form $A(x+\lambda)^n + B(x+\mu)^n = 0$, are*

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = \dots\dots = \begin{vmatrix} a_{n-4} & a_{n-3} & a_{n-2} \\ a_{n-3} & a_{n-2} & a_{n-1} \\ a_{n-2} & a_{n-1} & a_n \end{vmatrix} = 0$$

and $G^2 + 4H^3 \neq 0$, where G and H have the usual meaning; if $G^2 + 4H^3 = 0$ then $\lambda = \mu$ and the equation has $n-1$ equal roots.

Also, in the first case above

$$A = \frac{a\mu - b}{\mu - \lambda} \text{ and } B = \frac{a\lambda - b}{\lambda - \mu}.$$

B. R. Srinivasan, Madras. *Formula for the n-th prime.*

A recurrence formula for p_n, the nth prime as a symmetric function of $p_1, p_2, \dots, p_{n-1}$ is obtained; also the existence of a simple formula that expresses p_n as a function of n is proved. The precise results are :

$$\text{I.} \qquad p_{n+1} = I\left\{\frac{xs'_n(x) - x}{s_n(x) - x}\right\}, \ |x| \leqslant \tfrac{1}{2} = \lim_{x\to 0} \frac{xs'_n(x) - x}{s_n(x) - x},$$

where

$$s_n(x) = \frac{1}{1-x} - \sum_1^n \frac{1}{1-x^{p_r}} + \sum_{p_r \neq p_s} \frac{1}{1-x^{p_r p_s}} + \\ + \frac{(-1)^n}{1 - x^{p_1 p_2 \ldots p_n}}, \ (n \neq 0)$$

$$s_0(x) = \frac{x}{1-x}.$$

$$\text{II.} \qquad p_{n+1} = I + I\left\{\frac{\log\,[s_n(x) - x]}{\log x}\right\}, 0 < x \leqslant \tfrac{1}{2}$$

$$= \lim_{x\to 0} \frac{\log\,[s_n(x) - x]}{\log x}.$$

III. p_{n+1} = *least of the integers* $p_1 p_2 p_3 \ldots\ldots p_n F\left(\sum_1^n \frac{k_r}{p_r}\right)$ *except for unity, where F denotes fractional part and* k_n *is any integer,* $0 < k_r < p_r$.

IV. *The n-th prime can be expressed as the limit of the ratio of logarithms of rational algebraic functions in n arguments.*

M. V. Subba Rao, Tirupati. *Congruence properties of* $\sigma_r(n)$.

The function $\sigma_r(n)$, which represents the sum of the rth powers of the divisors of n, can be split up into its various *components*, the component corresponding to the divisor $\delta, 1 \leqslant \delta \leqslant \sqrt{n}$, being defined as $\delta^r + \delta'^r$ (where $\delta\delta' = n$) if $\delta' \neq \delta$ and as δ^r if $\delta' = \delta$. In this paper the notion of *strong divisibility* of $\sigma_r(n)$ by k is introduced by saying that k strongly divides $\sigma_r(n)$, written as $\sigma_r(n) \equiv 0 \pmod{k}$ if and only if k divides every component of $\sigma_r(n)$. Obviously strong divisibility implies the usual divisibility, but the other way need not be true.

It is shown here that

(i) *If* $k > 2$, $r \geqslant 1$, *a necessary condition that*

(A) $\sigma_r(kn + l) \equiv 0 \pmod{k}$ *for all integers* n *is that*

(B) $l^r + 1 \equiv 0 \pmod{k}$.

(ii) *This is also a sufficient condition for* (A) *to hold if and only if* k *satisfies*

(C) $w^{2r} \equiv 1 \pmod{k}$ *for all* $(w, k) = 1$.

These results include and go beyond those obtained by Hansraj Gupta and K. G. Ramanathan [*Math. Student*, XIII, (1945), 25-29 and 30 respectively] who showed that

(D) $\sigma_r(kn + l) \equiv 0 \pmod{k}$ *for all integers* n, $k \geqslant 2$, $(k, l) = 1$, *holds if and only if* (B) *and* (C) *simultaneously hold.* This brings out the interesting result that (A) and (D) are equivalent to each other.

ANALYSIS

B. R. Bhonsle, Jabalpur. *Some recurrence relations and series for the generalized Laplace transform.*

Hari Shankar, Moradabad. *On the lower order of 'a-points' of a meromorphic function.*

Let $w(z)$ be a meromorphic function of order ρ, lower order λ and of genus P. As usual $n\,(r, a)$ denotes the number of roots of the equation $w(z) = a$, $(0 \leqslant |a| \leqslant \infty)$. Put $N(r) = n(r, 0) + n(r, \infty)$, and denote the lim sup and lim inf of the ratio $\log N(r)/\log r$ by ρ_1 and λ_1 respectively, as $r \to \infty$. The following theorem is proved. *If* $w(z)$ *is a meromorphic function of non-integral order* ρ *then*

$$\lambda - P \leqslant \frac{\lambda_1(\rho - P)}{\lambda_1(\rho - P) + \rho(P + 1 - \rho)}$$

COROLLARY 1. *If* $0 < \rho < 1$, *then* $\lambda \leqslant \lambda_1/(1+\lambda_1 - \rho)$.

COROLLARY 2. *If* $\lambda = \rho$ *and* $\rho > 0$ *non-integer, then*

$$\lambda_1 = \lambda = \rho = \rho_1.$$

S. K. LAKSHMANA RAO, Bangalore. *On the Relative Extrema of the Turan Expression for the General Hermite Function.*

Let $H_n(x)$ denote the general Hermite function and

$$\Delta_n(x) = (H_n(x))^2 - H_{n+1}(x)\, H_{n-1}(x)$$

its Turan expression. If $M_{1,n}$, $M_{2,n}$, $M_{3,n}$, and $m_{1,n}$, $m_{2,n}$, m_{3n}, ... denote the successive relative maxima and minima of $e^{-x^2}\Delta_n(x)$ as x decreases from $+\infty$ to 0, it is shown that (i) $M_{r,n} < M_{r+1,n}$, (ii) $m_{r,n} < m_{r+1,n}$ (iii) $M_{r,n+1} > M_{r,n}$ and (iv) $m_{r,n} < m_{r,n+1}$. the last two holding for a fixed r such that $[n] \geqslant r \geqslant 1$. Corresponding results for the classical orthogonal polynomials have been established by G. Szego, J. Todd and O. Szasz in the different cases.

V. LAKSHMIKANTHAM, Hyderabad. *On the functional boundedness of solutions of differential equations.*

In this paper, some asymptotic problems of solutions of non-linear equations are considered in a sufficiently general way so as to include the previous results by A. Winter, B. Viswanatham and the present author, as special cases. Accordingly, the functional boundedness of solutions of differential equations is defined as follows :

A solution $x(t)$ of a differential equation is said to be functionally bounded if it satisfies a functional order relation of the type

$$V(x(t), t) = O(L(t)) \text{ as } t \to \infty.$$

This reduces to ordinary boundedness for the choice of $V(x, t) = x$ or $L(t)x$. Functional boundedness of solutions of perturbed differential systems is also considered.

M. R. PARAMESWARAN, Madras. *On the translativity of Hausdorff and Quasi-Hausdorff methods of summability.*

In this paper are considered absolute regularity and translativity of conservative Hausdorff methods for the class $F(l)$ of sequences $[s(n)]$ defined by: $s = [s(n)] \in F(l)$ if $s(n) - s(n-1) = o\,(1)$

and further $\{s(n) - s(n-1)\}$ is Borel-summable to l. The same questions are also treated for 'quasi-Hausdorff methods for the class of bounded sequences, [cf.: Kuttner, *Proc. London Math. Soc.* 6 (1956); Ramanujan, *Proc. Nat. Inst. Sci. India* 24A (1958)].

The proof makes use of an idea due to Ramanujan [*Quart. J. Math.* 8 (1957)] that the properties of Hausdorff and quasi-Hausdorff methods are closely related to the corresponding properties of the Euler-Knopp and Taylor methods respectively. The following results are obtained for conservative methods (H, μ), (H^*, ν).

I. *If* $s \in F(l)$ *is summable by* (H, μ) *and* $\lim \mu_n \neq 0$, *then* $s(n) - s(n-1) \rightarrow l$.

II. (H, μ) *is absolutely regular for* $s \in F(l)$ *if and only if either* $s(n) - s(n-1) \rightarrow l$ *or* $\lim \mu_n = 0$.

III. (H, μ) *is translative in the wide sense for all* $s \in F(l)$.

IV. (H, μ) *sums (i) almost all, or (ii) NO divergent sequences of* 0*'s and* 1*'s according as* $\lim \mu_n$ *is zero or non-zero.*

V. *Multiplicative Hausdorff methods are absolutely regular for* $s \in F(0)$ *if absolutely regular in the wide sense for* s.

