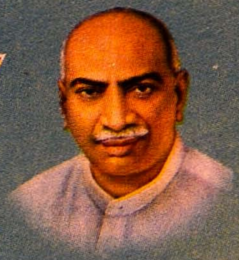

MADURAI KAMARAJ UNIVERSITY

(University with Potential for Excellence)

DISTANCE EDUCATION

B.Sc. (Mathematics)

THIRD YEAR

REAL ANALYSIS AND COMPLEX ANALYSIS

UNIT : 1-5

PAPER - VI

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B.Sc.,Mathematics **MAJOR PAPER VI**

REAL ANALYSIS AND COMPLEX ANALYSIS

Dear Student,

We welcome you as a student of the third year B.Sc., degree course in Mathematics.This Paper-VI deals with REAL ANALYSIS AND COMPLEX ANALYSIS. The learning material for this paper will be supplemented by contact seminars.

Learning through the Distance Education mode, as you are all aware, involves self-learning and self-assessment and in this regard you are expected to put in disciplined and dedicated effort.On our part, we assume of our guidance and support.

Best wishes.

B.Sc., MATHEMATICS – THIRD YEAR

PAPER VI REAL ANALYSIS AND COMPLEX ANALYSIS

SYLLABUS

UNIT 1 : Sequences – Definition and examples – Convergent and divergent sequences – Cauchy sequences (definitions only) introduction of countable and uncountable sets- Holder's and Minkowski's inequalities – Metric space – Definition and examples.

UNIT 2 : Open sets and closed sets (definition and examples only) – Completeness-definition and examples – Cantor's intersection theorem and Baire's category theorem.

UNIT 3 : Contiunuity – Definition and E: es – Homemorphism (Discontinuous functions on R are not included).

UNIT 4 : Connected – Definition and examples – Connected subsets of R- connectedness and Continuity – Intermediate value theorem.

UNIT 5 : Compactness – Definition and examples – Compact subsets of R – Equivalent characterization of Compactness.

UNIT 6 : Analytic function -C.R.-equations – Sufficient conditions – Harmonic Functions.

UNIT 7 : Elementary Transformation – Bilinear Transforming – Cross ratio-fixed points - Special Bilinear Transformation – Real axis to real axis – Unit circle to unit circle and real axis to unit circle onlv.

UNIT 8 : Cauchy's Fundamental theorem – Cauchy's integral formulae and formulae of derivatives – Morera's theorem – Cauchy's inequality – Lioville's theorem – Fundamental theorem of algebra.

UNIT 9 ; Taylor's theorem, Laurant's theorem – singular points – Poles – Argument principle – Rouche's theorem.

UNIT 10 : Calculus of residues – Evaluation of Definite integral.

Text Books: 1. Modern Analysis by S. Arumugam and A. Thangapandi Isaac.

New Gamma Publishing house, 2005.

2. Complex Analysis by S. Arumugam , Thangapandi Isaac

and A. Somasundaram, Sci. Tech Publilcations, Jan 2003.

Space for hints

0. PRELIMINARIES

INTRODUCTION

In this chapter, we introduce the notions of sets, functions and some properties of the real number system, which are needed for the rest of the book. The following concepts are useful to understand the concepts in the subsequent chapters.

0.1 SETS AND FUNCTIONS

The concepts of sets and functions are indispensable to almost all branches of mathematics. The usual material of elementary set theory is so current take it for granted.

We freely use the following notations of set theory.

(i) A is a subset of B written as $A \subseteq B$.

(ii) Union of two sets A and B written as $A \cup B$.

(iii) Intersection of two sets A and B written as $A \cap B$.

(iv) Complement of a subset A of X written as A^c.

(v) Difference of two sets A and B written as A-B.

(vi) Cartesian product of two sets A and B written as $A \times B$.

(vii) A function f from a set A to a set B written as $f: A \to B$.

(viii) The empty set ϕ which contains no element.

Certain letters are reserved to denote particular sets which occur often in our discussion. They are

N....... the set of all **natural numbers.**

Q .... the set of all **rational numbers.**

Q^+....... the set of all **positive rational numbers.**

R .... the set of all **real numbers.**

C .... the set of all **complex numbers.**

R^n .... the set of all ordered n-tuples $(x_1, x_2, \ldots\ldots, x_n)$ of real numbers.

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C^n .... the set of all ordered n-tuples $(z_1, z_2, \ldots, z_n)$ of complex numbers.

0.2 INTERVALS IN R

We use order structure in the real number system **R** to define certain subsets of **R** called ***intervals.***

Let a ,b$\in$ **R** and a < b.

(i) $(a,b) = \{x \,/\, x \in \mathbf{R}, a<x<b\}$ is called the **open interval** with a and b as end points.

(ii) $[a,b] = \{x \,/\, x \in \mathbf{R}, a \leq x \leq b\}$ is called the **closed interval** with a and as end points.

(iii) $(a,b] = \{x/x \in \mathbf{R}, a<x\leq b\}$ is called the **open-closed interval** with a and b as end points.

(iv) $[a,b) = \{x/x \in \mathbf{R}, a \leq x<b\}$ is called the **closed-open interval** with a and b as end points.

(v) $[a,\infty) = \{x/x \in \mathbf{R} \text{ and } x \geq a\}$.

(vi) $(a,\infty) = \{x/x \in \mathbf{R} \text{ and } x>a\}$.

(vii) $(-\infty,a] = \{x/x \in \mathbf{R} \text{ and } x \leq a\}$.

(viii) $(-\infty,a) = \{x/x \in \mathbf{R} \text{ and } x<a\}$.

(ix) $(-\infty,\infty) = \mathbf{R}$.

Any subset of **R** which is one of the above forms is called an ***interval***. Any interval of the form (i), (ii), (iii) or (iv) is called ***a finite interval*** or ***bounded interval*** and an interval of the form (v), (vi), (vii) or (ix) is called an ***infinite interval*** or ***unbounded interval.***

0.3 BOUNDED SETS

Definition 0.3.1 A subset A of **R** is said to be ***bounded above*** if there exists an element $\alpha \in \mathbf{R}$ such that $a \leq \alpha$ for all $a \in A$. Then α is called an ***upper bound*** of A.

A is said to be ***bounded below*** if there exists an element $\beta \in \mathbf{R}$ such that $a \geq \beta$ for all $a \in A$. β is called a ***lower bound*** of A.

A is said to be ***bounded*** if it is both bounded above and bounded below.

Space for hints

Note 0.3.2

1. Let A **R.** If $\alpha \in \mathbf{R}$ is an upper bound of A then any $x>\alpha$ is also upper bound of A. Thus a set, which is bounded above, has infinite number of upper bounds. Similarly a set, which is bounded below, has infinite number of lower bounds.

2. Let A **R** and $x \in \mathbf{R}$. Then x is not an upper bound of A iff there exists at least one element $a \in A$ such that $x<a$. Similarly x is not a lower bound of A iff there exists at least one element $a \in A$ such that $x>a$.

Examples 0.3.3

1. Let $A=\{2,3,5\}$. Any element $x \in \mathbf{R}$ such that $x \leq 2$ is a lower bound of A and any element $x \in R$ such that $x \geq 5$ is an upper bound of A.

2. Let $A=\mathbf{N}$. Any real number x is not an upper bound of **N**, since there exists a natural number n such that $n \geq x$. Hence **N** is not bounded above. However **N** is bounded below. Any real number $x \leq 1$ is a lower bound of **N**.

3. Let $A=\mathbf{Z}$. Then **Z** is neither bounded above nor bounded below.

4. Let $A=\{x/x \leq 2\} = (-\infty,2]$. A is bounded above but not bounded below. Any real number $x \geq 2$ is an upper bound of A.

5. Let $A=(1,0)$, $A = \{x/x \in \mathbf{R} \text{ and } 0<x<1\}$. Here any number $y \geq 1$ is an upper bound of A. Hence $[1,\infty)$ is the set of all upper bounds of A. We notice that the least upper bound of A is 1.

0.4 LEAST UPPER BOUND AND GREATEST LOWER BOUND

Definition 0.4.1 Let A **R** and $u \in \mathbf{R}$. u is called the ***least upper bound*** *(l.u.b)* or ***supremum*** *(sup)* of A if

(i) u is an upper bound of A.

(ii) If $v < u$ then v is not an upper bound of A.

Space for hints

Let $A \subseteq$ **R** and $l \in$ **R**. l is called the ***greatest lower bound(glb)*** or ***infimum (inf)*** of A if

(i) l is a lower bound of A.

(ii) If $m > l$ then m is not a lower bound of A.

Examples 0.4.2

1. Let $A = \{1,3,5,6\}$. Then *glb* of $A = 1$ and *lub* of $A = 6$. In this case both *glb* and *lub* belong to A.

2. Let $A=(0,1)$. Then *glb* of $A = 0$ and *lub* of $A = 1$. In this case both *glb* and *lub* do not belong to A.

3. Let $A = [a,b)$. then *glb* of A = a and *lub* of A =b. In this case *glb* $\in A$ and *lub* $\notin A$.

4. Let A = **N**. Then **N** is not bounded above and hence **N** does not have any *lub*. However *glb* of **N** = 1.

Exercises 0.4.3

1. Find the *lub* and *glb* of each of following sets, if they exist. State whether *lub* and *glb* belong to the given sets or not.

(i) $A = \{1, 1/2, 1/3, 1/4, \ldots, 1/n, \ldots\}$

(ii) $A = \{1/2, 2/3, 3/4, 4/5, \ldots\}$

(iii) $A = [-3,1)$

(iv) $A = \{1\}$ (v) $A = (0, 100)$ (vi) $A = [-\infty, 1)$ (vii) $A = \{x \mid x^2 \leq 2\}$

(viii) $A = (-1,3) \cup (5,6)$ (ix) $A = \{\pm 1, \pm 1/2, \pm 1/3, \ldots, \pm 1/n, \ldots\}$

(x) $A = \{1, -2, 3, -4, 5, \ldots\}$ (xi) A = **Q** (xii) A = 2**N**

0.5 BOUNDED FUNCTIONS

Definition 0.5.1

Let $f: A \to$ **R** be any function. Then the range of f is a subset of **R**. f is said to be a ***bounded function*** if its range is a bounded subset of **R**. Hence f is a bounded function iff there exists a real number m such that $|f(x)| \leq m$ for all $x \in A$.

Space for hints

Examples 0.5.2

1. f: [0,1] → **R** given by $f(x) = x+2$ is a bounded function where as f: **R**→ **R** given by $f(x) = x+2$ is not a bounded function.

2. f: **R**→ **R** given by $f(x) = 1$ if x is rational

   and $f(x) = 0$ if x is irrational

   Then f is a bounded function.

3. f: **R**→ **R** defined by $f(x) = \sin x$ is abounded function since $|\sin x| \leq 1$ for all $x \in$ **R**.

Space for hints

UNIT-1

1.0 INTORDUCTION

A great deal of analysis is concerned with sequences and series. Consider the following collection of real numbers given by 1, 1/2, 1/3, 1/4,........, 1/n,...... In this collection the first element is 1, the second element is ½, the third element is 1/3 and so on. This is an example of sequence of real numbers. We may think of a sequence as any arrangement of elements where we can say which element is first, which is second, which is third and so on. In other words the elements of sequence are labelled with the elements of **N** preserving their order. In general such a labeling can be done by means of a function f whose domain is **N**. If the range of f is a subset of an arbitrary set X, we get a sequence of elements of X. Now we deal with sequences of real numbers.

1.1 SEQUENCES

Definition 1.1.1

Let $f: \mathbf{N} \rightarrow \mathbf{R}$ be a function and let $f(n) = a_n$. Then $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called the ***sequence*** in **R** determined by the function f and is denoted by (a_n). a_n is called the n^{th} term of the sequence. The range of the function f, which is a subset of **R**, is called the ***range of the sequence***.

Examples 1.1.2

1. The function $f: \mathbf{N} \rightarrow \mathbf{R}$ given by $f(n) = n$ determines the sequence 1,2,3,....,n,....

2. The function $f: \mathbf{N} \rightarrow \mathbf{R}$ given by $f(n) = n^2$ determines the sequence 1,4,9,....., n^2,....

3. The function $f: \mathbf{N} \rightarrow \mathbf{R}$ given by $f(n) = (-1)^n$ determines the sequence -1,1,-1,1,..... Thus the terms of a sequence need not be distinct. The range of this sequence is {1,-1}. However we note that the sequence $((-1)^n)$ and $((-1)^{n+1})$ are different. The first sequence starts with -1 and the second sequence starts with 1.

4. The sequence $((-1)^{n+1})$ is given by 1,-1,1,-1..... The range of this sequence is also {1,-1}. However we note that the sequence

$((-1)^n)$ and $((-1)^{n+1})$ are different. The first sequence starts with -1 and the second sequence starts with 1.

Space for hints

5. The constant function f: $\mathbf{N} \to \mathbf{R}$ given by $f(n)=1$ determines the sequence 1,1,1,..... Such a sequence is called ***a constant sequence.***

6. The function f: $\mathbf{N} \to \mathbf{R}$ given by

$$f(n) = n/2 \text{ if n is even and}$$

$$f(n) = \tfrac{1}{2}(1-n) \text{ if n is odd}$$

Determines the sequence 0,1,-1,2,-2,.....,n,-n..... The range of this sequence is **Z**.

7. The function f: $\mathbf{N} \to \mathbf{R}$ given by $f(n) = n/n+1$ determines the sequence ½,2/3,3/4,.....,n/n+1,....

8. The function f: $\mathbf{N} \to \mathbf{R}$ given by $f(n) = 1/n$ determines the sequence 1,1/2,1/3,....,1/n,....

9. The function f: $\mathbf{N} \to \mathbf{R}$ given by $f(n) = 2n+3$ determines the sequence 5,7,9,11,....

10. Let $x \in \mathbf{R}$. The function f: $\mathbf{N} \to \mathbf{R}$ given by $f(n) = x^{n-1}$ determines the ***geometric sequence*** $1, x, x^2, \ldots, x^n, \ldots$

11. The sequence (-n) is given by -1,-2,-3,...,-n,.... The range of this sequence is the set of all negative integers.

12. A sequence can also be described by specifying the first few terms and stating a rule for determining a_n in terms of the previous terms of the sequence. For example, let $a_1=1, a_2=1$ and $a_n=a_{n-1}+a_{n-2}$. Then $a_3=a_2+a_1=2$; $a_4=a_3+a_2=3$ and so on. We thus obtain the sequence 1,1,2,3,5,8,13,...... This sequence is called **Fibonacci's sequence**.

13. Let $a_1=\sqrt{2}$ and $a_{n+1}=\sqrt{(2+a_n)}$. This defines the sequence $\sqrt{2}, \sqrt{(2+\sqrt{2})}, \ldots\ldots$

1.2 CONVERGENT SEQUENCES

Consider the sequence 1,1/2,1/3,....,1/n,..... We observe that as n increases 1/n approaches zero. In fact by making the value of n sufficiently large, we can bring 1/n as close to 0 as we want. This is roughly what we mean when we say that the sequence (1/n) converges to 0 or 0 is the limit of this sequence. This idea is formulated mathematically in the following definition.

Space for hints

Definition 1.2.1

A sequence (a_n) is said to ***converge*** to a number l if given $\varepsilon>0$ there exists a positive integer m such that $|a_n - l| < \varepsilon$ for all $n \geq m$.

We say that l is the limit of the sequence and we write $\lim_{n\to\infty} = l$ or $(a_n) \to l$.

Note 1.2.2 $(a_n) \to l$ iff given $\varepsilon>0$ there exists a natural number m such that $a_n \in (l - \varepsilon, l + \varepsilon)$ for all $n \geq m$ (i.e), All but a finite number of terms of the sequence lie within the interval $(l - \varepsilon, l + \varepsilon)$.

Note 1.2.3 The above definition does not give any method of finding the limit of a sequence. In many cases, by observing the sequence carefully, we can guess whether the limit exists or not and also the value of the limit.

Examples 1.2.3

1. $\lim_{n\to\infty} 1/n = 0$ or $(1/n) \to 0$.

Proof. Let $\varepsilon>0$ be given.

Then $|1/n - 0| = 1/n < \varepsilon$ if $n > 1/\varepsilon$.

Hence if we choose m to be any natural number such that $m > 1/\varepsilon$ then

$|1/n - 0| < \varepsilon$ for all $n \geq m$.

Therefore $\lim_{n\to\infty} 1/n = 0$.

Note 1.2.4 If $\varepsilon = 1/100$, then m can be chosen to be any natural number greater than 100. In this example the choice of m depends on the given ε and $[1/\varepsilon] + 1$ is the smallest value of m that satisfies the requirements of definition.

Note 1.2.5 The constant sequence 1,1,1,........ converges to 1.

Proof. Let $\varepsilon > 0$ be given.

Let the given sequence be denoted by (a_n).

Then $a_n = 1$ for all n.

Space for hints

$a_n - 1 | = 1 - 1 = 0 < \varepsilon$ for all $n \in N$.

$| a_n - 1 | < \varepsilon$ for all $n \geq m$ where m can be chosen to be any natural number.

Therefore $\lim_{n \to \infty} a_n = 1$.

Note 1.2.6 In this example, the choice of m does not depend on the given ε.

3. $\lim_{n \to \infty} \frac{n+1}{n} = 1$.

Proof. Let $\varepsilon > 0$ be given.

Now, $\frac{n+1}{n} - 1 = 1 + \frac{1}{n} - 1 = |1/n|$.

If we choose m to be any natural number greater than $1/\varepsilon$

we have, $\frac{n+1}{n} - 1 | < \varepsilon$ for all $n \geq m$.

Therefore, $\lim_{n \to \infty} \frac{n+1}{n} = 1$.

4. $\lim_{n \to \infty} 1/2^n = 0$.

Proof. Let $\varepsilon > 0$ be given.

Then $|1/2^n - 0| = 1/2^n < 1/n$ (since $2^n > n$ for all $n \in N$).

$| 1/2^n - 0 | < \varepsilon$ for all $n \geq m$ where m is any natural number greater than $1/\varepsilon$

Therefore $\lim_{n \to \infty} 1/2^n = 0$.

5. The sequence $((-1)^n)$ is not convergent.

Proof. Suppose the sequence $((-1)^n)$ converges to l.

Then, given $\varepsilon > 0$, there exists a natural number m such that $| (-1)^n - l < \varepsilon$ for all $n \geq m$.

$| (-1)^m - (-1)^{m+1} | = (-1)^m - l + l - (-1)^{m+1}|$

Space for hints

$$\le |(-1)^m - l| + |(-1)^{m+1} - l|$$

$$< \varepsilon + \varepsilon = 2\varepsilon.$$

But $|(-1)^m - (-1)^{m+1}| = 2$.

$2 < 2\varepsilon$ i.e., $1 < \varepsilon$ which is a contradiction since $\varepsilon > 0$ is arbitrary.

The sequence $((-1)^n)$ is not convergent.

1.3 DIVERGENT SEQUENCES

Definition 1.3.1

A sequence (a_n) is said to ***diverge*** to ∞ if given any real number $k > 0$, there exists $m \in N$ such that $a_n > k$ for all $n \ge m$. In symbols we write $(a_n) \to \infty$ or $\lim_{n\to\infty} a_n = \infty$.

Note 1.3.2 $(a_n) \to \infty$ iff given any real number k . 0 there exists $m \in N$ such that $a_n \in (k,\infty)$ for all $n \ge m$.

Examples 1.3.3

1. $(n) \to \infty$.

Proof. Let $k > 0$ be any given real number.

Choose m to be any natural number such that $m > k$.

Then $n > k$ for all $n \ge m$.

$(n) \to \infty$.

2. $(n^2) \to \infty$.

Proof. Let $k > 0$ be any given real number.

Choose m to be any natural number such that $m > \sqrt{k}$.

Then $n > k$ for all $n \ge m$.

$(n^2) \to \infty$.

3. $(2^n) \to \infty$.

Proof. Let $k > 0$ be any given real number.

Space for hints

Then $2^n > k \Leftrightarrow n \log 2 > \log k$.

$$\Leftrightarrow n > (\log k)/\log 2$$

Hence if we choose m to be any natural number such that $m > (\log k)/\log 2$, then $2^n > k$ for all $n \geq m$.

$(2^n) \to \infty$.

Definition 1.3.4

A sequence (a_n) is said to ***diverge*** to $-\infty$ if given any real number $k < 0$ there exists $m \in N$ such that $a_n < k$ for all $n \geq m$. In symbols we write $\lim_{n\to\infty} a_n = -\infty$ or $(a_n) \to -\infty$.

Note 1.3.5 $(a_n) \to -\infty$ iff given any real number $k < 0$, there exists $m \in N$ such that $a_n \in (-\infty, k)$ for all $n \geq m$.

A sequence (a_n) is said to be divergent if either $(a_n) \to \infty$ or $(a_n) \to -\infty$.

1.4 CAUCHY SEQUENCES

Definition 1.4.1

A sequence (a_n) is said to be a ***Cauchy sequence*** if given $\varepsilon > 0$, there exists

$n_0 \in N$ such that $|a_n - a_m| < \varepsilon$ for all $n,m \geq n_0$.

Note 1.4.2 In the above definition the condition $|a_n - a_m| < \varepsilon$ for all $n, m \geq n_0$ can be written in the following equivalent form, namely,

$|a_{n+p} - a_n| < \varepsilon$ for all $n \geq n_0$ and for all positive integers p.

Examples 1.4.3

1. The sequence $(1/n)$ is a Cauchy sequence.

Proof. Let $(a_n) = (1/n)$. Let $\varepsilon > 0$ be given.

Now $|a_n - a_m| = |1/n - 1/m|$.

Therefore if we choose n_0 to be any positive integer greater than $1/\varepsilon$, we get $|a_n - a_m| < \varepsilon$ for all $n,m \geq n_0$.

Therefore $(1/n)$ is a Cauchy sequence.

Space for hints

2. The sequence ($(-1)^n$) is not a Cauchy sequence.

Proof. Let $(a_n) = ((-1)^n)$.

Therefore $| a_n - a_{n+1}| = 2$.

Therefore If $\varepsilon < 2$, we cannot find n0 such that $| a_n - a_{n+1}| < \varepsilon$ for all $n,m \geq n_0$

Therefore ($(-1)^n$) is not a Cauchy sequence.

3.(n) is not a Cauchy sequence.

Proof. Let $(a_n) = (n)$.

Therefore $| a_n - a_m| \geq 1$ if $n \neq m$.

Therefore if we choose $\varepsilon < 1$, we cannot find n_0 such that$| a_n - a_m| < \varepsilon$ for all $n,m \geq n_0$

Therefore (n) is not a Cauchy sequence.

1.5 INTRODUTION OF COUNTABLE AND UN-COUNTABLE SETS

COUNTABLE SETS

If a set A is finite, then we can count the number of elements in A.In other words , we can label the elements of a by using the natural numbers 1,2,3,…,n for some n and the number of elements in a is n.In this case, there exists a bijection f from A onto the set {1,2,3,…,n}. Hence if A and B are two finite sets having the same number of elements, then there exists a bijection from a to B.

Definition 1.5.1

Two sets A and B are said to be ***equivalent*** if there exists a bijection f from A to B.

Note 1.5.2 From what we have seen above, two finite setsA and B are equivalent iff they have the same number of elements. Hence a finite set cannot be equivalent to a proper subset of itself. However an infinite set can be equivalent to a proper subsets as seen in the following examples.

Example 1.5.3

Let $A = \mathbf{N}$ and $B = \{2,4,6,\ldots,2n,\ldots\}$.

Space for hints

Then f: A→ B defined by $f(n) = 2n$ is a bijection. Hence A is equivalent to B even though A has actually 'more' elements than B.

Example 1.5.4

N is equivalent to **Z.**

The function f:**N**→ **Z** defined by

f(n) = n/2 if n is even and f(n) = (1-n)/2 if n is odd.

Then f is a bijection. Hence **N** is equivalent to **Z.**

Definition 1.5.5

A set A is said to be ***countably infinite*** if A is equivalent to the set of natural numbers **N.**

A is said to be ***countable*** if it is finite or countably infinite.

Note 1.5.6 Let A be a countably infinite set. Then there is a bijection f from **N** to A. Let $f(1) = a_1, f(2) = a_2, \ldots\ldots, f(n) = a_n, \ldots\ldots$

Then $A = \{a_1, a_2, \ldots, a_n, \ldots\}$.

Thus all the elements of A can be labelled by using the elements of **N**

Example: {2,4,6,....,2n,...} is a countable set.

Example 1.5.7 **Z** is countable

Example 1.5.8 Let A = {1/2,2/3,3/4,.........}.

The function f: **N**→ A defined by $f(n)$ = n/(n+1) is bijection.

Hence A is countable.

Theorem 1.5.9 A subset of a countable set is countable.

Proof. Let A be a countable set and let $B \subseteq A$.

If A or B is finite, then obviously B is countable.

Hence let A and B be both infinite.

Since A is countably infinite, we can write $A = \{a_1, a_2, \ldots, a_n,$

Let a_{n_1} be the first element in A such that $a_n \in B$.

Let a_{n_2} be the first element in A which follows a_n such that $a \in B$

Space for hints

Proceeding like this we get B = $\{a_{n_1}, a_{n_2}, \ldots\ldots\}$. Thus all the elements of B can be labelled by using the elements of **N**. Hence B is countable.

Theorem 1.5.10 $\mathbf{Q}^+$ is countable.

Proof. Take all positive rational numbers whose numerator and denominator add up to 2.

We have only one number namely 1/1.

Next take all positive rational numbers whose numerator and denominator add up to 3.

We have 1/2 and 2/1.

Next take all positive rational numbers whose numerator and denominator add up to 4.

We have 1/3,2/2 and 3/1.

Proceedings like this, we can list all the positive rational numbers together

from the beginning omitting those which are already listed.

Thus we obtain the set {1,1/2,2,3,1/3,1/4,2/3,3/2,4,............}. This set c

ontains every positive rational number each occurring exactly once.

Thus $\mathbf{Q}^+$ is countable.

Theorem 1.5.11 **Q** is countable.

Proof. We know that $\mathbf{Q}^+$ is countable.

Let $Q^+ = \{r_1, r_2, \ldots, r_n, \ldots\}$.

Therefore $Q = \{0, r_1, r_2, \ldots, r_n, \ldots\}$.

Let f: N $\to$Q be defined by $f(1) = 0, f(2n) = r_n$ and $f(2n+1) = -r_n$.

Clearly f is a bijection and hence **Q** is countable.

Theorem 1.5.12

$N \times N$ is countable.

Proof.

Space for hints

$N \times N = \{ (a,b) / a,b \in N\}$

Take all ordered pairs (a,b) such that a + b = 2.

There is only one such pair namely (1,1).

Next take all ordered pairs (a,b) such that a + b = 3.

We have (1,2) and (2,1).

Next take all ordered pairs (a,b) such that a + b = 4.

We have (3, 1), (2,2) and (1,3).

