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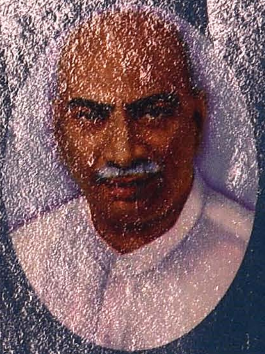

# B.Sc Mathematics

## Third Year

## ANCILLARY PAPER - III

# Classical Algebra & Lattice Theory

RECOGNISED BY DISTANCE EDUCATION COUNCIL



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# MADURAI KAMARAJ UNIVERSITY
**(UNIVERSITY WITH POTENTIAL FOR EXCELLENCE)**

## B.Sc Mathematics
**Final Year**

**ANCILLARY PAPER – III**

# CLASSICAL ALGEBRA & LATTICE THEORY

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## Welcome

Dear Students,

We welcome you as a student of the Final year B.Sc degree course.

This paper deals with the subject 'CLASSICAL ALGEBRA & LATTICE THEORY'. The learning material for this paper will be supplemented by contact lectures.

In this book the first seven units deal with Classical Algebra and the last three units deal with Lattice Theory.

Learning through the Distance Education mode, as you are all aware, involves self learning and self assessment and in this regard you are expected to put in disciplined and dedicated effort.

As our part, we assure of our guidance and support.

With best wishes,

# SYLLABUS

## B.Sc., Final Year

## Ancillary Paper – III
## CLASSICAL ALGEBRA & LATTICE THEORY

Unit 1: Sequence – Convergence – Divergence and oscillation – series – Convergence – Divergence.

Unit 2: Test for series of positive terms – Comparison test – D'Alembert's test – Ratio test – Cauchy's Root test – Rabbe's test

Unit 3: Cauchy's Integral test – Harmonic series – Absolute Convergence – Conditional convergence Alternating series – Leibnitz's test

Unit 4: Binomial theorem for a Rational index – Greatest term – Binomial Co-efficient – Approximate values

Unit 5: Exponential Theorem – Logarithmic Series – Modification of Logarithmic Series – Euler's Constant

Unit 6: Application of Exponential and of Logarithmic series to Limits and approximations.

Unit 7: Summation of Series using Binomial, Logarithmic and Exponential Series.

Unit 8: Partially ordered set – Definition of Lattice – Examples.

Unit 9: Distributive Lattice – Modulo Lattice – Examples – Simple properties.

Unit 10: Boolean Algebra - Examples

## Text Books:

**For Units 1 to 7**

1. Sequence and series – Arumugam & Issac
2. Calculus Volume I & II – S. Narayanan & T.K.Manickavasagam Pillai
3. Algebra Volume I – S. Narayanan & T.K.Manickavasagam Pillai

**For Units 8 to 10**

Modern Algebra by Arumugam & Issac, Sci.Tech Publications, Aug 2003

# SCHEME OF LESSONS

# CLASSICAL ALGEBRA & LATTICE THEORY

# UNIT-1

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## Unit Structure:

Section 1.1 : Sequence

Section 1.2 : Convergence.

Section 1.3 : Divergence and oscillation

Section 1.4 : Series

Section 1.5 : Series – Convergence and divergence

**Introduction:** In this unit we develop the theory of sequence of real numbers. Also we discuss the convergence and divergence of sequences and the properties, some important theorems on the sequences.

**Preliminaries:**

Certain letters are reserved to denote particular sets which occur often in our discussion. They are

**N.......... the set of all natural numbers.**

**Z........... the set of all integers**

**Q........... the set of all rational numbers**

**$Q^+$.......... the set of all positive rational numbers**

**R........... the set of all real numbers**

**C........... the set of all complex numbers**

**$R^n$.......... the set of all ordered n-tuples $(x_1,x_2,\ldots,x_n)$ of real numbers**

**$C^n$.......... the set of all ordered n-tuples $(x_1,x_2,\ldots,x_n)$ of complex numbers**

Next we shall see the definitions of least upper bound (l.u.b.) and greatest lower bound (g.l.b.)

Definition: A subset A of R is said to be bounded above if there exists an element $\alpha \in R$ such that $a \leq \alpha$ for all $a \in A$. Then $\alpha$ is an upper bound of A.

Definition: Let $A \subseteq R$ and $u \in R$. u is called the least upper bound (l.u.b.) or supremum(sup) if

(i) u is an upper bound of A

(ii) if $v < u$ then v is not an upper bound of A.

Definition: A subset A of R is said to be bounded below if there exists

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an element $\beta \in R$ such that $a \geq \beta$ for all $a \in A$. Then $\beta$ is a lower bound of A.

Definition: Let $A \subseteq R$ and $\ell \in R$. u is called the greatest lower bound (g.l.b.) or infimum(inf) if

(i) $\ell$ is a lower bound of A

(ii) if $m > \ell$ then m is not a lower bound of A.

Examples:

1. If A = {1,4,8,12} then glb of A = 1 and lub of A = 12.
2. If A = (0,1) then glb of A = 0 and lub of A = 1.

## SECTION-1.1 -SEQUENCES

### INTRODUCTION

A great deal of analysis is concerned with sequence and series.

Consider the following collection of real numbers given by

1,2,3,…,n,…

$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$

These are the examples of sequence of real numbers. (i.e.) a sequence is an arrangement of elements where we can say which element is first, which is second and so on. In other words the elements of a sequence are labeled with the elements of **N**, the set of all natural numbers, preserving their order.

In general such a labeling can be made by means of a function f whose domain is **N**.

Definition: Let f: **N**→**R** be a function and let f(n) = $a_n$. Then $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called the sequence in **R** determined by the function f and is denoted by $(a_n)$. $a_n$ is called the $n^{th}$ term of the sequence. The range of the function f, which is a subset of **R**, is called the **range of the sequence**.

Examples:

1.The function f: **N**→**R** given by f(n) = n determines the sequence 1,2,3,….,n,…

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2.The function f: $\mathbf{N} \to \mathbf{R}$ given by $f(n) = n^2$ determines the sequence $1,4,9,\ldots,n^2,\ldots$

3.The function f: $\mathbf{N} \to \mathbf{R}$ given by $f(n) = (-1)^n$ determines the sequence $-1,1,-1,1,\ldots$ The range of this sequence is $\{-1,1\}$.

4. The function f: $\mathbf{N} \to \mathbf{R}$ given by $f(n) = (-1)^{n+1}$ determines the sequence $1,-1,1,-1,1,\ldots$ The range of this sequence is $\{-1,1\}$.

5. The constant function f: $\mathbf{N} \to \mathbf{R}$ given by $f(n) = 1$ determines the sequence $1,1,1,\ldots$.

The range of this sequence is $\{1\}$. Such a sequence is called constant sequence.

6. The function f: $\mathbf{N} \to \mathbf{R}$ given by $f(n) = \begin{cases} \frac{n}{2} & \text{if n is even} \\ \frac{1-n}{2} & \text{if n is odd} \end{cases}$

determines the sequence $0,1,-1,2,-2,\ldots,n,-n,\ldots$ The range of this sequence is $\mathbf{Z}$, the set of integers.

7. Let $x \in \mathbf{R}$. The function f: $\mathbf{N} \to \mathbf{R}$ given by $f(n) = x^{n-1}$ determines the geometric sequence $1,x,x^2,\ldots,x^n,\ldots$

8. The function f: $\mathbf{N} \to \mathbf{R}$ given by $f(n) = \frac{1}{n}$ determines the sequence

$$1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n},\ldots$$

9. Let $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{(2+a_n)}$. This defines the sequence $\sqrt{2}, \sqrt{2+\sqrt{2}},\ldots$

**Bounded sequences:**

Definition: A sequence $(a_n)$ is said to be bounded above if there exist a real number k such that $a_n \le k$ for all $n \in N$. Then k is called an upper bound of the sequence$(a_n)$.

A sequence $(a_n)$ is said to be bounded below if there exist a real number k such that $a_n \ge k$ for all $n \in N$. Then k is called a lower bound of the sequence$(a_n)$.

A sequence $(a_n)$ is said to be bounded if it is both bounded above and below.

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<u>Note</u>: A sequence $(a_n)$ is bounded iff there exists a real number $k \geq 0$ such that $|a_n| \leq k$ for all n.

**Monotonic Sequences:**

<u>Definition</u>:

A sequence $(a_n)$ is said to be monotonic increasing if $a_n \leq a_{n+1}$ for all n $(a_n)$ is said to be monotonic decreasing if $a_n \geq a_{n+1}$ for all n. $(a_n)$ is said to be strictly monotonic increasing if $a_n < a_{n+1}$ for all n and $(a_n)$ is said to be strictly monotonic decreasing if $a_n > a_{n+1}$ for all n. $(a_n)$ is said to be monotonic if it is monotonic increasing or monotonic decreasing.

<u>Examples</u>:

1) 1,2,2,3,3,4,4,.... is a monotonic increasing sequence.
2) 1,2,3,4,.... Is a strictly monotonic increasing sequence.
3) $1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ....$ is a strictly monotonic decreasing sequence.
4) 1,–1,1,–1,1,... is neither monotonic increasing nor decreasing.

**Problems:**

1) $\left(\frac{2n-7}{3n+2}\right)$ is a monotonic increasing sequence.

Solution: $a_n - a_{n+1} = \frac{2n-7}{3n+2} - \frac{2(n+1)-7}{3(n+1)+2} = \frac{-25}{(3n+2)(3n+5)} < 0.$

Therefore $a_n < a_{n+1}$.

Hence the given sequence is monotonic increasing.

2) Show that if $(a_n)$ is monotonic sequence then $\left(\frac{a_1 + a_2 + .... + a_n}{n}\right)$ is also monotonic sequence.

Solution: Let $(a_n)$ be monotonic increasing sequence.

$\therefore\ a_1 \leq a_2 \leq a_3 \leq .....a_n \leq a_{n+1} \leq ....$---------------- (1)

Let $b_n = \frac{a_1 + a_2 + .... + a_n}{n}$.

$$\text{Now } b_{n+1} - b_n = \frac{a_1 + a_2 + .... + a_{n+1}}{n+1} - \frac{a_1 + a_2 + .... + a_n}{n}$$

$$= \frac{na_{n+1} - (a_1 + a_2 + .... + a_n)}{n(n+1)}$$

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$$\geq \frac{na_{n+1} - (a_n + a_n + .... + a_n)}{n(n+1)} \text{ ( by(1))}$$

$$= \frac{n(a_{n+1} - a_n)}{n(n+1)} \geq 0 \text{ ( by (1))}$$

$\therefore b_{n+1} \geq b_n$ .

$\therefore (b_n)$ is monotonic increasing sequence.

The proof is similar if $(a_n)$ is monotonic decreasing.

**CYP Questions:**

1) If $(a_n)$ and $(b_n)$ are two monotonic increasing(decreasing) sequences show that $(a_n+b_n)$ is also monotonic increasing(decreasing) sequences.

2) If $(a_n)$ is monotonic increasing sequence show that $(\lambda a_n)$ is monotonic increasing if $\lambda$ is positive and $(\lambda a_n)$ is monotonic decreasing if $\lambda$ is negative.

3) Write the first four terms of the following sequences

(i). $\left(\frac{(-1)^n}{n}\right)$ (ii). $\left(\frac{1-(-1)^n}{n^3}\right)$ (iii). $\left(\frac{2n^2+1}{2n^2-1}\right)$ (iv). $(n!)$

4) Determine which of the following sequences are monotonic.

(i) $(\log n)$ (ii) $((-1)^{n+1}n)$ (iii) $\left(\frac{1}{n!}\right)$ (iv) $\left(2+\frac{1}{n}\right)$

## SECTION-1.2 - CONVRGENT SEQUENCES

Definition: A sequence $(a_n)$ is said to converge to a number $\ell$ if given $\varepsilon>0$ there exists a positive integer m such that $|a_n - \ell| < \varepsilon$ for all $n \geq m$.

We say that $\ell$ is the limit of the sequence and we write

$$\lim_{n\to\infty} a_n = \ell \text{ or } a_n \to \ell .$$

Note: $a_n \to \ell$ iff given $\varepsilon > 0$ there exists a natural number m such that $a_n \in (\ell - \varepsilon, \ell + \varepsilon)$ for all $n \geq m$.

(i.e.) All but a finite number of terms of the sequence lie within the interval $(\ell - \varepsilon, \ell + \varepsilon)$.

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Theorem 1.2.1: A sequence cannot converge to two different limits.

Proof: Let $(a_n)$ be a convergent sequence.

If possible let $\ell_1$ and $\ell_2$ be two distinct limits of $(a_n)$.

Let $\varepsilon > 0$ be given.

Since $a_n \to \ell_1$, there exists a natural number $n_1$ such that

$$|a_n - \ell_1| < \frac{\varepsilon}{2} \text{ for all } n \geq n_1. \text{ ------------------------ (1)}$$

Since $a_n \to \ell_2$, there exists a natural number $n_2$ such that

$$|a_n - \ell_2| < \frac{\varepsilon}{2} \text{ for all } n \geq n_2. \text{ ------------------------ (2)}$$

Let $m = \max\{ n_1, n_2 \}$.

$$\begin{aligned} \text{Then } |\ell_1 - \ell_2| &= |\ell_1 - a_m + a_m - \ell_2| \\ &\leq |a_m - \ell_1| + |a_m - \ell_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ ( by (1) and (2) )} \\ &= \varepsilon. \end{aligned}$$

$\therefore |\ell_1 - \ell_2| < \varepsilon$ and this is true for every $\varepsilon > 0$.

Clearly this is possible iff $\ell_1 - \ell_2 = 0$.

Hence $\ell_1 = \ell_2$.

**Examples:**

1. $\lim\limits_{n\to\infty} \frac{1}{n} = 0.$ (or) $\left(\frac{1}{n}\right) \to 0$

Proof: Let $\varepsilon > 0$ be given.

Then $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$ if $n > \frac{1}{\varepsilon}$.

Hence if we choose m to be any natural number such that $m > \frac{1}{\varepsilon}$ then

$\left|\frac{1}{n} - 0\right| < \varepsilon$ for all $n \geq m$.

$$\therefore \lim_{n\to\infty} \frac{1}{n} = 0.$$

2. The constant sequence 1,1,1,1,.... Converges to 1.

Proof: Let $\varepsilon > 0$ be given.

Let the given sequence be denoted by $(a_n)$.

Then $a_n = 1$ for all n.

$\therefore |a_n - 1| = |1 - 1| = 0 < \varepsilon$ for all $n \in N$.

$\therefore |a_n - 1| < \varepsilon$ for all $n \geq m$ where m can be chosen to be any natural number.

$$\therefore \lim_{n\to\infty} a_n = 1.$$

3. $\lim\limits_{n\to\infty} \frac{n+1}{n} = 1.$

Proof: Let $\varepsilon > 0$ be given.

Then $\left|\frac{n+1}{n} - 1\right| = \left|1 + \frac{1}{n} - 1\right| = \left|\frac{1}{n}\right|.$

Therefore if we can choose m to be any natural number greater than $\frac{1}{\in}$ we have $\left|\frac{n+1}{n} - 1\right| < \in$ for all $n \geq m$.

$$\therefore \lim_{n\to\infty} \frac{n+1}{n} = 1.$$

4. $\lim\limits_{n\to\infty} \frac{1}{2^n} = 0.$

Proof: Let $\varepsilon > 0$ be given.

Then $\left|\frac{1}{2^n} - 0\right| = \frac{1}{2^n} < \frac{1}{n}$ (since $2^n > n$ for all $n \in N$)

$\therefore \left|\frac{1}{2^n} - 0\right| < \varepsilon$ for all $n \geq m$ where m is any natural number greater than $\frac{1}{\in}$.

$$\therefore \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

5. The sequence $((-1)^n)$ is not convergent.

Proof: Suppose the sequence $((-1)^n)$ converges to $\ell$.

Then, given $\varepsilon > 0$, there exists a natural number m such that $|(-1)^n - 1| < \varepsilon$ for all $n \geq m$.

$$\begin{aligned} \therefore |(-1)^m - (-1)^{m+1}| &= |(-1)^m - \ell + \ell - (-1)^{m+1}| \\ &\leq |(-1)^m - \ell| + |(-1)^{m+1} - \ell| \\ &< \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

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But $|(-1)^m - (-1)^{m+1}| = 2$.

$\therefore$ $2 < 2\varepsilon$ (i.e.) $1 < \varepsilon$, which is a contradiction since $\varepsilon > 0$ is arbitrary.

$\therefore$ The sequence $((-1)^n)$ is not convergent.

Theorem 1.2.2: Any convergent sequence is a bounded sequence.

Proof: Let $(a_n)$ be a convergent sequence.

Let $\lim_{n\to\infty} a_n = \ell$.

Let $\varepsilon > 0$ be given. Then there exists $m \in N$ such that

$|a_n - \ell| < \varepsilon$ for all $n \geq m$.

$\therefore |a_n| < |\ell| + \varepsilon$ for all $n \geq m$.

Now, let $k = \max\{|a_1|, |a_2|, \ldots, |a_{m-1}|, |\ell| + \varepsilon\}$

Then $|a_n| \leq k$ for all n.

$\therefore$ $(a_n)$ is a bounded sequence.

**CYP Questions:**

1) Prove that $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

2) Prove that $\lim_{n\to\infty} \left(1 + \frac{1}{n!}\right) = 1$.

3) Prove that the following sequences are not convergent.
(i) $((-1)^n n)$ (ii) $(n^2)$

## SECTION-1.3 -DIVERGENT AND OSCILLATING SEQUENCES

Definition. A sequence $(a_n)$ is said to diverge to $\infty$ if given any real number $k > 0$, there exists $m \in N$ such that $a_n > k$ for all $n \geq m$. We write $(a_n) \to \infty$ or $\lim_{n\to\infty} a_n = \infty$.

**Examples.**

1. Prove that $(n) \to \infty$.

Proof: Let $k > 0$ be any given real number.

Choose m to be any natural number such that $m > k$.

Then $n > k$ for all $n \geq m$.

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$\therefore (n) \to \infty$.

2. Prove that $(n^2) \to \infty$.

Proof: Let $k > 0$ be any given real number.

Choose m to be any natural number such that $m > \sqrt{k}$.

Then $n^2 > k$ for all $n \geq m$.

$\therefore (n^2) \to \infty$.

3. Prove that $(2^n) \to \infty$.

Proof: Let $k > 0$ be any given real number.

Then $2^n > k \Leftrightarrow n \log 2 > \log k$.

$$\Leftrightarrow n > \frac{\log k}{\log 2}$$

Hence if we choose m to be any natural number such that $m > \frac{\log k}{\log 2}$, then $2^n > k$ for all $n \geq m$.

$\therefore (2^n) \to \infty$.

Definition. A sequence $(a_n)$ is said to diverge to $-\infty$ if given any real number $k < 0$, there exists $m \in N$ such that $a_n < k$ for all $n \geq m$. We write $(a_n) \to -\infty$ or $\lim_{n \to \infty} a_n = -\infty$.

Note: A sequence $(a_n)$ is said to be divergent if either $(a_n) \to \infty$ or $(a_n) \to -\infty$.

Theorem 1.3.1: $(a_n) \to \infty$ iff $(-a_n) \to -\infty$

Proof: Let $(a_n) \to \infty$.

Let $k < 0$ be any given real number. Since $(a_n) \to \infty$ there exists $m \in N$ such that $a_n > -k$ for all $n \geq m$.

$\therefore -a_n < k$ for all $n \geq m$.

$\therefore (-a_n) \to -\infty$.

Similarly we can prove that $(-a_n) \to -\infty \Rightarrow (a_n) \to \infty$

Theorem 1.3.2 : If $(a_n) \to \infty$ and $a_n \neq 0$ for all $n \in N$ then $\left(\frac{1}{a_n}\right) \to 0$.

Proof: Let $\varepsilon > 0$ be given. Since $(a_n) \to \infty$, there exists $m \in N$ such that $a_n > \frac{1}{\in}$ for all $n \geq m$.

$$\therefore \frac{1}{a_n} < \varepsilon \text{ for all } n \geq m.$$

$$\therefore \left|\frac{1}{a_n}\right| < \varepsilon \text{ for all } n \geq m.$$

$$\therefore \left(\frac{1}{a_n}\right) \to 0.$$

Theorem 1.3.3: If $(a_n) \to 0$ and $a_n > 0$ for all $n \in N$ then $\left(\frac{1}{a_n}\right) \to \infty$.

Proof: Let $k > 0$ be any given real number. Since $(a_n) \to 0$, there exists $m \in N$ such that $|a_n| < \frac{1}{k}$ for all $n \geq m$.

$$\therefore a_n < \frac{1}{k} \text{ for all } n \geq m. \text{ ( since } a_n > 0)$$

$$\therefore \frac{1}{a_n} > k \text{ for all } n \geq m.$$

$$\therefore \left(\frac{1}{a_n}\right) \to \infty.$$

Theorem 1.3.4: Any sequence $(a_n)$ diverging to $\infty$ is bounded below but not bounded above.

Proof: Let $(a_n) \to \infty$. Then for any given real number $k > 0$ there exists $m \in N$ such that $a_n > k$ for all $n \geq m$. --------------- (1)

Therefore k is not an upper bound of the sequence $(a_n)$.

Therefore $(a_n)$ is not bounded above.

Now let $\ell = \min\{a_1, a_2, a_3, \ldots, a_n, k\}$

Form (1) we see that $a_n \geq \ell$ for all n.

$\therefore (a_n)$ is bounded below.

Definition: A sequence $(a_n)$ which is neither convergent nor divergent to $\infty$ or $-\infty$ is said to be an oscillating sequence. An oscillating sequence which is bounded is said to be finitely oscillating. An oscillating sequence which is unbounded is said to be infinitely oscillating.

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**THE ALGEBRA OF LIMITS.**

Theorem 1.3.6: If $(a_n) \to a$ and $(b_n) \to b$ then $(a_n + b_n) \to a + b$.

Proof: Let $\varepsilon > 0$ be given.

Now $|(a_n + b_n) - (a + b)| = |a_n + b_n - a - b|$

$$\leq |a_n - a| + |b_n - b| \text{ ------------ (1)}$$

Since $(a_n) \to a$, there exists a natural number $n_1$, such that

$$|a_n - a| < \frac{\varepsilon}{2} \text{ for all } n \geq n_1. \text{ ------------------------ (2)}$$

Since $(b_n) \to b$, there exists a natural number $n_2$, such that

$$|b_n - b| < \frac{\varepsilon}{2} \text{ for all } n \geq n_2. \text{ ------------------------ (3)}$$

Let $m = \max\{n_1, n_2\}$.

Then $|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b|$ (by (1))

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ for all } n \geq m \text{ (by (2) and (3))}$$

$$= \varepsilon.$$

$\therefore (a_n + b_n) \to a + b$.

Theorem 1.3.7: If $(a_n) \to a$ and $k \in R$ then $(ka_n) \to ka$.

Proof: If $k = 0$, then the sequence $(ka_n)$ becomes the constant sequence $0,0,\ldots,0$ and hence converges to $0 = 0.a = ka$.

Now let $k \neq 0$.

Then $|ka_n - ka| = |k|\,|a_n - a|$ ------------------ (1)

Let $\varepsilon > 0$ be given.

Since $(a_n) \to a$, there exists a natural number m, such that

$$|a_n - a| < \frac{\varepsilon}{|k|} \text{ for all } n \geq m. \text{ ------------------------ (2)}$$

$\therefore |ka_n - ka| = |k|\,|a_n - a|$ (by (1))

$$< |k| \frac{\varepsilon}{|k|} \text{ for all } n \geq m \text{ (by (2))}$$

$$= \varepsilon$$

$\therefore (ka_n) \to ka$.

Theorem 1.3.8: If $(a_n) \to a$ and $(b_n) \to b$ then $(a_nb_n) \to ab$.

Proof: Let $\varepsilon > 0$ be given.

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Now $|a_nb_n - ab| = |a_nb_n - a_nb + a_nb - ab|$

$\leq |a_nb_n - a_nb| + |a_nb - ab|$

$= |a_n||b_n - b| + |b||a_n - a|$ --------------- (1).

Since $(a_n) \to a$ and every convergent sequence is bounded, $(a_n)$ is a bounded sequence.

$\therefore$ there exists a real number $k > 0$ such that $|a_n| \leq k$ for all n ------ (2)

Now equation (1) becomes $|a_nb_n - ab| \leq k\,|b_n - b| + |b||a_n - a|$ ------- (3)

Since $(a_n) \to a$, there exists a natural number $n_1$, such that

$$|a_n - a| < \frac{\varepsilon}{2|b|} \text{ for all } n \geq n_1. \text{ -------------------------------- (4)}$$

Since $(b_n) \to b$, there exists a natural number $n_2$, such that

$$|b_n - b| < \frac{\varepsilon}{2k} \text{ for all } n \geq n_2. \text{ -------------------------------- (5)}$$

Let $m = \max\{n_1, n_2\}$.

Then $|a_nb_n - ab| \leq k\,|b_n - b| + |b||a_n - a|$ (by (3))

$$< k\left(\frac{\varepsilon}{2k}\right) + |b|\left(\frac{\varepsilon}{2|b|}\right) \text{ for all } n \geq m \text{ ( by (4) \& (5))}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore (a_nb_n) \to ab$.

<u>Theorem 1.3.9</u>: If $(a_n) \to a$ and $a_n \neq 0$ for all $n \in N$ and $a \neq 0$ then $\left(\frac{1}{a_n}\right) \to \frac{1}{a}$.

<u>Proof</u>: Let $\varepsilon > 0$ be given.

Now $\left|\frac{1}{a_n} - \frac{1}{a}\right| = \left|\frac{a_n - a}{a_n a}\right| = \frac{1}{|a_n||a|}|a_n - a|$ ----------- (1)

Also $a \neq 0$. Hence $|a| \neq 0$.

Since $(a_n) \to a$, there exists a natural number $n_1$, such that

$$|a_n - a| < \frac{1}{2}|a| \text{ for all } n \geq n_1.$$

Hence $|a_n| > \frac{1}{2}|a|$ for all $n \geq n_1$. --------------------- (2)

By using (1) & (2) we get

$$\left|\frac{1}{a_n}-\frac{1}{a}\right| = \frac{1}{|a_n||a|}|a_n - a| < \frac{2}{|a|^2}|a_n - a| \text{ for all } n \geq n_1 \text{.....(3)}$$

Again since $(a_n) \to a$, there exists a natural number $n_2$, such that

$$|a_n - a| < \frac{1}{2}\varepsilon|a|^2 \text{ for all } n \geq n_2. \text{---------------- (4)}$$

Let $m = \max\{n_1, n_2\}$.

$$\therefore \left|\frac{1}{a_n}-\frac{1}{a}\right| < \frac{2}{|a|^2}\frac{1}{2}|a|^2\varepsilon = \varepsilon \text{ for all } n \geq m \text{ ( by (3) \& (4) )}$$

$$\therefore \left(\frac{1}{a_n}\right) \to \frac{1}{a}.$$

Theorem 1.3.10: If $(a_n) \to a$ and $a_n \geq 0$ for all $n \in N$ then $a \geq 0$.

Proof: Suppose $a < 0$. Then $-a > 0$.

Choose $\varepsilon$ such that $0 < \varepsilon < -a$ so that $a + \varepsilon < 0$.

Now, Since $(a_n) \to a$, there exists a natural number m, such that

$|a_n - a| < \varepsilon$ for all $n \geq m$.

Therefore $a - \varepsilon < a_n < a + \varepsilon$ for all $n \geq m$.

Since $a + \varepsilon < 0$, we have $a_n < a + \varepsilon < 0$ for all $n \geq m$, which is a contradiction, since $a_n \geq 0$. Hence $a \geq 0$.

Theorem 1.3.11: If $(a_n) \to a$ and $(b_n) \to b$ and $a_n \leq b_n$ then $a \leq b$.

Proof: Since $a_n \leq b_n$, we have $b_n - a_n \geq 0$ for all n.

Also $b_n - a_n \to b - a$. (by theorem 1.3.6).

$\therefore$ $b - a \geq 0$. (by theorem 1.3.10).

Hence $a \leq b$.

Theorem 1.3.12: If $(a_n) \to \ell$ and $(b_n) \to \ell$ and $a_n \leq c_n \leq b_n$ for all n , then $(c_n) \to \ell$.

Proof: Let $\varepsilon > 0$ be given.

Since $(a_n) \to \ell$, there exists a natural number $n_1$, such that

$\ell - \varepsilon < a_n < \ell + \varepsilon$ for all $n \geq n_1$.

Also since $(b_n) \to \ell$, there exists a natural number $n_2$, such that

$\ell - \varepsilon < b_n < \ell + \varepsilon$ for all $n \geq n_2$.

Let $m = \max\{n_1, n_2\}$.

SPACE FOR HINT

$\therefore \ell - \varepsilon < a_n \le c_n \le b_n < \ell + \varepsilon$ for all $n \ge m$.

$\therefore \ell - \varepsilon < c_n < \ell + \varepsilon$ for all $n \ge m$.

$\therefore | c_n - \ell | < \varepsilon$ for all $n \ge m$.

$\therefore (c_n) \to \ell$.

Theorem 1.3.13: If $(a_n) \to \infty$ and $(b_n) \to \infty$ then $(a_n + b_n) \to \infty$.

Proof: Let $k > 0$ be any given real number.

Since $(a_n) \to \infty$, there exists $n_1 \in N$ such that $a_n > \frac{1}{2}k$ for all $n \ge n_1$.

Also since $(b_n) \to \infty$, there exists $n_2 \in N$ such that $b_n > \frac{1}{2}k$ for all $n \ge n_2$.

Let $m = \max\{n_1, n_2\}$.

Then $a_n + b_n > \frac{1}{2}k + \frac{1}{2}k = k$ for all $n \ge m$.

$\therefore (a_n + b_n) \to \infty$.

Theorem 1.3.14: If $(a_n) \to \infty$ and $(b_n) \to \infty$ then $(a_n b_n) \to \infty$.

Proof: Let $k > 0$ be any given real number.

Since $(a_n) \to \infty$, there exists $n_1 \in N$ such that $a_n > \sqrt{k}$ for all $n \ge n_1$.

Also since $(b_n) \to \infty$, there exists $n_2 \in N$ such that $b_n > \sqrt{k}$ for all $n \ge n_2$.

Let $m = \max\{n_1, n_2\}$.

Then $a_n b_n > \sqrt{k}\,\sqrt{k} = k$ for all $n \ge m$.

$\therefore (a_n b_n) \to \infty$.

Theorem 1.3.15: Let $(a_n) \to \infty$. Then (i) if $c > 0$, $(ca_n) \to \infty$,

(ii) if $c > 0$, $(ca_n) \to -\infty$.

Proof:

(i) Let $c > 0$. Let $k > 0$ be any given real number.

Since $(a_n) \to \infty$, there exists $m \in N$ such that $a_n > \frac{k}{c}$ for all $n \ge m$.

$\therefore c\, a_n > k$ for all $n \ge m$.

$\therefore (ca_n) \to \infty$.

(ii) Let $c < 0$. Let $k < 0$ be any given real number. Then $\frac{k}{c} > 0$.

Since $(a_n) \to \infty$, there exists $m \in N$ such that $a_n > \frac{k}{c}$ for all $n \ge m$.

$\therefore c\, a_n < k$ for all $n \geq m$. ( since $c < 0$)

$\therefore (ca_n) \to -\infty$.

Theorem 1.3.16: If $(a_n) \to \infty$ and $(b_n)$ is bounded then $(a_n + b_n) \to \infty$.

Proof: Since $(b_n)$ is bounded, there exists a real number $m < 0$ such that

$$b_n > m \text{ for all } n. \quad \text{---------------------} (1)$$

Let $k > 0$ be any real number.

Since $m < 0$, $k - m > 0$.

Since $(a_n) \to \infty$, there exists $n_0 \in N$ such that

$$a_n > k - m \text{ for all } n \geq n_0. \text{ ---------------- } (2)$$

$\therefore\ a_n + b_n > k - m + m = k$ for all $n \geq n_0$. ( by (1) and (2) )

$\therefore (a_n + b_n) \to \infty$.

**Problems:**

1) Show that $\lim\limits_{n\to\infty} \dfrac{3n^2 + 2n + 7}{6n^2 + 5n + 6} = \dfrac{1}{2}$.

Solution: Let $a_n = \dfrac{3n^2 + 2n + 7}{6n^2 + 5n + 6} = \dfrac{3 + \frac{2}{n} + \frac{7}{n^2}}{6 + \frac{5}{n} + \frac{6}{n^2}}$

Now, $\lim\limits_{n\to\infty} 3 + \dfrac{2}{n} + \dfrac{7}{n^2} = 3 + 2 \lim\limits_{n\to\infty} \dfrac{1}{n} + 7 \lim\limits_{n\to\infty} \dfrac{1}{n^2}$

$= 3 + 0 + 0 = 3.$

Similarly $\lim\limits_{n\to\infty} 6 + \dfrac{5}{n} + \dfrac{6}{n^2} = 6 + 5 \lim\limits_{n\to\infty} \dfrac{1}{n} + 6 \lim\limits_{n\to\infty} \dfrac{1}{n^2}$

$= 6 + 0 + 0 = 6.$

$$\therefore\ \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^2 + 2n + 7}{6n^2 + 5n + 6} = \lim_{n\to\infty} \frac{3 + \frac{2}{n} + \frac{7}{n^2}}{6 + \frac{5}{n} + \frac{6}{n^2}}$$

$$= \frac{\lim\limits_{n\to\infty} 3 + \frac{2}{n} + \frac{7}{n^2}}{\lim\limits_{n\to\infty} 6 + \frac{5}{n} + \frac{6}{n^2}} = \frac{3}{6} = \frac{1}{2}.$$

2) Show that $\lim\limits_{n\to\infty} \left( \dfrac{1^2 + 2^2 + \ldots\ldots + n^2}{n^3} \right) = \dfrac{1}{3}$.

SPACE FOR HINT

Solution: W.K.T. $1^2+2^2+.....+n^2=\frac{n(n+1)(2n+1)}{6}$

$$\therefore \lim_{n\to\infty}\left(\frac{1^2+2^2+.....+n^2}{n^3}\right)=\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}$$

$$=\lim_{n\to\infty}\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$=\frac{1}{6}1.2=\frac{1}{3}$$

3) Show that $\lim_{n\to\infty}\frac{n}{\sqrt{(n^2+1)}}=1$.

Solution: $\lim_{n\to\infty}\frac{n}{\sqrt{(n^2+1)}}=\lim_{n\to\infty}\frac{1}{\sqrt{\left(1+\frac{1}{n^2}\right)}}$

$$=\frac{1}{\lim_{n\to\infty}\sqrt{\left(1+\frac{1}{n^2}\right)}}\text{ (by theorem 1.3.9)}$$

$$=\frac{1}{\sqrt{\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)}}$$

$$=1.$$

4) Show that if $(a_n)\to 0$ and $(b_n)$ is bounded then $(a_nb_n)\to 0$.

Solution: Since $(b_n)$ is bounded, there exists $k>0$ such that $|b_n|\le k$ for all n.

$\therefore |a_nb_n|\le k|a_n|$.

Now let $\varepsilon>0$ be given.

Since $(a_n)\to 0$, there exists $m\in N$ such that $|a_n|<\frac{\varepsilon}{k}$ for all $n\ge m$.

$\therefore |a_nb_n|<\varepsilon$ for all $n\ge m$.

$\therefore (a_nb_n)\to 0$.

5) Show that $\lim_{n\to\infty}\frac{\sin n}{n}=0$.

Solution: Since $|\sin n|\le 1$ for all n, $(\sin n)$ is a bounded sequence.

Also $\left(\frac{1}{n}\right)\to 0$.

SPACE FOR HINT

By the above problem, $\left(\frac{\sin n}{n}\right) \to 0.$

6) Show that $\lim\limits_{n\to\infty}\left(a^{1/n}\right) = 1$ where $a > 0$ is any real number.

Solution: Case (i) Let $a = 1$. Then $a^{1/n} = 1$ for each n.

Hence $\left(a^{1/n}\right) \to 1.$

Case (ii) Let $a > 1$. Then $a^{1/n} > 1$.

Let $a^{1/n} = 1 + h_n$ where $h_n > 0$.

$\therefore a = (1+h_n)^n$

$= 1+nh_n + \ldots\ldots + h_n^n.$

$> 1+nh_n.$

$\therefore\ h_n < \frac{a-1}{n}$

$\therefore\ 0 < h_n < \frac{a-1}{n} \to 0$ as $n \to\infty$.

$\therefore (a^{1/n}) = (1+h_n) \to 1$ as $n \to\infty$.

Case (iii) Let $0 < a < 1$. Then $1/a > 1$.

$\therefore \left(1/a\right)^{1/n} \to 1.$ ( by case (ii))

$\therefore \left(\frac{1}{a^{1/n}}\right) \to 1.$

$\therefore (a^{1/n}) \to 1$

7) Show that $\lim\limits_{n\to\infty}\left(n^{1/n}\right) = 1.$

Solution: We know that $n^{1/n} \geq 1$ for all n.

Let $n^{1/n} = 1 + h_n$ where $h_n > 0$.

Then $n = (1+h_n)^n$

$= 1+nc_1h_n + nc_2h_n^2 + \ldots\ldots + h_n^n.$

$> \frac{1}{2}n(n-1)h_n^2\ .$

$\therefore\ h_n^2 < \dfrac{2}{n-1}$

$\therefore\ h_n < \sqrt{\dfrac{2}{n-1}} \to 0$ as $n \to \infty$.

$\therefore\ h_n \to 0$ as $n \to \infty$. ( since $h_n \geq 0$)

$\therefore (n^{1/n}) = (1 + h_n) \to 1$ as $n \to \infty$.

Theorem 1.3.17:

(i) A monotonic increasing sequence which is bounded above converges to its l.u.b.

(ii) A monotonic increasing sequence which is not bounded above diverges to $\infty$.

(iii) A monotonic decreasing sequence which is bounded below converges to its g.l.b.

(iv) A monotonic decreasing sequence which is not bounded below diverges to $-\infty$.

Proof: (i) Let $(a_n)$ be a monotonic increasing sequence which is bounded above. Let k be the l.u.b. of the sequence.

Then $a_n \leq k$ for all n. ------------------------ (1)

Now let $\varepsilon > 0$ be given.

$\therefore k - \varepsilon < k$ .

$\therefore k - \varepsilon$ is not an upper bound of $(a_n)$.

Hence there exists $a_m$ such that $a_m > k - \varepsilon$.

Since $(a_n)$ is monotonic increasing, $a_n \geq a_m$ for all $n \geq m$.