VI. *Statements* II, IV *and* V *are true for quasi-Hausdorff methods also; for these methods, statements* I, III *hold for bounded sequences.*

M. R. Parameswaran, Madras. *A Tauberian theorem for oscillation of sequences.*

Let $s^\alpha = \{s_n^\alpha\}$, $As = (1-x) \sum_{n=0}^{\infty} s_n x^n$ denote the Cesàro-transform (of order α) and the Abel-transform respectively of the real sequence $s = \{s_n\}$. Let osc. s, osc. As denote the oscillations of $\{s_n\}$, As as $n \rightarrow \infty$, $x \rightarrow 1 - 0$ respectively. In this note is proved the

Theorem. *Let s be a real sequence and let α, β be real numbers such that* osc. $As^\beta < \infty$ *and*

$$\lim_{\delta \rightarrow +0} \liminf_{n \leqslant m \leqslant n(1+\delta)} \frac{s_m^\alpha - s_n^\alpha}{\delta} \geqslant 0$$

Then osc. s^{α} = osc. $As < \infty$.

This generalizes V. Ramaswami's theorem [*J. London Math. Soc.* 10 (1935)] which has $\alpha = \beta = 0$. S. Minakshisundaram [*J. Indian Math. Soc.* 3 (1938-39)] has given an interesting proof of this special case. This note adapts Minakshisundaram's method, with a proof simplified by the use of " product theorem " for oscillations :

Osc. $As < \infty$ *implies* osc. $A\,(Hs) \leqslant$ osc. As *where* H *is any regular positive Hausdorff method.*

M. S. Ramanujan, Aligarh : *On the " Total translativity " of Hausdorff Methods.*

Among the well-known Cesàro, Hölder and Euler methods it is known that only the Cesàro methods are totally translative. Considering a wider class of methods, viz. the Hausdorff methods which satisfy Conditions A of Kuttner [*Proc. London Math. Soc.* (3), 6 (1956), 117-138] the following result is proved.

Theorem. *Let* $\lambda = (H, \mu_n)$ $(\mu_n \neq 0$, *for all* $n)$ *define a multiplicative Hausdorff transformation which satisfies Conditions A of Kuttner. Then the method is totally translative if, and only if, it is the Cesàro's methods or a multiple of it by a scalar.*

The proof consists in an application of Hurwitz's condition for the triangular matrices to be totally regular and thus getting the most plausible form of the μ_n making the method totally translative.

S. R., Sinha, Allahabad. *On the non-absolute Summability (A) of the conjugate series of a Fourier series.*

Plessner proved [*Mitt. Math. Sem. Univ. Giess.* 10 (1936) 1-36] the following theorem for the conjugate series, when the conjugate function exists as a Cauchy integral :

Let $V(x, \theta) = \sum_{1}^{\infty} (b_n \cos n\theta - a_n \sin n\theta)\, x^n$, $(0 \leqslant x < 1)$.

Then, if for any θ, *the condition*

$$\int_0^t \psi(t)\, dt = o(t), \text{ as } t \to 0$$

is satisfied then

$$\lim_{x\to 1}\left[V(x,\theta)-(1/2\pi)\int_{\epsilon}^{\pi}\psi(t)\cot(t/2)\,dt\right]=0.$$

The object of the present paper is to obtain an analogue in the following form, of the above theorem of Plessner for the case of non-absolute summability (A).

THEOREM 1. *If* $\psi_1(t)\in BV\,(0,\pi)$, *then*

$$\left[V(x,\theta)-(1/2\pi)\int_{\epsilon}^{\pi}\psi(t)\cot(t/2)\,dt\right]\in BV_x \text{ in } (0,1).$$

An evident conclusion from the above theorem will be the following

THEOREM 1 (a). *If* $\psi_1(t)\in BV(0,\pi)$, *then the non-bounded variation of the conjugate function is a necessary and sufficient condition for the non-absolute summability* (A) *of the conjugate series of a Fourier series.*

MISS SULAXANA KUMARI, Gorakhpur. *On the Riesz summability of Fourier series.*

Supposing that $f(\theta)$ is integrable (L) over $(-\pi,\pi)$ and periodic outside this range with period 2π, and $\phi(t)=\frac{1}{2}\{f(x+t)+f(x-t)-2s\}$,

$$\phi_\alpha(t)=\frac{a}{t^\alpha}\int_0^t(t-u)^{\alpha-1}\phi(u)\,du,\ \alpha>0;\ \lambda(w)=\exp\{(\log w)^{1+\beta}\},\ \beta>0;$$

$$f(\theta)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)=A+\sum_{n=1}^{\infty}A_n(\theta).$$

The following summability and convergence criteria for the above Fourier series of $f(\theta)$, at $\theta=x$, have been obtained in this paper.

THEOREM 1. *If for* $\alpha>0$, $\beta>0$,

(i) $\phi_\alpha(t)=O\left\{\left(\log\frac{1}{t}\right)^{-\alpha\beta}\right\}$, *as* $t\to 0$, *and*

(ii) $\int_0^t |\phi(u)|\, du = o(t)$, *as* $t \to 0$, *then the Fourier series of* $f(\theta)$, *at* $\theta = x$, *is summable* $(R, \lambda(w), \delta)$, *for every* $\delta > 0$, *to the sum* s.

THEOREM 2. *If the conditions* (i) *and* (ii) *are satisfied, and if*

$$A_n(x) > -k\, n^{n-1}(\log n)^{\beta}, \text{ for } k > 0,$$

then the Fourier series of $f(\theta)$, *at* $\theta = x$, *converges to the sum* s.

Theorems 3 and 4 give analogous results for the case of conjugate series.

C. B. L. VERMA, Jabalpur. *On a relationship between the Laplace transform and generalized Laplace transform of a given function.*

A large number of theorems, in which any two functions are connected with each other through a chain of relations under Laplace transform and/or its generalizations have been obtained by various authors from time to time. But it is also interesting to observe the relationship that exists between the Laplace transform $\phi(p)$ and a generalized Laplace transform $\psi(p)$ of a given function $f(t)$. In this paper an attempt has been made to investigate this relationship in the case of the Laplace transform and its generalizations as given by Meijer and Verma, and with the aid of the theorems established, some line integrals involving Bessel and hypergeometric functions have been evaluated.

C. B. L. VERMA, Jabalpur. *On a property of generalized Laplace transform involving Meijer's G-Function.*

In this paper is proved a theorem on generalized Laplace transform defined by Meijer in the form

$$\phi(p) = p \int_0^\infty e^{-\frac{1}{2}px}(px)^{-k-\frac{1}{2}}\, W_{k+\frac{1}{2},m}(px)\, f(x)\, dx\, ;\; R(p) > 0.$$

Connecting this transform, with the classical Laplace transform the author has utilized the theorem to evalute some infinite integrals involving product of Meijer's G-Function and other hypergeometric functions.

The theorem deduced is the following :

If $\psi(p)$ is the Meijer transform of $x^{l-2}\, h(x)$ and if $h(p)$ and $\phi(p)$ are the Laplace transforms of $x^{l-2k-3}\, \phi(x)$ and $f(x)$ respectively then

$$\psi(p) = p^{-2k-1} \int_0^\infty G_{2,3}^{3,1}\left(x \,\middle|\, \begin{matrix} 2k-l+1, 0 \\ -1, k+m, k-m. \end{matrix}\right) x f\left(\frac{x}{p}\right) dx,$$

provided $R(p) > 0$, $R(l-2k) > 0$, $R(l-k \pm m) > 0$ and the integral involved is convergent.

GEOMETRY

Sahib Ram Mandan, Kharagpur. *Tetrads of Moebius tetrahedra.*

The present paper is one in continuation of the two published lately [*Amer. Math. Month.* 64 (1957), 471-78, 65 (1958), 247-51] by the author and deals with nets of quadrics associated with tetrads of Moebius tetrahedra. There the treatment is synthetic, while here it is analytic. A good work has been done by Edge [*Proc. Lond. Math. Soc.* (2), 41 (1936), 338-60] for such a net associated with a pair of Moebius tetrads.

We may indicate here the main topics with which the paper is concerned. It is divided into three sections. In §1 it is shown, after giving a simple algebraic confirmation of some known results, that there exist two conjugate tetrads of Moebius tetrahedra.

In §2 eighteen nets of quadrics are established as related to the two conjugate tetrads of Moebius tetrahedra, nine nets for either tetrad of them. It is also shown there how they are interrelated. Associated Plücker surfaces are also mentioned here.

In §3 two reciprocal sets of thirty six each new associated quadrics are discovered and their distribution in the above nets is shown.

Sahib Ram Mandan, Kharagpur. *On four intersecting spheres.*

The following results have been arrived at :

The radical tetrahedron of four intersecting spheres coincides with a diagonal tetrahedron of the desmic system of intersection of those spheres. The pairs of opposite vertices of the system referred to this tetrahedron form pairs of conjugate points for the orthogonal sphere of the given four spheres and are the centres of similitude of the tetrad of associated spheres.

The planes of perspectivity of the eight pairs of complementary tetrahedra of intersection of four intersecting spheres form two tetrahedra desmic with their radical tetrahedron and are the radical planes of the corresponding pairs of complementary spheres of intersection, and are the planes of similitude of the associated tetrad of spheres.

The diagonal tetrahedra of either desmic system of intersection of four intersecting spheres form the other desmic system of intersection of those spheres.

The eight centres of similitude (other than the orthogonal centre of four intersecting spheres) of the eight pairs of complementary spheres of intersection of the four given spheres form two tetrahedra desmic with their central tetrahedron and thus form a set of eight associated points.

The desmic system of intersection of four intersecting spheres is inscribed in the one conjugate to *that of centres* for those spheres and reciprocally their other desmic system of intersection is circumscribed to that of centres for them.