Proceedings like this and listing all the ordered pairs together from the beginning,

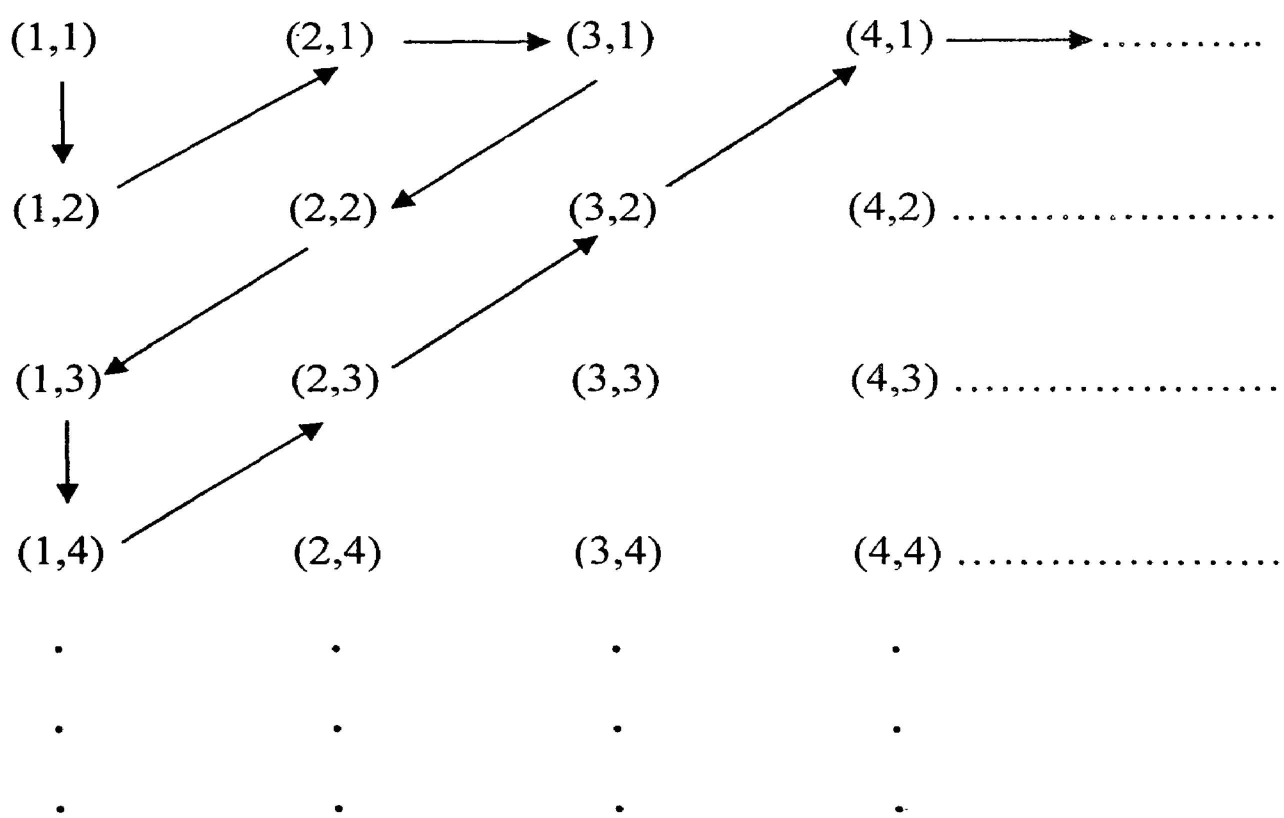
we get the set { (1,1), (1,2), (2,1), (3,1), (2,2) , (1,3) ,.......}.

This set contains every ordered pair belonging to N × N exactly once.

Thus N × N is countable.

Note 1.5.13

The above process of arranging the elements of N× N as a sequence can be represented by means of diagram as follows. This process is known as **Cantor's diagonalisation process.**



Theorem 1.5.14

If A and B are countable sets then A × B is also countable.

Space for hints

Proof.

We assume that A and B are countably infinite.

Let A = { $a_1, a_2, \ldots a_n, \ldots$ }; B = { $b_1, b_2, \ldots, b_n, \ldots$ }

Now define f: N× N →A × B by f (i, j) = (a_i, b_j)

We claim that f is a bijection.Suppose x = (p,q) ∊ N × N and y = (u,v) ∊ N× N.

Now f(x) = f(y) => (a_p, b_q) =(a_u, b_v)

=> $a_p = a_u$, $b_q = b_v$.

=>p = u and q = v

=>(p,q) = (u,v)

=>x=y

Therefore f is 1 -1.

Now suppose (a_m, a_n) ∊ A × B

Then (m, n) ∊ N × N and f(m,n) =(a_m, a_n).

Therefore f is onto. Hence f is a bijection

Hence A× B is equivalent to N × N which is countable.

Theorem 1.5.15

Let A be a countably infinite set and f be a mapping of A onto a set B.

Then B is countable.

Proof.

Let A be a countably infinite set and f: A→ B be an onto map.

Let b ∊B. Since f is onto, there exists at least one pre – image for b.

Choose one element a∊ A such that f(a) = B.

Now, define g: B ⟶ A by g(b) = a.

Clearly g is 1-1.

Therefore B is equivalent to a subset of the countable set A.

Therefore B is countable. (by Theorem 1.5.9)

Space for hints

Theorem 1.5.16

Countable union of countable sets is countable.

***Proof*.** Let S = { $A_1, A_2, \ldots\ldots A_n, \ldots$ }be a countable family of countable sets.

Case 1

Let each A_i be countably infinite.

Let $A_1 = \{ a_{11}, a_{12}, \ldots\ldots a_{1n}, \ldots \}$

$A_2 = \{ a_{21}, a_{22}, \ldots\ldots a_{2n}, \ldots \}$

....

....

$A_n = \{ a_{n1}, a_{n2}, \ldots\ldots a_{nn}, \ldots \}$

....

....

Now we define a map f: $N \times N \rightarrow \bigcup A_n$ by $f(i,j) = a_{ij}$.

Clearly f is onto.

Also by Theorem 1.5.12, $N \times N$ is countably infinite.

Hence by Theorem 1.5.16 , $\bigcup A_n$ is countably infinite.

Case 2

Let each A_i is countable.

For each i , choose a set B_i such that B_i is a countably infinite set and $A_i \subseteq B_i$.

Then $\bigcup A_i \subseteq \bigcup B_i$.

Now, $\bigcup B_i$ is countable (by case 1))

Therefore $\bigcup A_i$ is countable.(By Theorem 1.5.9)

PROBLEMS

Problem 1.5.17

Any countably infinite set is equivalent to a proper subset of itself.

Space for hints

Solution.

Let A be countably infinite set.

Hence $A = \{ a_1, a_2, \ldots\ldots a_n, \ldots \}$

$B = \{ a_2, a_3, \ldots\ldots a_n, \ldots \}$

Clearly B is a proper subset of A.

Define a map $f: A \longrightarrow B$ by $f(a_n) = a_{n+1}$.

Clearly f is a bijection. Hence A is equivalent to B.

Problem 1.5.18

Any infinite set contains a countably infinite subset.

Solution.

Let A be an infinite set.

Choose any element $a_1 \in A$.

Now , since A is infinite set, we can choose another elements

$a_2 \in A - \{ a_1 \}$.

Now, suppose we have chosen $a_1, a_2, \ldots\ldots a_n$ from A.

Since A is infinite , $A - \{ a_1, a_2, \ldots\ldots a_n \}$ is also infinite.

Therefore we can choose a_{n+1} from $A - \{ a_1, a_2, \ldots\ldots a_n \}$.

Now, $B = \{ a_1, a_2, \ldots\ldots a_n, a_{n+1}, \ldots\ldots \}$ is countably infinite subset of A.

Problem 1.5.19

Any inifinte set is equivalent to a proper subset of itself.

Solution.

Let A be an infinite set.

By above problem, A contains a countably infinite subset $B = \{ a_1, a_2, \ldots\ldots a_n \ldots\ldots \}$.

Clearly $A = (A - B) \cup B$.

Now consider the following subset C of A given by

$C = (A - B) \cup \{ a_1, a_2, \ldots\ldots a_n \ldots\ldots \} = A - \{ a_1 \}$.

Space for hints

Clearly C is proper subset of A.

Consider the function f: A $\rightarrow$ C defined by f(x) = x if $x \in A - B$ and $f(a_n) = a_{n+1}$.

Obviously f is a bijection. Hence A is equivalent to C.

Exercises 1.5.20

1. Let A = {1,2,3,....n,.....} and B = { 1,4,9, ...n^2,.....} . Show that A and B are equivalent.

2. Show that N and A = {101,102,103,.......} are equivalent.

3. Show that f: [0,1] $\rightarrow$ [a,b] defined by f(x) = a+ (b-a) x is a bijection. Hence deduce that any two closed intervals, [a,b] and [c,d] are equivalent.

4. Show that for any two sets A and B, the set $A \times B$ is equivalent to the set $B \times A$.

5. Prove that the set of all even integers is countably inifite.

UNCOUNTABLE SETS

Definition 1.5.21

A set which is not countable is called ***uncountable.***

All the infinite sets we have considered in the previous section are countable.

We shall now give an example of an uncountable set.

Theorem 1.5.22

(0,1] is uncountable.

Proof.

Every real number in (0,1] can be written uniquely as a non – terminating decimal $0.a_1a_2, \ldots\ldots a_n \ldots$ Where $0 \le a_i \le 9$ for each i subject to the following restriction that any terminating decimal. $a_1a_2\ldots \ldots a_n\, 000\ldots$ is written as $.a_1a_2a_3\ldots(a_n-1)999\ldots$
For example. .54 = .53999.....

1 = .999...

Space for hints

Suppose (0,1] is countable.

Then the elements of (0,1] can be listed as $\{x_1, x_2 \ldots \ldots x_n, \ldots\ldots)$

where $x_1 = .a_{11}\, a_{12}, \ldots\ldots a_{1n} \ldots$

$x_2 = .\ a_{21}\, a_{22}, \ldots\ldots a_{2n} \ldots$

....

....

$x_n = .\ a_{n1}\, a_{n2}, \ldots\ldots a_{nn} \ldots\}$

....

....

Now , for each positive integer n choose an integer b_n such that $0 \leq b_n \leq 9$ and $b_n \neq 0$ and $b_n \neq a_{nn}$.

Let $y = .\ b_1\, b_2\, b_3 \ldots$

Clearly $y \in (0,1]$.

Also y is different form each x_i for each I which is contradicicition.

Hence (0,1] is countable.

Corollary 1.5.23

Any subset A of R which contains (0,1] is uncountable.

Proof.

Suppose A is countable.

Therefore by Theorem 1.5.9, any subset of A is countable.

Hence we get (0,1] is countable which is contradicition.

Therefore A is uncountable.

Corollary 1.5.24

R is uncountable.

Proof.

The results follows directly by taking A =R.

Corollary 1.5.25

The set S of irrational numbers is uncountable.

Proof.

Suppose S is countable.

We know that Q is countable.

Therefore $S \cup Q = R$ is countable which is a contradiction ,

by Theorem 1.5.16.

Therefore S is uncountable.

Exercises1.5.26

1. Prove that C is uncountable.

2. Prove that the set of all irrational numbers lying in the interval (o,1] is uncountable.

3. Prove that any interval in R which contains more than one point is uncountable.

INEQUALITIES OF HOLDER AND MINKOWSKI

Theorem 1.5.27(Holder's Inequality) If p > 1 and q is such that 1/p +1/q =1 ,then

$$\sum_{i=1}^{n} |a_i b_i| \leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^{n} |b_i|^q\right]^{\frac{1}{q}}$$ where $a_1,a_2,\ldots\ldots a_n$ and $b_1,b_2,\ldots\ldots b_n$ are real numbers.

Proof.

First we shall prove the inequality

$x^{1/p} y^{1/q} \leq x/p + y/q$ where $x \geq 0$ and $y \geq 0$.

This inequality is trivial if $x = 0$ or $y = 0$.

Now , let $x,y > 0$.

Consider $f(t) = t^{\lambda} - \lambda t + \lambda - 1$ where $\lambda = 1/p$ and $t \geq 0$.

Then $f'(t) =- \lambda t^{\lambda-1} - \lambda = \lambda(t^{\lambda-1} - 1)$.

Space for hints

Space for hints

Therefore f(1) = f '(1) = 0 .

Also f ' (t) > 0 for 0< t< 1 and f'(t) < 0 for t > 1.

Therefore $f(t) \leq 0$ for all $t \geq 0$ and in particular $f(x/y) \leq 0$.

Therefore $(x/y)^{\lambda} - \lambda (x/y) + \lambda - 1 \leq 0$.

Therefore $(x/y)^{1/p} - 1/p\,(x/y) + 1/p - 1 \leq 0$.

Multiplying by y ,we get $x^{1/p}\, y^{(1-1/p)} - x/p - (1 - 1/p)y \leq 0$.

Therefore $x^{1/p}\, y^{(1-1/p)} - x/p - y/q \leq 0$. (Since $1 - 1/p = 1/q$).

Therefore $x^{1/p}\, y^{1/q} \leq x/p + y/q$.

Now to prove Holder's inequality, we apply the above inequality to the numbers

$$x_j = \frac{|a_j|^p}{\sum_{i=1}^{n} |a_i|^p}\,;\; y_j = \frac{|b_j|^q}{\sum_{i=1}^{n} |b_i|^q} \quad \text{for each } j = 1,2,\ldots,n.$$

We get $$\frac{|a_j|^p}{[\sum_{i=1}^{n} |a_i|^p]^{\frac{1}{p}}} \frac{|b_j|^q}{[\sum_{i=1}^{n} |b_i|^q]^{\frac{1}{q}}} \leq \frac{x_j}{p} + \frac{y_j}{q} \quad \text{for all } j = 1,2,\ldots,n$$

Adding these n inequalities we get

$$\frac{\sum_{i=1}^{n} |a_i|}{[\sum_{i=1}^{n} |a_i|^p]^{\frac{1}{p}}} \frac{|b_i|}{[\sum_{i=1}^{n} |b_i|^q]^{\frac{1}{q}}} \leq \sum_{j=1}^{n} \left(\frac{x_j}{p} + \frac{y_j}{q}\right) \rightarrow (1)$$

$$\sum_{j=1}^{n} \left(\frac{x_j}{p} + \frac{y_j}{q}\right) = \frac{1}{p} \sum_{j=1}^{n} x_j + \frac{1}{q} \sum_{j=1}^{n} y_j$$

$$= 1/p + 1/q \qquad \left(\text{since } \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j = 1\right)$$

Using this in (1) we get ,

Space for hints

$$\sum_{i=1}^{n}|a_i||b_i| \le \left[\sum_{i=1}^{n} |a_i|^p\right]^{\frac{1}{p}}\left[\sum_{i=1}^{n} |b_i|^q\right]^{\frac{1}{q}}$$

Therefore $\sum_{i=1}^{n}|a_i b_i| \le \left[\sum_{i=1}^{n} |a_i|^p\right]^{\frac{1}{p}}\left[\sum_{i=1}^{n} |b_i|^q\right]^{\frac{1}{q}}$

Note 1.5.28

If we put p= q = 2 in Holder's inequality we get the following inequality which is known as Cauchy – Schwarz inequality.

$$\sum_{i=1}^{n}|a_i b_i| \le \left[\sum_{i=1}^{n} |a_i|2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n} |b_i|^2\right]^{\frac{1}{2}}$$

Theorem: 1.5.29 (Minkowski's inequality)

If $p \ge 1$. $[\sum_{i=1}^{n}|a_i + b_i|^p]^{\frac{1}{p}} \le \left[\sum_{i=1}^{n} |a_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} |b_i|^p\right]^{\frac{1}{p}}$ where , $a_1, a_2\ldots \ldots\ldots a_n$ and $b_1, b_2\ldots \ldots\ldots b_n$ are real numbers.

Proof.

This inequlity is trivial when p =1. Let p>1.

Clearly, $[\sum_{i=1}^{n}|a_i + b_i|^p]^{\frac{1}{p}} \le [\sum_{i=1}^{n}(|a_i| + |b_i|)^p]^{\frac{1}{p}} \to (1)$

$$Now, \sum_{i=1}^{n}[|a_i| + |b_i|]^p = \sum_{i=1}^{n}[|a_i| + |b_i|]^{p-1}(|a_i| + |b_i|)$$

$$= \sum_{i=1}^{n}|a_i|(|a_i| + |b_i|)^{p-1} + \sum_{i=1}^{n} |b_i|(|a_i| + |b_i|)^{p-1}$$

$$\le [\sum_{i=1}^{n}|a_i|^p]^{\frac{1}{p}}[\sum_{i=1}^{n} |a_i| + |b_i|)^{(p-1)q}]^{\frac{1}{q}} + [\sum_{i=1}^{n} |b_i|^p]^{\frac{1}{p}}[\sum_{i=}^{n} (|a_i| + |b_i|)^{(p-1)q}]^{\frac{1}{q}}$$

Where 1/p +1/q = 1 (using Holder's inequality)

Now, since 1/p +1/q = 1 we have p + q = pq

Space for hints

Hence (p-1) q = p.

Therefore dividing by

$$[\sum_{i=1}^{n}(|a_i|+|b_i|)^p]^{\frac{1}{q}}$$,we get

$$[\sum_{i=1}^{n}(|a_i|+|b_i|)^p]^{1-\frac{1}{q}} \le [\sum_{i=1}^{n}|a_i|^p]^{\frac{1}{p}} + [\sum_{i=1}^{n}|b_i|^p]^{\frac{1}{p}}$$

$$\therefore \quad [\sum_{i=1}^{n}(|a_i|+|b_i|)^p]^{\frac{1}{p}} \le [\sum_{i=1}^{n}|a_i|^p]^{\frac{1}{p}} + [\sum_{i=1}^{n}|b_i|^p]^{\frac{1}{p}} \quad \rightarrow(2)$$

From (1) and (2) we get the required inequality.

1.6 METRIC SPACES

The concept of convergence of sequences of real numbers depends on the absolute value of the difference between any two real numbers. We observe that this absolute value is nothing but the distance between the two numbers when they are considered as points on the real line.In this section, we define the concept of metric spaces and provide many examples.

Definition 1.6.1

Let M be a non- empty set. Let d: M × M → **R** be a function.

d is called a ***metric*** on M if

(i) $d(x,y) \ge 0$, $\forall$ x,y ∈ M

(ii) $d(x,y) = 0 \Leftrightarrow x = y$, $\forall$ x,y ∈ M

(iii) $d(x,y) = d(y,x)$, $\forall$ x,y ∈ M (Symmetry)

(iv) $d(x,z) \le d(x,y) + d(y,z)$, $\forall$ x,y,z ∈ M (Triangle inequality).

d is also called a distance function. The set M together with a metric d is called a ***Metric Space***. We denote a Metric Space by (M,d).

Example 1.6.2

In **R**, we define $d(x,y) = |x-y|$. Then d is a Metric on **R**.

This is called the ***usual Metric*** on **R**.

Proof. (i) $d(x,y) = |x-y| \geq 0$, $\forall x,y \in \mathbf{R}$.

(ii) $d(x,y) = 0 \Leftrightarrow |x-y| = 0$

$\Leftrightarrow x = y$

Thus $d(x,y) = 0 \Leftrightarrow x = y$, $\forall x,y \in \mathbf{R}$.

(iii) $d(x,y) = |x - y|$

$= |y - x|$

$= d(y,x)$

Thus $d(x,y) = d(y,x)$, $\forall x,y \in \mathbf{R}$.

(iv) let $x,y,z \in \mathbf{R}$

$d(x,z) = |x-z|$

$= |x-y+y-z|$

$\leq |x-y| + |y-z|$

$= d(x,y) + d(y,z)$

Therefore, $d(x,z) \leq d(x,y) + d(y,z)$ $\forall x,y,z \in \mathbf{R}$.

Therefore d is a Metric on **R**.

Hence (**R**,d) is a Metric Space.

Note 1.6.3

Whenever we consider R as a Metric Space,

the underlying metric is taken to be the usual metric unless otherwise stated.

Example 1.6.4

In $\mathbb{C}$, we define $d(z,w) = |z-w|$. Then d is a metric on $\mathbb{C}$.

This is called the ***usual metric*** on $\mathbb{C}$.

Proof. Let $z = x+iy$ and $w = u+iv$ be two complex numbers.

(i) $d(z,w) = |z-w|$.

$= |(x+iy) - (u+iv)|$

Space for hints

Space for hints

$= |(x-u) + i(y-v)|$

$= \sqrt{(x-u)^2+(y-v)^2}$

$\geq 0.$

Therefore, $d(z,w) \geq 0,\ \forall\ z,w \in \mathbb{C}$.

(ii) $d(z,w) = 0 \Leftrightarrow |z-w| = 0$

$\Leftrightarrow z = w$

$d(z,w) = 0 \Leftrightarrow z = w,\ \forall\ z,w \in \mathbb{C}$.

(iii) $d(z,w) = |z-w|$.

$= |w - z|$

$= d(w,z)$.

Therefore, $d(z,w) = d(w,z),\ \forall\ z,w \in \mathbb{C}$.

(iv) Let $x,y,z \in \mathbb{C}$.

Let $x = x_1+ix_2,\ y = y_1+iy_2,\ z = z_1+iz_2$

$d(x,z) = |x - z|$

$= |(x_1+ix_2) - (z_1+iz_2)|$

$= |(x_1+ix_2) - (y_1+iy_2) + (y_1+iy_2) + (z_1+iz_2)|$

$\leq |(x_1+ix_2) - (y_1+iy_2)| + |(y_1+iy_2) - (z_1+iz_2)|$

$= |x - y| + |y - z|$

$= d(x,y) + d(y,z)$

Therefore, $d(x,z) \leq d(x,y) + d(y,z),\ \forall\ x,y,z \in \mathbb{C}$.

$(\mathbb{C},d)$ is a metric space.

Example 1.6.5 On any non-empty set M ,we define d as follows.

$$d(x,y) = 0 \text{ if } x = y \text{ and } d(x,y) = 1 \text{ if } x \neq y.$$

Then d is a metric on M. This is called the d***iscrete metric*** on M.

Proof. Let M be any non – empty set.

We define d as follows

Space for hints

$d(x,y) = 0$ if $x = y$ and $d(x,y) = 1$ if $x \neq y$.

(i) $d(x,y)$ is either 0 or 1, from the definition

Therefore, $d(x,y) \geq 0$, $\forall\, x,y \in M$.

(ii) $d(x,y) = 0 \Leftrightarrow x = y$, from the definition.

(iii) From the definition of d, $d(x,y) = d(y,x)$, $\forall\, x,y \in M$.

(iv) case (i): $x = z$.

Then $d(x,z) = 0$.

Also, $d(x,y) + d(y,z) \geq 0$

Therefore, $d(x,y) + d(y,z) \geq d(x,z)$

Thus $d(x,z) \leq d(x,y) + d(y,z)$, $\forall\, x,y \in M$.

Case (ii): $x \neq z$.

Then $d(x,z) = 1$

Suppose $y = x$ and $y = z$

Then $x = z$, a contradiction.

Therefore $x \neq y$ or $y \neq z$.

Hence $d(x,z) \leq d(x,y) + d(y,z)$, $\forall\, x,y \in M$.

Therefore, d is a Metric on M.

Minkowski's Inequality 1.6.6

$$\text{If } p \geq 1,\ [\sum_{i=1}^{n} |a_i+b_i|^p]^{1/p} \leq [\sum_{i=1}^{n} |a_i|^p]^{1/p} + [\sum_{i=1}^{n} |b_i|^p]^{1/p},$$

where $a_1,a_2,\ldots\ldots,a_n$ and $b_1,b_2,\ldots\ldots,b_n$ are real numbers.

Example 1.6.7

In $\mathbf{R}^n$ define $d(x,y) = [\sum_{i=1}^{n} (x_i - y_i)^2]^{1/2}$, where $x = (x_1,x_2,\ldots\ldots,x_n)$ and $y = (y_1,y_2,\ldots\ldots,y_n)$.

Then d is a metric on $\mathbf{R}^n$. This is called the ***usual metric on R^n***.

Proof.

Space for hints

(i) $d(x,y) = [\sum_{i=1}^{n} (x_i - y_i)^2]^{1/2} \geq 0$, since $(x_i - y_i)^2 \geq 0$.

(ii) $d(x,y) = 0 \Leftrightarrow [\sum_{i=1}^{n} (x_i - y_i)^2]^{1/2} = 0.$

$\Leftrightarrow (x_i - y_i)^2 = 0, \ \forall \ i = 1,2,\ldots\ldots,n.$

$\Leftrightarrow x_i = y_i , \ \forall \ i = 1,2.\ldots\ldots,n.$

$\Leftrightarrow (x_1,x_2,\ldots.x_n) = (y_1,y_2,\ldots\ldots\ldots y_n)$,

$\Leftrightarrow x = y , \ \forall \ i = 1,2,\ldots\ldots,n.$

Hence $d(x,y) = 0 \Leftrightarrow x = y , \ \forall \ x,y \in \mathbf{R^n}$

(iii) $d(x,y) = [\sum_{i=1}^{n} (x_i - y_i)^2]^{1/2}$

$= [\sum_{i=1}^{n} (y_i - x_i)^2]^{1/2}$

$= d(y,x)$

Thus $d(x,y) = d(y,x) , \forall \ x,y \in \mathbf{R^n}$

(iv) Let $a_i = x_i - y_i$ and $b_i = y_i - z_i$ and $p=2$ in Minkowski's inequality

Then we get

$$[\sum_{i=1}^{n} (x_i - z_i)^2]^{1/2} \leq [\sum_{i=1}^{n} (x_i - y_i)^2]^{1/2} + [\sum_{i=1}^{n} (y_i - z_i)^2]^{1/2}$$

$\Rightarrow d(x,z) \leq d(x,y) + d(y,z)$
since x, y, z are arbitrary,

$d(x,z) \leq d(x,y) + d(y,z) \ \forall \ x,y \in \mathbf{R^n}$

Therefore d is a Metric on $\mathbf{R^n}$.

Note 1.6.8

$\mathbf{R^n}$ with usual Metric is called ***the n-dimensional Euclidean Space.***

Example 1.6.9

Let $x,y \in R^2$. Then $x = (x_1,x_2)$ and $y = (y_1,y_2)$, where $x_1,y_1,x_2,y_2 \in R$.

We define $d(x,y) = | x_1 - y_1 | + x_2 - y_2 |$. Then d is a metric on $\mathbf{R^2}$.

Proof.

(i) $d(x,y) = | x_1 - y_1 | + | x_2 - y_2 | \geq 0$.

(ii) $d(x,y) = 0 \Leftrightarrow | x_1 - y_1 | + | x_2 - y_2 | = 0$.

$\Leftrightarrow | x_1 - y_1 | = 0$ and $| x_2 - y_2 | = 0$.

$\Leftrightarrow x_1 = y_1$ and $x_2 = y_2$

$\Leftrightarrow (x_1 , x_2) = (y_1, y_2)$

$\Leftrightarrow x = y$

Therefore $d(x,y) = 0 \Leftrightarrow x = y \quad \forall \; x,y \in \mathbf{R}^2$.

(iii) $d(x,y) = | x_1 - y_1 | + | x_2 - y_2 |$

$= | y_1 - x_1 | + | y_2 - x_2 |$

$= d(y,x)$

$d(x,y) = d(y,x), \quad \forall \; x,y \in \mathbf{R}^2$

(iv) Let $x,y, z \in \mathbf{R}^2$

$d(x,z) =| x_1 - z_1 | + | x_2 - z_2 |$

$=| x_1 - y_1 + y_1 - z_1 | + | x_2 - y_2 + y_2 - z_2 |$

$\leq \{| x_1 - y_1 | +| y_1 - z_1 |\}+\{| x_2 - y_2 |+ |y_2 - z_2|\}$

$= \{| x_1 - y_1 |+ | x_2 - y_2 | \}+\{|y_1 - z_1|+ |y_2 - z_2|\}$

$= d(x,y) + d(y,z)$

Since $x,y, z \in \mathbf{R}^2$ are arbitrary,

$d(x,z) \leq d(x,y) + d(y,z) \quad , \forall \quad x,y, z \in \mathbf{R}^2$

Hence d is a Metric on $\mathbf{R}^2$.