Hence $a_n > k - \varepsilon$ for all $n \geq m$. ---------------- (2)

$\therefore k - \varepsilon < a_n \leq k < k + \varepsilon$ for all $n \geq m$. ( by (1) and (2) )

$\therefore | a_n - k | < \varepsilon$ for all $n \geq m$.

$\therefore\ (a_n) \to k$.

(ii) Let $(a_n)$ be a monotonic increasing sequence which is not bounded above.

Let $k > 0$ be any real number.

Since $(a_n)$ is not bounded, there exists $m \in N$ such that $a_m > k$.

Also $a_n \geq a_m$ for all $n \geq m$.

$\therefore$ $a_n > k$ for all $n \geq m$.

$\therefore$ $(a_n) \to \infty$.

Proof (iii) and (iv) are similar to that of (i) and (ii) respectively.

Note: A monotonic sequence cannot be an oscillating sequence.

**Problems:**

1) Let $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots + \frac{1}{n!}$. Show that $\lim_{n\to\infty} a_n$ exists and lies between 2 and 3.

Solution: Clearly $(a_n)$ is monotonic increasing sequence.

Also $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots + \frac{1}{n!}$

$$\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots + \frac{1}{2^{n-1}}$$

$$= 1 + \left( \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} \right)$$

$$= 1 + 2\left(1 - \frac{1}{2^n}\right)$$

$$= 3 - \frac{1}{2^{n-1}} < 3.$$

$\therefore$ $a_n < 3$.

$\therefore$ $(a_n)$ is bounded above.

$\therefore$ $\lim_{n\to\infty} a_n$ exists.

Also $2 < a_n < 3$ for all n.

$\therefore$ $2 < \lim_{n\to\infty} a_n < 3$.

Note: The limit of the above sequence is denoted by e.

2) Show that the sequence $\left(1 + \frac{1}{n}\right)^n$ converges.

Solution: Let $a_n = \left(1 + \frac{1}{n}\right)^n$.

By the binomial theorem,

SPACE FOR HINT

$$a_n = 1 + nc_1\frac{1}{n} + nc_2\frac{1}{n^2} + nc_3\frac{1}{n^3} + \ldots\ldots + nc_n\frac{1}{n^n}.$$

$$= 1 + 1 + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \ldots\ldots$$

$$+ \frac{n(n-1)(n-2)\ldots(n-(n-1))}{n!}\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots\ldots$$

$$+ \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\frac{n-1}{n}\right)$$

$$< 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots + \frac{1}{n!}$$

$< 3$ (by the above problem)

$\therefore$ $(a_n)$ is bounded above.

Also $a_{n+1} = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \ldots\ldots$

$$+ \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\ldots\ldots\left(1-\frac{n}{n+1}\right)$$

$$> 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots\ldots$$

$$+ \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\frac{n-1}{n}\right)$$

$= a_n$.

$\therefore$ $a_{n+1} > a_n$ .

$\therefore$ $(a_n)$ is monotonic increasing sequence.

$\therefore$ $(a_n)$ is a convergent sequence.

3) Show that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots\ldots+\frac{1}{n!}\right) = e.$

Solution: Let $a_n = \left(1+\frac{1}{n}\right)^n$ and $b_n = \left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots\ldots+\frac{1}{n!}\right)$.

Then $a_n = \left(1+\frac{1}{n}\right)^n = 1 + nc_1\frac{1}{n} + nc_2\frac{1}{n^2} + nc_3\frac{1}{n^3} + \ldots\ldots + nc_n\frac{1}{n^n}$.

SPACE FOR HINT

$$= 1 + 1 + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \ldots\ldots$$
$$+ \frac{n(n-1)(n-2)\ldots(n-(n-1))}{n!}\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots..$$
$$+ \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots.\left(1-\frac{n-1}{n}\right)$$

$$< 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots.. + \frac{1}{n!} = b_n \text{ for all n.}$$

(i.e.) $a_n < b_n$ for all n.

$$\therefore \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n \text{ -------------- (1)}$$

Now let m > n.

$$a_m = \left(1+\frac{1}{m}\right)^m = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{m}\right) + \frac{1}{3!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right) + \ldots..$$

$$+ \frac{1}{n!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\ldots.\left(1-\frac{n-1}{m}\right) + \ldots + \frac{1}{m!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\ldots.\left(1-\frac{m-1}{m}\right)$$

$$> 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{m}\right) + \frac{1}{3!}\left(1-\frac{1}{m}\right) + \ldots.. + \frac{1}{n!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\ldots.\left(1-\frac{n-1}{m}\right)$$

Fixing n and taking limit as $m \to \infty$ we get

$$\lim_{m\to\infty} a_m \ge 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots. + \frac{1}{n!} = b_n$$

Now taking limit as $n \to \infty$ we get

$$\lim_{m\to\infty} a_m \ge \lim_{n\to\infty} b_n \text{ --------------------------- (2)}$$

From (1) and (2) we get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = e.$$

$$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots..+\frac{1}{n!}\right) = e.$$

4) Let $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots.. + \frac{1}{n}$. Show that $(a_n)$ diverges to $\infty$.

Solution: Clearly $(a_n)$ is a monotonic increasing sequence.

SPACE FOR HINT

Let $m = 2^n - 1$.

$$a_m = 1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{m} = 1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{2^n-1}$$

$$= 1+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+.....+\left(\frac{1}{2^{n-1}}+.....+\frac{1}{2^n-1}\right)$$

$$> 1+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+.....+\left(\frac{1}{2^n}+.....+\frac{1}{2^n}\right)$$

$$= 1+(n-1)\frac{1}{2} = \frac{1}{2}(n+1)$$

$\therefore a_m > \frac{1}{2}(n+1)$

$\therefore (a_n)$ is not bounded above.

Hence $(a_n) \to \infty$.

5) Discuss the behavior of the geometric sequence $(r^n)$ .

Solution:

Case(i) Let $r = 0$.

Then $(r^n)$ reduces to the constant sequence 0,0,.... and hence converges to 0.

Case(ii) Let $r = 1$.

Then $(r^n)$ reduces to the constant sequence 1,1,.... and hence converges to 1.

Case(iii) Let $0 < r < 1$.

In this case $(r^n)$ is a monotonic decreasing sequence and $r^n > 0$ for all $n \in N$.

$\therefore (r^n)$ is a monotonic decreasing sequence and bounded below and hence $(r^n)$ converges.

Let $(r^n) \to \ell$.

Since $r^n > 0$ for all n, $\ell > 0$ ---------- (1).

Claim: $\ell = 0$.

Let $\varepsilon > 0$ be given.

Since $(r^n) \to \ell$, there exists $m \in N$ such that $\ell < r^n < \ell + \varepsilon$ for all $n \geq m$

Fix $n > m$.

Then $\ell < r^{n+1}$ .-------------------------- (2)

Also $r^{n+1} = r.r^n < r(\ell+\varepsilon)$ -------------- (3)

$\therefore \ell < r^{n+1} < r(\ell+\varepsilon)$ ( by (2) and (3))

$\therefore \ell < \left(\frac{r}{1-r}\right)\varepsilon$

Since this is true for every $\varepsilon > 0$, we get $\ell \leq 0$ ----- (4)

$\therefore \ell = 0$ ( by (1) and (4) )

Case(iv) Let $-1 < r < 0$.

Then $r^n = (-1)^n |r|^n$ where $0 < |r| < 1$.

By case (iii) $(|r|^n) \to 0$.

Also $((-1)^n)$ is a bounded sequence.

$\therefore ((-1)^n |r|^n)$ converges to 0.

$\therefore (r^n) \to 0$.

Case(v) Let $r = -1$.

In this case $(r^n)$ reduces to $-1, 1, -1, 1, \ldots$ which oscillates finitely.

Case(vi) Let $r > 1$.

Then $0 < \frac{1}{r} < 1$.

By case(iii) $\left(\frac{1}{r^n}\right) \to 0$

$\therefore (r^n) \to \infty$.

Case(vii) Let $r < -1$.

Then the terms of the sequence $(r^n)$ are alternatively positive and negative.

Also $|r| > 1$.

By case (vi) $(|r|^n)$ is unbounded.

$\therefore (r^n)$ oscillates infinitely.

Thus (i) $(r^n)$ converges if $-1 < r \leq 1$.

(ii) $(r^n)$ diverges if $r > 1$.

(iii) $(r^n)$ oscillates if $r \leq -1$.

6) Let $(a_n)$ and $(b_n)$ be two sequences of positive terms such that

$a_{n+1} = \frac{1}{2}(a_n+b_n)$ and $b_{n+1} = \sqrt{(a_n b_n)}$. Prove that $(a_n)$ and $(b_n)$ converges to the same limit.

SPACE FOR HINT

Solution: By hypothesis $a_{n+1}$ and $b_{n+1}$ are respectively AM and GM of $a_n$ and $b_n$.

Also we know that AM ≥ GM.

$\therefore\ a_{n+1} \geq b_{n+1}$. ------------------------------------ (1)

Moreover AM and GM of two numbers lie between the two numbers.

$\therefore\ a_n \geq a_{n+1} \geq b_n$ for all $n \in N$.----------------- (2)

and $a_n \geq b_{n+1} \geq b_n$ for all $n \in N$.----------------- (3)

$\therefore\ a_n \geq a_{n+1} \geq b_{n+1} \geq b_n$ for all $n \in N$.( by (1) , (2) & (3))

$\therefore$ $(a_n)$ is a monotonic decreasing sequence and $(b_n)$ is a monotonic increasing sequence.

Also $a_n \geq b_n \geq b_1$ for all $n \in N$ and $b_n \leq a_n \leq a_1$ for all $n \in N$.

$\therefore$ $(a_n)$ is a monotonic decreasing sequence and bounded below by $b_1$ and $(b_n)$ is a monotonic increasing sequence and bounded above by $a_1$.

$\therefore (a_n) \to \ell$ (say) and $(b_n) \to m$ (say)

Now $a_{n+1} = \frac{1}{2}(a_n + b_n)$.

Taking limit as $n \to \infty$, we get $\ell = \frac{1}{2}(\ell + m)$.

$\therefore\ \ell = m$.

**Cauchy's first limit theorem:**

Theorem 1.3.18:

Statement: If $(a_n) \to \ell$ then $\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right) \to \ell$.

Proof:

Case(i) Let $\ell = 0$.

Let $b_n = \frac{a_1 + a_2 + \ldots + a_n}{n}$.

Let $\varepsilon > 0$ be given.

Since $(a_n) \to 0$, there exists a natural number m, such that

$|a_n - 0| < \frac{\varepsilon}{2}$ for all $n \geq m$.

(i.e.) $|a_n| < \frac{\varepsilon}{2}$ for all $n \geq m$. ------------------ (1)

Now let $n \geq m$.

Then $|b_n| = \left|\frac{a_1 + a_2 + \ldots + a_m + a_{m+1} + \ldots + a_n}{n}\right|$

$\leq \left|\frac{a_1 + a_2 + \ldots + a_m}{n}\right| + \left|\frac{a_{m+1} + \ldots + a_n}{n}\right|$

$\leq \frac{|a_1| + |a_2| + \ldots + |a_m|}{n} + \frac{|a_{m+1}| + \ldots + |a_n|}{n}$

$= \frac{k}{n} + \frac{|a_{m+1}| + \ldots + |a_n|}{n}$, where $k = |a_1| + |a_2| + \ldots + |a_m|$

$< \frac{k}{n} + \left(\frac{n-m}{n}\right)\frac{\varepsilon}{2}$ (by (1))

$< \frac{k}{n} + \frac{\varepsilon}{2}$ ( since $\left(\frac{n-m}{n}\right) < 1$) --------------- (2)

Since $\left(\frac{k}{n}\right) \to 0$, there exists $n_0 \in N$ such that $\frac{k}{n} < \frac{\varepsilon}{2}$ for all $n \geq n_0$.--- (3)

Let $n_1 = \max\{m, n_0\}$

Then $|b_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ (by (2) & (3) )

$= \varepsilon$ for all $n \geq n_1$.

$\therefore (b_n) \to 0$.

Case(ii) Let $\ell \neq 0$.

Since $(a_n) \to \ell$ , $(a_n - \ell) \to 0$.

$\therefore \left(\frac{(a_1 - l) + (a_2 - l) + \ldots + (a_n - l)}{n}\right) \to 0$. (by case(i))

$\therefore \left(\frac{a_1 + a_2 + \ldots + a_n - nl}{n}\right) \to 0$.

$\therefore \left(\frac{a_1 + a_2 + \ldots + a_n}{n} - l\right) \to 0$.

$\therefore \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right) \to \ell$.

<u>Note</u>: The converse of the above theorem is need not be true.

For example, consider the sequence $(a_n) = ((-1)^n)$.

SPACE FOR HINT

Then $b_n = \frac{a_1 + a_2 + \ldots + a_n}{n} = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$.

Then $(b_n) \to 0$ and $(a_n)$ is not convergent.

**Cesaro's theorem**

Theorem 1.3.19

Statement:

$$\text{If } (a_n) \to a \text{ and } (b_n) \to b \text{ then } \left(\frac{a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1}{n}\right) \to ab.$$

Proof: Let $c_n = \frac{a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1}{n}$.

Put $a_n = a + r_n$ so that $(r_n) \to 0$.

$$\text{Then } c_n = \frac{(a + r_1)b_n + (a + r_2)b_{n-1} + \ldots + (a + r_n)b_1}{n}$$

$$= \frac{a(b_n + b_{n-1} + \ldots + b_1)}{n} + \frac{r_1 b_n + r_2 b_{n-1} + \ldots + r_n b_1}{n}$$

$$= \frac{a(b_1 + b_2 + \ldots + b_n)}{n} + \frac{r_1 b_n + r_2 b_{n-1} + \ldots + r_n b_1}{n}$$

Since $(b_n) \to b$, by Cauchy's first limit theorem,

$$\left(\frac{b_1 + b_2 + \ldots + b_n}{n}\right) \to b.$$

$$\therefore \left(\frac{a(b_1 + b_2 + \ldots + b_n)}{n}\right) \to ab.$$

Hence it is enough to prove that $\left(\frac{r_1 b_n + r_2 b_{n-1} + \ldots + r_n b_1}{n}\right) \to 0.$

Since $(b_n) \to b$, $(b_n)$ is a bounded sequence.

$\therefore$ There exists a real number $k > 0$ such that $|b_n| \le k$ for all n.

$$\therefore \left|\frac{r_1 b_n + r_2 b_{n-1} + \ldots + r_n b_1}{n}\right| \le k \left|\frac{r_1 + r_2 + \ldots + r_n}{n}\right|$$

Since $(r_n) \to 0$, $\left(\frac{r_1 + r_2 + \ldots + r_n}{n}\right) \to 0$ (by Cauchy's first limit theorem)

$$\therefore \left(\frac{r_1 b_n + r_2 b_{n-1} + \ldots + r_n b_1}{n}\right) \to 0.$$

$$\therefore \left(\frac{a_1b_n + a_2b_{n-1} + \ldots + a_nb_1}{n}\right) \to ab.$$

**Cauchy's second limit theorem:**

Theorem 1.3.20:

Statement: Let $(a_n)$ be a sequence of positive terms. Then

$\lim\limits_{n\to\infty} a_n^{1/n} = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ provided the limit on the right hand side exists, whether finite or infinite.

Proof:

Case(i) Let $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell$, finite.

Let $\varepsilon > 0$ be any given real number.

Then there exists $m \in N$ such that

$$\ell - \frac{1}{2}\varepsilon < \frac{a_{n+1}}{a_n} < \ell + \frac{1}{2}\varepsilon \text{ for all } n \geq m.$$

Choose $n \geq m$,

Then $\ell - \frac{1}{2}\varepsilon < \frac{a_{m+1}}{a_m} < \ell + \frac{1}{2}\varepsilon$

$$\ell - \frac{1}{2}\varepsilon < \frac{a_{m+2}}{a_{m+1}} < \ell + \frac{1}{2}\varepsilon$$

….. ….. ….. …..

….. ….. ….. …..

$$\ell - \frac{1}{2}\varepsilon < \frac{a_n}{a_{n-1}} < \ell + \frac{1}{2}\varepsilon$$

Multiplying these inequalities, we get

$$\left(l - \frac{1}{2}\varepsilon\right)^{n-m} < \frac{a_n}{a_m} < \left(l + \frac{1}{2}\varepsilon\right)^{n-m},$$

$$\therefore a_m \frac{\left(l - \frac{1}{2}\varepsilon\right)^n}{\left(l - \frac{1}{2}\varepsilon\right)^m} < a_n < a_m \frac{\left(l + \frac{1}{2}\varepsilon\right)^n}{\left(l + \frac{1}{2}\varepsilon\right)^m}$$

$\therefore k_1 \left(l - \frac{1}{2}\varepsilon\right)^n < a_n < k_2\left(l + \frac{1}{2}\varepsilon\right)$, where $k_1, k_2$ are some constants.

SPACE FOR HINT

$$\therefore k_1^{1/n}\left(l-\frac{1}{2}\varepsilon\right) < a_n^{1/n} < k_2^{1/n}\left(l+\frac{1}{2}\varepsilon\right). \text{ ------------- (1)}$$

Now, $\left(k_1^{1/n}\left(l-\frac{1}{2}\varepsilon\right)\right) \to \ell - \frac{1}{2}\varepsilon$ ( since $(k_1^{1/n}) \to 1$)

$\therefore$ There exists $n_1 \in N$ such that

$$(\ell-\frac{1}{2}\varepsilon)-\frac{1}{2}\varepsilon < k_1^{1/n}\left(l-\frac{1}{2}\varepsilon\right) < (\ell-\frac{1}{2}\varepsilon)+\frac{1}{2}\varepsilon \text{ for all } n \geq n_1. \text{ --- (2)}$$

Similarly, there exists $n_2 \in N$ such that

$$(\ell+\frac{1}{2}\varepsilon)-\frac{1}{2}\varepsilon < k_2^{1/n}\left(l-\frac{1}{2}\varepsilon\right) < (\ell+\frac{1}{2}\varepsilon)+\frac{1}{2}\varepsilon \text{ for all } n \geq n_2. \text{ ---(3)}$$

Let $n_0 = \max\{m, n_1, n_2\}$

Then $\ell - \varepsilon < k_1^{1/n}\left(l-\frac{1}{2}\varepsilon\right) < a_n^{1/n} < k_2^{1/n}\left(l+\frac{1}{2}\varepsilon\right) < \ell + \varepsilon$ for all $n \geq n_0$.

(by (1),(2) & n(3))

$\therefore \ell - \varepsilon < a_n^{1/n} < \ell + \varepsilon$ for all $n \geq n_0$.

Hence $(a_n^{1/n}) \to \ell$.

Case(ii) Let $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$.

Then $\lim_{n\to\infty} \left(\frac{1}{a_{n+1}}\right) \Big/ \left(\frac{1}{a_n}\right) = 0$.

$\therefore$ By case(i), $\left(\frac{1}{a_n}\right)^{1/n} \to 0$.

$\therefore (a_n^{1/n}) \to \infty$.

**Problems:**

1) Show that $\lim_{n\to\infty} \frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}\right) = 0$.

Solution: Let $a_n = \frac{1}{n}$.

Then $(a_n) \to 0$.

By Cauchy's first limit theorem,

SPACE FOR HINT

$$\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)\to 0$$

$$\text{(i.e.)}\ \left(\frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots..+\frac{1}{n}\right)\right)\to 0.$$

2) Show that $\lim\limits_{n\to\infty} n^{1/n} = 1.$

Solution: Let $a_n = n$.

$$\therefore \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1.$$

3) Show that $\frac{1}{n}[(n+1)(n+2)\ldots.(n+n)]^{1/n} \to 4/e.$

Solution: Let $a_n = \frac{1}{n}[(n+1)(n+2)\ldots.(n+n)]^{1/n}$

$$= \left[\frac{(n+1)(n+2)\ldots.(n+n)}{n^n}\right]^{1/n}$$

$$= \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots.\left(1+\frac{n}{n}\right)\right]^{1/n}$$

Let $b_n = \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots.\left(1+\frac{n}{n}\right)$.

Then $a_n = b_n^{1/n}$.

$$\text{Now } \frac{b_{n+1}}{b_n} = \frac{\left(1+\frac{1}{n+1}\right)\left(1+\frac{2}{n+1}\right)\ldots.\left(1+\frac{n+1}{n+1}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots.\left(1+\frac{n}{n}\right)}$$

$$= (2n+1)(2n+2)\frac{n^n}{(n+1)^{n+2}}$$

$$= \frac{2(2n+1)}{n+1}\,\frac{n^n}{(n+1)^n}$$

$$= 2\left(\frac{2+1/n}{1+1/n}\right)\frac{1}{(1+1/n)^n}$$

Taking limit as $n\to\infty$, we get

$$\lim_{n\to\infty}\frac{b_{n+1}}{b_n} = 4/e.$$

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By Cauchy's second limit theorem, $\lim_{n\to\infty} b_n^{1/n} = 4/e$.

$\therefore (a_n) \to 4/e$.

(i.e.) $\frac{1}{n}[(n+1)(n+2)\ldots.(n+n)]^{1/n} \to 4/e$.

**SUBSEQUENCES**

Definition: Let $(a_n)$ be a sequence. Let $(n_k)$ be a strictly increasing sequence of natural numbers. Then $(a_{n_k})$ is called a subsequence of $(a_n)$.

Note: The terms of a subsequence occur in the same order in which they occur in the original sequence.

Example:

1) $(a_{2n})$ is a subsequence of any sequence $(a_n)$.
2) 1,1,1,.... is a subsequence of the sequence 1,0,1,0,1....
3) Any sequence $(a_n)$ is a subsequence of itself.

Note: A subsequence of a non-convergent sequence can be a convergent sequence.

Theorem 1.3.21: If a sequence $(a_n)$ converges to $\ell$, then every sub sequence $(a_{n_k})$ of $(a_n)$ also converges to $\ell$.

Proof: Let $\varepsilon > 0$ be given.

Since $(a_n) \to \ell$, there exists a natural number m, such that

$$|a_n - \ell| < \varepsilon \text{ for all } n \geq m. \text{ ----------------- } (1).$$

Now choose $n_{k_0} \geq m$.

Then $k \geq k_0 \Rightarrow n_k \geq n_{k_0}$ (since $(n_k)$ is a monotonic increasing)

$\Rightarrow n_k \geq m$

$\Rightarrow |a_{n_k} - \ell| < \varepsilon$ (by (1))

Thus $|a_{n_k} - \ell| < \varepsilon$ for all $k \geq k_0$.

$\therefore (a_{n_k}) \to \ell$.

Theorem 1.3.22: If the subsequences $(a_{2n-1})$ and $(a_{2n})$ of a sequence $(a_n)$ converge to the same limit $\ell$ then $(a_{2n})$ also converges to $\ell$.

Proof: Let $\varepsilon > 0$ be given.

Since $(a_{2n-1}) \to \ell$ there exists a natural number $n_1$, such that

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$|a_{2n-1} - \ell| < \varepsilon$ for all $2n - 1 \geq n_1$.

Similarly there exists a natural number $n_2$, such that

$|a_{2n} - \ell| < \varepsilon$ for all $2n \geq n_2$.

Let $m = \max\{n_1, n_2\}$

Then $|a_n - \ell| < \varepsilon$ for all $n \geq m$.

$\therefore (a_n) \to \ell$.

**LIMIT POINTS**

Definition: Let $(a_n)$ be a sequence of real numbers. a is called a limit point or a cluster point of the sequence $(a_n)$ if given $\varepsilon > 0$, there exists infinite number of terms of the sequence in $(a - \varepsilon, a + \varepsilon)$ . If the sequence $(a_n)$ is not bounded above then $\infty$ is a limit point of the sequence. If the sequence $(a_n)$ is not bounded below then $-\infty$ is a limit point of the sequence.

Examples:

1) Consider the sequence 1,0,1,0,... . For this sequence 1 is a limit point since given $\varepsilon > 0$, the interval $(1 - \varepsilon, 1 + \varepsilon)$ contains infinitely many terms of the sequence $a_1, a_3, a_5, \ldots$...
2) For the constant sequence 1,1,1,...., 1 is the limit point.
3) The sequence $(a_n) = (n)$ is not bounded above and hence $\infty$ is a limit point.

Theorem 1.3.23: Let $(a_n)$ be a sequence. A real number a is a limit point of $(a_n)$ iff there exists a subsequence $(a_{n_k})$ of $(a_n)$ converging to a.

Proof: Suppose there exists a subsequence $(a_{n_k})$ of $(a_n)$ converging to a. Let $\varepsilon > 0$ be given.

Then there exists a natural number $k_0$, such that $a_{n_k} \in (a - \varepsilon, a + \varepsilon)$ for all $k \geq k_0$.

$\therefore$ $(a - \varepsilon, a + \varepsilon)$ contains infinitely many terms of the sequence $(a_n)$.

$\therefore$ a is a limit point of the sequence $(a_n)$.

Conversely suppose a is a limit of $(a_n)$.

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Then for each $\varepsilon > 0$ the interval $(a - \varepsilon, a + \varepsilon)$ contains infinitely many terms of the sequence. In particular we can find $n_1 \in N$ such that $a_{n_1} \in (a - 1, a + 1)$.

Also we can find $n_2 > n_1$ such that $a_{n_2} \in (a - \frac{1}{2}, a + \frac{1}{2})$.

Proceeding like this we can find $n_1 < n_2 < n_3 \ldots$ such that $a_{n_k} \in (a - \frac{1}{k}, a + \frac{1}{k})$.

Clearly $(a_{n_k})$ is a subsequence of $(a_n)$ and $|a_{n_k} - a| < 1/k$.

For any $\varepsilon > 0$, $|a_{n_k} - a| < \varepsilon$ if $k > 1/\varepsilon$.

$\therefore (a_{n_k}) \rightarrow a$.

**CAUCHY'S SEQUENCES**

Definition: A sequence $(a_n)$ is said to be a Cauchy sequence if given $\varepsilon > 0$ there exists $n_0 \in N$ such that $| a_n - a_m | < \varepsilon$ for all $n,m \geq n_0$.

Note : The condition $| a_n - a_m | < \varepsilon$ for all $n,m \geq n_0$ can also be written as $| a_{n+p} - a_n | < \varepsilon$ for all $n \geq n_0$ and for all positive integers p.

Examples:

1) The sequence $(1/n)$ is a Cauchy sequence.

Solution: Let $(a_n) = (1/n)$.

Let $\varepsilon > 0$ be given.

Now $| a_n - a_m | = \left|\frac{1}{n} - \frac{1}{m}\right|$.

Therefore if we choose $n_0$ to be any positive integer greater than $1/\varepsilon$, we get $| a_n - a_m | < \varepsilon$ for all $n,m \geq n_0$.

$\therefore$ The sequence $(1/n)$ is a Cauchy sequence.

2) The sequence $((-1)^n)$ is not a Cauchy sequence.

Proof: Let $(a_n) = (-1)^n$.

$\therefore | a_n - a_{n+1} | = 2$.

$\therefore$ If $\varepsilon < 2$, we cannot find $n_0$ such that $| a_n - a_{n+1} | < \varepsilon$ for all $n \geq n_0$.

$\therefore$ The sequence $((-1)^n)$ is not a Cauchy sequence.

Theorem 1.3.24: Any convergent sequence is a Cauchy sequence.

Proof: Let $(a_n) \rightarrow a$.

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Then given $\varepsilon > 0$, there exists $n_0 \in N$ such that $|a_n - a| < \frac{\varepsilon}{2}$ for all $n \geq n_0$.

$$\therefore |a_n - a_m| = |a_n - a + a - a_m|$$
$$\leq |a_n - a| + |a - a_m|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \text{ for all } n,m \geq n_0.$$
$$= \varepsilon.$$

$\therefore (a_n)$ is a Cauchy sequence.

Theorem 1.3.25: Any Cauchy sequence is a bounded sequence.

Proof: Let $(a_n)$ be a Cauchy sequence.

Let $\varepsilon > 0$ be given.

Then there exists $n_0 \in N$ such that $|a_n - a_m| < \varepsilon$ for all $n,m \geq n_0$.

$\therefore |a_n| < |a_{n_0}| + \varepsilon$ for $n \geq n_0$.

Now let $k = \max\{|a_1|, |a_2|, \ldots, |a_{n_0}| + \varepsilon\}$

Then $|a_n| \leq k$ for all n.

Hence $(a_n)$ is a bounded sequence.

Theorem 1.3.26: Let $(a_n)$ be a Cauchy sequence. If $(a_n)$ has a subsequence $(a_{n_k})$ converging to a, then $(a_n) \to a$.

Proof: Let $\varepsilon > 0$ be given.

Then given $\varepsilon > 0$, there exists $n_0 \in N$ such that $|a_n - a_m| < \frac{\varepsilon}{2}$ for all $n,m \geq n_0$.

Since $(a_{n_k}) \to a$, there exists $k_0 \in N$ such that $|a_{n_k} - a| < \frac{\varepsilon}{2}$ for all $k \geq k_0$.

Choose $n_k$ such that $n_k \geq n_{k_0}$ and $n_0$.

$$\text{Then } |a_n - a| = |a_n - a_{n_k} + a_{n_k} - a|$$
$$\leq |a_n - a_{n_k}| + |a_{n_k} - a|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for all } n \geq n_0.$$

Hence $(a_n) \to a$.

SPACE FOR HINT

**Cauchy's general principle of convergence**

Theorem 1.3.27:

Statement: A sequence $(a_n)$ in R is convergent iff it is a Cauchy sequence.

Proof: We already proved that any convergent sequence is a Cauchy sequence.

Conversely, let $(a_n)$ be a Cauchy sequence in R.

Since any Cauchy sequence is a bounded sequence, $(a_n)$ is a bounded sequence.

$\therefore$ There exists a subsequence $(a_{n_k})$ of $(a_n)$ such that $(a_{n_k}) \to a$.

By the above theorem $(a_n) \to a$.

Definition: Let $(a_n)$ be a bounded sequence. Then the lub of the set of all limit points of $(a_n)$ is called the upper limit or limit superior of the sequence and is denoted by $\overline{\lim}\, a_n$ or lim sup $a_n$.

The glb of the set of all limit points of $(a_n)$ is called the lower limit or limit inferior of the sequence and is denoted by $\underline{\lim}\, a_n$ or lim inf $a_n$.

If a sequence $(a_n)$ is not bounded above, then its upper limit is defined to be $\infty$ and if $(a_n)$ is not bounded below, then its lower limit is defined to be $-\infty$.

**CYP Questions:**

1) Prove that any sequence $(a_n)$ diverging to $-\infty$ is bounded above but not bounded below.
2) If $(a_n) \to a$ and $(a_n) \to a$ where $b_n \neq 0$ for all $n \in N$ and $b \neq 0$ then prove that $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$.
3) If $(a_n) \to a$, prove that $(|a_n|) \to |a|$.
4) If $(a_n) \to a$ and $a_n \geq 0$ for all n and $a \neq 0$, then prove that $\left(\sqrt{a_n}\right) \to \sqrt{a}$.
5) Show that

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{(2n^2+1)}}+\frac{1}{\sqrt{(2n^2+2)}}+\dots+\frac{1}{\sqrt{(2n^2+n)}}\right)=\frac{1}{\sqrt{2}}$$

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6) Let $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}$. Show that $(a_n)$ converges.

7) Show that if $|r| < 1$ then $(nr^n) \to 0$.

8) Let $(a_n)$ be any sequence and $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \ell$. If $\ell > 1$, then prove that $(a_n) \to 0$.

9) Prove that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

10) Prove that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

11) Prove that every bounded sequence has a convergent subsequence.

12) Prove that every bounded sequence has atleast one limit point

13) Prove that $(a_n)$ converges to $\ell$ iff $(a_n)$ is bounded and $\ell$ is the only limit point of the sequence.

## SECTION-1.4 -SERIES OF POSITIVE TERMS

### INFINITE SERIES

Definition: Let $(a_n) = a_1, a_2, \ldots, a_n\ldots$ be a sequence of real numbers. Then the expression $a_1 + a_2 + \ldots + a_n\ldots$ is called an infinite series of real numbers and is dented by $\sum_1^{\infty} a_n$ or $\sum a_n$.

Let $s_1 = a_1$, $s_2 = a_1+a_2$, $s_3 = a_1+a_2+a_3$, ....., $s_n = a_1+a_2+\ldots+a_n$,....

Then $(s_n)$ is called the sequence of partial sums of the given series $\sum a_n$.

The series $\sum a_n$ is said to converge, diverge or oscillate according as the sequence of partial sums $(s_n)$ converges, diverges or oscillates.

**Examples:**

1) Consider the series $1 + 1 + 1 + \ldots$.

Here $s_n = n$. Clearly the sequence $(s_n)$ diverges to $\infty$.

Hence the given series diverges to $\infty$

SPACE FOR HINT

2) Consider the geometric series $1 + r + r^2 + \dots + r^n + \dots$

Here $s_n = 1 + r + r^2 + \dots + r^{n-1} = \frac{1-r^n}{1-r}$.

Case(i) $0 \le r < 1$.

Then $(r^n) \to 0$.

$\therefore s_n = \frac{1}{1-r}$

$\therefore$ The given series converges to $\frac{1}{1-r}$.

Case(ii) $r > 1$.Then $s_n = \frac{r^n - 1}{1-r}$.

Also $(r^n) \to \infty$ when $r > 1$.

Hence the series diverges to $\infty$.

Case(iii) $r = 1$.

Then the series becomes $1 + 1 + 1 + \dots$

$\therefore (s_n) = (n)$ which diverges to $\infty$.

Case(iv) $r = -1$

Then the series becomes $1 - 1 + 1 - 1 + \dots$

$$\therefore s_n = \begin{cases} 0 \text{ if } n \text{ is even} \\ 1 \text{ if } n \text{ is odd} \end{cases}$$

$\therefore (s_n)$ oscillates finitely.

Hence the given series oscillates finitely.

Case(v) $r < -1$.

$\therefore (r^n)$ oscillates infinitely.

Hence the given series oscillates infinitely.

3) Consider the series $1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \dots$

Then $s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{(n-1)!}$

The sequence $(s_n) \to e$.

$\therefore$ The given series converges to the sum e.

Note 1) Let $\sum a_n$ be a series of positive terms. Then $(s_n)$ is a monotonic increasing sequence. Hence $(s_n)$ converges or diverges to $\infty$ according as

$(s_n)$ is bounded or unbounded. Hence the series $\sum a_n$ converges or diverges to $\infty$. Thus a series of positive terms cannot oscillate.

Note 2) Let $\sum a_n$ be a convergent series of positive terms converging to the sum s. Then s is the lub of $(s_n)$. Hence $s_n \le s$ for all n.

Also given $\varepsilon > 0$ there exists $m \in N$ such that $s - \varepsilon < s_n$ for all $n \ge m$.

Hence $s - \varepsilon < s_n \le s$ for all $n \ge m$.

Theorem 1.4.1: Let $\sum a_n$ be a convergent series converging to the sum s. Then $\lim_{n\to\infty} a_n = 0$.

Proof: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - s_{n-1}$

$= \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}$

$= s - s = 0.$

**Cauchy's general principle of convergence**

Theorem 1.4.2: The series $\sum a_n$ is convergent iff given $\varepsilon > 0$ there exists $n_0 \in N$ such that $|a_{n+1} + a_{n+2} + \ldots + a_{n+p}| < \varepsilon$ for all $n \ge n_0$ and for all positive integers p.

Proof: Let $\sum a_n$ be a convergent series.

Let $s_n = a_1 + a_2 + \ldots + a_n$.

$\therefore (s_n)$ is a convergent sequence.

Since any convergent sequence is a Cauchy sequence, $(s_n)$ is a Cauchy sequence.

$\therefore$ There exists $n_0 \in N$ such that $|s_{n+p} - s_n| < \varepsilon$ for all $n \ge n_0$ and for all $p \in N$.

$\therefore |a_{n+1} + a_{n+2} + \ldots + a_{n+p}| < \varepsilon$ for all $n \ge n_0$ and for all $p \in N$.

Conversely if $|a_{n+1} + a_{n+2} + \ldots + a_{n+p}| < \varepsilon$ for all $n \ge n_0$ and for all $p \in N$ then $(s_n)$ is a Cauchy sequence in R and hence $(s_n)$ is convergent.

$\therefore$ The given series converges.

**Problems:**

1) Apply Cauchy's general principle of convergence to show that the series $\sum(1/n)$ is not convergent.

Solution: Let $s_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$

SPACE FOR HINT

Suppose the series $\sum(1/n)$ is convergent.

$\therefore$ By Cauchy's general principle of convergence, given $\varepsilon > 0$ there exists $m \in N$ such that $|s_{n+p} - s_n| < \varepsilon$ for all $n \geq m$ and for all $p \in N$.

$$\therefore \left|\left(1+\frac{1}{2}+\ldots+\frac{1}{n+p}\right)-\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)\right| < \varepsilon \text{ for all } n \geq m \text{ and for all } p \in N.$$

$$\therefore \left|\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+p}\right| < \varepsilon \text{ for all } n \geq m \text{ and for all } p \in N.$$

In particular if we take $n = m$ and $p = m$ then we get

$$\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{m+m} > \frac{1}{2m}+\frac{1}{2m}+\ldots+\frac{1}{2m} = \frac{1}{2}$$

$\therefore \frac{1}{2} < \varepsilon$ which is a contradiction since $\varepsilon > 0$ is arbitrary.

$\therefore$ The given series is not convergent.

2) Apply Cauchy's general principle of convergence to show that the series $1-\frac{1}{2}+\frac{1}{3}-\ldots..+(-1)^n\frac{1}{n}+\ldots..$ is convergent.

Solution: Let $s_n = 1-\frac{1}{2}+\frac{1}{3}-\ldots..+(-1)^n\frac{1}{n}$.

$$\therefore |s_{n+p} - s_n| = \left|\frac{1}{n+1}-\frac{1}{n+2}+\ldots+\frac{(-1)^p}{n+p}\right|.$$

Now $\frac{1}{n+1}-\frac{1}{n+2}+\ldots+\frac{(-1)^p}{n+p}$

$$= \left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\ldots+\begin{cases}\frac{1}{n+p-1}-\frac{1}{n+p} \text{ if } p \text{ is even} \\ \frac{1}{n+p} \text{ if } p \text{ is odd}\end{cases}$$

$> 0$

$$\therefore |s_{n+p} - s_n| = \frac{1}{n+1}-\frac{1}{n+2}+\ldots+\frac{(-1)^p}{n+p}$$

SPACE FOR HINT

$$= \frac{1}{n+1} - \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \ldots$$

$$< \frac{1}{n+1}$$

$$< \varepsilon \text{ provided } n > \left(\frac{1}{\varepsilon} - 1\right)$$

$\therefore$ By Cauchy's general principle of convergence the given series is convergent.