The perpendicular from the orthogonal centre of four intersecting spheres upon the Newtonian plane of their associated tetrad of spheres passes through the circum-centre of their radical tetrahedron.

Each sphere of anti-similitude of the associated tetrad of spheres is orthogonal to eight of the spheres of intersection and to two of the four given intersecting spheres.

In the end, corresponding results for four mutually orthogonal and real spheres are deduced.

Finally, umbilical projection is suggested as a process to get these results rather quickly.

SAHIB RAM MANDAN, Kharagpur. *Harmonic Inversion.*

The purpose of this paper is to introduce the idea of Harmonic Inversion w.r.t. a pair of complementary sub-spaces in an n-dimensional space and then deduce a number of properties as an immediate consequence of the definition of this operation, e.g. a pair of mutually self-polar simplexes invert harmonically into another such pair. The existence of Moebius simplexes, that are mutually interlocked, i.e. inscribed as well as circumscribed to each other, is established in a space of an odd number of dimensions. Successive inversions w.r.t. the vertices of a simplex and its respective opposite prime faces form a cycle, whereas all the inversions w.r.t. all the pairs of opposite elements of a simplex together with identity form a group leading to a set of 2^n associated points such that any quadric variety, for which the simplex is self-polar, passing through any one of these points, passes through all of them. Finally it is shown that all the reflections, w.r.t. all the axial elements of a rectangular system of axes, together with the inversion w.r.t. the origin of the system and identity form a group.

MRS. NIRMALA PRAKASH and RAM BEHARI, Delhi. *Parallellism of Vectoroids.*

In this paper the parallelism of a bundle of vectoroids $U = \{u^A\}$ along a curve $C = C(t) \{C \in M_n\}$ has been defined and the condition for a set of vectoroids to be a parallel field has been obtained. It is shown that parallelism is preserved even when a new connection M^A_{Bi} given by the relation $M^A_{Bi} = L^A_{Bi} + 2\delta^A_B \mu_i$ is introduced in the bundle.

Further, by considering the set of closed curves at an arbitrary point $P \epsilon M_n$ as the set of automorphisms which map the set of vectoroids onto itself by parallel displacements, Holonomy groups for vectoroids have been obtained.

S. C. SAXENA, Delhi. *Generalized Riemann space and unified field theory.*

S. C. SAXENA and RAM BEHARI, Delhi. *Special types of Käehlerian manifolds.*

APPLIED MATHEMATICS

G. BANDYOPADHYAY, Kharagpur. *Interrelation between two rigorous solutions in unified field theory.*

This note establishes a method of passing over from a certain rigorous solution of a restricted type to another special rigorous solution of more general type. The process has its counterpart in physical interpretation. The paper also brings out a peculiar consequence of unified theory, viz. that the magnetic field which can be superimposed on a certain gravitational and electric field has an upper bound.

O. P. BHUTANI, Kharagpur. *Viscous flow through pipes (elliptic).*

In the first part of the paper an exact solution of pulsating laminar flow superposed on the steady motion in an elliptic pipe is presented under the assumption of parallel flow to the axis of the pipe. The asymptotic expressions for the velocity distribution have been obtained for the extreme values of the frequency of the pressure gradient. For this type of flow it has been found that the ratio of the total mean mass flow for the elliptic and circular cross-sections having the same area remains as in the case of steady flow.

In the second part, making use of the Laplace transform the Navier Stooke's equation and the equation for the temperature distribution for one dimensional unsteady flow in the case of natural heat convection inside the vertical tube, have been transformed to standard Mathieu equations and the contour integral involving Mathieu functions have been solved completely.

MADAN MOHAN GAIND, Kharagpur. *A two dimensional mixed boundary value problem of elasticity for inverse of an ellipse.*

The stress distribution in a thin elastic plate in the form of an inverse of ellipse subject to mixed boundary conditions is solved by reducing the problem to the solution of a non-homogeneous Hilbert problem which determines a sectionally holomorphic function having given lines of discontinuity.

P. C. JAIN, Delhi. *Isotropic temperature fluctuations in isotropic turbulence.*

In stationary, homogeneous and isotropic turbulence in an incompressible fluid where there is no over-all heat transfer, the theory due to S. Chandrasekhar [*Proc. Roy. Soc. Ser. A* 229 (1955)] of taking correlations at two points and at two different times, has been used to obtain a differential equation in temperature fluctuation $m(r, t)$ in terms of $Q(r, t)$—the defining scalar of the second order isotropic velocity correlation tensor Q_{ij}. This equation and the differential equation in Q obtained by S. Chandrasekhar (*loc. cit*). are the basic equations of the present paper. The equation in $m(r, t)$ and $p(r, t)$ has been solved for the special cases of very small and very large Peclet numbers and the results so obtained have been compared with those given by S. Corrsin [*J. Aero. Sci.* 18 (1951)]. An attempt has also been made to solve the basic equations in a very special case.

J. N. KAPUR, Delhi. *The internal ballistics of a supergun.*

In a recent paper, Jain and Sodha [*Appl. Sci. Res. Sec. A.* 7 (1958), 69-74] have discussed the internal ballistics of a supergun (hoch-druck pumpe) designed by Coenders [*German Research in*

World War II (New York) 1947, p. 191] during the second world war. In the present paper, it has been shown that their theory is the same as that for the burning of moderated charges in an orthodox gun [Kapur, *Proc. Nat, Inst. Sci. India* 22A (1956), 73-92 and 24A (1958)] and that their implicit assumption that the powder in the rth chamber is ignited only when the powders in the first $(r-1)$ chambers have been completely burnt out is not likely to lead to an efficient supergun. An alternative theory without this restriction and including their theory as a particular case has been discussed and its relation with the general theories of composite and moderated charges [Kapur, *loc. cit.* and same journal 22A (1956), 63-81, 23A, (1957) 469-482] has been given.

R. Manohar, Aligarh. *Pointwise bounds for the solutions of certain boundary value problems.*

Diaz and Greenberg [*J. Math. Phys.* 27 (1948) 193-201] find upper and lower bounds for the solution of a biharmonic boundary value problem analytically using Schwartz inequality. On the other hand pointwise bounds can also be obtained by using the method of hypercircle of Synge [*The hypercircle method in mathematical physics*, Cambridge (1957)]. It has been shown that the bounds obtained by hypercircle method are closer than the bounds given by Diaz and Greenberg. However, it is possible to improve the results of Diaz and Greenberg. No doubt the hypercircle method is more general but for all practical purposes the results obtained by either methods are same. Applications of these methods to other boundary value problems yield same sets of inequalities for the particular problem considered. For the derivaties also, the bounds determined by the two methods are the same.

R. S. Mishra, Gorakhpur. *A Study of Einstein's equations of unified field—II.*

R. S. Nanda, Kharagpur. *Steady canal flow with suction or injection.*

Heat transfer by laminar flow of a viscous incompressible fluid with suction or injection in the case of flow through a straight channel is considered. Exact solutions of the Navier-Stooke's equations and the energy equations have been obtained. Two types of problems are considered, one in which the two plates are kept at different temperatures and the other in which one of the plates is insulated. It is found that the temperature at any point of the fluid increases as the suction velocity increases.

K. PADMAVALLI, Madras. *A note on a Poincare problem.*

In a paper which has been submitted for publication in the *Journal of the Indian Mathematical Society*, poincare problem where the boundary consists of two st. line segments at right angles to each other, had been studied. In the present paper the effect on the solution, of replacing this boundary by a closed continuous curve enclosing each of these line segments in a narrow strip of the plane, is investigated.

P. S. RAU, Tirupati. *On the curvatures of a dynamical trajectory.*

If $q^1(t)$, $q^2(t), \ldots, q^N(t)$, which are N functions of time t belonging to class C_2 represent the coordinates of a dynamical system with N-degrees of freedom whose kinetic energy is $T = \frac{1}{2} \sum_{ji} m_{ij}\, q_i\, q_j$, then the configurations of the dynamical system can be put in 1-1 correspondence with the points $q(t) = [q^1(t), q^2(t), \ldots, q^N(t)]$ of the N-dimensional Riemannian manifold whose metric is $m_{ij}\, dq^i\, dq^j$. Hence the problem of motion of the dynamical system under given forces is equivalent to that of the trajectory of a single generalized particle, viz. $q(t)$ on this manifold. The question arises : How are the curvatures of the trajectory related to the force under which the dynamical system describes the trajectory ?

If v represents the magnitude of the velocity vector $\dot{q}(t) = [\dot{q}^1(t), \dot{q}^2(t), \ldots, \dot{q}^N(t)]$, K_i is the ith curvature of the trajectory, $i = 1, 2, \ldots, N-1$, $Q = (Q_1, Q_2, \ldots, Q_N)$ is the force and D^iQ represents the ith covariant derivative of Q defined by affine transference along the trajectory, then the following results are established :—

1. *Component of Q along the first normal is $K_1 . v^2$.*

2. *Component of* $D^{r-1}Q$ *along the r-th normal is*

$$K_1 K_2, \ldots, K_r v^2; \quad r = 2, 3, \ldots, N-1.$$

S. K. SHARMA, Kharagpur. *Visco-elastic steady flow.*

Using a modified form of the stress-strain relations for visco-elastic materials, some problems on steady flow have been solved in a closed form. An extra normal effect lacking in both the Newtonian and the non-Newtonian approaches has been found. The analysis is shown to have application in the study of gels formed by lyophillic solutions, an extreme illustration of which is table jelly. The results are found to be in good agreement with experiments.