Example 1.6.10

Consider $\mathbf{R}^n$. Let $p > 1$. We define $d(x,y) = [\sum_{i=1}^{n} (x_i - y_i)^p]^{1/p}$

where $x = (x_1,x_2,\ldots x_n)$, $y = (y_1,y_2,\ldots\ldots y_n)$. Then d is a Metric on $\mathbf{R}^n$.

Space for hints

Space for hints

Proof.

(i) $d(x,y) = [\sum_{i=1}^{n} (x_i - y_i)^p]^{1/p} \geq 0$, since $|x_i - y_i| \geq 0$

(ii) $d(x,y) = 0 \Leftrightarrow [\sum_{i=1}^{n} (x_i - y_i)^p]^{1/p} = 0$.

$\Leftrightarrow |x_i - y_i|^p = 0, \ \forall\ i = 1,2,\ldots\ldots,n.$

$\Leftrightarrow |x_i - y_i| = 0, \ \forall\ i = 1,2,\ldots\ldots,n.$

$\Leftrightarrow x_i = y_i, \ \forall\ i = 1,2,\ldots\ldots,n.$

$\Leftrightarrow (x_1,x_2,\ldots x_n) = (y_1,y_2,\ldots\ldots\ldots y_n)$.

$\Leftrightarrow x = y$

Therefore $d(x,y) = 0 \Leftrightarrow x = y, \ \forall\ x,y \in \mathbf{R}^n$.

(iii) $d(x,y) = [\sum_{i=1}^{n}$ [illegible] $]^{1/p}$

$= [\sum_{i=1}^{n} (y_i - x_i)^p]^{1/p}$

$= d(y,x)$

Thus $d(x,y) = d(y,x), \ \forall\ x,y \in \mathbf{R}^n$.

(iv) Let $a_i = x_i - y_i$ and $b_i = y_i - z_i$ in Minkowski's inequality,

We get

$$[\sum_{i=1}^{n} (x_i - z_i)^p]^{1/p} = [\sum_{i=1}^{n} (x_i - y_i + y_i - z_i)^p]^{1/p}$$

$$\leq [\sum_{i=1}^{n} (x_i - y_i)^p]^{1/p} + [\sum_{i=1}^{n} (y_i - z_i)^p]^{1/p}$$

Thus $d(x,z) \leq d(x,y) + d(y,z)$.

Since x,y,z are arbitrary, $d(x,z) \leq d(x,y) + d(y,z), \ \forall\ x,y,z \in \mathbf{R}^n$.

Therefore d is a Metric on $\mathbf{R}^n$.

Example 1.6.10

In $\mathbf{R}^n$, we define $d(x,y) = \max\{|x_i - y_i|, i=1,2,\ldots,n\}$, where $x = (x_1,x_2,\ldots.x_n)$, $y = (y_1,y_2,\ldots\ldots\ldots y_n)$. Then d is a Metric on $\mathbf{R}^n$.

Proof.

(i) $d(x,y) = \max \{| x_1 - y_1 |, | x_2 - y_2 |,\ldots\ldots,| x_n - y_n |\} \geq 0$.

(ii) $d(x,y) = 0 \Leftrightarrow \max \{| x_1 - y_1 |, | x_2 - y_2 |,\ldots\ldots,| x_n - y_n |\} = 0$.

$\Leftrightarrow x_i - y_i = 0$ for all $i= 1,2,\ldots..,n$.

$\Leftrightarrow x_i = y_i$ for all $i= 1,2,\ldots..,n$.

$\Leftrightarrow (x_1,x_2,\ldots.x_n) = (y_1,y_2,\ldots\ldots\ldots y_n)$

$\Leftrightarrow x = y$

(iii) $d(x,y) = \max \{ | x_i - y_i | \}$

$= \max \{ | y_i - x_i | \}$

$= d(y,x)$

(iv) Now, let $x,y,z \in R^n$. Since each $x_i,y_i,z_i \in \mathbf{R}$

We have $| x_i - z_i | = | x_i - y_i + y_i - z_i|$

$=| x_i - y_i| + | y_i - z_i |$ for all $i= 1,2,\ldots..,n$.

Therefore $\max| x_i - z_i| \leq \max| x_i - y_i | + \max| y_i - z_i|$

Therefore $d(x,z) \leq d(x,y) + d(y,z)$.

Hence d is metric on $\mathbf{R}^n$.

Example 1.6.11

Let $p \geq 1$. Let l_p denote the set of all sequences (x_n) such that

$\sum_1^{\infty} |x_n|^p$ is convergent. Define $d(x,y) = [\sum_{n=1}^{\infty} (x_n - y_n)^p]^{1/p}$

where $x = (x_n)$ and $y = (y_n)$. Then d is a Metric on l_p.

Proof. Let $a,b \in l_p$.

First we prove d(a,b) is a real number.

By Minkowski's inequality we have ,

$$[\sum_{i=1}^{n} |a_i+b_i|^p]^{1/p} \leq [\sum_{i=1}^{n} |a_i|^p]^{1/p} + [\sum_{i=1}^{n} |b_i|^p]^{1/p} \ldots\ldots(1)$$

Since $a,b \in l_p$ the right hand side of (1) has a finite limit as $n\to\infty$.

Space for hints

Space for hints

Therefore $(\sum_{i=1}^{\infty} |a_i+b_i|^p]^{1/p})$ is convergent series.

Similarly we can prove that $(\sum_{i=1}^{\infty} |a_i-b_i|^p]^{1/p})$ is also

a convergent series and hence d(a,b) is a real number.

Now, taking limit as $n\to\infty$ in (1) we get

$$[\sum_{i=1}^{\infty} |a_i+b_i|^p]^{1/p} \leq [\sum_{i=1}^{\infty} |a_i|^p]^{1/p} + [\sum_{i=1}^{\infty} |b_i|^p]^{1/p} \quad \text{..........(2)}$$

Obviously $d(x,y) \geq 0$,

$d(x,y) = 0$ iff $x = y$

and $d(x,y) = d(y,x)$

Now, let $x,y,z \in l_p$.

Taking, $a_i = x_i - y_i$ and $b_i - y_i - z_i$ in (2) we get

$$[\sum_{i=1}^{\infty} |x_i-z_i|^p]^{1/p} \leq [\sum_{i=1}^{\infty} |x_i- y_i|^p]^{1/p} + [\sum_{i=1}^{\infty} y_i-z_i|^p]^{1/p}$$

Therefore $d(x,z) \leq d(x,y) + d(y,z)$.

Hence d is a metric on l_p.

Note 1.6.12

In particular, l_2 is a metric space with the metric defined by

$$d(x,y) = [\sum_{n=1}^{\infty} (x_n - y_n)^2]^{1/2}.$$

Example 1.6.13

Let M be the set of all bounded real valued functions defined on a non-empty set E.

Define $d(f,g) = \sup \{ | f(x) - g(x) | / x \in E \}$.Then d is a metric on M.

Proof.

(i) $d(f,g) = \sup \{ | f(x) - g(x) | \} \geq 0$.

(ii) $d(f,g) = 0 \Leftrightarrow \sup \{ | f(x) - g(x) | \} \geq 0$.

$\Leftrightarrow | f(x) - g(x) | = 0$ for all $x \in E$.

$\Leftrightarrow f(x) = g(x)$ for all $x \in E$.

$\Leftrightarrow f = g$.

(iii) Also, $d(f,g) = \sup \{ | f(x) - g(x) | \}$

$= \sup \{ | g(x) - f(x) | \}$

$= d(g,f)$.

(iv) Now, let $f,g,h \in M$.

We have $| f(x) - h(x) | \leq | f(x) - g(x) | + | g(x) - h(x) |$

Therefore $\sup \{ | f(x) - h(x) | \} \leq \sup \{ | f(x) - g(x) | \} + \sup \{ | g(x) - h(x) | \}$

Therefore $d(f,h) \leq d(f,g) + d(g,h)$

Hence d is a metric on M.

Example 1.6.14

Let M be the set of all sequences in **R**. Let $x,y \in M$ and

let $x = (x_n)$ and $y = (y_n)$. Define $d(x,y) = \sum_{n=1}^{\infty} \frac{| x_n - y_n |}{2^n (1+ | x_n - y_n |)}$

Then d is a metric on M.

Proof. Let $x,y \in M$. First we prove that

(i) $d(x,y)$ is a real number ≥ 0.

We have $\frac{| x_n - y_n |}{2^n (1+ | x_n - y_n |)} \leq 1/2^n$ for all n.

Also $\sum_{n=1}^{\infty} 1/2^n$ is a convergent series.

[**Comparison Test: 1.**Let $\sum c_n$ be a convergent series of positive terms.

Space for hints

Let $\sum$ a_n be another series of positive terms. If there exists m∊ **N** such that

$a_n \leq c_n$ for all $n \geq m$, then $\sum$ a_n is also convergent.

2. Let $\sum$ d_n be a divergent series of positive terms.

Let $\sum$ a_n be another series of positive terms. If there exists m∊ **N** such that

$a_n \geq d_n$ for all $n \geq m$, then $\sum$ a_n is also divergent.]

Therefore $\sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)}$ is a convergent series. (by comparison test)

Therefore d(x,y) is a real number and $d(x,y) \geq 0$.

(ii) $d(x,y) = 0 \Leftrightarrow \frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)} = 0$

$\Leftrightarrow |x_n - y_n| = 0$ for all n.

$\Leftrightarrow x_n = y_n$ for all n.

$\Leftrightarrow x = y.$

(iii) Also, $d(x,y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)}$

$= \sum_{n=1}^{\infty} \frac{|y_n - x_n|}{2^n(1+|y_n - x_n|)}$

$= d(y,x).$

(iv) Now, let x,y,z ∊ M. Then

$$\frac{|x_n - y_n|}{(1+|x_n - y_n|)} = \frac{(1+|x_n - y_n|)-1}{1+|x_n - y_n|}$$

$$= 1 - \frac{1}{1+|x_n - z_n|} \leq 1 - \frac{1}{1+|x_n - y_n|+|y_n - z_n|}$$

$$= \frac{|x_n - y_n|+|y_n - z_n|}{1+|x_n - y_n|+|y_n - z_n|}$$

Space for hints

$$= \frac{|x_n - y_n|}{1+|x_n - y_n|+|y_n - z_n|} + \frac{|y_n - z_n|}{1+|x_n - y_n|+|y_n - z_n|}$$

$$\leq \frac{|x_n - y_n|}{1+|x_n - y_n|} + \frac{|y_n - z_n|}{1++|y_n - z_n|}$$

Multiplying both sides of this inequality by $\frac{1}{2^n}$ and taking the sum from n = 1 to ∞ we get

$$\sum_{n=1}^{\infty} \frac{|x_n - z_n|}{2^n(1+|x_n - z_n|)}$$

$$\leq \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)} + \sum_{n=1}^{\infty} \frac{|x_n - z_n|}{2^n(1+|y_n - z_n|)}$$

Therefore $d(x,z) \leq d(x,y) + d(y,z)$. This is true for all $x,y,z \in M$.

Therefore d is a metric on M.

Example 1.6.15

Let l^{∞} denote the set of all bounded sequences of real numbers. Let $x = (x_n)$ and $y= (y_n) \in l^{\infty}$ define d on l^{∞} as $d(x,y) = \text{lub}|x_n - y_n|$. Then d is a metric on l^{∞}.

Proof.

(i) $d(x,y) = \text{lub}|x_n - y_n| \geq 0$

(ii) $d(x,y) = 0 \Leftrightarrow \text{lub}|x_n - y_n| = 0$

$\Leftrightarrow |x_n - y_n| = 0$ for $1 \leq n < \infty$

$\Leftrightarrow x_n = y_n$ for $1 \leq n < \infty$

$\Leftrightarrow (x_n) = (y_n)$

$\Leftrightarrow x = y$

(iii) Now, $d(x,y) = \text{lub}|x_n - y_n|$

$= \text{lub}|y_n - x_n|$

$= d(y,x)$.

Space for hints

(iv) Let $z = (z_n) \in l^{\infty}$.

Now $| x_n - z_n | = | x_n - y_n + y_n - z_n |$

$\leq | x_n - y_n | + | y_n - z_n$

$\leq \text{lub} | x_n - y_n | + \text{lub} | y_n - z_n |$

$= d(x,y) + d(y,z)$.

Therefore Lub $| x_n - z_n | \leq d(x,y) + d(y,z)$.

Therefore $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in l^{\infty}$.

Therefore d is a metric on l^{∞}.

PROBLEMS

Problem 1.6.16

Let d_1 and d_2 be two metrics on a set M. Define $d(x,y)=d_1(x,y)+d_2(x,y)$.

Prove that d is a metric on M.

Solution.

Let d_1 and d_2 be two metrics on a set M.

Define $d(x,y)=d_1(x,y)+d_2(x,y)$.

(i) Since d_1 and d_2 are metrics on M, $d_1(x,y)\geq 0$ and $d_2(x,y)\geq 0$ for all $x,y\in M$.

This implies that $d(x,y)\geq 0$.

Since x and y are arbitrary, $d(x,y)\geq 0$ for all $x, y\in M$.

(ii) $d(x,y) = 0 \Leftrightarrow d_1(x,y)+d_2(x,y)=0$.

$\Leftrightarrow d_1(x,y)=0$ and $d_2(x,y)=0$.

$\Leftrightarrow x = y$, since d_1 and d_2 are metrics on M.

This is true for all x and y, since x and y are arbitrary.

Hence $d(x,y) = 0 \Leftrightarrow x = y$ for all $x,y \in M$.

(iii) $d(x,y) = d_1(x,y)+d_2(x,y)$.

$= d_1(y,x)+d_2(y,x)$, since d_1 and d_2 are metrics on M.

$=d(y,x)$.

This is true for all x and y, since x and y are arbitrary.

Hence $d(x,y) = d(y,x)$ for all $x,y \in M$.

(iv) Let $x,y,z \in M$.

Since d_1 is a metric on M, $d_1(x,z) \leq d_1(x,y) + d_1(y,z)$ for all $x,y,z \in M$.

Since d_2 is a metric on M, $d_2(x,z) \leq d_2(x,y)+d_2(y,z)$ for all $x,y,z \in M$.

$d_1(x,z)+ d_2(x,z))\leq d_1(x,y)+d_1(y,z)+ d_2(x,y)+d_2(y,z)$

This implies that $d(x,z) \leq d(x,y)+d(y,z)$.

This is true for all $x,y,z \in M$, since x,y,z are arbitrary.

Hence $d(x,z) \leq d(x,y)+d(y,z)$ for all $x,y,z \in M$.

Thus d is a metric on M.

Problem 1.6.17

Determine whether d(x,y) defined on R by $d(x,y) = (x-y)^2$ is a metric or not.

Solution.

Let $x,y \in \mathbf{R}$.

(i) $d(x,y) = (x-y)^2 \geq 0, \ \forall x,y \in \mathbf{R}$.

(ii) $d(x,y) = 0 \Leftrightarrow (x-y)^2 = 0$

$\Leftrightarrow x = y$.

(iii) $d(x,y) = (x - y)^2$

$= (y - x)^2$

$= d(y,x)$.

Therefore, $d(x,y) = d(y,x), \ \forall x,y \in \mathbf{R}$.

(iv) Let $x = -5$, $y = -4$ and $z = 4$

$d(x,z) = d(-5,4)$

Space for hints

Space for hints

$= (-5 - 4)^2$

$= 81.$

$d(x,y) = d(-5,-4)$

$= (-5 + 4)^2$

$= 1.$

$d(y,z) = d(-4,4)$

$=(-4 - 4)^2$

$= 64.$

Therefore triangle inequality does not hold.

Therefore d is not a metric on **R.**

Problem 1.6.18

If d is a metric on M, is d^2 a metric on M ?

Solution.

Consider d(x,y) defined on **R** by $d(x,y) = | x - y |$.

d is a metric on **R** , by example 1.6.2.

$d^2(x,y) = | x - y |^2$

$= (x - y)^2$

d^2 is not a metric ,by problem 1.6.17.

Problem 1.6.19

If d is a metric on M, prove that $\sqrt{d}$ is a metric on M.

Solution.

Let x,y, z ∊ M

Let d be a metric on M.

(i) Since d is a metric $d(x,y) \geq 0$.

Therefore $\sqrt{d(x,y)} \geq 0$.

(ii) Since d is a metric on M,

$d (x,y) = d(y,x)$, $\forall$ x,y ∊**M.**

Space for hints

therefore $\sqrt{d(x,y)} = \sqrt{d(y,x)}$, $\forall$ x,y $\in$**M.**

(iii) $\sqrt{d(x,y)} = 0 \Leftrightarrow$ d(x,y) =0

$\Leftrightarrow$ x=y ($\therefore$ d is a metric on M).

$\therefore \sqrt{d(x,y)} = 0 \Leftrightarrow$ x=y, $\forall$ x,y $\in$**M.**

(iv) Since d is a metric , d(x,z) $\leq$ d(x,y) + d(y,z).

$\sqrt{d(x,y)} \leq \sqrt{d(x,y)} + d(y,z)$

$\leq \sqrt{d(x,y)} + \sqrt{d(y,z)}$ $(\because \sqrt{a+b} \leq \sqrt{a} + \sqrt{b})$

Hence $\sqrt{d(x,z)} \leq \sqrt{d(x,y)} + \sqrt{d(y,z)}$

Hence $\sqrt{d}$ is a metric on M.

Problem 1.6.20

Let (M,d)be a metric space. Define $d_1(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$.

Prove that d_1 is a metric on M.

Solution.

Let (M,d) be a metric space.

Define $d_1(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$.

(i) Since d is a metric, d(x,y) $\geq$ 0 for all x,y $\in$ M

$$\therefore \frac{d(x,y)}{1+d(x,y)} \geq 0$$

Therefore $d_1(x,y)$) $\geq$ 0.for all x,y $\in$M

$$(ii)\ d_1(x,y) = 0 \Leftrightarrow \frac{d(x,y)}{1+d(x,y)} = 0$$
$$\Leftrightarrow d(x,y) = 0$$
$$\Leftrightarrow x = y \quad (\because \quad d \text{ is a metric on } M.)$$

Space for hints

$$(iii)\ d_1(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} \quad (\because\ d \text{ is a metric })$$

$$\Leftrightarrow\ d_1(y,x)$$

Hence $d_1(x,y) = d_1(y,x),\ \forall\, x,y \in M$

(iv) let x,y,z∈M

$$d_1(x,z) = \frac{d(x,z)}{1+d(x,z)} =$$

$$= \frac{1+d(x,z)-1}{1+d(x,z)}$$

$$\Leftrightarrow \frac{1+d(x,z)}{1+d(x,z)} - \frac{1}{1+d(x,z)}$$

$$= 1 - \frac{1}{1+d(x,z)}$$

$$\leq 1 - \frac{1}{1+d(x,y)+d(y,z)}$$

$$= \frac{1+d(x,y)+d(y,z)-1}{1+d(x,y)+d(y,z)}$$

$$= \frac{d(x,y)+d(y,z)}{1+d(,y)+d(y,z)}$$

$$= \frac{d(x,y)}{1+d(x,y)+d(y,z)} + \frac{d(y,z)}{1+d(x,y)+d(y,z)}$$

$$\leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}$$

$$= d_1(x,y) + d_1(y,z)$$

$$\therefore\ d_1(x,z) \leq d_1(x,y) + d_1(y,z) \quad \forall x,y,z \in M.$$

Therefore d_1 is a metric on M.

Problem 1.6.21

Let (M,d) be a metric space. Define $d_1(x,y) = \min\{1, d(x,y)\}$.

Prove that d_1 is a metric on M.

Space for hints

Solution.

(i) $d_1(x,y) = \min\{1, d(x,y)\} \geq 0$.

$$\therefore d_1(x,y) \geq 0, \forall x, y \in M$$

(ii) $d_1(x,y) = 0 \Leftrightarrow \min\{1, d(x,y)\} = 0$.

$\Leftrightarrow d(x,y) = 0$.

$\Leftrightarrow x = y$ (since d is a metric on M).

Therefore $d_1(x,y) = 0 \Leftrightarrow x = y \ \forall x, y \in M$.

(iii) $d_1(x,y) = \min\{1, d(x,y)\}$

$= \min\{1, d(y,x)\}$ (since d is a metric on M).

$= d_1(y,x)$

$d_1(x,y) = d_1(y,x)$, $\forall x, y \in M$.

(iv) Now, let $x, y, z \in M$.

Then $d_1(x,z) = \min\{1, d(x,z)\} \leq 1$.

If $d_1(x,y) = 1$ or $d_1(y,z) = 1$, then obviously $d_1(x,z) \leq d_1(x,y) + d_1(y,z)$

Let $d_1(x,y) < 1$ and $d_1(y,z) < 1$. Then

$d_1(x,y) + d_1(y,z) = \min\{1, d(x,y)\} + \min\{1, d(y,z)\}$

$= d(x,y) + d(y,z)$

$\geq d(x,z)$

$\geq \min\{1, d(x,z)\}$

$= d_1(x,z)$

$d_1(x,z) \leq d_1(x,y) + d_1(y,z)$, $\forall x, y, z \in M$.

Therefore d_1 is a metric on M.

Problem 1.6.22

Let M be a non empty set. Let $d: M \times M \to R$ be a function such that

(1) $d(x,y) = 0$ iff $x = y$

(2) $d(x,y) \leq d(x,z) + d(y,z)$ for all $x, y, z \in M$

Space for hints

Prove that d is a metric on M.

Solution.

Let M be a non empty set. Let $d: M \times M \to R$ be a function such that

(1) $d(x,y) = 0$ iff $x = y$

(2) $d(x,y) \le d(x,z) + d(y,z)$ for all $x,y,z \in M$

(i) Put $y = x$ in (2)

We have $d(x,x) \le d(x,z) + d(x,z)$

This implies that $0 \le 2d(x,z)$ (by (1))

Therefore $d(x,z) \ge 0$. $\forall$ x, z $\in$M.

(ii) Put $z = x$ in (2)

We have $d(x,y) \le d(x,x) + d(y,x) = 0 + d(y,x)$ (by (1))

$d(x,y) \le d(y,x)$, $\forall$ x, y $\in$M.

since this is true for all x, y $\in$ M ,we have

$d(y,x)$, $\le d(x,y)$.

Hence $d(x,y) = d(y,x)$.

(iii) Now (2) can be written as

$d(x,y) \le d(x,z) + d(z,y)$ (since $d(x,y) = d(y,x)$).

which is the triangle inequality.

Therefore d is a metric on M.

Problem 1.6.23

If $(M_1, d_1), (M_2, d_2), \ldots\ldots (M_n, d_n)$ are metric spaces,

then $M_1 \times M_2 \times \ldots\ldots \times M_n$ is a metric space with metric d defined by

$$d(x,y) = \sum_{i=1}^{n} d_i(x_i, y_i) \quad \text{where } x = (x_1, x_2, \ldots x_n),\ y = (y_1, y_2, \ldots\ldots y_n)$$

Solution.

$$d(x,y) = \sum_{i=1}^{n} d_i(x_i, y_i)$$

Space for hints

(i) $d(x,y) = \sum_{i=1}^{n} d_i(x_i, y_i) \geq 0, \forall x, y \in M$

(ii) $d(x,y) = 0 \Leftrightarrow \sum_{i=1}^{n} d_i(x_i, y_i) = 0$

$\Leftrightarrow d_i(x_i, y_i) = 0$ for all i = 1,2,……,n.

$\Leftrightarrow x_i = y_i$ for all i = 1,2,……,n.

$\Leftrightarrow (x_1,x_2,\ldots.x_n) = (y_1,y_2,\ldots\ldots\ldots y_n)$

$\Leftrightarrow x = y, \ \forall x, y \in M.$

(iii) $d(x,y) = \sum_{i=1}^{n} d_i(x_i, y_i)$

$= \sum_{i=1}^{n} d_i(y_i, x_i)$

$= d(y,x)$

Therefore $d(x,y) = d(y,x)$, $\forall x, y \in M$.

(iv) Let $\forall x, y, z \in M$.

Then $d(x,z) = \sum_{i=1}^{n} d_i(x_i, z_i)$

$\leq \sum_{i=1}^{n} [d_i(x_i, y_i) + d_i(y_i, z_i)]$

$= \sum_{i=1}^{n} d_i(x_i, y_i) + \sum_{i=1}^{n} d_i(y_i, z_i)$

$= d(x,y) + d(y,z)$

Therefore $d(x,z) = d(x,y) + d(y,z)$, $\forall x, y, z \in M$.

Hence d is a metric on M.

Problem 1.6.24

Let M = { a, b, c }.We define d on M as follows :

d(a,b) = d(b,a) = 3 : d(b,c) = d(c,b) =4

d(c,a) = d(a,c) = 5 and d(a,a) = d(b,b) = d(c,c) = a .

Space for hints

Prove that d is a metric on M.

Solution.

(i) $d(a,b) = 3 \geq 0$; $d(b,c) \geq 4$; $d(c,a) = 5 \geq 0$

(ii) $d(a,b) = .0$,$\Leftrightarrow$ $a = b$

(iii) $d(a,b) = d(b,a) = 3$; $d(b,c) = d(c,b) = 4$

$d(c,a) = d(a,c) = 5$ (given)

(iv) $d(a,b) = 3$

$d(b,c) = 4$

Therefore $d(a,b) + d(b,c) = 7$

$\Rightarrow 5 < 7$

Therefore $d(a,c) < d(a,b) + d(b,c)$

Therefore d is a metric on M.

Problem 1.6.25

If d is a metric on M, prove that (i) 2d is a metric on M.

(ii) nd is a metric on M where $n \in N$.

Solution.

(i) Let (M,d) be a metric space.

Since d is metric on M, we have $d(x,y) \geq 0, \ \forall\, x, y \in M$.

$d(x,y) \geq 0, \ \forall\, x, y \in M$.

Therefore $2d(x,y) \geq 0, \ \forall\, x, y \in M$.

(ii) $d(x,y) = 0 \Leftrightarrow x = y$ (since d is a metric on M)

Therefore $2d(x,y) = 0, \Leftrightarrow x = y$

(iii) since d *is metric on* M,

$d(x,y) = d(y,x), \quad \forall\, x, y \in M$.

Therefore $2d(x,y) = 2d(y,x), \quad \forall\, x, y \in M$.

(iv) since d is metric on M,

$d(x,z) \leq d(x,y) + d(y,z), \quad \forall\, x, y, z \in M.$

Therefore $2d(x,z) \leq 2d(x,y) + 2d(y,z), \quad \forall\, x, y, z \in M.$

Therefore 2d is a metric on M.