**CYP Questions:**

1) Let $\sum a_n$ converges to a and $\sum b_n$ converges to b. Then prove that $\sum (a_n \pm b_n)$ converges to $a \pm b$ and $\sum ka_n$ converges to ka.

2) Show that the series $\sum\left(\frac{1}{2^n}\right)$ converges to the sum 1.

3) Show that the series $1 + 2 + 3 + \ldots.$ diverges to $\infty$.

SPACE FOR HINT

# UNIT – 2

## Unit Structure:

Section 2.1 : Test for series of positive terms – Comparison test

Section 2.2: D'Alember's test – Ratio test – Root test – Rabbe's test

**Introduction:** In this unit we discuss the convergence and divergence of series by using various tests like Comparison test, Kummer's test, ratio test , raabe's test and Cauchy's root test. This unit contains many solved examples.

## SECTION 2.1 : TEST FOR SERIES OF POSITIVE TERMS – COMPARISON TEST

### COMPARISON TEST:

Theorem 2.1.1:

(i) Let $\sum c_n$ be a convergent series of positive terms. Let $\sum a_n$ be another series of positive terms. If there exists $m \in N$ such that $a_n \leq c_n$ for all $n \geq m$ then $\sum a_n$ is also convergent.

(ii) Let $\sum d_n$ be a divergent series of positive terms. Let $\sum a_n$ be another series of positive terms. If there exists $m \in N$ such that $a_n \geq d_n$ for all $n \geq m$ then $\sum a_n$ is also divergent.

Proof: (i) Since the convergence or divergence of a series is not altered by the removal of a finite number of terms, without loss of generality we may assume that $a_n \leq c_n$ for all n.

Let $s_n = c_1 + c_2 + \ldots + c_n$ and $t_n = a_1 + a_2 + \ldots + a_n$.

Since $a_n \leq c_n$ we have $t_n \leq s_n$ .

Since $\sum c_n$ is convergent, $(s_n)$ is a convergent sequence.

$\therefore (s_n)$ is a bounded sequence.

$\therefore$ There exists a real positive number k such that $s_n \leq k$ for all n.

$\therefore$ $t_n \leq s_n \leq k$ for all n.

$\therefore$ $t_n \leq k$ for all n.

SPACE FOR HINT

Hence $(t_n)$ is a bounded above.

Since $a_n \geq 0$, $(t_n)$ is monotonic increasing sequence.

$\therefore (t_n)$ converges.

$\therefore \sum a_n$ converges.

(ii) Let $s_n = d_1 + d_2 + \ldots + d_n$ and $t_n = a_1 + a_2 + \ldots + a_n$.

Let $\sum d_n$ diverges and $a_n \geq d_n$ for all n.

$\therefore t_n \geq s_n$.

Since $\sum d_n$ diverges, $(s_n)$ diverges to $\infty$.

$\therefore (s_n)$ is not bounded above.

$\therefore (t_n)$ is not bounded above.

Also $(t_n)$ is monotonic increasing and hence $(t_n)$ diverges to $\infty$.

$\therefore \sum a_n$ diverges to $\infty$.

Theorem 2.1.2:

(i) If $\sum c_n$ converges and if $\lim_{n\to\infty}\left(\frac{a_n}{c_n}\right)$ exists and is finite then $\sum a_n$ also converges.

(ii) If $\sum d_n$ diverges and if $\lim_{n\to\infty}\left(\frac{a_n}{d_n}\right)$ exists and is greater than zero then $\sum a_n$ also diverges.

Proof: (i) Let $\lim_{n\to\infty}\left(\frac{a_n}{c_n}\right) = k$.

Let $\varepsilon > 0$ be given.

Then there exists $n_1 \in N$ such that $\frac{a_n}{c_n} < k + \varepsilon$ for all $n \geq n_1$.

$\therefore a_n < (k+\varepsilon)c_n$ for all $n \geq n_1$.

Also since $\sum c_n$ is a convergent series, $\sum (k+\varepsilon)c_n$ is also convergent.

By Comparison test, $\sum a_n$ is convergent.

(ii) Let $\lim_{n\to\infty}\left(\frac{a_n}{d_n}\right) = k > 0$.

Choose $\varepsilon = \frac{1}{2}k$.

Then there exists $n_1 \in N$ such that $k - \frac{1}{2}k < \frac{a_n}{d_n} < k + \frac{1}{2}k$ for all $n \geq n_1$.

$\therefore \frac{a_n}{d_n} > \frac{1}{2}k$ for all $n \geq n_1$.

$\therefore a_n > \frac{1}{2}k\, d_n$ for all $n \geq n_1$.

Since $\sum d_n$ is a divergent series, $\sum \frac{1}{2}kd_n$ is also a divergent series.

By Comparison test, $\sum a_n$ is divergent.

Theorem 2.1.3:

(i) Let $\sum c_n$ be a convergent series of positive terms. Let $\sum a_n$ be another series of positive terms. If there exists $m \in N$ such that $\frac{a_{n+1}}{a_n} \leq \frac{c_{n+1}}{c_n}$ for all $n \geq m$, then $\sum a_n$ is also convergent.

(ii) Let $\sum d_n$ be a divergent series of positive terms. Let $\sum a_n$ be another series of positive terms. If there exists $m \in N$ such that $\frac{a_{n+1}}{a_n} \geq \frac{d_{n+1}}{d_n}$ for all $n \geq m$, then $\sum a_n$ is also divergent.

Proof: (i) Since $\frac{a_{n+1}}{a_n} \leq \frac{c_{n+1}}{c_n}$, we have $\frac{a_{n+1}}{c_{n+}} \leq \frac{a_n}{c_n}$.

$\therefore \left(\frac{a_n}{c_n}\right)$ is a monotonic decreasing sequence.

$\therefore \frac{a_n}{c_n} \leq k$ for all n where $k = \frac{a_1}{c_1}$.

$\therefore a_n \leq kc_n$ for all $n \in N$.

Since $\sum c_n$ is convergent, $\sum kc_n$ is also a convergent series of positive terms.

By Comparison test, $\sum a_n$ is convergent.

SPACE FOR HINT

Proof of (ii) is similar to that of (i).

Theorem 2.1.4: The harmonic series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

Proof:

Case(i) Let $p = 1$. Then the series becomes $\sum \frac{1}{n}$ which diverges.

Case(ii) Let $p < 1$. Then $n^p < n$ for all n.

$$\therefore \frac{1}{n^p} > \frac{1}{n}.$$

By comparison test, $\sum \frac{1}{n^p}$ diverges.

Case(iii) Let $p > 1$

Let $s_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p}$.

Then $s_{2^{n+1}-1} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{(2^{n+1}-1)}$

$$= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \dots$$

$$+ \left(\frac{1}{(2^n)^p} + \frac{1}{(2^{n+1})^p} + \dots + \frac{1}{(2^{n+1}-1)^p}\right)$$

$$< 1 + 2\left(\frac{1}{2^p}\right) + 4\left(\frac{1}{4^p}\right) + \dots + 2^n\left(\frac{1}{(2^n)^p}\right)$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2p-2}} + \dots + \frac{1}{2^{(p-1)n}}$$

$$\therefore s_{2^{n+1}-1} < 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \dots + \left(\frac{1}{2^{p-1}}\right)^n$$

Since $p > 1$, $p - 1 > 0$.

Hence $\frac{1}{2^{p-1}} < 1$.

$$\therefore 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \dots + \left(\frac{1}{2^{p-1}}\right)^n$$

$$< \frac{1}{1-\frac{1}{2^{p-1}}} = k(\text{say})$$

$\therefore s_{2^{n+1}-1} < k.$

Let n be any positive integer. Choose $m \in N$ such that $n \le 2^{m+1} - 1$.

Since $(s_n)$ is a monotonic increasing sequence, $s_n \le s_{2^{m+1}-1}$.

Hence $s_n < k$ for all n.

$\therefore$ $(s_n)$ is a monotonic increasing sequence and bounded above.

$\therefore$ $(s_n)$ is convergent.

$\therefore$ $\Sigma \frac{1}{n^p}$ converges.

**Problems**

1) Discuss the convergence of the series $\Sigma \frac{1}{\sqrt{(n^3+1)}}$.

Solution: Clearly $\frac{1}{\sqrt{(n^3+1)}} < \frac{1}{n^{3/2}}$

Also $\Sigma \frac{1}{n^{3/2}}$ is convergent.

By Comparison test, $\Sigma \frac{1}{\sqrt{(n^3+1)}}$ is convergent.

2) Discuss the convergence of the series $\Sigma \frac{\sqrt{(n+1)}-\sqrt{n}}{n^p}$.

Solution: Let $a_n = \frac{\sqrt{(n+1)}-\sqrt{n}}{n^p} = \frac{\sqrt{(n+1)}-\sqrt{n}}{n^p} \text{ x } \frac{\sqrt{(n+1)}+\sqrt{n}}{\sqrt{(n+1)}+\sqrt{n}}$

$$= \frac{n+1-n}{n^p\left(\sqrt{(n+1)}+\sqrt{n}\right)} = \frac{1}{n^p\left(\sqrt{(n+1)}+\sqrt{n}\right)}$$

Let $b_n = \frac{1}{n^{p+1/2}}$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{p+1/2}}{n^p\left(\sqrt{(n+1)}+\sqrt{n}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{\left(\sqrt{(1+1/n)}+1\right)}$$

SPACE FOR HINT

$= \frac{1}{2}$

Also $\sum b_n$ is convergent if $p + ½ > 1$ and divergent $p + ½ \le 1$

$\therefore \sum a_n$ is convergent if $p > ½$ and divergent if $p \le ½$.

3) Discuss the convergence of the series $\sum \frac{1^2+2^2+...+n^2}{n^4+1}$.

Solution: Let $a_n = \frac{1^2+2^2+...+n^2}{n^4+1}$

$= \frac{n(n+1)(2n+1)}{6(n^4+1)}$

Now let $b_n = \frac{1}{n}$.

$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2(n+1)(2n+1)}{6(n^4+1)}$

$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{6\left(1+\frac{1}{n^4}\right)}$

$= \frac{1}{3}$.

Also $\sum b_n$ is divergent.

$\therefore \sum a_n$ is divergent.

4) Show that $\sum \frac{1}{4n^2-1} = \frac{1}{2}$.

Solution: Let $a_n = \frac{1}{4n^2-1}$

Then $a_n < \frac{1}{n^2}$

Also $\sum \frac{1}{n^2}$ is convergent.

$\therefore$ By Comparison test, the given series converges

By partial fraction, we get

$$a_n = \frac{1}{4n^2-1} = \frac{1}{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

SPACE FOR HINT

$\therefore s_n = a_1 + a_2 + \ldots + a_n$

$$= \frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\ldots\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right]$$

$$= \frac{1}{2}\left[1-\frac{1}{2n+1}\right]$$

$$\therefore \lim_{n\to\infty} s_n = \frac{1}{2}.$$

$$\therefore \sum \frac{1}{4n^2-1} = \frac{1}{2}.$$

**CYP Questions:**

1) Discuss the convergence of the series $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \ldots$

2) Discuss the convergence of the series $\sum_{1}^{\infty} (\log\log n)^{-\log n}$.

3) Discuss the convergence of the series $\sum \frac{n^p}{\sqrt{(n+1)} - \sqrt{n}}$.

4) Discuss the convergence of the series $\sum_{2}^{\infty} (\log n)^{-\log\log n}$.

## SECTION 2.2: D'ALEMBER'S TEST – RATIO TEST – ROOT TEST – RABBE'S TEST

**KUMMERS TEST**

Theorem 2.2.1:

Statement: Let $\sum a_n$ be a given series of positive terms and $\sum \frac{1}{d_n}$ be a series of positive terms diverging to $\infty$. Then

(a) $\sum a_n$ converges if $\lim_{n\to\infty}\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) > 0$ and

(b) $\sum a_n$ diverges if $\lim_{n\to\infty}\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) < 0$

Proof: (a) Let $\lim_{n\to\infty}\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) = \ell > 0$

SPACE FOR HINT

Case(i) Let $\ell$ be finite.

Then given $\varepsilon > 0$, there exists $m \in N$ such that

$$\ell - \varepsilon < d_n \frac{a_n}{a_{n+1}} - d_{n+1} < \ell + \varepsilon \text{ for all } n \geq m.$$

$$(\ell - \varepsilon)\, a_{n+1} < d_n a_n - d_{n+1} a_{n+1} < (\ell - \varepsilon)\, a_{n+1} \text{ for all } n \geq m.$$

$$\therefore d_n a_n - d_{n+1} a_{n+1} > (\ell - \varepsilon)\, a_{n+1} \text{ for all } n \geq m.$$

Taking $\varepsilon = \frac{1}{2}\ell$, we get $d_n a_n - d_{n+1} a_{n+1} > \frac{1}{2}\ell a_{n+1}$ for all $n \geq m$.

Now let $n \geq m$,

$$\therefore d_m a_m - d_{m+1} a_{m+1} > \frac{1}{2}\ell a_{m+1}$$

$$d_{m+1} a_{m+1} - d_{m+2} a_{m+2} > \frac{1}{2}\ell a_{m+2}$$

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$$d_{n-1} a_{n-1} - d_n a_n > \frac{1}{2}\ell a_n.$$

Adding all these inequalities, we get

$$d_m a_m - d_n a_n > \frac{1}{2}\ell (a_{m+1} + \ldots + a_n)$$

$\therefore d_m a_m - d_n a_n > \frac{1}{2}\ell (s_n - s_m)$ where $s_n = a_1 + a_2 \ldots + a_n$

$$\therefore d_m a_m > \frac{1}{2}\ell (s_n - s_m).$$

$\therefore s_n < \dfrac{2d_m a_m + l s_m}{l}$, which is independent of n.

$\therefore$ The sequence $(s_n)$ of partial sums is bounded.

$\therefore \sum a_n$ is convergent.

Case(ii) Let $\ell = \infty$.

Then given any real number $k > 0$ there exists a positive integer m such that $d_n \frac{a_n}{a_{n+1}} - d_{n+1} > k$ for all $n \geq m$.

$\therefore d_n a_n - d_{n+1} a_{n+1} > k\, a_{n+1}$ for all $n \geq m$. --------- (1)

SPACE FOR HINT

Now let $n \geq m$. Write (1) for m, m+1,……,(n−1) and adding we get

$$d_m a_m - d_n a_n > k(a_{m+1} + \ldots + a_n)$$

$$= k(s_n - s_m).$$

$$\therefore d_m a_m > k(s_n - s_m).$$

$$\therefore s_n < \frac{d_m a_m}{k} + s_m$$

$\therefore$ The sequence $(s_n)$ is bounded and hence $\sum a_n$ is convergent.

(b) Let $\lim_{n \to \infty} \left( d_n \frac{a_n}{a_{n+1}} - d_{n+1} \right) = \ell < 0.$

Suppose $\ell$ is finite.

Choose $\varepsilon > 0$ such that $\ell + \varepsilon < 0$.

Then there exists $m \in N$ such that

$$\ell - \varepsilon < d_n \frac{a_n}{a_{n+1}} - d_{n+1} < \ell + \varepsilon < 0 \text{ for all } n \geq m.$$

$$\therefore d_n a_n < d_{n+1} a_{n+1} \text{ for all } n \geq m.$$

Now let $n \geq m$. Then

$\therefore d_m a_m < d_{m+1} a_{m+1}$

$d_{m+1} a_{m+1} < d_{m+2} a_{m+2}$

……………………

……………………

$d_{n-1} a_{n-1} < d_n a_n$.

$\therefore d_m a_m < d_n a_n$

$\therefore a_n > \dfrac{d_m a_m}{d_n}$

By hypothesis, $\sum \frac{1}{d_n}$ is divergent.

Hence $\sum \frac{d_m a_m}{d_n}$ is divergent.

By Comparison test, $\sum a_n$ is divergent.

The proof is similar if $\ell = -\infty$.

SPACE FOR HINT

Note: The above test fails if $\lim_{n\to\infty}\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) = 0$.

<u>Cor: 1</u> **(D' Alembert's ratio test)**

Let $\sum a_n$ be a series of positive terms. Then $\sum a_n$ converges if

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} > 1$ and diverges if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$

<u>Proof</u>: The series 1 + 1 + 1 + ….. is divergent.

Put $d_n = 1$ in Kummer's test.

Then $d_n \frac{a_n}{a_{n+1}} - d_{n+1} = \frac{a_n}{a_{n+1}} - 1$

$\therefore \sum a_n$ converges if $\lim_{n\to\infty}\left(\frac{a_n}{a_{n+1}} - 1\right) > 0$.

$\therefore \sum a_n$ converges if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} > 1$

Similarly $\sum a_n$ diverges if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$.

<u>Cor: 2</u> **(Raabe's test)**

Let $\sum a_n$ be a series of positive terms. Then $\sum a_n$ converges if

$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1$ and diverges if $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) < 1$.

<u>Proof</u>: The series $\sum \frac{1}{n}$ is divergent.

Put $d_n = n$ in Kummer's test.

Then $d_n \frac{a_n}{a_{n+1}} - d_{n+1} = n\frac{a_n}{a_{n+1}} - (n+1) = n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1$

$\therefore \sum a_n$ converges if $\lim_{n\to\infty}\left(\frac{a_n}{a_{n+1}} - 1\right) > 0$.

$\therefore \sum a_n$ converges if $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1$

Similarly $\sum a_n$ diverges if $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) < 1$.

SPACE FOR HINT

Cor: 3 **( De Morgan and Bertrand's test)**

Let $\sum a_n$ be a series of positive terms.

Then $\sum a_n$ converges if $\lim\limits_{n\to\infty} \log n\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right]>1$ and

diverges if $\lim\limits_{n\to\infty} \log n\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right]<1$.

Proof: The series $\sum\frac{1}{n\log n}$ is divergent.

Put $d_n = n\log n$ in Kummer's test.

Then $d_n\frac{a_n}{a_{n+1}}-d_{n+1}=(n\log n)\frac{a_n}{a_{n+1}}-(n+1)\log(n+1)$

$$=\log n\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right]+(n+1)\log n-(n+1)\log(n+1)$$

$$=\log n\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right]-(n+1)\log\left(\frac{n+1}{n}\right)$$

$$\therefore \lim_{n\to\infty}\left(d_n\frac{a_n}{a_{n+1}}-d_{n+1}\right)$$

$$=\lim_{n\to\infty}(\log n)\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right]-\lim_{n\to\infty}\log\left(1+\frac{1}{n}\right)^{n+1}$$

$$=\lim_{n\to\infty}(\log n)\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right]-1.$$

$\therefore \sum a_n$ converges if $\lim\limits_{n\to\infty} \log n\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right]>1$ and

diverges if $\lim\limits_{n\to\infty} \log n\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right]<1$.

Note: General form of Kummer's test.

Let $\sum a_n$ be a given series of positive terms and $\sum\frac{1}{d_n}$ be a series of positive terms diverging to $\infty$. Then

(a) $\sum a_n$ converges if $\lim\inf\left(d_n\frac{a_n}{a_{n+1}}-d_{n+1}\right)>0$and

(b) $\sum a_n$ diverges if $\lim\sup\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) < 0$

**Gauss Test**

Theorem 2.2.2:

Statement: Let $\sum a_n$ be a series of positive terms such that

$\frac{a_n}{a_{n+1}} = 1 + \frac{\beta}{n} + \frac{r_n}{n^p}$ where $p > 1$ and $(r_n)$ is a bounded sequence. Then

the series $\sum a_n$ converges if $\beta > 1$ and diverges if $\beta \le 1$.

Proof: $\frac{a_n}{a_{n+1}} = 1 + \frac{\beta}{n} + \frac{r_n}{n^p}$, $p > 1$

$$\therefore n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{\beta}{n} + \frac{r_n}{n^p}\right) = \beta + \frac{r_n}{n^{p-1}}$$

Since $p > 1$, $\lim_{n\to\infty} \frac{1}{n^{p-1}} = 0$.

Also $(r_n)$ is a bounded sequence.

Hence $\lim_{n\to\infty} \frac{r_n}{n^{p-1}} = 0$.

$$\therefore \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \beta.$$

$\therefore$ By Raabe's test $\sum a_n$ converges if $\beta > 1$ and diverges if $\beta \le 1$.

If $\beta = 1$, Raabe's test fails. In this case we apply Kummer's test by taking $d_n = n \log n$.

$$\text{Now } d_n \frac{a_n}{a_{n+1}} - d_{n+1} = n\log n\left(1 + \frac{1}{n} + \frac{r_n}{n^p}\right) - (n+1)\log(n+1)$$

$$= -(n+1)\log\left(1 + \frac{1}{n}\right) + \frac{r_n \log n}{n^{p-1}}$$

$$= -\log\left(1 + \frac{1}{n}\right)^{n+1} + \frac{r_n \log n}{n^{p-1}}$$

It is clear that $\left(\frac{\log n}{n^{p-1}}\right) \to 0$.

Since $(r_n)$ is a bounded sequence, $\left(\frac{r_n \log n}{n^{p-1}}\right) \to 0$.

SPACE FOR HINT

$$\therefore \lim_{n\to\infty}\left(d_n \frac{a_n}{a_{n+1}} - d_{n+1}\right) = -\log e = -1 < 0.$$

$\therefore$ By Kummer's test $\sum a_n$ diverges.

**Problems:**

1) Test the convergence of the series $\frac{1}{3}+\frac{1.2}{3.5}+\frac{1.2.3}{3.5.7}+\ldots$

Solution: Let $a_n = \frac{1.2.3....n}{3.5.7....(2n+1)}$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{1.2.3....n}{3.5.7....(2n+1)} \text{x} \frac{3.5.7....(2n+1)(2n+3)}{1.2.3....n(n+1)}$$

$$= \frac{2n+3}{n+1} = \frac{2+3/n}{1+1/n}.$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 2 > 1.$$

$\therefore$ By D'Alembert's ratio test $\sum a_n$ is convergent.

2) Test the convergence of the series $\sum \frac{n^n}{n!}$

Solution: Let $a_n = \frac{n^n}{n!}$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{n^n}{n!} \text{x} \frac{(n+1)!}{(n+1)^{(n+1)}}$$

$$= \frac{(n+1)n^n}{(n+1)^{(n+1)}} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{e} < 1.$$

$\therefore$ $\therefore$ By D'Alembert's ratio test $\sum a_n$ is divergent.

3) Test the convergence of the series $\sum \frac{2^n n!}{n^n}$,

Solution: Let $a_n = \frac{2^n n!}{n^n}$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{2^n n!}{n^n} \text{x} \frac{(n+1)^{(n+1)}}{2^{n+1}(n+1)!}$$

SPACE FOR HINT

$$= \frac{(n+1)^{(n+1)}}{2(n+1)n^n} = \frac{1}{2}\left(1+\frac{1}{n}\right)^n$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{e}{2} > 1.$$

$\therefore$ By D'Alembert's ratio test $\sum a_n$ is convergent.

4) Test the convergence of the series $\sum\sqrt{\frac{n}{n+1}}x^n$ where x is any positive real number.

Solution: Since x is positive the given series is a series of positive terms.

Now, $\frac{a_n}{a_{n+1}} = \sqrt{\frac{n(n+2)}{(n+1)^2}}\frac{1}{x}$

$$= \sqrt{\frac{(1+2/n)}{(1+1/n)^2}}\frac{1}{x}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{x}$$

$\therefore$ By ratio test, $\sum a_n$ converges if $x < 1$ and diverges if $x > 1$.

If $x = 1$, the ratio test fails.

When $x = 1$, $a_n = \sqrt{\frac{n}{n+1}} = \frac{1}{\sqrt{(1+1/n)}}$.

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1.$$

$\therefore$ The series diverges.

5) Test the convergence of the series $\sum\frac{n^2+1}{5^n}$.

Solution: Let $a_n = \frac{n^2+1}{5^n}$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{n^2+1}{5^n} x \frac{5^{n+1}}{(n+1)^2+1}$$

$$= \frac{5(n^2+1)}{(n+1)^2+1}$$

$$= \frac{5(n^2+1)}{n^2+2n+2}$$

SPACE FOR HINT

$$= \frac{5\left(1+\frac{1}{n^2}\right)}{1+\frac{2}{n}+\frac{2}{n^2}}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 5 > 1.$$

$\therefore$ By ratio test the series converges.

6) Test the convergence of the series

$$\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{2^2}+\frac{1}{3^2}\right)+\left(\frac{1}{2^3}+\frac{1}{3^3}\right)+\ldots\ldots$$

Solution: Let $a_n = \frac{1}{2^n}+\frac{1}{3^n} = \frac{2^n+3^n}{2^n 3^n}$.

$$\therefore \frac{a_n}{a_{n+1}} = \frac{2^n+3^n}{2^n 3^n} \text{ x } \frac{2^{n+1}3^{n+1}}{2^{n+1}+3^{n+1}}$$

$$= \frac{6(2^n+3^n)}{2^{n+1}+3^{n+1}}$$

$$= \frac{2(1+(2/3)^n}{(1+(2/3)^{n+1}}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 2 > 1.$$

$\therefore$ By ratio test the series converges.

7) Test the convergence of the series

$$\frac{1}{3}x+\frac{1.2}{3.5}x^2+\frac{1.2.3}{3.5.7}x^2+\ldots\ldots$$

Solution: Let $a_n = \frac{1.2.3....n}{3.5.7....(2n+1)}x^n$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{1.2.3....n}{3.5.7....(2n+1)}x^n \text{ x } \frac{3.5.7....(2n+1)(2n+3)}{1.2.3....n(n+1)}\frac{1}{x^{n+1}}$$

$$= \frac{2n+3}{n+1}\left(\frac{1}{x}\right) = \frac{2+3/n}{1+1/n}\left(\frac{1}{x}\right).$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{2}{x}.$$

$\therefore$ By D'Alembert's ratio test the series $\sum a_n$ converges if $\frac{2}{x} > 1$ and diverges if $\frac{2}{x} < 1$.

(i.e.) the series $\sum a_n$ converges if $x < 2$ and diverges if $x > 2$.

If $x = 2$, the ratio test fails.

In this case, $\frac{a_n}{a_{n+1}} = \frac{2n+3}{2n+2} = 1 + \frac{1}{2n+2}$

$$\therefore \frac{a_n}{a_{n+1}} - 1 = \frac{1}{2n+2}$$

$$\therefore n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{n}{2n+2} = \frac{1}{2+2/n}$$

$$\therefore \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{1}{2} < 1$$

$\therefore$ By Raabe's test, the series diverges.

**Cauchy's Root test**

Theorem 2.2.3:

Statement: Let $\sum a_n$ be a given series of positive terms. Then $\sum a_n$ is convergent if $\lim_{n\to\infty} a_n^{1/n} < 1$ and divergent if $\lim_{n\to\infty} a_n^{1/n} > 1$.

Proof:

Case(i) Let $\lim_{n\to\infty} a_n^{1/n} = \ell < 1$.

Choose $\varepsilon > 0$ such that $\ell + \varepsilon < 1$.

Then there exists $m \in N$ such that $a_n^{1/n} < \ell + \varepsilon$ for all $n \geq m$.

$\therefore$ $a_n < (\ell + \varepsilon)^n$ for all $n \geq m$.

Since $\ell + \varepsilon < 1$, $\sum (l + \varepsilon)^n$ is convergent.

$\therefore$ By Comparison test $\sum a_n$ is convergent.

Case(ii) Let $\lim_{n\to\infty} a_n^{1/n} = \ell > 1$.

Choose $\varepsilon > 0$ such that $\ell - \varepsilon > 1$.

Then there exists $m \in N$ such that $a_n^{1/n} > \ell - \varepsilon$ for all $n \geq m$.

$\therefore$ $a_n > (\ell - \varepsilon)^n$ for all $n \geq m$.

SPACE FOR HINT

Since $l-\varepsilon > 1$, $\sum(l-\varepsilon)^n$ is divergent.

$\therefore$ By Comparison test $\sum a_n$ is divergent.

Note: The following is the more general form of Cauchy's root test.

Let $\sum a_n$ be a given series of positive terms. Then $\sum a_n$ is convergent if $\lim\limits_{n\to\infty} \sup a_n^{1/n} < 1$ and divergent if $\lim\limits_{n\to\infty} \sup a_n^{1/n} > 1$.

**Cauchy's condensation test**

Theorem 2.2.4:

Statement: Let $a_1+a_2+a_3+....+a_n+......$ -------------------- (1)

be a series of positive terms and whose terms are monotonic decreasing. Then this series converges or diverges according as the series

$$ga_g + g^2a_g^2 + ......+ g^na_g^n + .........\quad \text{----------------------- (2)}$$

converges or diverges where g is any positive integer > 1.

Proof: Let $s_n = a_1+a_2+a_3+....+a_n$ and

$$t_n = ga_g + g^2a_g^2 + ......+ g^na_g^n.$$

Then $s_g^n = (a_1+a_2+a_3+....+a_g) + (a_{g+1}+a_{g+2}+....+a_g^2) +$

$$.........................\ .........................$$

$$+ (a_{g^{n-1}+1} + a_{g^{n-1}+2} + ......+ a_g^n).$$

$\leq ga_1 + (g^2-g)a_g + .....+ (g^n-g^{n-1})\ a_g^{n-1}$. (since the terms of the series are monotonic decreasing)

$= ga_1 + g(g-1)a_g + g^2(g-1)a_g^2 + .....+ g^{n-1}(g-1)\ a_g^{n-1}$.

$= ga_1 + (g-1)(a_g + g^2a_g^2 + .....+ g^{n-1}\ a_g^{n-1})$.

$= ga_1 + (g-1)\ t_{n-1}$.

$\therefore s_g^n \leq ga_1 + (g-1)\ t_{n-1}$.

$\therefore$ If the series (2) converges then (1) converges.

Now $s_g^n \geq ga_g + (g^2-g)\ a_g^2 + .....+ (g^n-g^{n-1})\ a_g^n$.

$$= ga_g + \frac{g-1}{g}(g^2a_g^2 + .....+ g^n\ a_g^n)$$

$$= ga_g + \frac{g-1}{g}(t_n - ga_g)$$

SPACE FOR HINT

$$= a_g + \frac{g-1}{g} t_n.$$

$\therefore$ If the series (2) diverges then (1) diverges.

**Problems:**

1) Test the convergence of the series $\sum \frac{1}{(\log n)^n}$.

Solution: Let $a_n = \frac{1}{(\log n)^n}$

$$\therefore \sqrt[n]{a_n} = \frac{1}{\log n}$$

$$\therefore \lim_{n\to\infty} \sqrt[n]{a_n} = 0 < 1.$$

$\therefore$ By Cauchy's root test $\sum \frac{1}{(\log n)^n}$ converges.

2) Test the convergence of the series $\sum \left(1+\frac{1}{n}\right)^{-n^2}$.

Solution: Let $a_n = \left(1+\frac{1}{n}\right)^{-n^2}$

$$\therefore \sqrt[n]{a_n} = \left(1+\frac{1}{n}\right)^{-n}$$

$$\therefore \lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{e} < 1.$$

$\therefore$ By Cauchy's root test $\sum \left(1+\frac{1}{n}\right)^{-n^2}$ converges.

3) Test the convergence of the series $\sum \frac{n^3+a}{2^n+a}$.

Solution: Let $a_n = \frac{n^3+a}{2^n+a}$ and $b_n = \frac{n^3}{2^n}$

$$\therefore \frac{a_n}{b_n} = \frac{n^3+a}{2^n+a} \times \frac{2^n}{n^3}$$

$$= \frac{(n^3+a)2^n}{n^3(2^n+a)} = \left(\frac{n^3+a}{n^3}\right)\left(\frac{2^n}{2^n+a}\right)$$

SPACE FOR HINT

$$= \left(1+\frac{a}{n^3}\right)\left(\frac{1}{1+(a/2^n)}\right).$$

$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = 1.$

$\therefore$ By Comparison test, the given series is convergent or divergent according as $\sum \frac{n^3}{2^n}$ is convergent or divergent.

$$\sqrt[n]{b_n} = \left(\frac{n^3}{2^n}\right)^{1/n} = \frac{n^{3/n}}{2}.$$

Also $\lim_{n\to\infty} n^{3/n} = 1.$

$\therefore \lim_{n\to\infty} \sqrt[n]{b_n} = \frac{1}{2}$

$\therefore \sum b_n$ is convergent.

$\therefore \sum a_n$ is convergent.

4) Test the convergence of the series $\sum \frac{1}{n\log n}$.

Solution: By Cauchy's condensation test the series $\sum \frac{1}{n\log n}$ converges or diverges with the series $\sum \frac{2^n}{2^n \log 2^n} = \sum \frac{1}{n\log 2} = \frac{1}{\log 2}\sum \frac{1}{n}$

We know that the series $\sum \frac{1}{n}$ is divergent.

$\therefore$ The given series diverges.

5) Test the convergence of the series

$$\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{2^2}+\frac{1}{3^2}\right)+\left(\frac{1}{2^3}+\frac{1}{3^3}\right)+\ldots\ldots$$

Solution:

$$a_n^{1/n} = \begin{cases} \left(\frac{1}{3^{n/2}}\right)^{1/n} & \text{if n is even} \\ \left(\frac{1}{2^{(n+1)/2}}\right)^{1/n} & \text{if n is odd} \end{cases}$$

SPACE FOR HINT

$$= \begin{cases} \dfrac{1}{\sqrt{3}} & \text{if n is even} \\ \dfrac{1}{2^{1/2(1+1/n)}} & \text{if n is odd} \end{cases}$$

The sequence $\left(\frac{1}{2^{1/2(1+1/n)}}\right)$ converges to $\left(\frac{1}{\sqrt{2}}\right)$ as $n \to \infty$.

$\therefore \frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{3}}$ are the only limit points of the given sequence

$\lim\sup a_n^{1/n} = \frac{1}{\sqrt{2}} < 1.$

$\therefore$ By Cauchy's root test the given series is convergent.

**CYP Questions:**

1) Test the convergence of the series $1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots$ where x is any positive real number.

2) Test the convergence of the series $\sum \frac{n^p}{n!}$, $(p > 0)$.

3) Test the convergence of the series $\sum \frac{x^n}{n}$.

4) Test the convergence of the series

$$1 + \frac{\alpha\beta}{r}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{r(r+1)2!}x^2 + \ldots$$

5) Prove that the series $\sum e^{-\sqrt{n}} x^n$ converges if $0 < x < 1$ and diverges if $x > 1$.

6) Test the convergence of the series $\sum \frac{1}{n(\log n)^p}$.

7) Test the convergence of the series $\sum \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

SPACE FOR HINT

# UNIT – 3

## Unit Structure:

Section 3.1: Cauchy's Integral test

Section 3.2: Harmonic series

Section 3.3: Absolute Convergence

Section 3.4: Conditional convergence

**Introduction:** In this unit we discuss the convergence by cauchy's integral test and the Harmonic series. Also we discuss the absolute convergence and conditional convergence of the series and some important theorems on the series.

## SECTION-3.1 - CAUCHY'S INTEGRAL TEST

### CAUCHY'S INTEGRAL TEST

Theorem 3.1.1: Let f be a non-negative monotonic decreasing integrable function defined on $[1,\infty)$. Let $I_n = \int_1^n f(x)dx$ . Then the series $\Sigma f(n)$ converges iff the sequence $(I_n)$ converges. Further the sum of the series lies between $I = \lim_{n\to\infty} I_n$ and $I + f(I)$.

Proof: Let $f(n) = a_n$ .

Since f is monotonic decreasing $f(n-1) \geq f(x) \geq f(n)$ where $n-1 \leq x \leq n$

$$\therefore a_{n-1} \geq f(x) \geq a_n.$$

$$\therefore \int_{n-1}^{n} a_{n-1}dx \geq \int_{n-1}^{n} f(x)dx \geq \int_{n-1}^{n} a_n dx.$$

$$\therefore (n-(n-1))a_{n-1} \geq \int_{n-1}^{n} f(x)dx \geq (n-(n-1))a_n.$$

$$\therefore a_{n-1} \geq \int_{n-1}^{n} f(x)dx \geq a_n. \quad \text{------------------------} (1)$$

Replacing n by 2,3,4,....,n in (1) and adding we get

$$a_1+a_2+a_3+\ldots.+a_{n-1} \geq \int_1^n f(x)dx \geq a_2+a_3+a_4+\ldots.+a_n$$

SPACE FOR HINT

$\therefore\ s_n - a_n \geq I_n \geq s_n - a_1$ where $s_n = a_1 + a_2 + a_3 + \ldots + a_n$

$\therefore\ a_1 \geq s_n - I_n \geq a_n.$

Since f is non-negative, $f(n) = a_n \geq 0$.

$\therefore\ a_1 \geq s_n - I_n \geq a_n \geq 0.$

Let $s_n - I_n = A_n$.

$\therefore\ a_1 \geq A_n \geq 0.$ -------------------------------- (2)

$\therefore$ $(A_n)$ is a bounded sequence.

Also $A_{n+1} - A_n = s_{n+1} - s_n - I_{n+1} + I_n$

$$= a_{n+1} - \int_n^{n+1} f(x)dx$$

$$\leq a_{n+1} - \int_n^{n+1} a_{n+1} dx = 0$$

$\therefore A_{n+1} \leq A_n$ .

$\therefore$ $(A_n)$ is a bounded monotonic decreasing sequence.

$\therefore\ \lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty} (s_n - I_n)$ exists.

$\therefore\ \lim\limits_{n\to\infty} s_n$ exists iff $\lim\limits_{n\to\infty} I_n$ exists and $\lim\limits_{n\to\infty} A_n = s - I$ ------- (3)

where s is the sum of the series and $I = \lim\limits_{n\to\infty} I_n$.

$\therefore$ The series $\sum f(n)$ converges iff the sequence $(I_n)$ converges.

In this case, from(2), $a_1 \geq \lim\limits_{n\to\infty} A_n \geq 0$.

By (3), $a_1 \geq s - I \geq 0$.

$\therefore\ I + a_1 \geq s \geq I$.

$\therefore\ I + f(1) \geq s \geq I$.

**Problems:**

1) Show that $\lim\limits_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots.. + \frac{1}{n} - \log n\right)$ exists and lies between 0 and 1.(This limit is known as Euler's constant)

Solution: Consider the function $f(x) = \frac{1}{x}$ defined on $[1,\infty)$.

Then f(x) is non-negative and monotonic decreasing.

Let $I_n = \int_1^n \frac{1}{x} dx = (\log x)_1^n = \log n$.

SPACE FOR HINT

Let $f(x) = a_n = \frac{1}{n}$.

$$\therefore s_n - I_n = 1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}-\log n$$

By Cauchy's integral test $s_n - I_n$ converges and its limit lies between 0 and $a_1$.