AVTAR SINGH, Kharagpur. *Stress distribution within transversely isotropic bodies of revolution bounded by one or two cones due to rotation or gravity.*

In this paper the problem of axially symmetric stress distribution within semi-infinite transversely isotropic solids bounded by one or two cones, due to (i) rotation about the axis of symmetry and, (ii) gravity, has been considered. The boundaries are taken to be stress-free and the displacements, within such solids, can be expressed in terms of two stress functions which satisfy two second order partial differential equations. These equations have been solved by the method of similarity solutions.

By various choices of the semi-vertex angles of the two bounding cones, various types of solids can be obtained, such as, circular plate of infinite radius and linearly varying thickness ; semi-infinite conical shell of linearly varying thickness, semi-infinite solid cone, semi-infinite solid, semi-infinite solid with conical depression or conical exclusion.

A. C. SRIVASTAVA, Kharagpur. *Flow of non-Newtonian fluids between two infinite plates—one rotating and the other at rest.*

The equations of motion for the flow of a non-Newtonian fluid between two infinite plates, one of which is rotating and the other is at rest, have been approximately solved. It is found that under certain conditions depending on the distance d between two plates and the angular velocity Ω, the plate experiences a suction, but if d is decreased and Ω is increased sufficiently, it experiences a thrust which increases with μ. This latter phenomenon is not exhibited by Newtonian fluids. For a particular liquid and speed of rotation the normal thrust on the non-rotating plate varies as d which agrees with the experimental results of Ward and Lord. The theory developed also explains some experimental results of Pooper and Reiner [*Brit. J. Appl. Phys.* 7 (1956), 452-453].

P. C. VAIDYA and K. B. SHAH, Ahmedabad. *Electromagnetic field of radially flowing radiation in an expanding universe.*

In this paper is presented a rigorous solution of the field equations of general relativity which has the following characteristics : (1) It is regular everywhere. (2) For $0 < r \leqslant R(t)$, it represents the field of electromagnetic radiation travelling radially away from the origin through a distribution of matter of non-zero density and pressure. (3) The boundary $r = R(t)$ is the wavefront of flowing radiation. (4) The field passes off continuously at $r = R(t)$ with the expanding universe of zero curvature, the pressure and density of the interior matter being continuous with the pressure and density of cosmic fluid. It is further found that the electromagnetic field in the radiation zone disappears as soon as one switches over from the expanding universe of zero curvature to flat universe for the background. This is due to the existence of a relation between the density of flowing radiation and the rate of expansion of the universe. This solution is quite distinct from either the Einstein-Straus solution [*Rev. Mod. Phys.*, 17 (1945), 120], or the McVittie solution [*Monthly No . Roy. Astron. Soc.* 93 (1933), 325], or the generalization of the latter for a radiating star [Vaidya and Shah, *Proc. Nat. Inst. Sci.*, India, 23 (1957), 534], in as much as the source of radiation in the present case does not produce a static gravitational field because when the electromagnetic field is switched off, the entire solution reduces to flat space-time.

D. N. Verma, Delhi. *A supplement to Allen's relaxation method of 'Blocking outward'.*

Introduction of 'skew block relaxation operations' is suggested with reference to the equation

$$\frac{d^2 y}{d x^2} + w(x) = 0,$$

corresponding to any given number of points of subdivision (in an equal-width mesh). These are defined by the property that they affect residuals at two symmetrically situated points only, in magnitudes whose sum is zero. A superposition of these operations forms a natural supplement to Allen's method of 'blocking outwards' the residuals, the two making into a systematic procedure for complete liquidation. An improvement of this method is also suggested, consisting in the "pooling inward" (towards the centre) of the residuals from the anti-symmetric pattern in which Allen's method leaves them, and then using just one skew block operation to achieve final liquidation of all residuals. The methods are compared with some other systematic relaxation and numerical solutions.

P. D. S. Verma, Kharagpur. *Deformation energy for hypoelastic materials.*

Using the conditions under which the deformation energy of the moving volume remains a scalar invariant of the stress tensor and the rate of strain tensor for an isotropic medium, an expression for the deformation energy for isotropic hypoelastic materials of grade zero has been found. It happens to be the same as in the ordinary or classical theory of elasticity.

STATISTICS AND PROBABILITY

S. R. Adke and S. W. Dharmadhikari, Poona. *Gain due to sequential sampling from gamma population.*

It is well known that the use of the sequential probability ratio test usually results in a considerable saving in the average number of observations required to reach a decision, over that required by the classical Neyman-Pearson test of the same strength. A. Wald has evaluated this gain when observations are drawn from a normal population with a known variance to test a simple hypothesis about the mean against a simple alternative. The present authors obtain corresponding results for the gamma population defined by

$$f(x, \sigma) = \exp[-x \mid \sigma]\, x^{l-1}/\sigma^l\, \Gamma(l).$$

Assuming that l is known, the test of a simple hypothesis on σ against a simple alternative is considered. It is shown that the percent gain in the number of observations is independent of l, the parameter specifying the gamma function. Numerical results are also presented.

A. K. BHATTACHARJI, Kharagpur. *A note on a stochastic model for dependent binomial events.*

The paper is concerned with the derivation of a stochastic process generated by a series of binomial trials which are not mutually independent. Starting with some postulates defining the stochastic model, the probability of success at an abrupt trial has been obtained using the method of generating functions. Dandekar's modified binomial distribution is found to be a particular case of the stochastic model considered.

M. V. JAMBUNATHAN, Mysore. *The use of repeated ogives in the computation of moments.*

The method of computing the moments of a frequency distribution employing repeated ogives or successive cumulations was first given by Hardy, and a modification of this method was suggested by Elderton (*Frequency Curves and Correlation*) wherein there is appreciable saving of arithmetical work. Subsequently Dwyer (*Annals of Math. Stat.*, IX, 1938) and the present writer (11*th Conference of the Indian Math. Soc.* 1939 and 3*rd Session of the Ind. Stat. Conf.* 1940) obtained symmetric formulae giving the moments in terms of the entries under a single column of cumulated frequencies. The multipliers needed in the formulae had to be *built up* step by step. This paper furnishes a new method of proof and also derives explicit expressions for the multipliers, namely,

$$a_j = (-1)^{j-1} \left\{ \binom{r-1}{j-1} \nabla O^r + \binom{r-2}{j-1} \nabla^2 O^r + \dots + \nabla^{r-j+1} O^r \right\}$$

The paper also brings out the interesting fact that the entries of the table required for the three different formulae constitute the three sides of a triangle. Hardy's formula uses the entries S_1, S_2, S_3, ..., along the horizontal side, Elderton's method uses entries along the sloping side consisting of S_1, ΔS_1, $\Delta^2 S_1$, ..., while the Dwyer-Jambunathan formula employs S_r, ∇S_r, $\nabla^2 S_r$, ..., which constitute the third side (the vertical side) of the triangle. The multipliers in the three cases are the ascending differences of zero, the descending differences of zero, and the set of coefficients a's given by the formula above. These three exhaust all possible "straight entry" formulae.

MRS. MARAKATHA KRISHNAN, Madras. *Approximations to the non-central F_1'-distribution.*

Suppose two independent variates $\chi_1'^2$ and $\chi_2'^2$ follow two non-central χ^2-distributions with ν_1 and ν_2 degrees of freedom and non-central parameters λ_1 and λ_2 respectively. Then the distribution of the ratio $F_1' = \dfrac{\chi_1'^2/\nu_1}{\chi_2'^2/\nu_2}$ has been obtained by P. C. Tang (*Statistica-Research Memoirs*, 2, 1938). Evaluating the probability integral and percentage points of the F_1' distribution involve a considerable amount of labour. Two approximate methods of evaluating the above probability integral are discussed here.

Method 1: The distribution of F_1' is approximated by that of an F-distribution with ν_1' and ν_2' degrees of freedom, multiplied by a scale factor k, both F_1' and kF having the same first three moments. ν_1', ν_2' and k can be got in terms of ν_1, ν_2, λ_1, λ_2; and the probability integral $\int_x^\infty p(F_1')\, dF_1'$ is approximately given by $\int_{x/k}^\infty p(F)\, dF$ which can be calculated using the tables on incomplete beta functions (K. Pearson, 1934).

Method 2: Using P. B. Patnaik's method of approximating a non-central χ'^2-distribution by a weighted χ^2-distribution (*Biometrika* 1949) in both numerator and denominator of F_1', another approximation of F_1' is given by the distribution of $gF(\nu_1'', \nu_2'')$, where the constant g, and the degrees of freedom ν_1'', ν_2'' can be got in terms of ν_1, ν_2, λ_1, λ_2; and the approximate value of the probability integral $\int_{x/g}^{\infty} p(F)dF$ can be evaluated.

On comparing the true values of the probability integral with the approximate values in certain cases, it is seen that the approximation is very close in both methods.