(ii) (i)since d is metric on M,

$d(x,y) \geq 0, \ \forall\, x, y \in M.$

Therefore $n\, d(x,y) \geq 0, \ \forall\, x, y \in M.$

ii) $d(x,y) = 0 \Leftrightarrow x = y$ (since d is a metric on M)

Therefore $nd(x,y) = 0 \Leftrightarrow x = y, \ \forall\, x, y \in M.$

iii) $d(x,y) = d(y,x)$ (since d is a metric on M)

Therefore $nd(x,y) = nd(y,x), \ \forall\, x, y \in M.$

iv) $d(x,z) \leq d(x,y) + d(y,z), \ \forall\, x, y, z \in M.$ (since d is a metric on M)

Therefore $nd(x,z) \leq nd(x,y) + nd(y,z), \ \forall\, x, y, z \in M.$

Therefore nd is a metric on M.

Space for hints

Space for hints

UNIT – 2

The real number system has two types of properties. The first type are algebraic properties, dealing with addition , multiplication and so on. The other type ,called topological properties,have to do with the notion of distance between numbers and with the concept of limit.There are special types of sets that play a distinguished role in analysis. These are the open sets and closed sets that are in the discussion of continuity.

2. 1 OPEN SETS AND CLOSED SETS

Definition 2.1.1

Let (M,d) be a metric space. Let A be a subset of M. A is called a ***bounded set*** in M

if there exists a positive real number k such that $d(x,y) \leq k, \forall x, y \in A$.

Example 2.1.2

Any finite subset of a metric space (M,d) is bounded.

Proof.

Case 1. $A = \phi$.

In this case A is obviously bounded.

Case 2. $A \neq \phi$

Since A is finite ,$\{ d(x,y): x,y \in A\}$ is a finite set of real numbers.

Let $k = \max \{ d(x,y): x,y \in A\}$

Then $d(x,y) \leq k, \forall x, y \in A$.

Therefore A is bounded.

Example 2.1.3

In R with usual metric , [2,5] and [0,1] are bounded sets.

Proof. $d(x,y) \leq 3, \forall x, y \in [2,5]$.

$d(x,y) \leq 1, \forall x, y \in [0,1]$

Therefore [2,5] and [0,1] are bounded sets.

Example 2.1.4

In R with usual metric, $(0,\infty)$ is unbounded.

Example 2.1.5

In R with discrete metric, $(0,\infty)$ is a bounded subset of R.

Proof. In a discrete metric space M, either $d(x,y) = 0$ (or) 1, $\forall$ x, y $\in$M.

Therefore $d(x,y) \leq 1$, $\forall$ x, y $\in$ $(0,\infty)$

Therefore $(0,\infty)$ is a bounded subset of R.

Note 2.1.6

Any subset of a discrete metric space M is abounded subset of M.

Definition 2.1.7

Let (M,d) be a metric space. Let A be a subset of M. Then the diameter d(A) of is defined by

$d(A) = \text{l.u.b}\{ d(x,y) / x, y \in R\}$.

Note 2.1.8

A non – empty set A in a metric space M is a bounded set iff d(A) is finite.

Note 2.1.9

Let A,B $\subseteq$ M .Then A $\subseteq$ B => $d(A) \leq d(B)$.

Example 2.1.10

The diameter of any non empty subset in a discrete metric space is 1.

Example 2.1.11

In R with usual metric, the diameter of any interval is equal to the length of the interval.

Exercises 2.1.12

Now we find the diameter of the following subsets of R with usual metric.

(i) If $A = \{1,3,5,7,9\}$,then $d(A) = 8$.

(ii) If $A = \{0, 1,2, 3,\ldots\ldots..100\}$,then $d(A) = 100$.

Space for hints

Space for hints

(iii) If $A = [-3,5]$, then $d(A) = 8$.

(iv) If $A = [-1/2, 1/2]$ then $d(A) = 1$.

(v) $d(\mathbf{N}) = \infty$

(vi) $d(\mathbf{Q}) = \infty$

(vi) Let $A = [1,2] \cup [5,6]$. Then $d(A) = 5$.

OPEN BALL (OPEN SPHERE) IN A METRIC SPACE

Definition 2.1.13 Let (M,d) be a Metric space. Let $a \in M$ and let r be a positive

real number. The ***open ball*** (or) ***the open sphere*** with centre 'a' and

radius 'r' is denoted by $B_d(a,r) = \{ x \in M ; d(a,x) < r\}$.

Note 2.1.14

(i) $B_d(a,r)$ is a subset of M.

(ii) $B_d(a,r)$ can be written as $B(a,r)$ when the metric d under consideration is clear.

(iii) $B(a,r)$ is always non empty, since it contains at least its centre a.

(iv) $B(a,r)$ is a bounded set,

Example 2.1.15

Consider $\mathbf{R}$ with usual metric. Let $a \in \mathbf{R}$.

Then $B(a,r) = \{ x \in R : d(a,x) < r\}$.

$= \{ x \in R : |a - x| < r\}$.

$= \{ x \in R : - r < a - x < r\}$.

$= \{ x \in R : -a - r < - x < r - a\}$.

$= \{ x \in R : -(a + r) < - x < - (a - r)\}$.

$= \{ x \in R : (a + r) > x > (a - r)\}$.

$= \{ x \in R : (a - r) < x < (a + r) \}$.

$= (a - r, a + r)$.

Space for hints

Example 2.1.16

In $\mathbf{R}^2$ with usual metric, B(a,r) is the interior of the circle with centre 'a' and radius 'r'.

Example 2.1.17

Let (M,d) be a discrete metric space. Then B(a,r) = M if r>1 and

B(a,r) = {a} if $r \leq 1$.

Proof. Let a $\in$M and let r be any positive real number.

Case 1. r>1

B(a,r) = { x $\in$M : d(a,x) < r}

= M.

Therefore for every x $\in$M, d(a,x) < r.

Hence B(a,r) = M.

Case2. $r \leq 1$

If $x \neq a$,then d(a,x) = 1 for every x $\in$M.

$\Rightarrow$ d(a,x) = 1 $\geq$ r.

$\Rightarrow$ B(a,r) = { x $\in$M : d(a,x) $\leq r \leq 1$ }.

= {a},since $x \notin$ B(a,r).

Examples 2.1.18

1. In **R** with usual metric, B(a,r) = (a-r,a+r).

2. In **R** with usual metric, B(-1,1) = (-2,0).

3. In **R** with usual metric,B(1,1) = (0,2).

4. In [0,1] with usual metric, B(1/2,1) = [0,1].(Prove this using Example 2.1.15)

OPEN SETS IN A METRIC SPACE

Definition 2.1.19 Let (M,d) be a metric space. Let A be a subset of M.

Then A is called ***an open set*** in M if for every x$\in$A, there exists a positive

real number r such that B(x,r) $\subseteq$ A.

Space for hints

Example 2.1.20

Prove that in R with usual metric, $(0,1)$ is an open set.

Proof. Let $x \in (0,1)$.

Define $r = \min\{x-0, 1-x\}$.

Then r is a positive real number.

We know that ,in **R** with usual metric, $B(a,r) = (a-r, a+r)$.

This shows that $B(x,r) = (x-r,x+r) \subseteq (0,1)$.

This is true for all $x \in (0,1)$.Therefore $(0,1)$ is an open subset of **R** with usual metric.

Example 2.1.21

In R with usual metric, $[0,1)$ is not an open set,since no open ball with centre 0 is contained in $[0,1)$.

Example 2.1.21

Let $A = [0,1) \subseteq M = [0,2)$. Prove that A is open in M.

Proof. Let $x \in [0,1)$.

Case 1 $x = 0$.

Then $B(0.1/2) = \{ y \in [0,2) / d(0,y) < ½ \}$

$= \{ y \in [0,2) / |y| < ½ \}$

$= \{ y \in [0,2) / -1/2 < y < ½ \}$

$= \{ y \in [0,2) / y \in (-1/2, ½) \}$

$= [0, ½) \subseteq [0,1)$.

Case 2 $x \neq 0$.

Let $r = \min \{x, 1- x\}$

Clearly $r > 0$.

$B(x.r) = (x-r,x+r) \subseteq [0,1)$.

Since x is arbitrary, for every $x \in A$, there exist a positive real number r such that $B(x,r) \subseteq A = [0,1)$.

Therefore, A is open.

Example 2.1.22

Any open interval (a,b) is an open set in **R** with usual metric.

Proof.

Let (a,b) be an open interval in **R**.

Let $x \in (a,b)$.

Define $r = \min\{x-a, b-x\}$

Clearly $r > 0$.

Then $B(x,r) \subseteq (a,b)$

Therefore, (a,b) is an open set.

Note 2.1.23 In **R** with usual metric $(-\infty, a)$ and (a,∞) are open sets.

Example 2.1.24

In **R** with usual metric, the set {0} is not an open set, since an open ball with centre 0 is not contained in {0}.

Example 2.1.25

In **R** with usual metric, any finite non-empty subset A of **R** is not an open set.

Proof.

Let A be a subset of **R,** where **R** is a metric space with usual metric.

Also, A is finite. Let $y \in A$ and let $r > 0$.

Then $B(y,r)$ is an open interval in **R,** which is an infinite set.

Therefore, $B(y,r)$ is not a subset of A.

Therefore, A is not an open set.

Example 2.1.26

Q is not open in **R** with usual metric.

Space for hints

Space for hints

Proof.

Consider **R** with usual metric.

Let $x \in$ **Q.**

For any $r > 0$, $B(x,r) = (x-r,x+r)$.

This open interval contains both rational and irrational numbers

Thererfore $B(x,r)$ is not a subset of **Q**.

Hence **Q** is not open.

Example 2.1.27

Z is not open in **R**.

Proof.

Let $x \in$ **Z.**

For any $r > 0$, $B(x,r) = (x-r,x+r)$.

This set contain rational and irrational numbers.

This is not a subset of **Z.**

Therefore **Z** is not open.

Example 2.1.28

The set of all *irrational* numbers is not open in **R** with usual metric.

Example 2.1.29

In a discrete metric space M, every subset A is open.

Proof.

If $A = \phi$, then trivially A is open.

Let $A \neq \phi$.

Let $x \in A$.

Then $B(x,r) \subseteq A$ only when $r \leq 1$.

Therefore, for every $x \in A$ we can find $r \leq 1$ such that $B(x,r) \subseteq A$.

Therefore, A is open.

Space for hints

CLOSED SETS IN A METRIC SPACE

Definition 2.1.30 Let (M,d) be a metric space. Let $A \subseteq M$. Then A is said to be ***closed*** in M if the complement of A is open in M.

Example 2.1.31

In **R** with usual metric ,any closed interval [a,b] is a closed set.

Proof.

$[a,b]^c = R - [a,b] = (-\infty,a) \cup (b,\infty)$.

Also $(-\infty,a)$ and (b,∞) are open in **R.**

Therefore $[a,b]^c$ is open in **R.**

Therefore [a,b] is closed in **R.**

Example 2.1.32

In R with usual metric [a,b) is neither closed nor open.

Proof.

[a,b) is not open in **R** since any open ball with centre 'a' is not a proper subset of [a,b).

Now, $[a,b)^c = \mathbf{R} - [a,b) = (-\infty,a) \cup [b,\infty)$.This set is not open since any open ball with centre 'b' is not a proper subset of $[b, \infty)$. Therefore [a,b) is not closed in **R**.

Hence [a,b) is neither open nor closed in **R**.

Definition 2.1.33 Let (M,d) be a metric space and let A be a subset of M. Let $x \in A$. x is called an ***interior point*** of A if there exists a positive real number r such that $B(x,r) \subseteq A$.The set of all interior points of A is denoted by ***Int A***. A point $x \in M$ is called ***a limit point*** or ***a cluster point*** or ***an accumulation point*** of A if every open ball with centre x contains at least one point of A different from x. The set of all limit point of A is denoted by D(A).

Example 2.1.34

In **R** with usual metric, Int((2,3)) = (2,3) and Int([2,3]) = (2,3).

Space for hints

Example 2.1.35

In **R** with usual metric (a,b] is neither closed nor open.

Proof.

(a,b] is not open in **R** since b is not an interior point of (a,b].

Now, $(a,b]^c = \mathbf{R} - (a,b] = (-\infty,a] \cup (b,\infty)$ and this set is not open since a is not an interior point.

Therefore, (a,b] is not closed in **R**.

Hence (a,b] is neither closed nor open in **R**.

Theorem 2.1.36 An arbitrary union of open sets is open.

Example 2.1.37

Z is closed in **R**.

Proof.

$$\mathbf{Z}^c = \bigcup_{n=-\infty}^{\infty} (n,n+1).$$

The open interval (n,n+1) is open and union of open sets is open.

$\mathbf{Z}^c$ is open.

Hence **Z** is closed.

Example 2.1 38

Q is not closed in **R**.

Proof.

$\mathbf{Q}^c$ = the set of irrationals which is not open in **R**.

Therefore, **Q** is not closed in **R.**

Example 2.1.39

The set of irrational numbers is not closed in **R**.

Proof.

The complement of irrationals i.e., the set of rationals is not open in **R.**

Therefore, the set of irrational numbers is not closed in **R**.

Space for hints

Example 2.1.40

In **R** with usual metric ,every singleton set is closed.

Proof.

Let $a \in$ **R**.

Then $\{a\}^c =$ **R**- $\{a\} = (-\infty,a) \cup (a,\infty)$.

Since $(-\infty,a)$ and (a,∞) are both open sets, by Theorem 2.1.36, $(-\infty,a) \cup (a,\infty)$ is open.

Therefore $\{a\}^c$ is open in **R.**

Hence $\{a\}$ is closed in **R.**

Example 2.1.41

Every subset of a discrete metric space is closed.

Proof.

Let (M,d) be a discrete metric space.

Let $A \subseteq M$.

Since every subset of a discrete metric space is open, A^c is open.

Therefore A is closed.

Note 2.1.42

Every subset of a discrete metric space is both open and closed.

Definition 2.1.43 Let (M,d) be a metric space. Let $a \in M$.

Let r be any positive real number.

Then the ***closed ball*** or the ***closed sphere*** with centre a and radius r,

denoted by $B_d[a,r]$,is defined by

$$B_d[a,r]=\{ x \in M \ / \ d(a,x)\leq r\}$$

When the metric under consideration is clear, we write B[a,r] instead of $B_d[a,r]$.

Example 2.1.44

In **R** with usual metric, $B[a,r] = [a-r,a+r]$.

Space for hints

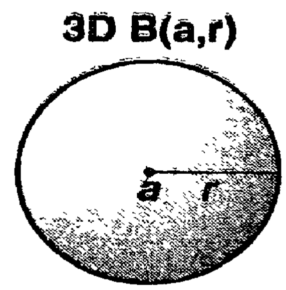
Example 2.1.45

In $\mathbb{R}^2$ with usual metric let $a = (a_1,a_2) \in \mathbb{R}^2$.

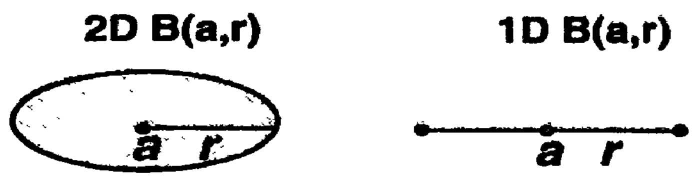
Then $B[a,r] = \{ (x,y) \in \mathbb{R}^2 / d(a_1,a_2)) \leq r \}$.

$$= \{ (x,y) \in \mathbb{R}^2 / (x-a_1)^2 + (y-a_2)^2 \leq r^2 \}.$$

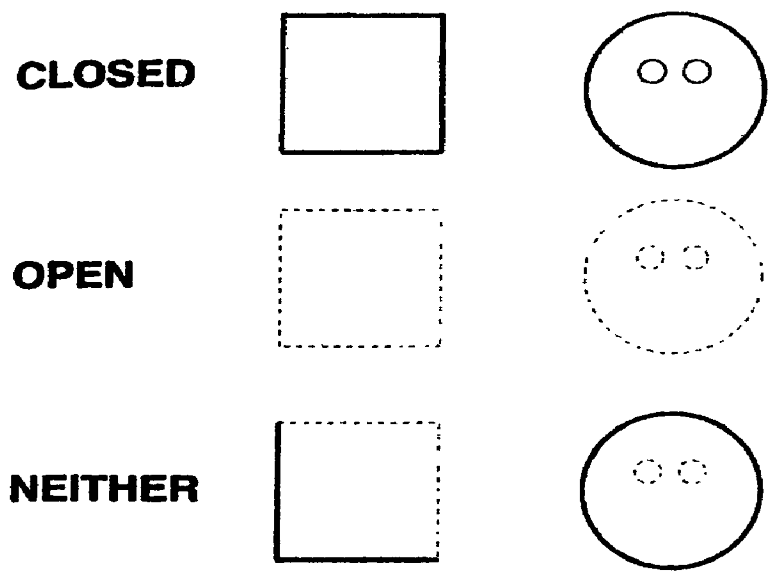
Hence B[a,r] is the set of all points which lie within and on the circumference of the circle with centre *a* and radius *r*.



The boundary of the ball $B(a, r)$ is the sphere. The closed ball includes its boundary.

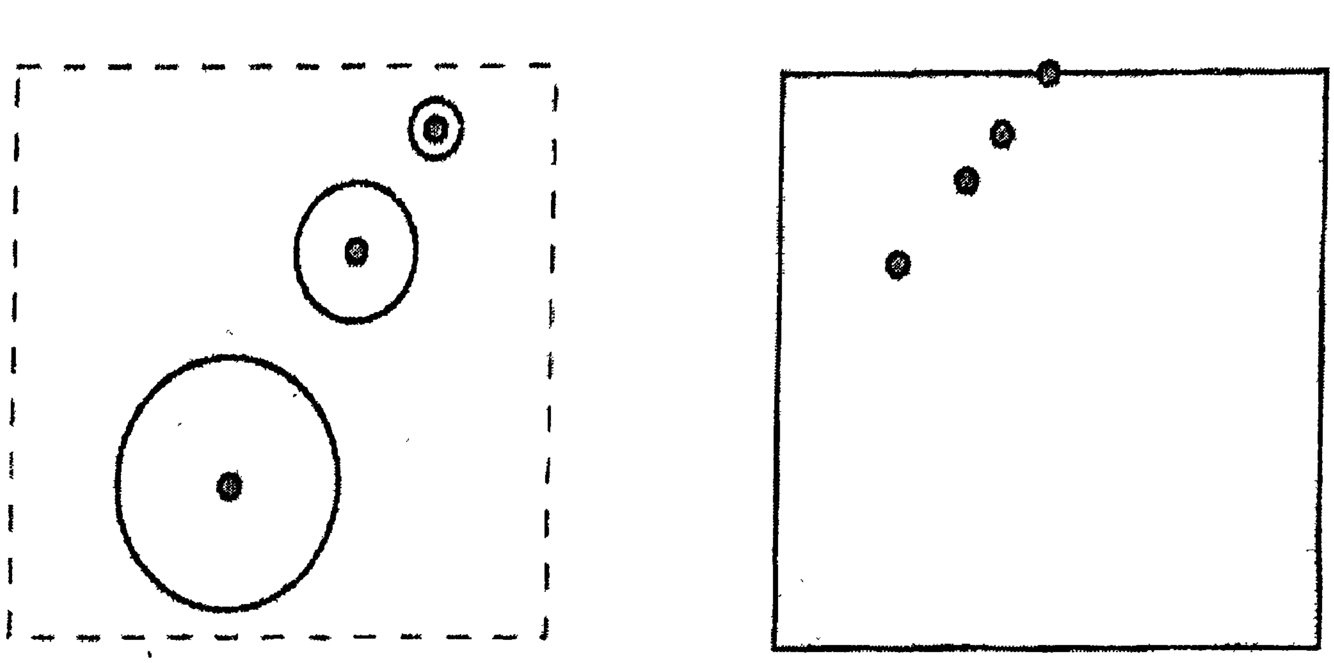


The boundary of a 2D ball or "disc" is a circle. The boundary of a 1D ball or interval is two points.

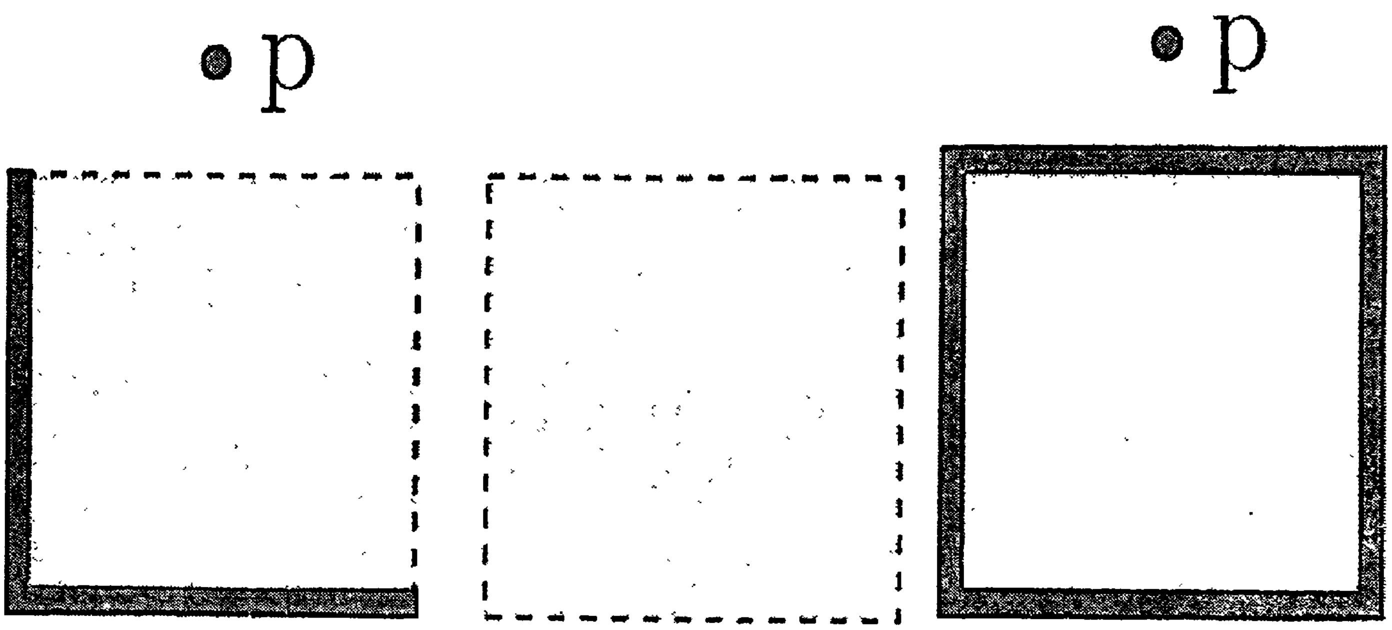


Some sets are closed or open but most are neither

Space for hints



An open set includes a ball about every point. A closed set includes all of its accumulation points.



A set, its interior, its closure, and an isolated point p.

2.2 COMPLETENESS IN METRIC SPACES

Definition 2.2.1

Let (M,d) be a metric space. Let $(x_n) = x_1, x_2, x_3, \ldots, x_n, \ldots$ be a sequence of points in M. Let $x \in M$. We say that (x_n) ***converges to x*** if given $\varepsilon > 0$, there exists a positive integer n_0 such that $d(x_n, x) < \varepsilon$ for all $n \geq n_0$. Also x is called the limit point of the sequence (x_n).

Note 2.2.2

If (x_n) converges to x, we write $\lim_{n\to\infty} x_n = x$ (or) $(x_n) \to x$.

Space for hints

Note 2.2.3

$(x_n) \to x$ iff for each open ball $B(x,\varepsilon)$ with centre x, there exists a p i-tive integer n_0 such that $x_n \in B(x,\varepsilon)$, for all $n \geq n_0$.

Note 2.2.4

$(x_n) \to x$ iff the sequence of real numbers $(d(x_n,x)) \to 0$.

Theorem 2.2.5 For a convergent sequence (x_n), the limit is unique.

Proof.

Let $\varepsilon > 0$ be given.

Suppose there exist two positive integers n_1 and n_2 such that

$$d(x_n,x) < \varepsilon/2, \text{ for all } n \geq n_1 \text{ and}$$

$$d(x_n,y) < \varepsilon/2, \text{ for all } n \geq n_2.$$

Let m be a positive integer such that $m \geq n_1,n_2$.

Then $d(x,y) \leq d(x,x_m) + d(x_m,y)$.

$$< \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon.$$

Therefore, $d(x,y) < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, $d(x,y) = 0$.

Therefore $x = y$.

Hence for a convergent sequence (x_n), the limit is unique.

Note 2.2.6

If $(x_n) \to x$, then x is called the limit of the sequence (x_n).

Theorem 2.2.7 Let M be a metric space and $A \subseteq M$. Then

(i) $x \in \overline{A}$ iff there exists a sequence (x_n) in A such that $(x_n) \to x$.

(ii) x is a limit point of A iff there exists a sequence (x_n) of distinct points in A such that $(x_n) \to x$.

Proof.

Let M be a metric space and $A \subseteq M$.

Space for hints

Let $x \in \overline{A}$.

We know that, for any subset A of a metric space M, $\overline{A} = A \cup D(A)$.

Therefore, $x \in A \cup D(A)$.

Then $x \in A$ or $x \in D(A)$.

Suppose $x \in A$.

Then there is a constant sequence x,x,....,x,.... Converging to x.

Suppose $x \in D(A)$.

We know the following result.

"Let (M,d) be a metric space. Let $A \subseteq M$. Then x is a limit point of A iff each open ball with centre x contains an infinite number of points of A."

By the above result, there exists an open ball B(x,1/n) containing infinite number of points of A.

Therefore, we can choose $x_n \in B(x,1/n) \cap A$ such that $x_n \neq .x_1,x_2,\ldots,x_{n-1}$ for each n.

Therefore, (x_n) is a sequence of distinct points in A.

Also, $d(x_n , x) < 1/ n$ for all n.

$$\Rightarrow \lim_{n\to\infty} d(x_n,x) < \lim_{n\to\infty} 1/n$$

$$\Rightarrow \lim_{n\to\infty} d(x_n,x) < 0 \rightarrow (I)$$

$$d(x_n,x) \geq 0 \quad \forall n.$$

Therefore $\lim_{n\to\infty} d(x_n,x) \geq 0 \rightarrow (2)$

Therefore $\lim_{n\to\infty} d(x_n,x) = 0$

Therefore $(x_n) \rightarrow x$.

Space for hints

Conversely, suppose there exists a sequence (x_n) in A such that $(x_n) \to x$.

Then for any $r > 0$, there exists a positive integer n_0 such that $d(x_n, x) < r$, for all $n \geq n_0$.

$\Rightarrow x_n \in B(x,r)$, for all $n \geq n_0$.

Clearly, $x_n \in A$

Therefore, $x_n \in B(x,r) \cap A$.

Hence $x_n \in B(x,r) \cap A \neq \phi$. ------> (3)

We know the following result.

$x \in \overline{A} \Leftrightarrow B(x,r) \cap A \neq \phi$, for all $r>0$.

By the above result, $x \in \overline{A}$.

From(3), x is a limit point of A.

Therefore $x \in D(A)$.

Definition 2.2.8

Let (M,d) be a metric space. Let (x_n) be a sequence of points in M. (x_n) is

said to be a ***Cauchy sequence*** in M if given $\varepsilon > 0$ there exists positive

integer n_0 such that $d(x_m,x_n) < \varepsilon$ for all $m,n \geq n_0$.

Theorem 2.2.9 Let (M,d) be a metric space. Then any convergent sequence

in M is a Cauchy sequence.

