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DIRECTORATE OF DISTANCE EDUCATION

577

B.Sc., THIRD YEAR

PAPER - X

(ANCILLARY)

M A T H E M A T I C S

THEORY OF EQUATIONS AND NUMERICAL ANALYSIS

Madurai Kamaraj University
Madurai – 625 021

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SYLLABUS
PAPER - X
(ANCILLARY)
Theory of Equations and Numerical Analysis

UNIT A: THEORY OF EQUATIONS

Unit I: Theory of equations – imaginary roots-rational roots-Relation between the roots and coefficients of equations-symmetric function of the roots-sum of the power of the roots of an equation-Newton's Theorem.

Unit II: Transformation of equations-Roots multiplided by a given number-reciprocal roots-reciprocal equations-standard forms to increase and decrease the roots of a given equation by given quantity-Removal of terms.

Unit III: Descarte's rule of signs-Roll's theorem-Multiple roots-Strurn's Theorem-General solutions of the cubic equation-cardon's method.

UNIT – B: NUMBERICAL METHODS

Unit IV: Numerical solution of Algebraic and Transcendental equations-Iteration method-Newton-Raphson method-Method of False Position.

Unit V: Solutions of simultaneous linear equations-Gauss's method-Gauss's Jordan method-Iteration method-Gauss's Seidel method.

Unit VI: Interpolation-Newton's forward and backward formula-divided differences and their properties-Newton's divided difference formula-Gauss formula-stirling's formula-Bessel's formula-Laplace-Everett formula-Lagrange's formula-simple problems-inverse interpolation using Lagrange's formula-successive approximations-simple problems.

Unit VII: Finite Differences Forward, Backward differences-Operators-Relations-Properties-finding missing Terms-Inverse Operators-Factorial Notation.

Unit VIII: Numerical Differentiation-finding the first and second derivatives-maximum and minimum value of a function for the given data.

Unit IX: Numerical integration-newton's-cote's formula-Trapezoidal Rule-simpson's one-third rule-simpson's Three eights rule-Weddle's Rule

Unit X: Difference Equation-Solution of first and second order equation with constant coefficients

Books for Reference

Numerical Analysis By Manikavasagam Pillai & Narayanan.

CONTENTS

Theory of Equations and Numerical Analysis

Lessons Compiled By:
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Model Question Papers
Ancillary Paper – X
Theory of Equations And Numerical Analysis
Model Paper – I

Sec – A **8 x 5 = 40marks**

Answer any Eight out of 12 questions

1. Solve the equation $x^4-5x^3+4x^2+8x-8=0$ given that $1-\sqrt{5}$
2. Solve: $4x^4-20x^3+33x^2-20x+4=0$
3. Find the nature of the roots of the equation $4x^3-21x^2+18x+30=0$.
4. If α, β, γ are the roots of the equation $x^3+ax^2+bx+c=o$, find the equation whose roots are $\alpha\beta, \beta\gamma, \gamma\alpha$.
5. Represent the function $x^4-12x^3+42x^2-30x+9$ in the factorial notation.
6. Find by the method of iteration a real root of $2x-\log_{10}x=7$.
7. Solve the equation $14x-5y=5.5$

   $2x+7y=19.3$

   by Jacobi method.
8. Show that $\nabla(ay_i)=a\nabla y_i$ a being a constant.
9. Find the cubic polynomial for the following table

x :	0	1	2	3
y :	1	0	1	10

10. Derive simpson's $\frac{1}{3}$ rule.
11. Solve the equation $y_{k+2}+y_{k+1}+y_k=k.2^k$.
12. Using Lagrange's interpolation formula, fit a polymial to the data

x :	0	1	3	4
y :	-12	0	6	12

Sec – B

Answer any 6 out of 10 Questions **6 x 10 = 60 marks**

13. Solve the equation $x^4+2x^3-21x^2-22x+40=0$ whose roots are in Arithmetic progression.
14. Show that the equation $x^6+3x^2-5x+1=0$ has atleast four imaginary roots
15. If the sum of the two roots of the equation $x^4+px^3+qx^2+rx+s=0$ in equal to the sum of the other two roots, prove that $p^3+8r=4pq$.
16. Discuss the reality of the roots of $x^4+4x^3-2x^2-12x\alpha=0\ \forall\ \alpha$.
17. Solve the equation $x^4+4x^3+5x^2+2x-6=0$ by removing its second term.
18.

x :	21	25	29	33	37
y :	18.4708	17.8144	17.1070	16.3432	15.5154

Use Gauss's forward and backward formula to find y for x=30 for the above data.

19. Evaluate $I = \int_0^1 \frac{dx}{1+x}$ correct to four decimal places by both Trapezodal and simpson's $\frac{1}{3}$ rules with h=0.125

20. From the below table of values of x & y obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for x =1.2

x :	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y:	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

21. Solve the following equation by method of differences

i) $u_{x+1} - u_x = x^2$, $u_0=1$

ii) $u_{x+2} - 7u_{x+1} + 12u_x = \cos x$ with $u_0=0=u$.

22. a) Write a short note on Inverse of Interpolation

b) Derive Lagrange's interpolation formula.

Model Paper – II

Theory of Equations and Numerical Analysis

Sec-A

Answer any Eight out of 12 questions

8 x 5 = 40 marks

1. If α, β, γ are the roots of $x^3+qx+r=0$ find the value of $\frac{1}{\beta+\gamma} + \frac{1}{\gamma+\alpha} + \frac{1}{\alpha+\beta}$
2. Remove the fractional coefficients form

   $X^3 - \frac{1}{4}x^2 + \frac{1}{3}x - 1 = 0$
3. Fine the nature of the roots of

   $X^4+4x^3-30x^2+10=0$
4. Solve the equation $8x^3-84x^2+262x-231=0$, if the roots are in A.P.
5. Find the multiple roots of the equation

   $27x^4 - 72x^2+64x-16=0$
6. Find a real root of the equation $x^3+x^2-1=0$ using iteration method.
7. Give the geometrical meaning of Newton's method
8. Using Guass Elimination method, find the inverse of the matrix. $\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$

9. Prove that $\mu\,\sigma = \frac{1}{2}\Delta + \frac{1}{2}\Delta E^{-1}$.

10. Evaluate $\int_{7.47}^{7.52} f(x)dx$ from the following table

X:	7.47	7.48	7.49	7.50	7.51	7.52
F(x):	1.93	1.95	1.98	2.01	2.03	2.06

11. Solve $y_{x+1} - 2y_x \cos\alpha + y_{x-1} = 0$

12. Derive Trapezoidal Rule.

Section – B

Answer Any 6 out of 10 questions **6 x 10 = 60 marks**

13. Solve the equation $3x^4 - 40x^3 + 130x^2 - 120x + 27 = 0$. Given that the product of two of its roots in equal to the product of the other two.

14. Find the equation whose roots are the cubes of the roots of $2x^5 + 3x^4 + 4x^3 - 2x^2 - 3x + 1 = 0$

15. Find the number of real roots of the equation $x^4 + 4x^3 - 4x - 13 = 0$ by sturm's theorem

16. Prove Newton-Gregory's backward interpolation formula.

17. From the following table, Using Lagrange's formula find Y when x=0.5

x:	0.4846555	0.4937452	0.5027498	0.5116682
y:	0.46	0.47	0.48	0.49

18. Find the minimum value of f(x) for the data.

x:	0	1	2	3	4	5	6	7
f(x):	890	844	769	668	541	389	401	462
x :	8	9						
f(x):	495	530						

19. Prove the following identity

$$u_1 x + u_2 x^2 + u_3 x^3 + \ldots = \frac{x}{1-x}u_1 + \frac{x^2}{(1-x^2)}x\Delta u_1 + \frac{x_3}{1-x^3}\Delta^2 u_1 + \ldots$$

20. Using Numerical differentiation find the value of sec 31° for the following table:

θ :	31	32	33	34
tan θ:	0-6008	0.6249	0.6494	0.6745

21. Evalute $\int_0^6 \sqrt{x(1-x)}\,dx$, Using weddles rule.

22. From the following table, obtain f(x) as a polynomial in powers of (x-5)

x:	0	2	3	4	5	6
f(x):	4	26	58	112	466	922.

Model Paper –3

Theory of equation and Numerical Analysis

Sec-A **8 x 5 = 40marks**

Answer any Eight out of 12 questions

1. If $\alpha, \beta; \gamma$ are the roots of $x^3 - 5x+7=0$ find $\sum\frac{\alpha^2-\beta\gamma}{\beta+\gamma}$
2. Solve the equation $x^4-6x^3+11x^2-10x+2=0$ if $2+i\sqrt{3}$ is a root of the equation.
3. Solve the equation $x^3-12x^2+39x-28=0$ given that the roots are in A.P.
4. Find the multiple roots of the equation $27x^4-72x^2+64x-16=0$.
5. Using method of iteration find the real root of the equation $2x-\log_{10}x=7$ correct to four decimal places.
6. Find the missing values in the following data:

X:	0	5	10	15	20	25
Y:	6	10	-	17	-	31

7. From the table of values.

x:	1.46	1.47	1.48	1.49
f(x):	0.885604	0.885633	0.885747	0.885945

Find f(1.4684)

8. Prove that $y_{n-k} = y_n - Kc_1 \nabla y_n + kc2 \nabla y_n + \ldots\ldots + (-1)^k \nabla^k y_n$
9. Find the value of $\frac{dy}{dx}$ at x=1.05 for the data.

X:	1.00	1.05	1.10	1.15	1.20	1.25	1.30
Y:	1.00000	1.02470	1.04881	1.07238	1.09544	1.11803	1.1401

10. Derive simpson's $\frac{1}{3}$ rule.
11. Obtain the Newton forward interpolation formula.
12. Solve the equation $Y_{k+2}+5y_{k+1}+6y_k=ex$

Sec –B

Answer any out of 10 question **6x10=60marks**

13. Show that the equation $\frac{A^2}{x-a}+\frac{B^2}{x-b}+\frac{C^2}{x-c}+\ldots\ldots\ldots\ldots\ldots+\frac{H^2}{x-h}=k$ cannot have an imaginary root. If A,B,C...H, a,b,c...h are real and distinct.
14. If α, β, γ are the roots of $x^3-7x+7=0$ find $\frac{1}{\alpha^4}+\frac{1}{\beta^4}+\frac{1}{\gamma^4}$
15. Show that the roots of $x^3+px^2+qx+r=0$ are in A.P if $2p^3-9pq+27r=0$
16. Solve using Cardon's method:
    $x^3-7x+6=0$
17. Solve the equation $x^4-8x^3+19x^2-12x+12=0$ by removing its second term.

18. Find the value of $e^{1.85}$ given
$e^{1.7}=5.4739$, $e^{1.8}=6.0496$, $e^{1.9}=6.6859$, $e^{2.0}=7.3891$,
$e^{2.1}=8.1662$, $e^{2.2}=9.0250$ $e^{2.3}=9.9742$

19. Using Lagrange's interpolation formula find Y(10) given that Y(5) = 12, Y(6) = 13, Y(9) = 14 Y(11)=16

20. From the following find the maximum value of the function:

X:	0	1	2	3	4	5
Y:	0	0.25	0	2.25	16.00	56.25

21. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ Using Trapezoidal rule with h=0.2.Hence determine the value of π .

22. Using the Gauss-Jordom method solve the following. Equations 10x+y+z=12, 2x+10y+z=13,x+y+5z=7

UNIT – I
THEORY OF EQUATIONS AND NUMERICAL ANALYSIS
UNIT – A
THEORY OF EQUATIONS

1.1 An Expression of the Form

$f(x)=a\,x^n+a_1x^{n-1}+a_2x^{n-2}+.....+a_n,$

Where n is a positive integer and $a_0,a_1,a_2,......a_n$ are constants is called a polynomial in x of the n^{th} degree if $a_0\neq 0$.

$f(x)=\ a_0x^n+a_1x^{n-1}+a_2x^{n-2}+........+a_n=0$ is called an algebraic equation or polynomial Equation of the nth degree, if $a_0\neq 0$.

An equation is not altered if all its terms be divided by any quantity.

Dividing the equation by a_0.

We can make the coefficient of x^n in the above equation equal to unity.

Then the equation can be written in the form

$X^n+P_1x^{n-1}+P_2x^{n-2}+........+P_{n-1}x+P_n = 0$

Equations of the first, second ,Third , fourth...... etc. degree are know as linear, quadratic , cubic biquadratic ...... equations respectively.

The term independent of x is called the absolute term.

Any value of x for which the polynomial f(x) vanishes is called a root of the equation f(x)=0.

The main object of the theory of equations is to find the roots of the equation f(x)=0(i.e),to solve the equation.

In this chapter unless otherwise stated f(x)represents always a polynomial in x we can easily see that f(x)is a continuous function of x for all values of x.

Remainder Theorem:

If f(x) is a polynomial then f(a) is the remainder when f(x) is divided by x-a.

Divide the polynomial f(x) by x-a until a remainder is obtained which does not involve x.

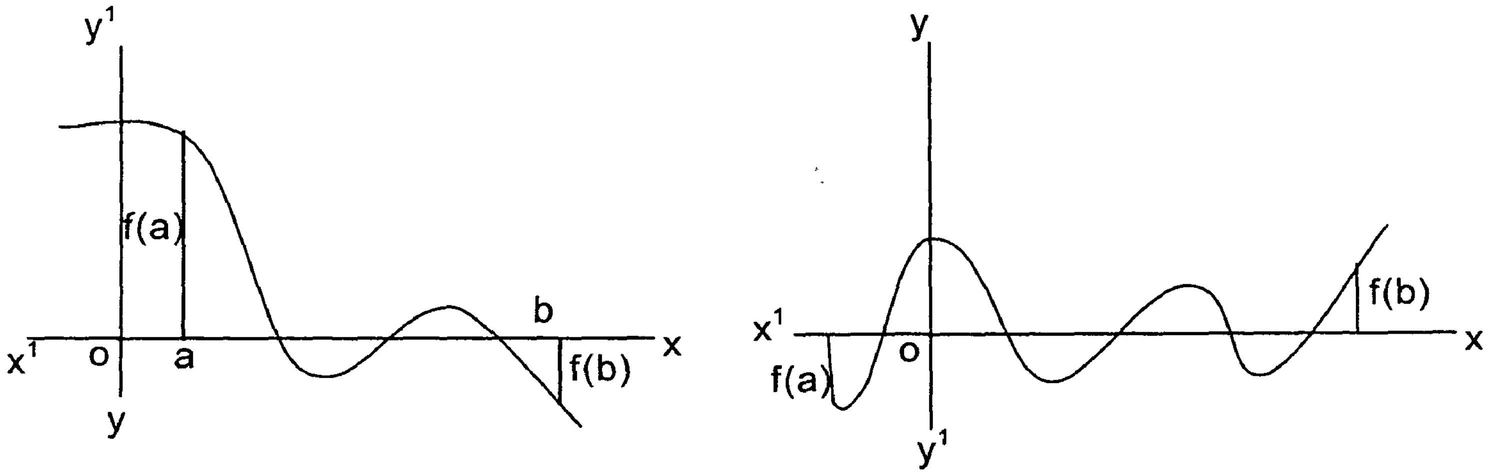
Let the quotient be Q(x) and remainder R.
Then f(x)=(x-a)Q(x)+R.
Substituting x=a in the above equation.
We get f(a)=R.

Cor:

If f(a)=0, the polynomial f(x) has the factor x-a,(ie) if a be the root of the equation f(x) = 0 then x-a is a factor of the polynomial f(x).

Theorem:

If f(a) and f(b) are of different signs, then atleast one root of the equation f(x)=0 must lie between a and b.



As x changes gradually from a to b, the function f(x) changes gradually from f(a) and f(b) and therefore must pass through all intermediate values, but since f(a) and f(b) have different signs, the value zero must be between them,

(ie) f(x) assumes the value zero for atleast for one value of x between a and b.

This theorem can be proved by means drawing the graph of the function y=f(x).

Since f(a) and f(b) have different signs, the graph y=f(x) must cross the x-axis atleast once between a and b.

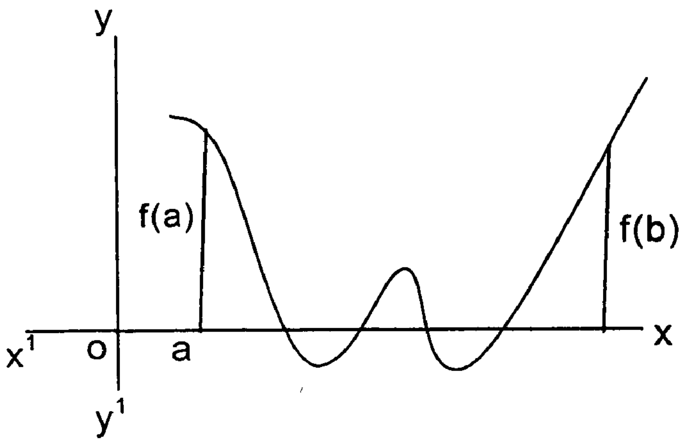
At the point where the graph crosser the x-axis there is a real root of f(x) =0.

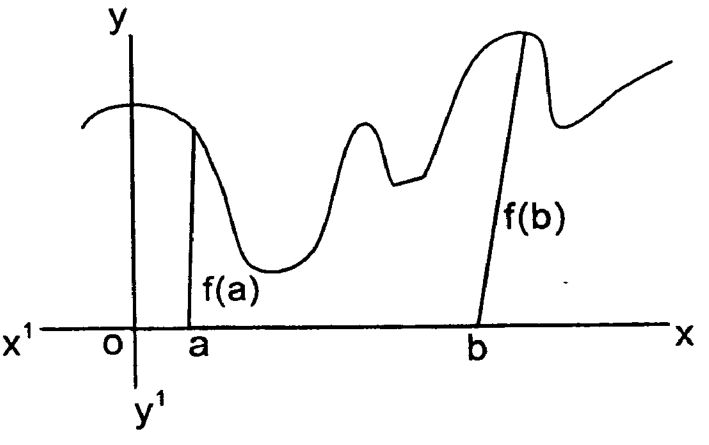
$\therefore$ There is atleast one real root between a and b.

Theorem:

If f(a) and f(b) have the same sign, it does not follow that f(x) =0 has no root between a and b.

It is evident that when two points are connected by a curve, the portions of the curve between these points must cut the axis at an odd number of times when the points are on opposite sides of the axis and an even number of times or not at all. When the points on the same side of the axis.





Hence we get the following results:-

1) If f(a) and f(b) have like signs, an even number of roots of f(x) = 0 lie between a and b or else there is no root between a and b.
2) If f(a) and f(b) have unlike signs, an odd number of roots of f(x)=0 lie between a and b.

Theorem:

If f(x)=0 is an equation of odd degree, it has atleast one real root whose sign is opposite to that of the last term.

Let f(x)be $x^n+p_1x^{n-1}+..........+p_n$.
Substituting $-\infty$,o,$+\infty$, for x in f(x).

We get $f(-\infty) = -\infty$ since n is odd.
$f(o) = p_n$.
$f(+\infty) = +\infty$.

Hence if P_n in positive f(x) = 0 has atleast one root lying between $-\infty$ and o and if P_n is negative f(x)=0 has atleast one root lying between o and $+\infty$.

If f(x)=0 is of even degree and the absolute term is negative equation has atleast one positive root and atleast one negative root.

Let f(x) be $x^n + p\,x^{n-1} + \ldots\ldots + p_n$.

Here n is even and P_n is negative.

$f(-\infty) = +\infty$. Since n in even.

$f(o) = P_n =$ a negative quantity

$f(+\infty) = +\infty$.

Hence f(x)=0 has atleast one root lying between $-\infty$ and 0, and atleast another lying between 0 and $+\infty$.

We have proved that every equation except one of an even degree with a positive last term has a real root. Such an equation of even degree may have even number of real roots or no real root. We shall assume that every equation f(x)=0 where f(x) is a polynomial in x has a root real or imaginary. The proof of this theorem is beyond the scope of this book.

Theory:

Every equation f(x)=0 of the n^{th} degree has n roots and no more.

Let f(x) be the polynomial.

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots + a_n$$

We assume that every equation f(x) = 0 has atlest one root real or imaginary.

Let α_1 be a root of f(x) = 0.

Then f(x) is exactly divisible by $x - \alpha_1$,

So that $f(x) = (x - \alpha_1)\,\phi(x)$

Where $\phi_1(x)$ is an rational integral function of degree n-1.

Again $\phi_1(x)=0$ has a root real or imaginary And Let that root be α_2.

Then $\phi_1(x)$ is exactly, divisible by $x - \alpha_2$.

So that $\phi_1(x) = (x - \alpha_2)\,\phi_2(x)$

Where $\phi_2(x)$ is a rational integral function of degree n–2.

$$\therefore f(x) = (x - \alpha_1)(x - \alpha_2)\,\phi_2(x)$$

By continuing in this way,

We obtain

$$f(x) = (x - \alpha_1)(x - \alpha_2)\ldots\ldots(x - \alpha_n)\,\phi_n(x)$$

Where $\phi_n(x)$ is of degree n-1,(ie) zero.

$\therefore \phi_n(x)$ is a constant.

Equating the coefficients of x^n on both sides.

We get $\phi_n(x)$ = Coefficients of x^n

$= a_0$

$\therefore f(x) = a_0(x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_n)$.

Hence the equation f(x)=0 has n roots, since f(x) vanishes when x has any one of the values $\alpha_1, \alpha_2 \ldots\ldots \alpha_n$.

If x is given any value different from any one of these n roots then no factor of f(x) can vanish and the equation is not satisfied.

Hence f(x) = 0 cannot have more than n roots.

Example:1 (U.Q)

If α be a real root of the cubic equation $x^3+px^2+qx+r=0$, of which the coefficients are real, show that the other two roots of the equation are real, if $p^2 \geq 4q+2p\alpha+3\alpha^2$.

Solution:

Since α is a root of the equation.

Let $x^3 + px^2 + qx + r$ is exactly divisible by $x - \alpha$

$\therefore$ Let $x^3 + px^2 + qx + r \equiv (x-\alpha)(x^2+ax+b)$

$= x^3 + ax^2 + bx - \alpha x^2 - a\alpha x - \alpha b$

$x^3 + px^2 + qx + r = x^3 + (a-\alpha)x^2 + x(b-a\alpha) - \alpha b$

Equating the coefficients of power of x on both sides,

We get $p = -\alpha + a$

$q = -a\alpha + b$

$r = -b\alpha$

$\therefore a = p + \alpha$ and $b = q + a\alpha = q + \alpha(p+\alpha)$

$\therefore b = q + \alpha p + \alpha^2$

The other two roots of the equation are the roots of $x^2 + (p+\alpha)x + q + p\alpha + \alpha^2 = 0$ which is real if $(p+\alpha)^2 - 4(q+p\alpha+\alpha^2) \geq 0$

(ie) $p^2 - 2p\alpha - 4q - 3\alpha^2 \geq 0$ [$\because b^2 - 4ab \geq 0$]

(ie) $p^2 \geq 4q + 2p\alpha + 3\alpha^2$.

Example: 2

If $x_1, x_2, x_3 \ldots\ldots x_n$ are the roots of the equation $(a_1 - x)(a_2 - x) \ldots\ldots\ldots (a_n - x) + k = 0$ then show that $a_1, a_2 \ldots\ldots$ an are the real roots of the equation

$$(x_1 - x)(x_2 - x) \ldots\ldots\ldots (x_n - x) - k = 0.$$

Solution:

Since $x_1, x_2 \ldots\ldots x_n$ are the roots of the equation

$$(a_1 - x)\, a_2 - x) \ldots\ldots\ldots (a_n - x) + k = 0.$$

We have

$$(a_1 - x)\, a_2 - x) \ldots\ldots\ldots (a_n - x) + k \equiv (x_1 - x)(x_2 - x) \ldots\ldots\ldots (x_n - x)$$

[Hint: check that the coefficient of x^n on both side are equal].

$$\therefore (x_1 - x)(x_2 - x) \ldots\ldots\ldots (x_n - x) - k \equiv (a_1 - x)(a_2 - x) \ldots\ldots\ldots (a_n - x)$$

So $(a_1 - x)\, a_2 - x) \ldots\ldots\ldots (a_n - x) - b = 0 \Rightarrow (x - a_1)(x - a_2) \ldots\ldots\ldots (x - a_n) = 0$

$\therefore a_1, a_2, a_3 \ldots\ldots\ldots a_n$ are the roots of $(x_1 - x)(x_2 - x) \ldots\ldots\ldots (x_n - x) - k = 0$

Example: 3 (U.Q)

Show that if a,b,c are real, the roots of $\dfrac{1}{x+a} + \dfrac{1}{x+b} + \dfrac{1}{x+c} = \dfrac{3}{x}$ are real.

Solution:

Let $\dfrac{1}{x+a} + \dfrac{1}{x+b} + \dfrac{1}{x+c} = \dfrac{3}{x}$

Simplifying, we get

$$\frac{(x+b)(x+c) + (x+a)(x+c) + (x+a)(x+b)}{(x+a)(x+b)(x+c)} = \frac{3}{x}$$

$$\Rightarrow x\,(x+b)(x+c) + x\,(x+a)(x+c) + x\,(x+a)(x+b) - 3\,(x+a)(x+b)(x+c) = 0$$

Let f(x) be the expression on the left hand side. It can easily be seen that f(x) is a quadratic function of x.

$\therefore f(-a) = -a\,(b-a)(c-a)$

$f(-b) = -b\,(c-b)(a-b)$

$f(-c) = -c\,(a-c)(b-c)$

with out loss of generality

Let us assume that a>b>c and a,b,c are all positive.

Then a − b, b − c, a − c are positive.

$\therefore$ f(−a) = −ve

f(−b) = +ve

f(−c) = −ve

$\therefore$ The equation has atleast one real root between −a and −b, and another between −a and −b, and another between −b and −c.

The equation can have only two roots since f(x) = 0 is a quadratic equation.

$\therefore$ The roots of the equations are real.

Example:

The equation $(x-1)^3 + (2x-1)^3 + \ldots\ldots\ldots + (nx-1)^3 = 0$ has for its root $\frac{2}{n+1}$; find the quadratic equation satisfied the other two roots .

Solution:

Let $x^3 - 3x^2 + 3x - 1 + 8x^3 - 12x^2 + 6x - 1 + \ldots\ldots\ldots + n^3 x^3 - 3n^2 x^2 + 3nx - 1 = 0$.