B. RAJA RAO, Poona. *A double inequality on Mills' ratio for the class of distributions admitting sufficient statistics.*

This paper originates from the author's investigation into the properties of the members of the class Ω of distributions admitting sufficient statistics. Precisely it establishes the monotonicity, and obtains a double inequality on Mills' ratio R_x for the continuous and differentiable distributions belonging to the class Ω. A very elegant inequality on R_x for this class is obtained and it is worth recording since it satisfactorily locates this ratio. The Beta distributions is treated in a more detailed form and the closeness with which the double inequality can locate R_x is studied by means of a table. Some recurrence relations and monotonicity properties of functions involving R_x for the Beta distribution are established.

A. B. L. SRIVASTAVA, Kharagpur. *The distribution of regression coefficient in samples from bivariate non-normal populations.*

Assuming the parent population to be represented by the bivariate Edgeworth surface, the sampling distribution of regression coefficient b_{21} has been derived by the method of characteristic function, and also the distribution of the t-statistic used for testing its significance has been obtained. The formulae of this paper give the corrective terms which can be used to show how the 'normal theory' values of mean, variance, probability points of b_{21} and the critical region of the t-test for b_{21} are affected by non-normality of the parent population if we have some idea of its third and fourth order semi-invariants.

M. N. VARTAK, Bombay. *Relations among the blocks of the Kronecker products of designs.*

TOPOLOGY

R. VENKATARAMAN, Madurai. *Symmetrically ordered sets.*

Adopting usual terminology, a class of ordered sets called symmetrically ordered is defined and characterized. If A is any ordered set of the ordertype of an ordinal number θ, the ordertype of the ordered set of all integer-valued functions defined on A, each of which has utmost a finite number of non-zero values, ordered according to "last differences", is denoted by $(\omega^* + \omega)^{[\theta]}$. Let I be an ideal of ordered set P. If I has an ultimate segment which is similar to the dual of an initial segment of the co-ideal determined by set complementation of I, I is said to have symmetric character. An ordered set, every proper ideal of which has symmetric character is called symmetrically ordered. That every symmetrically ordered set is a segment (viz. a subset which includes with every pair of its elements all intermediate elements as well) of the ordered type $(\omega^* + \omega)^{[\theta]}$, for a suitable ordinal, θ, is established by proving:

1. $(\omega^* + \omega)^{[\theta]}$, for θ any ordinal is symmetrically ordered.

2. If θ is any ordinal, the ordertype of any initial segment of it, is

$$A_\theta = \omega + (\omega^* + \omega)\omega + \dots + (\omega^* + \omega)^{[a]}\,\omega + \dots, \; a < \theta.$$

3. Every symmetrically ordered set with a first element is isomorphic to an initial segment of A_θ, for a suitable ordinal θ.

LIST OF DELEGATES

S. K. Abhyankar, S. R. Adke, Afzal Ahmed, Agarwal Bhagwandas, M. K. Agarwal, A. S. Apte, M. A. Apte, B. B. Bagi, V. K. Balachandran, Balagurunathan, T. J. Balvani, G. Bandyopadyay, S. P. Bandyopadyay, M. Bhaskaran, M. L. Bhatia, M. N. Bhat, P. L. Bhatnagar, B. R. Bhonsle, M. L. Chandratreya, G. L. Chandratreya, A. C. Choudhari, S. S. Cheema, J. Datta, J. De, Deshmukh, K. K. Deshpande, S. W. Dharmadhikari, D. K. Dhavan, V. Ganapati Iyer, J. M. Gandhi, M. M. Garde, H. G. Gharpury, Rev. A. Gonsalves, V. D. Gopalakrishnan, K. R. Gunjikar, H. Gupta, J. D. Gupta, O. P. Gupta, P. D. Gupta, G. K. Hebalkar, S. K. Hindi, V. S. Huzurbazar, Jaganatha Chari, R. K. Jaggi, P. C. Jain, V. V. Kale, K. M. Kamalamma, A. R. Kamat, L. S. Kamat, Kameswara Rao, S. C. Kanbur, D. D. Kapadia, D. R. Kaprekar, J. N. Kapur, L. N. Kaul, S. N. Kawalgikar, D. W. Kerkar, Khanna Girija, M. N. Khatri, W. F. Kibble, Kishan Rao, K. S. Krishnan, V. S. Krishnan, Mrs. Krishnan, N. R. Kulkarni, R. T. Kulkarni, V. Lakshmikanthan, M. M. Lal, S. Leelamma, I. S. Luthar, Mrs. Luthar, B. S. Madhava Rao, S. Mahadevan, Mrs. Mahadevan, Malik, S. S. Malurkar, S. D. Manerikar, R. Manohar, Markandeswar Rao, K. P. Mathew, A. N. Mehra, B. B. Mehra, N. K. Mehta, S. Minakshisundaram, R. S. Mishra, Mukunda Lal, P.S.V. Naidu, V. V. Narlikar, Mrs. Narlikar, Narayan Singh, Mrs. Nirmala Prakash, G. C. Nivas, B. Y. Oke, K. Padmavalli, J. N. Panda, M. Parameswara Iyer, M. R. Parameswaran, R. P. Paranjpye, J. N. Patnaik, N. H. Phadke, B. N. Prasad, M. Raghavacharyalu, B. Raja Rao, D. V. Rajalakshman, M. Rajagopalan, Ram Behari, J. Ramakanth, B. S. Ramakrishnan, S. Ramakrishnan, S. V. Ramamurti, M. S. Ramanujan, M. Ranganathan, P. V. Ranganathan, A. V. Rangarajan, P. S. Rau, M. Ray, Mrs. Ray, K. S. Reddy, T. R. Sahane, Sahib Ram, N. Sankaran, K. V. Sankaranarayanan, K. Savitri, S. C. Saxena, N. G. Shabde, C. C. Shah, K. B. Shah, K. M. Shah, K. N. Shah, Shamihoke, Shantinarayan, D. L. Sharma, H. G. S. Sharma, K. B. L. Shrivastava, J. A. Siddiqi, M. K. Singal, V. N. Singh, S. R.

Sinha, S. V. Sirdesai, B. S. R. Somayajula, P. R. Sreenath, Srinivasa Rao, A. K. Srinivasan, Mrs. Srinivasan, P. K. Srinivasan, C. N. Srinivasiengar, B. R. Srinivasan, M. V. Subba Rao, S. Subramanian, L. V. Subramanian, S. S. Subramanian, Sundar Lal, S. Swaminathan, S. Swetaranyam, P. Tiwari, Iqbal Unnisa, Wazir Unnisa, P. C. Vaidya, C. S. Venkataraman, R. Venkataraman, Mrs. Venkataraman, T. Venkatarayadu, P. S. Venkatesan, M. Venkateswara Rao, V. Venugopal Rao, C. B. L. Verma, P. D. S. Verma, B. Viswanathan, S. Viswanathan, P. N. Vijavrergia.

THE RECEPTION COMMITTEE

R. P. Paranjpye (*President*), M. V. Bhide (*Vice-President*), V. S. Huzurbazar (*local secretary and treasurer*), S. R. Adke, M. N. Bhat, W. H. Golay, G. K. Hebalkar, N. R. Kulkarni, B. S. Madhava Rao, S. V. Sirdesai (*members*).

Elements of Calculus. By Shaukat Abbas, Educational Book Depot, Hyderabad, W. Pak., xiv + 304 pp. Price Rs. 5/8.

THE book under review is meant to be a text-book for the Intermediate classes of the Pakistan Universities. It covers both differential and integral calculus. As is to be expected we do not find any novelty of treatment which could mark out this book from among those of similar scope.

From natural considerations the notions of limit and continuity are not seriously studied at the Intermediate level. The author has taken some pains to try to explain these notions, but in a way which does little credit to him. Thus we find him stating on page 13 "If we can find a small positive number ϵ such that $|x|$ can be less than ϵ, we say that 'x tends to zero' ". Again, in page 37, he states " Hence $x \sin 1/x$ is discontinuous at $x = 0$ ".

The book contains many solved examples. Defects and mistakes in printing are by no means wanting.

One feels on the whole that the author has made an earnest attempt to write a "good" book without however any remarkable success.

R. RAMACHANDRAN

Algebra. By J. W. Archbold, Sir Isaac Pitman & Sons, London, (1958) xix + 440 pp. sh. 45.

THE book under review is written mainly to meet the requirements of the B. A. and B.Sc. syllabus of the London University. The contents fall into three broad divisions : The first six chapters dealing with numbers, induction, summation of finite series, inequalities and complex numbers form a broad introduction to the following chapters. Chapters 7-14 are concerned mainly with polynomials, polynomial equations, factorization, rational functions, symmetric functions and the cubic and quartic equations. Chapters 15-25 (barring chapter 15 on determinants of orders 2, 3 and 4) contain the elements of linear and abstract algebra, under the heads: groups, ring and fields, vector spaces, matrices, rank, symmetric group, determinants of order n, characteristic equation some matrix types, quadratic forms and discriminants and resultants. The reviewer feels that the above could have been arranged in the more natural way as vector spaces, determinants of order n, matrices rank, etc. followed by the chapters on abstract algebra.

The book is well written and the treatment is, throughout, lucid and rigourous. There are many graded examples some of them chosen from among the mathematical notes appearing in the Gazette or Monthly enhance the usefulness of the book for further reading. The book contains a very good and exhaustive index. The author's efforts to bring within these 440 pages considerable matter of algebra, both classical and modern, is laudable.