Proof.

Let (M,d) be a metric space.

Let (x_n) be a convergent sequence in M converging to $x \in M$.

Let $\varepsilon > 0$ be given.

Since (x_n) is converging to x, there exists a positive integer n_0 such that

$d(x_n,x) < \varepsilon/2$, for all $n \geq n_0$.

$$d(x_n,x_m) \leq d(x_n,x) + d(x,x_m)$$

$$< \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon$$

Thus $d(x_n,x_m) < \varepsilon$, for all $m,n \geq n_0$.

Hence (x_n) is a Cauchy sequence.

Remark 2.2.10 the converse of the above theorem is not true

Consider the metric space (0,1] with usual metric.

(1/n) is a Cauchy sequence in (0,1].

Definition 2.2.11

A metric space M is said to be ***complete*** if every Cauchy sequence in M

converges to a point in M.

Example 2.2.12

(0,1] with usual metric is not complete, since $(1/n) \rightarrow 0 \notin (0,1]$.

Example 2.2.13

R with usual metric is complete.

Example 2.2.14

C with usual metric is complete.

Proof.

Let (z_n) be a Cauchy sequence in C.

Let $z_n = x_n + iy_n$ where xn,yn ϵ R.

We claim that (x_n) and (y_n) are Cauchy sequence in R.

Let $\varepsilon > 0$ be given.

Since (z_n) is a Cauchy sequence, there exists a positive integer n_0

such that $| z_n - z_m | < \varepsilon$, for all $m,n \geq n_0$.

Now, $|x_n - x_m| \leq |z_n - z_m|$ and $|y_n - y_m| \leq |z_n - z_m|$.

Hence $|x_n - x_m| < \varepsilon$, for all $m,n \geq n_0$.

$|y_n - y_m| < \varepsilon$, for all $m,n \geq n_0$.

Space for hints

Therefore, (x_n) and (y_n) are Cauchy sequences in R.

Since R is complete, there exist $x,y \in R$ such that $(x_n) \to x$ and $(y_n) \to y$.

Let $z = x + iy$.

Claim : $(z_n) \to z$

We have $|z_n - z| = , |(x_n + i\, y_n) - (x + iy)|$

$= |(x_n - x) + i(y_n - y)|$

$\leq |(x_n - x)| + |(y_n - y)|$ ---------------→ (1)

Now let $\varepsilon > 0$ be given.

Since $(x_n) \to x$ and $(y_n) \to y$ there exists positive integers n_1 and n_2

such that $|x_n - x| < \frac{1}{2}\varepsilon$ all $n \geq n_1$ and $|y_n - y| < \frac{1}{2}\varepsilon$ for all $n \geq n_2$.

Let $n_3 = \max\{n_1, n_2\}$

From (1) we get $|z_n - z| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$ for all $n \geq n_3$.

Therefore $(z_n) \to z$.

Therefore C is complete.

Example 2.2.15

Any discrete metric space is complete.

Proof.

Let (M,d) be a discrete metric space.
Let (x_n) be a Cauchy sequence in M. Then there exists a positive

interger n_o such that $d(x_n, x_m) < ½$ for all $n, m \geq n_0$.

Since d is the discrete metric, distance between any two points is either 0 or 1.

Therefore $d(x_n, x_m) = 0$ for all $n, m \geq n_0$.

Therefore $x_n = x_{n0} = x$ (say) for all $n, \geq n_0$.

Therefore $d(x_n, x) = x$, for all $n \geq n_0$.

Space for hints

Therefore $(x_n) \to x$.

Hence M is complete.

Example 2.2.16

$\mathbf{R}^n$ with usual metric is complete.

Proof.

Let (x_p) be a Cauchy sequence in $\mathbf{R}^n$.

Let $x_p = (x_{p1}, \ldots\ldots x_{pn})$

Let $\varepsilon > 0$ be given.

Then there exists a positive integer n_0 such that $d(x_p, x_q) < \varepsilon$ for all $p,q \geq n_0$.

Therefore $[\sum_{k=1}^{n} (x_{p_k} - x_{q_k})^2]^{\frac{1}{2}} < \varepsilon$ for all $p,q \geq n_0$.

Therefore $\sum_{k=1}^{n} (x_{p_k} - x_{q_k})^2 < \varepsilon^2$ for all $p,q \geq n_0$

Therefore for each $k = 1,2\ldots\ldots n$ we have

$| x_{p_k} - x_{q_k}) | < \varepsilon$ for all $p,q \geq n_0$

Therefore (x_{p_k}) is a Cauchy sequence in R for each $k = 1,2\ldots\ldots n$.

Since R is complete, there exists $y_k \in R$ such that $(x_{p_k}) \to y_k$.

Let $y = (y_1, y_2 \ldots\ldots y_n)$. We claim that $(x_p) \to y$

Since $(x_{p_k}) \to y_k$ there exists a positive integer m_k such that

$| x_{p_k} - y_k | < \frac{\varepsilon}{\sqrt{n}}$ *for all* $p \geq m_k$

Let $m_0 = \max\{m_1, m_2, \ldots\ldots\ldots m_n\}$

Then $d(x_p, y) = [\sum_{k=1}^{n} (x_{p_k} - y_k)^2]^{\frac{1}{2}}$

$$< [n(\frac{\varepsilon}{\sqrt{n}})^2]^{\frac{1}{2}} \text{ for all } p \geq m_0$$

Space for hints

$=\epsilon$ for all $p \geq m_0$

Thus $d(x_p,y) < \epsilon$ for all $p \geq m_0$.

Therefore $(x_p) \rightarrow y$ Hence $\mathbf{R}^n$ is complete.

Example 2.2.17

l_2 is complete.

Proof. Let (x_p) be a Cauchy sequence in l_2.

Let $x_p = (x_{p1},......x_{pn},......)$.

Let $\epsilon > 0$ be given.

Then there exists a positive integer n_0 such that $d(x_p,x_q) < \epsilon$ for all p,q $\geq n_0$.

$(ie)\ [\sum_{n=1}^{m}(x_{p_n} - x_{q_n})^2]^{\frac{1}{2}} < \varepsilon$ for all $p,q \geq n_0$.

$\sum_{n=1}^{m}(x_{p_n} - x_{q_n})^2 < \varepsilon^2$ for all $p,q \geq n_0$ $\rightarrow (1)$

For each n = 1,2,3,.... We have

$| x_{p_n} - x_{q_n}) | < \varepsilon$ for all $p,q \geq n_0$

Therefore (x_{p_n}) is a Cauchy sequence in R for each k = 1,2......n.

Since R is a complete, there exists $y_n \in R$ such that $(x_{pn}) \rightarrow y_n . \rightarrow (2)$

Let $y = (y_1, y_2 ..... y_n........)$. We claim that $(x_p) \rightarrow y$

For any fixed positive integer m, we have

$\sum_{n=1}^{m}(x_{p_n} - x_{q_n})^2 < \varepsilon^2$ for all $p,q \geq n_0$ (using (1))

Fixing q and allowing $p \rightarrow \infty$ in this finite sum we get

Space for hints

$$\sum_{n=1}^{m}(y_n - x_{q_n})^2 < \varepsilon^2 \quad \text{for all } q \geq n_0 \quad \text{(using (2))}$$

Since this is true for every positive integer m,

$$\sum_{n=1}^{\infty}(y_n - x_{q_n})^2 < \varepsilon^2 \quad \text{for all } q \geq n_0 \quad \rightarrow(3)$$

$$[\sum_{n=1}^{m}| y_n |^2]^{\frac{1}{2}} = [\sum_{n=1}^{m}| y_n - x_{q_n} + x_{q_n} |^2]^{\frac{1}{2}}$$

$$\leq [\sum_{n=1}^{\infty}| y_n - x_{q_n} |^{2}]^{\frac{1}{2}} + [\sum_{n=1}^{\infty}|x_{q_n} |^2]^{\frac{1}{2}}$$ (by Minkowski's inequality)

$$\leq \varepsilon + [\sum_{n=1}^{\infty}|x_{q_n} |^2]^{\frac{1}{2}} \quad \forall q \geq n_0$$ (by using (3))

Since $x_q \in l_2$ we have

$[\sum_{n=1}^{\infty}|x_{q_n} |^2]^{\frac{1}{2}}$ *converges.*

$\therefore [\sum_{n=1}^{\infty}|x_{q_n} |^2]^{\frac{1}{2}}$ *converges.*

Thererfore y $\in l_2$.

Also (3) gives

$$d(y - x_q) \leq \varepsilon \quad \forall\, q \geq n_0$$

Therefore $(x_p) \rightarrow y$.

Therfore l_2 is complete.

Note 2.2.18

A subspace of a complete metric space need not be complete.

Space for hints

For example R with usual metric is complete. But the subspace (0,1] is not complete.

Theorem 2.2.19

A subset A of a complete metric space M is complete iff A is closed.

***Proof*.** Suppose A is complete .

To prove that A is closed, we shall prove that A contains all its limit points.

Let x be a limit point of A.

Then by theorem 2.2.7

" Let M be a metric space and A $\subseteq$ M.Then x is a limit point of A iff there

exists a sequence (x_n) of distinct points in A such that $(x_n) \to x$."

Since A is complete , $x \in A$.

Therefore A contains all its limit points.

Hence A is closed.

Conversely, Let A be a closed subset of M.

Let (x_n) be a Cauchy sequence in A.

Then (x_n) is a Cauchy sequence ini M also and since M is complete there

exists $x \in M$ such that $(x_n) \to x$. Thus (x_n) is a sequence in A converging to x.

$\therefore x \in \overline{A}$. .(By theorem 2.2.7).

Hence $x \in A$.

Thus every Cauchy sequence (x_n) in A converges to a point in A.

Therefore A is complete.

Note 2.2.20

[0,1] with usual metric is complete, Since it is a closed subset of the complete metric space R.

Note 2.2.21

Consider Q. Since $\overline{Q} = R$, Q is not a closed subset of R.

Hence Q is not complete.

Solved problems

Problem 2.2.22

Let A, B be subsets of R. Prove that $\overline{A \times B} = \overline{A} \times \overline{B}$

Solution.

Let $(x, y) \in \overline{A \times B}$

Therefore there exists a sequence $(x_n, y_n) \in A \times B$ such that $(x_n, y_n) \rightarrow (x,y)$(by theorem 2.2.7)

"let M be a metric space and $A \subseteq M$. Then $x \in \overline{A}$ iff there exists a sequence (x_n) in A such that $(x_n) \rightarrow x$".

Therefore $(x_n) \rightarrow x$ and $(y_n) \rightarrow y$.

Also (x_n) is a sequence in A and (y_n) is a sequence in B.

By theorem "Let M be a metric space and $A \subseteq M$. Then

(i) $x \in \overline{A}$ iff there exists a sequence (x_n) in A such that $(x_n) \rightarrow x$.

(ii) x is a limit point of A iff there exists a sequence (x_n) of distinct points in A such that $(x_n) \rightarrow x$."

$x \in \overline{A}$ *and* $y \in \overline{B}$,

$(x,y) \in \overline{A} \times \overline{B}$.

Hence $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$ ....(1)

Space for hints

Space for hints

Now, let $x \in \overline{A}$ and $y \in \overline{B}$.

We know the following:

Let M be a metric space and $A \subseteq M$. Then

(i) $x \in \overline{A}$ iff there exists a sequence (x_n) in A such that $(x_n) \to x$.

(ii) x is a limit point of A iff there exists a sequence (x_n) of distinct points in A such that $(x_n) \to x$.

By this theorem, there exists a sequence (x_n) in A such that $(x_n) \to x$ and a sequence (y_n) in B such that $(y_n) \to y$.

Therefore $((x_n, y_n))$ is a sequence in $A \times B$ which converges to (x,y).

Again by theorem, $(x,y) \in \overline{A \times B}$.

Therefore $\overline{A} \times \overline{B} \subseteq \overline{A \times B}$ ....(2)

From (1) and (2), $\overline{A} \times \overline{B} = \overline{A \times B}$.

Problem 2.2.23

If A and B are closed subsets of **R**, prove that $A \times B$ is a closed subset of $\mathbf{R} \times \mathbf{R}$.

Solution. A is closed if and only if $A = \overline{A}$ ....(*)

Since A and B are closed subsets of R, $A = \overline{A}$ and $B = \overline{B}$.

By the previous problem, $\overline{A} \times \overline{B} = \overline{A \times B}$.

Hence $\overline{A \times B} = A \times B$.

By (*), $A \times B$ is closed.

Thus $A \times B$ is a closed subset of $\mathbf{R} \times \mathbf{R}$.

2.3 CANTOR'S INTERSECTION THEOREM

Space for hints

Theorem 2.3.1(Cantor's intersection theorem)

Statement: Let M be a metric space. M is complete $\Leftrightarrow$ for every sequence (F_n) of non empty closed subsets of M such that $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots\ldots \supseteq F_n \supseteq \ldots\ldots$ and $(d(F_n)) \to 0$, $\bigcap_{n=1}^{\infty} F_n \neq \phi$.

Proof.

Let M be a metric space.

Suppose M is a complete metric space.

Let (F_n) be a sequence of non-empty closed subsets of M such that

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots\ldots \supseteq F_n \supseteq \ldots\ldots \qquad \ldots\ldots\ldots\ldots(1)$$

$$\text{and } (d(F_n)) \to 0 \qquad \ldots\ldots\ldots\ldots(2)$$

Claim: $\bigcap_{n=1}^{\infty} F_n \neq \phi$.

For each positive integer n, choose a point $x_n \in F_n$.

By (1) , $x_n, x_{n+1}, x_{n+2}, \ldots\ldots$ all lie in F_{n-1}.

Therefore $x_m \in F_n$ for all $m \geq n$. ....(3)

Since $(d(F_n)) \to 0$, given $\varepsilon > 0$, there exists a positive integer n_0 such that $\bar{d}(d(F_n), 0) < \varepsilon$, for all

$n \geq n_0$, where $\bar{d}$ is a metric on M.

$\Rightarrow d(F_n) < \varepsilon$, for all $n \geq n_0$.

In particular,

$d(F_{n_0}) < \varepsilon$ .... (4)

Therefore, $\bar{d}(x,y) < \varepsilon$ for all $x,y \in F_n$.

From (3) and (4), $x_m \in F_{n_0}$, for all $m \geq n_0$.

Space for hints

Hence m,n $\geq n_0 \Rightarrow x_m, x_n \in F_{n_0}$.

$$\Rightarrow \bar{d}(x_m,x_n) < \varepsilon \quad \text{by (4)}.$$

Therefore by definition, (x_n) is Cauchy sequence in M.

Since M is complete, there exists a point x in M such that $(x_n) \rightarrow x$.

Now we claim that $x \in \bigcap_{n=1}^{\infty} F_n$.

We know the following result.

"Let M be a metric space and $A \subseteq M$. Then $x \in \bar{A}$ iff there exists a sequence (x_n) in A such that $(x_n) \rightarrow x$." ....(*)

For any positive integer n, $x_n, x_{n+1}, x_{n+2}, \ldots\ldots$ is a sequence in F_n and this sequence converges to x.

By (*), $x \in \overline{F_n}$.

Since F_n is closed, $\overline{F_n} = F_n$.

Therefore $x \in F_n$ for each n.

$\Rightarrow x \in \bigcap_{n=1}^{\infty} F_n$.

Hence $\bigcap_{n=1}^{\infty} F_n \neq \phi$.

Let (F_n) be a sequence of non-empty closed subsets of M such that

$F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots\ldots\supseteq F_n \supseteq \ldots..$ and $(d(F_n)) \rightarrow 0$.

Conversely, assume that $\bigcap_{n=1}^{\infty} F_n \neq \phi$.

Claim: M is complete.

Let (x_n) be a Cauchy sequence in M.

Let $F_1 = \{ x_1, x_2, \ldots\ldots, x_n, \ldots\ldots\}$

$F_2 = \{ x_2, x_3, \ldots\ldots, x_n, \ldots\ldots\}$

....

....

Space for hints

$F_n = \{ x_n, x_{n+1}, \ldots\ldots\ldots\ldots \}$

Clearly, $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots\ldots \supseteq F_n \supseteq \ldots\ldots$ ....(5)

We know the following result.

"Let M be a metric space. Then $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$."

By the above result (5) can be written as follows $\overline{F_1} \supseteq \overline{F_2} \supseteq \ldots\ldots\ldots \supseteq \overline{F_n} \supseteq \ldots\ldots\ldots$

Therefore $(\overline{F_n})$ is a sequence of non empty closed subsets of M.

Also, $(\overline{F_n})$ is a decreasing sequence of closed sets.

Since (x_n) is a Cauchy sequence, given $\varepsilon > 0$, there exists a positive integer n_0 such that

$\overline{d}(x_m, x_n) < \varepsilon$, for all $m,n \geq n_0$.

Therefore for any integer $n \geq n_0$, the distance between any two points of F_n is less than ε.

That is $\overline{d}(x,y) < \varepsilon$ for all $x,y \in F_n$.

$\Rightarrow$ l.u.b $\{\overline{d}(x, y) : x,y \in F_n\} < \varepsilon$.

$\Rightarrow d(Fn) < \varepsilon$..........(6)

Hence $d(Fn) < \varepsilon$ for all $n \geq n_0$.

We know that $d(A) = d(\overline{A})$ for any subset A of a metric space.

By this result, $d(F_n) = d(\overline{F_n})$.

From (6), $d(\overline{F_n}) < \varepsilon$ for all $n \geq n_0$............(7).

$| d(\overline{F_n}) - 0 | < \varepsilon$ for all $n \geq n_0$.

$d(\overline{F_n}) \to 0$.

Space for hints

By the assumption, $\bigcap_{n=1}^{\infty} F_n \neq \phi$.

Let $x \in \bigcap_{n=1}^{\infty} F_n$

Since $\overline{Fn}$ is the smallest closed set containing F_n and since $x_n \in F_n$, $x_n \in \overline{Fn}$.

Therefore $\overline{d}(x_n, x) \leq d\left(\overline{F_n}\right)$.

$\Rightarrow \overline{d}(x_n, x) < \varepsilon$, for all $n \geq n_0$ from (7).

$\Rightarrow (x_n) \rightarrow x$. Hence every Cauchy sequence in M is a Convergent saequence.

Therefore by the definition, M is complete.

Remark 2.3.2

In the above theorem , $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

For,

Suppose that $\bigcap_{n=1}^{\infty} F_n$ contain two distinct points say x and y.

Then $d(F_n) \geq \overline{d}(x_n, x)$ for all n.

Therefore $(d(F_n))$ does not tend to zero.

This is a contradiction.

Thus $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

Remark 2.3.3

In the above theorem , $\bigcap_{n=1}^{\infty} F_n$ may be empty if each F_n is not closed.

For example, consider $F_n = (0,1/n)$ in **R**.

Clearly $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots\ldots \supseteq F_n \supseteq \ldots\ldots$ and $(d(F_n)) = (1/n) \rightarrow 0$ as $n \rightarrow \infty$.

But $\bigcap_{n=1}^{\infty} F_n = \phi$.

Space for hints

Remark 2.3.4

In the above theorem, $\bigcap_{n=1}^{\infty} F_n$ may be empty if the hypothesis $(d(F_n)) \to 0$ is omitted.

For example,consider $F_n = [n,\infty)$ in **R.**

Clearly (F_n) sequence of non empty closed subsets such that

$F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots\ldots \supseteq F_n \supseteq \ldots.$

Also $\bigcap_{n=1}^{\infty} F_n = \phi$.

Here $d(F_n) = \infty$ for all n and therefore the hypothesis $(d(F_n)) \to 0$ is not true.

2.4 BAIRE'S CATEGORY THEOREM

Definition 2.4.1 Let (M,d) be a metric space. Let A be a subset of M. Then A is called

a ***nowhere dense*** set in M if Int $\overline{A} = \phi$.

Definition 2.4.2 Let (M,d) be a metric space. Let A be a subset of M.

A is said to be of ***first category*** in M if A can be expressed as a countable union of nowhere dense sets. A set which is not of first category

is said to be of ***second category***.

Remark 2.4.3 If A is of first category, then $A = \bigcup_{n=1}^{\infty} E_n$,where E_n is nowhere dense in M.

Example 2.4.4

Consider **R** with usual metric. Let $A = \{1,1/2,1/3,\ldots 1/n,\ldots\}$.

We know that $\overline{A} = A \cup D(A)$.

Clearly $A \cup D(A) = \{0,1,1/2,1/3,\ldots,1/n,\ldots\}$

Therefore $\overline{A} = \{0,1,1/2,1/3,\ldots,1/n,\ldots\}$.

Clearly Int $\overline{A} = \phi$.

Space for hints

Therefore in **R** with usual metric ,A = {1,1/2,1/3,...1/n,...} is nowhere dense.

Example 2.4.5

In any discrete metric space,any non-empty subset A is not nowhere dense.

Proof. Let M be a discrete metric space.

Let A be a non-empty subset of M.

Since M is discrete, A is both open and closed.

Since A is closed, A = $\overline{A}$.

$\Rightarrow$ Int A = Int $\overline{A}$.

We know that, in a discrete metric space, Int A = A.

But A is non-empty.

$\Rightarrow$ Int A = Int $\overline{A} \neq \phi$.

$\Rightarrow$ A is not nowhere dense.

Example 2.4.5

In $\mathbb{R}$ with usual metric, any finite subset A is nowhere dense.

Proof. Let A be any finite subset of **R.**

In R with usual metric, every singleton set is closed......(1)

In any metric space, the union of finite number of closed sets is closed...(2)

$$A = \bigcup_{x \in A} \{x\}$$

It is given that A is finite.

From (1) and (2), A is closed.

Therefore A = $\overline{A}$. $\Rightarrow$ Int A = Int $\overline{A}$ ......(3)

Let x$\in$ A and let r > 0.

Then B(x,r) = (x-r,x+r) , an infinite set.

Therefore B(x,r) is not a subset of A.

Hence x is not an interior of A.

$\Rightarrow$ Int A = Int $\overline{A}$ = ϕ.

Hence A is nowhere dense in **R**.

Example 2.4.6

In R with usual metric, every singleton set {x} is nowhere dense.

Proof.

Consider **R** with usual metric.

Let A = {x}.

$A^c = (-\infty,x) \cup (x,\infty)$

$(-\infty,x)$ and (x,∞) are open sets. An arbitrary union opn sets is open.

$\Rightarrow A^c$ is open.

$\Rightarrow$ A is closed.

$\Rightarrow A = \overline{A}$.

$\Rightarrow$ Int A = Int $\overline{A}$ = ϕ, since Int A = ϕ.

$\Rightarrow$ A is nowhere dense.

Note 2.4.7

If A and B are sets of first category in a metric space M, then $A \cup B$ is also of first category.

Theorem 2.4.8 (Baire's Category Theorem)

Statement: Any complete metric space is of second category.

Proof. Let M be a complete metric space.

Claim: M is not of first category.

Let (A_n) be a sequence of nowhere dense sets in M.

It is enough if we prove that $\bigcup_{n=1}^{\infty} A_n \neq M$.

Suppose that $A_1, A_2, A_3, \ldots$ are nowhere dense subsets of M.

Space for hints

Space for hints

We know the following result (*) :

" Let M be a metric space and $A \subseteq M$. Then the following are equivalent.

(i) A is nowhere dense in M.

(ii) $\overline{A}$ does not contain any nonempty open set.

(iii) Each nonempty open set has a nonempty open subset disjoint from $\overline{A}$.

(iv) Each nonempty open set has a nonempty open subset disjoint from A.

(v) Each nonempty open set contains open sphere disjoint from A.

M is open and A_1 is nowhere dense.

By (*) , there exists open ball B_1 of radius less than 1 such that B_1 is disjoint from A_1.

Let F_1 denote the concentric closed ball whose radius is ½ times that of B_1.

Int F_1 is open. A_2 is nowhere dense.

Again by (*) , Int F_1 contains an open ball B_2 of radius less than ½ such that B_2 is disjoint from A_2.

Let F_2 denote the concentric closed ball whose radius is ½ times that of B_2.

Int F_2 is open and A_3 is nowhere dense.

By (*) , Int F_2 contains an open ball B_3 of radius less than ¼ such that B_3 is disjoint from A_3.

Let F_3 denote the concentric closed ball whose radius is ½ times that of B_3.

Proceeding like this , we get sequence of nonempty closed balls Fn such that $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots\ldots \supseteq F_n \supseteq \ldots\ldots$ and $d(F_n) < 1/2^n$.

Hence $(d(F_n)) \to 0$, as $n \to \infty$.

It is given that M is complete.

By Cantor's intersection theorem, there exists a point x in M such that $x \in \bigcap_{n=1}^{\infty} F_n$.

Each F_n is disjoint from A_n.

Hence $x \notin A_n$ for all n.

$\Rightarrow x \notin \bigcup_{n=1}^{\infty} A_n$

$\Rightarrow \bigcup_{n=1}^{\infty} A_n \neq M$.

Hence M is of second category.

Corollary 2.4.9

The set **R** is of second category.

Proof. Since **R** is complete, **R** is od second category.(Baire's category theorem).

PROBLEMS

Problem 2.4.10

Show that a metric space which is of second category need not be complete.

Solution.

Consider M = **R-Q.**

We know that **Q** is of first category.

Claim: (1) M is of second category.

(2) M is not complete.

We know that a subset A of a complete metric space is complete $\Leftrightarrow$ A is closed.

Here M is not closed.

Therefore by the above result, M is not complete.

Now suppose that M is of first category.

Space for hints

Space for hints

Then $M \cup Q = (\mathbf{R}-\mathbf{Q}) \cup \mathbf{Q}$

$= \mathbf{R}$, which is of first category.

This is a contradiction to the fact that **R** is of second category.

Therefore M is of second category.

Problem 2.4.11

Prove that any nonempty open interval in **R** is of second category.

Solution.

Let (a,b) be a nonempty open interval in **R.**

Claim: (a,b) is of second category.

Suppose that (a,b) is of first category.

$[a,b] = (a,b) \cup \{a\} \cup \{b\}$.

{a} and {b} are of first category.

We know that $A \cup B$ is of first category if A and B are of first category.

Therefore [a,b] is of first category.

We know that a subset A of a complete metric space is complete iff A is closed.

By this result, [a,b] is complete.

By Baire's category theorem,[a,b] is of second category.

This is a contradiction.

Hence (a,b) is of second category.

Problem 2.4.12

Prove that a closed set A in a metric space M is nowhere dense iff A^c is everywhere dense.

Solution. Let M be a metric space.

Let A be a closed subset of M.

Then $A = \overline{A}$. ......(1).

Claim: A^c is everywhere dense.

Space for hints

Assume that A is nowhere dense in M.

$\Rightarrow$ Int $\overline{A} = \phi$

$\Rightarrow$ Int A = ϕ, from (1).

It is enough if we prove that $\overline{A^c} = M$.

Clearly, $\overline{A^c} \subseteq M$. ....(2)

Let x $\in$ M.

Let G be any open set such that x $\in$ G.

Since Int A = ϕ, G $\not\subset$ A.

$\Rightarrow G \cap A^c \neq \phi$.

We know that, x $\in \overline{A}$ iff $G \cap A^c \neq \phi$ for every open set containing x.

By the above result, x $\in \overline{A^c}$

Therefore M $\subseteq \overline{A^c}$ ....(3)

Hence M = $\overline{A^c}$, from (2) and (3).