The given equation is $x^3 \sum n^3 - 3x^2 . \sum n^2 + 3x . \sum n - n = 0$

$$\text{(or) } x^3 \left[\frac{n(n+1)}{2}\right]^2 - 3x^2 \frac{n(n+1)(2n+1)}{6} + 3x\left[\frac{n(n+1)}{2}\right] - n = 0$$

Since $\frac{2}{n+1}$ is a root of the above equation $x - \frac{2}{n+1}$ will divide it without reminder & hence by synthetic division,

We have

$$\frac{2}{n+1} \quad \frac{n^2(n+1)^2}{4} \quad - \frac{n(n+1)(2n+1)}{2} \quad - \frac{3}{2} n(n+1) - n$$

$$\begin{array}{ccc} & \frac{n^2(n+1)}{2} - n(n+1) & n \\ \hline -\frac{n(n+n1)^2}{2} & \frac{1}{2} n(n+1) & \end{array}$$

Hence the required quadratic equation is $\frac{n^2(n+1)^2}{4} x^2 - \frac{n(n+1)^2}{2} x + \frac{1}{2} n(n+1) = 0$

Example:

Prove that (1 – a) (1 – b) (1 – c).....= n, if 1, a, b, c be the roots of the equation . $x^n - 1 = 0$.

Solution:

We have $x^n - 1 \equiv (x-1)\,[(x-a)(x-b)(x-c)\ldots\ldots]$

Differentiate both sides w.r. to x treating the R.H.S as product of two functions.

$$nx^{n-1} = 1\,[(x-a)(x-b)(x-c)\ldots..] + (x-1)\frac{d}{dx}[(x-a)(x-b)(x-c)\ldots..]$$

Putting x = 1 in both sides of the above identity

We get

$n = 1\,(n-a)(1-a)(1-c)\ldots\ldots$

Example: 4

If $\alpha_1, \alpha_2 \ldots\ldots. \alpha_n$ are the roots of $x^n + nax - b = 0$ show that $(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\ldots\ldots(\alpha_1 - \alpha_n) = n\left(\alpha_1^{n-1} + a\right)$

Solution:

$x^n + nax - b = (x-\alpha_1)(x-\alpha_2)(x-\alpha_2)\ldots(x-\alpha_n)$ Differentiating both sides

We get

$$nx^{n-1} + na = (x-\alpha_2)\ldots.(x-\alpha_n) + (x-\alpha_1)\frac{d}{dx}[(x-\alpha_2)\ldots..(x-\alpha_n)$$

Putting x = α_1 in both sides the second factor in R.H.S. in zero

$$n\alpha_1^{n-1} + na = (\alpha_1 - \alpha_2)\ldots\ldots.(\alpha_1 - \alpha_n)$$

(or)

$$n\left(\alpha_1^{n-1} + a\right) = (\alpha_1 - \alpha_2)\ldots\ldots.(\alpha_1 - \alpha_n)$$

Example: 5

Prove that the equation

$$\frac{A^2}{x-a} + \frac{B^2}{x-b} + \frac{C^2}{x-c} + \ldots\ldots. + \frac{k^2}{x-k} = x + l.$$

where A,B,C....a, b, c.......and are all real numbers has all its root real

Solution:

Let us suppose $\alpha + i\beta$ is an imaginary root of the above equation.

In that case $\alpha - i\beta$ will also be a root.

Substituting for x both these roots,

We get

$$\frac{A^2}{(\alpha - a)+i\beta} + \frac{B^2}{(\alpha - b)+i\beta} + \frac{C^2}{(\alpha - c)+i\beta} + \ldots\ldots + \frac{k^2}{(\alpha - k)+i\beta} = (\alpha + l)+i\beta$$

$$\& \quad \frac{A^2}{(\alpha - a)-i\beta} + \frac{B^2}{(\alpha - b)-i\beta} + \frac{C^2}{(\alpha - c)-i\beta} + \ldots\ldots + \frac{k^2}{(\alpha - k)-i\beta} = (\alpha + l)-i\beta$$

Substracting the above two relations

We get

$$-2i\beta\left[\frac{A^2}{(\alpha - a)^2+\beta^2} + \frac{B^2}{(\alpha - b)^2+\beta^2} + \frac{C^2}{(\alpha - c)^2+\beta^2} + \ldots\ldots + \frac{k^2}{(\alpha - k)^2+\beta^2}\right. = 2i\beta$$

$$\text{(or)} \quad -2i\beta\left[\frac{A^2}{(\alpha - a)^2+\beta^2} + \frac{B^2}{(\alpha - b)^2+\beta^2} + \frac{C^2}{(\alpha - c)^2+\beta^2} + \ldots\ldots + \frac{k^2}{(\alpha - k)^2+\beta^2} + 1\right] = 0$$

The expression within the brackets in the sum of '+ve' quantities and as such cannot be zero & hence $-2i\beta = 0$ or $\beta = 0$.

$\therefore$ The given equation cannot have imaginary roots.

Hence all its roots are real.

Exercise

1. If $x^3+3px+q$ has a factor of the form $x^2-2ax+a^2$ show that $q^2+4p^3 = 0$.
2. If px^3+qx+r has a factor of the form x^2+ax+1 prove that $p^2=pq+r^2$
3. If a, b, c are all positive, show that all the roots of $\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} = \frac{1}{x}$ are real.
4. If a < b < c < d, show that all the roots of the equation $(x - a)(x - b)(x - c) = k(x - b)(x - d)$ are real for all values of k.

1.2 Imaginary Roots

In an Equation with real coefficients, Imaginary roots ocuur in pairs:

Let the equation be $f(x) = 0$. and Let $\alpha + i\beta$ be an imaginary root of the equation. We assure that $\alpha - i\beta$ is also a root of $f(x) = 0$

We have

$[x-(\alpha+i\beta)]\,[x-(\alpha-i\beta)] = [(x-\alpha)-i\beta]\,[(x-\alpha)+i\beta)$

$= x^2 - x\alpha + i\beta x - \alpha x + \alpha^2 - i\alpha\beta + i\beta x + i\beta\alpha - i\beta^2$

$= x^2 - 2x\alpha + \alpha^2 + \beta^2$

$= (x-\alpha)^2 + \beta^2$

$\therefore [x-(\alpha+i\beta)]\,[x-(\alpha-i\beta)] = (x-\alpha)^2 + \beta^2$ (1)

when f(x) is divided by $(x-\alpha)^2 + \beta^2$

Let the quotient be ϕ (x) and reminder be Ax + b.

Here ϕ (x) is of degree n – 2.

$\therefore f(x) = [(x-\alpha)^2 + \beta^2]\,\phi(x) + Ax + B$ (2)

Substituting $(\alpha - i\beta)$ for x in the equation (2)

We get

$f(\alpha-i\beta) = [(\alpha+i\beta-\alpha)^2 + \beta^2]\,\phi(\alpha-i\beta) + A(\alpha-i\beta)+B$

$= A(\alpha-i\beta)+B$

but $f(\alpha-i\beta)=0$ since $\alpha-i\beta$ is a root of f(x) = 0

(ie) $A((\alpha+i\beta) + B = 0$

(ie) Equating to zero the real and imaginarily part.

$A\alpha + B = 0$ & $AB = 0$

Since $\beta \neq 0$, A=0 and hence B = 0

$\therefore f(x) = [(x-\alpha)^2 + \beta^2]\,\phi(x)$

$f(\alpha-i\beta) = [(\alpha-i\beta-\alpha)^2 + \beta^2]\,\phi(\alpha-i\beta) = 0$

$\therefore \alpha-i\beta$ is also a root of f(x) =0.

Example: 1

From a rational cubic equation which shall have for roots $1, 3-\sqrt{-2}$.

Solution:

Since $3-\sqrt{-2}$ is a root of the equation.

$3+\sqrt{-2}$ is also a root.

So, we have to form an equation whose roots are 1, $3+\sqrt{-2}$, $3-\sqrt{-2}$

Hence the required equation is

$(x-1)\,(x-3-\sqrt{-2})\,(x-3+\sqrt{-2}) = 0$

$(x - 1) [(x - 3)^2 - \sqrt{-2}\,)^2] = 0$

$(x - 1) [(x - 3)^2 + 2)] = 0$

$(x - 1) [x^2 - 6x + 9 + 2] = 0 \qquad \left[\because \sqrt{-2} = i\sqrt{2},\ (i\sqrt{2})^2 = -2\right]$

$(x - 1) [x^2 - 6x + 11] = 0$

(ie) $x^3 - 6x^2 + 11x - x^2 + 6x - 11 = 0$

$\therefore x^3 - 7x^2 + 17x - 11 = 0$

Example: 2

Find the equation with rational coefficients whose roots are $1+5\sqrt{-1}$, $5-\sqrt{-1}$.

Solution:

Since $1+5\sqrt{-1}$, $5-\sqrt{-1}$.is a roots of the equation , $1-5\sqrt{-1}$, $5+\sqrt{-1}$ is also a root.

So we have to form an equation whose roots are $1+5\sqrt{-1}$, $5-\sqrt{-1}$, $1-5\sqrt{-1}$, $5+\sqrt{-1}$.

Hence the required equation is

$[x-(1+5\sqrt{-1})][x-(1-5\sqrt{-1})]\ [x-(5+\sqrt{-1})]\ [x-(5-\sqrt{-1})] = 0$

$[(x-1)^2 - (5\sqrt{-1})^2]\ [(x-5)^2 - (\sqrt{-1})^2] = 0$

$[x^2 - 2x + 1 + 25]\ [x^2 - 10x + 25 + 1] = 0$

(ie) $(x^2 - 2x + 26)\ (x^2 - 10x + 26) = 0$

(ie) $x^4 - 10x^3 + 26x^2 - 2x^3 + 20x^2 - 52x + 26x^2 - 260x + 676 = 0$

(ie) $x^4 - 12x^3 + 72x^2 - 312x + 676 = 0$

Example: 3

Solve the equation $x^4 + 4x^3 + 5x^2 + 2x - 2 = 0$ of which one root is $-1+\sqrt{-1}$.

Solution:

Imaginary roots occur in pairs.

Hence $-1-\sqrt{-1}$ is also a root of the equation.

$\therefore$ The expression on the left side of equation has the factors

$(x+1-\sqrt{-1})\ (x+1+\sqrt{-1})$.

$\therefore$ The expression on the left side is exactly divisible by $(x+1)^2+1$, (ie) $x^2 + 2x + 2$.

Dividing $x^4 + 4x^3 + 5x^2 + 2x - 2$ by $x^2 + 2x + 2$

We have

$$
\begin{array}{r|l}
 & x^2+2x-1 \\
\hline
 & x^4+4x^3+5x^2+2x-2 \\
x^2+2x+2 & x^4+2x^3+2x^2 \\
 & (-)\quad(-) \\
\hline
 & \quad 2x^3+3x^2+2x \\
 & \quad 2x^3+4x^2+4x \\
 & \quad (-)\ (-)\quad(-) \\
\hline
 & \qquad -x^2-2x-2 \\
 & \qquad -x^2-2x-2 \\
 & \qquad (+)\ (+)\ (+) \\
\hline
 & \qquad\quad 0 \\
\hline
\end{array}
$$

We get the quotient x^2+2x-1.

$\therefore\ x^4+4x^3+5x^2+2x-2 = (x^2+2x+2)(x^2+2x-1)$

$\therefore$ Thus the other roots are $-1 \pm \sqrt{2}$

$$
\left[\begin{array}{l}
\because x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} \Rightarrow \dfrac{-2 \pm \sqrt{4+4}}{2} \\
x = -1 \pm \sqrt{2}
\end{array}\right]
$$

Example: 4

Show that $\dfrac{a^2}{x-\alpha} + \dfrac{b^2}{x-\beta} + \dfrac{c^2}{x-\gamma} - x + \delta = 0$ has only real roots if a,b,c, $\alpha, \beta, \gamma, \delta$ are all real.

Solution:

If possible Let $P + iq$ be a root.

Then $P - iq$ is also a root.
Substituting these values for x,
We have

$$\frac{a^2}{P+iq-\alpha}+\frac{b^2}{P+iq-\beta}+\frac{c^2}{P+iq-\gamma}-P-iq+\delta=0 \quad (1)$$

$$\frac{a^2}{P-iq-\alpha}+\frac{b^2}{P-iq-\beta}+\frac{c^2}{P-iq-\gamma}-P-iq+\delta=0 \quad (2)$$

substracting (2) from (1)
we get

$$\left.\begin{array}{l}\frac{a^2}{P-iq-\alpha}+\frac{b^2}{P-iq-\beta}+\frac{c^2}{P-iq-\gamma}-P+iq+\delta \\ -\frac{a^2}{P+iq-\alpha}-\frac{b^2}{P+iq-\beta}-\frac{c^2}{P+iq-\gamma}+P+iq-\delta\end{array}\right\}=0$$

$$\frac{a^2(p+iq-\alpha)-a^2(p-iq-\alpha)}{((p-\alpha)-iq)((p-\alpha)+iq)}+\frac{b^2(p+iq-\beta)-b^2(p-iq-\beta)}{((p-\beta)-iq)((p-\beta)+iq)}$$

$$+\frac{c^2(p+iq-\gamma)-c^2(p-iq-\gamma)}{((p-\gamma)-iq)((p-\gamma)+iq)}+2iq=$$

Simplify,

$$\text{(ie)} -\frac{2a^2iq}{(p-\alpha)^2+q^2}-\frac{2b^2iq}{(p-\beta)^2+q^2}-\frac{2c^2iq}{(p-\gamma)^2+q^2}-2iq=0$$

$$\text{(ie)} -2iq\left[\frac{a^2}{(p-\alpha)^2+q^2}+\frac{b^2}{(p-\beta)^2+q^2}+\frac{c^2}{(p-\gamma)^2+q^2}+1\right]=0$$

This is only possible when $q = 0$, since the other factor cannot be zero. In that case the roots are real

Example:

Solve the equation $x^4+2x^3-5x^2+6x+2=0$ given that $1+\sqrt{-1}$ is a root.

Solution:

Let $f(x) = x^4+2x^3-5x^2+6x+2$

Since $1 + i$ is a root of $f(x)$.

$1 - i$ is also a root of $f(x)$.

The factor f(x) is

$$[x - (1 + i)]\,[x - (1 - i)] = 0$$

$$[x - 1 - i]\,[x - 1 + i] = 0$$

$$(x - 1)^2 - (i)^2 = 0$$

$$x^2 - 2x + 1 + 1 = 0$$

$$x^2 - 2x + 2 = 0$$

$$
\begin{array}{r|l}
 & x^2+4x+1 \\
\hline
x^2-2x+2 & x^4 + 2x^3 - 5x^2 + 6x + 2 \\
 & x^4 - 2x^3 + 2x^2 \\
 & (-)\ (+)\ (-) \\
\cline{2-2}
 & \quad 4x^3 - 7x^2 + 6x \\
 & \quad 4x^3 - 8x^2 + 8x \\
 & \quad (-)\ (+)\ (-) \\
\cline{2-2}
 & \qquad x^2 - 2x + 2 \\
 & \qquad x^2 - 2x + 2 \\
 & \qquad (-)\ (+)\ (-) \\
\cline{2-2}
 & \qquad\quad 0 \\
\cline{2-2}
\end{array}
$$

$$f(x) = (x^2-2x + 2)\,(x^2+4x+1)$$

$$= (x^2-2x + 2)\,(-2+\sqrt{3}\,)\,(-2-\sqrt{3}\,)$$

$\therefore$ The four roots are $1 - \sqrt{i}$, $1 + \sqrt{i}$, $-2+\sqrt{3}$, $-2-\sqrt{3}$.

Exercise

1. Find the equation with rational coefficients whose roots are
   - i) $4\sqrt{3}$, $5+2\sqrt{-1}$
   - ii) $\sqrt{-1} - \sqrt{5}$
   - iii) $-\sqrt{3} + \sqrt{-2}$

2. Solve $x^4 - 4x^2 + 8x + 35 = 0$ given that $2+i\sqrt{3}$ is a root of it.
3. Solve the equation $3x^3 - 4x^2 + x + 88 = 0$ which has a root $2-\sqrt{-7}$

4. Given that $-2+\sqrt{-7}$ is a root of the equation $x^4 + 2x^2 - 16x + 77 = 0$.
5. Show that the equation

$\frac{a^2}{x-a^1} + \frac{b^2}{x-b^1} + \frac{c^2}{x-c^1} + \ldots\ldots\ldots + \frac{k^2}{x-k^1} = x - m$ where a,b,c......... k are all different cannot have an imaginary root.

6. Solve the equation $x^5 - x^4 + 8x^2 - 9x - 15 = 0$, one root being $-\sqrt{3}$ and another $1+2\sqrt{-1}$.

Answer:

1. i) $x^4 - 10x^3 + 19x^2 + 480x - 1392 = 0$
   ii) $x^4 - 8x^2 + 36 = 0$
   iii) $x^4 - 2x^2 + 25 = 0$
2. $2 \pm i\sqrt{3}\ -2 \pm i$
3. $2 \pm \sqrt{-7}, -\frac{8}{3}$
4. $-2 \pm i\sqrt{7}, 2 \pm i\sqrt{3}$.
5. -1.

1.3 Rational Root

In an Equation with rational coefficients irrational roots occur in Pairs:

Let $f(x) = 0$ denote the equation.

and suppose that $a+\sqrt{b}$ is a root of the equation where a and b_1 are rational and $\sqrt{b}$ is irrational. We assert then $a - \sqrt{b}$ is also a root of $f(x) = 0$

$[x - (a+\sqrt{b})]\ [x - (a - \sqrt{b})]$

$= [(x - a) + \sqrt{b}]\ [x - a) - \sqrt{b}$

$= (x - a)^2 - (\sqrt{b})^2 = (x - a)^2 - b.$ (1)

If $f(x)$ is divided by $(x - a)^2 - b$.

Let the quotient be $\phi(x)$ and reminder be $Ax+B$.

$\therefore$ Here $\phi(x)$ is a polynomial of degree $n - 2$.

$\therefore$ f(x) = $\{(x-a)^2 - b\}\ \phi(x) + Ax+B$ (2)

substituting a + $\sqrt{b}$ for x in (2)

we get

$$f(a+\sqrt{b}) = \{(a+\sqrt{b} - a)^2 - b\}\ \phi(a+\sqrt{b}) + A(a+\sqrt{b}) + B$$

$$= A(a+\sqrt{b}) + B$$

but $f(a+\sqrt{b}) = 0 \Rightarrow A(a+\sqrt{b}) + B = 0$

Equating rational and irrational parts.

We get

$A\,a+B = 0$ and $A=0$

$\therefore B = 0$

$\therefore f(x) = [(x-a)^2 - b]\ \phi(x)$

$f(a-\sqrt{b}) = [(a-\sqrt{b}-a)^2 - b]\ \phi(a-\sqrt{b}) = 0$

Hence $a-\sqrt{b}$ is also a root of f(x) = 0

Example: 1

Frame an equation with rational coefficients one of whose roots is $\sqrt{5}+\sqrt{2}$.

Since the roots are $\sqrt{5}+\sqrt{2}$

Then the other roots are $\sqrt{5}-\sqrt{2}$, $-\sqrt{5}+\sqrt{2}$, $-\sqrt{5}-\sqrt{2}$.

Hence the required equation is

$(x-\sqrt{5}-\sqrt{2})(x-\sqrt{5}+\sqrt{2})(x+\sqrt{5}-\sqrt{2})(x+\sqrt{5}+\sqrt{2}) = 0$

(ie) $[(x-\sqrt{5})^2 - \sqrt{2})]\ [(x+\sqrt{5})^2 - \sqrt{2})] = 0$

(ie) $[(x^2 - 2x\sqrt{5}+3)]\ [(x^2+2x\sqrt{5}+3)] = 0$

(ie) $(x^2+3)^2 - 4x^2 .5 = 0$

(ie) $x^4 + 6x^2 + 9 - 20x^2 = 0$

(ie) $x^4 - 14x^2 + 9 = 0$.

Example:2

Solve the equation $x^4-5x^3+4x^2+8x-8 = 0$ given that one of the roots is $1-\sqrt{5}$.

Solution:

Since the irrational roots occur is pairs $1+\sqrt{5}$ is also a root.

The factors corresponding to these roots are $(x-1+\sqrt{5})\,(x-1-\sqrt{5})$

(ie) $(x-1)^2 - 5$

(ie) x^2-2x-4

Dividing $x^4 - 5x^3 + 4x^2 + 8x - 8$ by $x^2 - 2x - 4$

We have

$$
\begin{array}{r|l}
 & x^2-3x+2 \\
\hline
x^2-2x-4 & x^4 + 5x^3 - 4x^2 + 8x - 8 \\
 & x^4 - 2x^3 - 4x^2 \\
 & (-)\ (+)\ (+) \\
\cline{2-2}
 & \quad -3x^3 + 8x^2 + 8x \\
 & \quad -3x^3 + 6x^2 + 12x \\
 & \quad (+)\ (-)\ (-) \\
\cline{2-2}
 & \qquad 2x^2 - 4x - 8 \\
 & \qquad 2x^2 - 4x - 8 \\
 & \qquad (-)\ (+)\ (+) \\
\cline{2-2}
 & \qquad\quad 0 \\
\cline{2-2}
\end{array}
$$

We get the quotient $x^2 - 3x+2$.

$$
\begin{aligned}
\therefore\ x^4 - 5x^3 + 4x^2 + 8x - 8 &= (x^2 - 2x - 4)\,(x^2 - 3x + 2) \\
&= (x^2 - 2x - 4)\,(x - 1)\,(x - 2)
\end{aligned}
$$

$\therefore$ The roots of the equation are $1 \pm \sqrt{5}$, 1,2.

Example: 3

Solve the equation $x^6 - 4x^5 - 11x^4 + 40x^3 + 11x^2 - 4x - 1 = 0$ given that one root is $\sqrt{2}-\sqrt{3}$.

Solution:-

Let $f(x) = x^6 - 4x^5 - 11x + 40x^3 + 11x^2 - 4x - 1 = 0$

Then the other roots are since $\sqrt{2}-\sqrt{3}, \sqrt{2}+\sqrt{3}, -\sqrt{2}+\sqrt{3}, -\sqrt{2}-\sqrt{3}$ are also a root of f(x).

$$[x - \sqrt{2}+\sqrt{3}]\,[x - \sqrt{2}-\sqrt{3}]\,[x + \sqrt{2}-\sqrt{3}]\,[x + \sqrt{2}+\sqrt{3}] = 0.$$

$$[(x - \sqrt{2})^2 - (\sqrt{3})^2]\;\;[(x + \sqrt{2})^2 - (\sqrt{3})^2 = 0$$

$$[x^2 + 2 - 2\sqrt{2}x - 3]\;\;[x^2 + 2 + 2\sqrt{2}x - 3] = 0$$

$$[x^2 - 2\sqrt{2}x - 1]\;\;[x^2 + 2\sqrt{2}x - 1] = 0$$

$$x^4 + 2\sqrt{2}x^3 - x^2 - 2\sqrt{2}x^3 - 8x^2 + 2\sqrt{2}x - x^2 - 2\sqrt{2}x + 1 = 0.$$

$$x^4 - 10x^2 + 1 = 0.$$

$$
\begin{array}{r|l}
 & x^2 - 4x - 2 \\
\hline
X^4 - 10x^2 + 4 & x^6 - 4x^5 - 11x^4 + 40x^3 + 11x^2 - 4x - 1 \\
 & x^6 \quad -10x^4 + a \quad + x^2 \\
 & (-) \quad (+) \quad (-) \quad (-) \\
\cline{2-2}
 & -4x^5 - x^4 + 40x^3 + 10x^2 - 4x \\
 & -4x^5 - 0 + 40x^3 + 0 - 4x \\
 & (+)\;(+)\;(-)\;(-)\;(+) \\
\cline{2-2}
 & -x^4 + 10x^2 - 1 \\
 & -x^4 + 10x^2 - 1 \\
 & (-)\;(+)\;(+) \\
\cline{2-2}
 & 0 \\
\cline{2-2}
\end{array}
$$

$f(x) = (x^4 - 10x^2 + 1)(x^2 - 4x - 1)$

$\therefore$ The six roots of f(x) are $\sqrt{2} \pm \sqrt{3}$, $-\sqrt{2} \pm \sqrt{3}$, $2 \pm \sqrt{5}$.

Example: 4

Solve $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$ given that $2 + \sqrt{3}$ is a root of the equation.

Solution:

Since $2+\sqrt{3}$ is a root, $2 - \sqrt{3}$ is also a root of the equation.

$\therefore [x - (2+\sqrt{3})]\,[x - (2 - \sqrt{3})] = (x - 2)^2 - 3$

$= x^2 - 4x + 1$

when f(x) is divided by $(x^2 - 4x+1)$ the remainder is zero.

$\therefore x^4 - 10x^3 + 26x^2 - 10x + 1 = (x^2 - 4x + 1)\,(x^2 + ax + 1)$

$= x^4 - ax^3 + x^2 - 4x^3 - 4ax^2 - 4x + x^2 + ax + 1$

$= x^4 - (a+4)\,x^3 + (2 - 4a)x^2 + (a-4)\,x + 1$

Equating coefficients of x^3 on both sides,

$a - 4 = 0 \Rightarrow a = -6.$

Hence $f(x) = (x^2 - 4x + 1)\,(x^2 - 6x + 1)$

$x^2 - 6x + 1 = 0$, we get $x = \dfrac{+6 \pm \sqrt{36 - 4}}{2}$

$= \dfrac{+6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2}$

$\therefore$ The four roots are $2 \pm \sqrt{3}$, $3 \pm 2\sqrt{2}$

Example 5:-

Solve the equation $x^4 - 8x^3 + 16x^2 - 28x + 5 = 0$ given that one root $3 - \sqrt{10}$.

Solution:

Let $f(x) = x^4 - 8x^3 + 16x^2 - 28x - 5$

Since $3 - \sqrt{10}$ is a root of f(x) .

$3 + \sqrt{10}$ is also a root of f(x).

$[x - 3 + \sqrt{10}\,]\,[x - 3\ \sqrt{10}\,] = 0$

$[(x - 3)^2 - (\sqrt{10}\,)^2] = 0$

$[x^2 - 6x + 9 - 10] = 0$

$x^2 - 6x - 1 = 0.$

$$
\begin{array}{r|l}
 & x^2-2x-5 \\
\hline
x^2-6x-1 & x^4-8x^3+16x^2-28x+5 \\
 & x^4-6x^3-x^2 \\
 & (-)\ (+)\ (+) \\
\hline
 & -2x^3+17x^2-28x \\
 & -2x^3+12x^2+2x \\
 & (+)\ (-)\ (-) \\
\hline
 & 5x^2-30x-5 \\
 & -5x^2+30x+5 \\
\hline
 & 0 \\
\hline
\end{array}
$$

$f(x) = (x^2-6x-1)\ (x^2-2x-5)$

$$x^2-2x-5 = 0 \Rightarrow x = \frac{2 \pm \sqrt{4+20}}{2}$$

$$= \frac{2 \pm \sqrt{20}}{2}$$

$$x = 1 \pm \sqrt{6}$$

$\therefore$ The three roots are of $f(x) = 0$ are $3 \pm, \sqrt{10}\ 1 \pm \sqrt{6}$

Exercise

1. One root of the equation $3x^5-4x^4-42x^3+56x^2+27x-36 = 0$ is $\sqrt{2} + \sqrt{5}$ Find the remaining roots.
2. Find the equation with rational coefficients whose roots are i) $\sqrt{-1} - \sqrt{4}$. ii) $\sqrt{3} + \sqrt{2}$.
3. Solve $6x^4-13x^3-35x^2-x+3 = 0$ given that $2-\sqrt{3}$ is a root.
4. Solve $x^3-11x^2+37x-35 = 0$ given that $3+\sqrt{2}$ is a root.
5. Frame an equation with rational coefficients, one of whose roots in $\sqrt{2} + \sqrt{5}$.