But $a_1 = f(1) = 1$.

$$\therefore \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}-\log n\right) \text{ exists and lies between 0 and 1.}$$

2) Discuss the convergence of the series $\sum_{n=2}^{\infty}\frac{1}{n(\log n)^{\alpha}}$ where $\alpha \geq 0$.

Solution: Let $a_n = \frac{1}{n(\log n)^{\alpha}}, \alpha \geq 0, n \geq 2$.

Consider the function $f(x) = \frac{1}{x(\log x)^{\alpha}}$ so that $f(n) = a_n$.

Clearly f(x) is non-negative and monotonic decreasing on $[2, \infty)$.

Case(i) Let $\alpha \neq 1$.

$$\therefore I_n = \int_2^n \frac{dx}{x(\log x)^{\alpha}}$$

$$= \left[\frac{1}{1-\alpha}(\log x)^{1-\alpha}\right]_2^n$$

$$= \frac{(\log n)^{1-\alpha}}{1-\alpha} - \frac{(\log 2)^{1-\alpha}}{1-\alpha}$$

$\therefore (I_n)$ converges if $\alpha > 1$ and diverges if $\alpha < 1$.

Hence by Cauchy's integral test, the given series converges if $\alpha > 1$ and diverges if $\alpha < 1$.

Case(ii) Let $\alpha = 1$.

$$\therefore I_n = [\log(\log x)]_2^n$$

$$= \log(\log n) - \log(\log 2) \to \infty \text{ as } n \to \infty.$$

$\therefore (I_n)$ diverges and hence the given series diverges.

3) Using the integral test discuss the convergence of the series $\sum ne^{-n^2}$

Solution: Let $a_n = ne^{-n^2}$.

Consider the function $f(x) = xe^{-x^2}$ so that $f(n) = = a_n$.

Clearly f(x) is non-negative and monotonic decreasing on $[1, \infty)$.

$$I_n = \int_1^n xe^{-x^2}dx$$

$$= \frac{1}{2}\left(e^{-1} - e^{-n^2}\right)$$

$$\therefore (I_n) \to \frac{1}{2}e^{-1} \text{ as } n \to \infty.$$

By Cauchy's integral test, the given series is convergent and its sum lies between $\frac{1}{2}e^{-1}$ and $\frac{3}{2}e^{-1}$.

**ALTERNATING SERIES**

**Definition**: A series whose terms are alternatively positive and negative is called an alternating series.

(i.e.) the alternating series is of the form

$$a_1 - a_2 + a_3 - a_4 + \ldots\ldots = \sum(-1)^{n+1}a_n \text{ where } a_n > 0 \text{ for all n.}$$

Examples:

1) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots. = \sum(-1)^{n+1}\left(\frac{1}{n}\right)$.

2) $1 - 2 + 3 - 4 + \ldots. = \sum(-1)^{n+1}n$.

**Leibnitz's Test:**

Theorem 3.1.2:

Statement: Let $\sum(-1)^{n+1}a_n$ be an alternating series whose terms $a_n$ satisfy the following conditions

(i) $(a_n)$ is a monotonic decreasing sequence.

(ii) $\lim_{n\to\infty} a_n = 0$.

Then the given alternating series converges.

Proof: Let $(s_n)$ denote the sequence of partial sums of the given series.

Then $s_{2n} = a_1 - a_2 + a_3 - a_4 + \ldots\ldots a_{2n-1} - a_{2n}$.

$$s_{2n+2} = s_{2n} + a_{2n+1} - a_{2n+2}.$$

$$\therefore s_{2n+2} - s_{2n} = a_{2n+1} - a_{2n+2} \geq 0. \text{ (by (i))}$$

$$\therefore s_{2n+2} \geq s_{2n}.$$

SPACE FOR HINT

$\therefore$ $(s_{2n})$ is a monotonic increasing sequence.

Also $s_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots\ldots - (a_{2n-2} - a_{2n-1})$

$\leq a_1$.

$\therefore$ $(s_{2n})$ is bounded above.

$\therefore$ $(s_{2n})$ is a convergent sequence.

Let $(s_{2n}) \to s$.

Now $s_{2n+1} = s_{2n} + a_{2n+1}$.

$$\therefore \lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} a_{2n+1}.$$

$$= s + 0 \text{ ( by (ii))}$$

$\therefore (s_{2n+1}) \to s$.

Thus the subsequences $(s_{2n})$ and $(s_{2n+1})$ converges to the same limit.

$\therefore (s_n) \to s$.

$\therefore$ The given series converges.

**Problems:**

1) Show that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ converges.

Solution: Let $a_n = \frac{1}{n}$.

Then the given series is $\sum(-1)^{n+1}\left(\frac{1}{n}\right) = \sum(-1)^{n+1} a_n$

Since $\frac{1}{n} > \frac{1}{n+1}$, we have $a_n > a_{n+1}$ for all n.

$\therefore (a_n)$ is monotonic decreasing sequence.

Also $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

$\therefore$ By Leibnitz's test, the given series converges.

2) Show that the series $\sum\left(\frac{(-1)^{n+1}}{\log(n+1)}\right)$ converges.

Solution: Let $a_n = \frac{1}{\log(n+1)}$

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\log(n+1)} = 0$.

Since $\frac{1}{\log n} > \frac{1}{\log(n+1)}$, we have $a_n > a_{n+1}$ for all $n \geq 2$.

$\therefore$ By Leibnitz's test, the given series converges.

3) Show that the series $\sum(-1)^{n+1}\left(\frac{n}{3n-2}\right)$ oscillates.

Solution: Let $a_n = \frac{n}{3n-2}$.

Then $a_n > a_{n+1}$ for all n.

Also $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n}{3n-2} = \lim\limits_{n\to\infty} \frac{1}{3-2/n} = \frac{1}{3} \neq 0$.

$\therefore$ The given series oscillates.

4) Show that the series $\sum(-1)^{n+1}\left(\frac{1+2+3+....+n}{(n+1)^3}\right)$ converges.

Solution: Let $a_n = \frac{1+2+3+....+n}{(n+1)^3}$

$$= \frac{n(n+1)}{2(n+1)^3} = \frac{n}{2(n+1)^2}$$

Then $a_n > a_{n+1}$ for all n.

Also $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n}{2(n+1)^2} = \lim\limits_{n\to\infty} \frac{1}{2n(1+1/n)^2} = 0$.

$\therefore$ By Leibnitz's test, the given series converges.

**CYP Questions:**

1) Discuss the convergence of the following series using Cauchy's integral test.

(i) $\sum_{1}^{\infty}\frac{1}{n^2+1}$ (ii) $\sum_{1}^{\infty}\frac{1}{n(\log n)^2}$ (iii) $\sum_{1}^{\infty}\frac{1}{n\log n(\log\log n)^2}$

2) Test the convergence of the following series by using Leibnitz's test.

(i) $1-\frac{1}{5}+\frac{1}{9}-\frac{1}{13}+.....$

(ii) $\frac{1}{1.2}-\frac{1}{3.4}+\frac{1}{5.6}-\frac{1}{7.8}+......$

(iii) $\sum\frac{(-1)^{n+1}(n+1)}{2n}$

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(iv) $\sum \frac{(-1)^{n+1} n}{5n+1}$

(v) $\sum (-1)^n \left(1+\frac{1}{n}\right)$

## SECTION-3.2 - HARMONIC SERIES

Definition: The series $\sum \frac{1}{n^p}$ is said to be harmonic series.

Theorem 3.2.1: The harmonic series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

Proof:

Case(i) Let $p = 1$. Then the series becomes $\sum \frac{1}{n}$ which diverges.

Case(ii) Let $p < 1$. Then $n^p < n$ for all n.

$$\therefore \frac{1}{n^p} > \frac{1}{n}.$$

By comparison test, $\sum \frac{1}{n^p}$ diverges.

Case(iii) Let $p > 1$

Let $s_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p}$.

$$\text{Then } s_{2^{n+1}-1} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{(2^{n+1}-1)}$$

$$= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \dots$$

$$+ \left(\frac{1}{(2^n)^p} + \frac{1}{(2^{n+1})^p} + \dots + \frac{1}{(2^{n+1}-1)^p}\right)$$

$$< 1 + 2\left(\frac{1}{2^p}\right) + 4\left(\frac{1}{4^p}\right) + \dots + 2^n\left(\frac{1}{(2^n)^p}\right)$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2p-2}} + \dots + \frac{1}{2^{(p-1)n}}$$

$$\therefore s_{2^{n+1}-1} < 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \dots + \left(\frac{1}{2^{p-1}}\right)^n$$

Since $p > 1$, $p - 1 > 0$.

Hence $\frac{1}{2^{p-1}} < 1$.

$$\therefore 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \ldots + \left(\frac{1}{2^{p-1}}\right)^n$$

$$< \frac{1}{1 - \frac{1}{2^{p-1}}} = k(\text{say})$$

$\therefore s_{2^{n+1}-1} < k$.

Let n be any positive integer. Choose $m \in N$ such that $n \leq 2^{m+1} - 1$.

Since $(s_n)$ is a monotonic increasing sequence, $s_n \leq s_{2^{m+1}-1}$.

Hence $s_n < k$ for all n.

$\therefore$ $(s_n)$ is a monotonic increasing sequence and bounded above.

$\therefore$ $(s_n)$ is convergent.

$\therefore$ $\sum \frac{1}{n^p}$ converges.

**CYP Questions:**

1) Show that the sum of the series $\sum \frac{1}{n^p}$ lies between $\frac{1}{p-1}$ and $\frac{p}{p-1}$ if $p > 1$.

## SECTION-3.3 - ABSOLUTE CONVERGENCE:

Definition: A series $\sum a_n$ is said to be absolutely convergent if the series $\sum |a_n|$ is convergent.

Examples:

1) The series $\sum \frac{(-1)^n}{n^2}$ is absolutely convergent, for, $\sum \left|\frac{(-1)^n}{n^2}\right| = \sum \frac{1}{n^2}$, which is convergent.

SPACE FOR HINT

2) The series $\sum \frac{(-1)^n}{n}$ is not absolutely convergent, for,

$$\sum \left| \frac{(-1)^n}{n} \right| = \sum \frac{1}{n}, \text{ which is divergent.}$$

Note: If $\sum a_n$ is a convergent series of positive terms, then $\sum a_n$ is absolutely convergent.

Theorem 3.3.1: Any absolutely converge series is convergent.

Proof: Let $\sum a_n$ be absolutely convergent.

$\therefore \sum |a_n|$ is convergent.

Let $s_n = a_1 + a_2 + a_3 + \ldots + a_n$ and $t_n = |a_1| + |a_2| + |a_3| + \ldots + |a_n|$

Since $\sum |a_n|$ is convergent, $(t_n)$ is convergent and hence it is a Cauchy sequence.

$\therefore$ Given $\varepsilon > 0$, there exists $n_1 \in N$ such that

$$|t_n - t_m| < \varepsilon, \text{ for all } n, m \geq n_1. \text{-----------------} (1)$$

Let $m > n$.

Then $|s_n - s_m| = |a_{n+1} + a_{n+2} + a_{n+3} + \ldots + a_m|$

$\leq |a_{n+1}| + |a_{n+2}| + |a_{n+3}| + \ldots + |a_m|$

$= |t_n - t_m|$

$< \varepsilon$, for all $n, m \geq n_1$ (by (1))

$\therefore (s_n)$ is a Cauchy sequence in R and hence it is convergent.

$\therefore \sum a_n$ is convergent.

Note: The converse of the above theorem need not be true. For example

The series $\sum \frac{(-1)^n}{n}$ is convergent. However $\sum \frac{1}{n}$ is divergent so that the series is not absolutely convergent.

**CYP Questions:**

1) Test the convergence of the series $\sum \frac{(-1)^n}{n^p}$.

2) Test the convergence of the series $\sum \frac{(-1)^n}{n^3}$

SPACE FOR HINT

## SECTION-3.4 - CONDITONAL CONVERGENCE:

Definition: A series $\sum a_n$ is said to be **conditionally convergent** if it is convergent but not absolutely convergent.

Example: The series $\sum \frac{(-1)^n}{n}$ is conditionally convergent.

Theorem 3.4.1: In an absolutely convergent series, the series formed by its positive terms alone is convergent and the series formed by its negative terms alone is convergent and conversely.

Proof: Let $\sum a_n$ be the absolutely convergent series.

$$\text{Define } p_n = \begin{cases} a_n & \text{if } a_n > 0 \\ 0 & \text{if } a_n \le 0 \end{cases} \text{ and } q_n = \begin{cases} 0 & \text{if } a_n \ge 0 \\ -a_n & \text{if } a_n < 0 \end{cases}$$

(i.e.) $p_n$ is a positive terms of the given series and $q_n$ is the modulus of a negative term.

$\therefore \sum p_n$ is the series formed with the positive terms of the given series and $\sum q_n$ is the series formed with the moduli of the negative terms of the given series

Clearly $p_n \le |a_n|$ and $q_n \le |a_n|$ for all n.

Since the given series is absolutely convergent, $\sum |a_n|$ is a convergent series of positive terms.

By Comparison test, $\sum p_n$ and $\sum q_n$ are convergent.

Conversely, let $\sum p_n$ and $\sum q_n$ converge to p and q respectively.

By the definition of $p_n$ and $q_n$ we have $|a_n| = p_n + q_n$.

$$\begin{aligned} \therefore \sum |a_n| &= \sum (p_n + q_n) \\ &= \sum p_n + \sum q_n \\ &= p + q. \end{aligned}$$

$\therefore \sum a_n$ is absolutely convergent.

Theorem 3.4.2: If $\sum a_n$ is an absolutely convergent series and $(b_n)$ is a bounded sequence, then the series $\sum a_n b_n$ is an absolutely convergent series.

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Proof: Since $(b_n)$ is a bounded sequence, there exists a real number $k > 0$ such that $|b_n| \le k$ for all n.

$\therefore |a_n b_n| = |a_n||b_n| \le k \cdot |a_n|$ for all n.

Since $\sum a_n$ is an absolutely convergent, $\sum |a_n|$ is convergent

$\therefore \sum k|a_n|$ is convergent.

$\therefore$ By Comparison test, $\sum |a_n b_n|$ is convergent.

$\therefore \sum a_n b_n$ is absolutely convergent.

**Problems:**

1) Test the convergency of the series $\sum \frac{(-1)^n}{n^p}$

Solution: Case(i) Let $p > 1$.

$$\text{Then } \sum \left|\frac{(-1)^n}{n^p}\right| = \sum \frac{1}{n^p} \text{ is convergent.}$$

$\therefore$ The given series is absolutely convergent and hence convergent.

Case(ii) Let $0 < p \le 1$.

Since $\frac{1}{n^p} > \frac{1}{(n+1)^p}$, $\left(\frac{1}{n^p}\right)$ is a monotonic decreasing sequence and $\left(\frac{1}{n^p}\right) \to 0$.

$\therefore$ By Leibnitz's test the given series converges.

In this case the convergence is not absolute since $\sum \frac{1}{n^p}$ diverges when $0 < p \le 1$.

Case(iii) Let $p = 0$.

Then the series reduces to $-1 + 1 - 1 + 1 - \ldots$..which oscillates finitely.

Case(iv) Let $p < 0$.

Then the sequence $\left(\frac{1}{n^p}\right)$ is unbounded. Hence the given series oscillates infinitely.

2) Show that the series $\sum (-1)^n [\sqrt{(n^2+1)} - n]$ is conditionally convergent.

Solution: Let $a_n = \sqrt{(n^2+1)} - n = \dfrac{1}{\sqrt{(n^2+1)}+n}$

Then $a_n > a_{n+1}$ for all n.

Also $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{(n^2+1)}+n} = 0.$

$\therefore$ By Leibnitz's test, the given series converges.

Next we prove that $\sum |(-1)^n[\sqrt{(n^2+1)} - n]|$ is divergent.

$|(-1)^n[\sqrt{(n^2+1)} - n]| = \sqrt{(n^2+1)} - n = a_n = \dfrac{1}{\sqrt{(n^2+1)}+n}$.

Let $b_n = \dfrac{1}{n}$

$$\therefore \frac{a_n}{b_n} = \frac{n}{\sqrt{(n^2+1)}+n} = \frac{1}{\sqrt{\left(1+\frac{1}{n^2}\right)}+1}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2}.$$

$\therefore$ By Comparison test, $\sum a_n$ is divergent.

$\therefore$ The given series is not absolutely convergent.

$\therefore$ The given series is conditionally convergent.

3) Show that the series $\sum \dfrac{x^{n-1}}{(n-1)!}$ converges absolutely for all values of x

Solution: Let $a_n = \dfrac{x^{n-1}}{(n-1)!}$

$$\therefore \left|\frac{a_n}{a_{n+1}}\right| = \frac{n}{|x|}$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \infty \text{ for all } x \neq 0.$$

$\therefore$ By ratio test the series $\sum \left|\dfrac{x^{n-1}}{(n-1)!}\right|$ is convergent for all $x \neq 0$

and the convergence is true for $x = 0$.

$\therefore$ The series converges absolutely for all x.

SPACE FOR HINT

4) Test the convergence of the series $\sum \frac{(-1)^n \sin n\alpha}{n^3}$

Solution: Since $|\sin\theta| \le 1$, we have $\left|\frac{(-1)^n \sin n\alpha}{n^3}\right| \le \frac{1}{n^3}$

Since $\sum \frac{1}{n^3}$ is convergent, the series $\sum \frac{(-1)^n \sin n\alpha}{n^3}$ is absolutely convergent.

<u>Theorem 3.4.3</u>: Let $(a_n)$ be a bounded sequence and $(b_n)$ be a monotonic decreasing bounded sequence. Then the series $\sum a_n(b_n - b_{n+1})$ is absolutely convergent.

<u>Proof</u>: Since $(a_n)$ and $(b_n)$ are bounded sequences there exists a real number $k > 0$ such that $|a_n| \le k$ and $|b_n| \le k$ for all n.--------(1)

Let $s_n$ denote the partial sum of the series $\sum |a_n(b_n - b_{n+1})|$.

Then $s_n = \sum_{r=1}^{n} |a_r(b_r - b_{r+1})|$

$= \sum_{r=1}^{n} |a_r|(b_r - b_{r+1})$ ($(b_n)$ is monotonic decreasing)

$\le k\sum_{r=1}^{n}(b_r - b_{r+1})$

$= k(b_1 - b_{n+1})$

$\le k(|b_1| - |b_{n+1}|)$

$\le k(k+k) = 2k^2$.

$\therefore (s_n)$ is a bounded sequence.

$\therefore \sum |a_n(b_n - b_{n+1})|$ is convergent.

$\therefore \sum a_n(b_n - b_{n+1})$ is absolutely convergent.

**Dirichlet's test**

<u>Theorem 3.4.4:</u>

<u>Statement</u>: Let $\sum a_n$ be a series whose sequence of partial sums $(s_n)$ is bounded. Let $(b_n)$ be a monotonic decreasing sequence converging to 0. Then the series $\sum a_n b_n$ converges.

<u>Proof</u>: Let $t_n$ denote the partial sum of the series $\sum a_n b_n$.

$$\therefore t_n = \sum_{r=1}^{n} a_r b_r$$

$$= s_1 b_1 + \sum_{r=1}^{n} (s_r - s_{r-1}) b_r \quad (\because s_r - s_{r-1} = a_r)$$

$$= \sum_{r=1}^{n-1} s_r (b_r - b_{r+1}) + s_n b_n \quad \text{---------------(1)}$$

Since $(s_n)$ is bounded and $(b_n)$ is a monotonic decreasing bounded sequence $\sum_{r=1}^{n-1} s_r (b_r - b_{r+1})$ is a convergent sequence.

Also since $(s_n)$ is bounded and $(b_n) \to 0$, $(s_n b_n) \to 0$

$\therefore$ From (1) it follows that $(t_n)$ is convergent.

$\therefore \sum a_n b_n$ is convergent.

**Abel's test**

Theorem 3.4.5:

Statement: Let $\sum a_n$ be a convergent series. Let $(b_n)$ be a bounded monotonic sequence. Then the series $\sum a_n b_n$ is convergent.

Proof: Since $(b_n)$ is bounded monotonic sequence, $(b_n) \to b$ (say)

$$\text{Let } c_n = \begin{cases} b - b_n \text{ if } (b_n) \text{ is monotonic increasing} \\ b_n - b \text{ if } (b_n) \text{ is monotonic decreasing} \end{cases}$$

$$\therefore a_n c_n = \begin{cases} a_n b - a_n b_n \text{ if } (b_n) \text{ is monotonic increasing} \\ a_n b_n - a_n b \text{ if } (b_n) \text{ is monotonic decreasing} \end{cases}$$

$$\therefore a_n b_n = \begin{cases} b a_n - a_n c_n \text{ if } (b_n) \text{ is monotonic increasing} \\ b a_n + a_n c_n \text{ if } (b_n) \text{ is monotonic decreasing} \end{cases} \text{------(1)}$$

Clearly $(c_n)$ is a monotonic decreasing sequence converging to 0. Also since $\sum a_n$ is a convergent series its sequence of partial sums is bounded.

$\therefore$ By Dirichlet's test $\sum a_n c_n$ is convergent.

Also $\sum a_n$ is convergent.

$\therefore \sum b a_n$ is convergent.

$\therefore$ By (1) $\sum a_n b_n$ is convergent.

SPACE FOR HINT

**Problems:**

1) Show that the convergence of $\sum a_n$ implies the convergence of $\sum \frac{a_n}{n}$.

Solution: Let $\sum a_n$ be convergent.

The sequence $\left(\frac{1}{n}\right)$ is a bounded monotonic sequence.

$\therefore$ By Abel's test $\sum \frac{a_n}{n}$ is convergent.

2) Show that the series $\sum \frac{\sin n\theta}{n}$ converges for all values of $\theta$ and $\sum \frac{\cos n\theta}{n}$ converges if $\theta$ is not a multiple of $2\pi$.

Solution: First consider the series $\sum \frac{\sin n\theta}{n}$

Let $a_n = \sin n\theta$ and $b_n = \frac{1}{n}$

Clearly $(b_n)$ is a monotonic decreasing sequence converging to 0.

Now $s_n = \sin\theta + \sin 2\theta + \ldots + \sin n\theta$.

$$= \frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\left[2\sin\theta\sin\frac{\theta}{2} + \ldots\ldots + 2\sin n\theta\sin\frac{\theta}{2}\right]$$

$$= \frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\left[\left(\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right) + \ldots\ldots + \left(\cos\left(\frac{2n-1}{2}\right)\theta - \cos\left(\frac{2n+1}{2}\right)\theta\right)\right]$$

$$= \frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\left[\cos\frac{\theta}{2} - \cos\left(\frac{2n+1}{2}\right)\theta\right]$$

$$\therefore |s_n| = \left|\frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\right|\left|\cos\frac{\theta}{2} - \cos\left(\frac{2n+1}{2}\right)\theta\right|$$

$$\leq \frac{1}{2}\left|\operatorname{cosec}\frac{\theta}{2}\right|\left[\left|\cos\frac{\theta}{2}\right| + \left|\cos\left(\frac{2n+1}{2}\right)\theta\right|\right]$$

$$\leq \frac{1}{2}\left|\operatorname{cosec}\frac{\theta}{2}\right| x2 = \left|\operatorname{cosec}\frac{\theta}{2}\right|$$

$$\therefore |s_n| \leq \left|\operatorname{cosec}\frac{\theta}{2}\right|$$

SPACE FOR HINT

$\therefore$ $(s_n)$ is a bounded sequence when $\theta$ is not a multiple of $2\pi$.

$\therefore$ By Dirichlet's test $\sum a_n b_n = \sum \frac{\sin n\theta}{n}$ converges when $\theta$ is not a multiple of $2\pi$.

When $\theta$ is a multiple of $2\pi$ the series $\sum \frac{\sin n\theta}{n}$ reduces to $0+0+0+\ldots\ldots$ which trivially converges to 0.

$\therefore \sum \frac{\sin n\theta}{n}$ converges for all values of $\theta$.

Next consider the series $\sum \frac{\cos n\theta}{n}$

$s_n = \cos\theta + \cos 2\theta + \ldots. + \cos n\theta$.

$$= \frac{1}{2} \operatorname{cosec} \frac{\theta}{2} \left[ \sin\left(\frac{2n+1}{2}\right)\theta - \sin \frac{\theta}{2} \right]$$

$$\therefore |s_n| \leq \left| \operatorname{cosec} \frac{\theta}{2} \right|$$

$\therefore$ $(s_n)$ is a bounded sequence when $\theta$ is not a multiple of $2\pi$.

$\therefore$ By Dirichlet's test $\sum a_n b_n = \sum \frac{\cos n\theta}{n}$ converges when $\theta$ is not a multiple of $2\pi$.

When $\theta$ is a multiple of $2\pi$ the series $\sum \frac{\cos n\theta}{n}$ reduces to $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$ which diverges.

$\therefore$ The series $\sum \frac{\cos n\theta}{n}$ converges except when $\theta$ is not a multiple of $2\pi$.

SPACE FOR HINT

**CYP Questions.**

(1) Discuss the convergence of the following series.

(i) $\Sigma \frac{(-1)^n}{\log(n+1)}$

(ii) $\Sigma(-1)^n \left\{\frac{1}{n^2} + \frac{1}{(n+1)^2}\right\}$

(2) Prove that $\sum_{n=2}^{\infty} \frac{\sin n}{\log n}$ is convergent.

(3) Discuss the convergence of the series $\Sigma\left(1+\frac{1}{2}+\ldots.+\frac{1}{n}\right)\frac{\sin n\theta}{n}$

SPACE FOR HINT

# UNIT – 4

**Unit Structure:**



**Introduction:** In this unit we discuss the Binomial theorem for a rational index, particular cases of Binomial expansion, greatest term of the expansion, and finding the approximate value by using Binomial expansion.

## SECTION - 4.1 - BINOMIAL THEOREM FOR A RATIONAL INDEX

If n is a rational number and $-1 < x < 1$(i.e. $|x| < 1$) the sum of the series $1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots + \frac{n(n-1)\ldots.(n-r+1)}{r!}x^r + \ldots.$ is the real positive value of $(1+x)^n$.

Let $f(n) = 1 + \frac{n_1}{1!}x + \frac{n_2}{2!}x^2 + \ldots\ldots + \frac{n_r}{r!}x^r + \ldots.$

We can sum this series only when it is absolutely convergent.

Denoting this series by $u_1 + u_2 + u_3 + \ldots..$

Then $\frac{u_{r+1}}{u_r} = \frac{n_r}{r!}x^r \div \frac{n_{r-1}}{(r-1)!}x^{r-1}$

$$= \frac{n-r+1}{r}x$$

$$\therefore \left|\frac{u_{r+1}}{u_r}\right| = \left|\frac{n-r+1}{r}\right| |x|$$

$$= \left|\frac{n+1}{r} - 1\right| |x|$$

$$\therefore \lim_{r\to\infty} \frac{u_{r+1}}{u_r} = |x|$$

SPACE FOR HINT

$\therefore$ The series $|u_1| + |u_2| + |u_3| + \ldots$ is convergent if $|x| < 1$.

(i.e.) the series $u_1 + u_2 + u_3 + \ldots$ is absolutely convergent if $|x| < 1$.

Similarly the series $f(m) = 1 + \frac{m_1}{1!}x + \frac{m_2}{2!}x^2 + \ldots\ldots + \frac{m_r}{r!}x^r + \ldots$

$$f(m+n) = 1 + \frac{(m+n)_1}{1!}x + \frac{(m+n)_2}{2!}x^2 + \ldots\ldots + \frac{(m+n)_r}{r!}x^r + \ldots$$

are also absolutely convergent if $|x| < 1$.

By Vanderminde's theorem, $f(m)f(n) = f(m+n)$ for all values of m and n, provided $|x| < 1$

Note: $f(m)f(n)f(p) \ldots$ S factors $= f(m+n+p+\ldots s \text{ terms})$

$$\left\{f\left(\frac{r}{s}\right)\right\}^s = f(r).$$

**Some important particular cases of the Binomial expansion.**

1) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

2) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots + (n+1)x^n + \ldots$

3) $(1-x)^{-3} = \frac{1}{2}\{1.2 + 2.3x + 3.4x^2 + 4.5x^3 + \ldots + (n+1)(n+2)x^n + \ldots\}$

4) $(1-x)^{-4} = \frac{1}{6}\{1.2.3 + 2.3.4x + 3.4.5x^2 + 4.5.6x^3 + \ldots$

$+ (n+1)(n+2)(n+3)x^n + \ldots\}$

5) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots +$

6) $(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \ldots\ldots$

7) $(1-x)^{-1/3} = 1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \frac{1.4.7}{3.6.9}x^3 + \ldots\ldots$

**Problems:**

1) Find the general term in the expansion of $(1+x)^{2/3}$

Solution: The $(r+1)^{th}$ term $= \dfrac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)\ldots\left(\frac{2}{3}-r+1\right)}{r!}x^r$

$$= \frac{2(-1)(-4)(-7)\ldots(-3r+5)}{3^r(r!)}x^r$$

SPACE FOR HINT

The number of factors in the numerator is r and (r – 1) of these are negative.

$$\therefore \text{The } (r+1)^{th} \text{ term} = (-1)^{r-1}\frac{2.1.4.7....(3r-5)}{3^r(r!)}x^r$$

2) Expand $(1+3x)^{5/2}$ given $|x| < \frac{1}{3}$.

Solution: The function can be expanded in ascending powers of x if

$|3x| < 1$ (i.e.) if $|x| < \frac{1}{3}$.

$$(1+3x)^{5/2} = 1 + \frac{5}{2}(3x) + \frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2!}(3x)^2 + \frac{\frac{5}{2}\left(\frac{5}{2}-1\right)\left(\frac{5}{2}-2\right)}{3!}(3x)^3$$

$$+ \frac{\frac{5}{2}\left(\frac{5}{2}-1\right)\left(\frac{5}{2}-2\right)\left(\frac{5}{2}-3\right)}{4!}(3x)^4 + \ldots\ldots$$

$$= 1 + 5.\left(\frac{3x}{2}\right) + \frac{5.3}{2!}\left(\frac{3x}{2}\right)^2 + \frac{5.3.1}{3!}\left(\frac{3x}{2}\right)^3 + \frac{5.3.1.(-1)}{4!}\left(\frac{3x}{2}\right)^4$$

$$+ \ldots\ldots$$

The terms which follow are alternatively positive and negative and the general term is

$$\frac{5.3.1.(-1)(-3)(-5)......(-2r-7)}{r!}\left(\frac{3x}{2}\right)^r$$

$$\text{(i.e.) } (-1)^{r-3}\frac{5.3.1.1.3.5)......(2r+7)}{r!}\left(\frac{3x}{2}\right)^r, \ r > 3.$$

3) Find the first term with a negative coefficient in the expansion of $(1+2x)^{14/3}$.

Solution: The $(r+1)^{th}$ term in the expansion of $(1+2x)^{14/3}$

$$= \frac{\frac{14}{3}\left(\frac{14}{3}-1\right)....\left(\frac{14}{3}-r+1\right)}{r!}(2x)^r$$

The first negative term will occur for the least value of r such that

$$\frac{14}{3} - r + 1 < 0$$

$$\text{(i.e.) } r > 5\frac{2}{3}$$

SPACE FOR HINT

Thus the first negative term is obtained by taking r = 6 and its value is

$$= \frac{\frac{14}{3}\left(\frac{14}{3}-1\right)\ldots\left(\frac{14}{3}-5\right)}{6!}(2x)^6$$

$$= \frac{\frac{14}{3}\cdot\frac{11}{3}\cdot\frac{8}{3}\cdot\frac{5}{3}\cdot\frac{2}{3}\left(-\frac{5}{3}\right)}{6!}2^6x^6$$

$$= -\frac{14.11.8.5.2.1}{6!}\left(\frac{2}{3}\right)^6 x^6.$$

**Sign of terms in the Binomial expansion**

Let us denote the $r^{th}$ term by $u_r$

Then we get

$$\frac{u_{r+1}}{u_r} = \frac{n-r+1}{r}x = -\left(1-\frac{n+1}{r}\right)x$$

If $x > 0$ and $r > n + 1$, $\frac{u_{r+1}}{u_r}$ is positive.

If $x < 0$ and $r > n + 1$, $\frac{u_{r+1}}{u_r}$ is negative.

After a certain stage, the terms are alternately positive and negative if x is positive and are all of the same sign if x is negative.

**CYP Questions:**

1) Find the general term in the expansion of the following function in ascending powers of x. In each case, state when the expansion is valid.

(i) $(1+x)^{-1/2}$ (ii) $(1+x)^{-1/3}$ (iii) $(1-x)^{-6}$

(iv) $(8+3x)^{5/3}$ (v) $\{(27-x^3)^2\}^{1/3}$.(vi) $\frac{1}{\sqrt{1-2x}}$

2) Find the named terms in the expansion of the following functions, when the expansions are valid.

(i) $4^{th}$ term in $(1+4x)^{-3}$ (ii) $6^{th}$ term in $(1+7x)^{11/2}$

(ii) $5^{th}$ term in $(8-5x^2)^{-11/2}$ (iv) $10^{th}$ term in $(4-7x)^{-21/2}$

3) Find the first negative term in the expansion of

(i) $\left(1+\frac{5}{4}x\right)^{28/5}$ (ii) $\left(1+\frac{3}{4}x\right)^{23/4}$

SPACE FOR HINT

# SECTION 4.2 : BINOMIAL THEOREM – GREATEST TERM

**Numerically greatest term**

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots + \frac{n(n-1)\ldots.(n-r+1)}{r!}x^r + \ldots.$$

The numerical value of any term is not affected by changing x into – x and so x may be assumed to be positive.

$$u_{r+1} = \frac{n(n-1)(n-2)\ldots.(n-r+2)(n-r+1)}{r!}x^r$$

$$u_r = \frac{n(n-1)(n-2)\ldots.(n-r+2)}{r!}x^{r-1}$$

$$\therefore \frac{u_{r+1}}{u_r} = \frac{n-r+1}{r}x$$

We also know that x must be numerically less than unity, unless n is a positive integer

$\therefore u_{r+1} \geq u_r$, numerically if $|n-r+1|x \geq r$

If $r < n+1$, $|n-r+1| = n-r+1$

$r > n+1$, $|n-r+1| = r-n-1$.

In the first case

$u_{r+1} \geq u_r$, if $(n-r+1)x \geq r$

(i.e.) $r \leq \dfrac{(n+1)x}{x+1}$

In the second case

$u_{r+1} \geq u_r$ if $(r-n-1)x \geq r$

(i.e.) $r \geq \dfrac{-(n+1)x}{1-x}$

since $1-x$ is positive.

**Problems:**

1) Find the greatest term in the expansion of $(1+x)^{13/2}$ when $x = \frac{2}{3}$.

SPACE FOR HINT

Solution: Here $u_{r+1} = \dfrac{\frac{13}{2}\left(\frac{13}{2}-1\right)\left(\frac{13}{2}-2\right)\ldots\left(\frac{13}{2}-r+1\right)}{r!}x^r$

and $u_r = \dfrac{\frac{13}{2}\left(\frac{13}{2}-1\right)\left(\frac{13}{2}-2\right)\ldots\left(\frac{13}{2}-r+2\right)}{(r-1)!}x^{r-1}$

$$\therefore \frac{u_{r+1}}{u_r} = \frac{\left(\frac{13}{2}-r+1\right)}{r}x = \frac{(13-2r+2)}{2r}x = \frac{15-2r}{2r}x$$

$$= \frac{15-2r}{2r}\frac{2}{3} = \frac{15-2r}{3r}$$

$\therefore u_{r+1} \geq u_r$, if $\dfrac{15-2r}{3r} \geq r$

(i.e.) $\therefore u_{r+1} \geq u_r$, if $15 - 2r \geq 3r$

(i.e.) $\therefore u_{r+1} \geq u_r$, if $r \leq 3$.

Hence the third and the fourth terms are numerically equal, both being the greatest.

The numerical value of the 3rd term $= \dfrac{\frac{13}{2}\left(\frac{13}{2}-1\right)}{2!}\left(\frac{2}{3}\right)^2$

$$= \frac{13.11}{2!}\left(\frac{1}{3}\right)^2$$

2) Find the greatest term in the expansion of $(1-x)^{31/3}$ when $x = \dfrac{2}{7}$.

Solution: Here $u_{r+1} = \dfrac{\frac{31}{3}\left(\frac{31}{3}-1\right)\left(\frac{31}{3}-2\right)\ldots\left(\frac{31}{3}-r+1\right)}{r!}x^r$

and $u_r = \dfrac{\frac{31}{3}\left(\frac{31}{3}-1\right)\left(\frac{31}{3}-2\right)\ldots\left(\frac{31}{3}-r+2\right)}{(r-1)!}x^{r-1}$

$$\therefore \frac{u_{r+1}}{u_r} = \frac{\left(\frac{31}{3}-r+1\right)}{r}x = \frac{(31-3r+3)}{3r}x = \frac{34-3r}{3r}x$$

$$= \frac{34-3r}{3r}\frac{2}{7} = \frac{68-6r}{21r}.$$

SPACE FOR HINT

$\therefore\ u_{r+1} \geq u_r$, if $\dfrac{68-6r}{21r} \geq r$

(i.e.) $\therefore\ u_{r+1} \geq u_r$, if $68 - 6r \geq 21r$

(i.e.) $\therefore\ u_{r+1} \geq u_r$, if $27r \leq 68$.

(i.e.) $\therefore\ u_{r+1} \geq u_r$, if $r \leq \dfrac{68}{27} = 2\dfrac{14}{27}$.

$\therefore$ If $r = 2$, $u_{r+1} \geq u_r$.

$\therefore$ The third term is the greatest term.

The value of the third term = $\dfrac{\frac{31}{3}\left(\frac{31}{3}-1\right)}{2!}\left(\dfrac{2}{7}\right)^2 = \dfrac{248}{63}$

3) Find the coefficient of $x^n$ in the expansion of $\dfrac{1+3x+2x^2}{(1-x)^4}$ in ascending powers of x.

Solution: $\dfrac{1+3x+2x^2}{(1-x)^4} = (1+3x+2x^2)(1-x)^{-4}$

$$= (1+3x+2x^2)\left\{1+4x+\frac{4.5}{2!}x^2+\frac{4.5.6}{3!}x^3+\ldots+\frac{(n+1)(n+2)(n+3)}{n!}x^n+\ldots\right\}$$

Coefficient of $x^n$

$$= \frac{(n+1)(n+2)(n+3)}{6} + 3.\ \frac{n(n+1)(n+2)}{6} + \frac{2(n-1)n(n+1)}{6}$$

$$= \frac{(n+1)}{6}\left\{6n^2+9n+6\right\}$$

$$= \frac{(n+1)(2n^2+3n+2)}{2}.$$

4) If n is a positive integer, prove that the coefficient of $x^{3n}$ in the expansion of $\dfrac{1+x}{(1+x+x^2)^3}$ in a series of ascending powers of x is $\dfrac{1}{2}(n+1)(3n+2)$.