The printing and get up are fine. The book should prove a useful addition to any library and a valuable companion to a mathematically minded reader.

M. S. RAMANUJAN

Proceedings of the Third Congress on Theoretical and Applied Mechanics. The Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, pp. 362.

THE Third Congress on Theoretical and Applied Mechanics was held from December 24 to 27, 1957 at the Indian Institute of Science, Bangalore, and the Proceedings under review consist of the papers presented at that Congress.

The proceedings are divided into the following three parts:

Part I: Elasticity, Plasticity and Rheology, Part II: Fluid Mechanics, Part III: Vibrations, Thermodynamics, Mathematics of Physics, Statistics and Computation, and contain in all thirtysix papers of which fifteen have been contributed by the visiting scientists from U.S.A., U.S.S.R., Poland, Hungary, Japan, Burma and Australia.

Part I starts with the Presidential address by Dr. S. R. Sen Gupta on some modern trends in the Design of frame struct res, theory of design based on plastic failure, and contains in all sixteen papers dealing with a wide variety of topics in Elasticity, Plasticity and Rheology.

Part II consists of nine papers on Fluid mechanics covering problems on ground-water flow along a plane impermeable base, turbulence, flows of visco-elastic and non-Newtonian fluids, shock waves, etc.

Part III contains eleven papers on a wide variety of subjects like stability criteria for and response function of forced vibrations in non-linear systems, solution of Fredholm integral equation of second kind, discrete models and matrix methods in engineering mechanics, theory of the synthesis of mechanisms for the reproduction of certain kinds of algebraic and transcendental curves, use of analogue computors to solve some elasticity problems, transport of heat by convection and boiling in liquids enclosed in vertical tubes.

The Proceedings, apart from giving the representative sample of the type of work which is engaging attention in India, contain papers which are of high academical value. The get-up is attractive and the printing is nice except for some misprints here and there. Prof. B. R. Seth (Executive Editor) and his colleagues on the editorial committee deserve congratulations for bringing out these in such an attractive form.

P. L. Bhatnagar

Nomography—By L. Ivan Epstein, Interscience Publishers, New York—(1958) pp. 134—Price $ 4.50

The author has taken pains to develop the subject from elementary level and has given the theoretical background in constructing the nomographic charts. The author himself has stated that there are other books on this subject dealing with the practical applications of these charts in industrial use; here the author has dealt with only basic principles of nomographic chart construction and the allied subjects to understand the same, viz. Determinants, Matrix method, Projective transformation, etc.

I feel that this kind of treatment of this subject will appeal only to a man who knows a bit of this subject, and for a beginner he will lack the precise approach, as he could not get at a glance what are the necessary conditions, etc. required to tackle a problem by the use of nomograms. Moreover the book is wanting in concrete illustrations of applied problems to impress upon the mind of a reader the usefulness of this subject. The author expects the student to do the spade work at many places and it is very likely that he may lose interest. As such I do not think it will be of much use to students who want to learn this subject from their practical point of view. The broad generalization of the theory behind may not have the catching effect on an untrained student mind.

No doubt the author deserves congratulations for the efforts he has taken to present the theoretical background to this subject.

R. THRIVIKRAMAN

Pure Geometry for Degree classes: *Parts I and II* By N. ch. Pattabhi Ramacharyulu, Light House, Agraharam, Eluru (1959); Part I 198 pp. Rs. 4/-; Part II pp. 88 Rs. 1.25.

THESE books are intended for the new three year degree course in mathematics for the South Indian Universities. Part I treats about ranges and pencils, Properties of triangles and circles, Inversion, Conic sections, Properties of Parabola and central conics. Part II deals with the elementary solid geometry of planes, solids, sphere, cylinder, cone, simple cases of conical and orthogonal projection.

The topics are dealt with in a clear manner and the book contains well chosen worked examples and exercises for students at the end of each chapter. The printing is good.

S. MAHADEVAN

Structure of Rings. By Jacobson, N. American Mathematical Society Colloquium Publications, vol. 37, pp. 263 (1956).

THIS book deals with the structure of rings which do not necessarily satisfy the chain conditions for one sided ideals. The author who is one of those chiefly responsible for the development of this new structure theory, has given in this volume a thorough-going treatment of the subject. The tools employed not only yield better insight into the older structure theorems, but also apply to a much wider class of rings. This theory is applied, in the last chapter of the book, for instance, to the study of algebras satisfying a polynomial identity. We give first a brief summary of the contents.

The first chapter introduces the notion of the (Jacobson) radical and (generalized) semi-simplicity. The notion of a primitive ring is also introduced.

The main result of chapter II is a generalization of a classical theorem of Burnside (on irreducible representations of a multiplicatively closed system). This theorem is first stated in algebraic language and later in a topological way (by introducing the familiar topology on a set of operators) which says that an irreducible ring of linear transformations is dense in the bicommutant.

In the third chapter, the author uses the new techniques to give short proofs of the classical theorems of Artin-Wedderburn (on semi-simple rings with minimum condition).

The aim of the fourth chapter is to prove a fundamental structure theorem of primitive rings with non-zero minimal one-sided ideals. This theorem asserts that such a ring is isomorphic to a dense subring of linear transformations of a vector space over a division ring, containing non-zero linear transformations of finite rank. Other equivalent formulations are also given. The question of uniqueness is also stated.

Chapter V introduces the notion of tensor product of modules and algebras. The problem of studying the structure of the tensor product of two algebras of known structure is treated. The Brauer group of similarity classes of central simple algebras is also introduced.

Chapter VI is devoted to semi-simple modules and Galois theory. Among various other things, several theorems on extensions or derivations and isomorphisms of algebras are proved.

Chapter VII specialises to division rings (which are not necessarily of finite rank over their centres). The Galois theory of such rings is considered. A proof of Wedderburn's theorem that every finite division ring is commutative and the Cartan-Brauer-Hua theorem (as also its analogue for derivations) are included.

In Chapter VII the author discusses several types of nil radicals, especially the upper and lower nil radicals of Baer. A very short proof of a theorem of Levitzki, that in a ring with maximum condition every nil ideal is nil potent, is given.

In Chapter IX, a natural topology is put on the set of primitive ideals of a ring. The problem of representing a ring as a ring of continuous function on a certain topological space is considered.

The final chapter deals with some applications of the preceding structure theory. First, several theorems (e.g. a theorem of Herstein, which generalizes a theorem of the author) on the commutativity of certain types of rings are proved. Next, Kaplansky's solution of the analogue of the Kurosch problem for algebraic algebras with polynomial identities is presented.

The presentation of the material is masterly. Though the book is almost self-contained, the mathematical maturity expected of the reader is much more than that of a new comer to this field. The author has given very illustrative examples and they lie scattered all through the book. But in several places, the author has preferred to omit the motivation.

The reviewer found several minor misprints, none of them serious. There is a small error in one of the examples that follow Chapter I. In (1)(*b*), the author defines the equality $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. This will not be transitive if the ring has zero divisors. The correct equality should read: $\frac{a}{b} = \frac{c}{d}$ if and only if, there exists a $t \notin Y$ such that $adt = bct$.

There is no doubt that this book will be of immense help to workers in this field and will be an outstanding reference book for several years to come.

R. SRIDHARAN

Mathematics of Engineering Systems (*Linear and Nonlinear*): By Derek F. Lawden, Methuen & Co. (1954) pp. 380, 30 sh.

OF this book the author says in his preface, " This volume gives an account of a number of mathematical methods which may be used to analyse the behaviour of a large diversity of physical systems. These methods, and their applications were found widely dispersed amongst texts dealing with many branches of engineering, research reports and papers. They have been gathered together in this book and developed in logical sequence to form a modern course in applied mathematics, suitable for students in electronics, electrical engineering, applied physics and instrument technology. "

The book has five chapters. The first, an introductory chapter, " revises the more important results (of advanced calculus?) and derives others which are required for use in the later chapters ". In 50 and odd pages, the author very rapidly touches on practically all the topics included in Hardy's *Pure Mathematics* !

Chapters 2 and 3, covering roughly half the book, deal with linear differential equations with constant coefficients. The former, subtitled " classical methods ", introduces the operator notation and develops the usual methods for obtaining the solutions in terms of the roots of the characteristic polynomials. After a brief mention of systems of such equations, the notion of the stability of linear systems is introduced and the well-known Hurwitz's criteria for stability are given. Chapter 3, subtitled " modern methods ", treats the same problem of the analysis of linear systems in terms of their response to the unit step function, the unit impulse and sinusoidal functions. The major part of the treatment is confined to the steady state analysis and the stability problem is again touched on, this time, from the point of view of the Nyquist diagram.

The chapter ends with a very sketchy introduction to the use of Laplace transform methods in this context.

Chapter 4 deals with Fourier analysis and summarises the main results in about 50 pages and contains numerous worked examples. The chapter is again a very rapid survey of nonlinear differential equations. The method of isoclines and the perturbation method are outlined and brief references are made to limit cycles, subharmonic oscillations, hard and soft oscillators and other notions of nonlinear theory.