Therefore A^c is everywhere dense in M.

Conversely, let A^c be everywhere dense in M.

By definition, M = $\overline{A^c}$

Claim: Int A = ϕ.

Let G be any non-empty open set in M since $\overline{A^c}$ = M, we have $G \cap A^c \neq \phi$.

$\Rightarrow$ G $\not\subset$ A.

$\Rightarrow$ The only open set which is contained in A is the empty set.

$\Rightarrow$ Int A = ϕ.

$\Rightarrow$ Int $\overline{A} = \phi$.

$\Rightarrow$ A is nowhere dense in M.

Space for hints

UNIT-3

3.1 CONTINUITY

One of the main aims in considering the metric spaces is the study of continuous

functions. Early mathematicians considered defining a real-valued continuous function with an interval domain as one that maps every subinterval in its domain onto an interval or a point.In this unit,We provide the definition of continuous functions and give several characterizations of continuous functions.

Definition 3.1.1 Let (M_1,d_1) and (M_2,d_2) be two metric spaces.

Let $f:M_1 \to M_2$ be a function. The function f is said to have ***limit as $x \to a$*** if

given $\varepsilon > 0$, there exists δ 0 such that $0 < d_1(x,a) < \delta \Rightarrow d_2((f(x),f(a)) < \varepsilon$.

We write $\lim_{x \to \infty} f(x) = f(a) = l$ (say).

Definition 3.1.2.

Let (M_1,d_1) and (M_2,d_2) be two metric spaces. Let $a \in M_1$.

A function $f: M_1 \to M_2$ is said to be ***continuous at 'a'*** if given $\varepsilon>0$,

there exists $\delta> 0$ such that $d_1(x,a) <\delta \Rightarrow d_2(f(x),f(a))<\varepsilon$.

f is said to be ***continuous*** if it is continuous at every point of M_1.

Remarks 3.1.3

1. f is continuous at a iff $\lim_{x \to \infty} f(x) = f(a)$.

2. The condition $d_1(x,a) <\delta \Rightarrow d_2(f(x),f(a))<\varepsilon$ can be rewritten as

(i) $x \in B_{d_1}(a,\delta) \Rightarrow f(x) \in B_{d_2}(f(a),\varepsilon)$ (or)

(ii) $f(B_{d_1}(a,\delta)) \subseteq B_{d_2}(f(a),\varepsilon)$

Example 1.3.4 Let (M_1,d_1) and (M_2,d_2) be two metric spaces.

Then any constant function $f: M_1 \to M_2$ is continuous.

Proof. Let (M_1,d_1) and (M_2,d_2) be two metric spaces.

Space for hints

Let $f:M_1 \rightarrow M_2$ be a function defined by $f(x) = a$ where 'a' is a fixed element.

Let $x \in M_1$. Let $\varepsilon > 0$ be given.

Since $x \in M_1$ and M_1 is open, there exists $B(x,\delta)$ for any $\delta > 0$ in M.

Since f is constant,

$f(B(x, \delta)) = \{a\} \subseteq B(a,\varepsilon)$

Therefore f is continuous at x.

Since x is arbitrary, f is continuous.

Example 1.3.5

Let (M_1,d_1) be a discrete metric space. Let (M_2,d_2) be any metric space.

Then any function $f: M_1 \rightarrow M_2$ is continuous.

Proof.

Let (M_1,d_1) be a discrete metric space. Let (M_2,d_2) be any metric space.

Claim: f is continuous .

Let $x \in M_1$ and $\varepsilon > 0$ be given .

Since M_1 is discrete , for any $\delta < 1$,$B(x,\delta) = \{x\}$.

Therefore $f(B_{d_1}(x,\delta)) = \{f(x)\} \subseteq B_{d_2}(f(x),\varepsilon)$

Therefore f is continuous at x.

Since x is arbitrary, f is continuous.

Theorem 3.1.6*

Let (M_1,d_1) and (M_2,d_2) be two metric spaces. Let $a \in M_1$.

A function $f: M_1 \rightarrow M_2$ is continuous at 'a' iff $(x_n) \rightarrow a \Rightarrow (f(x_n)) \rightarrow f(a)$.

Proof.

Let (M_1,d_1) and (M_2,d_2) be two metric spaces.

Let $a \in M_1$.

Suppose $f: M_1 \rightarrow M_2$ is continuous at 'a'.

Space for hints

Let (x_n) be a sequence in M_1 such that $(x_n) \to a$.

Claim: $(f(x_n)) \to f(a)$.

Let $\varepsilon > 0$ be given.

Since f is continuous, there exists $\delta > 0$ such that

$d_1(x,a) < \delta \Rightarrow d_2(f(x),f(a)) < \varepsilon. \to (1)$

Since $(x_n) \to a$, there exists a positive integer n_0 such that

$d_1(x_n,a) < \delta$ for all $n \geq n_0$

From (1), $d_2(f(x),f(a)) < \varepsilon$, for all $n \geq n_0$.

Therefore by definition, $(f(x_n)) \to f(a)$.

Conversely, suppose $(x_n) \to a \Rightarrow (f(x_n)) \to f(a)$.

Suppose f is not continuous at a.

Then there exists $\varepsilon > 0$, such that for al l $\delta > 0$, $f(B(a,\delta) \not\subset B(f(a),\varepsilon)$.

In particular,

$$f(B(a,\frac{1}{n}) \not\subset B(f(a),\varepsilon).$$

Choose x_n such that

$$x_n \in B(a,\frac{1}{n}) \quad and \quad f(x_n) \notin B(f(a),\varepsilon).$$

Therefore $d_1(x_n,a) < 1/n \Rightarrow d_2(f(x),f(a)) \geq \varepsilon$.

Therefore $(x_n) \to a$ and $(f(x_n))$ does not converge to f(a)

This is contradiction to the assumption.

Therefore f is continuous at 'a'.

Corollary 3.1.7

A function $f: M_1 \to M_2$ is continuous iff $(x_n) \to x \Rightarrow (f(x_n)) \to f(x)$.

Now we provide some characterizations for continuous functions using open sets. Theorem 3.1.8 and Theorem 3.1.11 are very important.

Space for hints

Theorem 3.1.8*

Let (M_1,d_1) and (M_2,d_2) be any two metric spaces.

A function $f: M_1 \to M_2$ is continuous iff $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

(i.e) f is continuous iff inverse image of every open set is open.

Proof.

Let (M_1,d_1) and (M_2,d_2) be any two metric spaces.

Suppose that f is continuous.

Claim: $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

Let G be an open set in M_2.

If $f^{-1}(G)$ is empty, then it is open.

Assume that $f^{-1}(G) \neq \phi$.

Let $x \in f^{-1}(G)$.

Therefore $f(x) \in G$.

Since G is open, there exists a positive real number ε, such that $B(f(x),\varepsilon) \subseteq G. \to (1)$

By the definition of continuity, there exists an open ball $B(x,\delta) \subseteq B(f(x),\varepsilon)$.

From (1), $f(B(x,\delta)) \subseteq G$.

Therefore $B(x,\delta) \subseteq f^{-1}(G)$.

Since $x \in f^{-1}(G)$ is arbitrary, $f^{-1}(G)$ is open in M_1.

Conversely, assume that $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

Claim: f is continuous

Let $x \in M_1$.

Now, $B(f(x),\varepsilon)$ is an open set in M_2.

By the assumption, $f^{-1}(B(f(x),\varepsilon))$ is open in M_1.

Also, $x \in f^{-1}(B(f(x),\varepsilon))$.

Space for hints

Therefore there exists $\delta > 0$, such that $B(x,\delta) \subseteq f^{-1}(B(f(x),\varepsilon))$.

$f(B(x,\delta)) \subseteq (B(f(x),\varepsilon))$.

Therefore f is continuous at x.

Since $x \in M_1$ is arbitrary, f is continuous at every point of M_1.

Therefore f is continuous.

Note 3.1.9

If f: $M_1 \to M_2$ is continuous and G is open in M_1, then it is not necessary that f(G) is open in M_2.

(ie)Under a continuous map the image of an open set need not be an open set.

Proof.

For example,

Let M_1= R with discrete metric.

Let M_2 = R with usual metric.

Letf: $M_1 \to M_2$ be defined by f(x) = x.

Since M_1 is discrete, every subset of M_1 is open. For any open set G in M_2 , $f^{-1}(G)$ is open in M_1.

By the above theorem f is continuous.

Let A = {x} be a subset of M_1.

Since M_1 is discrete, A is open in M_1.

But f(A) ={x} is not open in M_2.

Note 3.1.10

In the above example, f is a continuous bijection whereas

f^{-1}: $M_2 \to M_1$ is not continuous.

Proof.

{x} is open in M_1.

$((f^{-1})^{-1}\{x\})=\{x\}$ which is not open in M_2.

Therefore f^{-1} is not continuous.

Space for hints

Thus if f is a continuous bijection, f^{-1} need not be continuous.

Theorem 3.1.11*

Let (M_1,d_1) and (M_2,d_2) be two metric spaces.

A function f: $M_1 \rightarrow M_2$ is continuous iff

$f^{-1}(F)$ is closed in M_1 whenever F is closed in M_2.

Proof.

Let (M_1,d_1) and (M_2,d_2) be two metric spaces.

Suppose f: $M_1 \rightarrow M_2$ is continuous.

Let $F \subseteq M_2$ be closed in M_2.

Therefore F^c is open in M_2.

Therefore by the above theorem, "f is continuous iff $f^{-1}(G)$ is open in M_1

whenever G is open in M_2" ....(*)

By the result,

$f^{-1}(F^c)$ is open in M_1.

But we know that,

$f^{-1}[F^c] = [f^{-1}(F)]^c$.

Therefore $[f^{-1}(F)]^c$ is open in M_1.

Therefore $f^{-1}(F)$ is closed in M_1.

Conversely, suppose $f^{-1}(F)$ is closed in M_1 whenever F is closed in M_2.

Claim: f is continuous.

Let G be an open set in M_2.

Therefore G^c is closed in M_2.

Therefore by the assumption,

$f^{-1}(G^c)$ is closed in M_1.

Therefore $[f^{-1}(G)]^c$ is closed in M_1.

Therefore $f^{-1}(G)$ is open in M_1.

Space for hints

By (*), f is continuous. (by Theorem 3.1.8).

Theorem 3.1.12

Let (M_1,d_1) and (M_2,d_2) be two metric spaces. Then a function

$f: M_1 \rightarrow M_2$ is continuous iff $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$.

Proof.

Let (M_1,d_1) and (M_2,d_2) be two metric spaces.

Suppose that f is continuous.

Claim: $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$.

Let $A \subseteq M_1$.

Then $f(A) \subseteq M_2$.

$\overline{f(A)}$ is closed set in M_2.

We know the following result:

f is continuous iff $f^{-1}(F)$ is closed in M_1

whenever F is closed in M_2. ....(1)

By the above theorem,

$f^{-1}(\overline{f(A)})$ is closed in M_1.

It is clear that $f(A) \subseteq \overline{f(A)}$

$\Rightarrow A \subseteq f^{-1}(\overline{f(A)})$.

But $\overline{A}$ is the smallest closed set containing A.

Since $\overline{A}$ is the smallest closed sset containing A and

$f^{-1}(\overline{f(A)})$ is a closed set containing A, we have

$\overline{A} \subseteq f^{-1}(\overline{f(A)})$.

$\Rightarrow f(\overline{A}) \subseteq \overline{f(A)}$.

Conversely, assume that $f(\overline{A}) \subseteq \overline{f(A)}$ for all A M_1.

It is enough if we prove that $f^{-1}(F)$ is closed in M_1. (By (1))

Space for hints

Let F be a closed set in M_2.

Then $f^{-1}(F)$ is a subset of M_1.

By the assumption, $f\,\overline{(f^{-1}(F))} \subseteq \overline{ff^{-1}(F)}$.

$= \overline{F}$

= F. (since F is closed)

Therefore , $f\overline{(f^{-1}(F))} \subseteq F$.

$\Rightarrow \overline{(f^{-1}(F))} \subseteq f^{-1}(F)$ . ...(A)

Also, $f^{-1}(F) \subseteq \overline{(f^{-1}(F))}$ ....(B)

From (A) and (B)

$f^{-1}(F) = \overline{(f^{-1}(F))}$.

Therefore $f^{-1}(F)$ is closed in M_1.

Therefore by (1), f is continuous.

SOLVED PROBLEMS

Problem 3.1.13

Let f be a continuous real valued function defined on a metric space M. Let $A = \{ x \in M / f(x) \geq 0\}$.

Prove that A is closed.

Solution.

Let $A = \{ x \in M / f(x) \geq 0\}$.

$= \{ x \in M / f(x) \in [0,\infty)\}$.

$= f^{-1}([0,\infty))$.

Also, $[0,\infty)$ is a closed subset of R.

Since f is continuous, $f^{-1}([0,\infty))$ is closed in M.(By Theorem 3.1.11)

Therefore A is closed.

Space for hints

Problem 3.1.14

Show that the function f: R→ R defined by

" $f(x) =$ 0 if x is irrational and 1 if x is rational "

is not continuous by each of the following methods.

(i) By the usual ε, δ method.

(ii) By exhibiting a sequence (x_n) such that $(x_n) \to x$ and $(f(x_n))$ does not converge to f(x).

(iii) By exhibiting an open set G such that $f^{-1}(G)$ is not open.

(iv) By exhibiting a closed subset F such that $f^{-1}(F)$ is not closed.

(v) By exhibiting a subset A of R such that $f(\overline{A}) \not\subset \overline{f(A)}$.

Solution.

(i) To prove that,

f is not continuous at x ,we have to show that there exists an $\varepsilon > 0$ such that $\forall \delta > 0$, $f(B(x, \delta)) \not\subset B(f(x),\varepsilon)$.

Let $\varepsilon = ½$.

For any $\delta > 0$, $B(x, \delta) = (x-\delta, x+\delta)$ contains both rational and irrational numbers.

If x is rational, choose $y \in B(x, \delta)$ such that y is irrational and if x is irrational, choose $y \in B(x, \delta)$ such that y is rational.

Then $| f(x) - f(y) | = 1$ (by definition of f)

(i.e) $d(f(x),f(y)) = 1$.

Therefore $f(y) \notin B(f(x),1/2)$

Thus $y \in B(x, \delta)$ and $f(y) \notin B(f(x),1/2)$

Therefore $f(B(x, \delta)) \not\subset B(f(x),\varepsilon)$.

Hence f is not continuous at x.

Space for hints

(ii) Let $x \in R$.

Suppose that x is rational.

Then $f(x) = 1$.

Let (x_n) be a sequence of irrational numbers such that $(x_n) \to x$.

Then $(f(x_n)) \to 0$ and $f(x) = 1$.

Therefore $(f(x_n))$ does not converge to $f(x)$.

Similarly, if x is irrational numbers, $(f(x_n))$ does not converge to $f(x)$.

(iii) Let $G = (1/2, 3/2)$.

Clearly G is open in R.

Now $f^{-1}(G) = \{x \in R / f(x) \in G\}$.

$= \{x \in R / f(x) \in (1/2, 3/2)\}$.

$= Q$.

But Q is not open in R.

Thus $f^{-1}(G)$ is not open in R.

Therefore f is not continuous.

(iv) Choose $F = [1/2, 3/2]$.

Then $f^{-1}(F) = Q$ which is not closed in R.

Therefore f is not continuous.

(v) Let $A = Q$.

Then $\overline{A} = R$.

$f(\overline{A}) = f(R) = \{0,1\}$ (by definition of f).

Also, $f(A) = f(Q) = \{1\}$.

$\overline{f(A)} = \overline{\{1\}} = \{1\}$.

$f(\overline{A}) \not\subset \overline{f(A)}$.

Therefore f is not continuous (by Theorem 3.1.12).

Space for hints

Problem 3.1.15

Let M_1,M_2,M_3 be metric spaces.

If f: $M_1 \to M_2$ and g: $M_2 \to M_3$ are continuous functions ,

then prove that $g \circ f : M_1 \to M_3$ is also continuous.

That is composition of two continuous functions is continuous.

Solution. : Suppose that f:$M_1 \to M_2$ and g: $M_2 \to M_3$ are continuous functions ,

Let G be open in M_3.

Since g is continuous, g^{-1} (G) is open in M_2. (By Theorem 3.1.8)

Now, since f is continuous, $f^{-1}(g^{-1}$ (G)) is open in M_1. (By Theorem 3.1.8)

(i.e) $(g \circ f)^{-1}(G)$ is open in M_1.

Therefore g o f is continuous. (By Theorem 3.1.8)

Problem 3.1.16

Let M be a metric space . Let f: M $\to$R and

g : M $\to$R be two continuous functions. Prove that f+g: M$\to$R is continuous.

Solution. Let M be a metric space .

Let f: M $\to$R and g : M $\to$R be two continuous functions.

Let (x_n) be sequence converging to x in M.

Since f and g are continuous functions, $(f(x_n)) \to f(x)$ and

$(g(x_n)) \to g(x)$. (By Theorem 3.1.6).

Therefore $(f(x_n)+g(x_n)) \to f(x) + g(x)$

(i.e) $((f+g)(x_n) \to (f+g)(x)$

Therefore f+g is continuous (By Theorem 3.1.6).

Problem 3.1.17

Let f, g be continuous real valued functions on a metric space M.

Let A = { x / x$\in$ M and f(x)< g(x)}. Prove that A is open.

Space for hints

Solution. Let f, g be continuous real valued functions on a metric space M.

Since f and g are continuous real valued functions of M,

f – g is also a continuous real valued function on M.

Now A= { xϵ M/ f(x)< g(x)}.

= { xϵ M / f(x)- g(x)< 0}.

={ xϵ M / (f-g)(x)< 0)}.

={ xϵ M / (f-g)(x)ϵ(-∞,0)}.

=$(f-g)^{-1}$ {(-∞,0)}.

Now, (-∞,0) is open in R, and f- g is continuous.

Hence $(f-g)^{-1}$ {(-∞,0)} is open in M (By Theorem 3.1.8).

Therefore A is open in M.

Problem 3.1.18

If f : R→R and g : R→R are both continuous functions on R and if h: R^2→R^2 is defined by h(x,y) = (f(x),g(y)), prove that h is continuous on R^2.

Solution.

Let (x_n,y_n) be a sequence in R^2 converging to (x,y).

Claim: $(h(x_n,y_n)) \to h(x, y)$

Since $((x_n,y_n)) \to (x, y)$ in R^2 , $(x_n) \to x$, $(y_n) \to y$ in R .

Also f and g are continuous.

By Theorem 3.1.6, We have the following:

"Let (M_1,d_1) and (M_2,d_2) be two metric spaces. Let aϵM_1 .

A function f: $M_1 \to M_2$ is continuous at 'a' iff $(x_n) \to a \Rightarrow (f(x_n)) \to$

f(a)."

By this Theorem, $(f(x_n)) \to f(x)$ and $(g(y_n)) \to g(y)$.

$\Rightarrow (f(x_n), g(y_n)) \to (f(x),g(y))$.

$\Rightarrow (h(x_n,y_n)) \to h(x,y)$.

Space for hints

Again by the above Theorem 3.1.6, h is continuous on $\mathbf{R}^2$.

Problem 3.1.19

Let (M,d) be a metric space. Let a$\in$ M. Show that the function f: M$\rightarrow$**R** defined by f(x) = d(x,a) is continuous.

Solution. Let (M,d) be a metric space. Let a$\in$ M and let x$\in$M.

Let (x_n) be a sequence in M such that $(x_n) \rightarrow x$.

Define f by f(x) = d(x,a) .

Claim: $(f(x_n)) \rightarrow f(x)$.

Let $\varepsilon > 0$ be given.

Now $|f(xn) - f(x) = |d(x_n,a) - d(x-a)| \leq d(x_n,x)$.

Since $(x_n) \rightarrow x$, there exists a positive integer n_1 such that $d(x_n,x)<\varepsilon$ for all $n \geq n_1$.

Therefore $|f(x_n) - f(x)| < \varepsilon$ for all $n \geq n1$.

Therefore $(f(x_n)) \rightarrow f(x)$.

Therefore f is continuous. (By this Theorem: A function $f: M_1 \rightarrow M_2$ is continuous iff $(x_n) \rightarrow x \Rightarrow (f(x_n)) \rightarrow f(x)$).

Problem 3.1.20

Let f be a function form R^2 onto R defined by f(x,y) = x for all (x,y) $\in$ R^2.

Show that f is continuous R^2.

Solution. Let f be a function form R^2 onto R defined by f(x,y) = x for all (x,y) $\in$ R^2.

Let (x,y) $\in$ R^2.

Let (x_n,y_n)be a sequence in R^2 converging to (x,y).

Then $(x_n) \rightarrow x$ and $(y_n) \rightarrow y$.

Therefore $(f(x_n,y_n)) = (x_n) \rightarrow x = f(x,y)$.

Therefore $(f(x_n,y_n)) = f(x,y)$.

Therefore f is continuous. (By this Theorem: A function

$f: M_1 \to M_2$ is continuous iff $(x_n) \to x \Rightarrow (f(x_n)) \to f(x)$).

Problem 3.1.21

Define $f: l_2 \to l_2$ as follows: If $s \in l_2$ is the sequence $s_1, s_2, \ldots$, let f(s) be the sequence $0, s_1, s_2, \ldots$ Show that f is continuous on l_2.

Solution. Define $f: l_2 \to l_2$ as follows: If $s \in l_2$ is the sequence $s_1, s_2, \ldots$,

let f(s) be the sequence $0, s_1, s_2, \ldots$

Let $y = (y_1, y_2, \ldots y_n \ldots) \in l_2$.

Let (x_n) be a sequence in l_2 converging to y.

Let $x_n = (x_{n_1}, x_{n_2} \ldots x_{n_k}, \ldots)$.

Then $(x_n) \to y_1$, $x_{n_2} \to y_2$, $(x_{n_k}) \to y_k, \ldots$

Therefore $(f(x_n)) = ((0, x_{n_1}, x_{n_2} \ldots x_{n_k}, \ldots) \to$

$(0, y_1, y_2, \ldots y_n, \ldots) = f(y)$

Therefore $(f(x_n) \to f(y)$.

Therefore f is continuous. (By this Theorem: A function

$f: M_1 \to M_2$ is continuous iff $(x_n) \to x \Rightarrow (f(x_n)) \to f(x)$).

Problem 3.1.22.

Let G be an open subset of R. Prove that the characteristic function on G defined by

$\chi_G(x) = \{1$ if $x \in G$

$\{0$ if $x \notin G$ is continuous at every point of G.

Solution.

Let G be an open subset of R.

$\chi_G(x) = \{1$ if $x \in G$

$\{0$ if $x \notin G$ is continuous at every point of G.

Space for hints

Let x ∈ G so that $\chi_G(x) = 1$.

Let $\varepsilon > 0$ be given.

Since G is open and x ∈ G, we can find a $\delta > 0$ such that $B(x, \delta) \subseteq G$.

Therefore $\chi_G(B(x, \delta)) \subseteq \chi_G(G)$.

$= \{1\}$.

$\subseteq B(1,\varepsilon)$.

Thus $\chi_G(B(x, \delta)) \subseteq B(\chi_G(x),\varepsilon)$.

Therefore χ_G is continuous at x.

Since x ∈ G is arbitrary, χ_G is continuous on G.

3.2 HOMEOMORPHIS*

Definition 3.2.1

Let (M_1,d_1) and (M_2,d_2) be two metric spaces. A function

$f: M_1 \rightarrow M_2$ is called a ***homeomorphism*** if

a) f is 1-1 and onto

b) f is continuous.

c) f^{-1} is continuous.

M_1 and M_2 are said to be ***homeomorphic*** if there exists a homeomorphism $f: M_1 \rightarrow M_2$.

Definition 3.2.2

A function $f: M_1 \rightarrow M_2$ is said to be an ***open map*** if f(G) is open in M_2 for every open set G in M_1.

Definition 3.2.3

A function $f: M_1 \rightarrow M_2$ is said to be a ***closed map*** if

f(F) is closed in M_2 for every closed set F in M_1.

Remark 3.2.4

Let a function $f: M_1 \rightarrow M_2$ be a 1-1 and onto function.

Space for hints

Then f^{-1} is continuous iff f is an open map.

Proof. Let a function f: $M_1 \rightarrow M_2$ be a 1-1 and onto function.

f^{-1} is continuous iff for any open set G in M_1,

$(f^{-1})^{-1}(G)$ is open M_2.(By Theorem 3.1.8)

But $(f^{-1})^{-1}(G) = f(G)$.

Therefore f^{-1} is continuous iff for every open set G in M_1, f(G) is open in M_2.

Therefore f^{-1} is continuous iff f is an open map.

Remark 3.2.5

f^{-1} is continuous iff f is a closed map.

Remark 3.2.6

Let f: $M_1 \rightarrow M_2$ be a 1-1, onto map. Then the following are equivalent.

(i) f is a homeomorphism.

(ii) f is a continuous open map.

(iii) f is a continuous closed map.

Remark 3.2.7

Let f: $M_1 \rightarrow M_2$ be a homeomorphism. $G \subseteq M_1$ is open in M_1 iff f(G) is open in M_2.

Proof.

Let f:$M_1 \rightarrow M_2$ be a homeomorphism.

Suppose G is open in M_1.

Since f is a homeomorphism, f^{-1} is continuous.

By Remark (3.2.6), f is an open map.

$\Rightarrow$ f(G) is open in M_2.

Conversely, suppose that f(G) is open in M_2.

We know that, f is continuous iff $f^{-1}(X)$ is open in M_1 whenever X is open in M_2.

Space for hints

By the above theorem, $f^{-1}(f(G))$ is open in M_1.

$\Rightarrow$ G is open in M_1.

Remark 3.2.8

Let f: $M_1 \to M_2$ be a 1-1 onto map. Then f is a homeomorphism iff it satisfies the following condition.

F is closed in M_1 iff f(F) is closed in M_2.

Proof.

Define f: $[0,1] \to [0,2]$ by $f(x) = 2x$.

Clearly f is 1-1 and onto.

$f^{-1}(x) = \frac{1}{2} x$.

f and f^{-1} are both continuous.

Therefore f is a homeomorphi.

Example 3.2.9

The metric spaces $(0,\infty)$ and R with usual metrics are homeomorphic.

Proof.

Define f: $(0,\infty) \to R$ by $f(x) = \log_e x$.

Here $f^{-1}(x) = e^x$.

f and f^{-1} are both continuous.

Hence f is a homeomorphism.

Example 3.2.10

The metric spaces (0,1) and $(0,\infty)$ with usual metrices are homeomorphic.

Proof.

Define f: $(0,1) \to (0,\infty)$ by $f(x) = x / (1-x)$.

We claim that f is 1-1 and onto.

Let $f(x) = f(y)$.

Therefore $x/(1-x) = y/(1-y)$.

Therefore $x(1-y) = y(1-x)$.

$\Rightarrow$ x-xy = y-xy.

Therefore x = y.

Hence f is 1-1.

Let y $\in (0,\infty)$.

Therefore f(x) = y $\Rightarrow$ x/(1-x) = y.

$\Rightarrow$ y-xy = x.

$\Rightarrow$ x(1+y) = y.

$\Rightarrow$ x = y/(1+y).

Therefore y/(y+1) $\in (0,1)$ is the pre-image of y under f.

Clearly f and f^{-1} are continuous.

Therefore f is a homeomorphism.