Solution: $\dfrac{1+x}{(1+x+x^2)^3} = \dfrac{(1+x)(1-x)^3}{((1-x)(1+x+x^2))^3}$

$$= \frac{(1+x)(1-x)^3}{(1-x^3)^3} = (1-2x+2x^3-x^4)(1-x^3)^{-3}$$

SPACE FOR HINT

$$= (1-2x+2x^3-x^4)\left\{1+3x^3+\frac{3.4}{2}x^6+\ldots+\frac{(n+1)(n+2)}{2}x^{3n}+\ldots\right\}$$

Coefficient of $x^{3n} = \frac{(n+1)(n+2)}{2} + 2.\ \frac{n(n+1)}{2}$

$$= \frac{1}{2}(n+1)(3n+2).$$

5) If n is a positive integer, prove that the coefficient of $x^n$ in the expansion of $\frac{(3x-2)^n}{(1-x)^2}$ is $1-2n$.

Solution: $\frac{(3x-2)^n}{(1-x)^2} = \frac{\{1-3(1-x)\}^n}{(1-x)^2}$

Numerator $= \{1-3(1-x)\}^n$

$$= 1-3n(1-x)+\frac{n(n-1)}{2!}3^2(1-x)^2-\frac{n(n-1)(n-2)}{3!}3^3(1-x)^3$$

$+\ \ldots\ldots$

$= 1-3n(1-x)+\frac{n(n-1)}{2!}3^2(1-x)^2 -$ terms containing $(1-x)\}^n$ and higher powers of $(1-x)$.

$\therefore$ Given expression $= \frac{1}{(1-x)^2} - \frac{3n(1-x)}{(1-x)^2} +$ an expression containing powers of $(1-x)$ upto $n-2$.

$= \frac{1}{(1-x)^2} - \frac{3n}{(1-x)} +$ an expression containing powers of $(1-x)$ upto $n-2$.

$= (1-x)^{-2} - 3n(1-x)^{-1} +$ terms not containing $x^n$.

$= 1+2x+3x^2+\ldots\ldots+(n+1)x^n+\ldots.-3n(1+x+x^2+\ldots.)$ + terms not containing $x^n$.

$\therefore$ Coefficient of $x^n$ in $\frac{(3x-2)^n}{(1-x)^2}$ is $(n+1)-3n = 1-2n$.

6) Prove that if n is a positive integer, the coefficient of $x^n$ in $(1+x+x^2)^n$ is $1+\frac{n(n-1)}{(1!)^2}+\frac{n(n-1)(n-2)(n-3)}{(2!)^2}+\ldots\ldots$

Solution: $(1+x+x^2)^n = \{(1+x(1+x)\}^n$

$$= 1 + nc_1\, x(1+x) + nc_2\, x^2(1+x)^2 + \ldots\ldots$$

SPACE FOR HINT

$$+ nc_{n-1}x^{n-1}(1+x)^{n-1} + nc_n x^n (1+x)^n.$$

Coefficient of $x^n$ in $nc_n x^n(1+x)^n = nc_n$.

Coefficient of $x^n$ in $nc_{n-1}x^{n-1}(1+x)^{n-1} = nc_{n-1}.(n-1)c_1$.

Coefficient of $x^n$ in $nc_{n-2}x^{n-2}(1+x)^{n-2} = nc_{n-2}.(n-2)c_2$.

..................

$\therefore$ Coefficient of $x^n = nc_n + nc_{n-1}.(n-1)c_1 + nc_{n-2}.(n-2)c_2 + \ldots$

$$= 1 + \frac{n}{1!}\frac{(n-1)}{1!} + \frac{n(n-1)}{2!}\frac{(n-2)(n-3)}{2!} + \ldots\ldots$$

$$= 1 + \frac{n(n-1)}{(1!)^2} + \frac{n(n-1)(n-2)(n-3)}{(2!)^2} + \ldots\ldots$$

7) Expand $\frac{1}{(1-2x)(1+3x)}$ in a series of ascending powers of x, and find when the expansion is valid.

Solution: By partial fraction

$$\frac{1}{(1-2x)(1+3x)} = \frac{2}{5(1-2x)} + \frac{3}{5(1+3x)}$$

$$= \frac{2}{5}(1-2x)^{-1} + \frac{3}{5}(1+3x)^{-1}$$

$$= \frac{2}{5}\{1 + 2x + 2^2x^2 + 2^3x^3 + \ldots + 2^nx^n + \ldots\}$$

$$+ \frac{3}{5}\{1 - 3x + 3^2x^2 - 3^3x^3 + \ldots + (-1)^n 3^n x^n + \ldots\}$$

$$= 1 - x + 7x^2 \ldots + \frac{1}{5}\{2^{n+1} + (-1)^n 3^{n+1}\}x^n + \ldots$$

$$= 1 + \frac{1}{5}\sum_{n=1}^{\infty}\{2^{n+1} + (-1)^n 3^{n+1}\}x^n .$$

The expansion is valid if and only if $|2x| < 1$ and $|3x| < 1$ and both thses conditions are satisfied if $|x| < \frac{1}{3}$.

8) Find the coefficient of $x^n$ in the expansion of $\frac{x+1}{(x-1)^2(x-2)}$.

Solution: By partial fraction

$$\frac{x+1}{(x-1)^2(x-2)} = \frac{3}{(x-2)} - \frac{3}{(x-1)} - \frac{2}{(x-1)^2}$$

SPACE FOR HINT

$$= -\frac{3}{2\left(1-\frac{x}{2}\right)} + \frac{3}{1-x} - \frac{2}{(x-1)^2}$$

$$= -\frac{3}{2}\left(1-\frac{x}{2}\right)^{-1} + 3\,(1-x)^{-1} - 2\,(1-x)^{-2}$$

$$= -\frac{3}{2}\left\{1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+....\right\} + 3\{1+x+x^2+....\}$$

$$-2\{1+2x+3x^2+4x^3+.....+(n+1)x^n+....\}$$

$\therefore$ Coefficient of $x^n = -\frac{3}{2}.\frac{1}{2^n}+3-2(n+1)$

$$= 1-2n-\frac{3}{2^{n+1}}.$$

9) Prove that the coefficient of $x^n$ in the expansion of $\frac{1+x}{(1+x^2)(1-x)^2}$ is $\frac{1}{2}\{(2n+3)+(-1)^{p-1}\}$, where $p = \frac{n}{2}$ or $\frac{n+1}{2}$ according as n is even or odd.

Solution: $\frac{1+x}{(1+x^2)(1-x)^2} = \frac{\frac{1}{2}(x-1)}{1+x^2} + \frac{\frac{1}{2}}{1-x} + \frac{1}{(1-x)^2}$

$$= \frac{1}{2}(x-1)(1+x^2)^{-1} + \frac{1}{2}(1-x)^{-1} - (1-x)^{-2}$$

$$= \frac{1}{2}(x-1)\{1-x^2+x^4.....\} + \frac{1}{2}\{1+x+x^2+....\}$$

$$+1+2x+3x^2+4x^3+....$$

Put n = 2p.

If p is even, coefficient of $x^{2p} = -\frac{1}{2}+\frac{1}{2}+(2p+1) = 2p+1$

If p is odd , coefficient of $x^{2p} = \frac{1}{2}+\frac{1}{2}+(2p+1) = 2p+2$

$\therefore$ When $\frac{n}{2} = p$, coefficient of $x^n = n+1$ or $n+2$

Let $n = 2p-1$.

If p is even, coefficient of $x^{2p-1} = -\frac{1}{2}+\frac{1}{2}+2p = 2p$

SPACE FOR HINT

If p is odd , coefficient of $x^{2p-1}$ = $\frac{1}{2}+\frac{1}{2}+2p = 2p+1$

$\therefore$ When $p = \frac{n+1}{2}$, coefficient of $x^n = n + 1$ or $n + 2$.

When p is even, coefficient of $x^n = n+1 = \frac{1}{2}\{(2n+3)+(-1)^{p-1}\}$,

When p is odd, coefficient of $x^n = n+2 = \frac{1}{2}\{(2n+3)+(-1)^{p-1}\}$.

**CYP Questions:**

1) Find the greatest term in each of the following expansions:

(i) $(1+x)^{-6}$ when $x = \frac{4}{13}$.

(ii) $(1-2x)^{21/2}$ when $x = \frac{2}{3}$.

(iii) $(1-x)^{-12}$ when $x = \frac{7}{9}$.

(iv) $\left(1+\frac{2}{3}x\right)^{-2}$ when $x = \frac{3}{4}$.

## SECTION 4.3 : BINOMIAL CO-EFFICIENT & APPROXIMATE VALUES

**Sum of coefficients:**

If f(x) can be expanded as an ascending series in x, we can find the sum of the first (n + 1) coefficients.

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

$$\therefore \frac{f(x)}{1-x} = (a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots).$$
$$(1 + x + x^2 + x^3 + \ldots)$$

$\therefore$ Coefficient of $x^n$ in $\frac{f(x)}{1-x} = a_0 + a_1 + a_2 + a_3 + \ldots + a_n$.

Thus to find the sum of the first (n+1) coefficients in the expansion of f(x), we have only to find the coefficient of $x^n$ in the expansion of $\frac{f(x)}{1-x}$.

**Problems:**

1) Find the sum of the coefficients of the first (r + 1) terms in the expansion of $(1-x)^{-3}$

Solution: The sum of the coefficients of the first (r + 1) terms in the expansion of $(1-x)^{-3}$

$$= \text{the coefficient of } x^r \text{ in the expansion of } \frac{(1-x)^{-3}}{1-x}$$

$$= \text{the coefficient of } x^r \text{ in the expansion of } (1-x)^{-4}$$

$$= \text{the coefficient of } x^r \text{ in } 1+4x+\frac{4.5}{2!}x^2+\frac{4.5.6}{3!}x^3 +\ldots+\frac{(r+1)(r+2)(r+3)}{r!}x^r+\ldots$$

$$= \frac{(r+1)(r+2)(r+3)}{6}$$

2) If n is a pc ve integer, prove that $1+3n+\frac{3.4}{1.2}\frac{n(n-1)}{2!}$

$$+\frac{3.4}{1.2}\frac{n(n-1)}{2!}+\frac{4.5}{1.2}\frac{n(n-1)(n-2)}{3!}+\ldots+\frac{(n+1)(n+2)}{2}=2^{n-3}(n^2+7n+8)$$

Solution: We know that

$$(x+1)^n = x^n + nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}+\ldots\ldots$$

$$(1-x)^{-3} = 1+3x+\frac{3.4}{1.2}x^2+\frac{4.5}{1.2}x^3+\ldots..$$

$\therefore$ The given series is the coefficient of $x^n$ in $\frac{(x+1)^n}{(1-x)^3}$

$$= \text{the coefficient of } x^n \text{ in } \frac{\{2-(1-x)\}^n}{(1-x)^3}$$

$$= \text{the coefficient of } x^n \text{ in } 2^n - 2^{n-1}n(1-x)+\frac{n(n-1)}{2}2^{n-2}(1-x)^2$$

SPACE FOR HINT

$$+\frac{\text{terms involving } (1-x)^3 \text{ and powers of } (1-x), \text{ higher than the third}}{(1-x)^3}$$

$$= \text{the coefficient of } x^n \text{ in } \frac{2^n}{(1-x)^3} - \frac{n.2^{n-1}}{(1-x)^2} + \frac{n.(n-1)2^{n-3}}{(1-x)} + \text{terms}$$

involving powers of $(1 - x)$ less than $n - 3$.

$$= 2^n \frac{(n+1)(n+2)}{2} - n.2^{n-1}(n+1) + n(n-1)2^{n-3}$$

$$= 2^{n-3}\{4(n+1)(n+2) - 4n(n+1) + n(n-1)\}$$

$$= 2^{n-3}(n^2 + 7n + 8)$$

**Approximate Values.**

**Problems.**

1) Find correct to six places of decimals the value of $\frac{1}{(9998)^{1/4}}$

Solution: $\frac{1}{(9998)^{1/4}} = \frac{1}{(10000-2)^{1/4}}$

$$= \frac{1}{(10^4-2)^{1/4}}$$

$$= \frac{1}{10\left(1-\frac{2}{10^4}\right)^{1/4}}$$

$$= \frac{\left(1-\frac{2}{10^4}\right)^{-1/4}}{10}$$

$$= \frac{1+\frac{1}{4}\frac{2}{10^4}+\frac{\frac{1}{4}.\frac{5}{4}}{2!}\frac{4}{10^8}+....}{10}$$

$$= \frac{1}{10}+\frac{1}{2}.\frac{1}{10^5}+\frac{5}{8}.\frac{1}{10^9}+....$$

$= 0.1 + 0.000005 + 0.0000000005$

$= 0.1000050005$

$= 0.100005$ correct to six places of decimals

2) Find correct to six places of decimals the value of $(1.01)^{1/2} - (0.99)^{1/2}$.

Solution: $(1.01)^{1/2} = (1 + 0.01)^{1/2} = (1 + x)^{1/2}$ where $x = 0.01$

SPACE FOR HINT

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \ldots..$$

$$(0.99)^{1/2} = (1 - 0.01)^{1/2} = (1 - x)^{1/2}$$

$$= 1 - \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 - \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \ldots..$$

$\therefore (1.01)^{1/2} - (0.99)^{1/2} =$

$$= 2\left\{\frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{5!}x^5 + \ldots\right\}$$

$$= 2\left\{\frac{1}{2}x + \frac{1}{16}x^3 + \frac{7}{256}x^5 + \ldots\right\}$$

$$= x + \frac{1}{8}x^3 + \frac{7}{128}x^5 + \ldots$$

$$= (0.01) + \frac{1}{8}(0.01)^3 + \frac{7}{128}(0.01)^5 + \ldots$$

$= 0.01 + \frac{1}{8}(0.000001) +$ terms not affecting the 6th decial places

$= 0.01 + 0.010000125$

$\therefore (1.01)^{1/2} - (0.99)^{1/2} = 0.010000$ correct to six places of decimals.

3) When x is small, prove that

$$\frac{(1-3x)^{-2/3} + (1-4x)^{-3/4}}{(1-3x)^{-1/3} + (1-4x)^{-1/4}} = 1 + \frac{3}{2}x + 4x^2 \text{ approximately.}$$

Solution:

$$\frac{(1-3x)^{-2/3} + (1-4x)^{-3/4}}{(1-3x)^{-1/3} + (1-4x)^{-1/4}} =$$

$$\frac{1 + \frac{2}{3}.3x + \frac{\frac{2}{3}.\frac{5}{3}}{2!}(3x)^2 + \frac{\frac{2}{3}.\frac{5}{3}.\frac{8}{3}}{3!}(3x)^3 + \ldots 1 + \frac{3}{4}.4x + \frac{\frac{3}{4}.\frac{7}{4}}{2!}(4x)^2 + \frac{\frac{3}{4}.\frac{7}{4}.\frac{11}{4}}{3!}(4x)^3 + \ldots}{1 + \frac{1}{3}.3x + \frac{\frac{1}{3}.\frac{4}{3}}{2!}(3x)^2 + \frac{\frac{1}{3}.\frac{4}{3}.\frac{7}{3}}{3!}(3x)^3 + \ldots 1 + \frac{1}{4}.4x + \frac{\frac{1}{4}.\frac{5}{4}}{2!}(4x)^2 + \frac{\frac{1}{4}.\frac{5}{4}.\frac{9}{4}}{3!}(4x)^3 + \ldots}$$

Since $x^3$ and higher powers of x may be neglected the expression,

$$\frac{(1-3x)^{-2/3} + (1-4x)^{-3/4}}{(1-3x)^{-1/3} + (1-4x)^{-1/4}} = \frac{2 + 5x + 15\frac{1}{2}x^2}{2 + 2x + 4\frac{1}{2}x^2}$$

SPACE FOR HINT

$$= \frac{2+5x+15\frac{1}{2}x^2}{2\left(1+x+\frac{9}{8}x^2\right)}$$

$$= \frac{2+5x+15\frac{1}{2}x^2}{2}\left(1+x+\frac{9}{8}x^2\right)^{-1}$$

$$= \left(1+\frac{5}{2}x+\frac{31}{4}x^2\right)\left\{1+x\left(1+\frac{9}{4}x\right)\right\}^{-1}$$

$$= \left(1+\frac{5}{2}x+\frac{31}{4}x^2\right)\left\{1-x\left(1+\frac{9}{4}x\right)+x^2\left(1+\frac{9}{4}x\right)^2-....\right\}$$

$$= \left(1+\frac{5}{2}x+\frac{31}{4}x^2\right)\left\{1-x-\frac{9}{4}x^2+x^2\right\}$$

($x^3$ and higher powers of x neglected)

$$= \left(1+\frac{5}{2}x+\frac{31}{4}x^2\right)\left\{1-x-\frac{5}{4}x^2\right\}$$

$$= 1+\frac{5}{2}x+\frac{31}{4}x^2-x-\frac{5}{2}x^2-\frac{5}{4}x^2$$

$$= 1+\frac{3}{2}x+4x^2$$

4) Prove that $\sqrt{x^2+16}-\sqrt{x^2+9}=\frac{7}{2x}$ nearly for sufficiently large values of x.

Solution: $\sqrt{x^2+16}-\sqrt{x^2+9}=(x^2+16)^{1/2}-(x^2+9)^{1/2}$

$$= x\left(1+\frac{16}{x^2}\right)^{1/2}-x\left(1+\frac{9}{x^2}\right)^{1/2}$$

$$= x\left(1+\frac{1}{2}\frac{16}{x^2}-...\right)-x\left(1+\frac{1}{2}\frac{9}{x^2}-...\right)$$

$$= x+\frac{8}{x}-x-\frac{9}{2x} \quad (\text{ since } \frac{1}{x} \text{ is small})$$

$$= \frac{7}{2x} \text{ nearly}$$

5) If $\sqrt{N}=a+x$ when x is very small, then prove that

$$\sqrt{N}=a.\frac{3N+a^2}{N+3a^2} \text{ approximately.}$$

Solution: $N=(a+x)^2$

$$a.\frac{3N+a^2}{N+3a^2}=a.\frac{3(a+x)^2+a^2}{(a+x)^2+3a^2}$$

$$= a.\frac{4a^2+6ax+3x^2}{4a^2+2ax+x^2}$$

SPACE FOR HINT

$$= a.\frac{4a^2+6ax}{4a^2+2ax} \text{ approximately}$$

$$= a.\frac{1+\frac{3x}{2a}}{1+\frac{x}{2a}}$$

$$= a.\left(1+\frac{3x}{2a}\right)\left(1+\frac{x}{2a}\right)^{-1}$$

$$= a.\left(1+\frac{3x}{2a}\right)\left(1-\frac{x}{2a}\right)$$

$$= a.\left(1+\frac{3x}{2a}-\frac{x}{2a}\right) \text{ approximately}$$

( by omitting higher powers of x)

$$= a + x$$

$$= \sqrt{N}$$

6) Show that the error in taking $\frac{1+x}{2+x}+\frac{2+x}{4}$ as an approximation to $\sqrt{1+x}$ is approximately equal to $\frac{x^4}{2^7}$ when x is small.

Solution: $\sqrt{1+x} = (1+x)^{1/2}$

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \ldots..$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^5 + \ldots..$$

Now $\frac{1+x}{2+x}+\frac{2+x}{4} = \frac{2+x}{4}+\frac{1+x}{2\left(1+\frac{x}{2}\right)}$

$$= \frac{1}{2}+\frac{x}{4}+\frac{1+x}{2}\left(1+\frac{x}{2}\right)^{-1}$$

$$= \frac{1}{2}+\frac{x}{4}+\frac{1+x}{2}\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{16}-\ldots.\right)$$

$$= \frac{1}{2}+\frac{x}{4}+\frac{1}{2}+\frac{x}{2}-\frac{x}{4}-\frac{x^2}{4}+\frac{x^2}{8}+\frac{x^3}{8}-\frac{x^3}{16}-\frac{x^4}{16}+\frac{x^4}{32}+\frac{x^5}{32}\ldots.$$

$$= 1 + \frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{x^4}{32}+\ldots..$$

Neglecting $x^5$ and higher powers, the difference between the two expressions is

$$\frac{5x^4}{128}-\frac{x^4}{32} = \frac{x^4}{128} = \frac{x^4}{2^7}$$

SPACE FOR HINT

$\therefore$ The error is approximately equal to $\frac{x^4}{2^7}$.

**CYP Questions:**

1) If n is a positive integer and
$\frac{(1+x)^n}{(1-x)^3} = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots\ldots$
show that $a_0 + a_1 + a_2 + \ldots + a_{n-1} = \frac{1}{3}n(n+2)(n+7)2^{n-4}$.
2) If $(1-x)^{-3} = 1 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots\ldots$ find the value of $1 + a_1 + a_2 + \ldots + a_n$.
3) Find the sum of the first n + 1 coefficients in the expansion of $\frac{2x-4}{(1+x)(1-2x)}$ in ascending powers of x.
4) Find to three decimal places $(998)^{1/3}$.
5) Find the value of $(1.02)^{3/2} - (0.98)^{3/2}$.
6) When x is small, show that $\frac{(1-x)^{-5/2}(16+8x)^{1/2}}{(1+x)^{-1/2}+(2+x)} = 1 + \frac{23}{40}x^2$ approximately.
7) If x is large, prove that $(x^3+6)^{1/3} - (x^3+3)^{1/3} = \frac{1}{x^2} - \frac{3}{x^5}$ nearly.

SPACE FOR HINT

# UNIT-5

## Unit Structure:

Section 5.1: Exponential Theorem

Section 5.2: Logarithmic Series

Section 5.3: Modification of Logarithmic Series

Section 5.4: Euler's Constant .

**Introduction:** In this unit we discuss the exponential series and its application, logarithmic series and its modifications.Also we discuss the about the Euler's constant.

## SECTION 5.1: EXPONENTIAL THEOREM

**The Exponential Limit**

Find $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

Solution:

Case(i): Let n be a positive integer.

By the Binomial theorem,

$$\left(1+\frac{1}{n}\right)^n = 1+nC_1\frac{1}{n}+nC_2\frac{1}{n^2}+nC_3\frac{1}{n^3}+......+nC_n\frac{1}{n^n}$$

$$=1+1+\frac{n(n-1)}{2!}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\frac{1}{n^3}+.....$$

$$.......+\frac{n(n-1)(n-2).....(n-(n-1))}{n!}\frac{1}{n^n}$$

$$=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+.......$$

$$......+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)....\left(1-\frac{n-1}{n}\right)$$

$$<1+\frac{1}{1!}+\frac{1}{2!}+.......+\frac{1}{n!}\quad\left(\because 1-\frac{n-1}{n}<1\right)$$

$$=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+........+\frac{1}{1.2.3......n}$$

$$<1+1+\frac{1}{2}+\frac{1}{2^2}+.......+\frac{1}{2^{n-1}}$$

SPACE FOR HINT

$$< 1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots+\frac{1}{2^{n-1}}$$

$$= 1+2\left(1-\frac{1}{2^n}\right)$$

$$< 3.$$

$\therefore$ The limit of $\left(1+\frac{1}{n}\right)^n$ cannot be infinity.

$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ = a finite number.

This finite number is usually denoted by the letter e.

Case(ii) Let n be a positive fraction.

$\therefore$ n lies between two consecutive integers.

Let those numbers be m and m + 1.

$$\therefore\ m < n < m+1.$$

$$\therefore \frac{1}{m+1} < \frac{1}{n} < \frac{1}{m}$$

$$\therefore 1+\frac{1}{m+1} < 1+\frac{1}{n} < 1+\frac{1}{m}$$

Since m < n < m + 1

$$\left(1+\frac{1}{m+1}\right)^m < \left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{m}\right)^{m+1}$$

As $n \to \infty$, we have $m \to \infty$ and $m + 1 \to \infty$.

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m+1} = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\left(1+\frac{1}{m}\right)$$

$$= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m \lim_{m\to\infty}\left(1+\frac{1}{m}\right)$$

= e . 1 (since m is a positive integer, by case(i))

= e.

$$\lim_{m\to\infty}\left(1+\frac{1}{m+1}\right)^m = \lim_{m\to\infty}\frac{\left(1+\frac{1}{m+1}\right)^{m+1}}{\left(1+\frac{1}{m+1}\right)}$$

SPACE FOR HINT

$$= \frac{\lim_{m\to\infty}\left(1+\frac{1}{m+1}\right)^{m+1}}{\lim_{m\to\infty}\left(1+\frac{1}{m+1}\right)} = \frac{e}{1} = e.$$

$$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Case(iii) Let n be a negative integer.

Let $n = -m$ where m is positive.

$$\left(1+\frac{1}{n}\right)^n = \left(1-\frac{1}{m}\right)^{-m} = \left(\frac{m-1}{m}\right)^{-m} = \left(\frac{m}{m-1}\right)^m$$

$$= \left(1+\frac{1}{m-1}\right)^{m-1}\left(1+\frac{1}{m-1}\right).$$

As $n \to \infty$, we have $m - 1 \to \infty$

$$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{m-1\to\infty}\left(1+\frac{1}{m-1}\right)^{m-1} \lim_{m-1\to\infty}\left(1+\frac{1}{m-1}\right)$$

$= e.1$ (by cases(i) and (ii))

$= e.$

We can easily seen that $e = 1+\frac{1}{1!}+\frac{1}{2!}+.......+\frac{1}{n!}+.....\infty$

Now to show that e is an incommensurable number.

If it is not an incommensurable number, let it be $\frac{a}{b}$, where a and b are positive integers.

Then $\frac{a}{b} = 1+\frac{1}{1!}+\frac{1}{2!}+.......+\frac{1}{b!}+\frac{1}{(b+1)!}+....$

Multiply both sides by b!, we get

$$a(b-1)! = b!+\frac{b!}{1!}+\frac{b!}{2!}+.......+\frac{b!}{b!}+\frac{b!}{(b+1)!}+....$$

$$= b!+\frac{b!}{1!}+\frac{b!}{2!}+.......+\frac{b!}{b!}+\frac{1}{b+1}+\frac{1}{(b+1)(b+2)}+........$$

$$= \text{integer} + \frac{1}{b+1}+\frac{1}{(b+1)(b+2)}+........$$

Consider the series

$$\frac{1}{b+1}+\frac{1}{(b+1)(b+2)}+\frac{1}{(b+1)(b+2)(b+3)}+\ldots\ldots$$

$$<\frac{1}{b+1}+\frac{1}{(b+1)^2}+\frac{1}{(b+1)^3}+\ldots\ldots$$

$$<\frac{\frac{1}{b+1}}{1-\frac{1}{b+1}}=\frac{1}{b}.$$

Also this series greater than $\frac{1}{b+1}$.

$\therefore a(b-1)! =$ integer + a fraction, since the sum of series lies between $\frac{1}{b}$ and $\frac{1}{b+1}$.

$\therefore$ an integer = integer + a fraction, which is absurd.

$\therefore$ e cannot be expressed in the form $\frac{a}{b}$.

$\therefore$ e is incommensurable.

Note: The value of e is calculated to more than 500 decimal places.

The value of e is 2.7182818284……

**The Exponential Theorem:**

Theorem 5.1.1: For all values of x,

$$\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}+\ldots\right)^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots$$

Proof: Let $f(x) = 1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots$

In this series $u_{r+1}=\frac{x^r}{r!}$.

Then $u_r=\frac{x^{r-1}}{(r-1)!}$.

Therefore $\frac{u_{r+1}}{u_r}=\frac{x}{r}$.

SPACE FOR HINT

$$\therefore \lim_{r\to\infty} \frac{u_{r+1}}{u_r} = \lim_{r\to\infty} \frac{x}{r} = 0.$$

$\therefore$ The series f(x) is convergent for all values of x.

Thus the following series

$$f(m) = 1+\frac{m}{1!}+\frac{m^2}{2!}+....+\frac{m^r}{r!}+.....$$

$$f(n) = 1+\frac{n}{1!}+\frac{n^2}{2!}+....+\frac{n^r}{r!}+.....$$

$$f(m+n) = 1+\frac{(m+n)}{1!}+\frac{(m+n)^2}{2!}+....+\frac{(m+n)^r}{r!}+.....$$

are convergent for all values of m and n.

$\therefore$ Series for f(m) and f(n) can be multiplied.

$$f(m).f(n) = \left\{1+\frac{m}{1!}+\frac{m^2}{2!}+....+\frac{m^r}{r!}+.....\right\}.\left\{1+\frac{n}{1!}+\frac{n^2}{2!}+....+\frac{n^r}{r!}+.....\right\}$$

$$= 1+\left(\frac{m}{1!}+\frac{n}{1!}\right)+\left(\frac{m^2}{2!}+\frac{mn}{1!.1!}+\frac{n^2}{2!}\right)+\left(\frac{m^3}{3!}+\frac{m^2}{2!}\frac{n}{1!}+\frac{m}{1!}\frac{n^2}{2!}+\frac{m^3}{3!}\right)+......$$

The (r+1)th term in the above product is

$$\frac{m^r}{r!}+\frac{m^{r-1}}{(r-1)!}\frac{n}{1!}+\frac{m^{r-2}}{(r-2)!}\frac{n^2}{2!}+...+\frac{m^2}{2!}\frac{n^{r-2}}{(r-2)!}+\frac{m}{1!}\frac{n^{r-1}}{(r-1)!}+\frac{n^r}{r!}$$

$$= \frac{1}{r!}\left\{m^r+\frac{r!}{(r-1)!}m^{r-1}n+\frac{r!}{(r-2)!2!}m^{r-2}n^2+...+\frac{r!}{(r-1)!}mn^{r-1}+n^r\right\}$$

$$= \frac{1}{r!}\left\{m^r+rC_1m^{r-1}n+rC_2m^{r-2}n^2+...+rC_1mn^{r-1}+n^r\right\}$$

$$= \frac{1}{r!}(m+n)^r$$

$$\therefore f(m).f(n) = 1+\frac{(m+n)}{1!}+\frac{(m+n)^2}{2!}+....+\frac{(m+n)^r}{r!}+.....$$

$$= f(m+n)$$

Case(i) Let x be a positive integer.

Then by the above result, we have

f(m).f(n).f(p)...... x factors = f(m + n + p +.... x terms).

Put m = n = p = 1.

$\therefore$ f(1).f(1)...... x factors = f(1 + 1 + 1 + .... x terms)

SPACE FOR HINT

(i.e.) $\{f(1)\}^x = f(x)$.

Case(ii) Let x be a positive fraction $\frac{p}{q}$, where p and q are positive integers.

$$\text{Then } \left\{f\left(\frac{p}{q}\right)\right\}^q = f\left(\frac{p}{q}\right).f\left(\frac{p}{q}\right)\text{.......q factors}$$

$$= f\left(\frac{p}{q}+\frac{p}{q}+\text{.....q terms}\right)$$

$$= f(p)$$

$$= \{f(1)\}^p.$$

$$\therefore \{f(1)\}^{p/q} = f\left(\frac{p}{q}\right)$$

$$\therefore \{f(1)\}^x = f(x).$$

Case(iii) Let x be a negative integer.

Let x = – s where s is a positive integer.

$f(s).f(-s) = f(s-s) = f(0) = 1.$

$$\therefore f(-s) = \frac{1}{f(s)}.$$

$$\text{Thus } f(x) = f(-s) = \frac{1}{f(s)} = \frac{1}{\{f(1)\}^s}$$

$$= \{f(1)\}^{-s} = \{f(1)\}^x.$$

$\therefore f(x) = \{f(1)\}^x$ for all values of x.

$$\text{But } f(1) = 1+\frac{1}{1!}+\frac{1}{2!}+\text{.......}+\frac{1}{n!}+\text{.....}$$

$$\therefore 1+\frac{x}{1!}+\frac{x^2}{2!}+\text{....}+\frac{x^n}{n!}+\text{.....} = \left(1+\frac{1}{1!}+\frac{1}{2!}+\text{....}+\frac{1}{n!}+\text{.....}\right)^x$$

Since $e = 1+\frac{1}{1!}+\frac{1}{2!}+\text{.......}+\frac{1}{n!}+\text{.....}$, we get

$$e^x = \left(1+\frac{1}{1!}+\frac{1}{2!}+\text{....}+\frac{1}{n!}+\text{.....}\right)^x$$

$$= 1+\frac{x}{1!}+\frac{x^2}{2!}+\text{....}+\frac{x^n}{n!}+\text{.....}$$

SPACE FOR HINT

Cor: $a^x = 1+\frac{x}{1!}\log_e a+\frac{x^2}{2!}(\log_e a)^2+....+\frac{x^r}{r!}(\log_e a)^r+.....$ where a is a positive number.

Proof:

$$a^x = e^{\log_e a^x} = e^{x\log_e a}$$

Substituting in the Exponential theorem, we the expression for $a^x$.

**Problems:**

1) Show that the coefficient of $x^n$ in the series

$$1+\frac{b+ax}{1!}+\frac{(b+ax)^2}{2!}+......+\frac{(b+ax)^n}{n!}+..... \text{ is } \frac{e^b a^n}{n!}$$

Solution:

$$\text{The series } 1+\frac{b+ax}{1!}+\frac{(b+ax)^2}{2!}+......+\frac{(b+ax)^n}{n!}+.....$$

$$= e^{b+ax}$$

$$= e^b.e^{ax}$$

$$= e^b\left\{1+\frac{ax}{1!}+\frac{(ax)^2}{2!}+......+\frac{(ax)^n}{n!}+.....\right\}$$

$\therefore$ Coefficient of $x^n = \frac{e^b a^n}{n!}$.

2) Show that the coefficient of $x^n$ in the series

$$1+\frac{1+2x}{1!}+\frac{(2+2x)^2}{2!}+......+\frac{(2+2x)^n}{n!}+..... \text{ is } \frac{2^n e}{n!}.$$

Solution:

$$\text{The series } 1+\frac{1+2x}{1!}+\frac{(2+2x)^2}{2!}+......+\frac{(2+2x)^n}{n!}+.....$$

$$= e^{1+2x}$$

$$= e^1.e^{2x}$$

$$= e\left\{1+\frac{2x}{1!}+\frac{(2x)^2}{2!}+......+\frac{(2x)^n}{n!}+.....\right\}$$

$\therefore$ Coefficient of $x^n = \frac{2^n e}{n!}$.

3) Find the coe fficient of $x^n$ in the expansion of $\frac{1+2x-3x^2}{e^x}$

Solution: $\frac{1+2x-3x^2}{e^x} = (1+2x-3x^2)e^{-x}$

$$= (1+2x-3x^2)\left\{1-x+\frac{x^2}{2!}-......+(-1)^n\frac{x^n}{n!}+.....\right\}$$

$\therefore$ The coefficient of $x^n = 1.\frac{(-1)^n}{n!}+2.\frac{(-1)^{n-1}}{(n-1)!}-3\frac{(-1)^{n-2}}{(n-2)!}$

$$= \frac{(-1)^n}{n!}\{1-2n-3n(n-1)\}$$

$$= \frac{(-1)^n}{n!}\{1+n-3n^2\}$$

**CYP Questions:**

1) Find the coefficient of $x^n$ in the expansion of

$\Sigma(-1)^n\frac{(2+3x)^n}{n!}$

2) Expand $\frac{1+2x+3x^2}{e^x}$ as a power series in x and write down the coefficient of $x^n$.

## SECTION 5.2: THE LOGARITHMIC SERIES

Theorem 5.2.1: If $-1 < x < 1$, then

$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}+.....+(-1)^n\frac{x^n}{n}+....$$

Proof: $(1+x)^y = e^{y\log(1+x)}$

$$= 1 + y\log(1+x) + \frac{y^2}{2!}\{\log(1+x)\}^2+\frac{y^3}{2!}\{\log(1+x)\}^3+\ldots \text{ ----- (1)}$$

By the Binomial theorem,

if $|x|<1, (1+x)^y = 1+yx+\frac{y(y-1)}{2!}x^2+\frac{y(y-1)(y-2)}{3!}x^3+.....$ ------ (2)

Since this series is absolutely convergent, it can be re-arranged I ascending powers of y.

SPACE FOR HINT

Then the coefficients of corresponding powers of y in (1) and (2) are equal.

If we equate the coefficients of y, we have

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots + (-1)^n \frac{x^n}{n} + \ldots.$$

Cor: We can obtain the series for $\{\log(1+x)\}^2$ by equating the coefficients of $y^2$ in the series (1) and (2).

$$\therefore \frac{1}{2!}\{\log(1+x)\}^2 = \text{Coefficient of } y^2 \text{ in}$$

$$\frac{y(y-1)}{1.2}x^2 + \frac{y(y-1)(y-2)}{1.2.3}x^3 + \frac{y(y-1)(y-2)(y-3)}{1.2.3.4}x^4 \ldots\ldots$$

$$= \text{Coefficient of } y^2 \text{ in}$$

$$\frac{y-1}{1.2}x^2 + \frac{(y-1)(y-2)}{1.2.3}x^3 + \frac{(y-1)(y-2)(y-3)}{1.2.3.4}x^4 + \ldots\ldots$$

$$\therefore \frac{1}{2!}\{\log(1+x)\}^2 = \frac{x^2}{1.2} - \frac{(1+2)}{1.2.3}x^3 + \frac{(1.2+2.3+3.4)}{1.2.3.4}x^4 + \ldots\ldots$$

$$= \frac{x^2}{2} - \frac{1}{3}\left(1+\frac{1}{2}\right)x^3 + \frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^4 + \ldots\ldots$$

$$\therefore \{\log(1+x)\}^2 = 2\left\{\frac{x^2}{2} - \frac{1}{3}\left(1+\frac{1}{2}\right)x^3 + \frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^4 + \ldots\ldots\right\}$$

## SECTION 5.3: MODIFICATION OF THE LOGARITHMIC SERIES

From the above theorem we have

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots + (-1)^n \frac{x^n}{n} + \ldots. \quad \text{------ (1)}$$

Replace x by $-x$ in (1) we get

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} + \ldots\ldots - \frac{x^n}{n} - \ldots.$$

$$= -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots + \frac{x^n}{n} + \ldots.\right)$$

$$\therefore -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots + \frac{x^n}{n} + \ldots \text{ --------------- (2)}$$

Adding (1) and (2), we get

$$\log(1+x) - \log(1-x) = 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \ldots$$

$$\text{(i.e.) } \log\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right) \text{ ---------------------- (3)}$$

Put $x = \dfrac{1}{2n+1}$ in (3), we get

$$\log\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}} = 2\left(\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\right)$$

$$\log\frac{\frac{2n+1+1}{2n+1}}{\frac{2n+1-1}{2n+1}} = 2\left(\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\right)$$

$$\log\frac{2n+2}{2n} = 2\left(\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\right)$$

$$\log\frac{n+1}{n} = 2\left(\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\right) \text{ -------- (4)}$$

this expansion is valid if $\left|\dfrac{1}{2n+1}\right| < 1$

(i.e.) when $n > 0$ or $n < -1$.