Answer:

1) $\pm\sqrt{2} \pm \sqrt{5}, \frac{4}{3}$

2) $2 \pm \sqrt{3}, -\frac{3}{2} \pm \frac{1}{3}$

3) $5, 3 \pm \sqrt{2}$

4) $x^4 - 14x^2 + 9 = 0$

1.4 RELATION BETWEEN THE ROOTS AND COEFFICIENTS OF EQUATIONS

Let the equation be

$f(x) = x^n + p_1x^{n-1} + p_2x^{n-1} + \dots\dots\dots + p_{n-1}\, x + p_n = 0$

If this equation has the roots $\alpha_1, \alpha_2 \dots\dots \alpha_n$,

Then

We have $x^n + p_1x^{n-1} + p_2\, x^{n-2} + \dots\dots\dots + p_{n-1}\, x + p_n$

$= (x-\alpha_1)\,(x-\alpha_2)\dots\dots\dots\,(x-\alpha_n)$

$= x^n - \sum\alpha_1\, x^{n-1} + \sum\alpha_1,\alpha_2 x^{n-2} - \dots\dots + (-1)^n\, \alpha_1\, \alpha_2 \dots\dots \alpha_n$

Where $\sum\alpha_1$ = sum of the roots.

$\sum\alpha_1,\alpha_2$ = Sum of the products of the roots taken in pairs.

$\sum\alpha_1,\alpha_2,\alpha_3$ = Sum of the products of the roots taken three at a time

$x^n + p_1x^{n-1} + p_2\, x^{n-2} + \dots\dots + p_{n-1}\, x + p_n$

$= x^n - S_1\, x^{n-1} + S_2\, x^{n-2} - \dots\dots\dots + (-1)^n\, S_n$.

Where S_r is the sum of the products of the quantities $\alpha_1, \alpha_2 \dots\dots \alpha_n$ taken r at a time . Equating the coefficients of like powers on both sides,

We have $-P_1 = S_1$ = Sum of the roots.

$(-1\,)^2\, P_2 = S_2$ = Sum of the products of the roots taken two at a time.

,

,

,

,

,

$(-1)^n\, P_n = S_n$ = product of the roots.

If the equation is $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots\dots + a_{n-1}x + a_n = 0$ divided each term of the equation by a_0.

The equation becomes

$$x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \ldots\ldots + \frac{a_{n-1}}{a_0}x + \frac{a_n}{a_0} = 0 \text{ and So}$$

We have

$$\Sigma\, \alpha_1 = -\frac{a_1}{a_0}$$

$$\Sigma\, \alpha_1, \alpha_2 = \frac{a_2}{a_0}$$

$$\Sigma\, \alpha_1, \alpha_2, \alpha_3 = -\frac{a_3}{a_0}$$

$$\Sigma\, \alpha_1, \alpha_2 \ldots\ldots \alpha_n = (-1)^n \frac{a_n}{a_0}.$$

These n equations are of no help in the general solution of an equation but they are often helpful in the solution of numerical equations when some special relation in known to exist among the roots. The method in illustrated in the examples given below.

Example:- (U.Q)

Show that the roots of the equation $x^3+px^2+qx+r = 0$ are in Arithmetical progression if $2p^3 - 9pq + 27r = 0$

Show that the above condition is satisfied by the equation $x^3 - 6x^2 + 13x - 10 = 0$. Hence of otherwise solve the equation.

Solution:

Let the roots of the equations

$$x^3 + px^2 + qx + r = 0 \text{ be}$$

$$\alpha - \delta,\ \alpha,\ \alpha + \delta.$$

We have from the relation of the roots and coefficients

$$\alpha - \delta + \alpha + \alpha + \delta = -p$$

$$(\alpha - \delta)\alpha + (\alpha - \delta)(\alpha + \delta) + \alpha(\alpha + \delta) = q$$

$$(\alpha - \delta)\alpha(\alpha + \delta) = -r$$

Simplifying these equations

We get

$$3\alpha = -P \qquad (1)$$

$$3\alpha^2 - \delta^2 = q \qquad (2)$$

$$\alpha^3 - \alpha\,\delta^2 = -r \qquad (3)$$

From (1), $\alpha = -\frac{p}{3}$

From (2) $-\ \delta^2 = q - 3\alpha^2 \ - \delta^2 = q - 3\left(-\frac{p}{3}\right)^2 = q - \frac{3p^2}{9_3}$

$$\delta^2 = \frac{p^2}{3} - q$$

Substituting these values in (3)

We get,

$$\left(-\frac{p}{3}\right)^3 - \left(-\frac{p}{3}\right)\left(\frac{p^2}{3} - q\right) = -r$$

$$\frac{-p^3}{27} + \frac{+p^3}{9} - \frac{pq}{3} = -r$$

(ie) $-p^3 + 3p^3 - 9pq = -27r$

$\therefore\ 2p^3 - 9pq + 27r = 0$

In the equation $x^3 - 6x^2 + 13x - 10 = 0$

$P = -6 \quad q = 13 \quad r = -10$

$\therefore\ 2p^3 - 9pq + 27r = 2(-6)^3 - 9(-6)\,13 + 27(-10) = 0$

$\therefore$ The condition is satisfied and so the roots of the equation are in arithmetical progression. In this case the equations (1),(2), (3) become.

$3\alpha = -(-6) = 6$

$3\alpha^2 - \delta^2 = 13$

$\alpha^3 - \alpha\,\delta^2 = -(-10) = 10.$

$\therefore \alpha = 2;\ 3(2)^2 - \delta^2 = 13$

$12 - \delta^2 = 13 \Rightarrow \delta^2 = -1$

$\therefore\ \delta = \pm\ i$

$\therefore$ The roots are 2–i, 2, 2+i .

Example: 2(U.Q)

Find the condition that the roots of the equation $x^3 + px^2 + qx + r = 0$ may be in

i) Geometrical progression

ii) Harmonic progression

Solution:

i) Let the roots in G.P be $\frac{\alpha}{\beta}, \alpha, \alpha\beta$ where β is the common ratio.

Product of the root $= \frac{\alpha}{\beta} \cdot \alpha . \alpha\beta = -r$

(ie) $\alpha^3 = -r$ (1)

$x = \alpha$ is a root of $f(x) = 0$.

$\therefore \alpha^3 + p\alpha^2 + q\alpha r + r = 0$

$-r + \alpha(p\alpha + q) + r = 0$

$\therefore \alpha(p\alpha + q) = 0 : \alpha \neq 0.$

$\therefore \alpha = -q/p$

Putting in (1)

We get $\frac{-q^3}{p^3} = -r$

$\therefore p^3 r = q^3$

ii) Let the roots of the equation be H.P

Put $x = \frac{1}{y}$

Then the equation becomes

$$\left(\frac{1}{y}\right)^3 + p\left(\frac{1}{y}\right)^2 + q\left(\frac{1}{y}\right) + r = 0$$

(ie) $1 + py + qy^2 + ry^3 = 0xy$

$\therefore ry^3 + qy^2 + py + 1 = 0$ (1)

The roots of the equation (1) are A.P.

Let $\alpha - d$, α, $\alpha + d$ be the roots of (1)

Sum of the roots $= \alpha - d + \alpha - \alpha + d = \frac{q}{r}$

$\therefore 3\alpha = -\frac{q}{r}$

$\alpha = -\frac{q}{3r}$

α Satisfies the equation (1)

$$r\left(-\frac{q}{3r}\right)^3 + q\left(-\frac{q}{3r}\right)^2 + p\left(-\frac{q}{3r}\right) + 1 = 0$$

(ie) $\frac{-q^3}{27r^2} + \frac{-q^3}{9r^2} - \frac{pq}{3r} + 1 = 0$

(ie) $-q^3 + 3q^3 - 9pqr + 27r^2 = 0$

(ie) $2q^3 - 9pqr + 27r^2 = 0$

Example:3

Solve $x^3 - 15x^2 + 71x - 105 = 0$ given that the roots of the equation are A.P.

Solution:

Let the roots be $\alpha - d, \alpha, \alpha + d$

Sum of the roots = $\alpha - d + \alpha + \alpha + d = (-P) = 15$

$3\alpha = 15$

$\alpha = 5$

Since x = 5 is a root x – 5 is a factor of f(x) $\therefore$ $x^3 - 15x^2 + 71x - 105 = (x - 5)(x^2 + ax + 21)$

Equating coefficients of x^2,

$\therefore a - 5 = -15$

$\therefore a = -10$

Solving

$x^2 - 10x + 21 = 0$

$(x - 3)(x - 7) = 0$

$\therefore x = 3, 7$

Aliter:

Product of roots = 105

$(5 - d).5.(5 + d) = 105$

$25 - d^2 = 105/5$

$25 - d^2 = 21$

$d^2 = 4$

$d = \pm 2$.

The roots are 3,5,7.

Example:4

Solve $x^3 - 19x^2 + 114x - 216 = 0$ given that the roots are in G.P

Solution:

Let the roots be $\frac{\alpha}{r}, \alpha, \alpha r$

Product of the roots $= \left(\alpha/_r\right) \cdot \alpha \cdot \alpha r = \alpha^3 = -r$

$\alpha^3 = -(-216) = 216$

$\alpha = 6$

$\therefore$ x – 6 is a root of f(x).

$\therefore x^3 - 19x^2 + 114x - 216 = (x-6)(x^2 + ax + 36)$

Equating coefficients of x^2

$a - 6 = 19$

$\therefore a = -13$

Solving $x^2 - 13x + 36 = 0$

$(x-4)(x-9) = 0$

we get x = 4 (or) 9

$\therefore$ The roots are 4,6,9.

Example: 5

Solve $6x^3 - 11x^2 + 6x - 1 = 0$given that the roots are in Harmonic progression.

Solution:-

Put $x = \frac{1}{y}$

Then the given equation becomes,

$6\left(\frac{1}{y}\right)^3 - 11\left(\frac{1}{y}\right)^2 + 6\left(\frac{1}{y}\right) - 1 = 0$

(ie) $6/_{y^3} - 11/_{y^2} + 6/_y - 1 = 0$

(ie) $6 - 11y + 6y^2 - y^3 = 0$

$\therefore y^3 - 6y^2 + 11y - 6 = 0$ **(1)**

The roots of (1) are in A.P.

Let the roots of (1) be

$\alpha - d,\ \alpha,\ \alpha + d$

$\therefore$ Sum of roots

$\alpha - d + \alpha + \alpha + d = 6$

$3\alpha = 6$

$\therefore \alpha = 2$

$y - 2$ is a factor of $y^3 - 6y^2 + 11y - 6$.

$\therefore y^3 - 6y^2 + 11y - 6 = (y - 2)(y^2 + ay + 3)$

Equating coefficients of y^2

$\therefore a - 2 = -6$

(ie) $a = -4$

Solving $y^2 - 4y + 3 = 0$

We get $(y - 1)(y - 3) = 0$

$y = 1$ (or) 3

$\therefore$ The roots of (1) are 1,2,3.

$\therefore$ Hence the roots of given equation are

$1, \frac{1}{2}, \frac{1}{3}$

Example:6

Find the condition that the roots of the equation $ax^3 + 3bx^2 + 3cx + d = 0$ may be in geometric progression.

Solve the equation $27x^3 + 42x^2 - 28x - 8 = 0$ whose roots are in geometric progression.

Solution:

Let the roots of the equation be $\frac{k}{r}, k, kr$. Relation of the roots and coefficients,

$\therefore \frac{k}{r} + k + kr = -\frac{3b}{a}$ (1)

$\frac{k^2}{r} + k^2 + k^2 r = \frac{3c}{a}$ (2)

$\frac{k}{r} . k . kr = k^3 = . \frac{d}{a}$ (3)

From (1), $k\left(\frac{1}{r} + 1 + r\right) = -\frac{3b}{a}$

From (2), $k^2\left(\frac{1}{r}+1+r\right)=-\frac{3c}{a}$

Dividing one by the other,

We get

$$K=-\frac{c}{b}$$

Substituting this value of k is (3)

We get $\left(-\frac{c}{b}\right)^3=-\frac{d}{a}$.

$$\therefore\ ac^3=b^3\ d$$

In the equation $27x^3+42x^2-28x-8=0$

$$\frac{k}{r}+k+kr=-\frac{42}{27} \qquad (4)$$

$$\frac{k^2}{r}+k^2+k^2\ r=-\frac{28}{27} \qquad (5)$$

$$k^3=\frac{8}{27} \qquad (6)$$

$$k=\frac{2}{3}$$

Substituting the value of k in (4)

we get

$$\frac{2}{3}\left(\frac{1}{r}+r+1\right)=-\frac{42}{27}$$

(ie) $\frac{1}{r}+1+r\quad=\quad-\frac{\not{42}^{\,\not{21}\,7}}{\not{27}_{\,\not{9}\,3}}\times\frac{\not{3}}{\not{2}}$

(ie) $\frac{3}{7}\left(\frac{1}{r}+1+r\right)=0$

(ie) $3\,(1+\gamma+\gamma^2)=0$

(ie) $3\gamma^2 + 3\gamma + 3 = 0$

(ie) $(3\gamma + 1)(\gamma + 3) = 0$

$$\therefore \quad \gamma = -\frac{1}{3} \quad \text{or} \quad -3$$

For both the values of γ, the roots are $-2, \frac{2}{3}, -\frac{2}{9}$

Example: 7(U.Q)

Solve the equation $81x^3 - 18x^2 - 36x + 8 = 0$ whose roots are in harmonic progression.

Solution:

Let the roots be α, β, γ

Then $\frac{2}{\beta} = \frac{1}{\alpha} + \frac{1}{\gamma}$

(ie) $2\alpha\gamma = \beta\gamma + \alpha\beta$

From the relation between the coefficients and the roots,

We have $\alpha + \beta + \gamma = \frac{18}{81}$

$$\alpha\beta + \beta\gamma + \gamma\alpha = -\frac{36}{81}$$

$$\alpha\beta\gamma = -\frac{8}{81}$$

From (1) and (3)

We get

$$2\gamma\alpha + \gamma\alpha = -\frac{36}{81}$$

(ie) $3\gamma\alpha = -\frac{36}{81}$

$$\therefore \gamma\alpha = -\frac{4}{27}$$

Substituting this value of $\gamma\alpha$ in (4)

We get $\beta\left(\frac{4}{27}\right) = -\frac{8}{81}$

$$\therefore \beta = \frac{2}{3}$$

From (2), we have

$$\alpha + \gamma = \frac{18}{81} - \frac{2}{3} = -\frac{4}{9}$$

From (5) and (6)

We get $x^2 - \frac{4}{9}x - \frac{4}{27} = 0 \qquad \left[\because x^2 - (\alpha + \gamma)x + \alpha\gamma = 0\right]$

(ie) $27x^2 - 12x - 4 = 0$

$(9x - 2)(3x + 2) = 0$

$x = \frac{2}{9}$ and $-\frac{2}{3}$

$\alpha = \frac{2}{9}$ and $\gamma = -\frac{2}{3}$

$\therefore$ The roots are $\frac{2}{9}, \frac{2}{3}$ and $-\frac{2}{3}$

Example: 8(U.Q)

If the sum of two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ equals the sum of the other two, prove that $p^3 + 8r = 4pq$.

Solution:

Let the roots of the equation be α, β, γ and δ.

Then $\alpha + \beta = \gamma + \delta$.

From the relation of the coefficient and the roots,

We have $\alpha + \beta + \gamma + \delta = -p$ (2)

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\delta + \beta\delta + \gamma\delta = q$ (3)

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r \quad (4)$$

$$\alpha\beta\gamma\,\delta = s \quad (5)$$

From (1) and (2)

We get $2(\alpha + \beta) = -p$ (6)

(3) can be written as

$$\alpha\beta + \gamma\delta + (\alpha + \beta)(\gamma + \delta) = q$$

$$\text{(ie)}\ (\alpha\beta + \gamma\delta) + (\alpha + \beta)^2 = q \quad (7)$$

(4) Can be written as

$$\alpha\beta\,(\gamma + \delta) + \gamma\delta\,(\alpha + \beta) = -r$$

$$\text{(ie)}\ (\alpha + \beta) + (\alpha\beta + \gamma\delta) = -r \quad (8)$$

From (6) and (7)

$$\text{We get}\ \alpha\beta + \gamma\delta + \frac{p^2}{4} = q$$

$$\therefore \alpha\beta + \gamma\delta = q - \frac{p^2}{4} \quad (9)$$

From (8)

$$\text{We get}\ -\frac{p}{2}(\alpha\beta + \gamma\delta) = -r$$

$$\text{(ie)}\ \alpha\beta + \gamma\delta = \frac{2r}{p} \quad (10)$$

Equating (9) and (10)

$$\text{We get}\ q - \frac{p^2}{4} = \frac{2r}{p}$$

(ie) $4pq - p^3 = 8r$

(ie) $p^3 + 8r = 4pq$

Example: 9

Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ given that two of its roots are equal in magnitude and opposite in sign.

Solution:

Let the roots of the equation be $\alpha, \beta, \gamma, \delta$.

Here $\gamma = -\delta$

(ie) $\gamma + \delta = 0$ (1)

From the relations of the roots and coefficients

$\alpha + \beta + \gamma + \delta = 2$ (2)

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 4$ (3)

$\alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \alpha\gamma\delta = -6$ (4)

$\alpha\beta\gamma\delta = -21$ (5)

From (1) and (2)

We get $\alpha + \beta = 2$ (6)

(3) Can be written as $\alpha\beta + \gamma\delta + (\alpha + \beta)(\gamma + \delta) = 4$

$\therefore \alpha\beta + \gamma\delta = 4$ (1)

(4) Can be written as $\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -6$

(ie) $\gamma\delta(\alpha + \beta) = -6$ (8)

From (6) and (8)

We get $\gamma\delta = -3$ (9)

but $\gamma + \delta = 0$

$\therefore \gamma = \sqrt{3}, \qquad \delta = -\sqrt{3}$

From (7) and (9)

We get $\alpha\beta = 7$

$\therefore$ α and β are the roots of $x^2 - 2x + 7 = 0$

$$\alpha = 1 + \sqrt{-6}, \qquad \beta = 1 - \sqrt{-6}$$

$\therefore$ The roots of the equation are

$$\pm\sqrt{3}, \quad 1 \pm \sqrt{-6}.$$

Example: 10

Find the condition that the general biquadratic equation $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$ may have two pairs of equal roots.

Solution:

Let the roots be $\alpha, \alpha, \beta, \beta$

From the relations of Coefficients and roots

$$2\alpha + 2\beta = -\frac{4b}{a} \qquad (1)$$

$$\alpha^2 + \beta^2 + 4\alpha\beta = \frac{6c}{a} \qquad (2)$$

$$2\alpha\beta^2 + 2\alpha^2\beta = -\frac{4d}{a} \qquad (3)$$

$$\alpha^2\,\beta\,\delta = \frac{e}{a} \qquad (4)$$

From (1) we get $\alpha + \beta = -\dfrac{2b}{a}$ (5)

From (3) we get $2\alpha\beta(\alpha + \beta) = -\dfrac{4d}{a}$

$$\therefore\ \alpha\beta = \frac{d}{b} = 0 \qquad (6)$$

From (5) and (6)

We get that α, β are the roots of the equation

$$x^2 + \frac{2b}{a}x + \frac{d}{b} = 0$$

$$\therefore ax^4 + 4bx^3 + 6cx^2 + 4dx + e \equiv a\left(x^2 + \frac{2b}{a}x + \frac{d}{b}\right)^2$$

Comparing Coefficients.

$$6c = a\left(\frac{4b^2}{a^2} + \frac{2d}{b}\right) \text{ and } e = \frac{ad^2}{b^2}$$

$$\therefore 6c = a\left(\frac{4b^3 + 2da^2}{a^2 b}\right)$$

$$\therefore 6c\,ab = 4b^3 + 2da^2$$

$$\therefore 3abc = a^2 d + 2b^3 \quad \& \quad eb^2 = ad^2$$

Example: 11

Solve $2x^3 - x^2 - 22x - 24 = 0$ given that two of its roots are in the ratio 3:4

Solution:

Let the roots be 3k,4k, and γ

Sum of the roots $= 7k + \gamma = \frac{1}{2}$ (1)

and

$$\Sigma\alpha\beta = 12k^2 + 4k\gamma + 3k\gamma$$

$$= -\frac{22}{2} = -11 \qquad (2)$$

Using $\gamma = \frac{1}{2} - 7k$ in (2)

$$12k^2 + 4k\left(\frac{1}{2} - 7k\right) + 3k\left(\frac{1}{2} - 7k\right) = -11$$

(ie) $12k^2 + 2k - 28k^2 + \frac{3k}{2} - 21k^2 = -11$

$-37k^2 + \frac{7k}{2} = -11$

$37k^2 - \frac{7k}{2} = 11$

(ie) $74k^2 - 7k - 22 = 0$

(ie) $74k^2 - 7k - 22 = 0$

(ie) $(37k - 22)(2k + 1) = 0$

$\therefore k = -\frac{1}{2}$ or $\frac{22}{37}$

Taking $k = -\frac{1}{2}$,

The roots are $-\frac{3}{2}, -2, \frac{1}{2} + \frac{7}{2}$

(ie) $-\frac{3}{2}, -2, 4$

Taking $k = \frac{22}{37}$, $3k = \frac{66}{37}$ does not satisfy the equation.

Example: 12

Form the third degree equation, two of whose roots are $1 - I$ and 2.

Solution:

Since $1 - I$ is a root $1 + I$ is also a root.

$\therefore$ The equation of degree three is

$[x - (1 - i)]\,[x - (1 + I))]\,[x - 2] = 0$

$[(x - 1)^2 - i^2]\,(x - 2) = 0$

$(x^2 - 2x + 2)\,(x - 2) = 0$

(i.e.) $x^3 - 4x^2 + 6x - 4 = 0$

Example: 13

Solve $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$ whose roots are in A.P.

Solution:

Let the roots be $a - 3d, a - d, a + d, a + 3d$

Sum of roots = $4a = -2$

$$\therefore a = -\tfrac{1}{2}$$

Product of roots = $(a^2 - 9d^2)(a^2 - d^2) = 40$

(or) $\left(\frac{1}{4} - 9d^2\right)\left(\frac{1}{4} - k\right) = 40$

$9k^2 - \frac{5}{2}k + \frac{1}{16} = 40$ or $144k^2 - 40k - 639 = 0$

$(4k - 9)(36k + 71) = 0$

$\therefore \quad k = \frac{9}{4}$ or $-\frac{71}{36}$

(i.e.) $d^2 = \frac{9}{4}$ or $-\frac{71}{36}$

$\therefore d = \pm \tfrac{3}{2}$

$\therefore$ The roots are $-5, -2, 1, 4$.

Example: 14

If α, β, γ are the roots of $x^3 + Px^2 + qx + r = 0$ find the condition if i) $\alpha + \beta = 0$ ii) $\alpha\beta = -1$

Solution:

i) Since $\alpha + \beta + \gamma = -P$

$$0 + \gamma = -P$$

$$\therefore \gamma = -P$$

γ satisfies the equation

$\therefore -P^3 + P^3 - Pq + r = 0$

$\therefore\ r = Pq.$

ii) Since $\alpha\beta\gamma = -r$

$-\gamma = -r$

$\therefore\ \gamma = r$ satisfies the equation substituting.

$r^3 + Pr^2 + qr + r = 0$

(i.e.) $r^2 + Pr + q + 1 = 0$

Exercise

1. Solve $x^3 - 12x^2 + 39x - 28 = 0$ whose roots are in A.P.
2. The roots of the equation $8x^3 - 14x^2 + 7x - 1 = 0$ are in geometrical progression. Find them.
3. Show that the roots of the equation. $x^4 - Px^3 + qx^2 - rx + \frac{r^2}{P^2} = 0$ are in simple proportion. Hence solve $x^4 - 12x^3 + 47x^2 - 72x + 36 = 0$.
4. Solve the equation $6x^3 - 11x^2 - 3x + 2 = 0$ given that the roots of the equation are in harmonic progression.
5. Show that the four roots $\alpha, \beta, \gamma, \delta$ of the equation $x^4 + Px^3 + qx^2 + rx + S = 0$ will be connected by the relation $\alpha\beta + \gamma\delta = 0$ if $P^2 S + r^2 = 4qS$.
6. Solve the equation $x^4 + 4x^3 - 2x^2 - 12x + 9 = 0$ given that it has two pairs of equal roots.
7. If the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has three equal roots show that each of them is equal to $\frac{6c - ab}{3a^2 - 8b}$

1.5 SYMMETRIC FUNCTION OF THE ROOTS

If a function involving all the roots of an equation is unaltered in value if any two of the roots are interchanged. It is called a symmetric function of the roots.

Let $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots\ \alpha_n$ be the roots of the equation

$f(x) = x^n + P_1 x^{n-1} + P_2 x^{n-2} + \ldots\ldots + P_n = 0$

we have learned that

$$S_1 = \Sigma\, \alpha_1 = -P_1$$

$$S_2 = \Sigma\, \alpha_1\, \alpha_2 = P_2$$

$$S_3 = \Sigma\, \alpha_1\, \alpha_2\, \alpha_3 = -P_3$$

Without knowing the value of the roots separately in terms of the coefficients, by using the above relations between the coefficients and the roots of an equation. We can express any symmetric function of the roots in terms of the coefficients of the equation.

Example: 1

If α, β, γ are the roots of the equation $x^3 + Px^2 + qx + r = 0$, find the value of i) $\Sigma\, \alpha^2 \beta$ ii) $\Sigma\, \alpha^2$ iii) $\Sigma\, \alpha^3$

Solution:

$$\alpha + \beta + \gamma = -P$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = q$$

$$\alpha\beta\gamma = -r$$

$$\Sigma\alpha^2\beta = \alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma\alpha^2 + \gamma^2\beta$$

i) $\Sigma\alpha^2\beta = ((\Sigma\alpha\beta)\ ((\Sigma\alpha) - 3\,\alpha\beta\gamma$

$$= (\alpha\beta + \beta\gamma + \gamma\alpha)\ (\alpha + \beta + \gamma) - 3\,\alpha\beta\gamma$$

$$= q\,(-P) - (3)\,(-r) = 3r - Pq.$$

$$\Sigma\alpha^2\beta = 3r - Pq$$

ii) $\Sigma\alpha^2 = \alpha^2 + \beta^2 + \gamma^2$

$$= (\alpha + \beta + \gamma)^2 - 2\,(\alpha\beta + \alpha\gamma + \beta\gamma)$$

$$= (\text{à a})^2 - 2$$

$$= P^2 - 2q$$

ii) $\sum \alpha^3 = \alpha^3 + \beta^3 + \gamma^3$

$$= ((\alpha+\beta+\gamma)^3 - 3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\alpha\gamma) + 3\alpha\beta\gamma$$

$$\sum \alpha^3 = -P^3 + 3Pq - 3r.$$

Example: 2

If $\alpha, \beta, \gamma, \delta$ be the roots of the bi quadratic equation $x^4 + Px^3 + qx^2 + rx + S = 0$.