Example 3.2.11

R with usual metric is not homeomorphic to R with discrete metric.

Proof.

Let M_1 = R with usual metric.

Let M_2 = R with discrete metric.

Let f: $M_1 \rightarrow M_2$ be any 1-1 onto map.

Now, {a} is open in M_2.

But $f^{-1}(\{a\}) = \{f^{-1}(a)\}$ is not open in M_1.

Hence f is not continuous.

Thus any bijection f: $M_1 \rightarrow M_2$ is not a homeomorphism.

Hence M_1 is not homeomorphic to M_2.

Definition 3.2.12

Let (M_1,d_1) and (M_2,d_2) be two metric spaces.

Let f: $M_1 \rightarrow M_2$ be a 1-1 onto map. f is said to be an ***isometry*** if

$d_1(x,y) = d_2(f(x),f(y))$ for all x,y $\in M_1$.

An isometry is a distance preserving map.

Space for hints

Space for hints

M_1 and M_2 are said to be ***isometric*** if there exists an isometry f from M_1 onto M_2.

Example 3.2.13

$\mathbf{R}^2$ with usual metric and $\mathbf{C}$ with usual metric

are isometric and $f : \mathbf{R}^2 \to \mathbf{C}$ defined by $f(x,y) = x+iy$ is the required isometry.

Proof.

Let d_1 denote the usual metric on $\mathbf{R}^2$ and d_2 denote the usual metric on $\mathbf{C}$.

Let $a = (x_1,y_1)$ and $b = (x_2,y_2) \in \mathbf{R}^2$.

Then $d_1(a,b) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= | (x_1 - x_2) + i(y_1 - y_2) |.$

$= | (x_1 + iy_1) - i(x_2 + iy_2) |.$

$= d_2 (f(a),f(b)).$

Therefore f is an isometry.

Example 3.2.14 Let d_1 be the usual metric on [0,1] and d_2 be the usual metric on [0,2].

The map f:[0,1] $\to$ [0,2] defined by $f(x) = 2x$ is not isometry.

Proof. Let d_1 be the usual metric on [0,1] and d_2 be the usual metric on [0,2].

Consider the map f:[0,1] $\to$ [0,2] defined by $f(x) = 2x$.

Let $x,y \in [0,1]$.

Then $d_2 (f(x),f(y)) = | f(x) - f(y) |.$

$= | 2x - 2y |.$

$= 2| x - y |.$

$= 2d_1(x,y).$

Therefore $d_1(x,y) \neq d_2(f(x),f(y))$.

Hence f is not an isometry.

Remark 3.2.15

Space for hints

Since f is an isometry preserves distances,

the image of an open ball B(x,r) is the open ball B($f(x)$,r).

Remark 3.2.16

Under an isometry, the image of an open set is also an open set.

Remark 3.2.17

If f is an isometry, then f^{-1} is an isometry.

Remark 3.2.18

Under isometry, the inverse image of an open set is open.

Remark 3.2.19

An isometry is a homeomorphism.

Remark 3.2.20

A homeomorphism from one metric space to another need not be an isometry.

For example, f:[0,1] $\longrightarrow$ [0,2] defined by $f(x) = 2x$ is a homeomorphism.

But f is not an isometry.

UNIT- 4

Space for hints

4.1 CONNECTED SPACES

In this unit, we discuss the connectedness of metric spaces. Geometrically ,it is evident that an interval cannot be written as a disjoint union of nonempty open intervals.We define the connected space and then study its properties.Also, we characterize all connected subsets of **R**.

Definition 4.1.1

Let (M_1,d) be a metric space. M is said to be ***connected*** if

M cannot be represented as the union of two disjoint non-empty open sets.

If M is not connected it is said to be ***disconnected.***

Theorem 4.1.2

Let M_1 be a subspace of a metric space M.

Let $A_1 \subseteq M_1$. Then A_1 is open in M_1 iff there exists an open set A

in M such that $A_1 = A \cap M_1$

Example 4.1.3

Let M = R and $M_1 = [1,2] \cup [3,4]$ with usual metric. Then M_1 is disconnected.

Proof.

Claim: [1,2] and [3,4] are open sets in M_1.

$[1,2] = (1/2,5/2) \cap ([1,2] \cup [3,4])$ and $[3,4] = (5/2,9/2)\ ) \cap ([1,2] \cup [3,4])$.

By Theorem 4.1.2, [1,2] and [3,4] are open sets in M_1.

Thus, M_1 can be written as the union of two disjoint non-empty

open sets namely, [1,2] and [3,4].

Hence M_1 is disconnected.

Example 4.1.4

Any discrete metric space M with more than one point is disconnected.

Proof.

Let A be a proper non-empty subset of M.

Since M has more than one point such a set exists.

Then A^c is also non-empty.

Since M is discrete ,every subset of M is open.

Therefore A and A^c are open .

Thus $M = A \cup A^c$ where A and A^c are two disjoint non-empty open sets.

Therefore M is not connected.

Theorem 4.1.5

Let (M,d) be a metric space. Then the following are equivalent.

(i) M is connected

(ii) M cannot be written as the union of two disjoint non-empty closed sets.

(iii) M cannot be written as the union of two non-empty sets A and B such that

$A \cap \overline{B} = \overline{A} \cap B = \phi$.

(iv) M and ϕ are the only sets which are both open and closed in M.

Proof.

(i) => (ii)

Assume that M is connected.

Suppose (ii) is not true.

Therefore $M = A \cup B$ where A and B are closed $A \neq \phi$, $B \neq \phi$ and $A \cap B = \phi$.

Therefore $A^c = B$ and $B^c = A$

Since A and B are closed, A^c and B^c are open.

Therefore B and A are open.

Therefore $M = B \cup A = A \cup B$

Space for hints

Space for hints

Thus M can be written as the union of two disjoint non-empty open sets.

Therefore M is not connected which is a contradiction to the assumption.

Therefore (i)=> (ii)

(ii)=>.(iii)

Suppose (iii) is not true.

Then M=A$\cup$B where A $\neq\phi$, B$\neq\phi$, $A \cap \overline{B} = \overline{A} \cap B = \phi$.

Claim: A and B are closed.

Let $x \in \overline{A}$.

Since $\overline{A} \cap B = \phi$, $x \notin B$.

Since A$\cup$B = M , x$\in$ A.

$\therefore \overline{A} \subseteq A$.

But $\overline{A}$ is the smallest closed set containing A .

$\therefore A \subseteq \overline{A}$.

Hence $\overline{A} = A$.

We know that A is closed iff $\overline{A} = A$ ....(1)

Therefore A is closed.

Let $x \in \overline{B}$.

Since $A \cap \overline{B} = \phi$, $x \notin A$

Since A$\cup$B = M, x$\in$ B

$\therefore \overline{B} \subseteq B$

But $\overline{B}$ is the smallest closed set containing B.

$\therefore B \subseteq \overline{B}$.

Hence $\overline{B} = B$.

Space for hints

Therefore B is closed.(By (1))

Now $A \cap B = \overline{A} \cap B \quad (\because \quad A = \overline{A})$

$= \phi$

Thus $M = A \cup B$ where A, B are non-empty disjoint closed sets.

This is a contradiction to the assumption that

M cannot be written as the union of two disjoint non-empty closed sets.

Therefore (ii) => (iii)

(iii)=> (iv)

Suppose (iv) is not true.

Then there exists $A \subseteq M$ such that $A \neq M$ and $A \neq \phi$ and A is both open and closed.

Let $B = A^c$.

Then B is both open and closed.

Also $B \neq \phi$.

Therefore $M = A \cup B$.

Since A is closed, $\overline{A} = A$.

Therefore $\overline{A} \cap B = A \cap A^c = \phi$,

Also $A \cap \overline{B} = A \cap B = A \cap A^c = \phi$

Therefore $M = A \cup B$ where $A \cap \overline{B} = \overline{A} \cap B = \phi$ which is a contradiction to the assumption (iii)

Therefore (iii)=> (iv)

(iv) => (i)

Suppose (i) is not true.

Therefore M is not connected.

Therefore $M = A \cup B$ where A and B are closed $A \neq \phi$, $B \neq \phi$ and $A \cap B = \phi$.

Space for hints

Then $B^c = A$.

Since B is open , A is closed.

Also, $A \neq \phi$ and $A \neq M$. (since $B \neq \phi$).

Therefore A is a proper non-empty subset of M

which is both open and closed which is a contradiction to (iv).

Therefore (iv) => (i).

Theorem 4.1.6

A metric space M is connected iff there does not exist a

continuous function f from M onto the discrete metric space $\{0,1\}$.

Proof.

First assume that M is connected.

Suppose there exists a continuous function f from M onto $\{0,1\}$.

We know that , every subset of a discrete metric space is both open and closed.

Therefore $\{0\}$, $\{1\}$ are open.

We know the following result,

" Let (M_1,d_1) and (M_2,d_2) be any two metric spaces.

A function $f: M_1 \rightarrow M_2$ is continuous iff $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

(i.e) f is continuous iff inverse image of every open set is open."→(*)

Therefore $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are open.

Let $A = f^{-1}(\{0\})$, $B = f^{-1}(\{1\})$.

Since f is onto, A and B are non- empty.

Also $A \cap B = \phi$ and $A \cup B = M$.

Thus $M = A \cup B$ where $A \neq \phi$, $B \neq \phi$ and $A \cap B = \phi$ and A and B are open.

This is a contradiction to the fact that M is connected.

Therefore there does not exist a continuous function f

from M onto the discrete metric space $\{0,1\}$.

Space for hints

Conversely, assume that there does not exist a continuous function f from M onto the discrete metric space $\{0,1\}$.

Claim: M is connected.

Suppose that M is not connected.

Then there exists disjoint non-empty open sets A and B in M such that $M = A \cup B$.

Define $f: M \to \{0,1\}$ by $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$.

Clearly, f is onto.

Also $f^{-1}(\phi) = \phi$, $f^{-1}(\{0\}) = A$, $f^{-1}(\{1\}) = B$ and $f^{-1}(\{0,1\}) = M$

Thus inverse image of every open set in $\{0,1\}$ is open in M.

Therefore by (*) , f is continuous .

Therefore there exists a continuous function f from M onto $\{0,1\}$.

This is a contradiction to the assumption .

Therefore M is connected.

Note 4.1.7

The above theorem can be restated as follows:

M is connected iff every continuous function $f: M \to \{0,1\}$ is not onto.

PROBLEMS

Problem 4.1.8

Let M be a metric space . Let A be a connected subset of M.

If B is a subset of M such that $A \subseteq B \subseteq \overline{A}$ then B is connected.

In particular $\overline{A}$ is connected.

Solution.

Let M be a metric space .

Let A be connected subset of M.

Let B be a subset of M such that $A \subseteq B \subseteq \overline{A}$.

Space for hints

Suppose B is not connected.

Then $B = B_1 \cup B_2$ where $B_1 \neq \phi$,$B_2 \neq \phi$, $B_1 \cap B_2 = \phi$,B_1 and B_2 are open in B.

Since B_1 and B_2 are open sets in B, there exist open sets G_1 and G_2 such that $B_1 = G_1 \cap B$ and

$B_2 = G_2 \cap B$.(By Theorem 4.1.2)

Therefore $B = B_1 \cup B_2$.

$$= (G_1 \cap B) \cup (G_2 \cap B)$$

$$= (G_1 \cup G_2) \cap B$$

Therefore $B \subseteq (G_1 \cup G_2)$

But $A \subseteq B \subseteq \overline{A}$

Therefore $A \subseteq (G_1 \cup G_2)$

Therefore $A = (G_1 \cup G_2) \cap A$.

$$= (G_1 \cap A) \cup (G_2 \cap A)$$

$(G_1 \cap A)$ and $(G_2 \cap A)$ are open in A.

Further, $(G_1 \cap A) \cap (G_2 \cap A) = (G_1 \cup G_2) \cap A$

$$= (G_1 \cap G_2) \cap B \text{ (Since } A \subseteq$$

$$= (G_1 \cap B) \cup (G_2 \cap B)$$

$$= B_1 \cap B_2$$

$$= \phi.$$

Therefore $(G_1 \cap A) \cap (G_2 \cap A) = \phi$

Since A is connected, either $(G_1 \cap A) = \phi$ or $(G_2 \cap A) = \phi$.

Without loss of generality assume that $(G_1 \cap A) = \phi$.

Since G_1 is open in M, $G_1 \cap \overline{A} = \phi$.

Therefore $(G_1 \cap B) = \phi$, since $B \subseteq \overline{A}$.

Therefore $B_1 = \phi$, which is a contradiction.

Space for hints

Therefore B is connected.

Problem 4.1.9

If A and B are connected subsets of a metric space M and if $A \cap B \neq \phi$, prove that $A \cup B$ is connected.

Solution.

Let f: $A \cup B \to \{0,1\}$ be a continuous function.

Since $A \cap B \neq \phi$, we can choose $x_0 \in A \cap B$.

Let $f(x_0) = 0$.

Since f: $A \cup B \to \{0,1\}$ is continuous.

$f/_A : A \to \{0,1\}$ is also continuous.

But A is connected.

Hence $f/_A$ is not onto (by theorem 4.1.6)

Therefore $f(x) = 0$ for all $x \in A$ or $f(x) = 1$ for all $x \in A$.

But $f(x_0) = 0$ and $x_0 \in A$.

Therefore $f(x) = 0$ for all $x \in A$.

Similarly, $f(x) = 0$ fo for all $x \in B$

Therefore $f(x) = 0$ for all $x \in A \cup B$.

Thus any continuous function f: $A \cup B \to \{0,1\}$ is not onto.

Therefore $A \cup B$ is connected. (by theorem 4.1.6)

4.2 CONNCETED SUBSETS OF R.

Theorem 4.2.1*

A subspace of R is connected $\Leftrightarrow$ it is an interval.

Proof.

Let A be a connected subspace of R.

Claim: A is an interval.

Space for hints

Suppose that A is not an interval.

Then there exist $a,b,c \in R$ such that $a < b < c$ and $a,c \in A$ but $b \notin A$. ϕ

Let $A_1 = (-\infty,b) \cap A$ and $A_2 = (b, \infty) \cap A$.

Since $(-\infty,b)$ and (b, ∞) are open sets in R, A_1 and A_2 are open sets in A.

Also, $A_1 \cap A_2 = \phi$ and $A_1 \cup A_2 = A$

Further $a \in A_1$ and $c \in A_2$.

Hence $A_1 \neq \phi$ and $A_2 \neq \phi$.

Thus A is the union of two disjoint non-empty open sets A_1 and A_2.

Hence A is not connected.

This is a contradiction to the fact that A is connected .

Therefore A is an interval.

Conversely, assume that A is an interval.

Claim: A is connected.

Suppose A is not connected.

Then $A = A_1 \cup A_2$, $A_1 \neq \phi$, $A_2 \neq \phi$, $A_1 \cap A_2 = \phi$ where A_1 and A_2 are closed sets in A.

Let $x \in A_1$ and $z \in A_2$.

Since $A_1 \cap A_2 = \phi$, $x \neq z$.

Without loss of generality assume that $x < z$.

Since A is an interval, we have $[x,z] \subseteq A$.

(i.e) $[x,z] \subseteq . A_1 \cup A_2$.

Therefore every element of $[x,z]$ is either in A_1 or in A_2 .

Define y by

$y = \text{lub} \{ [x,z] \cap A_1\}$

Clearly, $x \leq y \leq z$.

Since $[x,z] \subseteq A$, $y \in A$.

Space for hints

Let $\varepsilon > 0$ be given.

Then by the definition of l.u.b , there exists $t \in [x,z] \cap A_1$ such that $y - \varepsilon < t \leq y$.

Therefore $(y - \varepsilon,\ y + \varepsilon) \cap ([x,z] \cap A_1) \neq \phi$.

Therefore $y \in \overline{[x,z] \cap A_1}$

Therefore $y \in [x,z] \cap A_1$.

Therefore $y \in (y - \varepsilon,\ y + \varepsilon) \cap ([x,z] \cap A_1)$.

Hence $y \in (y - \varepsilon,\ y + \varepsilon) \cap ([x,z] \cap A_1) \neq \phi$.

Therefore y is a limit point of $[x,z] \cap A_1$ ....→(1)

We know that, for any set A, $\overline{A} = A \cup D(A)$.

From (1) , $y \in D([x,z] \cap A_1)$.

But $\overline{[x,z] \cap A_1} = [x,z] \cap A_1 \cup D([x,z] \cap A_1)$.

Since [x,z] and A_1 are closed sets, $[x,z] \cap A_1$ is also closed.

Therefore $y \in [x,z] \cap A_1$.

Hence $y \in A_1$ → (2)

By the definition of y, $y + \varepsilon \in A_2$, for all $\varepsilon > 0$ such that $y + \varepsilon \leq z$.

Hence $y \in (y - \varepsilon,\ y + \varepsilon)$.

Also y is a limit point of A_2.

Therefore $y \in D(A_2)$.

But $\overline{A_2} = A_2 \cup D(A_2)$.

Therefore $y \in \overline{A_2}$.

A_2 is closed, $A_2 = \overline{A_2}$.

Therefore $y \in A_2$. →(3)

From (2) and (3)

$y \in A_1 \cap A_2$.

$A_1 \cap A_2 \neq \phi$.

Space for hints

This is contradiction , since $A_1 \cap A_2 = \phi$.

Therefore A is connected.

Theorem 4.2.2

R is connected.

Proof.

R = (- ∞, ∞) is an interval.

Therefore R is connected.(By Theorem 4.2.1)

PROBLEMS

Problem 4.2.3

Give an example to show that a subspace of a connected metric space need not be connected.

Solution.

We know that R is connected.

$M_1 = [1,2] \cup [3,4]$ is a subspace of R with usual metric.

Since [1,2] and [3,4] are open sets in M_1, M_1 is disconnected.

Problem 4.2.4

Prove or disprove: if A and C are connected subsets of a metric space M

and if $A \subseteq B \subseteq C$, then B is connected.

Solution.

We disprove this statement by giving a counter example.

Let A = [1,2] ; B =[1,2] ∪ [3,4] , C = R.

Clearly, $A \subset B \subset C$.

Here A and C are connected.

But B is not connected.

4.3 CONNECTEDNESS AND CONTINUITY

Space for hints

Theorem 4.3.1

Let M_1 be a connected metric space. Let M_2 be any metric space.

Let $f : M_1 \rightarrow M_2$ be a continuous function . Then $f(M_1)$ is a connected subset of M_2.

(i.e) Any continuous image of a connected space is connected.

Proof.

Let M_1 be a connected metric space.

Let M_2 be any metric space.

Let $f : M_1 \rightarrow M_2$ be a continuous function .

Let $A = f(M_1)$.

Claim: A is connected.

Suppose A is not connected.

We know that , M is connected iff M and ϕ are the only sets which are both

open and closed in M_1.

Since A is not connected, by the above theorem, there exists

a proper non- empty subset B of A which is both open and closed in A.

We know that, f is continuous iff inverse image of open set is open.

By the above theorem, $f^{-1}(B)$ is a proper open subset of M_1.

Therefore $M_1 = f^{-1}(B) \cup [f^{-1}(B)]^c$.

Since $f^{-1}(B)$ is closed, $[f^{-1}(B)]^c$ is open.

Therefore M_1 can be written as the union of two

disjoint non-empty open sets.

Therefore M_1 is not connected.

This is a contradiction .

Therefore A is connected.

Theorem 4.3.2 (Intermediate value theorem)

Space for hints

Let f be a real valued continuous function defined on an interval I.

Then f takes every value between any two values it assumes.

Proof.

Let f be a real valued continuous function defined on an interval I.

Let a,b $\in$ I and f(a) $\neq$ f(b).

Without loss of generality, assume that f(a) < f(b).

Let c be such that f(a) < c < f(b).

We know that the following theorem

" A subspace of R is connected iff it is an interval"

Since I is an interval, by the above theorem, I is connected subset of R.

We know : let M_1 be a connected metric space . Let M_2 be any metric space.

Let f: $M_1 \to M_2$ be a continuous function. then f(M_1) is a connected subset of M_2.

By the above theorem, f(I) is connected.

Therefore f(I) is an interval. By Theorem 4.2.1)

Clearly, f(a), f(b) $\in$ f(I),

Hence [f(a), f(b)] $\subseteq$ f(I).

Since f(a) < c < f(b), c $\in$ f(I).

Therefore c = f(x) for some x $\in$ I.

PROBLEMS

Problem 4.3.3

Prove that if f is a non- constant real valued continuous function

on R then the range of f is uncountable.

Solution.

We know that R is connected.

Since f is a continuous function on R, f(R) is a

connected subset of R .(By Theorem 4.3.1)

Space for hints

Therefore f (R) is an interval in R.

Also, since f is a non- constant function the interval f(R) contains more than one point.

Therefore f(R) is uncountable.

(i.e) the range of f is uncountable.

Problem 4.3.4

Give an example to show that union of two connected sets need not be connected.

Solution.

Let A = [1,2] and B = [3,4].

We know that, A subspace o R is connected iff it is an interval.

By the above theorem, [1,2] and [3,4] are connected.

Let $C = A \cup B = [1,2] \cup [3,4]$.

We know the following theorem,

Let M_1 be a subspace of a metric space M. Let $A_1 \subseteq M_1$.

Then A_1 is open in M_1 iff there exists an open set A in M such that $A_1 = A \cap M_1$.

By the above theorem, [1,2] and [3,4] are open sets in C.

Hence C can be written as the union of two disjoint non-empty open sets.

Therefore, C is not connected.

Exercises:

1. Prove that if f: **R**→ **R** is a continuous function which assumes only rational values then f is a constant function.

(*Hint.* Use intermediate value theorem).

2. Prove that $A = \{(x,y) / x^2+y^2 = 1\}$ is a connected subset of R^2.

(Hint. Consider $f: [0,2\Pi] \rightarrow A$ given by f(x) = (cos x, sin x)).

UNIT- 5

Space for hints

5.1 COMPACT METRIC SPACES

Definition 5.1.1

Let M be a metric space. A family of open sets $\{G_\alpha\}$ in M is called an ***open cover*** for M if $\bigcup G_\alpha = M$.

A subfamily of $\{G_\alpha\}$ which itself is an open cover is called a ***sub-cover***.

A metric space M is said to be ***compact*** if every open cover for M has a finite subcover.

(i.e) for each family of open sets $\{G_\alpha\}$ such that $\bigcup G_\alpha = M$, there exists a finite subfamily $\{G_{\alpha_1}, G_{\alpha_2}, \ldots\ldots, G_{\alpha_n}\}$ such that $\bigcup_{i=1}^{n} G_{\alpha_i} = M$.

Example 5.1.2

R with usual metric is not compact.

Proof.

Consider the family of open intervals $\{G_n\} = \{(-n,n) / n \in N\}$.

We know that " Every open interval is an open set".

This is a family of open sets in R.

Clearly, $\bigcup_{n=1}^{\infty} (-n,n) = R$.

Therefore $\{G_n\} = \{(-n,n) / n \in N\}$ is an open cover for R.

This open cover has no finite subcover.

Therefore R is not compact.

Example 5.1.3

(0,1) with usual metric is not compact.

Proof.

Consider the family of open intervals $\{(1/n,1) / n = 2,3,\ldots.\}$.

Clearly, $\bigcup_{n=2}^{\infty}$ (1/n,1) = (0,1).

Therefore {(1/n,1) / n = 2,3,….} is an open cover for (0,1) and

this open cover has no finite subcover.

Hence (0,1) is not compact.

Example 5.1.4

[0,∞) with usual metric is not compact.

Proof.

Consider the family of intervals { [0,n) / n∈ N }.

Also $\bigcup_{n=1}^{\infty}$ [0,n) = [0,∞).

Therefore { [0,n) / n∈ N } is an open cover for [0,∞).

This open cover has no finite subcover.

Hence [0,∞) is not compact.

Example 5.1.5

Let M be an infinite set with discrete metric. Then M is not compact.

Proof.

Let x ∈ M.

Since M is a discrete metric space, {x} is open in M.

Also, $\bigcup_{x\in M}$ {x} = M.

Hence { {x} / x ∈ M } is an open cover for M and

since M is infinite this open cover has no finite subcover.

Hence M is not compact.

Theorem 5.1.6

Let M be a metric. Let A⊆ M.

A is compact iff given a family of open sets

{ G_α } in M such that ∪ G_α ⊇A there exists a

Space for hints

Space for hints

subfamily $G_{\alpha_1}, G_{\alpha_2}, \ldots\ldots\ldots, G_{\alpha_n}$ such that $\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq$ A.

Proof.

Let M be a metric space.

Let A $\subseteq$ M.

Suppose that A is compact.

Let $\{G_\alpha\}$ be a family of open sets in M such that $\cup G_\alpha \supseteq$ A.

Then $(\cup G_\alpha) \cap A = A$.

Therefore $\cup (G_\alpha \cap A) = A$. ....(1).

We know that, "Let M be a metric space and M_1 be a subspace of M. Let $A_1 \subseteq M_1$.

Then A_1 is open in M_1 iff there exists an open set A in M such that $A_1 = A \cap M_1$."

By the above result, $G_\alpha \cap A$ is an open set in A.

Therefore $\{G_\alpha \cap A\}$ is a family of open sets in A.

From (1), $\{G_\alpha \cap A\}$ is an open cover for A.

Since A is compact, this open cover

has a finite subcover namely, $G_{\alpha_1} \cap A, G_{\alpha_2} \cap A, \ldots\ldots\ldots, G_{\alpha_n} \cap A$.

Therefore $\bigcup_{i=1}^{n} (G_{\alpha_i} \cap A) = A$.

Therefore $(\bigcup_{i=1}^{n} G_{\alpha_i}) \cap A = A$.

Therefore $\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$.

Conversely, let $\{H_\alpha\}$ be an open cover for A.

Therefore each H_α is open in A.

Therefore $H_\alpha = G_\alpha \cap A$, where G_α is open in M.

Since family of $\{H_\alpha\}$ is open cover for A, $\bigcup H_\alpha = A$.

Therefore $\bigcup (G_\alpha \cap A) = A$.

Therefore $(\bigcup G_\alpha) \cap A = A$.

Therefore $\bigcup G_\alpha \supseteq A$.

Hence by hypothesis, there exists a finite subfamily, $G_{\alpha_1}, G_{\alpha_2}, \ldots\ldots, G_{\alpha_n}$ such that $\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$.

Therefore $(\bigcup_{i=1}^{n} G_{\alpha_i}) \cap A = A$.

Therefore $\bigcup_{i=1}^{n} (G_{\alpha_i} \cap A) = A$

Therefore $\bigcup_{i=1}^{n} H_{\alpha_i} = A$.

Thus $\{H_{\alpha_i}\}$, i= 1,2,….,n is a finite subcover of the open cover $\{H_\alpha\}$.

Therefore A is compact.

Theorem 5.1.7

Any compact subset A of a metric space M is bounded.

Proof.

Let M be a metric space.

Let A be a compact subset of M.

Claim: A is bounded.

Let $x_0 \in M$.

Consider $\{B(x_0, n) / n \in N\}$.

Clearly, $\bigcup_{n=1}^{\infty} B(x_0,n) = M$.