In (3), if we write $\dfrac{1+x}{1-x} = \dfrac{m}{n}$ so that $x = \dfrac{m-n}{m+n}$ we get the series

$$\log\frac{m}{n} = 2\left(\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \ldots\right) \text{ ------ (5)}$$

This expansion is valid when $\left|\dfrac{m-n}{m+n}\right| < 1$

$$\text{(i.e.) } \left(\frac{m-n}{m+n}\right)^2 - 1 < 0$$

$$\text{(i.e.) } \left(\frac{m-n}{m+n} - 1\right)\left(\frac{m-n}{m+n} + 1\right) < 0$$

SPACE FOR HINT

Multiplying through out by the factor $(m+n)^2$ which is positive for real m and n, this condition reduces to mn > 0, (i.e.) m,n either both positive or both negative.

If we write $x = \frac{m}{n}$ in (5), we get

$$\log x = 2\left(\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \ldots\right) \text{------------ (6)}$$

this expansion is valid for all positive values of x.

If we put $x = \frac{1}{2y-1}$ in (3), so that $\frac{1+x}{1-x} = \frac{y}{y-1}$ we get

$$\log\frac{y}{y-1} = 2\left(\frac{1}{2y-1} + \frac{1}{3(2y-1)^3} + \frac{1}{5(2y-1)^5} + \ldots\right) \text{------- (7)}$$

The condition $|x| < 1$ becomes $\left|\frac{1}{2y-1}\right| < 1$

(i.e.) $(2y-1)^2 > 1$

(i.e.) $y(y-1) > 0$

$\therefore$ The expansion (7) is valid when $y < 0$ or $y > 1$.

**Problems:**

1) Show that if $x > 0$,

$$\log x = \frac{x-1}{x+1} + \frac{1}{2}\frac{x^2-1}{(x+1)^2} + \frac{1}{3}\frac{x^3-1}{(x+1)^3} + \ldots..$$

Solution : $\text{RHS} = \frac{x-1}{x+1} + \frac{1}{2}\frac{x^2-1}{(x+1)^2} + \frac{1}{3}\frac{x^3-1}{(x+1)^3} + \ldots.$

$$= \frac{x}{x+1} + \frac{1}{2}\frac{x^2}{(x+1)^2} + \frac{1}{3}\frac{x^3}{(x+1)^3} + \ldots - \left\{\frac{1}{x+1} + \frac{1}{2}\frac{1}{(x+1)^2} + \frac{1}{3}\frac{1}{(x+1)^3} + \ldots\right\}$$

$$= -\log\left(1 - \frac{x}{x+1}\right) + \log\left(1 - \frac{1}{x+1}\right)$$

$$= -\log\frac{1}{x+1} + \log\frac{x}{x+1}$$

$$= \log\left\{\left(\frac{\frac{x}{x+1}}{\frac{1}{x+1}}\right)\right\} = \log x = \text{LHS.}$$

SPACE FOR HINT

The expansion is valid when $\left|\frac{x}{x+1}\right| < 1$and $\left|\frac{1}{x+1}\right| < 1$, $\left|\frac{x}{x+1}\right|$is always less than 1.

When $\left|\frac{1}{x+1}\right| < 1, |x+1| > 1$, (i.e.) $|x| > 0$.

$\therefore$ When $x > 0$, the expansion is valid.

2) Show that $\log\sqrt{12} = 1 + \left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3}+\ldots.$

Solution: RHS $= 1 + \left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3}+\ldots.$

$$= \frac{1}{2}.\frac{1}{4}+\frac{1}{4}.\frac{1}{4^2}+\frac{1}{6}.\frac{1}{4^3}+\ldots\ldots + 1+ \frac{1}{3}.\frac{1}{4}+\frac{1}{5}.\frac{1}{4^2}+\frac{1}{7}.\frac{1}{4^3}+\ldots\ldots$$

$$=\frac{1}{2}.\left(\frac{1}{2}\right)^2+\frac{1}{4}.\left(\frac{1}{2}\right)^4+\frac{1}{6}.\left(\frac{1}{2}\right)^6+\ldots\ldots$$

$$+1+\frac{1}{3}.\left(\frac{1}{2}\right)^2+\frac{1}{5}.\left(\frac{1}{2}\right)^4+\frac{1}{7}.\left(\frac{1}{2}\right)^6+\ldots\ldots$$

$$= \frac{1}{2}x^2+\frac{1}{4}x^4+\frac{1}{6}x^6+\ldots\ldots + 1 + \frac{1}{3}x^2+\frac{1}{5}x^4+\frac{1}{7}x^6+\ldots\ldots$$

where $x = \frac{1}{2}$

$$= \frac{1}{2}\left\{x^2+\frac{1}{2}x^4+\frac{1}{3}x^6+\ldots\ldots\right\} + \frac{1}{x}\left\{x+\frac{1}{3}x^3+\frac{1}{5}x^5+\frac{1}{7}x^7+\ldots\ldots\right\}$$

$$= -\frac{1}{2}\log(1-x^2)+\frac{1}{2x}\log\frac{1+x}{1-x}$$

$$= -\frac{1}{2}\log(1-\frac{1}{4})+\frac{1}{2\frac{1}{2}}\log\frac{1+\frac{1}{2}}{1-\frac{1}{2}}$$ , by putting $x = \frac{1}{2}$

$$= -\frac{1}{2}\log\frac{3}{4}+\frac{2}{2}\log\frac{\frac{3}{2}}{\frac{1}{2}}$$

$$= -\frac{1}{2}\log\frac{3}{4}+\log 3$$

$$= -\frac{1}{2}\log\frac{3}{4}+\frac{1}{2}\log 9$$

SPACE FOR HINT

$$= \frac{1}{2}\log\frac{9.4}{3} = \frac{1}{2}\log 12$$

$$= \log\sqrt{12} = \text{LHS.}$$

3) If a,b,c denote three consecutive integers, show that

$$\log_e b = \frac{1}{2}\log_e a + \frac{1}{2}\log_e c + \frac{1}{2ac+1} + \frac{1}{3}.\frac{1}{(2ac+1)^3} + ....$$

Solution: RHS = $\frac{1}{2}\log_e a + \frac{1}{2}\log_e c + \frac{1}{2ac+1} + \frac{1}{3}.\frac{1}{(2ac+1)^3} + ....$

$$= \frac{1}{2}\log_e a + \frac{1}{2}\log_e c + x + \frac{1}{3}.x^3 + .... \text{, where } x = \frac{1}{2ac+1}$$

$$= \frac{1}{2}\log_e a + \frac{1}{2}\log_e c + \frac{1}{2}\log\frac{1+x}{1-x}$$

$$= \frac{1}{2}\log_e a + \frac{1}{2}\log_e c + \frac{1}{2}\log\frac{1+\frac{1}{2ac+1}}{1-\frac{1}{2ac+1}}$$

$$= \frac{1}{2}\log_e a + \frac{1}{2}\log_e c + \frac{1}{2}\log\frac{\frac{2ac+1+1}{2ac+1}}{\frac{2ac+1-1}{2ac+1}}$$

$$= \frac{1}{2}\log_e a + \frac{1}{2}\log_e c + \frac{1}{2}\log\frac{2ac+2}{2ac}$$

$$= \frac{1}{2}\log_e ac + \frac{1}{2}\log\frac{ac+1}{ac}$$

$$= \frac{1}{2}\log\frac{ac+1}{ac}.ac$$

$$= \frac{1}{2}\log(ac+1) \text{ ------------------- (1)}$$

If a,b,c denote three consecutive integers, then b = a + 1 and b = c – 1

$\therefore$ a = b – 1 and c = b + 1

$\therefore ac = (b-1)(b+1) = b^2 - 1$

(i.e.) $ac + 1 = b^2$.

$\therefore$ From (1), $\frac{1}{2}\log(ac+1) = \frac{1}{2}\log b^2 = \log b = \text{LHS}$

SPACE FOR HINT

**CYP Questions:**

1) Show that $\log\frac{a+x}{a-x}=\frac{2ax}{a^2+x^2}+\frac{1}{3}\left(\frac{2ax}{a^2+x^2}\right)^3+\frac{1}{5}\left(\frac{2ax}{a^2+x^2}\right)^5+\ldots.$

2) Show that $\frac{a-b}{a}+\frac{1}{2}\left(\frac{a-b}{a}\right)^2+\frac{1}{3}\left(\frac{a-b}{a}\right)^3+\ldots.=\log_e a-\log_e b$.

3) If $n=\frac{1}{e}-\frac{1}{2}.\frac{1}{e^2}+\frac{1}{3}.\frac{1}{e^3}\ldots\ldots\infty$, show that $e^{n+1}-e-1=0$.

## SECTION 5.4: EULER'S CONSTANT

If $u_n=1+\frac{1}{2}+\frac{1}{3}+\ldots.+\frac{1}{n}-\log n$, then as $n\to\infty$, $u_n\to\gamma$, where $\gamma$ is a fixed number lying between 0 and 1.

$$u_n=1+\frac{1}{2}+\frac{1}{3}+\ldots.+\frac{1}{n}-\log n.$$

$$u_{n-1}=1+\frac{1}{2}+\frac{1}{3}+\ldots.+\frac{1}{n-1}-\log(n-1)$$

$$\therefore\ u_n-u_{n-1}=\frac{1}{n}-\log n+\log(n-1)$$

$$=\frac{1}{n}+\log\frac{n-1}{n}$$

$$=\frac{1}{n}+\log\left(1-\frac{1}{n}\right)$$

$$=\frac{1}{n}-\frac{1}{n}-\frac{1}{2}.\frac{1}{n^2}-\frac{1}{3}.\frac{1}{n^3}-\ldots.$$

$$=-\frac{1}{2}.\frac{1}{n^2}-\frac{1}{3}.\frac{1}{n^3}-\ldots.$$

$<0.$

$\therefore$ $u_n$ is a decreasing sequence.

For all values of x other than zero, $e^x>1+x$

Taking log on both sides, we get

$x>\log(1+x)$, where $x>-1$

$$\therefore\frac{1}{n}>\log\left(1+\frac{1}{n}\right)$$

SPACE FOR HINT

Also $-\log\left(1-\frac{1}{n}\right)=\frac{1}{n}+\frac{1}{2}.\frac{1}{n^2}+\frac{1}{3}.\frac{1}{n^3}+....$

$$\therefore -\log\left(1-\frac{1}{n}\right)>\frac{1}{n}$$

$$\therefore \log\left(1+\frac{1}{n}\right)<\frac{1}{n}<-\log\left(1-\frac{1}{n}\right)$$

$$\text{(i.e.) } \log\left(\frac{n+1}{n}\right)<\frac{1}{n}<\log\left(\frac{n}{n-1}\right)$$

Substituting $n-1, n-2, ...., 2$ in succession for n, we have

$$\log\left(\frac{n}{n-1}\right)<\frac{1}{n-1}<\log\left(\frac{n-1}{n-2}\right)$$

$$\log\left(\frac{n-1}{n-2}\right)<\frac{1}{n-2}<\log\left(\frac{n-2}{n-3}\right)$$

....... .......... ......... ......... ...

$$\log\frac{3}{2}<\frac{1}{2}<\log\frac{2}{1}.$$

Also $\log\frac{2}{1}<1=1.$

$$\therefore \log\left(\frac{n+1}{n}\right)+\log\left(\frac{n}{n-1}\right)+\log\left(\frac{n-1}{n-2}\right)+....+\log\frac{2}{1}$$

$$<1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{n}<\log\left(\frac{n}{n-1}\right)+\log\left(\frac{n-1}{n-2}\right)+....+\log\frac{2}{1}$$

(i.e.) $\log(n+1)<1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{n}<1+\log n$

$\therefore \log(n+1)-\log n<1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{n}-\log n<1$

(i.e.) $\log(n+1)-\log n<u_n<1.$

But $\log(n+1)-\log n>0.$

Hence $0<u_n<1$ and also we have proved that $u_n<u_{n-1}$.

Thus $u_n$ is a decreasing function which is always greater than 0.

Consequently $u_n \to$ a finite limit lying between 0 and 1. This finite limit is usually denoted by the letter $\gamma$. This number $\gamma$ is known as Euler's constant.

SPACE FOR HINT

**Problems:**

1) Sum the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+.....$ to $\infty$.

Solution: Let $S_{2n}$ be the sum of the series upto 2n terms and $S_{2n+1}$ upto 2n + 1 terms.

$$S_{2n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+.....+\frac{1}{2n-1}-\frac{1}{2n}$$

$$= \left(1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{2n}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+.....+\frac{1}{2n}\right)$$

By the above result

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}-\log n\right)=\gamma.$$

$$\therefore \lim_{2n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{2n}-\log 2n\right)=\gamma.$$

$$\therefore 1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{2n} = \gamma + \log 2n + \varepsilon, \text{ where } \varepsilon \to 0 \text{ as } 2n \to \infty.$$

Similarly $1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n} = \gamma + \log n + \varepsilon_1$, where $\varepsilon_1 \to 0$ as $n \to \infty$.

$$\therefore S_{2n} = (\gamma + \log 2n + \varepsilon) - (\gamma + \log n + \varepsilon_1)$$

$$= \log 2n - \log n + \varepsilon - \varepsilon_1$$

$$= \log\frac{2n}{n} + \varepsilon - \varepsilon_1$$

$$= \log 2 + \varepsilon - \varepsilon_1.$$

$$\therefore \lim_{n\to\infty} S_{2n} = \log 2.$$

Also $S_{2n+1} - S_{2n} \to 0$. Hence $S_{2n+1} \to \log 2$.

Thus $\log 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+(-1)^{n-1}\frac{1}{n}+....$ to $\infty$.

Note: We have proved that

$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}+.....+(-1)^n\frac{x^n}{n}+.... \text{ where } |x|<1.$$

When we put x = 1, in the above logarithmic series , we get

$$\log 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+(-1)^{n-1}\frac{1}{n}+....$$

$\therefore$ The logarithmic series is valid when x = 1.

SPACE FOR HINT

When x = −1 the series becomes

$$-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{n}-\ldots,\text{ which is divergent.}$$

∴ The logarithmic series is valid when x = −1.

This we get the logarithmic series

$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}+\ldots\ldots+(-1)^n\frac{x^n}{n}+\ldots$$ is valid in the range $-1 < x \le 1$.

2) Sum the series $\sum_{n=1}^{\infty}\frac{1}{(2n-1)2n(2n+1)}$

Solution: Let S be the sum of the given series and $u_n$ be the $n^{th}$ term.

Then $u_n = \dfrac{1}{(2n-1)2n(2n+1)}$

$$= \frac{1}{2}\frac{1}{2n-1} - \frac{1}{2n} + \frac{1}{2}\frac{1}{2n+1}$$

$$\therefore u_1 = \frac{1}{2}.\frac{1}{1} - \frac{1}{2} + \frac{1}{2}.\frac{1}{3}$$

$$u_2 = \frac{1}{2}.\frac{1}{3} - \frac{1}{4} + \frac{1}{2}.\frac{1}{5}$$

$$u_3 = \frac{1}{2}.\frac{1}{5} - \frac{1}{6} + \frac{1}{2}.\frac{1}{7}$$

..... ...... .....

..... ...... .....

$$u_n = \frac{1}{2}\frac{1}{2n-1} - \frac{1}{2n} + \frac{1}{2}\frac{1}{2n+1}$$

..... ...... .....

..... ...... .....

Adding the last fraction of a term with the first fraction of the next term. We get

$$S = \frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\ldots$$

$$= -\frac{1}{2}+1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\ldots$$

$$= -\frac{1}{2} + \log 2.$$

SPACE FOR HINT

3) Show that $\frac{5}{1.2.3}+\frac{7}{3.4.5}+\frac{9}{5.6.7}+......\infty = 3\log 2-1$.

Solution: Let S be the sum of the given series and $u_n$ be the $n^{th}$ term.

Then $u_n = \frac{2n+3}{(2n-1)2n(2n+1)}$.

By partial fraction, we get

$$u_n = 2.\frac{1}{2n-1} - 3.\frac{1}{2n} + 1.\frac{1}{2n+1}$$

$$\therefore u_1 = 2.\frac{1}{1} - 3.\frac{1}{2} + 1.\frac{1}{3}$$

$$u_2 = 2.\frac{1}{3} - 3.\frac{1}{4} + 1.\frac{1}{5}$$

$$u_3 = 2.\frac{1}{5} - 3.\frac{1}{6} + 1.\frac{1}{7}$$

..... ...... .....

..... ...... .....

$$u_n = 2.\frac{1}{2n-1} - 3.\frac{1}{2n} + 1.\frac{1}{2n+1}$$

..... ...... .....

..... ...... .....

$$S = 2 - 3.\frac{1}{2} + (1.\frac{1}{3} + 2.\frac{1}{3}) - 3.\frac{1}{4} + (1.\frac{1}{5} + 2.\frac{1}{5}) - 3.\frac{1}{6} + ....$$

$$= 2 - 3.\frac{1}{2} + 3.\frac{1}{3} - 3.\frac{1}{4} + 3.\frac{1}{5} - 3.\frac{1}{6} + ....$$

$$= 2 + 3(-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - .....)$$

$$= 2 + 3(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - ..... - 1)$$

$$= 2 + 3(\log 2 - 1)$$

$$= -1 + 3\log 2.$$

4) If k is a positive integer and $|x| < 1$, then

$$\sum_{n=1}^{\infty}\frac{x^n}{n+k} = \frac{x}{1+k} + \frac{x^2}{2+k} + \frac{x^3}{3+k} + \frac{x^4}{4+k} + ....$$

SPACE FOR HINT

$$= \frac{1}{x^k}\left(\frac{x^{k+1}}{k+1}+\frac{x^{k+2}}{k+2}+\frac{x^{k+3}}{k+3}+.....\infty\right)$$

$$= \frac{1}{x^k}\left\{\left(x+\frac{x^2}{2}+...+\frac{x^k}{k}\right)+\left(\frac{x^{k+1}}{k+1}+\frac{x^{k+2}}{k+2}+\frac{x^{k+3}}{k+3}+...\infty\right)-\left(x+\frac{x^2}{2}+...+\frac{x^k}{k}\right)\right\}$$

$$= \frac{1}{x^k}\left\{\left(x+\frac{x^2}{2}+...+\frac{x^k}{k}\frac{x^{k+1}}{k+1}+\frac{x^{k+2}}{k+2}+\frac{x^{k+3}}{k+3}+...\infty\right)-\left(x+\frac{x^2}{2}+...+\frac{x^k}{k}\right)\right\}$$

$$= \frac{1}{x^k}\left\{-\log(1-x)-\left(x+\frac{x^2}{2}+...+\frac{x^k}{k}\right)\right\}$$

$$= -\frac{1}{x^k}\left\{\log(1-x)+\left(x+\frac{x^2}{2}+...+\frac{x^k}{k}\right)\right\}$$

Similarly $\sum_{n=1}^{\infty}\frac{x^n}{n+1} = -\frac{1}{x}\{\log(1-x)+x\}$

$$\sum_{n=1}^{\infty}\frac{x^n}{n+2} = -\frac{1}{x}\left\{\log(1-x)+x+\frac{x^2}{2}\right\}$$

$$\sum_{n=1}^{\infty}\frac{x^n}{n+3} = -\frac{1}{x}\left\{\log(1-x)+x+\frac{x^2}{2}+\frac{x^3}{3}\right\}$$

5) Sum the series $\sum_{n=1}^{\infty}\frac{n^3+n^2+1}{n(n+2)}x^n$

Solution: Let S be the sum of the given series.

By partial fraction, we have

$$\frac{n^3+n^2+1}{n(n+2)} = (n-1)+\frac{1}{2}.\frac{1}{n}+\frac{3}{2}.\frac{1}{n+2}.$$

Then $S = \sum_{n=1}^{\infty}\frac{n^3+n^2+1}{n(n+2)}x^n = \sum_{n=1}^{\infty}\left\{(n-1)+\frac{1}{2}.\frac{1}{n}+\frac{3}{2}.\frac{1}{n+2}\right\}x^n$

$$= \sum_{n=1}^{\infty}(n-1)x^n + \frac{1}{2}\sum_{n=1}^{\infty}\frac{x^n}{n} + \frac{3}{2}\sum_{n=1}^{\infty}\frac{x^n}{n+2}$$

$$\sum_{n=1}^{\infty}(n-1)x^n = x^2+2x^3+3x^4+.....$$

$$= x^2(1+2x+3x^2+.....)$$

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$$= x^2(1-x)^{-2}$$

$$= \frac{x^2}{(1-x)^2}$$

$$\sum_{n=1}^{\infty}\frac{x^n}{n} = -\log(1-x)$$

and $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n+2} = -\frac{1}{x^2}\left\{\log(1-x)+x+\frac{x^2}{2}\right\}$

$$\therefore\ S = \frac{x^2}{(1-x)^2} - \frac{1}{2}\log(1-x) - \frac{3}{2x^2}\left\{\log(1-x)+x+\frac{x^2}{2}\right\}$$

6) Sum the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n(n+1)(n+2)}$

Solution: Let S be the sum of the series.

By partial fraction $\displaystyle\frac{1}{n(n+1)(n+2)} = \frac{1}{2}.\frac{1}{n} - \frac{1}{n+1} + \frac{1}{2}.\frac{1}{n+2}$

Then $\displaystyle S = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n(n+1)(n+2)}$

$$= \sum_{n=1}^{\infty}\left\{\frac{1}{2}.\frac{1}{n} - \frac{1}{n+1} + \frac{1}{2}.\frac{1}{n+2}\right\}(-1)^{n+1}x^n$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n} - \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n+1} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n+2}$$

We have $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n} = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \ldots.. = \log(1+x)$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n+1} = \frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4} - \ldots..$$

$$= \frac{1}{x}\left\{\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \ldots..\right\}$$

$$= \frac{1}{x}\{-\log(1+x)+x\}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n+2} == \frac{x}{3} - \frac{x^2}{4} + \frac{x^3}{5} - \ldots..$$

$$= \frac{1}{x^2}\left\{\frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots..\right\}$$

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$$= \frac{1}{x^2}\left\{\log(1+x) - x + \frac{x^2}{2}\right\}$$

$$\therefore S = \frac{1}{2}\log(1+x) - \frac{1}{x}\{-\log(1+x)+x\} + \frac{1}{2x^2}\left\{\log(1+x) - x + \frac{x^2}{2}\right\}$$

$$= \frac{1}{2}\log(1+x)(1+\frac{2}{x}+\frac{1}{x^2}) - \left(\frac{3}{4}+\frac{1}{2x}\right).$$

**Calculation of logarithms by means of the logarithmic series**

We know that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots + (-1)^n\frac{x^n}{n} + \ldots. \text{ where } |x| < 1.$$

Since this series is slowly convergent, the direct calculation of logarithms by means of this series is tedious.

The calculation is usually carried out in practice as follows

We have proved that

$$\log_e\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots.\right) \text{ when } -1 < x < 1.$$

$$\text{Let } y = \frac{1+x}{1-x} \text{ so that } x = \frac{y-1}{y+1}$$

$$\therefore \log_e y = 2\left(\frac{y-1}{y+1} + \frac{1}{3}\left(\frac{y-1}{y+1}\right)^3 + \frac{1}{5}\left(\frac{y-1}{y+1}\right)^5 + \ldots.\right) \text{where y lies between 0}$$

and $+\infty$.

Put $y = \frac{p}{q}$ in this series, where p and q are positive integers.

$$\therefore \log_e p - \log_e q = 2\left(\frac{p-q}{p+q} + \frac{1}{3}\left(\frac{p-q}{p+q}\right)^3 + \frac{1}{5}\left(\frac{p-q}{p+q}\right)^5 + \ldots.\right)$$

**Problems:**

1) Evaluate log 2 to 5 places of decimals.

Solution: Put p = 2, q = 1 in the series

SPACE FOR HINT

$$\log_e p - \log_e q = 2\left(\frac{p-q}{p+q} + \frac{1}{3}\left(\frac{p-q}{p+q}\right)^3 + \frac{1}{5}\left(\frac{p-q}{p+q}\right)^5 + ....\right)$$ , we get

$$\log_e 2 - \log_e 1 = 2\left(\frac{2-1}{2+1} + \frac{1}{3}\left(\frac{2-1}{2+1}\right)^3 + \frac{1}{5}\left(\frac{2-1}{2+1}\right)^5 + ....\right)$$

$$= 2\left(\frac{1}{3} + \frac{1}{3}.\frac{1}{3^3} + \frac{1}{5}.\frac{1}{3^5} + ....\right),$$

and $\log_e 1 = 0$.

$\frac{1}{3} = 0.333,333,3$

$\frac{1}{3^3} = 0.037,037$ $\qquad$ $\frac{1}{3}\frac{1}{3^3} = 0.012,345,7$

$\frac{1}{3^5} = 0.004,115,2$ $\qquad$ $\frac{1}{5}\frac{1}{3^5} = 0.000,832,0$

$\frac{1}{3^7} = 0.000,457,2$ $\qquad$ $\frac{1}{7}\frac{1}{3^7} = 0.000,055,3$

$\frac{1}{3^9} = 0.000,050,8$ $\qquad$ $\frac{1}{9}\frac{1}{3^9} = 0.000,005,6$

$\frac{1}{3^{11}} = 0.000,005,6$ $\qquad$ $\frac{1}{11}\frac{1}{3^{11}} = 0.000,000,5$

∴ Sum of the first 5 terms is 2(0.346,573,4) approximately.

(i.e.) 0.693,146,8

∴ log 2 = 0.69315 to 5 places of decimals.

We can calculate the error involved in taking only the first six terms.

The difference between log 2 and the sum of the first six terms

$$= 2.\left\{\frac{1}{13}.\frac{1}{3^{13}} + \frac{1}{15}.\frac{1}{3^{15}} + .....\right\}$$

$$< \frac{2}{13}\left\{\frac{1}{3^{13}} + \frac{1}{3^{15}} + .....\right\}$$

$$< \frac{2}{13}.\frac{1}{3^{13}}\left\{1 + \frac{1}{3^2} + \frac{1}{3^4} + .....\right\}$$

$$< \frac{2}{13}.\frac{1}{3^{13}}\frac{1}{1-\frac{1}{3^2}}$$

SPACE FOR HINT

$$= \frac{2}{13} \cdot \frac{1}{3^{13}} \cdot \frac{9}{8}$$

$$= \frac{1}{13} \cdot \frac{1}{3^{11}} \cdot \frac{1}{4}$$

$$= \frac{1}{52}(0.0000056)$$

$$= 0.0000011$$

Hence if we take log 2 = 0.69315, there is no error until the $6^{th}$ place of decimals.

Note: By means of this series by putting p = 3, q = 2, log 3 can be calculated.

By putting p = 5, q = 4, log 5 can be calculated.

**CYP Questions:**

1) Show that $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + .... = \log 2$.

2) Show that $\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + .... = 2 - \log 2$.

3) Show that $\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} + .... = \log 4 - 1$.

4) Show that $\frac{1}{1.2.3} - \frac{1}{2.3.4} + \frac{1}{3.4.5} - \frac{1}{4.5.6} + .... = 2\log 2 - \frac{5}{4}$.

5) Sum to infinity the series whose $n^{th}$ term is $\frac{x^n}{n+2}$.

6) Sum the series $\sum_1^{\infty} \frac{n^2+1}{n(n+2)} x^n$.

7) Sum the series $\sum_1^{\infty} \frac{(n+1)^3}{n(n+3)} x^n$.

8) Sum the series $\sum_1^{\infty} \frac{n^2}{(n+1)(n+2)} x^n$.

SPACE FOR HINT

# UNIT-6

## Unit Structure:

Section 6.1: Application of Exponential and Logarithmic series to Limits and approximations.

**Introduction:** We have already discuss about the exponential series and logarithmic series. In this unit we discuss the applications of exponential series and logarithmic series to find the limit and approximation values.

## SECTION - 6.1 - APPLICATION OF EXPONENTIAL AND LOGARITHMIC SERIES TO LIMITS AND APPROXIMATIONS.

The applications are shown in the following examples.

1) Evaluate $\lim_{x\to 0} \frac{e^x - e^{-x}}{\log(1+x)}$

Solution: $\lim_{x\to 0} \frac{e^x - e^{-x}}{\log(1+x)}$

$$= \lim_{x\to 0} \frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+....\right)-\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+....\right)}{x-\frac{x^2}{2}+\frac{x^3}{3}-....}$$

$$= \lim_{x\to 0} \frac{2x+\frac{2x^3}{3!}+\frac{2x^5}{5!}+....}{x-\frac{x^2}{2}+\frac{x^3}{3}-....}$$

$$= \lim_{x\to 0} \frac{2+\frac{2x^2}{3!}+\frac{2x^4}{5!}+....}{1-\frac{x}{2}+\frac{x^2}{3}-....}$$

$= 2.$

SPACE FOR HINT

2) Evaluate: $\lim_{n\to\infty}\left(1+\frac{3}{n^2}+\frac{5}{n^3}\right)^{n^2+7n}$

Solution: Let A = $\lim_{n\to\infty}\left(1+\frac{3}{n^2}+\frac{5}{n^3}\right)^{n^2+7n}$

Taking log on both sides, we have

$$\log A = \lim_{n\to\infty}\left(n^2+7n\right)\log\left(1+\frac{3}{n^2}+\frac{5}{n^3}\right)$$

$$= \lim_{n\to\infty}\left(n^2+7n\right)\left\{\left(\frac{3}{n^2}+\frac{5}{n^3}\right)-\frac{1}{2}\left(\frac{3}{n^2}+\frac{5}{n^3}\right)^2+\frac{1}{3}\left(\frac{3}{n^2}+\frac{5}{n^3}\right)^3-\ldots\right\}$$

$$= \lim_{n\to\infty}\left(n^2+7n\right)\left\{\left(\frac{3}{n^2}+\frac{5}{n^3}\right)-\frac{1}{2n^4}\left(3+\frac{5}{n}\right)^2+\frac{1}{3n^6}\left(3+\frac{5}{n}\right)^3-\ldots\right\}$$

$$= \lim_{n\to\infty}\left\{3+\frac{5}{n}+\frac{21}{n}+\frac{35}{n^2}-\frac{1}{2n^2}\left(3+\frac{5}{n}\right)^2-\frac{7}{2n}\left(3+\frac{5}{n}\right)^2+\ldots\right\}$$

Except the first, all the other terms will contain $\frac{1}{n}$ or higher powers of $\frac{1}{n}$.

$\therefore \log A = 3.$

$\therefore\ A = e^3.$

3) Prove that, if n is large

$$\left(n-\frac{1}{3n}\right)\log\frac{n+1}{n-1} = 2+\frac{8}{45n^4}+\ldots \text{ and}$$

$$\left(\frac{n+1}{n-1}\right)^{n-\frac{1}{3n}} = e^2\left(1+\frac{8}{45n^4}+\ldots\right)$$

Solution: Let A = $\left(\frac{n+1}{n-1}\right)^{n-\frac{1}{3n}}$.

Then $\log A = \left(n-\frac{1}{3n}\right)\log\frac{n+1}{n-1}$

$$= \left(n-\frac{1}{3n}\right)\log\frac{1+\frac{1}{n}}{1-\frac{1}{n}}$$

SPACE FOR HINT

$$= \left(n - \frac{1}{3n}\right) 2\left\{\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \frac{1}{7n^7} + \ldots\right\}$$

$$= 2\left\{1 + \frac{1}{3n^2} + \frac{1}{5n^4} + \frac{1}{7n^6} + \ldots - \frac{1}{3n^2} - \frac{1}{9n^4} - \ldots\right\}$$

$$= 2\left\{1 + \frac{4}{45n^4} + \ldots\right\}$$

$$= 2 + \frac{8}{45n^4} + \ldots$$

$$\therefore A = e^{2 + \frac{8}{45n^4} + \ldots}.$$

$$= = e^2 \ e^{\frac{8}{45n^4} + \ldots}.$$

$$= e^2\left(1 + \frac{8}{45n^4} + \ldots\right).$$

4) Show that if $e^x = 1 + xe^{yx}$, where $x^3$ and higher powers of x can be neglected, $y = \frac{1}{2!} + \frac{x}{4!}$.

Solution: Given that $e^x = 1 + xe^{yx}$.

We know that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$.

$$\therefore e^x - 1 = \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

$$= x\left\{1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots + \frac{x^{n-1}}{n!} + \ldots\right\}$$

By hypothesis $e^x = 1 + xe^{yx} \Rightarrow e^x - 1 = xe^{yx}$

$$\therefore xe^{yx} = x\left\{1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots + \frac{x^{n-1}}{n!} + \ldots\right\}$$

$$\therefore xe^{yx} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots + \frac{x^{n-1}}{n!} + \ldots$$

Taking log on both sides, we get

$$yx = \log\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots + \frac{x^{n-1}}{n!} + \ldots\right)$$

SPACE FOR HINT

$$= \left(\frac{x}{2!}+\frac{x^2}{3!}+....+\frac{x^{n-1}}{n!}+.....\right)-\frac{1}{2}\left(\frac{x}{2!}+\frac{x^2}{3!}+....+\frac{x^{n-1}}{n!}+.....\right)^2$$

$$-\frac{1}{3}\left(\frac{x}{2!}+\frac{x^2}{3!}+....+\frac{x^{n-1}}{n!}+.....\right)^3+.....$$

$$= \frac{x}{2}+\frac{x^2}{24}+ \text{terms in } x^3 \text{ and higher powers of x.}$$

$$\therefore y = \frac{1}{2!}+\frac{x}{4!}.$$

5) If $\log_e \frac{1}{1-x-x^2+x^3}$ be expanded in a series of ascending powers of x, show that the coefficient of $x^n$ will be $\frac{1}{n}$ or $\frac{3}{n}$ according as n is odd or even.

Solution: $1-x-x^2+x^3=(1-x)(1-x^2)$

$$\therefore \log_e \frac{1}{1-x-x^2+x^3} = -\log(1-x)-\log(1-x^2)$$

$$= x+\frac{x^2}{2}+\frac{x^3}{3}+.....+\frac{x^n}{n}+....$$

$$+x^2+\frac{x^4}{2}+\frac{x^6}{3}+.....+\frac{x^{2n}}{n}+....$$

If n is odd, the term containing $x^n$ will occur only in the expansion of $-\log(1-x)$.

$\therefore$ Coefficient of $x^n = \frac{1}{n}$ when n is odd.

If n is even, the term containing $x^n$ will occur in both the series.

If n = 2r, the coefficient of $x^{2r}$ in the first series is $\frac{1}{2r}$ and the coefficient of $x^{2r}$ in the second series is $\frac{2}{2r}$.

$\therefore$ Coefficient of $x^n = \frac{1}{2r}+\frac{2}{2r}=\frac{3}{2r}$.

$\therefore$ Coefficient of $x^n = \frac{3}{n}$ when n is even.

6) Sum the series $\log_3 e - \log_9 e + \log_{27} e - \log_{81} e + .....\infty$

SPACE FOR HINT

Solution: We know that $\log_x a.\log_a x = 1$.

$$\therefore \log_x a = \frac{1}{\log_a x}.$$

$$\therefore \log_3 e = \frac{1}{\log_e 3}.$$

$$\log_9 e = \frac{1}{\log_e 9} = \frac{1}{\log_e 3^2} = \frac{1}{2\log_e 3},$$

$$\log_{27} e = \frac{1}{\log_e 27} = \frac{1}{\log_e 3^3} = \frac{1}{3\log_e 3} \text{ and so on.}$$

$$\therefore \log_3 e - \log_9 e + \log_{27} e - \log_{81} e + \ldots\ldots\infty$$

$$= \frac{1}{\log_e 3} - \frac{1}{2\log_e 3} + \frac{1}{3\log_e 3} \ldots\infty$$

$$= \frac{1}{\log_e 3}(1 - \frac{1}{2} + \frac{1}{3} \ldots\ldots)$$

$$= \frac{\log_e 2}{\log_e 3}$$

7) Show by equating the coefficient of $x^n$ in the expansions of $\log_e(1 - x)$ and $\log_e(1 - 2x + x^2)$, that

$$2^n - n.2^{n-2} + \frac{n(n-3)}{1.2}2^{n-4} - \frac{n(n-4)(n-5)}{1.2.3}2^{n-6} + \ldots\ldots = 2$$

Solution: Since $2\log(1 - x) = -2\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots + \frac{x^n}{n} + \ldots\right)$,

the coefficient of $x^n$ in $2\log(1 - x) = -\frac{2}{n}$

$\log_e(1 - 2x + x^2) = \log_e(1 - (2x - x^2))$

$$= -(2x - x^2) - \frac{1}{2}(2x - x^2)^2 - \frac{1}{3}(2x - x^2)^3 \ldots\ldots - \frac{1}{n}(2x - x^2)^n \ldots\ldots$$

$$= -x(2-x) - \frac{1}{2}x^2(2-x)^2 - \frac{1}{3}x^3(2-x)^3 \ldots - \frac{1}{n}x^n(2-x)^n - \frac{1}{n+1}x^{n+1}(2-x)^{n+1} \ldots\ldots$$

The terms containing $x^n$ will not occur in terms after the term

$-\frac{1}{n}x^n(2-x)^n$.