Find i) $\sum \alpha^2$, ii) $\sum \alpha^2 \beta\gamma$ iii) $\sum \alpha^2 \beta^2$ iv) $\sum \alpha^3 \beta$ and v) $\sum \alpha^4$

Solution:

The relation between the roots and the coefficients are

$$\alpha+\beta+\gamma+\delta = -P$$

$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = q$$

$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -r$$

$$\alpha\beta\gamma\delta = S.$$

$$\sum \alpha^2 = \alpha^2+\beta^2+\gamma^2+\delta^2$$

$$= (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$$

$$= (\sum\alpha)^2 - 2\sum\alpha\beta$$

$$= P^2 - 2q$$

$$\sum \alpha^2\beta\gamma = \alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2+\alpha\beta^2\gamma+\alpha^2\beta\delta+\alpha\beta\delta^2$$
$$+ \alpha^2\gamma\delta+\alpha\gamma^2\delta+\alpha\gamma\delta^2+\beta^2\gamma\delta+\beta\gamma^2\delta+\beta\gamma\delta^2$$
$$= (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)(\alpha+\beta+\gamma+\delta) - 4\alpha\beta\gamma\delta$$

$$= (\sum\alpha\beta\gamma)(\sum\alpha) - 4\alpha\beta\gamma\delta$$

$$= Pr - 4S.$$

$$\sum\alpha^2\beta^2 = \alpha^2\beta^2+\alpha^2\gamma^2+\alpha^2\delta^2+\beta^2\gamma^2+\beta^2\delta^2+\gamma^2\delta^2$$

$$= (\sum\alpha\beta)^2 - 2\sum\alpha^2\beta\gamma - 6\alpha\beta\gamma\delta$$

$= q^2 - 2(Pr - 4S) - 6S.$

$= q^2 - 2Pr + 2S.$

$\Sigma\alpha^3\beta = (\Sigma\alpha^2)(\Sigma\alpha\beta) - \Sigma\alpha^2\beta\gamma$

$= (P^2 - 2q)q - (Pr - 4S)$

$= P^2q - 2q^2 - Pr + 4S.$

$\Sigma\alpha^4 = \alpha^4 + \beta^4 + \gamma^4 + \delta^4$

$= (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)^2 - 2(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\delta^2 + \delta^2\alpha^2)$

$= (P^2 - 2q)^2 - 2(q^2 - 2Pr + 2S)$

$\Sigma\alpha^4 = P^4 - 4P^2q + 2q^2 + 4Pr - 4S.$

Example: 3

If α, β, γ are the roots of the equation $x^3 + ax^2 + bx + c = 0$, from the equation whose roots are $\alpha\beta, \beta\gamma$ and $\gamma\alpha$.

Solution:

The relations between the roots and coefficients are

$$\alpha + \beta + \gamma = -a$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = b$$

$$\alpha\beta\gamma = -c$$

The required equation is

$$(x - \alpha\beta)(x - \beta\gamma)(x - \gamma\alpha) = 0$$

(i.e.) $(x^2 - x\beta\gamma - \alpha\beta x + \alpha\beta^2\gamma)(x - \gamma\alpha) = 0$

(i.e.) $x^3 - x^2\gamma\alpha - x^2\beta\gamma + x\alpha\beta\gamma^2 - \alpha\beta x^2 + \alpha^2\beta\gamma x + \alpha\beta^2\gamma x - \alpha^2\beta^2\gamma^2 = 0$

(i.e.) $x^3 - x^2(\alpha\beta + \beta\gamma + \gamma\alpha) + x(\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2) - (\alpha\beta\gamma)^2 = 0$

(i.e.) $x^3 - x^2(\alpha\beta + \beta\gamma + \gamma\alpha) + x\,\alpha\beta(\alpha + \beta + \gamma) - (\alpha\beta\gamma)^2 = 0$

(i.e.) $x^3 - bx^2 + acx - c^2 = 0$

Example: 4

If $\alpha, \beta \gamma$ are the roots of $x^3 + Px^2 + qx + r = 0$ form the equation whose roots are

$$\beta + \gamma - 2\alpha, \quad \gamma + \alpha - 2\beta, \quad \alpha + \beta - 2\gamma$$

Solution:

We have $\alpha + \beta + \gamma = -P$

$$\alpha\beta + \beta\gamma + \lambda\alpha = q$$

$$\alpha\beta\gamma = -r$$

In the required equation

S_1 = Sum of the roots = $\beta + \gamma - 2\alpha + \gamma + \alpha - 2\beta + \alpha + \beta - 2\gamma$

$$= 0$$

S_2 = Sum of the products of the roots taken two at a time

$$= (\beta + \gamma - 2\alpha)(\gamma + \alpha - 2\beta) + (\beta + \gamma - 2\alpha)(\alpha + \beta - 2\gamma) + (\alpha + \beta - 2\gamma)(\gamma + \alpha - 2\beta)$$

$$S_2 = (\alpha + \beta + \gamma - 3\alpha)(\alpha + \beta + \gamma - 3\beta) + \alpha + \beta + \gamma - 3\alpha)(\alpha + \beta + \gamma - 3\gamma) + (\alpha + \beta + \gamma - 3\gamma)(\alpha + \beta + \gamma - 3\beta)$$

$$S_2 = (-P - 3\alpha)(-P - 3\beta) + (-P - 3\alpha)(-P - 3\gamma) + (-P - 3\gamma)(-P - 3\beta)$$

$$S_2 = (P + 3\alpha)(P + 3\beta) + (P + 3\alpha)(P + 3\gamma) + (P + 3\gamma)(+P + 3\beta)$$

$$S_2 = P^2 + 3P\beta + 3\alpha\beta + 9\alpha\beta + P^2 + 3P\gamma + 3\alpha P + 9\alpha\gamma + P^2 + 3\gamma P + 3P\beta + 9\gamma\beta$$

$$S_2 = 3P^2 + 6P(\alpha + \beta + \gamma) + 9(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$\therefore S_2 = 3P^2 + 6P(-P) + 9q$$

$$S_2 = 9q - 3P^2$$

S_3 = Products of the roots

$$= (\beta + \gamma - 2\alpha)(\gamma + \alpha - 2\beta)(\alpha + \beta - 2\gamma)$$

$$= (\alpha + \beta + \gamma - 3\alpha)(\alpha + \beta + \gamma - 2\beta)(\alpha + \beta + \gamma - 3\gamma)$$

$$= (-P - 3\alpha)(-P - 3\beta)(-P - 3\gamma)$$

$= -\{P^3 + 3P^2(\alpha+\beta+\gamma) + 9P(\alpha\beta+\beta\gamma+\gamma\alpha) + 27\alpha\beta\gamma\}$

$= -\{P^3 + 3P^2(-P) + 9Pq - 27r\}$

$S_3 = 2P^3 - 9Pq + 27r$

Hence the required equation is $x^3 - S_1x^2 + S_2x - S_3 = 0$

$X^3 - (9q - 3P^2)x - (2P^3 - 3Pq + 27r) = 0$

Example:

If α, β, γ are the roots of $x^3 - 14x + 8 = 0$ find $\Sigma\alpha^2$ and $\Sigma\alpha^3$

Solution:

We have $\Sigma\alpha = 0$, $\Sigma\alpha\beta = -14$, $\alpha\beta\gamma = -8$

$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = 0 - 2(-14) = 28$

$\alpha^3 = 14\alpha - 8$ (since α satisfies $x^3 - 14x + 8$

$\Sigma\alpha^3 = 14\Sigma\alpha - 24 = 14(0) - 24 = -24$

Aliter: Use the identity for $\Sigma\alpha^3$.

$\Sigma\alpha^3 = (\Sigma\alpha^3) - 3(\Sigma\alpha)(\Sigma\alpha\beta) + 3\alpha\beta\gamma$

$= 0 - 3(0) + 3(-8) = -24$

OR if $\alpha+\beta+\gamma = 0$

$\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma = -24$

Exercises

1, If α, β, γ be the roots of the equation $x^3 + Px^2 + qx + r = 0$ find the value of

i) $\alpha^3 + \beta^3 + \gamma^3$

ii) $\dfrac{\beta^2+\gamma^2}{\beta\gamma} + \dfrac{\gamma^2+\alpha^2}{\gamma\alpha} + \dfrac{\alpha^2+\beta^2}{\alpha\beta}$

iii) $(\beta+\gamma-\alpha)^3 + (\gamma+\alpha-\beta)^3 + (\alpha+\beta-\gamma)^3$

2. Show that for the cubic equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$$

$$(\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2 = 18\ (a_1^2 - a_0 a_2)$$

3. If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 + Px^3 + qx^2 + rx + S = 0$, Evaluate i) $\Sigma\alpha^2\beta\gamma$ ii) $\Sigma(\beta+\gamma+\delta)^2$, iii) $\Sigma \dfrac{1}{\alpha^2}$

4. If α, β, γ are the roots of the equation $x^3 + qx + r = 0$, find the value of

i) $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$

ii) $\dfrac{1}{\beta+\gamma} + \dfrac{1}{\gamma+\alpha} + \dfrac{1}{\alpha+\beta}$

iii) $\left(\dfrac{1}{\beta} + \dfrac{1}{\gamma} - \dfrac{1}{\alpha}\right)\left(\dfrac{1}{\gamma} + \dfrac{1}{\alpha} - \dfrac{1}{\beta}\right)\left(\dfrac{1}{\alpha} + \dfrac{1}{\beta} - \dfrac{1}{\gamma}\right)$

5. Find the sum of the cubes of the roots of $x^4 - 22x^2 + 84x - 49 = 0$

1.6 SUM OF THE POWERS OF THE ROOTS OF AN EQUATION

Let $\alpha_1\,\alpha_2\,\alpha_3 \ldots\ldots \alpha_n$ be the roots of the equation f (x) = 0. The sum of the r^{th} powers of the roots.

(i.e.) $\alpha_1^r + \alpha_2^r + \ldots\ldots \alpha_n^r$ is usually denoted by S_r.

We can easily see that S_r constitutes a symmetric function of the roots and hence we can calculate the value of S_r by the methods described in the previous article. When r is greater than 4, the calculation of S_r by the previous method becomes tedious and in those cases, the following two methods can be used profitably.

We have

$$f(x) = (x-\alpha_1)(x-\alpha_2) \ldots\ldots\ldots (x-\alpha_n)$$

Taking Logarithms on both sides and differentiating we get

$$\frac{f^1(x)}{f(x)} = \frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \ldots\ldots + \frac{1}{x-\alpha_n}$$

$$\frac{x f^1(x)}{f(x)} = \frac{x}{x-\alpha_1} + \frac{x}{x-\alpha_2} + \ldots\ldots + \frac{x}{x-\alpha_n}$$

$$= \frac{1}{1-\frac{\alpha_1}{x}} + \frac{1}{1-\frac{\alpha_2}{x}} + \ldots\ldots + \frac{1}{1-\frac{\alpha_n}{x}}$$

$$= \left(1-\frac{\alpha_1}{x}\right)^{-1} + \left(1-\frac{\alpha_2}{x}\right)^{-1} + \ldots\ldots + \left(1-\frac{\alpha_n}{x}\right)^{-1}$$

$$= 1 + \frac{\alpha_1}{x} + \frac{\alpha_1^{\,2}}{x^2} + + \ldots\ldots + \frac{\alpha_1^{\,n}}{x^n} + \ldots\ldots$$

$$= +1 + \frac{\alpha_2}{x} + \frac{\alpha_2^{\,2}}{x^2} + \ldots\ldots + \frac{\alpha_2^{\,n}}{x^n} + \ldots\ldots$$

$$= + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= +1 + \frac{\alpha_n}{x} + \frac{\alpha_n^{\,2}}{x^2} + \ldots\ldots + \frac{\alpha_n^{\,n}}{x_n} + \ldots\ldots$$

$$= n + (\Sigma\alpha_1)\frac{1}{x} + (\Sigma\alpha_1^{\,2})\frac{1}{x^2} + \ldots\ldots + (\Sigma\alpha_1^{\,r})\frac{1}{x^r} + \ldots$$

$$= n + S_1\frac{1}{x} + S_2\frac{1}{x^2} + \ldots\ldots + S_r\frac{1}{x^r} + \ldots\ldots$$

$$= \therefore S_r = \text{Coefficient of } \frac{1}{x^r} \text{ in the expansion of } \frac{x f^1(x)}{f(x)}$$

Example: 1

Find the sum of the cubes of the roots of the equation $x^5 = x^2 + x + 1$.

Solution:

The equation can be written in the form

$$f(x) = x^5 - x^2 - x - 1 = 0$$

S_3 = Coefficient of $\frac{1}{x^3}$ in the expansion of

$$\frac{x(5x^3 - 2x - 1)}{x^5 - x^2 - x - 1}$$

$$= \text{Coefficient of } \frac{1}{x^3} \text{ in } \frac{5-\frac{2}{x^3}-\frac{1}{x^4}}{1-\frac{1}{x^3}-\frac{1}{x^4}-\frac{1}{x^5}}$$

$$= \text{Coefficient of } \frac{1}{x^3} \text{ in } \left(5-\frac{2}{x^3}-\frac{1}{x^4}\right)\left(1-\frac{1}{x^3}-\frac{1}{x^4}-\frac{1}{x^5}\right)^{-1}$$

$$= \text{Coefficient of } \frac{1}{x^3} \text{ in } \left(5-\frac{2}{x^3}-\frac{1}{x^4}\right)\left\{1+\frac{1}{x^3}-\frac{1}{x^4}-\frac{1}{x^5}+\left(\frac{1}{x^3}+\frac{1}{x^4}+\frac{1}{x^5}\right)^2+\ldots\ldots\right\}$$

$$= \text{Coefficient of } \frac{1}{x^3} \text{ in } \left(5-\frac{2}{x^3}-\frac{1}{x^4}\right)\left(1+\frac{1}{x^3}+\ldots\ldots\right)$$

$$= \frac{5}{x^3} - \frac{2}{x^3}$$

$S_3 = 3.$

Example: 2

Calculate the sum of the cubes of the roots of the equation $x^4 + 2x + 3 = 0$

Solution:

Let $f(x) = x^4 + 2x + 3 = 0$

$$S_r = \text{Coefficient of } \frac{1}{x^r} \text{ in the expansion of } \frac{x f^1(x)}{f(x)}$$

$$S_3 = \text{Coefficient of } \frac{1}{x^3} \text{ in the expansion of}$$

$$\frac{x(4x^3+2)}{x^4+2x+3}$$

$$= \text{Coefficient of } \frac{1}{x^3} \text{ in } \frac{4+\frac{2}{x^3}}{1+\frac{2}{x^3}+\frac{3}{x^4}}$$

$$= \text{Coefficient of } \frac{1}{x^3} \text{ in } \left(4+\frac{2}{x^3}\right)\left(1+\frac{2}{x^3}+\frac{3}{x^4}\right)^{-1}$$

$$= \text{Coefficient of } \frac{1}{x^3} \text{ in } \left(4+\frac{2}{x^3}\right)\left(1-\left(\frac{2}{x^3}+\frac{3}{x^4}\right)+\left(\frac{2}{x^3}+\frac{3}{x^4}\right)^2+\ldots\ldots\right)$$

$$= \text{Coefficient of } \frac{1}{x^3} \text{ in } \left(4+\frac{2}{x^3}\right)\left(1-\frac{2}{x^3}-\frac{3}{x^4}+\frac{4}{x^6}+\frac{9}{x^8}+\ldots\ldots\right)$$

$$= \text{Coefficient of } \frac{1}{x^3} \text{ in } \left(-\frac{8}{x^3}+\frac{2}{x^3}\right)$$

$S_3 = 6$

Example:

Calculate the sum of the cubes of the roots of the equation $x^3-6x^2+11x-6=0$

Solution:

Let $f(x) = x^3-6x^2+11x-6=0$

$$S_r = \text{Coefficient of } \frac{1}{x^r} \text{ in the expansion of } \frac{x f^1(x)}{f(x)}$$

$$S_3 = \text{Coefficient of } \frac{1}{x^3} \text{ in } \frac{x(3x^2-6x+11)}{x^3-6x^2+11x-6}$$

$$= \text{Coefficient of } \frac{1}{x^3} \text{ in } \frac{3-\frac{6}{x}+\frac{11}{x^2}}{1-\frac{6}{x}+\frac{11}{x^2}-\frac{6}{x^3}}$$

$$= \text{Coefficient of } \frac{1}{x^3} \text{ in } \left(3-\frac{6}{x}+\frac{11}{x^2}\right)\left(1-\frac{6}{x}+\frac{11}{x^2}-\frac{6}{x^3}\right)^{-1}$$

$$= \text{Coefficient of } \frac{1}{x^3} \text{ in } \left(3-\frac{6}{x}+\frac{11}{x^2}\right)\left(1-\left(\frac{6}{x}+\frac{11}{x^2}-\frac{6}{x^3}\right)+\left(\frac{6}{x}+\frac{11}{x^2}-\frac{6}{x^3}\right)^2\ldots\ldots\right)$$

$$= \text{Coefficient of } \frac{1}{x^3} \text{ in } \left(3-\frac{6}{x}+\frac{11}{x^2}\right)\left(1-\frac{6}{x}+\frac{11}{x^2}-\frac{6}{x^3}+\frac{36}{x^2}+\frac{121}{x^4}-\frac{36}{x^6}\ldots\ldots\ldots\right)$$

$$= \text{Coefficient of } \frac{1}{x^3} \text{ in } \left(\frac{18}{x^3}+\frac{66}{x^3}-\frac{66}{x^3}-\frac{216}{x^3}+\frac{6}{x^3}\ldots\ldots\right)$$

$= 36$

Exercise

1. Find the sum of the cubes of the roots of the equation.

i) $x^3-2x^2+x-1=0$

ii) $x^4-3x^3+5x^2-12x+4=0$

iii) $x^4-7x^2-4x-3=0$

1.7 NEWTON'S THEOREM

Newton's Theorem on the sum of the powers of the roots.

Let $\alpha_1, \alpha_2 \ldots\ldots \alpha_n$ be the roots of the equation
$f(x) = x^n + P_1 x^{n-1} + P_2 x^{n-2} + \ldots + P_n = 0.$

and let be $S_r = \alpha_1^r + \alpha_2^r + \ldots\ldots + \alpha_n^r$

so that $S_0 = n$

$f(x) = (x - \alpha_1)(x - \alpha_2) \ldots\ldots (x - \alpha_n).$

Taking logarithms on both sides and differentiating we get

$$\frac{f^1(x)}{f(x)} = \frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \ldots\ldots \frac{1}{x-\alpha_n}$$

(i.e.) $f^1(x) = \dfrac{f(x)}{x-\alpha_1} + \dfrac{f(x)}{x-\alpha_2} + \ldots\ldots + \dfrac{f(x)}{x-\alpha_n}$

By actual division,

We obtain

$$\frac{f(x)}{x-\alpha_1} = x^{n-1} + (\alpha_1 + P_1)\, x^{n-2} + (\alpha_1^2 + P_1\alpha_1 + P_2)\, x^{n-3} + \ldots\ldots + (\alpha_1^{n-1} + P_1\alpha_1^{n-2} + \ldots + P_{n-1})$$

$\vdots$

$$\frac{f(x)}{x-\alpha_2} = x^{n-1} + (\alpha_2 + P_1)\, x^{n-2} + (\alpha_2 + P_1\alpha_2 + P_2)\, x^{n-3} + \ldots\ldots + (\alpha_2^{n-1} + P_1\alpha_2^{n-2} + \ldots + P_{n-1})$$

$$\frac{f(x)}{x-\alpha_n} = x^{n-1} + (\alpha_n + P_1)\, x^{n-2} + (\alpha_n^2 + P_1\alpha_n + P_2)\, x^{n-3} + \ldots\ldots + (\alpha_n^{n-1} + P_1\alpha_n^{n-2} + \ldots + P_{n-1})$$

Adding all these fractions, we get

$$\frac{f(x)}{x-\alpha_1} + \frac{f(x)}{x-\alpha_2} + \ldots\ldots + \frac{f(x)}{x-\alpha_n}$$

$$= x^{n-1} + \alpha_1 x^{n-2} + P_1 x^{n-2} + \alpha_1^2 x^{n-3} + P_1\alpha_1 x^{n-3} + P_2 x^{n-3} + \ldots\ldots$$

$$+ \alpha_1^{n-1} + P_1\alpha_1^{n-2} + \ldots\ldots + P_{n-1} + x^{n-1} + \alpha_2 x^{n-2} + P_1 x^{n-2}$$

$$+ \alpha_2 x^{n-3} + P_1\alpha_2 x^{n-3} + P_2 x^{n-3} + \ldots\ldots + \alpha_2^{n-1} + P_1\alpha_2^{n-2} + \ldots\ldots + P_{n-1}$$

$$+ \ldots\ldots + x^{n-1} + \alpha_n x^{n-2} + P_1 x^{n-2} + \alpha_n^{\,2} x^{n-3} + P_1 \alpha_n x^{n-3} + P_2 x^{n-3}$$

$$+ \ldots\ldots + \alpha_n^{\,n-1} + P_1 \alpha_n^{\,n-2} + \ldots\ldots + P_{n-1}$$

$$= n x^{n-1} + (\alpha_1 + \alpha_2 + \ldots\ldots \alpha_n) x^{n-2} + n P_1 x^{n-2} + (\alpha_1^{\,2} + \alpha_2^{\,2} + \ldots\ldots + \alpha_n^{\,2}) x^{n-3}$$

$$+ P_1 (\alpha_1 + \alpha_2 + \alpha_3 + \ldots\ldots \alpha_n) + n P_2 x^{n-3} + \ldots + (\alpha_1^{\,n-1} + \ldots\ldots \alpha_n^{\,n-1})$$

$$+ P_1 (\alpha_1^{\,n-2} + \alpha_2^{\,n-2} + \ldots\ldots + \alpha_n^{\,n-2}) + \ldots\ldots + n P_{n-1}$$

$$= n x^{n-1} + S_1 x^{n-2} + n P_1 x^{n-2} + S_2 x^{n-3} + P_1 S_1 x^{n-3} + n P_2 x^{n-3} + \ldots\ldots\ldots\ldots$$

$$+ S_{n-1} + P_1 S_{n-2} + \ldots\ldots + n P_{n-1}$$

But $f'(x)$ is also equal to

$$= n x^{n-1} + (n-1) p_1 x^{n-2} + (n-2) p_2 x^{n-3} + \ldots\ldots + 2 p_{n-2} + p_{n-1}$$

Equating the coefficients in the two values of $f^1(x)$, we obtain the following relations.

$$S_1 + P_1 = 0$$

$$S_1 + P_1 S_1 + 2 P_2 = 0$$

$$S_3 + P_1 S_2 + P_2 S_1 + 3 P_3 = 0$$

$$S_4 + P_1 S_3 + P_2 S_2 + P_3 S_1 + 4 P_4 = 0$$

.. . .

.. . .

.. . .

.. . .

$$S_{n-1} + P_1 S_{n-2} + P_2 S_{n-3} + \ldots\ldots + P_{n-2} S_1 + (n-1) P_{n-1} = 0$$

From these (n– 1) relations we can calculate in succession the values of $S_1, S_2 \ldots\ldots S_{n-1}$ in terms of the coefficients $P_1, P_2 \ldots . P_{n-1}$.

We can extend our results to the sums of all positive powers of the roots, viz., $S_n, S_{n+1} \ldots . S_r$ where $r > n$.

We have $x^{r-n} f(x) = x^r + P_1 x^{r-1} + P_2 x^{r-2} + \ldots\ldots P_n x^{r-n}$

Replacing in this identity, x by the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ in succession and adding.

We have

$S_r + P_1 S_{r-1} + P_2 S_{r-2} + \ldots + P_n S_{r-n} = 0$

Now giving r the values n, n + 1, n+2 . . . . successively and observing that $S_0 = n$.

We obtain from the last equation

$S_n + P_1 S_{n-1} + P_2 S_{n-2} + \ldots + n P_n = 0$

$S_{n+1} + P_1 S_n + P_2 S_{n-1} + \ldots + P_n S_1 = 0$

$S_{n+2} + P_1 S_{n+1} + P_2 S_n + \ldots + P_n S_2 = 0$

and so on.

Thus we get

Consider $r < n$

$S_r + P_1 S_{r-1} + P_2 S_{r-2} + \ldots + r P_r = 0$

and $S_r + P_1 S_{r-1} + P_2 S_{r-2} + \ldots + P_n S_{r-n} = 0$ if $r = n$.

Cor:

To find the sum of the negative integral powers of the roots of f (x) = 0. Put $x = \frac{1}{y}$ and find the sums of the corresponding positive powers of the roots of the transformed equation.

Example: 1 (U. Q)

Show that the sum of the eleventh powers of the roots of $x^7 + 5x^4 + 1 = 0$ is zero.

Solution:

Since 11 is greater than 7,

The degree of the equation,

We have to use the latter equation in Newton's Theorem.

If we assume the equation as

$$x^7 + P_1 x^6 + P_2 x^5 + P_3 x^4 + P_4 x^3 + P_5 x^2 + P_6 x + P_7 = 0$$

We have $P_1 = P_2 = P_4 = P_5 = P_6 = 0$, $P_3 = 5$, $P_7 = 1$, $r = 11$, $n = 7$.

$$\therefore \quad S_{11} + P_1 S_{10} + P_2 S_9 + P_3 S_8 + P_4 S_7 + P_5 S_6 + P_6 S_5 + P_7 S_4 = 0$$

if $r \geq n$

Again (i.e.) $S_{11} + 5 S_8 + S_4 = 0$ (1)

$$S_8 + P_1 S_7 + P_2 S_6 + P_3 S_5 + P_4 S_4 + P_5 S_3 + P_6 S_2 + P_7 S_1 = 0$$

(i.e.) $S_8 + 5 S_5 + S_1 = 0$ (2)

Using the first equation in the Newton's Theorem.

$$S_5 + P_1 S_4 + P_2 S_3 + P_3 S_2 + P_4 S_1 + 5 P_5 = 0$$

(i.e.) $S_5 + 5 S_2 = 0$ (3)

Again $S_4 + P_1 S_3 + P_2 S_2 + P_3 S_1 + 4 P_4 = 0$

(i.e.) $S_4 + 5 S_1 = 0$ (4)

Again $S_2 + P_1 S_1 + 2 P_2 = 0$

(i.e.) $S_2 = 0$ (5)

Also $S_1 = 0$ (6)

From (4) (5) & (6)

We get $S_4 = 0$

From (3) and (5)

We get $S_5 = 0$

From (2), we get $S_5 = 0$

From (2), we get $S_8 = 0$

Substituting the values of S_4, S_8 in (1)

we get $S_{11} = 0$

Example: 2 (U. Q)

If a + b + c + d = 0, show that

$$\frac{a^5+b^5+c^5+d^5}{5} = \frac{a^2+b^2+c^2+d^2}{2} \quad \frac{a^3+b^3+c^3+d^3}{3}$$

Solution:

Since a + b + c + d = 0

We can consider that a, b, c, d are the roots of the equation.