The major drawback of the book, that strikes one even on a cursory first reading, is the inefficient allocation of space to the various topics. The bulk of chapter 2 would seem to be totally unnecessary. A unified treatment with the use of Laplace transforms right from the outset should have enabled the author to develop a more elegant and certainly more coherent picture of all the topics touched on in chapters 2 and 3, in substantially the same number of pages. Such treatments were, of course, already available when the present book was published; and although the author has included them in his bibliography, it is a pity, that he should have chosen to ignore them elsewhere in the treatment of his material. (It must be said, that in general, the bibliography which the author appends to the chapters are more reassuring than the chapters themselves.) Also, one feels, that the chapter on Fourier analysis could have been left out with profit and its space utilized for a more thorough and less hurried development of nonlinear differential equations—at least of the two dimensional autonomous systems.

By far the best aspect of the book, as it stands, is the variety of worked examples, mostly taken from the fields of electronics and servo systems. Some of them, especially the one on oscillators in chapter 2 and that dealing with Van der Pol's equation in chapter 5, have been developed in great detail and deserve special mention. Because of this, for those interested in the analysis of these specific problems, the book should be of great interest. Otherwise, in the reviewer's opinion, the merit of the book lies in *what* it refers to (by way of topics) rather than in *how* it deals with them.

R. NARASIMHAN

Tauberian theorems. By H. R. Pitt. Oxford University Press, (1958). pp. 10+174 Rs. 22.50.

THIS book is the second of a series of monographs being published under the auspices of the Tata Institute of Fundamental Research. Although some of its subject matter has been dealt with in other books, such as Hardy's *Divergent series*, the book contains much material which has hitherto been available only in the original papers. While a number of Tauberian theorems on particular kernels (such as the Cesàro or Abel kernels) are given, the emphasis throughout is on general Tauberian theorems. Indeed, so far as Tauberian theorems for Cesàro or Abel summability are concerned, the subject is in some respects more fully dealt with in Hardy's book; it is in its treatment of the general theorems that the essential value of the present book lies.

Chapter I is introductory in nature. Chapter II ("elementary Tauberian theorems") deals, broadly speaking, with those general Tauberian theorems which can be proved without the use of Fourier transforms or complex function theory. The first section deals with Tauberian conditions as such, and with inclusion relations between classes of functions satisfying different Tauberian conditions; in the remaining section, the Tauberian theorems are obtained. This chapter contains one or two minor inaccuracies. Thus the definition of the Tauberian class T given on page 7 requires that there should be a $\delta = \delta\ (\epsilon, x)$ defined for all x and $\epsilon < 0$ such that (inter alia) $\delta < 0$ [equation (2.1.6)]; but in proving (in Theorem 2) that the Tauberian class S is contained in T, we take $\delta = 0$ for $x < X(\epsilon)$. Of course, since we are concerned mainly with what happens when $x \to \infty$, this cannot be regarded as more than a point of detail.

A point which seems worth mentioning, since a reader might well be misled, concerns Theorem 10 of this chapter. It would be *natural to take the enunciation* of the theorem as requiring that the stated conditions should hold for an *any* $\epsilon > 0$. But in proving the theorem, we take *some* $\epsilon > 0$ and keep it fixed throughout. Thus we need suppose only that the conditions are satisfied for some particular $\epsilon > 0$. This remark applies also to the assumption that $s(v)$ belongs to T; we do not need the full force of assumption, but require only that the equations defining the class T should be satisfied for the particular value of ϵ considered. These remarks are of importance because of the assertion (made without proof) that Theorem 9 may be deduced from Theorem 10. It may be proved (though it is by no means obvious) that, if the conditions of Theorem 9 are satisfied and if we put

$$k(u, v) = \begin{matrix} c_n(u) \\ 0 \end{matrix} \quad (n < -v < n+1) \ (v < 0),$$

then the conditions of Theorem 10 are satisfied for a suitably chosen ϵ. But it is not necessarily true that they are satisfied for *any* $\epsilon > 0$. Thus Theorem 9 follows from Theorem 10 only if Theorem 10 is taken in the sense indicated.

In Chapter III ("Classical Tauberian theorems") the "standard" theorems for Cesàro, Abel and Borel summability are obtained. The treatment of these has some features of interest; use is made, as far as possible, of the general theorems of Chapter II, whereas in the past, except when Wiener's theorems have been used, *adhoc* proofs have usually been constructed for each particular method.

Chapter IV ("Wiener's theory") and Chapter V (" Mercerian theorems ") constitute the most important part of the book. Together, they may be described in general terms as dealing with the applications of Fourier transforms to summability theory. Since the pioneering work of Wiener in this field, much has been done in extending and generalizing Wiener's results, and in simplifying his proofs. The time was therefore ripe for the material of the various original papers to be gathered together in a convenient form. With the exception of Wiener himself, Pitt has probably done more than any other one man in the development of the theory and he was thus particularly fitted for this task. It would have been impossible, without expanding the book unduly, to put in everything that has been done in this field; but all the more important results are here. These two chapters reach a high standard of accuracy and a careful study of them has failed to reveal any mistakes more serious than a small number of minor misprints.

Chapter VI ("Tauberian theorems and the prime number theorem") deals with the various proofs of the prime number theorem. The first section deals with Ikehara's theorem. This is of interest for its own sake, and its extensions are therefore carried further than is required for the proof of the prime number theorem. There is an unfortunate slip in the proof of Theorem 7. Defining

$$F(u) = 2 \int_0^\infty e^{-t^a} \cos ut \, dt,$$

where $a > 1$, it is asserted that "It is plain that $F(u) > 0$". The result that $F(u) > 0$ for all u is, however, false if $a > 2$. Its falsity may be proved so simply that I give a proof here. Since sufficient conditions for the validity of Fourier's integral theorem are clearly satisfied, we have, for $t \geqslant 0$,

$$\frac{1}{\pi} \int_0^\infty F(u) \cos ut \, du = e^{-t^a},$$

and hence

$$\frac{2}{\pi} \int_0^\infty F(u) \, (1 - \cos ut)^2 \, du = \frac{1}{\pi} \int_0^\infty F(u) \, (3 - 4 \cos ut + \cos 2ut) \, du$$
$$= 3 - 4 e^{-t^a} + e^{-(2t)^a}$$

If we expand the expression on the right in powers of which the first term is $(4-2^a)\, t^a$, which is negative if $a > 2$. Thus the expression itself is negative for sufficiently small t; whereas if it were true that $F(u) > 0$ (or even that $F(u) \geqslant 0$), the integrand on the left would be everywhere non-negative. If $a \leqslant 2$, however, the assertion that $F(u) > 0$ for all u is true (though certainly not trivial). Thus the proof of Theorem 7 is, as it stands, valid only in the cases $0 < \alpha \leqslant \frac{1}{2}$ or $\alpha = 1$. (This remark does not, of course, imply that the *result* is necessarily false in the other cases.) Fortunately, no use is made of this theorem elsewhere in the book.

The next section deals with the various " classical " proofs of the prime number theorem. One would have welcomed a rather fuller treatment of Ingham's method, but no other adverse criticism could be made. A final, and very interesting, section deals with Selberg's " elementary " proof of the prime number theorem, and with the related Tauberian theorems.

This book is likely to come to be regarded as the standard work on the subject. While it is logically complete in itself, a student with no previous knowledge of Tauberian theorems might find it difficult reading; but for anyone aspiring to original work in this field, this book is really essential.

B. Kuttner

Introduction to the Physics of many-body systems, by D. Ter Haar, Interscience Publishers, New York 1 (1958), pp.viii + 125, $ 1. 95.

It has been long realised that the problem of nuclear structure and phenomena connected with it is essentially a many-body problem. In atomic structure also where we have to deal with systems involving many electrons there was a great simplification in that the electrons move in a common external field of the nucleus and the interaction between these electrons is treated as a perturbation. In the nucleus however the interactions between the nuclei are very strong and there is no common potential. It is only recently that methods have been devised using equivalent potentials and the problem has become tractable numerically. Thus interest in the physics of many-body systems has received a new impetuous especially with the recent successes of the Brueckner theory.

This monograph gives a brief survey of this rapidly expanding field with emphasis on the diversity and range of the problems engaging the attention of physicists rather than on detailed discussions and derivations. It is divided into two parts representing roughly two modes of reducing the system of a large number of interacting particles to a system of non-interacting or weakly interacting particles—a procedure necessary to evaluate the physical properties of the system.

The first part deals with the effective field theories and the second with theories of collective behaviour. In Chapters 2 and 3 of Part I is given a brief account of the Hartree-Fock theory of the self-consistent field, the statistical model of the atom and their relative merits. Chapter 4 deals with the adaptation of the Hartree-Fock theory to nucleus and a brief reference to the importance and success of Brueckner's theory. Chapter 5 deals with the effective mass approximation according to which a particle of mass m interacting with a field can be replaced by an equivalent free particle called quasi particle of mass m_0. Following this there is a discussion of other quasi particles like excitons and polarons that occur in solid state physics.

In Part II systems exhibiting collective behaviour are described. The methods of Tomanaga, Skinner, Yevick and Percus, Zubarev and of Bohm and Pines have been discussed. A special stress is laid on Tomanaga's method as it is applicable to a large number of cases. The treatment is classical and the method of quantization is also indicated. The application of these methods to sound waves in gases and crystals, electron plasmas, nuclear collective behaviour and liquid helium are dealt with in later chapters.