$\therefore$ the coefficient of $x^n$ in $-\frac{1}{n}x^n(2-x)^n = -\frac{2^n}{n}$

SPACE FOR HINT

the coefficient of $x^n$ in $-\frac{1}{n-1}x^{n-1}(2-x)^{n-1} = -\frac{1}{n-1}.-2^{n-2}.(n-1)C_1$

$$= 2^{n-2}$$

the coefficient of $x^n$ in $-\frac{1}{n-2}x^{n-2}(2-x)^{n-2} = -\frac{1}{n-2}.2^{n-4}.(n-2)C_2$

$$= -\frac{1}{n-2}.2^{n-4}.\frac{(n-2)(n-3)}{2!}$$

$$= -\frac{(n-3)}{2!}.2^{n-4}$$

$$\therefore -\frac{2}{n} = -\frac{2^n}{n} + 2^{n-2} - \frac{(n-3)}{2!}.2^{n-4} + \frac{(n-4)(n-5)}{3!}.2^{n-6} \ldots\ldots$$

$$\therefore 2 = 2^n - n.2^{n-2} + \frac{n(n-3)}{1.2}2^{n-4} - \frac{n(n-4)(n-5)}{1.2.3}2^{n-6} + \ldots..$$

8) Show that if $a + b + c = 0$

(i) $\frac{a^7+b^7+c^7}{7} = \frac{a^5+b^5+c^5}{5}.\frac{a^2+b^2+c^2}{2}$

(ii) $\frac{a^5+b^5+c^5}{5} = \frac{a^3+b^3+c^3}{3}.\frac{a^2+b^2+c^2}{2}$

Solution: We know that $\log(1+ax) = ax - \frac{a^2x^2}{2} + \frac{a^3x^3}{3} - \ldots..$

$$\log(1+bx) = bx - \frac{b^2x^2}{2} + \frac{b^3x^3}{3} - \ldots..$$

$$\log(1+cx) = cx - \frac{c^2x^2}{2} + \frac{c^3x^3}{3} - \ldots..$$

$\therefore \log(1+ax) + \log(1+bx) + \log(1+cx)$

$$= x(a+b+c) - \frac{x^2}{2}(a^2+b^2+c^2) + \frac{x^3}{3}(a^3+b^3+c^3) - \ldots..$$

Coefficient of $x^n$ in the RHS $= \frac{(-1)^n}{n}(a^n+b^n+c^n)$.

$\log(1+ax) + \log(1+bx) + \log(1+cx)$

$= \log\{(1+ax)(1+bx)(1+cx)\}$

$= \log\{1 + (a+b+c)x + (ab+bc+ca)x^2 + abcx^3\}$

$= \log\{1 + (ab+bc+ca)x^2 + abcx^3\}$ $(\because a+b+c=0)$

$$= (ab+bc+ca)x^2 + abcx^3 - \frac{1}{2}\{(ab+bc+ca)x^2 + abcx^3\}^2$$

$$+\frac{1}{3}\{(ab+bc+ca)x^2+abcx^3\}^3 \;.....$$

$$= (ab + bc + ca)x^2 + abcx^3 - \frac{1}{2}x^4\{(ab+bc+ca)+abcx\}^2$$

$$+\frac{1}{3}x^6\{(ab+bc+ca)+abcx\}^3 \;.....$$

Equating the coefficient of $x^5$ on both sides; we get

$$\frac{a^5+b^5+c^5}{5} = -abc(ab + bc + ca) \text{ --------------- } (1)$$

Equating the coefficient of $x^2$ and $x^3$ on both sides, we get

$$-\frac{a^2+b^2+c^2}{2} = ab + bc + ca \text{ ---------------------- } (2)$$

$$\frac{a^3+b^3+c^3}{3} = abc \text{ ---------------------------------- } (3)$$

From (1), (2) and (3), we get

$$\frac{a^5+b^5+c^5}{5} = \frac{a^3+b^3+c^3}{3}.\frac{a^2+b^2+c^2}{2}$$

Again equating the coefficient of $x^7$ on both sides, we get

$$\frac{a^7+b^7+c^7}{7} = (ab + bc + ca)^2.abc$$

$$= -abc(ab + bc + ca). -(ab + bc + ca)$$

$$= \frac{a^5+b^5+c^5}{5}.\frac{a^2+b^2+c^2}{2}.$$

9) Obtain the expansion of $\log\left(1+\frac{1}{n}\right)$ in ascending powers of $\frac{1}{2n+1}$ and show that $\log\left(1+\frac{1}{n}\right)$ lies between $\frac{2}{2n+1}$ and $\frac{2n+1}{2n(n+1)}$ when n is positive.

Solution: $\log\left(1+\frac{1}{n}\right) = \log\frac{n+1}{n}$

$$= \log\frac{2n+2}{2n}$$

SPACE FOR HINT

$$= \log \frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}} = 2\left(\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\ldots\right)$$

$$\therefore \log\left(1+\frac{1}{n}\right) > \frac{2}{2n+1}.$$

$$\text{Again } \log\left(1+\frac{1}{n}\right) = \frac{2}{2n+1}\left(1+\frac{1}{3(2n+1)^2}+\frac{1}{5(2n+1)^4}+\ldots\right)$$

$$< \frac{2}{2n+1}\left(1+\frac{1}{2(2n+1)^2}+\frac{1}{4(2n+1)^4}+\ldots\right)$$

$$< \frac{2}{2n+1}\cdot\frac{1}{1-\frac{1}{2(2n+1)^2}}$$

$$< \frac{2}{2n+1}\cdot\frac{2(2n+1)^2}{\{2(2n+1)^2-1\}}$$

$$< \frac{4(2n+1)}{8n^2+8n+1}$$

$$< \frac{4(2n+1)}{8n^2+8n} = \frac{2n+1}{2n(n+1)}$$

$$\therefore \frac{2}{2n+1} < \log\left(1+\frac{1}{n}\right) < \frac{2n+1}{2n(n+1)}.$$

**CYP Questions:**

1) Evaluate $\lim_{x\to 0}\frac{e^x-\log(e+ex)}{x^2}$.

2) Evaluate $\lim_{x\to 0}\frac{e^x+\log_e(1+x)-(1+2x)}{5x^3}$.

3) Evaluate $\lim_{x\to 0}\frac{\log(1+x)}{(1+x)^m-(1+x)^{-m}}$.

4) Evaluate $\lim_{n\to\infty}\left(1+\frac{3}{n^2}+\frac{1}{n^3}\right)^{n^2}$.

SPACE FOR HINT

5) Show that $\lim_{n\to\infty}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n = e$.

6) Prove that $\lim_{x\to 0}\left(\frac{1+x}{1-x}\right)^{1/x} = e^2$.

7) If $\alpha$ and $\beta$ be the roots of the equation $x^2 + px + q = 0$, prove that

$$\log(1 - px + qx^2) = (\alpha+\beta)x - \frac{\alpha^2+\beta^2}{2}x^2 + \frac{\alpha^3+\beta^3}{3}x^3 - \ldots..$$

8) Expand $\log\frac{1+x+x^2}{1-x+x^2}$ in powers of x if $|x| < 1$.

9) Show that if $a + b + c = 0$

$$\frac{a^7+b^7+c^7}{7} = \frac{a^4+b^4+c^4}{4}\cdot\frac{a^3+b^3+c^3}{3}.$$

10) Show that $\log_e(1+n) < \frac{1}{1}+\frac{1}{2}+\ldots.+\frac{1}{n} < 1+\log_e(1+n)$.

SPACE FOR HINT

# UNIT-7

## Unit Structure:

Section 7.1 : Summation of Series using Binomial, Logarithmic and Exponential Series.

**Introduction:** In this unit we find the summation of finie and infinte series by using various methods namely by partia; fraction, Binomial, logarithmic and Exponential Series.

## Section 7.1 : Summation of Series using Binomial, Logarithmic and Exponential Series.

**Problems:**

1) Sum the series $\frac{3}{1^2 2^2}+\frac{5}{3^2 4^2}+.....+\frac{2n+1}{n^2(n+1)^2}$.

Solution: Let $u_n = \frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2}-\frac{1}{(n+1)^2}$

$$\therefore\ u_{n-1} = \frac{1}{(n-1)^2}-\frac{1}{n^2}$$

$$u_{n-2} = \frac{1}{(n-2)^2}-\frac{1}{(n-1)^2}$$

..............................

$$u_2 = \frac{1}{2^2}-\frac{1}{3^2}$$

$$u_1 = \frac{1}{1^2}-\frac{1}{2^2}$$

$$\therefore\ u_1+u_2+......+u_n = \frac{1}{1^2}-\frac{1}{(n+1)^2}$$

$$= \frac{n^2+2n}{(n+1)^2}.$$

2) Sum to n terms the series $\frac{3}{1.2}.\frac{1}{2}+\frac{4}{2.3}.\frac{1}{2^2}+\frac{5}{3.4}.\frac{1}{2^3}+.......$

Solution: Let $u_n = \frac{n+2}{n(n+1)}\left(\frac{1}{2}\right)^n$

SPACE FOR HINT

Let $\dfrac{n+2}{n(n+1)} = \dfrac{A}{n} + \dfrac{B}{n+1}$.

Then A = 2 and B = −1 .

$$\therefore\ u_n = \left(\frac{2}{n} - \frac{1}{n+1}\right)\left(\frac{1}{2}\right)^n$$

$$= 2\left(\frac{1}{n} - \frac{\frac{1}{2}}{n+1}\right)\left(\frac{1}{2}\right)^n$$

$$\therefore \frac{1}{2}u_n = \frac{1}{n}.\left(\frac{1}{2}\right)^n - \frac{1}{n+1}.\left(\frac{1}{2}\right)^{n+1}$$

Hence $\dfrac{1}{2}u_{n-1} = \dfrac{1}{n-1}.\left(\dfrac{1}{2}\right)^{n-1} - \dfrac{1}{n}.\left(\dfrac{1}{2}\right)^n$

$$\frac{1}{2}u_{n-2} = \frac{1}{n-2}.\left(\frac{1}{2}\right)^{n-2} - \frac{1}{n-1}.\left(\frac{1}{2}\right)^{n-1}$$

..............................

..............................

$$\frac{1}{2}u_2 = \frac{1}{2}.\left(\frac{1}{2}\right)^2 - \frac{1}{3}.\left(\frac{1}{2}\right)^3$$

$$\frac{1}{2}u_1 = \frac{1}{2}.\left(\frac{1}{2}\right)^1 - \frac{1}{2}.\left(\frac{1}{2}\right)^2.$$

$$\therefore\ \frac{1}{2}(u_1 + u_2 + \ldots\ldots + u_n) = \frac{1}{2}.\left(\frac{1}{2}\right)^1 - \frac{1}{n+1}.\left(\frac{1}{2}\right)^{n+1}$$

$$= \left(\frac{1}{4} - \frac{1}{(n+1)2^{n+1}}\right)$$

$$\therefore\ u_1 + u_2 + \ldots\ldots + u_n = \frac{1}{2} - \frac{1}{(n+1)2^n}.$$

3) Sum to n terms the series $\dfrac{8}{1.2.3}\left(\dfrac{5}{7}\right) + \dfrac{9}{2.3.4}\left(\dfrac{5}{7}\right)^2 + \dfrac{10}{3.4.5}\left(\dfrac{5}{7}\right)^3 + \ldots..$

Solution: Let $u_n = \dfrac{n+7}{n(n+1)(n+2)}\left(\dfrac{5}{7}\right)^n$.

Let $\dfrac{n+7}{n(n+1)(n+2)} = \dfrac{A}{n(n+1)} + \dfrac{B}{(n+1)(n+2)}$

SPACE FOR HINT

Then $A = \frac{7}{2}$ and $B = -\frac{5}{2}$.

$$\therefore\ u_n = \frac{1}{2}\left(\frac{7}{n(n+1)} - \frac{5}{(n+1)(n+2)}\right)\left(\frac{5}{7}\right)^n$$

$$= \frac{1}{2}.\frac{1}{7}\left(\frac{1}{n(n+1)} - \frac{\frac{5}{7}}{(n+1)(n+2)}\right)\left(\frac{5}{7}\right)^n$$

$$= \frac{1}{14}\left(\frac{\left(\frac{5}{7}\right)^n}{n(n+1)} - \frac{\left(\frac{5}{7}\right)^{n+1}}{(n+1)(n+2)}\right)$$

$$14u_n = \frac{\left(\frac{5}{7}\right)^n}{n(n+1)} - \frac{\left(\frac{5}{7}\right)^{n+1}}{(n+1)(n+2)}$$

Hence $14u_{n-1} = \dfrac{\left(\frac{5}{7}\right)^{n-1}}{(n-1)n} - \dfrac{\left(\frac{5}{7}\right)^n}{n(n+1)}$

$$14u_{n-2} = \frac{\left(\frac{5}{7}\right)^{n-2}}{(n-2)(n-1)} - \frac{\left(\frac{5}{7}\right)^{n-1}}{(n-1)n}$$

..............................

..............................

$$14u_2 = \frac{\left(\frac{5}{7}\right)^2}{2.3} - \frac{\left(\frac{5}{7}\right)^3}{3.4}$$

$$14u_1 = \frac{\left(\frac{5}{7}\right)}{1.2} - \frac{\left(\frac{5}{7}\right)^2}{2.3}.$$

$$\therefore\ 14(u_1 + u_2 + \ldots\ldots + u_n) = \frac{\left(\frac{5}{7}\right)}{1.2} - \frac{\left(\frac{5}{7}\right)^{n+1}}{(n+1)(n+2)}.$$

$$\therefore\ S_n = u_1 + u_2 + \ldots\ldots + u_n = \frac{1}{28}.\left(\frac{5}{7}\right) - \frac{1}{14(n+1)(n+2)}.\left(\frac{5}{7}\right)^{n+1}$$

SPACE FOR HINT

To find the sum of this series up to infinity, we find the limit as n tends to infnity.

$$\text{(i.e.)}\ \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{28}\cdot\left(\frac{5}{7}\right) - \frac{1}{14(n+1)(n+2)}\cdot\left(\frac{5}{7}\right)^{n+1}$$

$$S_\infty = \frac{5}{196} - 0 = \frac{5}{196}.$$

**Application of the Binomial Theorem to the summation of series:**

We know that when $|x| < 1$, for all values of n

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots.$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots.$$

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots.$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots.$$

If $n = \frac{p}{q}$, then $(1+x)^{p/q} = 1 + \frac{p}{q}x + \frac{p(p-q)}{q.2q}x^2 + \frac{p(p-q)(p-2q)}{q.2q.3q}x^3 + \ldots$

$$= \sum \frac{p(p-q)(p-2q).....(p-\overline{r-1}q)}{q.2q.3q....rq}x^r$$

$$(1-x)^{p/q} = 1 - \frac{p}{q}x + \frac{p(p-q)}{q.2q}x^2 - \frac{p(p-q)(p-2q)}{q.2q.3q}x^3 + \ldots$$

$$= \sum(-1)^r \frac{p(p-q)(p-2q).....(p-\overline{r-1}q)}{q.2q.3q....rq}x^r$$

$$(1+x)^{-p/q} = \sum(-1)^r \frac{\frac{p}{q}(\frac{p}{q}+1)(\frac{p}{q}+2).....(\frac{p}{q}+\overline{r-1})}{r!}x^r$$

$$= \sum(-1)^r \frac{p(p+q)(p+2q).....(p+\overline{r-1}q)}{q.2q.3q....rq}x^r$$

$$(1-x)^{-p/q} = \sum(-1)^r \frac{\left(-\frac{p}{q}\right)(-\frac{p}{q}-1)(-\frac{p}{q}-2).....(-\frac{p}{q}-\overline{r-1})}{r!}x^r$$

$$= \sum \frac{p(p+q)(p+2q).....(p+\overline{r-1}q)}{q.2q.3q....rq}x^r$$

SPACE FOR HINT

**Problems:**

1) Sum the series to infinity

$$1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+\ldots..$$

Solution: The factors in the numerators form an Arithmetic Progression with common difference 2.

Therefore we divide each of these by 2.

Each of the factors in the denominators has 4 for a factor, removing 4 from each will leave a factorial.

$$\therefore 1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+\ldots$$

$$=1+\frac{\frac{3}{2}}{1}.\frac{2}{4}+\frac{\frac{3}{2}.\frac{5}{2}}{1.2}.\left(\frac{2}{4}\right)^2+\frac{\frac{3}{2}.\frac{5}{2}.\frac{7}{2}}{1.2.3}.\left(\frac{2}{4}\right)^3+\ldots.$$

$$=1+\frac{\frac{3}{2}}{1!}.\frac{1}{2}+\frac{\frac{3}{2}.\frac{5}{2}}{2!}.\left(\frac{1}{2}\right)^2+\frac{\frac{3}{2}.\frac{5}{2}.\frac{7}{2}}{3!}.\left(\frac{1}{2}\right)^3+\ldots.$$

Put $n=\frac{3}{2}$ and $x=\frac{1}{2}$. Then

$$1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+\ldots=1+\frac{\frac{3}{2}}{1!}.\frac{1}{2}+\frac{\frac{3}{2}.\frac{5}{2}}{2!}.\left(\frac{1}{2}\right)^2+\frac{\frac{3}{2}.\frac{5}{2}.\frac{7}{2}}{3!}.\left(\frac{1}{2}\right)^3+\ldots.$$

$$=1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+\ldots.$$

$$=(1-x)^{-n}$$

$$=(1-\frac{1}{2})^{-3/2}$$

$$=2\sqrt{2}$$

2) Sum the series to infinity

$$\frac{1.4}{5.10}-\frac{1.4.7}{5.10.15}+\frac{1.4.7.10}{5.10.15.20}-\ldots..$$

Solution: Let $S=\frac{1.4}{5.10}-\frac{1.4.7}{5.10.15}+\frac{1.4.7.10}{5.10.15.20}-\ldots..$

The factors in the numerators form an Arithmetic Progression with common difference 3.

SPACE FOR HINT

Therefore we divide each of these by 3.

Each of the factors in the denominators has 5 for a factor, removing 5 from each will leave a factorial.

$$\therefore S = \frac{\frac{1}{3}.\frac{4}{3}}{1.2}\left(-\frac{3}{5}\right)^2 + \frac{\frac{1}{3}.\frac{4}{3}.\frac{7}{3}}{1.2.3}\left(-\frac{3}{5}\right)^3 + \frac{\frac{1}{3}.\frac{4}{3}.\frac{7}{3}.\frac{10}{3}}{1.2.3.4}\left(-\frac{3}{5}\right)^4 + ....$$

Put $n = \frac{1}{3}$ and $x = -\frac{3}{5}$. Then

$$S = \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + ....$$

$$= 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + .... -(1+nx)$$

$$= (1-x)^{-n} - (1+nx)$$

$$= \left(1+\frac{3}{5}\right)^{-1/3} - 1 + \frac{1}{3}.\frac{3}{5} = \left(\frac{8}{5}\right)^{-1/3} - \frac{4}{5}$$

$$= \left(\frac{5}{8}\right)^{1/3} - \frac{4}{5} = \frac{1}{2}(5)^{1/3} - \frac{4}{5}.$$

3) Sum the series to infinity

$$\frac{15}{16} + \frac{15.21}{16.24} + \frac{15.21.27}{16.24.32} + .....$$

Solution: Let $S = \frac{15}{16} + \frac{15.21}{16.24} + \frac{15.21.27}{16.24.32} + .....$

The factors in the numerators form an Arithmetic Progression with common difference 6.

The factors in the denominators form an Arithmetic Progression with common difference 8.

$$\therefore S = \frac{\frac{15}{6}}{2}\left(\frac{6}{8}\right) + \frac{\frac{15}{6}.\frac{21}{6}}{2.3}\left(\frac{6}{8}\right)^2 + \frac{\frac{15}{6}.\frac{21}{6}.\frac{27}{6}}{2.3.4}\left(\frac{6}{8}\right)^3 + ....$$

The factors of the denominators do not begin with 1. Hence we introduce 1 to the denominator of each coefficient. The number of factors in the numerator is to be the same as that of the factors in the denominator. So we have to introduce an additional factor in the numerator also, which is $\frac{9}{6}$.

SPACE FOR HINT

$$\therefore \frac{9}{6}S = \frac{\frac{9}{6}\frac{15}{6}}{1.2}\left(\frac{6}{8}\right) + \frac{\frac{9}{6}.\frac{15}{6}.\frac{21}{6}}{1.2.3}\left(\frac{6}{8}\right)^2 + \frac{\frac{9}{6}.\frac{15}{6}.\frac{21}{6}.\frac{27}{6}}{1.2.3.4}\left(\frac{6}{8}\right)^3 + \ldots.$$

Since the index of x in every term must be the same as the number of factors in the numerator or denominator of the coefficient, we have

$$\frac{9}{6}\frac{6}{8}S = \frac{\frac{9}{6}\frac{15}{6}}{1.2}\left(\frac{6}{8}\right)^2 + \frac{\frac{9}{6}.\frac{15}{6}.\frac{21}{6}}{1.2.3}\left(\frac{6}{8}\right)^3 + \frac{\frac{9}{6}.\frac{15}{6}.\frac{21}{6}.\frac{27}{6}}{1.2.3.4}\left(\frac{6}{8}\right)^4 + \ldots.$$

Put $n = \frac{9}{6}$ and $x = \frac{6}{8}$. Then

$$\frac{9}{8}S = \frac{\frac{9}{6}\frac{15}{6}}{1.2}\left(\frac{6}{8}\right)^2 + \frac{\frac{9}{6}.\frac{15}{6}.\frac{21}{6}}{1.2.3}\left(\frac{6}{8}\right)^3 + \frac{\frac{9}{6}.\frac{15}{6}.\frac{21}{6}.\frac{27}{6}}{1.2.3.4}\left(\frac{6}{8}\right)^4 + \ldots.$$

$$\frac{9}{8}S = \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots.$$

$$= 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots. - (1+nx)$$

$$= (1-x)^{-n} - (1+nx)$$

$$= \left(1-\frac{6}{8}\right)^{-9/6} - 1 - \frac{9}{6}.\frac{6}{8} = \left(\frac{2}{8}\right)^{-3/2} - \frac{17}{8}$$

$$= \left(\frac{1}{4}\right)^{-3/2} - \frac{17}{8} = (4)^{3/2} - \frac{17}{8} = 8 - \frac{17}{8}$$

$$= \frac{47}{8}$$

$$\therefore \frac{9}{8}S = \frac{47}{8}$$

$$\therefore S = \frac{47}{9}.$$

**CYP Questions:**

1) Find the sum to infinity of the series $\frac{1}{24} - \frac{1.3}{24.32} + \frac{1.3.5}{24.32.40} - \ldots.$

2) Prove that $(1+x)^n = 2^n\left\{1 - n\frac{1-x}{1+x} + \frac{n(n+1)}{1.2}\left(\frac{1-x}{1+x}\right)^2 \ldots..\right\}$

3) Find the sum to infinity of the series $\frac{2}{6} + \frac{2.5}{6.12} + \frac{2.5.8}{6.12.18} + \ldots.$

SPACE FOR HINT

4) Find the sum to infinity of the series

$$\frac{5}{3.6}.\frac{1}{4^2}+\frac{5.8}{3.6.9}.\frac{1}{4^3}+\frac{5.8.11}{3.6.9.12}.\frac{1}{4^4}+....$$

5) Find $2+\sum_{n=1}^{\infty}\frac{1}{3}.\frac{1}{6^{n-1}}.\frac{1.3.5.....(2n-1)}{n!}$

**Application of the Exponential Theorem to the summation of series:**

We have proved that for all real values of x,

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+....+\frac{x^n}{n!}+...... \text{----------------- (1)}$$

Replace x by –x in the relation, we get

$$e^{-x} = 1-\frac{x}{1!}+\frac{x^2}{2!}-....+(-1)^n\frac{x^n}{n!}+......\text{-------- (2)}$$

When $x = 1$, $e^1 = 1+\frac{1}{1!}+\frac{1}{2!}+....+\frac{1}{n!}+.....$ ------------- (3)

When $x = -1$, $e^{-1} = 1-\frac{1}{1!}+\frac{1}{2!}-....+(-1)^n\frac{1}{n!}+.....$ ------------- (4)

Adding (1) and (2), we get

$$\frac{e^x+e^{-x}}{2} = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+..... \text{------------------------------ (5)}$$

Subtracting (2) from (1), we get

$$\frac{e^x-e^{-x}}{2} = \frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+..... \text{------------------------------ (6)}$$

When x = 1, the series (5) and (6) become

$$\frac{e+e^{-1}}{2} = 1+\frac{1}{2!}+\frac{1}{4!}+..... \text{---------------------------------- (7)}$$

$$\frac{e^1-e^{-1}}{2} = \frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+..... \text{---------------------------------- (8)}$$

We shall use this series to find the sums of certain series.

Problems:

1) Sum the series

$$1+\frac{1+3}{2!}+\frac{1+3+3^2}{3!}+\frac{1+3+3^2+3^3}{4!}+....... \text{ to } \infty.$$

Solution: Let $u_n$ be the $n^{th}$ term of the given series and S be the sum to infinity of the series.

SPACE FOR HINT

$$\therefore u_n = \frac{1+3+3^2+3^3+....+3^{n-1}}{n!}$$

$$= \frac{3^n-1}{3-1}\cdot\frac{1}{n!}$$

$$= \frac{1}{2}\left(\frac{3^n}{n!}-\frac{1}{n!}\right)$$

$$\therefore u_1 = \frac{1}{2}\left(\frac{3^1}{1!}-\frac{1}{1!}\right)$$

$$u_2 = \frac{1}{2}\left(\frac{3^2}{2!}-\frac{1}{2!}\right)$$

$$u_3 = \frac{1}{2}\left(\frac{3^3}{3!}-\frac{1}{3!}\right)$$

..... ...... .....

..... ...... .....

$$u_n = \frac{1}{2}\left(\frac{3^n}{n!}-\frac{1}{n!}\right)$$

..... ...... .....

..... ...... .....

$$S = \frac{1}{2}\left(\frac{3^1}{1!}+\frac{3^2}{2!}+...\frac{3^n}{n!}+....\right)-\frac{1}{2}\left(\frac{1}{1!}+\frac{1}{2!}+...\frac{1}{n!}+....\right)$$

$$= \frac{1}{2}(e^3-1)-\frac{1}{2}(e-1)$$

$$= \frac{1}{2}e(e^2-1).$$

2) Show that $\left(1+\frac{1}{2!}+\frac{1}{4!}+.....\right)^2 = 1+\left(\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+.....\right)^2$

Solution: Since $\frac{e+e^{-1}}{2} = 1+\frac{1}{2!}+\frac{1}{4!}+.....$ and

$\frac{e^1-e^{-1}}{2} = \frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+.....$, we have to show that

$$\left(\frac{e+e^{-1}}{2}\right)^2 = 1+\left(\frac{e-e^{-1}}{2}\right)^2$$

SPACE FOR HINT

$$LHS = 1 + \left(\frac{e-e^{-1}}{2}\right)^2 = 1 + \frac{1}{4}(e-e^{-1})^2 = 1 + \frac{1}{4}(e^2+e^{-2}-2ee^{-1})$$

$$= 1 + \frac{1}{4}(e^2+e^{-2}-4ee^{-1}+2ee^{-1})$$

$$= 1 + \frac{1}{4}(e^2+e^{-2}+2ee^{-1}) - \frac{1}{4}4ee^{-1}$$

$$= 1 + \frac{1}{4}(e+e^{-1})^2 - 1$$

$$= \left(\frac{e+e^{-1}}{2}\right)^2 = \text{RHS.}$$

3) Sum the series $\sum_{n=0}^{\infty} \frac{(n+1)^3}{n!} x^n$

Solution: Let $S = \sum_{n=0}^{\infty} \frac{(n+1)^3}{n!} x^n$

Let $(n + 1)^3 = A + Bn + Cn(n-1) + Dn(n-1)(n-2)$

Put $n = 0$, we get $1 = A$

Put $n = 1$, we get $8 = A + B + 0 + 0$

$8 = 1 + B$

$B = 7$

Put $n = 2$, we get

$27 = A + 2B + 2C$

$27 = 1 + 14 + 2C$

$2C = 12 \Rightarrow C = 6$

Compare the coefficient of n3 , we get

$1 = D$

$\therefore (n + 1)^3 = 1 + 7n + 6n(n-1) + n(n-1)(n-2)$

$$\therefore S = \sum_{n=0}^{\infty} \frac{1+7n+6n(n-1)+n(n-1)(n-2)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} + 7\sum_{n=0}^{\infty} \frac{x^n}{(n-1)!} + 6\sum_{n=0}^{\infty} \frac{x^n}{(n-2)!} + \sum_{n=0}^{\infty} \frac{x^n}{(n-3)!}$$

We know that $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots. + \frac{x^n}{n!} + \ldots\ldots = e^x$

SPACE FOR HINT

$$\sum_{n=0}^{\infty}\frac{x^n}{(n-1)!}=x+\frac{x^2}{1!}+\frac{x^3}{2!}+....+\frac{x^{n+1}}{n!}+.....$$

$$=x.\left(1+\frac{x}{1!}+\frac{x^2}{2!}+....+\frac{x^n}{n!}+.....\right)=x.e^x$$

Similarely $\sum_{n=0}^{\infty}\frac{x^n}{(n-2)!}=x^2+\frac{x^3}{1!}+\frac{x^4}{2!}+....+\frac{x^{n+2}}{n!}+.....=x^2.e^x$

$$\sum_{n=0}^{\infty}\frac{x^n}{(n-3)!}=x^3+\frac{x^4}{1!}+\frac{x^5}{2!}+....+\frac{x^{n+3}}{n!}+.....=x^3.e^x$$

$\therefore$ S = $(1+7x+6x^2+x^3)e^x$

4) Sum the series $\frac{1^2}{1!}+\frac{1^2+2^2}{2!}+\frac{1^2+2^2+3^2}{3!}+\ldots+\frac{1^2+2^2+\ldots+n^2}{n!}+\ldots$

Solution: Let $u_n$ be the $n^{th}$ term of the given series and S be the sum to infinity of the series.

Then $u_n=\frac{1^2+2^2+\ldots+n^2}{n!}=\frac{n(n+1)(2n+1)}{6}.\frac{1}{n!}$

Let n(n + 1 )(2n+1) = A + Bn +Cn(n – 1) + Dn(n – 1)(n – 2)

Then A = 0, B = 6, C = 9, D = –2.

$$\therefore S=\sum_{n=0}^{\infty}\frac{6n+9n(n-1)+2(n-1)(n-2)}{6}\frac{1}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{6n}{6}\frac{1}{n!}+\sum_{n=0}^{\infty}\frac{9n(n-1)}{6}\frac{1}{n!}+\sum_{n=0}^{\infty}\frac{2(n-1)(n-2)}{6}\frac{1}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{1}{(n-1)!}+\frac{3}{2}\sum_{n=0}^{\infty}\frac{1}{(n-2)!}+\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{(n-3)!}$$

$$=e+\frac{3}{2}e+\frac{1}{3}e=\frac{17e}{6}.$$

5) Sum the series $\frac{5}{1!}+\frac{7}{3!}+\frac{9}{5!}+\ldots$

Solution: Let $u_n$ be the $n^{th}$ term of the given series and S be the sum to infinity of the series.

Then $u_n=\frac{2n+3}{(2n-1)!}$

Put 2n + 3 = A(2n – 1) + B

Then A = 1 and B = 4

$$\therefore u_n = \frac{2n-1+4}{(2n-1)!}$$

$$= \frac{2n-1}{(2n-1)!} + \frac{4}{(2n-1)!}$$

$$= \frac{1}{(2n-2)!} + \frac{4}{(2n-1)!}$$

$$\therefore u_1 = 1 + \frac{4}{1!}$$

$$u_2 = \frac{1}{2!} + \frac{4}{3!}$$

$$u_3 = \frac{1}{4!} + \frac{4}{5!}$$

..... ...... .....

..... ...... .....

$$u_n = \frac{1}{(2n-2)!} + \frac{4}{(2n-1)!}$$

..... ...... .....

..... ...... .....

$$S = \left(1 + \frac{1}{2!} + \frac{1}{4!} + .....\right) + 4\left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + .....\right)$$

$$= \frac{e+e^{-1}}{2} + 4\frac{e^1 - e^{-1}}{2} = \frac{1}{2}\left(e + \frac{1}{e}\right) + 4\frac{1}{2}\left(e - \frac{1}{e}\right)$$

$$= \frac{5}{2}e - \frac{3}{2e}$$

6) Show that if $a^r$ be the coefficient of $x^r$ in the expansion $e^{e^x}$, then prove that $a^r = \frac{1}{r!}\left\{\frac{1^r}{1!} + \frac{2^r}{2!} + \frac{3^r}{3!} + .....\right\}$. Hence show that

(i) $\frac{1^3}{1!} + \frac{2^3}{2!} + \frac{3^3}{3!} + ..... = 5e$

(ii) $\frac{1^4}{1!} + \frac{2^4}{2!} + \frac{3^4}{3!} + ..... = 15e.$

Solution: $e^{e^x} = 1 + e^x + \frac{(e^x)^2}{2!} + \frac{(e^x)^3}{3!} + .... + \frac{(e^x)^n}{n!} + .....$

SPACE FOR HINT

$$= 1 + e^x + \frac{e^{2x}}{2!} + \frac{e^{3x}}{3!} + .... + \frac{e^{nx}}{n!} + .....$$

$$= 1 + \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + .... + \frac{x^r}{r!} + .....\right)$$

$$+ \frac{1}{2!}\left(1 + 2x + \frac{2^2x^2}{2!} + .... + \frac{2^r x^r}{r!} + .....\right)$$

$$+ \frac{1}{3!}\left(1 + 3x + \frac{3^2x^2}{2!} + .... + \frac{3^r x^r}{r!} + .....\right) + .....$$

$\therefore$ The coefficient of $x^r = \frac{1}{r!}\left\{\frac{1^r}{1!} + \frac{2^r}{2!} + \frac{3^r}{3!} + .....\right\}$ ----------------- (1)

Again $e^{e^x} = e^{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}.....} = e.\ e^{x+\frac{x^2}{2!}+\frac{x^3}{3!}.....}$

$$= e.\left\{1 + \left(\frac{x}{1!} + \frac{x^2}{2!} + ....\right) + \frac{1}{2!}\left(\frac{x}{1!} + \frac{x^2}{2!} + ....\right)^2 + \frac{1}{3!}\left(\frac{x}{1!} + \frac{x^2}{2!} + .....\right)^3 + ...\right\}$$

$\therefore$ The coefficient of $x^3 = e\left(\frac{1}{3!} + \frac{1}{2!}.2.\frac{1}{2!} + \frac{1}{3!}\right)$

$$= \frac{e}{3!}(1 + 3 + 1) = \frac{5e}{3!}.$$

From (1), the coefficient of $x^3 = \frac{1}{3!}\left\{\frac{1^3}{1!} + \frac{2^3}{2!} + \frac{3^3}{3!} + .....\right\}$

$$\therefore \frac{1}{3!}\left\{\frac{1^3}{1!} + \frac{2^3}{2!} + \frac{3^3}{3!} + .....\right\} = \frac{5e}{3!}.$$

$$\therefore \frac{1^3}{1!} + \frac{2^3}{2!} + \frac{3^3}{3!} + ..... = 5e$$

Similarly, by equating the coefficient of $x^4$, we get

$$\frac{1^4}{1!} + \frac{2^4}{2!} + \frac{3^4}{3!} + ..... = 15e.$$

SPACE FOR HINT

## CYP Questions:

1) Show that $(\log 2)-\frac{1}{2!}(\log 2)^2+\frac{1}{3!}(\log 2)^3 \ldots\ldots \text{to } \infty=\frac{1}{2}$

2) Sum the series $\sum_{n=1}^{\infty}\frac{n^2+3}{n+2}\cdot\frac{x^n}{n!}$.

3) Prove that the infinite series $\frac{2\frac{1}{2}}{1!}-\frac{3\frac{1}{3}}{2!}+\frac{4\frac{1}{4}}{3!}-\frac{5\frac{1}{5}}{4!}+\ldots=\frac{1+e}{e}$.

4) Prove that $\sum_{1}^{\infty}\frac{n^3}{(n-1)!}=15e$.

5) Show that $\sum_{0}^{\infty}\frac{5n+1}{(2n+1)!}=\frac{e}{2}+\frac{2}{e}$.

6) Show that $\frac{2^2}{1!}+\frac{2^4}{3!}+\frac{2^6}{5!}+\ldots\ldots=\frac{e^4-1}{e^2}$.

7) Prove that if n is a positive integer

$$1-\frac{n}{1^2}x+\frac{n(n-1)}{1^2.2^2}x^2-\frac{n(n-1)(n-2)}{1^2.2^2.3^2}x^3+\ldots.=$$
$$e^x\left\{1-\frac{n+1}{1^2}x+\frac{(n+1)(n+2)}{1^2.2^2}x^2-\frac{(n+1)(n+2)(n+3)}{1^2.2^2.3^2}x^3+\ldots.\right\}$$

8) Show that $n^n-n(n-1)^n+\frac{n(n-1)}{1.2}(n-2)^n\ldots\ldots=n!$.

SPACE FOR HINT

# UNIT-8

**Unit Structure:**

Section 8.1: Partially ordered set.

Section 8.2:Definition of Lattice –Examples

**Introduction:** In this unit we develop the notations of partially ordered sets and lattices and distinguish various types of lattices.

## SECTION - 8.1- PARTIALLY ORDERED SETS

Definition. A relation defined on a set S which is reflexive, anti symmetric and transitive is called a partial ordering on S. A set S with a partial ordering $\rho$ defined on it is called a partially ordered set or a poset and is denoted by (S, $\rho$).

Note. Throughout this unit we shall use the symbol $\leq$ to denote a partial ordering.

Examples

1. N, Z, R are posets with the usual relation $\leq$.
2. In N we define a relation $\leq$ as follows $A \leq B \Leftrightarrow$ a divides b. Then (N,$\leq$ ) is a poset.
3. In $\mathscr{P}(s)$ we define $A \leq B \Leftrightarrow A \subseteq B$. Then ($\mathscr{P}(s)$,$\leq$) is a poset.
4. Let P denote the set of all subgroups of a group G. In P we define $H \leq K \leftrightarrow H \subseteq K$. Then (P, $\leq$) is a poset.

   Similarly the set of all sub rings of a ring, the set of all subspaces of vector space etc, are posets with respect to the above relation.

Definition: Let (S, $\leq$) be a poset. Let a,b$\in$S. a and b are said to be comparable if either $a \leq b$ or $b \leq a$.

Remark: In a poset, there may be pairs of elements which are not comparable. In ($\mathscr{P}$(N),$\leq$), {1,2} and {1,3} are not comparable. However in (R, $\leq$), any two elements are comparable.

Definition: A partial ordering in which any two elements are comparable is called a linear ordering or a total ordering. A set S with a linear

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ordering defined in it is called a linearly ordered set or a totally ordered set or a chain.

Definition: Let (P, $\leq$) be a poset. Let a,b$\in$P. If a $\leq$ b and a $\neq$ b, we say that a < b . Also we say that b covers a if a < b and there is no element c$\in$P such that a < c < b.

For example, in $\mathcal{P}$(N) any singleton set covers $\phi$. In N with usual $\leq$, 3 covers 2.

## Representation of finite posets by diagrams

A finite poset P can be conveniently represented by a diagram as follows. The elements of P are represented by small circles. If a,b$\in$P and b covers a, then the circle for b is placed above the circle for a and the two circles are joined by a line segment . The resulting figure is a diagram for the poset.