$x^4 + P_1 x^3 + P_2 x^2 + P_3 x + P_4 = 0$ where $P_1 = 0$.

From Newton's theorem, on the sums of powers of the roots.

We get

$$S_5 + P_1 S_4 + P_2 S_3 + P_3 S_2 + P_4 S_1 = 0 \quad (1)$$

$$S_4 + P_1 S_3 + P_2 S_2 + P_3 S_1 + 4 P_4 = 0 \quad (2)$$

$$S_3 + P_1 S_2 + P_2 S_1 + 3 P_3 = 0 \quad (3)$$

$$S_2 + P_1 S_1 + 2 P_2 = 0 \quad (4)$$

$$S_1 + P_1 = 0 \quad (5)$$

From (5), we get $S_1 = 0$

From (4), we get $S_2 = -2 P_2$

From (5), we get $S_3 = -3 P_3$

From (1), we get $S_5 \; 3 P_2 P_3 - 2 P_3 P_2 = 0$

(i.e.) $S_5 = 5 P_2 P_3$.

$$\therefore \frac{S_5}{5} = \frac{S_2}{2} \; \frac{S_3}{3}$$

(i.e.) $$\frac{a^5+b^5+c^5+d^5}{5} = \frac{a^2+b^2+c^2+d^2}{2} \quad \frac{a^3+b^3+c^3+d^3}{3}$$

Example: 3

Find $\frac{1}{\alpha^5} + \frac{1}{\beta^5} + \frac{1}{\gamma^5}$ where α, β, γ are the roots of the equation $x^3 + 2x^2 - 3x - 1 = 0$

Solution:

Put $x = \frac{1}{y}$ in the equation,

Then, the equation becomes,

$$\frac{1}{y^3}+\frac{2}{y^2}-\frac{3}{y}-1 = 0$$

(i.e) $y^3 + 3y^2 - 2y - 1 = 0$

The roots of the equation are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

$\therefore \frac{1}{\alpha^5} + \frac{1}{\beta^5} + \frac{1}{\gamma^5} = S_5$ for the equation $y^3 - 3y^2 - 2y - 1 = 0$

From Newton's theorem on the sum of the powers of the roots of the equations, we get

$$S_5 + 3S_4 - 2S_3 - S_2 = 0$$

$$S_4 + 3S_3 - 2S_2 - S_1 = 0$$

$$S_3 + 3S_2 - 2S_1 - S_0 = 0$$

$$S_2 + 3S_1 - 4 = 0$$

$$S_1 + 3 = 0$$

$\therefore S_1 = -3, S_2 = -3(-3) + 4$

$$S_2 = 13$$

$$S_3 = -3(13) + 2(-3) - 3$$

$$S_3 = -42$$

$$S_4 = 149$$

$$S_5 = -518$$

$$\therefore \frac{1}{\alpha^5} + \frac{1}{\beta^5} + \frac{1}{\gamma^5} = -518.$$

Example: 4

Show that the sum of the m^{th} powers, where $m \leq n$, of the roots of the equation.

$$x^n - 2x^{n-1} - 2x^{n-2} - \ldots\ldots - 2x - 2 = 0, \text{ is } 3^m - 1.$$

Solution:

If $m \leq n$, we get from the Newton's theorem.

$$S_m - 2S_{m-1} - 2S_{m-2} - \ldots\ldots - m.2 = 0$$

$$S_{m-1} - 2S_{m-2} - \ldots - (m-1)2 = 0$$

Subtracting one from another,

We get $S_m - 3S_{m-1} - 2 = 0$

(i.e.) $S_m = 2 + 3S_{m-1}$

$$S_m = 2 + 3(2 + 3S_{m-2})$$

$$= 2 + 3\ 2 + 3^2 S_{m-2}$$

$$= 2 + 3.2 + 3^2 (2 + 3S_{m-2})$$

$$= 2 + 3.2 + 3^2.2 + 3^3 S_{m-3}$$

Continuing like this, we get

$$S_m = 2 + 3.2 + 3^2.2 + 3^3.2 + \ldots + 3^{m-1}. S_1 \text{ but } S_1 = 2.$$

$$\therefore S_m = 2 + 3.2 + 3^2.2 + 3^3.2 + \ldots\ldots + 3^{m-1}.2$$

$$= 2(1 + 3 + 3^2 + 3^3 + \ldots\ldots + 3^{m-1})$$

$$= 2. \frac{3^m - 1}{2}$$

$$\therefore S_m = 3^m - 1$$

Example: 5

Determine the value of $\phi(\alpha_1) + \phi(\alpha_2) + \ldots + \phi(\alpha_n)$ where $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ are the roots of f (x) and ϕ (x) in any rational and integral function of x.

Solution:

We have $\frac{f^1(x)}{f(x)} = \frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \ldots\ldots + \frac{1}{x-\alpha_n}$

and $\therefore \quad \frac{f^1(x)\,\phi(x)}{f(x)} = \frac{\phi(x)}{x-\alpha_1} + \frac{\phi(x)}{x-\alpha_2} + \ldots\ldots + \frac{\phi(x)}{x-\alpha_n}$

Performing the division and retaining only the remainders on both sides of the equation, we have

$$\frac{R_0\,x^{n-1} + R_1\,x^{n-2} + \ldots + R_{n-1}}{f(x)} = \frac{\phi(\alpha_1)}{x-\alpha_1} + \frac{\phi(\alpha_2)}{x-\alpha_2} + \ldots\ldots + \frac{\phi(\alpha_n)}{x-\alpha_n}$$

Hence

$$R_0\,x^{n-1} + R_1\,x^{n-2} + \ldots + R_{n-1} = \Sigma\,\phi(\alpha_1)\ (x-\alpha_2)\ldots\ldots(x-\alpha_n)$$

Equating the coefficients of x^{n-1} on both sides of the equation,

we get $\Sigma\,\phi(\alpha_1) = R_0$.

Example: 6

If the degree of $\phi(x)$ does not exceed n –2 prove that $\sum_{1}^{n} \frac{\phi(\alpha_r)}{f^1(\alpha_r)} = 0$

Solution:

We have partial fractions

$$\frac{\phi(x)}{f(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \ldots\ldots + \frac{A_n}{x-\alpha_n}$$

$$\phi(x) = A_1\,(x-\alpha_2)\,(x-\alpha_3)\ldots\ldots(x-\alpha_n) + A_2\,(x-\alpha_1)\,(x-\alpha_3)\ldots\ldots(x-\alpha_n) + \ldots\ldots + A_n\,(x-\alpha_1)\,(x-\alpha_2)\ldots\ldots(x-\alpha_{n-1})$$

$$\left[\because\ f(x) = (x-\alpha_1)\ (x-\alpha_2)\ldots\ldots(x-\alpha_n)\right]$$

Put $x = \alpha_1$, $\therefore\ \phi(\alpha_1) = A_1\,(\alpha_1-\alpha_2)\,(\alpha_1-\alpha_3)\ldots\ldots(\alpha_1-\alpha_n)$

$$f(x) = (x-\alpha_1)\ (x-\alpha_2)\ldots\ldots(x-\alpha_n)$$

$$f^1(x) = (x-\alpha_2)\ldots\ldots(x-\alpha_n) + (x-\alpha_1)\,(x-\alpha_3)\ldots\ldots(x-\alpha_n)$$

$$+ \ldots\ldots + (x - \alpha_1)\ (x - \alpha_2) \ldots\ldots (x - \alpha_{n-1})$$

$$f^1(\alpha_1) = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \ldots\ldots (\alpha_1 - \alpha_n)$$

$$\phi(\alpha_1) = A_1\ f^1(\alpha_1)$$

$$\therefore\ \phi(\alpha_1) = A_1\ f^1(\alpha_1)$$

Hence $$\frac{\phi(x)}{f(x)} = \frac{\phi(\alpha_1)}{f^1(\alpha_1)} \cdot \frac{1}{x-\alpha_1} + \frac{\phi(\alpha_2)}{f^1(\alpha_2)} \cdot \frac{1}{x-\alpha_2} + \ldots + \frac{\phi(\alpha_n)}{f^1(\alpha_n)}\ \frac{1}{x-\alpha_n}$$

$$= \sum_{r=1}^{n} \frac{\phi(\alpha_r)}{f^1(\alpha_r)}\ \frac{1}{x-\alpha_r}$$

$$\therefore\ \frac{x\,\phi(x)}{f(x)} = \sum_{r=1}^{n} \frac{\phi(\alpha_r)}{f^1(\alpha_r)}\ \frac{x}{x-\alpha_r}$$

$$= \sum_{r=1}^{n} \frac{\phi(\alpha_r)}{f^1(\alpha_r)}\ \frac{1}{\left(1-\frac{\alpha_r}{x}\right)}$$

$$= \sum_{r=1}^{n} \frac{\phi(\alpha_r)}{f^1(\alpha_r)} \left\{1 + \frac{\alpha_r}{x} + \left(\frac{\alpha_r}{x}\right)^2 + \ldots\right\}$$

$$\sum_{r=1}^{n} \frac{\phi(\alpha_r)}{f^1(\alpha_r)} = \text{term independent of x in } \frac{x\,\phi(x)}{f(x)}$$

$\phi(x)$ is of degree $n - 2$,

$f(x)$ is of degree n.

Hence $x\ \phi(x)$ is of degree $n - 1$.

$$\therefore \frac{x\,\phi(x)}{f(x)} = \frac{B_0\, x^{n-1} + B_1\, x^{n-2} + \ldots B_{n-1}}{x^n + p_1\, x^{n-1} + \ldots + p_n} = \frac{B_0\, x^{n-1} + B_1\, x^{n-2} + \ldots B_{n-1}}{1 + \frac{P_1}{x} + \frac{P_2}{x_2} + \ldots + \frac{P_n}{x_r}}$$

$$= \frac{\frac{B_0}{x} + \frac{B_1}{x^2} + \ldots + \frac{B_{n-1}}{x^n}}{1 + \frac{P_1}{x} + \frac{P_2}{x_2} + \ldots + \frac{P_n}{x_r}}$$

Hence in the expansion of $\frac{x\phi(x)}{f(x)}$ there is no term independent of x.

$$\therefore \sum_{r=1}^{n} \frac{\phi(\alpha_r)}{f^1(\alpha_r)} = 0$$

Exercise

1. If α, β, γ are the roots of $x^3 + qx + r = 0$, prove that

   i) $3 S_2 S_5 = 5 S_3 S_4$.

   ii) $\frac{\alpha^5+\beta^5+\gamma^5}{5} = \frac{\alpha^3+\beta^3+\gamma^3}{3} - \frac{\alpha^2+\beta^2+\gamma^2}{2}$

   iii) $\frac{\alpha^7+\beta^7+\gamma^7}{7} = \frac{\alpha^5+\beta^5+\gamma^5}{5} - \frac{\alpha^2+\beta^2+\gamma^2}{2}$

2. Find the sum of the fourth powers of the roots x^3-2x^2+x-1

3. Find the sum of the fifth powers of the roots of $x^4-3x^3+5x^2-12x+4=0$

4. In the equation $x^4-x^3-7x^2+x+6=0$, find the values of S_4 and S_6.

5. Find the sum of fifth powers of the roots of $x^4-7x^2-4x-3=0$

6. Find the sum of the sixth powers of the roots of the equation $x^7-x^4+1=0$

7. Show that the sum of ninth powers of the roots of $x^3+3x+9=0$ is zero.

8. Prove that the sum of the twentieth powers of the roots of the equation.

   $x^4+ax+b=0$, is $50\ a^4 b^2 - 4b^5$

UNIT – II

2.1: TRANSFORMATIONS OF EQUATIONS

If an equation is given. It is possible to transform this equation into another whose roots bear with the roots of the original equation a given relation. Such a transformation often helps us to solve equation easily or to discuss the nature of the roots of the equations we shall explain here the most important elementary transformations of equations.

ROOTS WITH SIGNS CHANGED:

To transform equation into another whose roots are numerically the same as those of the given equation but opposite in sign.

Let $\alpha_1, \alpha_2 \ldots\ldots\ \alpha_n$ be the roots of the equation.

$x^n + P_1 x^{n-1} + P_2 x^{n-2} + \ldots\ldots + P_n = 0$

Then we have

$x^n + P_1 x^{n-1} + P_2 x^{n-2} + \ldots\ldots + P_n = (X - \alpha_1)(X - \alpha_2) \ldots\ldots (X - \alpha_n)$

changing x into $-x$,

we have

$(-x)^n + P_1 (-x)^{n-1} + P_2 (-x)^{n-2} + \ldots\ldots + P_n$

$\equiv (-x - \alpha_1)(-x - \alpha_2) \ldots\ldots (-x - \alpha_n)$

$\therefore$ The roots of the equation

$x^n - P_1 x^{n-1} + P_2 x^{n-2} - \ldots\ldots \pm P_n = 0$ are $-\alpha_1, -\alpha_2 \ldots\ldots -\alpha_n$.

Therefore to effect the required transformation we have to substitute $-x$ for x in the given equation; that is to change the sign of every alternate term of the given equation beginning with the second.

Example: 1

Find the equation whose roots are the roots of $x^5 + 6x^4 + 6x^3 - 7x^2 + 2x - 1 = 0$ with the signs changed.

Solution:

The roots of the equation

$x^n - P_1 x^{n-1} + P_2 x^{n-2} - \ldots\ldots \pm P_n = 0$ are $-\alpha_1 - \alpha_2 \ldots\ldots \alpha_n$

$\therefore$ Then the transformed equation.

$\therefore x^5 - 6x^4 + 6x^3 + 7x^2 + 2x + 1 = 0$

Example: 2

Change the sign of the roots of the equation $x^7 + 4x^5 + x^3 - 2x^2 + 7x + 3 = 0$.

Solution:

The roots of the equation.

$x^n - P_1 x^{n-1} + P_2 x^{n-2} - \ldots\ldots \pm P_n = 0$ are $-\alpha_1 - \alpha_2 \ldots\ldots -\alpha_n$.

$\therefore$ Then the transformed equation

$$x^7 + 4x^5 + x^3 + 2x^2 + 7x - 3 = 0.$$

2.2 ROOTS MULTIPLIED BY A GIVEN NUMBER

To transform an equation into another whose roots are m times that of the given equation.

$$x^n + P_1 x^{n-1} + P_2 x^{n-2} + \ldots\ldots + P_n \equiv (X - \alpha_1)(X - \alpha_2)\ldots\ldots(X - \alpha_n)$$

instead of X substitute $\frac{y}{m}$.

We get

$$\left(\frac{y}{m}\right)^n + P_1\left(\frac{y}{m}\right)^{n-1} + P_2\left(\frac{y}{m}\right)^{n-2} + \ldots\ldots P_n$$

$$\equiv \left(\frac{y}{m} - \alpha_1\right)\left(\frac{y}{m} - \alpha_2\right)\ldots\ldots\left(\frac{y}{m} - \alpha_n\right)$$

Multiplying both sides by m^n,

$$y^n + m P_1 y^{n-1} + m^2 P_2 y^{n-2} + \ldots\ldots + m^n P_n$$

$$\equiv (y - m\,\alpha_1)(y - m\,\alpha_2)\ldots\ldots(y - m\,\alpha_n).$$

$\therefore$ The equation $y^n + m P_1 y^{n-1} + m^2 P_2 y^{n-2} + m^n P_n = 0$ has the roots $m\,\alpha_1, m\,\alpha_2 \ldots\ldots m\,\alpha_n$.

Hence to effect this transformation is useful for the purpose of removing the coefficient of the first term of an equation when it is other than unity and generally for removing the fractional coefficients from an equation.

Example: 1

Remove the fractional coefficients from the equation

$$x^3 - \frac{1}{4}x^2 + \frac{1}{3}x - 1 = 0$$

Solution:

Multiply the roots by 12.

We get the transformed equation as

$$x^3 - \frac{1}{4}.12\ x^2 + \frac{1}{3}.12^2\ x - 1.12^3 = 0$$

$$\text{(i.e) } x^3 - 3x^2 + 48\ x - 1728 = 0$$

Example: 2

Remove the fractional coefficients from the equation $x^3 + \frac{1}{4}x^2 - \frac{1}{16}x + \frac{1}{72} = 0.$

Solution:

To transform the equation into another whose roots are multiplied by m. We get

$$x^3 + \frac{m}{4}\ x^2 - \frac{m^2}{16}x + \frac{m^3}{72} = 0$$

$$\text{(i.e.) } x^3 + \frac{m}{2^2}\ x^2 - \frac{m^2}{2^4}x + \frac{m^3}{2^3.3^2} = 0$$

If m = 12, the fractions $\frac{m}{2}, \frac{m^2}{2^4}, \frac{m^3}{2^3.3^2}$ will be integers.

Hence we have to multiply the roots by 12.

The equation becomes

$$x^3 + \frac{12\ x^2}{2^2} - \frac{12^2 x}{2^4} + \frac{12^3}{2^3.3^2} = 0$$

$$\text{(i.e.) } x^3 + 3\ x^2 - 9\ x + 24 = 0$$

Example: 3

Change the equation $2x^4 - 3x^3 + 3x^2 - x + 2 = 0$ into another the coefficient of whose highest term will be unity.

Solution:

Multiply the roots by 2. Then the transformed equation becomes.

$$2x^4 - 3.2x^3 + 3.2^2\ x^2 - 2^3 x + 2.2^4 = 0$$

$$\text{(i.e.) } 2x^4 - 6x^3 + 12x^2 - 8x + 32 = 0$$

Dividing by 2,

We get

$$x^4 - 3x^3 + 6x^2 - 4x + 16 = 0$$

Example: 4

Transform the equation $3x^3 + 4x^2 + 5x - 6 = 0$ into one in which the coefficient of x^3 is unity.

Solution:

Multiply the roots by 3.

Then the transformed equation becomes.

$y^n + mP_1 y^{n-1} + m^2 P_2 y^{n-2} + \ldots\ldots + m^n P_n = 0$ has the roots $m\alpha_1$, $m\alpha_2, \ldots . m\alpha_n$.

$$3x^3 + 3.4x^2 + 3^2.5x - 3^3.6 = 0$$

$$3x^3 + 12x^2 + 45x - 162 = 0$$

Dividing by 3.

We get

$$X^3 + 4x^2 + 15x - 54 = 0$$

Example: 5

Remove the fractional coefficients from the equation.

$$x^3 + \frac{3}{2}x^2 + \frac{5}{18}x + \frac{1}{108} = 0$$

Solution:

Multiply the roots by 12

We get the transformed equation as $x^3 + 12.\frac{3}{2}x^2 + 12^2.\frac{5}{18}x + \frac{12^3}{108} = 0$

$$x^3 + 18x^2 + 40x + 16 = 0$$

Exercise

1. Find the equation whose roots are the roots of $x^5 + 8x^4 + 3x^3 - 7x^2 + 5x - 6 = 0$ with the signs changed.
2. Change the sign of the roots of the equation.
   $x^7 + 7x^6 + 4x^4 + 2x^3 - 3x + 6 = 0$
3. Transform the equation $4x^3 + 2x^2 + 6x - 12 = 0$ into one in which the coefficients of x^3 in unity and all coefficients are integral.

4. Remove the fractional coefficients from the equation.

$$2x^3+\frac{3}{2}x^2-\frac{1}{8}x-\frac{3}{16}=0$$

5. Transform the equation $3x^4-\frac{5}{2}x^3+\frac{7}{6}x^2-x+\frac{7}{18}=0$ into another with integral coefficients and for the coefficients of the first term unity.

6. Transform the equation $2x^4-\frac{7}{2}x^3+\frac{5}{6}x^2-x+\frac{7}{18}=0$ into another with integral coefficients and for the coefficients of the first term unity.

2.3 RECIPROCAL ROOTS

To transform an equation into another whose roots are the reciprocals of the roots of the given equation.

Let $\alpha_1, \alpha_2 \ldots\ldots \alpha_n$ be the roots of the equation $x^n+P_1x^{n-1}+P_2x^{n-2}+P_n=0$.

We have

$$x^n+P_1x^{n-1}+P_2x^{n-2}+\ldots\ldots+P_n\equiv(x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_n)$$

put $x=\frac{1}{y}$

$$\left(\frac{1}{y}\right)^n+P_1\left(\frac{1}{y}\right)^{n-1}+P_2\left(\frac{1}{y}\right)^{n-2}+\ldots\ldots+P_n$$

$$=\left(\frac{1}{y}-\alpha_1\right)\left(\frac{1}{y}-\alpha_2\right)\ldots\ldots\left(\frac{1}{y}-\alpha_n\right)$$

Multiplying throughout by y^n

We have

$$P_ny^n+P_{n-1}y^{n-1}+P_{n-2}y^{n-2}+\ldots\ldots+P_1y+1$$

$$=(\alpha_1\alpha_2\ldots\ldots\alpha_n)\left(\frac{1}{\alpha_1}-y\right)\left(\frac{1}{\alpha_2}-y\right)\ldots\ldots\left(\frac{1}{\alpha_n}-y\right)$$

Hence the equation

$P_ny^n+P_{n-1}y^{n-1}+P_{n-2}y^{n-2}+\ldots\ldots+P_1y+1=0$ has roots $\frac{1}{\alpha_1},\frac{1}{\alpha_2}\ldots\ldots\frac{1}{\alpha_n}$.

2.4 RECIPROCAL EQUATION

If an equation remains unaltered when x is changed into its reciprocal it is called a reciprocal equation.

Let $x^n + P_1 x^{n-1} + P_2 x^{n-2} + P_{n-1} x + P_n = 0$ (1)

be a reciprocal equation.

When x is changed into its reciprocal $\frac{1}{x}$.

we get the transformed equation

$$P_n x^n + P_{n-1} x^{n-1} + P_{n-2} x^{n-2} + \ldots\ldots P_1 x + 1 = 0$$

$$\text{(i.e.) } x^n + \frac{P_{n-1}}{P_n} x^{n-1} + \frac{P_{n-2}}{P_n} x^{n-2} + \ldots\ldots\ldots + \frac{P_1}{P_n} x + \frac{1}{P_n} = 0 \quad (2)$$

Since (1) is a reciprocal equation, it must be the same as (2)

$$\therefore \frac{P_{n-1}}{P_n} = P_1, \therefore \frac{P_{n-2}}{P_n} = P_2 \ldots\ldots \frac{P_1}{P_n} = P_{n-1} \text{ and } \frac{1}{P_n} = P_n$$

$\therefore \quad P^2{}_n = 1$

$\therefore \quad P_n = \pm 1$

Case 1: $P_n = 1$

Then $P_{n-1} = P_1$, $P_{n-2} = P_2$, $P_{n-3} = P_3$

In this case the coefficients of the terms equidistant from the beginning and the end are equal in magnitude and have the same sign.

Case ii) $P_n = -1$

We have $P_{n-1} = -P_1$, $P_{n-2} = -P_2$, ... $P_1 = -P_{n-1}$.

In this case the terms equidistant from the beginning and the end are equal in magnitude but different in sign.

Standard from of reciprocal Equations.

If α be a root of a reciprocal equation $\frac{1}{\alpha}$ must also be a root, for it in a root of the transformed equation and the transformed equation is identical with the first equation. Hence the roots of a reciprocal equation occur in pairs. $\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta} \ldots\ldots\ldots$

When the degree is odd one of its roots must be its own reciprocal.

$$\gamma = \frac{1}{\gamma}$$

(u) $\gamma^2 = 1$

(u) $\gamma = \pm 1$

If the coefficients have all like signs, then −1 is a root. If the coefficients of the terms equidistant from the first and last have opposite signs, then +1 is a root. In either case the degree of an equation can be depressed by unity if we divide the equation by x + 1 or by x −1. The depressed equation is always a reciprocal equation of even degree with like signs for its coefficients.

If the degree of a given reciprocal equation is even. Say n = 2m and if terms equidistant from the first and last have opposite signs, then

$$P_m = -P_m$$

(i.e.) $P_m = 0$, so that in this type of reciprocal equations, the middle term is absent. Such an equation may be written as

$$x^{2m} - 1 + P_1 x (x^{2m-2} - 1) + \ldots\ldots = 0$$

Dividing by $x^2 - 1$, this reduces to a reciprocal equations of like signs of even degree. Hence all reciprocal equations may be reduced to an even degree reciprocal equation with like sign, and so an even degree reciprocal equation with like signs is considered as the standard from of reciprocal equations.

A Reciprocal Equation of the standard form can always be depressed to another of half the dimensions.

It has been shown in the previous article that all reciprocal equations can be reduced to a standard form, in which the degree is even and the coefficients of terms equidistant from the beginning and the end are equal and have the same sign.

Let the standard reciprocal equation be

$$a_0 x^{2m} + a_1 x^{2m-1} + a_2 x^{2m-2} + \ldots\ldots + a_m x^n + \ldots\ldots + a_1 x + a_0 = 0.$$

Dividing by x^m and grouping the terms equally distant from the ends, we have

$$a_0\left(x^m+\frac{1}{x^m}\right)+a_1\left(x^{m-1}+\frac{1}{x^{m-1}}\right)+\ldots\ldots+a_{m-1}\left(x+\frac{1}{x}\right)+a_m=0$$

Let $x+\frac{1}{x}=z$ and $x^r+\frac{1}{x^r}=X_r$

We have the relation $X_{r+1}=Z.\,X_r-X_{r-1}$

Giving r in succession the values 1, 2, 3, ...

we have $X_2=z\,x_1-x_0=z^2-2$

$$\left[\because x_2=x^2+\frac{1}{x^2}=z^2 = x^2+\frac{1}{x^2}-z = z^2-2\right]$$

$$X^3=z\,x_2-x_1=z^3-3z$$

$$X_4=z\,x_3-x_2=z^4-4x^2+2$$

$$X_5=z\,x^4-x_3=z^5-5z^3+5z$$

and so on.

Substituting these values in the above equation, we get an equation of the m^{th} degree in z. To every root of the reduced equation in z. Correspond two roots of the reciprocal equation. Thus if k be a root of the reduced equation, the quadratic

$$X+\frac{1}{X}=k.$$

(i.e.) $x^2-k\,x+1=0$. gives the two corresponding roots $\frac{k\pm\sqrt{k^2-4}}{2}$ of the given reciprocal equation.

Example: 1

Find the roots of the equation

$$X^5+4x^4+3x^3+3x^2+4x+1=0$$

Solution:

This is a reciprocal equation of odd degree with like signs.