Since A is a subset of M,

$$\bigcup_{n=1}^{\infty} B(x_0,n) \supseteq A.$$

Space for hints

Space for hints

Since A is compact, there exists finite subfamily say, $B(x_0,n_1)$, $B(x_0,n_2)$,... $B(x_0,n_k)$ such that

$$\bigcup_{n=1}^{k} B(x_0,n_i) \supseteq A.$$

Let $n_0 = \max\{n_1,n_2,\ldots,n_k\}$.

$$\bigcup_{n=1}^{k} B(x_0,n_i) = B(x_0,n_0).$$

$$\Rightarrow B(x_0,n_0) \supseteq A.$$

We know that, every open ball is an open set and every open ball is bounded.

Therefore $B(x_0,n_0)$ is bounded.

We know that, subset of bounded set is bounded.

Therefore A is bounded.

Note 5.1.8

Converse of the above theorem is not true.

(0,1) is a bounded subset of R.

But (0,1) is not compact.

Theorem 5.1.9

Any compact subset A of a metric space (M,d) is closed.

Proof.

Let M be a metric space.

Let A be a compact subset of a metric space M.

Claim: A is closed.

It is enough to prove that A^c is open.

Let $y \in A^c$, $x \in A$.

Then $x \neq y$.

Therefore $d(x,y) = r_x > 0$. (since $d(x,y) = 0 \Leftrightarrow x = y$.).

Clearly, $B(x, r_x/2)$ $B(y, r_x/2) = \phi$.

Now, consider the collection $\{ B(x, r_x/2) \ / \ x \in A \}$.

Clearly, $\bigcup_{x\in A} (B(x, r_x/2)) \supseteq A$.

Since A is compact, there exists a finite number

of such open ball say $B(x_1, r_{x_1}/2), \ldots\ldots, B(x_n, r_{x_n}/2)$

such that $\bigcup_{i=1}^{n} B(x_i, r_{x_i}/2) \supseteq A$. ....(1).

Let $V_y = \bigcap_{i=1}^{n} B(y, r_{x_i}/2)$.

V_y is open set containing y.

Since $B(y, r_y/2) \bigcap B(x, r_x/2) = \phi$,

$V_y \cap B(x_i, r_{x_i}/2) = \phi$, for all I = 1,2,….,n.

Therefore $V_y \cap [\bigcup_{i=1}^{n} B(x_i, r_{x_i}/2)] = \phi$.

Therefore $V_y \cap A = \phi$. (by (1))

Therefore $V_y \subseteq A^c$.

Therefore $\bigcup_{y\in A^c} V_y = A^c$ and each V_y is open.

Therefore A^c is open.

Hence A is closed.

Note 5.1.10

The converse of the above theorem is not true.

For example, $[0,\infty)$ is a closed subset of R.

But it is not **compact.**

Theorem 5.1.11

A closed subspace of a compact metric space is compact.

Proof.

Let M be a compact metric space.

Space for hints

Space for hints

Let A be a non-empty closed subset of M.

Claim: A is compact.

Let $\{G_\alpha / \alpha \in I\}$ be a family of open sets in M such that $\bigcup_{\alpha \in I} G_\alpha \supseteq A$.

Therefore A^c is open, since A is closed.

Therefore $\{G_\alpha / \alpha \in I\}$ is open cover for M.

Since M is compact, it has a finite subcover say $G_{\alpha_1}, G_{\alpha_2}, \ldots\ldots, G_{\alpha_n}$, A^c.

Therefore $(\bigcup_{i=1}^{n} G_{\alpha_i}) \cup A^c = M$.

Therefore $\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$.

Therefore A is compact.

5.2 COMPACT SUBSETS OF R

We know that , every subset of a compact space is closed and bounded.

However the converse is not true.

For example,

Consider an infinite discrete metric space (M,d).

Let A be an infinite subset of M.

Then A is bounded since $d(x,y) \leq 1$, for all $x,y \in A$.

Also, A is closed since any subset of a discrete metric space is closed.

Hence A is closed and bounded.

However, A is not compact because in an infinite discrete metric space M,

$\{x\}$ is open in M, where $x \in M$.

Also, $\bigcup_{x \in M} \{x\} = M$.

Hence $\{ \{x\} / x \in M \}$ is an open cover for M and since M is infinite,

this open cover has no finite subcover.

Space for hints

Hence M is not compact.

In R with usual metric the converse is also true.

Theorem 5.2.1 (Heine Borel Theorem)

Any closed interval [a,b] is a compact subset of R.

Proof:

Let [a,b] be a subset of R.

Claim: [a,b] is a compact subset of R.

Let $\{G_\alpha / \alpha \in I\}$ be family of open sets in R such that $\bigcup_{\alpha \in I} G_\alpha \supseteq [a,b]$.

Let S = { x / x ∈ [a,b] and [a,x] can be covered by a finite number of G_α's } ....(1).

Clearly a ∈ S and hence S ≠ ϕ.

Also, S is bounded above by b.

Let c denote the l.u.b. of S and clearly c ∈ [a,b].

Therefore $c \in G_{\alpha_1}$, for some $\alpha_1 \in I$.

Since G_{α_1} is open, there exists ε > 0 such that $(c-\varepsilon, c+\varepsilon) \subseteq G_{\alpha_1}$.

Choose $x_1 \in [a,b]$ such that $x_1 < c$ and $[x_1,c] \subseteq G_{\alpha_1}$.

Now, since $x_1 < c$, $[a,x_1]$ can be covered by a finite number of G_α's. (by (1)).

These finite number of G_α's together with G_{α_1} covers [a,c].

By using (1), c ∈ S.

Now, we have to prove that c = b.

Suppose c ≠ b.

Then choose $x_2 \in [a,b]$ such that $x_2 > c$ and $[c,x_2] \subseteq G_{\alpha_1}$.

Space for hints

As before, $[a,x_2]$ can be covered by a finite number of G_α's. (by (1)).

Hence $x_2 \in S$.

But $x_2 > c$ which is a contradiction, since c is the l.u.b of S.

Therefore c = b.

Therefore [a,b] can be covered by a finite number of G_α's.

Therefore [a,b] is a compact subset of R.

Theorem 5.2.2

A subset A of R is compact iff A is closed and bounded.

Proof.

If A is compact, then A is closed and bounded.

Conversely, let A be a subset of R which is closed and bounded.

Since A is bounded we can find a closed interval [a,b] such that $A \subseteq [a,b]$.

Since A is closed in R, A is closed in [a,b] also.

Thus A is closed subset of the compact space [a,b].

Since any closed subset is a compact subset of R, A is compact.

5.3 EQUIVALENT CHARACTERISATIONS FOR COMPACTNESS

Definition 5.3.1

Finite intersection property(FIP): A family J of subsets of a set M is said to

have the ***finite intersection property*** if any finite members of J have non-empty intersection.

Example 5.3.2

In R the family of closed intervals $J = \{[-n,n] / n \in N\}$ has finite intersection property.

Theorem 5.3.3

A metric space M is compact iff any family of closed sets with finite intersection property has non- empty intersection.

Space for hints

Proof.

Let M be a metric space.

Suppose M is compact.

Let $\{A_\alpha\}$ be a family of closed subsets of M with finite intersection property.

Claim: $\cap A_\alpha \neq \phi$.

Suppose $\cap A_\alpha = \phi$.

Then $(\cap A_\alpha)^c = \phi^c$.

Therefore $\cup A_\alpha^c = M$.

Also, since each A_α is closed , A_α^c is open.

Therefore $\{ A_\alpha^c \}$ is open cover for M.

Since M is compact this open cover has a finite subcover

say A_1^c , A_2^c ,........ A_n^c .

Therefore $\bigcup_{i=1}^{n} A_i^c = M$.

$\therefore \ (\bigcap_{i=n}^{n} A_i)^c = M$

$\therefore \ \bigcap_{i=n}^{n} A_i = \phi$,

Which is contradiction, since $\{A_\alpha\}$ has FIP to the definition of finite intersection property.

Therefore $\cap A_\alpha \neq \phi$.

That is, any family of closed sets with finite intersection property has non-empty intersection.;

Conversely, suppose that each family of closed sets in M with

Space for hints

finite intersection property has non- empty intersection.

Claim: M is compact.

Let $\{ G_\alpha / \alpha \in I \}$ be an open cover for M.

$$\therefore \bigcup_{\alpha\in I} G_\alpha = M$$

$$\therefore (\bigcup_{\alpha\in I} G_\alpha)^c = M^c$$

$$\therefore \bigcap_{\alpha\in I} {G_\alpha}^c = \phi$$

Since G_α is open , ${G_\alpha}^c$ is closed for each α.

Therefore $J = \{ {G_\alpha}^c / \alpha \in I \}$ is a family of closed sets whose intersection is empty.

Hence by definition this family of closed sets does not have

the finite intersection property.

Hence there exists a finite sub-collection of J say,

$\{ G_1^c, G_2^c, \ldots\ldots, G_n^c\}$ such that $\therefore \bigcap_{i=1}^{n} G_i^c = \phi$

$$\Rightarrow (\bigcup_{i=1}^{n} G_i)^c = \phi$$

$$\Rightarrow \bigcup_{i=1}^{n} G = M$$

Therefore $\{G_1, G_2, \ldots. G_n\}$ is a finite sub cover of the given open cover.

Hence M is compact.

Definition 5.3.4

A metric space M is said to be ***totally bounded*** if for every $\varepsilon > 0$,

there exists a finite number of elements $x_1, x_2, \ldots x_n \in M$ such that

$B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \ldots\ldots \cup B(x_n, \varepsilon) = M$.

Space for hints

A non – empty subset A of a metric space M is said to be ***totally bounded***

if the subspace of A is a totally bounded metric space.

Theorem 5.3.5

Any compact metric space is totally bounded.

Proof.

Let M be a compact metric space.

Then $\{ B(x, \varepsilon) / x \in M \}$ is an open cover for M.

Since M is compact this open cover has a finite sub cover say

$B(x_1, \varepsilon), B(x_2, \varepsilon), \ldots\ldots\ldots, B(x_n, \varepsilon)$.

Therefore $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \ldots\ldots\ldots \cup B(x_n, \varepsilon) = M$.

Therefore M is totally bounded.

Theorem 5.3.6

Let A be a subset of a metric space M. If A is totally bounded, then A is bounded.

Proof.

Let A be a subset of a metric space M.

Also, let A be a totally bounded subset of M.

Let $\varepsilon > 0$ be given.

Then there exists a finite number of points $x_1, x_2, \ldots, x_n \in A$ such that

$B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \ldots\ldots\ldots \cup B(x_n, \varepsilon) = A$, where $B(x_i, \varepsilon)$is an open ball in A.

We know that an open ball is a bounded set.

Thus A is union of a finite number of bounded sets and hence A is bounded.

Note 5.3.7

The converse of the above theorem is not true.

For example,

Space for hints

Let M be an infinite set with discrete metric.

Clearly, M is bounded.

Now, B $(x_1,1/2) = \{x\}$

Since M is infinite, M cannot be written as the union of a finite number of open balls B $(x_1,1/2)$.

Therefore M is not totally bounded.

Definition 5.3.8

Let (x_n) be a sequence in a metric space M.

Let $n_1 < n_2 \ldots\ldots\ldots < n_k < \ldots\ldots$ be an increasing sequence of positive integers. Then (x_{n_k}) is called a ***subsequence*** of (x_n).

Theorem 5.3.9

A metric space (M,d) is tot bounded iff every sequence in M has a Cauchy subsequence.

Proof.

Let (M,d) be a metric space.

Suppose every sequence in M has a Cauchy subsequence.

Claim: M is toally bounded;

Let $\varepsilon > 0$ be given.

Choose $x_1 \in M$.

If B $(x_1, \varepsilon) = M$, then obviously M is totally bounded.

If B $(x_1, \varepsilon) \neq M$, choose $x_2 \in M$ - B (x_1, ε) so that $d(x_1,x_2) \geq \varepsilon$.

Now, if B $(x_1, \varepsilon) \cup$ B $(x_2, \varepsilon) = M$.

Then M is totally bounded.

In not, choose $x_3 = M -$ [B (x_1, ε) U B (x_2, ε)] and so on.

Suppose this process does not stop at a finite stage.

Then we obtain a sequence $x_1, x_2, \ldots x_n, \ldots$ Such that $d(x_n,x_m) \geq \varepsilon$ if $n \neq m$.

Space for hints

Clearly this sequence (x_n) cannot have a Cauchy subsequence.

Hence the above process stops at a finite stage and

we get a finite set of points $\{ x_1, x_2, \ldots x_n \}$

Such that $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \ldots\ldots\ldots \cup B(x_n, \varepsilon) = M$.

Therefore M is totally bounded.

Conversely, suppose M is totally bounded.

Let $S_1 = \{x_{1_1}, x_{1_2}, \ldots\ldots x_{1_n} \ldots\ldots\ldots\}$ be a sequence in M.

If one term of the sequence is infinitely repeated,

then S_1 contains a constant subsequence which is obviously a Cauchy sequence.

Hence we assume that no term of S_1 is infinitely repeated

so that the range of S is infinite.

Now , since M is totally bounded, M can be covered by a

finite number of open balls of radius ½.

Hence at least one of these ball must contain an infinite number

of terms of sequence S_1.

Therefore S_1 contains a subsequence S_2 ,

$S_2 = \{x_{2_1}, x_{2_2}, \ldots\ldots x_{2_n} \ldots\ldots\ldots\}$ all terms of which lie within an open ball of radius ½.

Similarly, S_2 contains a subsequence $S_3 = \{x_{3_1}, x_{3_2}, \ldots\ldots x_{3_n} \ldots\ldots\ldots\}$ all terms of which lie within an open ball of radius 1/3.

We repeat this process of forming successive subsequences and finally we take the diagonal

sequence.

$S = (\{x_{1_1}, x_{2_2}, \ldots\ldots x_{n_n} \ldots\ldots\ldots\}$ we claim that S is a Cauchy subsequence of S_1.

If m> n both x_{m_m} and x_{n_n} lie within a open ball of radius 1/n.

Therefore $d(x_{m_m}, x_{n_n}) < 2/n$.

Space for hints

Hence $d(x_{m_m}, x_{n_n}) < \varepsilon$ if $n,m > 2/\varepsilon$.

This shows that S is a Cauchy subsequence of S_1.

Thus every sequence in M contains a Cauchy subsequence.

Corollary 5.3.10

A non- empty subset of a totally bounded set is totally bounded.

Proof.

Let A be a totally bounded subset of a metric space M.

Let B be a non- empty subset of A.

Let (x_n) be a sequence in B.

Thererfore (x_n) is a sequence in A .

Since A is totally bounded , (x_n) has a cauchy subsequence. (by theorem 5.3.9)

Thus every sequence in B ha ... uchy subsequence.

Therefore B is totally bounded.

Definition 5.3.11

A metric space M is said to be ***sequentially compact*** if every sequence in

M has a convergent subsequence.

Theorem 5.3.12

Let (x_n) be a Cauchy sequence in a metric space M.

If (x_n) has a subsequence (x_{n_k}) converging to x, then (x_n) converges to x.

Proof.

Let (x_n) be a Cauchy sequence in a metric space M.

Let $\varepsilon > 0$ be given.

Since (x_n) is a Cauchy sequence, there exists a positive integer m_1 such that

$d(x_n,x_m) \geq \varepsilon/2$ for all $n, m \geq m_1$.$\rightarrow$(1)

Also, since $(x_{n_k}) \rightarrow x$, there exists a positive integer m_2 such that

$d(x_{n_k},x) \geq \varepsilon/2$ for all $n_k \geq m_2$.$\rightarrow$(2)

Let $m_0 = \max\{m_1, m_2\}$ and fix $n_k \geq m_0$.

Then $d(x_n,x) \leq d(x_n, x_{n_k}) + d(x_{n_k},x)$

$< \varepsilon/2 + \varepsilon/2$ for all $n \geq m_0$. (by (1) & (2)).

$= \epsilon$ for all $n \geq m_0$.

Hence $(x_n) \longrightarrow x$.

Theorem 5.3.13

In a metric space M the following are equivalent.

(i) M is compact.

(ii) Any infinite subset of M has a limit point.

(iii) M is sequentially compact.

(iv) M is totally bounded and complete.

Proof.

(i) => (ii)

Let us assume that a metric space M is compact.

Also, let A be an infinite subset of M.

Suppose A has no limit point in M.

Let $x \in M$.

Since x is not a limit point of A, there exists an open ball $B(x, r_x)$ such that

$B(x, r_x) \cap (A - \{x\}) = \phi$.

Therefore $B(x, r_x) \cap A = \begin{cases} \{x\} & \text{if } x \in A. \\ \phi & \text{if } x \notin A. \end{cases}$

Now, $\{ B(x, r_x) / x \in M \}$ is open cover for M.

Also, each $B(x, r_x)$ covers at most one point of the infinite set A .

Hence this open cover cannot have a finite subcover which is a contradiction be the assumption.

Hence A has at least one limit point.

(ii) => (iii)

Let us assume that, A is an infinite subset of a metric space M having a limit point.

Space for hints

Claim: M is sequentially compact.

Let (x_n) be a sequence in M.

If one term of the sequence is infinitely repeated then ,

(x_n) contains a constant subsequence which is convergent.

Otherwise (x_n) has an infinite number of terms.

By the assumption this infinite set has limit point , say x.

"let (M,d) be a metric space. Let $A \subseteq M$. Then x is a limit point of A iff

each open ball with centre x contains an infinite number of points of A".

By the above statement, for any $r > 0$ the open bal l $B(x, r)$ contains

infinite number of terms of the sequence (x_n).

Now , choose a $n_1 > 0$, such that $x_{n_1} \in B(x,1)$

Then choose $n_2 > n_1$, such that $x_{n_2} \in B(x,1/2)$.

In general for $k > 0$, choose n_k such that $n_k > n_{k-1}$ and $x_{n_k} \in B(x,1/k)$.

Clearly, (x_{n_k}) is a subsequence of (x_n) .

Also, $d(x_{n_k} ,x) < 1/k$.

Therefore $(x_{n_k}) \to x$.

Thus (x_{n_k}) is a convergent subsequence of (x_n)

Hence M is sequentially compact.

(iii)=.> (iv):

Let us assume that M is sequentially compact.

That is every sequence in M has a convergent subsequence.

But every convergent sequence is Cauchy sequence.

Thus every sequence in M has a Cauchy subsequence.

By the theorem ," A metric space (M,d) is totally bounded iff every sequence in M has a Cauchy subsequence".

Therefore M is totally bounded.

Now we prove that M is complete.

Let (x_n) be a Cauchy sequence in M.

And (x_n) contains a convergent subsequence (x_{n_k}).

Let $(x_{n_k}) \to x$ (say)

Then $(x_n) \to x$ (by theorem 5.3.12)

Therefore M is complete.

(iv) => (i)

Let M be complete and totally bounded metric space.

Claim: M is compact.

Suppose M is not compact.

Then there exists an open cover $\{G_\alpha\}$ for M which has no finite sub cover.

Let $r_n = 1/2^n$.

Since M is totally bounded , M can be covered by a finite number of open balls of radius r_1.

Since M cannot be covered by a finite number of G_α's at least one of these open balls say $B(x_1,r_1)$ cannot be covered by a finite number of G_α's.

Now, $B(x_1,r_1)$ is totally bounded.

Hence as before we can find $x_2 \in B(x_1,r_1)$ such that $B(x_2,r_2)$ cannot be covered by a finite number of G_α's.

Proceeding like this we obtain a sequence (x_n) in M such that $B(x_n,r_n)$ cannot be covered by a finite number of G_α's. and $x_{n+1} \in B(x_n,r_n)$ for all n.

Now , $d(x_n,x_{n+p}) \leq$, $d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \ldots + d(x_{n+p-1},x_{n+p})$

$$< r_n + r_{n+1} + \ldots + r_{n+p-1}$$

$$= 1/2^n + 1/2^{n+1} + \ldots + 1/2^{n+p-1}$$

$$= 1/2^{n-1}(1/2 + 1/2^2 + \ldots + 1/2^p)$$

$$< 1/2^{n-1}$$

$d(x_n,x_{n+p}) < 1/2^{n-1}$.

Therefore (x_n) is a Cauchy sequence inM.

Space for hints

Space for hints

Since M is complete there exists $x \in M$ such that $(x_n) \to x$.

Now , $x \in G_\alpha$ for some α.

Since G_α is open we can find $\epsilon > 0$ such that $B(x, \epsilon) \subseteq G_\alpha$ $\to$(1)

We have $(x_n) \to x$ and $(r_n) = 1/2^n \to 0$.

Hence we can find $n_1 > 0$, such that $d(x_n, x) < \epsilon/2$ and $(r_n) < \epsilon/2$ for all $n \geq n_1$

Now fix $n \geq n_1$.

Claim: $B(x_n, r_n) \subseteq B(x, \varepsilon)$

Let $y \in B(x_n, r_n)$

Therefore $d(y,x_n) < r_n < \varepsilon/2$ (since $n \geq n_1$)

Now $d(y,x) \leq d(y,x_n) + d(x_n,x)$

$< \varepsilon/2 + \varepsilon/2.$

$= \varepsilon,$

(i.e) $d(y,x) < \varepsilon$.

Therefore $y \in B(x,\varepsilon)$ (by the definition of open ball).

Therefore $B(x_n, r_n) \subseteq B(x, \varepsilon) \subseteq G_\alpha$ (by (1)).

Thus $B(x_n,r_n)$ is covered by the single set G_α which is a contradiction to the assumption.

Hence M is compact.

Theorem 5.3.14

R with usual metric is complete.

Proof.

Let (x_n) be a Cauchy sequence in R.

Then (x_n) is a bounded sequence and hence it contained in a closed interval [a,b].

Now, [a,b] is compact and hence it is complete.

Hence (x_n) converges to some point $x \in [a,b]$.

Thus every Cauchy sequence (x_n) in R converges to some point x in R and hence R is complete.

SOLVED PROBLEMS

Space for hints

Problem 5.3.15

Give an example of a closed and bounded subset of l_2 which is not compact.

Solution.

"We know that $d(x,y) = \left[\sum_{n=1}^{\infty} | x_n - y_n |^p\right]^{1/p}$ where $x = (x_n)$ and $y = (y_n)$ then d is metric on l_p"

Consider $\mathbf{0} = (0,0,0,\ldots\ldots) \in l_2$.

Consider the closed ball B[0,1].

Clearly, B[0,1] is bounded.

Also, B[0,1] is a closed set.

Claim: B[0,1] is not compact.

Consider $e_1 = (1,0,0,\ldots..)$; $e_2 = (0,1,0,\ldots.)$; .... $e_n = (0,0,0,\ldots..,1,0,\ldots.)$.

Now, $d(0,e_n) = 1$ and hence $e_n \in B[0,1]$ for all n.

Thus (e_n) is a sequence in B[0,1].

Also, $d(e_n,e_m) = \sqrt{2}$ if $n \neq m$.

Hence the sequence (e_n) doesn't contain a Cauchy subsequence.

Therefore B[0,1] is not totally bounded. (by previous theorem)

Therefore B[0,1] is not compact.

Problem 5.3.16

Prove that any totally bounded metric space is separable.

Solution.

Let M be a totally bounded metric space.

Claim: M contains a countable dense subset.

For each natural number 'n', let $A_n = \{x_{n_1}, x_{n_2}, \ldots\ldots, x_{n_k}\}$ be a subset of M such that

$\bigcup_{i=1}^{k} B(x_{n_i}, 1/n) = M$. .... (1) (by definition)

Space for hints

Let $A = \bigcup_{n=1}^{\infty} A_n$.

Since each A_n is finite, A is a countable subset of M. ....(2)

Claim: A is dense in M.

Separable: "A metric space M is said to be ***separable*** if there exists a countable dense subset in

M."

Let $B(x,\varepsilon)$ be any open ball of radius $\varepsilon > 0$.

Choose a natural number 'n' such that $1/n < \varepsilon$.

Now, $x \in B(x_{n_i}, 1/n)$ for some i (by using (1)).

Therefore $d(x_{n_i}, x) < 1/n < \varepsilon$.

Therefore $(x_{n_i}) \in B(x,\varepsilon)$ (by the definition of open ball).

Thus every open ball in M has non-empty intersection with A (limit point definition).

Therefore $B(x,\varepsilon) \cap A \neq \phi$.

We know that, " Let M be a metric space and A M. Then the following are equivalent.

(i) A is dense in M.

(ii) The only closed set which contains A in M.

(iii) The only open set disjoint from A is ϕ.

(iv) A intersects every non-empty open set.

(v) A intersects every open ball."

Thus A is a countable dense subset of M.

Hence M is separable.

Problem 5.3.17

Prove that any bounded sequence in R has a convergent subsequence.

Solution.

Let (x_n) be a bounded sequence in R.

Then there exists a closed interval [a,b] such that $x_n \in [a,b]$, for all n.

Space for hints

Thus (x_n) is a sequence in the compact metric space [a,b].

(by Heine Borel Theorem: Any closed interval [a,b] is a compact subset of R).

We know that, "In a metric space M the following ar equivalent.

(i) M is compact.

(ii) Any infinite subset of M has a limit point.

(iii) M is sequentially compact.

(iv) M is totally bounded and complete."

Then (x_n) has a convergent subsequence.

Problem 5.3.18

Prove that the closure of a totally bounded set is totally bounded.

Solution.

Let A be a totally bounded subset of M.

Claim: $\overline{A}$ is totally bounded.

We shall show that every sequence in $\overline{A}$ contains a Cauchy subsequence.

Let (x_n) be a sequence in $\overline{A}$.

Let $\varepsilon > 0$ be given.

Then since $x_n \in \overline{A}$, $B(x_n, 1/3\ \varepsilon) \cap A \neq \phi$.

Choose $y_n \in B(x_n, 1/3\ \varepsilon) \cap A$.

Therefore $d(y_n, x_n) < 1/3\ \varepsilon$ ....(1).

" A metric space (M,d) is totally bounded iff every sequence in M has Cauchy subsequence."

By using the above result,

Now (y_n) is sequence in A. Since A is totally bounded (y_n) contains

a Cauchy sequence say (y_{n_k}).

Hence there exists a natural number 'm' such that

$$d(y_{n_i}, y_{n_j}) < 1/3\ \varepsilon, \text{ for all } n_i, n_j \geq m \quad \text{................(2).}$$

Therefore $d(x_{n_i}, x_{n_j}) \leq d(x_{n_i}, y_{n_i}) + d(y_{n_i}, y_{n_j}) + d(y_{n_j}, x_{n_j})$.

Space for hints

$< 1/3\ \varepsilon + 1/3\ \varepsilon + 1/3\ \varepsilon.$

$= \varepsilon$, for all $n_i, n_j \geq m$ (by (1) and (2)).

Hence (x_{n_k}) is a Cauchy subsequence of (x_n).

Therefore $\overline{A}$ is totally bounded.

Problem 5.3.19

Let A be a totally bounded subset of R. Prove that $\overline{A}$ is compact.

Solution.

We know that " The closure of a totally bounded set is totally bounded."

Since A is totally bounded, $\overline{A}$ is also totally bounded. ....(1).

We know that, " A subset A of a complete metric space M is complete iff A is closed."

Also, since $\overline{A}$ is a closed subset of R and R is complete $\overline{A}$ is complete. ....(2)

From (1) and (2) $\overline{A}$ is totally bounded and complete.

" In a metric space M the following are equivalent.

(i) M is compact.

(ii) Any infinite subset of M has a limit point.

(iii) M is sequentially compact.

(iv) M is totally bounded and complete."

By using above theorem, it is proved that $\overline{A}$ is compact.

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