EXAMPLES

1. Consider the poset {1,2,3,4} with the usual $\leq$ . Here 1$\leq$ 2$\leq$3$\leq$4 and 2 covers 1, 3 covers 2 and 4 covers 3. Hence we obtain the diagram of figure 1 for the poset.

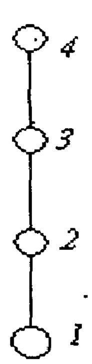


Fig 1

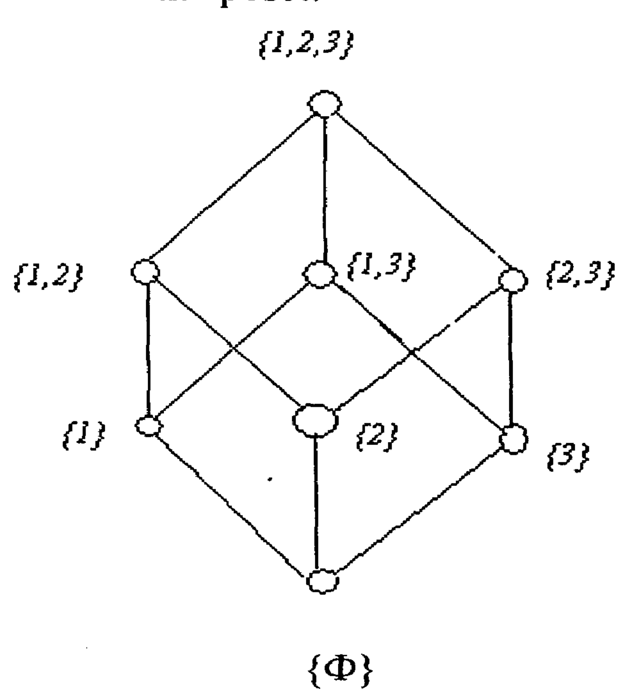


Fig 2

2. Consider the poset $\mathcal{P}$({1,2,3}) with the relation $\subseteq$. {1},{2},and {3} are covers for $\phi$. {1,2} is a cover for {1} and

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{2} and so on. {1,2,3} is a cover for {1,2},{1,3} and {2,3}. Hence we obtain the diagram of figure 2 for the poset.

3. Consider the set of all subgroups of the group $V_4$ ={e,a,b,c} given by {e}, {e,a}, {e,b}, {e,c} and $V_4$. We know that this is poset (refer example 4). The diagram of this poset is given in fig.3. The poset represented by this diagram is denoted by the symbol $M_5$.

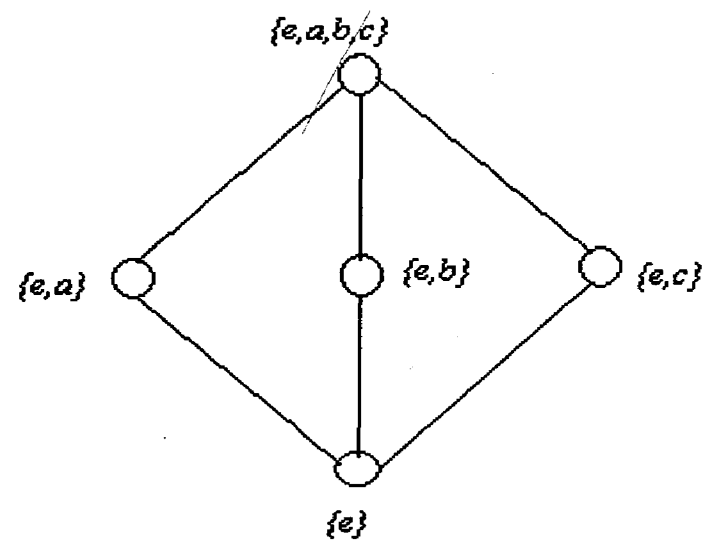


Fig 3

4. The poset consisting of all non-empty subsets of {1,2} is given by the diagram of Figure 4.

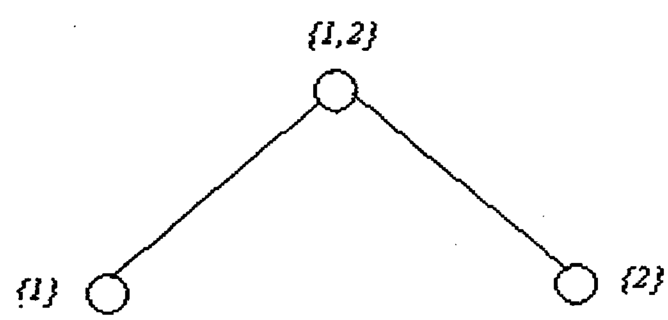


Fig 4

<u>Definition:</u> Let (P, ≤) be a poset. An element a∈P is called the least element or Zero element of P if a ≤ x for all x∈P.

An element a is called the greatest element or unit element of P if x ≤ a for all x ∈P.

<u>Note</u>. 1. A poset need not have a greatest element or a least element. For example the poset given in figure 4, does not have a least element.

The poset (N, ≤) does not have a greatest element.

2. The least element and the greatest element of a poset, if they exist, are unique.

   For, suppose a and b are two least elements of a poset P. Since a is a least element, $a \leq b$. Similarly $b \leq a$. Hence by the definition of poset, $a = b$. The proof is similar for the greatest element.

3. The least element of a poset is denoted by 0 and the greatest element is denoted by 1.

<u>Definition:</u> Let $(P, \leq)$ be a poset. Let A be a non-empty subset of P. An element $u \in P$ is called an upper bound of A if $a \leq u$ for all $a \in A$

An element $u \in P$ is called the least upper bound (l.u.b) of A if

(i) u is an upper bound of A.

(ii) if v is any other upper bound of A, then $u \leq v$.

An element $l \in P$ is called a lower bound of A if $l \leq a$ for all $a \in A$.

An element $l \in P$ is called the greatest lower bound (g.l.b) of A if

(i) $l$ is lower bound of A

(ii) if m is any other lower bound of A then $m \leq l$.

<u>Note 1</u>) The l.u.b and g.l.b of a set A, if they exist, are unique. For, let $u_1, u_2$, be two least upper bounds of A. Then $u_1$ is a l.u.b and $u_2$ is an upper bound for A. Hence $u_1 \leq u_2$. Similarly $u_2 \leq u_1$. Hence $u_1 = u_2$. The proof is similar for g.l.b.

<u>Note 2</u>) In a poset the l.u.b. and g.l.b. of a subset need not exist.

For example- consider the poset consisting of the set {{a}, {b}, {a,b}, {a,b,c},{a,b,d}} under set inclusion. The diagram for this poset is given in fig.5. In this poset the elements {a} and {b} do not have g.l.b. and their l.u.b is {a,b}. Also the element {a,b,c} and {a,b,d} do not have l.u.b. and their g.l.b. is given by {a,b}.

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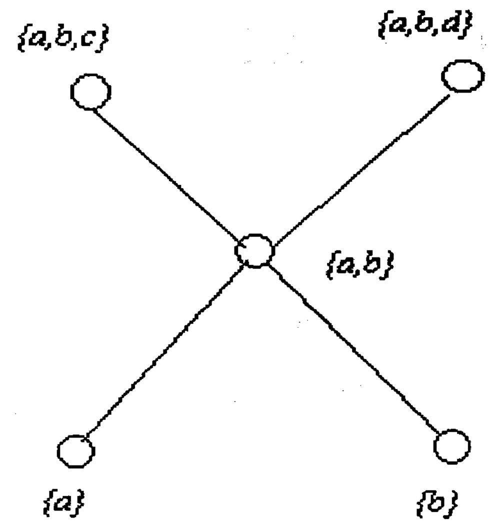


Fig 5

Note 3) Consider the poset ($\mathscr{P}(S)$, $\subseteq$). Let A,B$\in\mathscr{P}(S)$ Then l.u.b. of {A,B}= $A\cup B$ and g.l.b. of {A,B}=$A\cap B$.

Clearly $A\cup B$ is an upper bound of {A,B}. If C is any other upper bound of {A,B}, then $A\subseteq C$ and $B\subseteq C$ and hence $A\cup B\subseteq C$.

Hence $A\cup B$ is the l.u.b. of {A,B}.

Similarly $A\cap B$ is the g.l.b. of {A,B}.

**CYP Questions:**

1) Obtain the diagram for the following posets.

(a) $\mathscr{P}(\{1\})$

(b) {10,9,8,6,5} with usual $\leq$.

(c) {1,2,3,4,5,6,10,15,30}; $a\leq b \Leftrightarrow$ a divides b

(d) The set of all subgroups of $Z_6$.

(e) The set of all subgroups of $S_3$.

2) Find the least element and the greatest element, if they exist, for the following posets.

(a) $\mathscr{P}(A)$, where A is any non-empty set.

(b) The set of all finite subsets of any infinite set

(c) (N, $\leq$)

(d) (Z,$\leq$)

(e) The set of all non-empty subsets of a non-empty set A

(f) {2,5,8,20,40}, $a\leq b \Leftrightarrow$ a divides b

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## SECTION - 8.2- LATTICES

Definition: A lattice is a poset in which any two elements have a g.l.b. and a l.u.b .

We denote the l.u.b of a and b by $a \vee b$ (a cup b or join of a and b) and g.l.b. by $a \wedge b$ (a cap b or meet of a and b).

Examples

1) The poset N with the usual $\leq$ is a lattice.
   If $a,b \in N$, then $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$
2) The poset $(\mathcal{P}(S), \subseteq)$ is a lattice. Let $A,B \in \mathcal{P}(S)$.
   Then $A \vee B = A \cup B$ and $A \wedge B = A \cap B$.
3) The poset $(N, \leq)$ where $a \leq b$ iff "a divides b" is a lattice.
   Here $a \vee b$ = l.c.m. if a and b. $a \wedge b$ = g.c.d. of a and b.
4) The posets given in diagrams 1, 2 and 3 are lattices and the posets given in diagrams 4, 5 are not lattices.
5) Let G be a group. Let L be the set of all subgroups of G. In L we define $A \leq B$ iff $A \subseteq B$. Then L is a lattice.

   Proof. Clearly $(L, \leq)$ is a poset. Let $A,B \in L$. Then $A \cap B \in L$.

   We claim that $A \wedge B = A \cap B$.

   Clearly $A \cap B$ is a subgroup of A and B and hence $A \cap B \leq A,B$.
   Now, let $C \in L$ be such that $C \leq A, B$. Then C is a subgroup of A and B and hence $C \subseteq A \cap B$ i.e., $C \leq A \cap B$ and hence $A \cap B$ is the g.l.b. of A and B.

   Now, to find $A \vee B$ ,let H be the intersection of all subgroups of G containing $A \cup B$. Then H is the smallest subgroup of G containing A and B and hence $A \vee B = H$. Hence L is a lattice.
6) Let G be a group. Let L be the set of all normal subgroups of G. In L we define $A \leq B$ iff $A \subseteq B$. Then L is a lattice.
   Proof: Clearly $(L, \leq)$ is a poset. Now, if A and B are normal subgroups of G, $A \cap B$ and AB are also normal subgroups of G. It can be easily verified that $A \wedge B = A \cap B$ and $A \vee B = AB$. Hence L

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is a lattice.

Theorem 8.2.1: Let L be a lattice. Let $a,b \in L$. Then the following statements are equivalent.

1. $a \leq b$
2. $a \vee b = b$
3. $a \wedge b = a$

Proof. We shall prove that (1) and (2) are equivalent. Let $a \leq b$. Then b is an upper bound of {a,b} Also if c is any other upper bound of {a,b} then $b \leq c$.

$\therefore$ b is the l.u.b. of {a,b} (ie.) $b = a \vee b$.

Conversely, let $a \vee b = b$. We know that $a \leq a \vee b$ and hence $a \leq b$. Thus (1) and (2) are equivalent

Similarly we can prove that (2) and (3) are equivalent.

Theorem 8.2.2: Let L be a lattice. Let $a,b,c,d \in L$. Then $a \leq b$ and $c \leq d$

$\Rightarrow$ (i) $a \vee c \leq b \vee d$ and

(ii) $a \wedge c \leq b \wedge d$.

Proof. Let $a \leq b$ and $c \leq d$.

We know that $b \leq b \vee d$.

By transitivity of $\leq$, we get $a \leq b \vee d$ .......(1)

Also $c \leq d \Rightarrow c \leq b \vee d$ .......(2)

By (1) and (2), $b \vee d$ is an upper bound of {a,c}

But $a \vee c$ is the l.u.b. of {a,c}

$\therefore a \vee c \leq b \vee d$.

Similarly we can prove that $a \wedge c \leq b \wedge d$.

Aliter.

$a \leq b \Rightarrow a \vee b = b$

$c \leq d \Rightarrow c \vee d = d$

$$(a \vee c) \vee (b \vee d) = (a \vee b) \vee (c \vee d)$$
$$= b \vee d$$

Hence $a \vee c \leq b \vee d$.

The proof of (ii) is similar.

Cor. $a \leq b$ and $a \leq c$

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$\Rightarrow$ (i) $a \le b \vee c$

(ii) $a \le b \wedge c$

We now list the basic algebraic properties satisfied by the binary operations $\vee$ and $\wedge$ in a lattice. This leads us to an equivalent definition of a lattice.

Theorem 8.2.3. Let L be a lattice. Let $a,b,c \in L$ . Then we have.

$L_1$: $a \vee a = a$;

$L_1'$ : $a \wedge a = a$ (idempotenet)

$L_2$: $a \vee b = b \vee a$;

$L_2'$ :$a \wedge b = b \wedge a$ (commutative law)

$L_3$: $(a \vee b) \vee c = a \vee (b \vee c)$

$L_3'$ : $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (associative law)

$L_4$ : $a \wedge (a \vee b) = a$

$L_4'$ : $a \vee (a \wedge b) = a$ (absorption)

Proof :

($L_1$): $a \vee a$ = l.u.b. of $\{a,a\} = a$.

($L_1'$): $a \wedge a$ = g.l.b. of $\{a,a\} = a$.

($L_2$): $a \vee b$ = l.u.b of $\{a,b\}$

= l.u.b of $\{b,a\}$ = $b \vee a$.

($L_2'$): $a \wedge b$ = g.l.b. of $\{a,b\}$

= g.l.b. of $\{b,a\}$ = $b \wedge a$.

($L_3$): Clearly $(a \vee b) \vee c \ge a,b,c$.

$\therefore (a \vee b) \vee c$ is an upper bound of $\{a,b,c\}$

Moreover, if u is any element of L such that $u \ge a,b,c$ and

hence $u \ge a \vee b) \vee c$.

$\therefore$ $(a \vee b) \vee c$ is the l.u.b of $\{a,b,c\}$.

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Similarly $a \vee (b \vee c)$ is the l.u.b. of {a,b,c].

Since the l.u.b. of any subset of L is unique, we have

$(a \vee b) \vee c = a \vee (b \vee c)$.

$(L_3')$: Proof is similar to that of $L_3$

$(L_4)$: $a \vee b \geq a$

$\therefore a \wedge (a \vee b) = a$ (by Theorem 9.1)

$(L_4')$: $a \wedge b \leq a$

$\therefore a \vee (a \wedge b) = a$ (by Theorem 9.1)

Theorem 8.2.4. Let L be any non-empty set with two binary operations $\vee$ and $\wedge$ defined on it and satisfying $L_1, L_2, L_3, L_4$, $L_1'$, $L_2'$, $L_3'$, and $L_4'$. Then L is a lattice relative to a suitable definition of $\leq$ and $\vee$ and $\wedge$ are the l.u.b and g.l.b in this lattice.

Proof. First we shall prove that for any two elements $a,b \in L$, the conditions $a \vee b = b$ and $a \wedge b = a$ are equivalent

Suppose $a \vee b = b$

Then $a \wedge b = a \wedge (a \vee b)$

$= a$ (by $L_4$)

Similarly we can prove that

$a \wedge b = a \Rightarrow a \vee b = b$.

Now we define a relation $\leq$ in L by $a \leq b \Leftrightarrow a \vee b = b$.

We claim that $\leq$ is a partial ordering relation in L.

By $L_1$, $a \vee a = a$.

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$\therefore$ $a \le a$ and hence $\le$ is reflexive.

Now, let $a \le b$ and $b \le a$.

$\therefore$ $a \vee b = b$ and $b \vee a = a$.

$\therefore$ By $L_2$, $a = b$ and hence $\le$ is anti symmetric

Now, let $a \le b$ and $b \le c$.

$\therefore$ $a \vee b = b$ and $b \vee c = c$

$\therefore a \vee c = a \vee (b \vee c)$

$= (a \vee b) \vee c$ (by $L_3$)

$= b \vee c$

$= c$

$\therefore$ $a \le c$ and hence $\le$ is transitive.

$\therefore$ $\le$ is a partial ordering relation.

$\therefore$ $(L, \le)$ is a poset.

We shall now prove that $a \vee b$ is the l.u.b. of a and b and $a \wedge b$ is the g.l.b. of a and b

$a \wedge (a \vee b) = a$ (by $L_4$)

$\therefore$ $a \le a \vee b$

Also $b \wedge (a \vee b) = b$ (by $L_4$)

$\therefore$ $b \le a \vee b$

$\therefore$ $a \vee b$ is an upper bound of a and b.

Now, let c be any other upper bound of $\{a,b\}$.

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Then $a \leq c$ and $b \leq c$.

$\therefore\ a \vee c = c$ and $b \vee c = c$.

$\therefore\ (a \vee b) \vee c = a \vee (b \vee c)$ (by $L_3$)

$= a \vee c$

$= c.$

$\therefore\ a \vee b \leq c$

$\therefore\ a \vee b$ is the l.u.b. of a and b.

Similarly we can prove that $a \wedge b$ is the g.l.b of a and b. Hence the theorem

Definition: Let L be the lattice. A non-empty subset S of L is called a sub-lattice of L if $a, b \in S \Rightarrow a \vee b \in S$ and $a \wedge b \in S$.

Examples

1. $(N, \leq)$ is a lattice. Any non-empty subset of N is a sub-lattice of N.

2. $(\mathcal{P}(S), \subseteq)$ is a lattice. The set of all finite subsets of S is a sub-lattice of $\mathcal{P}(S)$.

3. Let L be a lattice. Let $a \in L$.

   Let $L_a = \{x / x \in L, x \leq a\}$ Then $L_a$ is a sub-lattice of L.

   Proof. Let $x, y \in L_a$. Then $x \leq a$ and $y \leq a$.

   $\therefore\ x \vee y \leq a$ and $x \wedge y \leq a$.

   $\therefore x \vee y$ and $x \wedge y \in L_a$.

   $\therefore\ L_a$ is a sub-lattice of L.

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4. Let L be a lattice. Let $a,b \in L$ and $a \leq b$. Then $\{a,b\}$ is a sub-lattice of L.

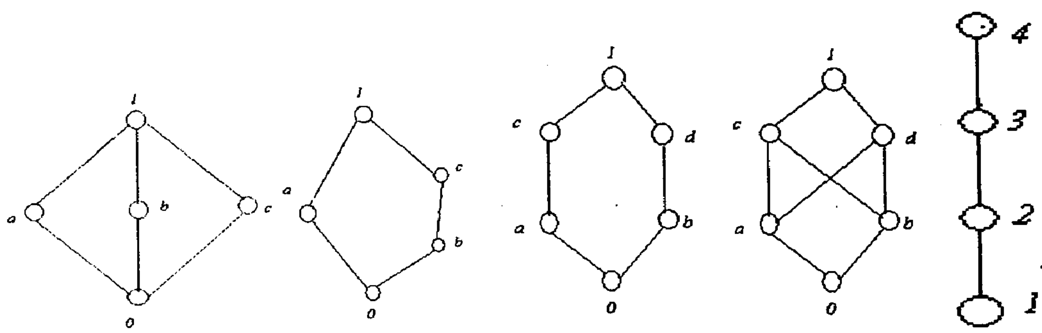
**CYP Questions:**

1) Determine which of the posets given in exercise in 8.1 are lattices and justify your answers.

2) Show that the set of all subspaces of a vector space V forms a lattice w.r.t the ordering defined by $A \leq B$ iff A is a subspace of B.
[Hint $A \vee B = A+B$ and $A \wedge B = A \cap B$]

3) Prove that any chain is a lattice.

4) Which of the following diagrams are lattices?




$M_5$ $N_5$

Fig 6

5) Find all the sub-lattices of the lattices $M_5$ and $N_5$ in Fig.6.

6) Let L be a lattice. Let $a,b \in L$ and $a \leq b$. Then prove that

$\{x/x \in L \text{ and } a \leq x \leq b\}$ is a sub lattice of L.

7) Show that the set of all normal subgroups of a group $G$ is a sub-lattice of the set of all subgroups of $G$.

8) Prove that the intersection of any two sub-lattices is a sub-lattice.

9) Prove that a lattice is a chain iff all its subsets are sub-lattices.

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# UNIT-9

## Unit Structure:

Section 9.1: Distributive Lattice.

Section 9.2: Modular Lattice – Examples – Simple properties.

**Introduction:** In this unit we develop the various types of lattices like distributive lattice, modular lattices and its properties.

### SECTION -9.1 - DISTRIBUTIVE LATTICES.

In this section we examine the validity of the following **distributive laws** in a lattice.

$L_5$ : $a\vee(b\wedge c) = (a\vee b)\wedge(a\vee c)$ and

$L_5'$ : $a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)$ .

First we shall prove that in any lattice $L_5$ is equivalent to $L_5'$.

Theorem 9.1.1: In any lattice

$L_5$ : $a\vee(b\wedge c) = (a\vee b)\wedge(a\vee c)$ and

$L_5'$ : $a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)$ are equivalent.

Proof. We shall first prove that $L_5 \Rightarrow L_5'$.

Let $a\vee(b\wedge c)$

$= (a\vee b)\wedge(a\vee c)$

$\therefore$ $(a\wedge b)\vee(a\wedge c)$

$= [(a\wedge b)\vee a)]\wedge[(a\wedge b)\vee c)]$ (by $L_5$)

$= [a\vee(a\wedge b)]\wedge[c\vee(a\wedge b)]$ (by $L_3$)

$= a\wedge[c\vee(a\wedge b)]$ (by $L_4'$)

$= a\wedge[(c\vee a)]\wedge(c\vee b)]$ (by $L_5$)

$= [a\wedge(c\vee a)]\wedge(c\vee b)$ (by $L_3'$)

$= a\wedge(c\vee b)$ (by $L_4$)

$= a\wedge(b\vee c)$ (by $L_2$)

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$\therefore L_5 \Rightarrow L_5'$.

Similarly we can prove that $L_5' \Rightarrow L_5$

Hence the theorem.

Definition: A lattice L is called a **distributive lattice** if

$a\vee(b\wedge c) = (a\vee b)\wedge(a\vee c)$ for all $a,b,c \in L$.

Note. In view of theorem 9.5, in any distributive lattice $L_5'$ is also valid. Also any lattice L satisfying $L_5'$ for all $a,b,c \in L$ is also a distributive lattice.

Examples

1. $(\mathscr{P}(S), \subseteq)$ is a distributive lattice.

Proof. Let $A,B,C \in \mathscr{P}(S)$

Then $A\vee(B\wedge C) = A\cup(B\cap C)$

$= (A\cup B)\cap(A\cup C)$

$= (A\vee B)\wedge(A\vee C)$

$(\mathscr{P}(S), \subseteq)$ is a distributive lattice.

2. Any chain is a distributive lattice.

Proof. Let L be any chain. Let $a,b,c \in L$. Since any two elements in L are comparable we assume without loss of generality that $a \le b \le c$.

$a\vee(b\wedge c) = a\vee b = b$

and $(a\vee b)\wedge(a\vee c) = b\wedge c = b$

$a\vee(b\wedge c) = (a\vee b)\wedge(a\vee c)$

Hence L is a distributive lattice.

3. $M_5$ is not a distributive lattice.(Figure 6)

Proof. $a\vee(b\wedge c) = a\vee 0 = a$ and

$(a\vee b)\wedge(a\vee c) = 1\wedge 1 = 1$

Thus $a\vee(b\wedge c) \ne (a\vee b)\wedge(a\vee c)$.

Hence $M_5$ is not a distributive lattice.

4. $N_5$ is not a distributive lattice(refer figure 9.6)

Proof. $b\vee(a\wedge c) = b\vee 0 = b$ and

$(b\vee a)\wedge(b\vee c) = 1\wedge c = c$.

$b\vee(a\wedge c) \ne (b\vee a)\wedge(b\vee c)$.

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Hence $N_5$ is not a distributive lattice.

5. The set of all subspaces of a vector space V forms a lattice.

If A and B are two subspaces of V, then

$A \vee B = A + B$ and $A \wedge B = A \cap B$.

This lattice need not be distributive

For example, take $V = V_2(R)$

Let $A = \{(x,0) / x \in R\}$,

$B = \{(0,x) / x \in R\}$ and

$C = \{(x,x) / x \in R\}$

Clearly A,B and C are subspaces of V.

Now $A \vee (B \wedge C) = A + (B \cap C) = A + \{0\} = A$ and

$(A \vee B) \wedge (A \vee C) = (A+B) \cap (A+C)$

$= V_2(R) \cap V_2(R)$

$= V_2(R)$

Hence $A \vee (B \wedge C) \neq (A \vee B) \wedge (A \vee C)$.

Problems

Problem 1. In any lattice L

$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$ and

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ for $a,b,c \in L$.

Solution. We know that $a \geq a \wedge b$ and $a \geq a \wedge c$

$a \vee a \geq (a \wedge b) \vee (a \wedge c)$ (Theorem 9.2)

$a \geq (a \wedge b) \vee (a \wedge c)$ ....(1)

Also we know that $b \vee c \geq a \wedge b$ and $b \vee c \geq a \wedge c$.

$(b \vee c) \vee (b \wedge c) \geq (a \wedge b) \vee (a \wedge c)$

$(b \vee c) \geq a \wedge b \vee (a \wedge c)$ ...(2)

$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$ (by (1) and (2))

Similarly $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.

Problem 2. In any distributive lattice L, $x \vee a = y \vee a$ and $x \wedge a = y \wedge a$

$\Rightarrow x = y$.

Solution.

$x = x \vee (x \wedge a)$ (by L'4 )

$= x \vee (y \wedge a)$ (hypothesis)

$= (x \vee y) \wedge (x \vee a)$ (L is distributive)

$= (y \vee x) \wedge (y \vee a)$ (hypothesis)

$= y \vee (x \wedge a)$ (L is distributive)

$= y \vee (y \wedge a)$ (hypothesis)

$= y$ (by L'4)

Problem 3. Show that in any distributive lattice

$$(a \vee b) \wedge (b \vee c) \wedge (c \vee a)$$
$$= (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$$

Solution. $(a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

$= [(a \wedge b) \vee (a \wedge c) \vee (b \wedge b) \vee (b \wedge c)] \wedge (c \vee a)$

(by L5 and L'5)

$= [(a \wedge b) \vee (a \wedge c) \vee b \vee (b \wedge c)] \wedge (c \vee a)$

$= [(a \wedge b) \vee (a \wedge c) \vee b] \wedge (c \vee a)$ (by L'4)

$= [b \vee (a \wedge b) \vee (a \wedge c)] \wedge (c \vee a)$ ( by L2)

$= [b \vee (a \wedge c)] \wedge (c \vee a)$ ( by L'4)

$= [b \wedge (c \vee a)] \vee [(a \wedge c) \wedge (c \vee a)]$ (by L'5)

$= [(b \wedge c) \vee (b \wedge a)] \vee (a \wedge c \wedge c) \vee (a \wedge c \wedge a)]$ (by L'5)

$= (b \wedge c) \vee (b \wedge a) \vee (a \wedge c) \vee (a \wedge c)$

$= (b \wedge c) \vee (b \wedge a) \vee (a \wedge c)$

$= (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$

**CYP Questions:**

1. Show that a lattice L is distributive iff

   $a \wedge (b \wedge c) \leq (a \wedge b) \vee (a \wedge c)$ for all $a,b,c \in L$.

2. Show that a lattice L is distributive iff $a \vee (b \wedge c) \geq (a \vee b) \wedge (a \vee c)$ for all $a,b,c \in L$.

   (Hint:Use solved problem 1)

## SECTION -9.2 - MODULAR LATTICES.

We now introduce another important family of lattices which are not distributive but which satisfy a weaker form of the distributive law.

Definition. A lattice L is said to be a modular lattice if



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$a\vee(b\wedge c)=(a\vee b)\wedge c$ where $a,b,c\in L$ and $a\leq c$.

Theorem 9.2.1: Any distributive lattice L is a modular lattice.

Proof. Let $a,b,c\in L$ and $a\leq c$.

Since $a\leq c$. $a\vee c=c$ ....(1)

Now, $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$ (L is distributive)

$=(a\vee b)\wedge c$ (by(1))

$\therefore$ L is modular.

The most important class of modular lattices is given in the following theorem.

Theorem 9.2.2: The lattice of normal subgroups of any group is a modular lattice.

Proof. Let G be the given group. Let A,B,C be normal sub-groups of G such that $A\subseteq C$.

We have to prove that

$$A\vee(B\wedge C)=(A\vee B)\wedge C \quad \text{....(1)}$$

We know that $A\vee B=AB$ and $A\wedge B=A\cap B$ (refer example 6 of 9.2)

$\therefore$ (1) reduces to $A(B\cap C)=(AB)\cap C$.

Now

$A\vee(B\wedge C)\leq(A\vee B)\wedge(A\vee C)$ (by solved problem 1 of 9.3)

$=(A\vee B)\wedge C$ ($\because A\subseteq C$)

$A(B\cap C)\subseteq(AB)\cap C$ .....(2)

Now, let $x\in(AB)\cap C$. Then $x\in AB$ and $x\in C$.

Now, $x\in AB\Rightarrow x=ab$ where $a\in A$ and $b\in B$.

Since $A\subseteq C$, $a\in A\Rightarrow a\in C$. Also $x\in C$.

$b=a^{-1}x\in C$. Also $b\in B$.

$b\in B\cap C$

$x=ab$ where $a\in A$ and $b\in B\cap C$

$x\in A(B\cap C)$

$(AB)\cap C\subseteq A(B\cap C)$ .....(3)

$A(B\cap C)=(AB)\cap C$ (by 2 and 3).

Examples

1. We have seen that the lattice of all normal subgroups of $V_4$ is

represented by the diagram $M_5$ (example 3 of 9.1).
Hence $M_5$ is a modular lattice. However $M_5$ is not a distributive lattice.(refer example 3 of 8.1).

2. $N_5$ is not a modular lattice (refer Fig.6)

Proof. Consider a,b,c $\in N_5$

Clearly $b \leq c. \vee$ (B

Now, $b\vee (a\wedge c) = b\vee 0 = b$ and

$(b \vee a) \wedge (b \vee c)= 1\wedge c = c$

$a\vee (b\wedge c)\neq (a\vee b) \wedge c$.

$N_5$ is not modular.

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# UNIT-10

## Unit Structure:

Section 10.1: Boolean Algebras – Examples

**Introduction:** In this unit we discuss about the complemented lattice, complemented distributive lattice, that is, Boolean algebra and its properties.

## SECTION 10.1: BOOLEAN ALGEBRAS – EXAMPLES

Definition. Let L be a lattice with 0 and 1. Let $a \in L$. An element $a' \in L$ is said to be a complement of a if $a \vee a' = 1$ and $a \wedge a' = 0$.

L is said to be *a complemented Lattice* if every element $a \in L$ has a complement.

Examples:

1. $(\mathscr{P}(S), \subseteq)$ is a complemented lattice. Here the least element 0 is $\Phi$ and the greatest element A is $S$. The complement of subset A of $S$ is the usual set theoretic complement A' of the set A since $A \cup A' = S$ and $A \cap A' = \Phi$.
2. In any lattice $0' = 1$ and $1' = 0$.
3. In $M_5$ of Fig 6, $a \vee b = 1$ and $a \wedge b = 0$. There fore b is a complement of a. c is also a complement of a. Thus the complement of an element need not be unique.
4. Consider the lattice given in Figure 9.1. Here the element 3 does not have a complement.
5. Let L denote the set of all subspaces of a finite dimensional inner product space V. L is a lattice with the usual ordering and if A and B are subspaces of V, then $A \vee B = A + B$ and $A \wedge B = A \cap B$

Also for any subspace A of V, the orthogonal complement $A^{\perp}$ is

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such that

$A \vee A^{\perp} = A + A^{\perp} = V$ and

$A \wedge A^{\perp} = A \cap A^{\perp} = \{0\}$.

Hence L is a complemented lattice.

Theorem 10.1.1: In a distributive lattice the complement of any element a, if it exists, is unique.

Proof. Let x and y be complements of $a \in L$.

$a \vee x = a \vee y = 1$ and

$a \wedge x = a \wedge y = 0$.

$x = y$ (refer solved problem 2 of 9.3).

Definition. A complemented distributive lattice is called a Boolean Algebra.

Examples

1. $(\mathscr{P}(S), \subseteq)$ is a Boolean algebra.
2. $M_5$ is a complemented lattice. However $M_5$ is not distributive and hence it is not a Boolean algebra.

Remark. Using Theorem 9.8 we see that in a Boolean algebra every element is uniquely complemented.

Solved Problems

Problem 1. Let $B$ be a Boolean algebra. Then

(1) $(a \vee b)' = a' \wedge b'$ and $(a \square b)' = a' \wedge b'$

(De Morgan's laws)

(2) $(a')' = a$

Solution (1) It is enough if we prove that

$(a \vee b) \vee (a' \wedge b') = 1$ and

$(a \vee b) \wedge (a' \wedge b') = 0$.

Now, $(a \vee b) \vee (a' \wedge b')$

$= [(a \vee b) \vee a'] \wedge [(a \vee b) \vee b']$ (by L5)

$= [(a \vee a') \vee b] \wedge [a \vee (b \vee b')]$ (by L2& L3)

$= (1 \vee b) \wedge (a \vee 1)$

$= 1 \wedge 1$

$= 1$.

Similarly $(a \vee b) \wedge (a' \wedge b') = 0$.

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(2) To prove that $(a')' = a$, it is enough if we show that $a' \vee a = 1$ and $a' \wedge a = 0$ which are true.

Problem 2. In a Boolean algebra if $a \vee x = b \vee x$ and $a \vee x' = b \vee x'$, then $a = b$.

Solution. $a \vee x = b \vee x$ and $a \vee x' = b \vee x'$

$(a \vee x) \wedge (a \vee x') = (b \vee x) \wedge (b \vee x')$

$a \vee (x \wedge x') = b \vee (x \wedge x')$ (by L5)

$a \vee 0 = b \vee 0$

$a = b$

problem 3. Show that in a Boolean algebra

$[a \vee (a' \wedge b)] \wedge [b \vee (b \wedge c)] = b$.

Solution. $[a \vee (a' \wedge b)] \wedge [b \vee (b \wedge c)]$

$= [(a \vee a') \wedge (a \vee b)] \wedge b$ (by L5 and L'4)

$= [1 \wedge (a \vee b)] \wedge b$

$= (a \vee b) \wedge b$

$= b$ (by L4)

Problem 4. Show that in a Boolean algebra B, the complement of any element is not itself.

Solution. Suppose $a \in B$. If possible let $a' = a$.

Then $a = a \vee a = a \vee a' = 1$

Also $a = a \wedge a = a \wedge a' = 0$

$\therefore 0 = 1$ which is a contradiction.

$\therefore a \neq a'$

Problem 5. Prove that a Boolean algebra can not have exactly three *elements*.

Solution. Let B be a Boolean algebra with three distinct elements 0, 1 and a.

By the above problem, $a' \neq a$.

Suppose $a' = 1$, then $(a')' = 1'$

$\therefore$ $a = 0$ which is a contradiction.

$\therefore$ $a' \neq 1$. Similarly $a' \neq 0$.

$a \in B$ does not have a complement which is a contradiction.

Hence there is no Boolean Algebra with three elements.

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**CYP Questions:**

1) Find which of the following lattices are complemented.

(i) The poset N with the usual $\leq$ is a lattice.

If a,b$\in$N, then $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$

(ii) The poset ($\mathscr{P}$(S), $\subseteq$) is a lattice. Let A,B$\in\mathscr{P}$(S).

Then $A \vee B = A \cup B$ and $A \wedge B = A \cap B$.

(iii) The poset (N, $\leq$) where $a \leq b$ iff "a divides b" is a lattice.

Here $a \vee b$ = l.c.m. if a and b. $a \wedge b$ = g.c.d. of a and b.

(iv) The diagrams 1, 2 ,3 4, 5 .

(v) Let G be a group. Let L be the set of all subgroups of G. In L we define $A \leq B$ iff $A \subseteq B$. Then L is a lattice.

(vi) Let G be a group. Let L be the set of all normal subgroups of G. In L we define $A \leq B$ iff $A \subseteq B$. Then L is a lattice.

2) Simplify the following expressions:-

(a) $(a\vee b)\wedge a'\wedge b'$

(b) $(a\wedge b\wedge c \vee a'\vee b'\vee c'$

(c) $(a\wedge b)\vee[c\wedge(a'\vee b')]$

(d) $[(a'\wedge b')'\vee c]\wedge(a\vee b')'$

(e) $(x\wedge y) \vee (x\wedge y') \vee (x'\vee y)\vee(x'\wedge y')$

3) Prove that in any Boolean algebra

$a\vee(a'\wedge b) = a\vee b$ for every pair of elements a and b

4. Prove that it $a\wedge x = b\wedge x$ and $a\wedge x' = b\wedge x'$, then $a = b$.

5. Prove that in any Boolean algebra, each of the identifies

$a\wedge x = a$ and $a\vee x = x$ for all x implies $a = 0$.

6. Prove that in a Boolean algebra, the following are equivalent.

(a) $a\wedge b = a$

(b) $a\vee b = b$

(c) $a\wedge b' = 0$

(d) $b\vee a' = 1$

(e) $a \leq b$

7. Find the complements of the following expressions.

(a) $x\vee y\vee z'$

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(b) $(x \vee y' \vee z') \wedge [x \vee (y \vee z')]$

(c) $(x' \vee y)' \wedge (x \vee y')$

8) prove that $(x \vee y) \wedge (x' \vee z) = (x' \wedge y) \vee (x \wedge z)$

9) Show that $(x \wedge y) \vee [(x \vee y') \wedge y]' = 1$

Answers. 1 (a) 0 (b) 1 (c) $(a \wedge b) \vee c$

(d) $a' \wedge b$ (e) 1 6(a) $x' \wedge y' \wedge z$ (b) $x' \wedge z$

(c) $x' \vee y$.