$\therefore (x+1)$ is a factor of $X^5+4x^4+3x^3+3x^2+4x+1$

The equation can be written as

$$X^5+x^4+3x^4+3x^3+3x^2+3x+x+1=0$$

(i.e.) $x^4 (x + 1) + 3x^3 (x + 1) + 3x (x + 1) + 1 (x + 1) = 0$

(i.e.) $(x+1) (x^4 + 3x^3 + 3x + 1) = 0$

$\therefore x + 1 = 0$ or $x^4 + 3x^3 + 3x + 1) = 0$

Dividing by x^2, we get $\left(x^2 + \frac{1}{x^2}\right) + 3\left(x + \frac{1}{x}\right) = 0$

Put $\left(x + \frac{1}{x}\right) = z.$

$$\therefore x^2 + \frac{1}{x^2} = z^2 - 2$$

$z^2 - 2 + 3z = 0$

$$z = \frac{-3 \pm \sqrt{17}}{2}$$

Hence $x + \frac{1}{x} = \frac{-3 \pm \sqrt{17}}{2}$

(i.e.) $\frac{x^2 + 1}{x} = \frac{-3 \pm \sqrt{17}}{2}$

$$2 (x^2 + 1) = x (-3 \pm \sqrt{17})$$

$$2 x^2 + 2 = - 3 x \pm \sqrt{17} x$$

(i.e.) $2x^2 + (3 + \sqrt{17}) x + 2 = 0$

(or) $2x^2 + (3 - \sqrt{17}) x + 2 = 0$

From these equations x can be found.

Example: 2

Solve the equation $6x^5 - x^4 - 43 x^3 + 43x^2 + x - 6 = 0$

Solution:

This is a reciprocal equation of odd degree with unlike signs.

Hence $x - 1$ is a factor of the left – hand side. The equation can be written as follows. Given

$$6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$$
$$6x^5 - 6x^4 + 5x^4 - 5x^3 - 38x^3 + 38x^2 + 5x^2 - 5x + 6x - 6 = 0$$

(i.e.) $6x^4(x-1) + 5x^3(x-1) - 38x^2(x-1) + 5x(x-1) + 6(x-1) = 0$

(i.e.) $(x-1)\ (6x^4 + 5x^3 - 38x^2 + 5x + 6) = 0$

$\therefore\ x = 1$ or $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$

We have to solve the equation

$$6x^4 + 5x^4 - 38x^2 + 5x + 6 = 0$$

Dividing by x^2

$$6x^2 + 5x - 38 + \frac{5}{x} + \frac{6}{x^2} = 0$$

(i.e.) $6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0$

Put $x + \frac{1}{x} = z$, $\therefore\ x^2 + \frac{1}{x^2} = z^2 - 2$

The equation becomes

$6(z^2 - 2) + 5z - 38 = 0 \Rightarrow 6z^2 + 5z - 50 = 0$

(ie) $(2z - 5)(3z + 10) = 0$

(i.e.) $z = -10/3$ or $5/2$

$\therefore\ x + \frac{1}{x} = \frac{-10}{3}$ or $x + \frac{1}{x} = \frac{5}{2}$

(i.e.) $3x^2 + 10x + 3 = 0$ (or) $2x^2 - 5x + 2 = 0$

(i.e.) $(x+3)(3x+1) = 0$ (or) $(2x-1)(x-2)$

(i.e.) $x = -3$ or $\frac{-1}{3}$ or 2 or $\frac{1}{2}$.

$\therefore$ The roots of the equations are

$1, -3, \frac{-1}{3}, 2$ and $\frac{1}{2}$.

Example: 4

1. Solve the equation $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0$.

Solution:

The given equation in a first type and even degree is a standard reciprocal equation.

Let $f(x) = 4x^4 - 20x^3 + 33x^2 - 20x + 4$.

Dividing the equation by x^2 and regrouping we get

$$4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0$$

$$\Rightarrow 4x^2 - 20x + 33 - \frac{20}{x} + \frac{4}{x^2} = 0$$

$$\Rightarrow 4\left(x^2 + \frac{1}{x^2}\right) - 20\left(x + \frac{1}{x}\right) + 33 = 0 \qquad (1)$$

Put $x + \frac{1}{x} = y$ & $x^2 + \frac{1}{x^2} = y^2 - 2$

$$\left[\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2\right]$$

(1) becomes

$$4(y^2 - 2) - 20y + 33 = 0$$

$$\Rightarrow 4y^2 - 8 - 20y + 33 = 0$$

$$\Rightarrow 4y^2 - 20y + 25 = 0$$

$$\Rightarrow (2y - 5)^2 = 0$$

$$\therefore y = \frac{5}{2}, \frac{5}{2}$$

Let $x + \frac{1}{x} = y$

$$\Rightarrow x + \frac{1}{x} = \frac{5}{2}$$

$$\frac{x^2 + 1}{x} = \frac{5}{2}$$

$$\Rightarrow 2x^2 + 2 = 5x$$

$\Rightarrow 2x^2 - 5x + 2 = 0$

$\Rightarrow (2x - 1)(x - 2) = 0$

$\Rightarrow x = \frac{1}{2}, 2$

Again, Let $x + \frac{1}{x} = \frac{5}{2}$

$X = \frac{1}{2}, 2$

The roots are $\frac{1}{2}, 2, \frac{1}{2}, 2$.

Example: 5

Solve the equation $6x^5 + 11x^4 - 33x^3 - 33x^2 + 11x + 6 = 0$

Solution:

Let $f(x) = 6x^5 + 11x^4 - 33x^3 - 33x^2 + 11x + 6$

This is a Reciprocal equation of first type and of odd degree.

Hence x+1 is a factor of f (x) by actual division.

−1	6	11	− 33	− 33	11	6
	0	−6	− 5	38	.5	−6
	6	5	−38	5	6	0

$\therefore$ $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$ is a S.R.E. Dividing by x^2 & regrouping.

We get

$6x^4 + 5x^4 - 38x^2 + 5x + 6 = 0 \qquad [\div x^2]$

$6x^2 + 5x - 38 + \frac{5}{x} + \frac{6}{x^2} = 0$

$\Rightarrow 6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0 \qquad (1)$

Put $x + \frac{1}{x} = y$ and $x^2 + \frac{1}{x^2} = y^2 - 2$

(1) becomes

$$6\left(x^2+\frac{1}{x^2}\right)+5\left(x+\frac{1}{x}\right)-38=0$$

$$\Rightarrow 6\,(y^2-2)+5\,(y)-38=0$$

$$\Rightarrow 6y^2-12+5y-38=0$$

$$\Rightarrow 6y^2+5y-50=0$$

$$y=\frac{-5\pm\sqrt{25+1200}}{12}$$

$$=\frac{-5\pm\sqrt{1225}}{12}=\frac{-5\pm 35}{12}$$

$$y=\frac{-5+35}{12},\ \frac{-5-35}{12}$$

$$=\frac{30}{12},\ \frac{-40}{12}$$

$$y=\frac{15}{6},\ \frac{-10}{3}$$

Let $x+\frac{1}{x}=y$.

$$\Rightarrow x+\frac{1}{x}=\frac{5}{2}$$

$$\Rightarrow 2x^2+2=5x$$

$$\Rightarrow 2x^2-5x+2=0$$

$$\Rightarrow (2x-1)\,(x-2)=0$$

$$\Rightarrow x=\frac{1}{2},\ 2$$

Again, Let $x+\frac{1}{x}=-\frac{10}{3}$

$$\Rightarrow \frac{x^2+1}{x}=-\frac{10}{3}$$

$$\Rightarrow 3x^2+3=-10\,x$$

$\Rightarrow 3x^2 + 10x + 3 = 0$

$\Rightarrow (3x + 1)(x+3) = 0 \Rightarrow x = -\frac{1}{3}, -3$

$\therefore$ The roots of f (x) = 0, $-1, \frac{1}{2}, 2, -\frac{1}{3}, -3$.

Exercise

1. Solve the following equation

   i) $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$

   ii) $x^4 + 3x^3 - 3x - 1 = 0$

   iii) $60x^4 - 736x^3 + 1433x^2 - 736x + 60 = 0$

   iv) $2x^6 - 9x^5 + 0x^4 - 3x^3 + 10x^2 - 9x + 2 = 0$

   v) $x^{10} - 3x^8 + 5x^6 - 5x^4 + 3x^2 - 1 = 0$

   vi) $x^6 + 2x^5 + 2x^4 - 2x^2 - 2x - 1 = 0$

   vii) $x^4 - x^3 - 8x^2 + x + 1 = 0$

2.5 STANDARD FORMS TO INCREASE AND DECREASE THE ROOTS OF A GIVEN EQUATION BY A GIVEN QUANTITY

Let the roots of the given equation

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_n = 0$$

be $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots\ldots \alpha_n$ and suppose.

We require the equation whose roots are $\alpha_1 - h, \alpha_2 - h, \alpha_3 - h \ldots\ldots \alpha_n - h$.
We have $f(x) \equiv a_0 (x - \alpha_1)(x - \alpha_2) \ldots\ldots (x - \alpha_n)$.

In this if we change x into y + h.
We have

$$f(y+h) = a_0 (y + h - \alpha_1)(y + h - \alpha_2) \ldots\ldots (y + h - \alpha_n)$$

The right – hand side vanishes when $y = \alpha_r - h$, (r =1, 2, ... n).

Hence if an equation is to be transformed into another whose roots are those of the first diminished by h. Substitute y + h for x in the given equation. Then we obtain the transformed equation as

$$a_0 (y+h)^n + a_1 (y+h)^{n-1} + a_2 (y+h)^{n-2} + \dots\dots + a_n = 0 \qquad (1)$$

$$a_0 \left[y^n + nc_1 y^{n-1} h + \dots\dots + h^n \right] + a_1 \left[y^{n-1} + nc_1 y^{n-2} h + \dots\dots + h^{n-1} \right]$$
$$+ a_2 \left[y^{n-2} + nc_1 y^{n-3} + \dots\dots h^{n-2} \right] + \dots\dots + a_n = 0$$

Expanding and collecting the coefficients of the powers of y.

Let the equation be

$$A_0 y_n + A_1 y^{n-1} + A_2 y^{n-2} + \dots\dots + A_{n-1} y + A_n = 0 \qquad (2)$$

Here $A_0, A_1, A_2 \dots\dots A_{n-1}, A_n$ are functions of $a_0, a_1 \dots\dots a_n$

Since $y = x - h$.
This equation (2) is equivalent to
$$A_0 (x-h)^n + A_1 (x-h)^{n-1} + A_2 (x-h)^{n-2} + \dots\dots + A_{n-1} (x-h) + A_n = 0 \qquad (3)$$

This equation (3) must be identical with the given equation (1). This forms suggests an easy rule for calculating $A_0, A_1, A_2 \dots\dots A_n$.

We can easily see that $A_0 = a_0$
If the polynomial on the left – hand side (3) is divided by x – h. The remainder is A_n and the quotient is

$$A_0 (x-h)^{n-1} + A_1 (x-h)^{n-2} + A_2 (x-h)^{n-3} + \dots\dots A_{n-2} (x-h) + A_{n-1}.$$

If the quotient again is divided by x – h. The remainder is A_{n-1} and the quotient is

$$A_0 (x-h)^{n-2} + A_1 (x-h)^{n-3} + \dots\dots + A_{n-3} (x-h) + A_{n-2}.$$

By continuing this process we can find all the coefficients of the transformed equation (2).

Instead of diminishing the roots is we desire to increase them, we take h negative.

Form of the Quotient and remainder when a polynomial is divided by a binomial.

Let the quotient when

$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_n$ is divided by $x - h$ be

$b_0 x^{n-1} + b_1 x^{n-2} + b_2 x^{n-3} + \ldots\ldots + b_{n-2} x + b_{n-1}$

This we shall represent by Q and the remainder by R. We have then the following equation.

$f(x) = (x - h) Q + R$

$\therefore (x - h) Q + R = (b_0 x^{n-1} + b_1 x^{n-2} + \ldots\ldots + b_{n-2} x + b_{n-1}) (x - h) + R$

$= b_0 x^n + (b_1 x^{n-1} + b_2 x^{n-2} + \ldots\ldots + b_{n-2} x^2 + b_{n-1} x -$
$b_0 h x^{n-1} - b_1 h x^{n-2} - b_2 h x^{n-3} + \ldots\ldots + b_{n-2} x h + b_{n-1} h + R$

$= b_0 x^n + (b_1 - hb_0) x^{n-1} + (b_2 - hb_1) x^{n-2} + \ldots\ldots + (b_{n-1} - hb_{n-2}) x + R - hb_{n-1}$

Equating the coefficients of corresponding powers of x on both sides.

We get the following series of equations to determine

$b_0, b_1, b_2 \ldots\ldots b_{n-1}, R \ldots$

$b_0 = a_0$

$b_1 - hb_0 = a_1$ (i.e.) $b_1 = a_1 + hb_0 = a_1 + ha_0$

$b_2 - hb_1 = a_2$ (i.e.) $b_2 = a_2 + hb_1$

$b_3 - hb_2 = a_3$, (i.e.) $b_3 = a_3 + hb_2$

. . . .

. . . .

. . . .

$b_{n-2} - hb_{n-3} = a_{n-2}$, (i.e) $b_{n-2} = a_{n-2} + hb_{n-3}$

$b_{n-1} - hb_{n-2} = a_{n-1}$, (i.e) $b_{n-1} = a_{n-1} + hb_{n-2}$

$R - hb_{n-1} = a_n$, (i.e.) $R = a_n + hb_{n-1}$

These equations supply a ready method of calculating in succession the coefficients $b_0, b_1, b_2 \dots\dots b_{n-1}$ of the quotient and the remainder R.

For this purpose we can write the series of operations as follows.

a_0	a_1	a_2	$a_3 \dots\dots$	a_{n-1}	a_n
	$a_0 h$	$b_1 h$	$b_2 h \dots\dots\dots$	$b_{n-2} h$	$b_{n-1} h$
	b_1	b_2	$b_3 \dots\dots\dots$	b_{n-1}	R

In the first line the successive coefficients of the given equations are written. The first term in the second line is obtained by multiplying a_0 by h. The product $a_0 h$ is placed under a_1 and then added to it in order to obtain the term b_1, in the third line. This term when obtained, is multiplied in turn by h and placed under a_2. The product is added to a_2 to obtain the second term b_2 in the third line. The repetition of this process furnishes in succession all the coefficients of the quotient, the last term thus obtained being the remainder.

Example: 1

Find the quotient and remainder when $3x^3 + 8x^2 + 8x + 12$ is divided by $x - 4$.

Solution:

The calculation is arranged as follows.

4	3	8	8	12
	0	12	80	352
	3	20	88	364 = R

The quotient is $3x^2 + 20x + 88$ and the remainder is 364.

Example: 2

Find the quotient and remainder when $2x^6 + 3x^5 - 15x^2 + 2x - 4$ is divided by $x + 5$.

Solution:

The calculation is arranged as follows:

–5	2	3	0	0	–15	2	–4
	0	–10	35	–175	875	–4300	21490
	2	–7	35	–175	860	–4298	21486 = R

The quotient is $2x^5 - 7x^4 + 35x^3 - 175x^2 + 860\,x - 4298$ and the remainder is 21486.

Example: 3

Diminish the roots of $x^4 - 5x^3 + 7x^2 - 4x + 5 = 0$ by 2.

Solution:

The coefficients in the transformed equations are the remainders when the polynomial is divided by $x - 2$ in succession.

The division of the polynomial by $x - 2$ can be exhibited as follows.

2	1	−5	7	−4	5
	0	+2	−6	2	−4
	1	−3	1	−2	1

The quotient is $x^3 - 3x^2 + x - 2$ and the remainder is 1.

∴ 1 is the absolute term in the transformed equation.

The coefficient of x is the remainder when $x^3 - 3x^2 + x - 2$ is divided by $x - 2$. The calculation is arranged as follows:

2	1	−3	1	−2
	0	2	−2	−2
	1	−1	−1	−4

The quotient is $x^2 - x - 1$ and the remainder is −4. When this is divided by $x - 2$. We get

2	1	−1	−1
	0	2	2
	1	1	1 = R

The quotient is $x + 1$ and remainder is 1.

When $x + 1$ is divided by $x - 2$, we get

2	1	1
	0	2
	1	3

The remainder is 3.

The coefficients of the transformed equation are 1, 3, 1, –4 and 1.

∴ The transformed equation is

$$x^4 + 3x^3 + x^2 - 4x + 1 = 0$$

All these operations can be combined and exhibited as follows.

2	1	–5	7	–4	5
	0	2	–6	2	–4
2	1	–3	1	–2	1
	0	2	–2	–2	
2	1	–1	–1	–4	
	0	2	2		
2	1	1	1		
	0	2			
2	1	1			
	0	2			
	1	3			

Example: 4

Increase by 7 the roots of the equation is the same as diminishing the roots by –7. Increase by 7 the roots of the equation $3x^4 + 7x^3 - 15x^2 + x - 2=0$

–7	3	7	–5	1	–2
	0	–21	98	–581	4060
–7	3	–14	83	–580	4058
	0	–21	245	–2296	
–7	3	–35	328	–2876	
	0	–21	392		
–7	3	–56	720		
	0	–21			
	3	–77			

∴ The transformed equation is $3x^4-77x^3+720x^2 - 2876x+4058 = 0$

Example: 5

Show that the equation $x^4-3x^3+4x^2-2x+1 = 0$ can be transformed into a reciprocal equation by diminishing the roots by unity. Hence solve the equation.

Solution:

The operation of diminishing the roots by 1 can be exhibited as follows.

1	1	−3	4	−2	1
	0	1	−2	2	0
1	1	−2	2	0	1
	0	1	−1	1	
1	1	−1	1	1	
	0	1	0		
1	1	0	1		
	0	1			
	1	1			

The transformed equation is $x^4+x^3+x^2+x+1=0$ which is a reciprocal equation.

The equation can be written as

$$\left(x^2+\frac{1}{x^2}\right) + \left(x+\frac{1}{x}\right) + 1 = 0$$

writing $x + \frac{1}{x} = z$, we get

$$Z^2 - 2 + z + 1 = 0$$

(i.e.) $z^2 + z - 1 = 0$

(i.e.) $z = \frac{-1 \pm \sqrt{5}}{2}$

(i.e.) $x + \frac{1}{x} = \frac{-1+\sqrt{5}}{2}$ (or) $x + \frac{1}{x} = \frac{-1-\sqrt{5}}{2}$

(i.e.) $2x^2\ (1 - \sqrt{5})\ x + 2 = 0$ (or) $2x^2 + (1 + \sqrt{5})\ x + 2 = 0$

(i.e.) $x = \frac{\sqrt{5}-1 \pm \sqrt{(1-\sqrt{5})^2 - 16}}{4}$ (or) $\frac{-(1+\sqrt{5}) \pm \sqrt{(1+\sqrt{5})^2 - 16}}{4}$

$\therefore$ The roots of the original equation are these roots increased by 1.

They are $\frac{\sqrt{5}+3 \pm \sqrt{-10-2\sqrt{5}}}{4}$, $\frac{3-\sqrt{5} \pm \sqrt{2\sqrt{5}-10}}{4}$

Example: 6

Transform $x^4 - 5x^3 + 7x^2 - 4x + 5 = 0$ into another equation whose roots are less by 3.

Solution:

The calculation is arranged as follows:

2	1	–5	7	–4	5
	0	2	–6	2	–4
2	1	–3	1	–2	1
	0	2	–2	–2	
2	1	–1	–1	–4	
	0	2	2		
2	1	1	1		
	0	2			
	1	3			

The transformed the equation is

$$\therefore x^4 + 3x^3 + x^2 - 4x + 1 = 0$$

Exercise

1. Diminish by 3 the roots of the equation

   $x^5 - 4x^4 + 3x^3 - 4x + 6 = 0$

2. Transform $x^4 + 5x^3 + 8x^2 - 4x + 5 = 0$ into another equation whose roots are less by .

3. Find the equation each of whose roots exceeds by 2 a root of the equation $x^3 - 4x^2 + 3x - 1 = 0$.

4. Find the equation whose roots are the roots of $x^4 - 5x^3 + 7x^2 - 17x + 11 = 0$. each diminished by 2.

5. Find the equation whose roots are the roots of $4x^5 - 2x^3 + 7x - 3 = 0$ each increased by 2.

6. Find the equation whose roots are those of the equation $2x^4 - 5x^3 + 11x^2 - 208x + 140 = 0$ diminished by 3. Hence or otherwise solve the given equation.

7. Find the equation whose roots are the roots of the equation $x^4 + 8x^3 + 12x^2 - 16x - 28 = 0$ each increased by 2. Hence solve the equation.

2.6 REMOVAL OF TERMS

One of the chief uses of this transformation is to remove a certain specified term from an equation. Such a step always helps to find the solutions of an equation.

Let the given equation be

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$$

Then if y = x – h.

We obtain the new equation

$$a_0(y+n)^n + a_1(y+n)^{n-1} + a_2(y+h)^{n-2} + ... + a_n = 0$$

which when arranged in descending powers of y, becomes

$$a_0y^n + (na_0h + a_1)\,y^{n-1} + \left\{\frac{n(n-1)}{2!}a_0h^2 + (n-1)\,a_1h + a_2\right\}y^{n-2} + ... = 0$$

If the term to be removed is the second,
we get

$$n\ a_0h + a_1 = 0$$

so that $h = \dfrac{-a_1}{na_0}$

If the term to be removed in the third, we get

$$\frac{n(n-1)}{2!}a_0h^2 + (n-1)\,a_1\,h + a_2 = 0$$

and so obtain a quadratic to find h and similarly we may remove any othei assigned term.

Example: 1

Find the relation between the coefficients in the equation $x^4+Px^3+qx^2+rx+S=0$ in order that the coefficients of x^3 and x may be removable by the same transformation.

Solution:

Let us reduce the roots of the equation by h.

Instead of x. Substitute x + h.

The transformed equation is

$$(x + h)^4 + P(x + h)^3 + q(x + h)^2 + r(x + h) + S = 0$$

$$x^4 + 4x^3 h + 6x^2 h^2 + 4 x h^3 + h^4 + P(x^3 + 3x^2 h + 3 x h^2 + h^3) + q(x^2 + 2 x h + h^2)$$

$$+ r(x + h) + S = 0$$

$$x^4 + 4x^3 h + 6x^2 h^2 + 4 x h^3 + h^4 + Px^3 + 3Px^2 h + 3Px h^2 + Ph^3 + q x^2 + 2q x h + q h^2) \; r x + hr + S = 0$$

$$\text{(i.e.) } \left.\begin{array}{c} x^4 + (4h+P)x^3+(6h^2+3Ph+q) x^2 + (4h^3+3Ph^2+2qh+r)x \\ h^4+Ph^3+qh^2+rh+S \end{array}\right\} = 0$$

The coefficients of x^3 and x in the transformed equation are zeros.

$$\therefore \; 4h + P = 0, \; 4h^3 + 3Ph^2 + 2qh + r = 0.$$

Eliminate h between these equations.

We get $h = -\frac{P}{4}$

$$\therefore \; 4\left(-\frac{P}{4}\right)^3 + 3P\left(-\frac{P}{4}\right)^2 + 2q\left(-\frac{P}{4}\right) + r = 0$$

$$\text{(i.e.) } \frac{-4P^3}{64} + \frac{3P^3}{16} - \frac{2Pq}{4} + r = 0$$

$$-P^3 + 3P^3 - 8Pq + 16r = 0$$

$$2P^3 - 8Pq + 16r = 0$$

$$\therefore \; P^3 - 4Pq + 8r = 0$$

Example: 2

Solve the equation $x^4 + 20x^3 + 143x^2 + 430x + 462 = 0$ by removing its second term.

Solution:

Let us assume that by diminishing the roots by h, the second term is removed.

Then the transformed equation becomes

$$(x + h)^4 + 20(x+h)^3 + 143(x+h)^2 + 430(x+h) + 462 = 0$$

$$\left.\begin{array}{l} x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 20\,(x^3 + 3x^2\,h + 3xh^2 + h^3 \\ \qquad + 143\,(x^2 + 2\,x\,h + h^2) + 430\,(x + h) + 462 \end{array}\right\} = 0$$

$$\left.\begin{array}{l} x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 20\,x^3 + 60\,x^2\,h + 60\,x\,h^2 + 20h^3 \\ \qquad + 143\,x^2 + 286\,x\,h + 143\,h^2 + 430\,x + 430h + 462 \end{array}\right\} = 0$$

(i.e.) $x^4 + x^3\,(4h + 20) + x^2\,(6h^2 + 60\,h + 143) + x\,(4h^3 + 60h^2 + 286\,h + 430)$
$+ h^4 + 20\,h^3 + 143\,h^2 + 430\,h + 462 = 0$

(i.e.) The coefficients of x^3 in the transformed equation are zero.

$4h + 20 = 0$

$h = -5$

Hence to remove the second term, increase the roots of the equation by 5.

–5	1	20	143	430	462
	0	–5	–75	–340	–450
–5	1	15	68	90	12
	0	–5	–50	–90	
–5	1	10	18	0	
	0	–5	–25		
–5	1	5	–7		
	0	–5			
	1	0			

∴ The transformed equation is $y^4 - 7y^2 + 12 = 0$

(i.e.) $(y^2 - 3)\,(y^2 - 4) = 0$

∴ The roots of the transformed equation are $\pm\sqrt{3}$, ± 2.

These roots are greater than the roots of the original equation by 5.

∴ The roots of the original equation are

$\sqrt{3} - 5,\ -\sqrt{3} - 5,\ 2 - 5,\ -2 - 5$

(i.e.) $-5 \pm \sqrt{3}$, -3, -7.

Example: 3

Solve the equations by removing the second term in each.