It would have been desirable if more attention had been paid to the recent work of Brueckner, Bethe, Goldstone and others. The book fulfils the main object of the author which is essentially to give a broad survey of a wide field " high-lighting the main points, indicating the trend of recent developments, and referring for more detailed discussions to the literature. "

ALLADI RAMAKRISHNAN

Modern Geometrical optics. By M. Herzberger Pure & Applied Mathematics Series-Vol. VIII, New York, Interscience (1958), pp. x + 504, $ 15.00.

THIS book is an attempt at providing a systematic presentation of the mathematical theory of geometrical optics and as the author says in his preface, it is "the result of more than fourteen years' continuous labour". To the average physicist who generally skips over geometrical optics and devotes himself only to a detailed study of physical optics, this book would come as an eye-opener. Not only are the mathematical techniques in this field both elegant and varied, but the physical laws also bear a close relation to other fields of study such as mechanics. In fact, Dr. Herzberger has in this book succeeded in transforming geometrical optics from an empirical field of study to an exact science. As he has himself stated, lens design has generally been in the past more an art than a science. The blame for this must be laid as much upon the average designer whose familiarity with higher mathematics was very limited as upon the physicist who rarely tried to interest himself in it from a theoretical point of view.

Dr. Herzberger has attempted in this book to develop "a mathematical model of an optical system that is complex enough so that all the characteristics of the geometrical optical image can be obtained from it". For doing this, he had necessarily to develop a calculus suited for the purpose. The book is divided into seven parts and concludes with an appendix containing tables and a brief historical survey. Parts I and II deal with the propagation of a single ray through an optical system while in Part III is discussed the theory of a manifold of rays based on the basic formulation of Hamilton and Lagrange, which contain within them the famous Fermat principle of least action. One wishes that the author had tried to give an account of the relationship between these optical laws and the corresponding laws in mechanics, at least in an appendix. The general laws of image formation are then considered, in particular for systems possessing an axis of symmetry. Finally, the theory of higher order aberrations, mainly developed by the author, are considered in detail. There is a short mathematical appendix giving the essentials of vector analysis, tensors, matrices and method of least squares which finds application in the book.

There is no doubt that this book will stimulate much original work in the theory of geometrical optics and it seems to be appropriate that it should be published as a monograph in the series on Pure and Applied Mathematics. A few minor deficiencies must, however, be pointed out. The notation used for vectors (viz. $\vec{r}$) and their products (e.g. $\vec{r}\ \vec{s}$ for dot product) differs completely from what has come to be the standard usage now-a-days (e.g. **r** and **r**. s, etc). As a consequence, the reviewer at least found considerable difficulty in automatically following the book and he had to pause at each stage to obtain the significance of the formulae concerned. There is an extensive bibliography, but surprisingly some work on concentric systems and on higher order errors in X-ray microscopes using Herzberger's own techniques do not find a place there (e.g. 'Theory of Image-formation in Combinations of X-ray focussing Mirrors' by Y.T. Thathachari, Proc. Ind. Acad. Sci., *A*, 37, 14(1953).

These are, of course, minor blemishes in what is otherwise an excellent attempt to systematize the body of knowledge comprised under Geometrical Optics into a coherent system. The book is warmly recommended to all libraries in Physics and Applied Mathematics.

G. N. RAMACHANDRAN

Introduction to algebraic geometry. By Serge Lang, Interscience tracts in pure and applied Mathematics, Number 5, Interscience publishers, Inc., New York (1958), pp. ix + 260, $ 7.25.

THIS book provides an excellent introduction to all the fundamental conceptions of modern algebraic geometry (excepting intersection theory) which have become by now more or less classical. We find here a treatment of the general notion of varieties, generic points, correspondences, Zariski's main theorem, normality, divisors and linear systems, differential forms, notion of a simple point, some fundamental aspects of algebraic groups and finally on the Riemann-Roch theorem of an algebraic curve. Thus there are some important topics in this book which are not to be found in Weil's " Foundations of algebraic geometry ". The author has a style which is rather informal and at the same time very clear.

The chapters dealing with projective normality, linear systems and differential forms can be very useful since the facts pertaining to these topics lie somewhat scattered and we find them treated here in a concise and clear manner.

The author intends this book as an introduction to Weil's "Foundations" and indeed the value of this book would have been greater had it not been for the fact that algebraic geometry is undergoing great changes, even in its foundations, with the introduction of sheaves and schemes. But a well-written book is always an asset and we welcome it therefore very warmly.

C. S. Seshadri

NEWS AND NOTICES

The following persons have been admitted to the life membership in the Society : B. N. Sahaney, J. A. Siddiqi, K. M. Sundaresan, and P. C. Vaidya.

The following persons have been admitted to the membership in the Society :

C. Adimoolam, Afzal Ahmad, Zafar Uddin Ahmad, P. Balasubramanian, V. Balakrishnan, R. L. Barajatya, M. N. Bhat, K. D. Bhattarai, J. I. S. Brinda, B. B. Chakraborty, K. Chandrasekar, K. R. Chaudhury, M. L. Chaudhury, M. N. Deogan, B. N. Dixit, S. Goel, S. C. Goel, Lata Gupta, K. C. Gupta, R. C. Gupta, V. K. Handa, R. K. Jain, P. Jothilingam, H. R. Krishna, M. R. Krishnamurty, N. R. Kulkurni, D. C. Kapur, R. N. Kesarwani, M. L. Kochar, Masood Khan, S. M. Luthra, R. P. Marwaha, Shaik Masood, Syed Md. Mazhar, K. N. Minakshi, D. N. Misra, J. N. Mittal, V. S. Nanda, P. P. Narayanaswami. Lakshmi Nataraj, N. S. Natarajan, S. Nijhawan, T. V. Panchapagesan, I. G. B. Panikkar, K. R. Parthasarathy, R. K. Parthia, Om Prakash Satya Prakash, L. Radakrishnan, M. Rajagopalan, G. K. Rajeswari, R. V. Ramachandran, Rammohan, R. Ranga Rao, S. N. Rao, H. N. Rawal, P. S. Rema, R. N. Sabharwal, M. M. Sarma, M. P. Sastry, N. K. Sharma, Bhupender Singh, Sahib Singh, B. R. Srinivasan, Bhama Srinivasan, V. K. Srinivasan, R. P. Srivastava, P. Subba Rao, D. S. Subramanian, V. V. Subramania Sastry, M. Sugunamma, M. A. Sundaram, B. Tamuli, T. Varadarajan, G. C. Varma, K. Vijayaraghavan, B. Viswanathan, N. Viswanathan.

Mr. Michael Canter of A. M. S. has been admitted as a member under the reciprocity agreement.

Dr. K. M. Saxena of D. G. B. College, Nanital, has been appointed as Professor to assist the Indian Aid Mission, Nepal.

Sri M. R. Parameswaran has been appointed as Reader, Madras University at Madurai.

Dr. V. Venugopal Rao has joined the Poona University as Reader in mathematics.

Prof. B. S. Madhava Rao has been appointed as Tilak Professor of Applied Mathematics, Poona University.

Sri M. N. Khatri, Research Student, M. S. University of Baroda, has been awarded a special allowance of Rs. 2,000/- by the India Government for 1960-61 to enable him to continue his research.

Dr. M. V. Jambunathan has been appointed as Statistical Officer, National Tuberculosis Inst. Bangalore.

Dr. N. Padma has gone to Connecticut College, New London, as visiting lecturer.

Dr. M. V. Subba Rao has gone to the University of Missouri as a visiting professor.

We regret to report the death on the 17th March 1960, of Prof. R. Vaidyanathaswami who recently retired from the Madras and Venkateswara Universities. He was an honorary member, editor of the Journal for many years and a former President of the Society. We offer our condolences to the bereaved family.

We regret to report the death on the 31st August 1960, of Dr. N. G. Shabde of Nagpur University. He served as professor and Principal of the College of Science, Nagpur. He was chairman S. S. C. Board Poona at the time of his death. He was an ardent member of the Council of the Society and we offer our condolences to the bereaved family.

Smt. K. N. Kamalamma has been awarded the Ph. D. degree of the University of Delhi for her thesis on 'Differential geometry of ruled surfaces of a rectilinear congruence'.

The University Grants Commission has appointed a Review Committee in Mathematics to effect improvements in teaching and research in mathematics.

Second South Asian Conference on mathematical education was held in the Tata Institute, Bombay from January 20-27, 1960.

This was preceded by the International Colloquium on Function theory.

The twenty-sixth Conference of the Indian Mathematical Society will be held from December 27-29, 1960, in Chandigarh under the auspices of the Panjab University.

The fifth Congress in Theoretical and Applied Mechanics was held in Roorkee from December 23-26, 1959. The subjects discussed included elasticity and plasticity, fluid mechanics, vibration and lubrication, thermodynamics, statistics and computation. This was preceded by the UNESCO symposium on Non-linear physical problems on December 21 and 22, 1959. This was held in Roorkee, the participants included K. G. Odqvist, President of the International Union of Theoretical and Applied Mechanics, E. Saibel, S. Kumar, A. N. Khosla, Sir H. Williams, B. R. Seth, S. N. B. Murthy and A. K. Chaudhury. The next Congress will be held in Delhi from December 23-26, 1960.

The Silver Jubilee of the National Institute of Sciences of India will be held in Delhi for a week from the 29th December. We offer the Institute hearty congratulations and good wishes.