$X^4 - 12x^3 + 48x^2 - 72x + 35 = 0$

Solution:

Let $f(x) = x^4 - 12x^3 + 48x^2 - 72x + 35$ (1)

Put $x = y + h$

(1) becomes

$(y + h)^4 - 12(y + h)^3 + 48(y + h)^2 - 72(y + h) + 35 = 0$

$y^4 + h^4 + 4y^3h + 6y^2h^2 - 12(y^3 + h^3 + 3y^2h + 3h^2y)$

$+ 48(y^2 + h^2 + 2hy) - 72y - 72h + 35 = 0$

$\Rightarrow y^4 + (4h - 12)y^3 + (6h^2 - 36h - 48)y^2 + (4h^3 - 36h - 12 + 96h)y$
$+ h^4 - 72h + 35 = 0$ (2)

Equating the coefficients y^3 to zero in (2) becomes $\Rightarrow 4h - 12 = 0$

$4h = 12 \Rightarrow h = 12/4 = 3$

we diminish the roots of (1) by (3)

3	1	–12	48	–72	35
	0	3	–27	63	–27
3	1	–9	27	–9	8
	0	3	–18	9	
3	1	–6	3	0	
	0	3	–9		
3	1	–3	–6		
	0	3			
	1	0			

The transformed equation by $y^4 - 6y^2 + 8 = 0$

$\Rightarrow y^4 - 6y^2 + 8 = 0$

$\Rightarrow y^4 - 4y^2 - 2y^2 + 8 = 0$

$\Rightarrow y^2 (y^2 - 2) - 2 (y^2 - 4) = 0$

$\Rightarrow (y^2 - 2) (y^2 - 4) = 0$

$\Rightarrow y^2 = 2, y^2 = 4$

$\Rightarrow y = \pm\sqrt{2}, y = \pm 2$

The roots of x = y + h are h = 3

$y = \sqrt{2}, y = -\sqrt{2}, y = 2, y = -2$

$x = \sqrt{2} + 3, x = -\sqrt{2} + 3, x = 2 + 3, x = -2 + 3$

$x = 5 \qquad x = 1$

$\therefore$ The roots are

$\sqrt{2} + 3, -\sqrt{2} + 3, 5, 1.$

Example: 4

Remove the second term from the equation $x^3 - 6x^2 + 10x - 3 = 0$

Solution:

Let $f(x) = x^3 - 6x^2 + 10x - 3 \qquad (1)$

Put x = y + h

(1) becomes

$(y + h)^3 - 6 (y + h)^2 + 10 (y + h) - 3 = 0$

$\Rightarrow y^3 + h^3 + 3y^2h + 3h^2y - 6 (y^2 + h^2 + 2hy) + 10y + 10h - 3 = 0$

$\Rightarrow y^3 + (3h - 6) y^2 + (3h^2 + 10 - 12h) y + h^3 - 6h^2 - 3 = 0 \qquad (2)$

Equating the coefficients y^2 to zero is (2) becomes $\Rightarrow 3h - 6 = 0$ $h = 6/3 = 2$

We now diminish the roots of (1) by 2.

2	1	–6	10	–3
	0	2	–8	4
2	1	–4	2	1
	0	2	–4	
2	1	–2	–2	
	0	2		
	1	0		

The transformed equation is $y^3 - 2y + 1 = 0$

$$\Rightarrow (y - 1)(y^2 + y - 1) = 0$$

$$\Rightarrow y = 1, \frac{-1 \pm \sqrt{5}}{2}$$

The roots of x = y + h are h = 2

$$y = 1, \quad y = \frac{-1+\sqrt{5}}{2}, \quad y = \frac{-1-\sqrt{5}}{2}$$

$$x = 1 + 2 \qquad x = \frac{-1+\sqrt{5}}{2} + 2 \qquad x = \frac{-1-\sqrt{5}}{2} + 2$$

$$x = 3 \qquad x = \frac{3+\sqrt{5}}{2}, \qquad x = \frac{3-\sqrt{5}}{2}$$

$$\therefore \text{The roots are } 3, \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$$

Exercise

1. Remove the second term from the equation.

   $x^5 + 5x^4 + 3x^3 + x^2 + x + 1 = 0$

2. Transform the equation $x^4 - 4x^3 - 18x^2 - 3x + 2 = 0$ into one which shall want the third term.

3. Solve the following equations by removing the second term in each:

   i) $x^4 + 4x^3 + 5x^2 + 2x - 6 = 0$

   ii) $x^4 + 16x^3 + 83x^2 + 152x + 84 = 0$

   iii) $x^3 - 12x^2 + 48x - 72 = 0$

   iv) $x^4 + 10x^3 + 83x^2 + 152x + 84 = 0$

   v) $x^3 + 6x^2 + 12x - 19 = 0$

   vi) $x^3 - 21x^2 + 144x - 320 = 0$

   vii) $x^4 - 8x^3 - x^2 + 68x + 60 = 0$

UNIT – III

DESCARTE'S RULE OF SIGNS

3.1 An equation f(x)=0 Cannot have more positive roots than these are changes of sign in f(x).

Let f(x) be a polynomial whose signs of the terms are

+ + – – – + – + + + – + –.

In this there are seven changes of sign including changes from + to – and from – to +. We shall show that if this polynomial be multiplied by a binomial (corresponding to a positive root) whose signs of the terms are + –, the resulting polynomial will have atleast one more change of sign than the original. Writing down only the signs of the terms in the multiplication, We have

+ + – – – + – + + + – + –

+ –

– – + + + – + – – – + – +

+ + – – – + – + + + – + –

+ ± – ± ± + – + ± ± – + – +

Here in the last the line the ambiguous sign ± in placed wherever there are two different signs to be added. Here we see in the product

1) An ambiguity replaces each continuation of sign in the original polynomial.
2) The sign before and after an ambiguity or a set of ambiguities are unlike and
3) A change of sign is introduced in the end.

Let us take the most unfavourable. Case and suppose that all the ambiguities are replaced by continuations, then the sign of the terms becomes

+ + – – – + – + + + – + – +

The number of changes of sign is 8. Thus even in the most unfavourable case there is one more change of sign than the number of changes of sign in the original polynomial. Therefore we may conclude in general that the effect of multiplication of a binomial factor $x - \alpha$ in to introduce at least one change of sign.

Suppose the product of all the factors corresponding to negative and imaginary roots of $f(x) = 0$ be a polynomial $F(x)$. The effect of multipuling $F(x)$ by each of the factors $x - \alpha$, $x - \beta$, $x - \gamma$ .... corresponding to the positive roots, α, β, γ in to introduce atleast one change of sign for each, so that when the complete product is formed containing all the roots. We have the resulting polynomial which has atleast as many changes of signs as it has positive roots. This is Descartes' rule of signs.

Descartes' rule of signs for negative roots:

Let $f(x) = (x - \alpha_1)(x - \alpha_2) \ldots\ldots(x - \alpha_n)$

By substituting $-x$ instead to x in the equation,
We get,

$f(-x) = (-x - \alpha_1)(-x-\alpha_2) \ldots\ldots (-x-\alpha_n)$

The roots of $f(-x) = 0$ are $-\alpha_1, -\alpha_2 \ldots\ldots -\alpha_n$.

$\therefore$ The negative roots of $f(x)=0$ become the positive roots of $f(-x)=0$.

Hence to find the maximum number of negative roots of $f(x) = 0$, it is enough to find the maximum number of positive roots of $f(-x)=0$.

So we can enuciate Descartes' rule for negative roots as follows:

No equation can have a greater number of negative roots then there are changes of sign in the term of the polynomial **f(– x)**.

Using Descartes' rule of signs we can ascertain whether an equation $f(x)=0$ has imaginary roots or not.

We can find the maximum number for possible roots and also for negative roots.

The degree of the equation will give the total numbers of roots of the equation. So if the sum of the maximum numbers of positive roots and negative roots is less than the degree of the equation, we are sure of the existence of imaginary roots. Take for example the equation

$X^7 + 8X^5 - X + 9 = 0$

The series of signs of the terms are as follows:

+ + – +

The number of changes of signs is 2 and the equation cannot have more than two positive roots.

Now change x into –x.

We get $-X^7 + 8X^5 - X + 9 = 0$

(ie) $X^7 + 8X^5 - X + 9 = 0$

The series of signs of the terms are

+ + – –

and the number of changes of sign is only one and so the equation cannot have more than one negative root.

Hence in the equation there cannot exit more than three real roots. Since it is a seventh degree equation, it has seven roots real or imaginary.

Therefore the given equation has atleast four imaginary roots.

An equation f(x)=0 is called complete. When all powers of x from n^{th} to the constant term are present. In an complete equation we can easily see that the sum of the number of changes of sign in f(x) and f(–x) is exactly equal to the degree of the equation.

Hence this rule can be used to detect the imaginary roots only in incomplete equations.

Example: 1

Determine completely the nature of the roots of the equation $x^5 - 6x^2 - 4x + 5 = 0$.

Solution:

The series of signs of the terms are + – – +.

Here there are two changes of sign.

Hence there cannot be more than two positive roots.

Changing x into –x,

The equation becomes,

$-x^5 - 6x^2 - 4x + 5 = 0$.

(ie) $x^5 + 6x^2 - 4x - 5 = 0$.

The series of the signs of the terms are

+ + – –.

Here there is only one change of sign.

∴ There cannot be more than one negative root.

So the equation has got at the most three real roots. The total number of roots of the equation is 5. Hence there are atleast two imaginary roots for the equation.

We can also determine the limits between which the real roots lie

$X =$	$-\alpha$	-2	-1	0	1	2	α
$X^5 - 6x^2 - 4x + 5 =$	–	–	+	+	–	+	+

The positive roots lie between 0 and 1 and 2 the negative root between –2 and –1.

Example :2

Show that the equation $x^7 - 3x^4 + 3x^3 - 1$ has atleast 4 imaginary roots.

Sol :-

Let $f(x) = x^7 - 3x^4 + 3x^3 - 1$

The sign of polynomials are

+ – + –

The number of changes of sign in f(x) in three Almost three (+ve) root of f(x)

Here

$$F(-x) = (-x)^7 - 3(-x)^4 + 3(-x)^3 - 1$$
$$= -x^7 - 3x^4 - 3(-x)^3 - 1$$

The sign of polynomials are

– – – –

f(–x) has no (+ve) real root.

f(x) has no (–ve) real root.

∴ f(x) has three real roots.

Degree of f(x) = 7

Imaginary roots = 7 – 3 =4

∴ f(x) = 0 lias at least 4 imaginary roots.

Example: 3

ST $x^6 + 3x^2 - 5x + 1 = 0$ has atleast 4 imaginary root.

Sol : -

Let $f(x) = x^6 + 3x^2 - 5x + 1$

The sign of polynomials are

+ + – +

The number of changes of sign in f(x) is two

At most two +ve root of f(x)

Here

$f(x) = (-x)^6 + 3(-x)^2 - 5(-x) + 1$

$= x^6 + 3x^2 + 5x + 1$

The sign of polynomials are

+ + + +

f(–x) has no (+ve) real root

f(x) has no (–ve) real root

∴ f(x) has two real root

Degree of f(x) = 6

Real root = 2

Imaginary root =4

∴ f(x) = 0 has 4 imaginary roots.

Example : 4

Find the number of real roots of the equation $x^3 + 18x - 6 = 0$.

Solution:

Let $f(x) = x^3 + 18x - 6$

The sign of polynomials are

+ + –

The number of changes in f(x) is only one.

Atmost one (+ve) root of f(x)

Here $f(-x) = (-x)^3 + 18(-x) - 6$

$= -x^3 - 18x - 6$

The sign of polynomials are

– – –

f(–x) has no. (+ve) real root
f(x) has no.(–ve) real root

∴ f(x) has only one real root

Deg. of f(x) =3

Real root = 1

Imaginary root = 3 – 1 = 2

∴ f(x) = 0 has atleast 2 imaginary root

Example : 5

Show that $12x^7 - x^4 + 10x^3 - 28 = 0$ has atleast four imaginary root

Solution:

Let $f(x) = 12x^7 - x^4 + 10x^3 - 28$

The sign of polynomials are

+ – + –

The number of changes is f(x) are three Atmost three (+ve) root of f(x)

Here $f(-x) = 12(-x)^7 - (-x)^4 + 10(-x^3) - 28$

$$f(-x) = -12x^7 - x^4 - 10x^3 - 28$$

The sign of polynomials are

– – – –

f(–x) has no. (+ve) real root

f(x) has no. (–ve) real root

∴ f(x) has three real root

Deg. of f(x) = 7

Real root = 3

Imaginary root = 7 – 3 = 4

∴ f(x)=0 has 4 imaginary root

Exercise

1. Show that $x^5 + 3x^2 + 5x - 5 = 0$ has at least four imaginary roots.
2. Prove that the equation $x^4 + 3x - 1 = 0$ has two real and two imaginary roots.
3. Find the number of imaginary roots of the equation $x^5 + 5 - 7 = 0$.
4. Find the number of real roots of $x^7 - x^6 - x^4 - 6x^2 + 7 = 0$.
5. Show that $x^9 + x^8 + x^4 + x^2 + 1 = 0$ has one real root which in negative and eight imaginary roots.
6. Show that the equation $x^n - 1 = 0$ has, when n is even, two real roots 1 and -1 and no other real root and when n is odd, the real root is 1 and no other real root.

3.2 ROLLES' THEOREM:

Between two consective real roots a and b of the equation $f(x) = 0$ where f(x) is a polynomial, there lies at least one real root of the equation $f^1(x)=0$.

Let f(x) be $(x - a)^m (x - b)^n \phi(x)$ where m and n are positive integers and $\phi(x)$ is not divisible by $(x - a)$ or by $(x - b)$. Since a and b are consecutive real roots of f(x), the sign of $\phi(x)$ in the interval $a \le x \le b$ in either positive throughout or negative throughout, for if it changes its sign between a and b, then there is a root of $\phi(x)=0$ that in of $f(x)=0$ lying between a and b, which is contrary to the hypothesis that a and b are consecutive roots.

$$\therefore f^l(x) = (x - a)^m n(x - b)^{n-1} \phi(x) + m(x-a)^{m-1} (x-b)^n \phi(x) + (x-a)^m (x-a)^n \phi(x)$$

$$= (x - a)^{m-1} (x - a)^{n-1} \chi(x)$$

where $\chi(x) = \{m(x-b) + n(x-a)\} \phi(x) + (x-a)(x-b) \phi^l(x)$

$$\therefore \chi(a) = m(a-b) \phi(a)$$

$$\chi(b) = m(b-a) \phi(b).$$

$\chi(a)$ and $\chi(b)$ have different signs since $\phi(a)$ and $\phi(b)$ have the same sign.

$\therefore \chi(x) = 0$ has at last one root between a and b.

Hence $f^l(x) = 0$ has atleast one root between a and b.

Cor.1

If all the roots of f(x) = 0 are real, then all the roots of $f'(x) = 0$ are also real.

If f(x) = 0 is a polynomial of degree n, $f'(x) = 0$ lies in each of the (n–1) intervals between the n roots of f(x) =0

Cor.2

If all the roots of f(x) =0 are real, then the roots of $f'(x) = 0$, $f''(x) = 0$, $f'''(x) = 0$ are real.

Cor.3

At the most only one real root of f(x)=0 can lie between two consecutive roots of $f'(x) = 0$, that is the real roots of $f'(x)$ separate those of f(x)=0.

Cor.4

If f(x) =0 has r real roots, then f(x) =0 cannot have more than (r+1) real roots.

Cor.5

f(x) =0 has atleast as many imaginary roots as $f'(x) = 0$

Example 1:

Find the nature of the roots of the equation $4x^3 - 21x^2 + 18x + 20 = 0$.

Solution:

Let us consider the function

$$f(x) = 4x^3 - 21x^2 + 18x + 20$$

We have $f'(x) = 12x^2 - 42x + 18$

$$= 6(2x - 1)(x - 3)$$

Hence the real roots of $f'(x) = 0$ are $\frac{1}{2}$ and 3. So the roots of f(x) =0, if any will be in the intervals between $-\alpha$ and $\frac{1}{2}$, $\frac{1}{2}$ and 3, 3 and +, α respectively,

X :	$-\alpha$	$\frac{1}{2}$	3	α
f(x)	–	+	–	+

$\therefore$ f(x) must vanish. Once in each of the above intervals.

Hence f(x) =0 has three real roots.

Example: 2

Show that the equation $3x^4 - 8x^3 - 6x^2 + 24x - 7 = 0$ has one positive, one negative and two imaginary roots.

Solution:

Let f(x) be $3x^4 - 8x^3 - 6x^2 + 24x - 7$

We have,

$$f'(x) = 12x^3 - 24x^2 - 12x + 24$$

$$= 12(x+1)(x-1)(x-2)$$

The roots of $f'(x) = 0$ are –1, +1, +2

X :	$-\alpha$	-1	$+1$	$+2$	$+\alpha$
f(x) :	+	–	+	+	+

$\therefore$ f(x) =0 has a real root lying between –1 and $-\alpha$, one between –1 and +1 and two imaginary roots.

We know that f(+1) =+, f(o) = –

$\therefore$ The real root lying between –1 and +1 lies between 0 and +1.

Hence it is a positive root.

The other real root lies between –1 and $-\alpha$ and so it is a negative root.

Example:3

Discuss the reality of the roots $x^4 + 4x^3 - 2x^2 - 12x + a = 0$ for all real values of a.

Solution:

Let f(x) be $x^4 + 4x^3 - 2x^2 - 12x + a$

$$f'(x) = 4x^3 + 12x^2 - 4x - 12$$

$$= 4(x+1)(x-1)(x+3)$$

$\therefore$ The roots of $f'(x) = 0$ are –3, –1, and I

x :	$-\alpha$	-3	-1	1	$+\alpha$
f(x) :	+	$a-9$	$7+a$	$a-9$	+

If $a - 9$ is negative and $7 + a$ is positive, the four roots of f(x) are real.

If $-7 < a < 9$, $f(x) = 0$ has four real roots.

If $a>9$, then f(x) is positive throughout and hence all the roots of $f(x) =0$ are imaginary.

If $a< -7$, the signs of f(x) at $-\alpha$, $-3, -1, 1, \alpha$ are respectively, +, −, −, −, +.

Hence $f(x) =0$ has two real roots and two imaginary roots.

Example: 4

Prove that all the roots of the equation $x^3 - 18x + 25 = 0$ are real.

Solution:

Let us consider the function $f(x) = x^3 - 18x +2x$

$$f'(x) = 3x^2 - 18$$
$$= 3(x^2 - 6)$$

Hence the real roots of $f'(x) = 0$ are $\pm\sqrt{6}$. So the roots of $f(x) = 0$

x :	$-\alpha$	$-\sqrt{6}$	$\sqrt{6}$	α
f(x) :	−	+	−	+

$\therefore$ f(x) must vanish, once in each of the above intervals.

Hence $f(x) =0$ has three real roots.

Example : 5

Discuss the nature of the roots of the equation $3x^4 +8x^3 - 30x^2 - 72x + k = 0$ for different values of k.

Solution:

Let $f(x) = 3x^4 +8x^3 - 30x^2 - 72x +k$

$$f'(x) = 12x^3 + 24x^2 - 60x -72$$
$$= 12 (x^3 +2x^2 - 5x - 6)$$
$$= 12(x+2) (x-1) (x+3)$$

	1	2	−5	−6
	0	2	8	6
−1	1	4	3	0
	0	−1	−3	
	1	3	0	

$\therefore$ The roots of $f'(x) = 0$ are $-2, 1, -3$.

x :	$-\infty$	-2	-3	1	∞
f(x) :	+	+8+k	−27 +k	91+k	+

If 8+k in positive, and $-27+k$ is positive. $91+k$ is positive.

The two roots of f(x) are real.

If $-8<a<27$, $-91<a<27$,

$\therefore f(x)=0$ has two real roots.

If $a>27$, then f(x) is '+ve' throughout & hence all the roots $f(x) = 0$ are imaginary.

$a< -8$, the sign of f(x) at $-\infty, -2, -3, 1, \infty$

+ + − + +

Hence $f(x) = 0$ has two roots real and two roots imaginary.

Exercise:

1. Find the nature of the roots of the equation
   (i) $4x^3 - 21x^2 + 18x + 30 = 0$
   (ii) $2x^3 - 9x^2 + 12x + 3 = 0$
   (iii) $x + 4x^3 - 20x^2 + 10 = 0$
2. Determine the value of a such that the equation $x^3 - 12x + a = 0$ has only one real root.
3. Find the range of the values of k for which the following equations have real roots
   (i) $x^3 + 4x^2 + 5x + 2 + k = 0$
   (ii) $2x^3 - 9x^2 + 12x - k = 0$
   (iii) $3x^4 - 4x^3 - 12x^2 + k = 0$

3.3 MULTIPLE ROOTS

If f(x) is a polynomial in x and the equation f(x)=0 has n roots equal to α, then f(x) must be of the form $(x-\alpha)^m\ \phi(x)$ where $\phi(\alpha) \neq 0$.

$\therefore$ $f(x) = (x-\alpha)^m\ \phi(x)$ where m and n are positive integers and $\phi(x)$ is not divisible by (x – a)

$$f'(x) = (x-\alpha)^m\ \phi'(x) + m(x-\alpha)^{m-1}\ \phi(x)$$

$$= (x-\alpha)^{m-1}\{(x-\alpha)\ \phi'(x) + m\phi(x)\}$$

Hence $(x-\alpha)^{m-1}$ is a common factor of f(x) and $f'(x)$ and it is easily seen that $(x-\alpha)^{m-1}$ will not be a common factor unless f(x) is divisible by $(x-\alpha)^m$.

Hence the multiple roots of f(x),

If any are to be deducted by finding the greatest common factors of f(x) and $f'(x)$ by the usual algebraic process.

We may then state a rule for finding the multiple roots of an equation f(x) = 0 as follows:

i) Find $f'(x)$
ii) Find the H.C.F of f(x) and $f'(x)$
iii) Find the roots of the H.C.F.

Each different root of the H.C.F will occur once more in f(x) than it does in the H.C.F.

Example 1:
Find the multiple roots of the equation $x^4 - 9x^2 + 4x + 12 = 0$

Solution:

Let $f(x) = x^4 - 9x^2 + 4x + 12$

$$f'(x) = 4x^3 - 18x + 4$$

$$= 2(2x^3 - 9x + 2)$$

2	1	0	−9	4	12
	0	2	4	−10	−12
2	1	2	−5	−6	0
	0	2	8	6	
	1	4	3	0	

The H.C.F of f(x) and $f'(x)$

2	4	0	−18	4
	0	8	16	−4
	4	8	−2	0

$$f(x) = (x-2)^2\,(x+1)(x+3)$$

$$f'(x) = (x-2)\left(\frac{-2 \pm \sqrt{6}}{2}\right)$$

H.C.F of f(x) and $f'(x)$ is $x-2$.

$\therefore$ $(x-2)^2$ is a factor of f(x)

$\therefore$ $f(x) = (x-2)^2\,(x+1)\,(x+3)$

$\therefore$ The roots of $f(x) = 0$ are 2, 2, − 1 and − 3.

Example: 2

Find the values of a for which $ax^3 - 9x^2 + 12x - 5 = 0$ has equal root and solve the equation in one case.

Solution:

Let $f(x) = ax^3 - 9x^2 + 12x - 5$

$$f'(x) = 3ax^2 - 18x + 12$$

Find the H.C.F

$$f(x) = ax^3 - 9x^2 + 12x - 5$$

$$f'(x) = 3ax^2 - 18x + 12$$

The process of finding the H.C.F is exhibited below.

1/3 x	$ax^3 - 9x^2 + 12x - 5$	$3ax^2 - 18x + 12$	−a
	$ax^3 - 6x^2 + 4x$	$3ax^2 - 8ax + 5a$	
	(−) (+) (−)	(−) (+) (−)	
	$-3x^2 + 8x - 5$	$(8a-18)x + 12 - 5a$	
	$-3(8a-18)x^2 + 8(8a-18)x - 5(8a+8)$		
	$3(8a-18)x^2 + 3(12-5a)x$		
	$(49a - 108)x - 5(8a - 18)$		

If f(x) and $f'(x)$ have a common linear factor

$$\frac{49a-108}{8a-18} \quad \frac{-5(8a-18)}{12-5a}$$

(ie) $588a + 540a - 245a^2 - 1296 = -320a^2 + 720a + 720a - 1620$

(ie) $75a^2 - 312a + 324 = 0$

(ie) $25a^2 - 104a + 108 = 0$

(ie) $(25a - 54)(a - 2) = 0$

$\therefore a = 2$ or $\frac{54}{25}$

If a = 2, the linear factor on both sides bocomes $-10x + 10$ and $-2x + 2$.

$f(x) = 2x^3 - 9x^2 + 12x - 5$

$= (x-1)^2 (2x-5)$

$f'(x) = (2x - 2)(x - 2) = 2(x-1)(x-2)$

$\therefore$ H.C.F is $(x - 1)$

$\therefore$ The roots of the equation : $1, 1, 5/2$.

Example: 3

Find the condition that the cubic equation $ax^3 + 3bx^2 + 3cx + d = 0$ has two equal roots and when the condition is satisfied, find the equal roots.

Solution:

Let α be the equal root.

Then α is also a root of $f'(x) = 0$ where

$f(x) = ax^3 + 3bx^2 + 3cx + d$

$f(\alpha) = a\alpha^3 + 3b\alpha^2 + 3c\alpha + d = 0$ (1)

and $f'(x) = 3a\alpha^2 + 6b\alpha + 3c = 0$

(ie) $\alpha^2 + 2b\alpha + c = 0$ (2)

Substracting the product of (2) and α from (1)
We get

$$(1) \Rightarrow a\alpha^3 + 3b\alpha^2 + 3c\alpha + d = 0$$

$$(2)\times\alpha \Rightarrow a\alpha^3 + 2b\alpha^2 + \alpha c = 0$$
$$(-) \quad (-) \quad\quad (-)$$

$$b\alpha^2 + 2C\alpha + D = 0 \quad (3)$$

From (2) and (3)
We have

$$\frac{\alpha^2}{2bd - 2c^2} = \frac{\alpha}{bc - ad} = \frac{1}{2ac - 2b^2}$$

$$\therefore \quad \frac{\alpha^2}{2(bd - c^2)} = \frac{\alpha}{bc - ad} = \frac{1}{2(ac - b^2)}$$

$\therefore (bc - ad)^2 = 4(bd - c^2)(ac - b^2)$

and $\alpha = \frac{1}{2}\left(\frac{bc - ad}{ac - b^2}\right)$

Example :4

Find the condition that the equations $ax^3 + 3bx + c = 0$, $a'x^3 + 3b'x + c' = 0$. Should have a common root. When this condition is satisfied, show that the common root is a double root of the equation

$$2((ab' - a'b)x^3 + (ac' - a'c)x^2 + (bc' - b'c) = 0$$

Solution:

Let α be the common root of the equation

Then $a\alpha^3 + 3b\alpha + c = 0$ (1)

$$a'\alpha^3 + 3b'\alpha + c' = 0 \quad (2)$$

Multiplying (1) by a' $\Rightarrow$ $aa'\alpha^3 + 3ba'\alpha + ca' = 0$

Multiplying (2) by a $\Rightarrow$ $aa'\alpha^3 + 3ab'\alpha + c'a = 0$

(–) (–) (–)

Subtracting, $\Rightarrow$ $3(a'b - ab')\alpha + ca' - ac' = 0$

$$\alpha = \frac{ac' - a'c}{3(ab' - a'b)} \quad (3)$$

Substituting in (1)
We get,

$$-\frac{a(ac' - a'c)^3}{27\,(ab' - a'b)^3} - \frac{3b(ac' - a'c)}{3(ab' - a'b)} + c = 0$$

(ie) $a(ac' - a'c)^3 + 27b(ac' - a'c)(ab' - a'b)^2 - 27c(ab' - a'b)^2 = 0$

If α is a double root of the equation

$$2(ab' - a'b)x^3 + (ac^1 - a'c)x^2 + (bc' - b'c) = 0 \quad (4)$$

α is also a root of

$$6(ab' - a'b)x^2 + 2x(ac' - a'c) = 0$$

(i.e) $6(ab' - a'b)\alpha^2 + 2\alpha(ac' - a'c) = 0$

(ie) $2\alpha\{(3)(ab' - a'b)\alpha + (ac' - a'c)\} = 0$

α is cannot be equal to zero

$\therefore \alpha = -\dfrac{(ac' - a'c)}{3(ab' - a'b)}$ which is the same as (3)

$\therefore \alpha$ is a double root of the equation (4)

Example : 5

Solve the equation $8x^3 - 20x^2 + 6x + 9 = 0$ given that it has a multiple root.

Solution:

Let $f(x) = 8x^3 - 20x^2 + 6x + 9$

$$f'(x) = 24x^2 - 40x + 6$$

Now $f'(x) = 0$

$$24x^2 - 40x + 6 = 0 \quad [\div 2]$$

$$12x^2 - 20x + 3 = 0$$

$$(6x - 1)(2x - 3) = 0$$

$$x = \tfrac{1}{6}, \tfrac{3}{2}$$

i) $f\left(\tfrac{1}{6}\right) = 8\left(\tfrac{1}{6}\right)^3 - 20\left(\tfrac{1}{6}\right)^2 + 6\left(\tfrac{1}{6}\right) + 9$

$$= 8 \times \tfrac{1}{216} - 20 \times \frac{1}{36} + 1 + 9$$

$$= \tfrac{1}{27} - \tfrac{5}{9} + 1 + 9$$

$$f\left(\tfrac{1}{6}\right) = \frac{256}{27} \neq 0$$

$$f\left(\tfrac{3}{2}\right) = 8\left(\tfrac{3}{2}\right)^3 - 20\left(\tfrac{3}{2}\right)^2 + 6\left(\tfrac{3}{2}\right) + 9$$

$$= 8.\frac{27}{8} - 20.\frac{9}{4} + 6.\frac{3}{2} + 9$$

$$= 27 - 45 + 9 + 9$$

$$= 45 - 45 = 0$$

$$\therefore f\left(\tfrac{3}{2}\right) = 0$$

$\therefore \frac{3}{2}$ is a root of f(x) = 0

but $\frac{3}{2}$ is also a root of $f'(x) = 0$

$\therefore \left(x - \frac{3}{2}\right)^2$ in a H.C.F of f(x) & f′(x)

$\frac{3}{2}$ is a multiple root of f(x) = 0

$\left(x - \frac{3}{2}\right)^2$ is a factor of f(x)

$(2x - 3)^2$ is a factor of f(x)

$(4x^2 + 9 - 12x)$ in a factor of f(x)

$$\begin{array}{r|l} & 2x + 1 \\ \hline 4x^2 - 12x + 9 & 8x^3 - 20x^2 + 6x + 9 \\ & 8x^3 - 24x^2 + 18x \\ & (-) \quad (+) \quad (-) \\ \hline & 4x^2 - 12x + 9 \\ & 4x^2 - 12x + 9 \\ \hline & 0 \end{array}$$

$\therefore$ (2x + 1) is a factor of f(x)

$2x = -1 \Rightarrow x = -\frac{1}{2}$ is a root of f(x)

$\therefore$ The multiple roots are $\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}$

Exercise:

1. Find the multiple roots of the equations
   i) $4x^4 + 24x^3 + 49x^2 + 45x + 25 = 0$
   ii) $4x^3 - 12x^2 - 15x - 4 = 0$
   iii) $x^4 - 6x^3 + 13x^2 - 24x + 36 = 0$

2. Find the value of k for which $x^3 + 4x^2 + 5x + 2 + k = 0$ has equal roots. Also find those roots

3.4 STRUM'S THEOREM

Let $f(x) = 0$ be an equation having no equal roots. Let $f_1(x)$ be the first derived function of $f(x)$

Let the process of finding the greatest common measure of $f(x)$ and $f_1(x)$ be performed

Let q_1 be the quotient and $f_2(x)$ the remainder with the sign changed.

Then $f(x) = q_1 f_1(x) - f_2(x)$

Similar operation can be performed between $f_1(x)$ and $f_2(x)$ and

We get $f_1(x) = q_2 f_2(x) - f_3(x)$

If we continue the operations

We get

$$f_2(x) = q_3 f_3(x) - f_4(x)$$
$$\vdots$$
$$f_{r-1}(x)\, q_r\, f_r(x) - f_{r+1}(x) \quad \text{and so on.}$$

The successive remainders with their signs changed. (ie) the functions $f_2(x)$, $f_3(x)$ ...... will go on diminishing in degree till we reach a numerical remainder, say $f_n(x)$.

These functions $f(x)$, $f_1(x)$, $f_2(x)$ ......$f_m(x)$ are called **Strum's function.**

The difference between the numbers of changes of sign in the series of Strum's functions when a is substituted for x and the number when b is substituted for x express exactly the number when b is substituted for x express exactly the number of real roots of the equation $f(x) = 0$ between a and b. This is known as Strum's Theorem and the proof of this theorem is beyond the scope of this book.

Substitute 0 and α in the series of Strum's functions and the difference between the number of changes of sign will give the **positive roots**

Substitute $-\alpha$ and 0 in the series of strum's functions and the difference between the number of changes of signs will give the **negative roots.**

The difference between the number of changes of sign when $-\alpha$ and $+\alpha$ are substituted in the series of Sturm's function will give the number of real roots of the equation.

Example:1

Find the number of real roots of the equation $x^4 - 14x^2 + 16x + 9 = 0$

Solution :

Let $f(x) = x^4 - 14x^2 + 16x + 9 = 0$

$f_1(x) = 4x^3 - 28x + 16 = 0$

$(\div) = x^3 - 7x + 4 = 0$

(x)	$x^3 - 7x + 4$ (−) (+) (+) $x^3 - \frac{12}{7}x^2 - \frac{9}{7}x$	$x^4 - 14x^2 + 16x + 9$ $x^4 - 7x^2 + 4x$ (−) (+) (+)	(x)
	$\frac{12}{7}x^2 - 7x + \frac{9}{7}x + 4$ $\frac{12}{7}x^2 - \frac{40x}{7} + 4 = 0 \div 7$ $12x^2 - 40x + 28 \div 3$	$-7x^2 + 12x + 9$ $7x^2 - 12x - 9 = f_2$	
	$3x^2 - 10x + 7$	(x3) $21x^2 - 36x - 27$ (x7) $21x^2 - 70x + 49$ (−) (+) (−)	
(x)	(x17) $\Rightarrow 51x^2 - 170x + 119$ (x3) $\Rightarrow 51x^2 - 114x$ (−) (+)	$34x - 76$ $(\div 2)$ $17x - 38 = f_3$	
	$-56x + 119$	$952x - 2128$ $-952x + 2023$	
		-105 $(\div(3))$ $35 = f_4$	x 56

$f_2(x) = 7x^2 - 12x - 9$

$f_3(x) = 17x - 38$

$f_4(x) = 35$

The signs of the Sturm's functions when $-\alpha$, 0 and $+\alpha$ are substituted are tabulated below.

Example: 2

Find the number and position of real roots of the equation $x^3 - 3x + 6 = 0$

Solution:

Let $f(x) = x^3 - 3x + 6$

$f_1(x) = 3x^2 - 3 = 0$

$= x^2 - 1$

x	$x^2 - 1$	$x^3 - 3x + 6$	x
	$x^2 - 3x$ (−) (+)	$x^3 - x$ (−) (+)	
	$3x - 1$	$-2x + 6 \div 2$	
		$F_2(x) = x - 3$	
		$X_3 \Rightarrow 3x - 9$	
		$3x - 1$	
		-8	

The Strum function are

$f(x) = x^3 - 3x + 16$

$f_1(x) = x^2 - 1$

$f_2(x) = x - 3$

$f_3(x) = -24$

When $-\alpha$, 0, $+\alpha$ are substituted in the series of Sturm's functions,

We get the signs.

	f(x)	$f_1(x)$	$f_2(x)$	$f_3(x)$	Numbers of changes of sign
$-\alpha$	−	+	−	−	2
0	+	−	−	−	1
$+\alpha$	+	+	+	−	1

The number of real roots is only one and we can see that it is negative. To fix the exact position of the negative root substitute −1, −2, −3 ......, until you get the same number of changes of sign.

		$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	Numbers of changes of sign
$-\alpha$	:	+	–	+	–	+	4
0	:	+	+	–	–	+	2
$+\alpha$	:	+	+	+	+	+	0

Hence the number of real roots is 4. Two of which are positive and the other two negative.

Example: 3

Find the Sturm's function of the polynomial $x^3 - 3x + 1 = 0$

Solution:

Let $f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3 \qquad (\div 3)$

$= x^2 - 1$

	$x^2 - 1$ (x^2)	$x^3 - 3x + 1$ $x^3 - x$ (–) (+)	x
x	$2x^2 - 2$	$-2x + 1$	
	$2x^2 - x$ – +	$(\div -)$ $2x-1 = f_2(x)$	
	$x - 2$	$2x - 1$ $2x - 4$ (–) (+)	2
	$x - 2 = f_3(x)$	3	

The Sturm's function

$f(x) = x^3 - 3x + 1$

$f_1(x) = 3x^2 - 3$

$f_2(x) = 2x - 1$

$f_3(x) = x - 2$

$f_4(x) = 3$

When $-\alpha$ is substituted in the Strums functions

		$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	Numbers of changes of sign
–1	:	+	+	–	–	1
–2	:	+	+	–	–	1
–3	:	–	+	–	–	2

$\therefore$ The negative root lies between –2 and –3

Example : 4

Find the number of real root of $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$, Using Strums function.

Solution:

Let $f(x) = x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1$

$f'(x) = 5x^4 - 20x^3 + 27x^2 - 18x + 5$

x	$5x^4 - 20x^3 + 27x^2 - 18x + 5$	$x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1$	
	$5x^4 + \quad 0 - 5x^2$	(x5)	
	$-20x^3 + 32x^2 - 18x + 5$	$5x^5 - 25x^4 + 45x^3 - 45x^2 + 25x - 5$	x
	$-20x^3 + 20x$	$5x^5 - 20x^4 + 27x^3 - 18x^2 + 5x$	
	(+) (−)	(−) (+) (−) (+) (−)	
	$f_3 = 32x^2 - 38x + 5.$	$-5x^4 + 18x^3 - 27x^2 + 20x - 5$	
	(x38)	$-5x^4 + 20x^3 - 27x^2 + 18x - 5$	
	$1216x^2 - 144x + 190$	(+) (−) (+) (−) (+)	
	$f_3(x) \times 26$	$-2x^3 + 2x$	
	$832x^2 - 988x + 130$	$(\div 2)$	
	$832x^2 - 60\ 8x$	$f_2(x) = x^3 - x \quad (\times 5)$	
	(−) (+)	$5x^3 - 5x \quad (4)$	
		$20x^3 - 20x$	
	$-380x + 130$	$x^3 - x\ (x\ 32)$	
	$(\div -10)$	$32x^3 - 32x$	
	$38x - 13$	$32x^3 - 38x^2 + 5x$	
	(x26)	(−) (+) (−)	
		$38x^2 + 37x \quad (\times 32)$	
	$988x - 388$		
	$988x - 722$	$1216x^2 - 1184x + 0$	
	(−) (+)	$1216x^2 - 144x + 1190$	
		(−) (+) (−)	
	$f_5 = 384$	$260x - 1190\ (\div 10)$	
		$f_4 = 26x - 19 \quad (\times 32)$	
		$= 832x - 608$	

The Sturm's function are

$f(x) = x_5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1$

$f_1(x) = 5x^4 - 20x^3 + 27x^2 - 18x + 5$

$f_2(x) = x^3 - x$

$f_3(x) = 32x^2 - 38x + 5$

$f_4(x) = 26x - 19$

$f_5(x) = -384$

when $-\alpha$, 0, and $+\alpha$ are substituted are

	f(x)	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	Numbers of changes of sign
$-\alpha$	–	+	–	+	–	–	4
0	–	+	+	+	–	–	2
α	+	+	+	+	+	–	1

Hence the number of real root are is 4 two of which are positive and other two negative.

Example : 5

Find the number of real roots of the equation $x^4 + 4x^3 - 4x - 13 = 0$

Solution:

Let $f(x) = x^4 + 4x^3 - 4x - 13$

$f'(x) = 4x^2 + 12x^2 - 4 \qquad (\div 4)$

$= x^3 + 3x^2 - 1$

x	$x^3 + 3x^2 + 0 - 1$	$x^4 + 4x^3 + 0 - 4x - 13$	x
	$x^3 + x^2 + 4x$	$x^4 + 3x^3 + 0 - x$	
	(–) (–) (–)	(–) (–) (–) (+)	
	$2x^2 - 4x - 1$	$x^3 + 0 - 3x - 13$	1
	$2x^2 + 4x + 8$	$x^3 + 3x^2 + 0 - 1$	
	(–) (–) (–)	(–) (–) (–) (+)	
	$-6x - 9$	$-3x^2 - 3x - 12$	
	$(\div -3)$	$\div 3$	
	$f_3(x) = 2x + 9$	$f_2(x) = x^2 + x + 4$	
		$(\times)\ 2$	
		$2x^2 + 2x + 8$	
		2x2 + 2x +8	x
		$2x^2 + 3x$	
		(–) (–)	
		$- x + 8$	
		$\times (-2)$	1
		$2x - 16$	
		$2x + 3$	
		(–) (–)	
		-19	

The Strums functions are

$$f(x) = x^4 + 4x^3 - 4x - 13$$

$$f_1(x) = x^3 + 3x^2 - 1$$

$$f_2(x) = x^2 + x + 4$$

$$f_3(x) = 2x + 3$$

$$f_4(x) = -19$$

Exercise :

1. Find the number of imaginary roots of the equation $x^4 + x^3 - x^2 - 2x + 4$
2. Find the number of distinct real roots of the equation $x^3 - 3x + 1 = 0$ & locate them.
3. Find the number and position of the roots of i) $x^5 - 5x + 1 = 0$
   ii) $x^3 - 7x + 7 = 0$
4. Find the range of values of c for which the equation $x^4 + 4x^3 - 8x^2 + c = 0$ has four real distinct roots.

3.5 GENERAL SOLUTIONS OF THE CUBIC EQUNATION

i) Cardon's Method:

1. Let the cubic equation be $x^3 + px + q = 0$ (1)

Let x be u + v

Substituting this value of x in equation (1)

We get

$$(u + v)^3 + p\,(u + v) + q = 0$$

(ie) $u^3 + v^3 + [3uv\,(u+v) + p\,(u+v)] + q = 0$

(ie) $u^3 + v^3 + q + (u+v)\,(3uv + p) = 0$

Choose u and v such that $3uv + p = 0$

Then the equation reduces to

$$u^3 + v^3 + q = 0 \qquad (2)$$

with the condition $3uv + p = 0$

$$\therefore u = -\frac{p}{3v}$$

Eliminate u from (2) and (3)

We get $\left(-\frac{p}{3v}\right)^3 + v^3 + q = 0$

$$\text{(ie) } -\frac{p^3}{27} + v^3 + q = 0$$

$$\text{(ie) } -\frac{p^3}{27} + v^6 + qv^3 = 0$$

$$\text{(ie) } v^6 + qv^3 - \frac{p^3}{27} = 0$$

From (4) and (5) relations,

We get that u^3 and v^3 are the roots of the equation

$$t^2 + qt - \frac{p^3}{27} = 0 \qquad \text{put } v^3 = t$$

u^3 and v^3 can be determined from this equation

$$t^2 + qt - \frac{p^3}{27} = 0$$

$$t = \frac{-q \pm \sqrt{q^2 - 4(1)\left(-\frac{p^3}{27}\right)}}{2(1)} = \frac{-q}{2} \pm \sqrt{\frac{q^2 + 4p^3/27}{4}}$$

$$t = -\frac{q}{2} \pm \left(\frac{q^2}{4} + \frac{p^3}{27}\right)^{1/2}$$

$$u^3 = -\frac{q}{2} \pm \left(\frac{q^2}{4} + \frac{p^3}{27}\right)^{1/2}, \; v^3 = -\frac{q}{2} \pm \left(\frac{q^2}{4} + \frac{p^3}{27}\right)^{1/2}$$

Roots of the equation (6) are real only, when $\frac{q^2}{4} + \frac{p^3}{27} \geq 0$.

In that case two roots of equation (1) are imaginary and one root real or two of the roots of the equation (1) are equal.

Let $\frac{q^2}{4} + \frac{p^3}{27}$

(ie) $4p^3 + 27q^2$ is positive.

Then u^3 and v^3 are real and

Let $u^3 = m^3$, $v^3 = n^3$

Here we obtain 3 values of u.viz. m.wm, w^2m and 3 values of viz., n, wn, w^2n, where w and w^2 are the cube roots of unity.

Hence we get 9 combinations for u + v. Out of the nine combinations the following 3 combinations values are only valid for u + v since

$u^3 v^3 = -\frac{p^3}{27}$ (ie) $uv = -\frac{p}{3}$

$m + n$; $mw + nw^2$ and $mw^2 + nw$

Hence they are the roots of the given equation(1).

The solution of the cubic equation depends on the roots of the equation (6)

The roots of the equation (6) are imaginary if $q^2 + \frac{4p^3}{27} < 0$ in that case both u^3 and v^3 are imaginary quantities.

This has no arithmetical meaning. Hence Cardon's method is not useful. So before trying to solve a cubical equation. Find the nature of its roots. If all the three roots are real we can not use cardon's method to get arithmetic values for the roots.

Example: 1

Solve the equation $x^3 - 6x - 9 = 0$

Solution:

Here p =– 6, and q =– 9

$4p^3 + 27q^2 = 4(-6)^3 + 27(-9)^2 = 1323 > 0$

Hence the equation has one real root and two imaginary roots and so Cardon's method is applicable.

$\therefore$ x = u + v where u^3 and v^3 are the roots of the equation.

$$t^2 + 9p - \frac{p^3}{27} = 0$$

(ie) $t^2 - 9p + 8 = 0 \quad (p = -6, q = -9)$

(ie) $(t-8)(t-1) = 0$

$\therefore u^3 = 8$ and $v^3 = 1$

$$\begin{array}{c|c} \multicolumn{2}{c}{8} \\ \hline -1 & -8 \end{array}$$

Hence 2 + 1 (ie) 3 is one of the roots of the equation.

The other roots are $2w + w^2$ and $2w^2 + w$. (or) since 3 is one of the roots of the equation, Dividing the given equation by x – 3.

We get the other roots of the given equation.
They are the roots of the equation

$x^3 + 3x + 3 = 0$

Hence the given equation has the 3 roots $x^3 + 3x + 3 = 0$

$$3, \frac{-3+i\sqrt{3}}{2}, \frac{-3-i\sqrt{3}}{2}$$

$$x = \left[\begin{array}{l} \dfrac{-3 \pm \sqrt{9-4(3)(1)}}{2} \\ \dfrac{-3 \pm \sqrt{9-12}}{12} \Rightarrow \dfrac{-3 \pm i\sqrt{3}}{2} \end{array}\right.$$

These are the same as 3, $2w + w^2$ and $2w^2 + w$.

Example: 2

Solve the equation $x^3 - 9x^2 + 108 = 0$.

Solution:

Transform this equation into one without the second term. (ie) the term without x^2 term.

This can be done by decreasing the roots by 3.

That equation is $x^3 - 27x + 54 = 0$ (1)

If α, β, γ are the roots of the equation (1), the roots of the given equation are $\alpha + 3$, $\beta + 3$, and $\gamma + 3$,.

Here u^3 and v^3 are the roots of the equation $t^2 + qt - \frac{p^3}{27} = 0$ where q = 54, p = –27.

(ie) $t^2 + 54t + \frac{(27)^3}{27} = 0$

(ie) $t^2 + 54t + (27)^2 = 0$

(ie) $(t + 27)^2 = 0$ $[(a+b)^2 = \because a^2 + 2ab + b^2]$

Hence two of the roots of equation are equal

$\therefore\ u^3 = -27$ and $v^3 = -27$

Hence $u = -3$ and $v = -3$

$\therefore$ The roots of the equation (1) are

$-6,\ -3\omega\ -3\omega^2$ and $-3\omega^2\ -3\omega$

Since ω and ω^2 are the cubic roots of the unity.

$1 + \omega + \omega^2 = 0.$

Hence these roots are $-6, 3, 3$.

$\therefore$ The roots of the given equation are -3, 6 and 6.

Example:

Solve the equation $x^3 - 12x + 65 = 0$

Solution:

This is the standard form

Put $x = u + v$

$x^3 = (u + v)^3$

$x^3 = u^3 + v^3 + 3uv(u+v)$

$= u^3 + v^3 + 3uvx$

$= x^3 - 3uvx - (u^2 + v^3) = 0$

Comparing the given equation.

We get,

$-3uv = -12$; $\qquad -(u^3 + v^3) = 65$

$uv = 4 \qquad u^3 + v^3 = -65$

$u^3v^3 = 4^3$

$u^3v^3 = 64$

form an equation whose roots are u^3 & v^3

$t^2 + 65t + 64 = 0$

$(t + 64)(t + 1) = 0$

$t = -64, -1$

One of the roots are u_1 & v_1

$u_1^3 = -64$, $\qquad v_1^3 = -1$

$u_1 = -4$, $\qquad v_1 = -1$

Hence the roots of the equation.

$u_1 + v_1,\ u_1w + v_1w^2,\ u_1w^2 + v_1w$

$u_1 = -4,\ v_1 = -1$

$$-4-1,\ -4\left(\frac{-1 \pm i\sqrt{3}}{2}\right)+(-1)\left(\frac{-1-i\sqrt{3}}{2}\right) -4\left(\frac{-1-i\sqrt{3}}{2}\right)+(-1)\left(\frac{-1+i\sqrt{3}}{2}\right)$$

$$\Rightarrow\ -5, \frac{4-4i\sqrt{3}+1+i\sqrt{3}}{2}, \frac{4+4i\sqrt{3}+1-i\sqrt{3}}{2}$$

Hence the three roots are

$$-5,\ \frac{5-i\,3\sqrt{3}}{2}, \frac{5+i\ 3\sqrt{3}}{2}$$

Example:

Solve the equation $x^3 - 9x + 28 = 0$

Solution:

This is the standard form

Put $x = u + v$

$x^3 = (u+v)^3$

$= u^3 + v^3 + 3uv(u+v)$

$= u^3 + v^3 + 3\,uvx$

$x^3 - 3uvx - (u^3 + v^3) = 0$

Comparing the given equation,

We get $-3uv = -9$; $\qquad -(u^3 + v^3) = 28$

$uv = 3 \qquad u^3 + v^3 = -28$

form an equation whose roots are u^3 & v^3

$t^2 + 28t + 27 = 0$

$(t+27)(t+1) = 0$

$t = -27,\ -1$

One of the roots are u_1 & v_1

$u_1^3 = -27, \qquad v_1^3 = -1$

$u_1 = -3, \qquad v = -1$

Hence the roots of the equation

$u_1 + v_1 \qquad u_1\,\omega + v_1\,\omega^2, \qquad u_1\,\omega^2 + v_1\,\omega$

$u_1 = -3, \qquad v_1 = -1$

$$-3 - 1, \quad -3\left(\frac{-1 + i\sqrt{3}}{2}\right) + (-1)\left(\frac{-1 - i\sqrt{3}}{2}\right)$$

$$\Rightarrow -4, \quad \left(\frac{3 - 3i\sqrt{3} + 1\ i\sqrt{3}}{2}\right) \left(\frac{3 + i\,3\sqrt{3} + 1 - i\sqrt{3}}{2}\right)$$

$$\Rightarrow -4, \quad \left(\frac{4 - i\,2\sqrt{3}}{2}\right)\left(\frac{4 + 2\,i\sqrt{3}}{3}\right)$$

Hence the three roots are

$\therefore -4, \qquad 2 - i\sqrt{3}, \qquad 2 + i\sqrt{3}$

Example:

Solve the equation $x^3 - 27x + 54 = 0$

Solution:

This is standard form

Put $x = u + v$

$$x^3 = (u + v)^3 = u^3 + v^3 + 3uv\,(u + v)$$
$$= u^3 + v^3 + 3uv\,x$$

$$\Rightarrow x^3 - 3uvx - (u^3 + v^3) = 0$$

Comparing the given equation, we get

$-3uv = -27; \qquad -(u^3 + v^3) = 54$

$uv = 9 \qquad u^3 + v^3 = -54$

form an equation whose roots are u^3 & v^3

$t^2 + 54t + 729 = 0$

$$\Rightarrow t = \frac{-54 \pm \sqrt{2916 - 2916}}{2} = \frac{-54}{2} = -27$$

$t = -27$ (twice)

One of the roots are u_1 & v_1

$$u_1^3 = -27 , v_1^3 = -27$$

$$u_1 = -3; \quad v_1 = -3$$

Hence the roots of the equation are $u_1 + v_1$, $u_1\omega + v_1\omega^2$, $u_1\omega^2 + v_1\omega$

$$u_1 = -3, v_1 = -3$$

$$-3-3, \quad -3\left(\frac{-1+i\sqrt{3}}{2}\right) + (-3)\left(\frac{-1-i\sqrt{3}}{2}\right)$$

$$-3\left(\frac{-1+i\sqrt{3}}{2}\right) + (-3)\left(\frac{-1-i\sqrt{3}}{2}\right)$$

$$\Rightarrow -6, \left(\frac{3-3i\sqrt{3}+3+3i\sqrt{3}}{2}\right), \left(\frac{3+3i\sqrt{3}+3-3i\sqrt{3}}{2}\right)$$

$$\Rightarrow -6, 3, 3.$$

Exercise:

1. Solve the equation

i) $x^3 = 6x^2 - 6x + 63 = 0$

ii) $x^3 = x^2 - 16x + 20 = 0$

iii) $x^3 = 3x^2 - 21x + 49 = 0$

iv) $x^3 = 6x^2 + 9x + 4 = 0$

v) $x^3 = 15x^2 - 33x + 847 = 0$

UNIT – IV

NUMERICAL METHODS

NUMERICAL SOLUTION OF ALGEBRAIC AND TRANSCENTAL EQUATIONS

4.1 Introduction:

The function f(x) of the form

$f(x) = a_o x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_{n-1}x + a_n$ ......(1)

where n is a positive integer and a_o, a_1 a_2 ....a_n are independent of x ((ie) are constants) with $a_o \neq 0$ although some of a_1, a_2....... a_n may be zero in known as a **polynomial** of degree n.

This may also be called as an **integral function** when x in f(x) occurs in an integral form only (ie) never in denominator or with negative power. The values of x making f(x) zero are known as **zeros** or **Roots** of the polynomial f(x), and every polynomial of degree n has n roots.

The Polynomial f(x) equated to zero gives the rational integral equation which is Algebraic or Transcendental according as f(x) is purely a polynomial in x as (1) or contains some other functions (transcendental) such as trigonometric, Logarithmic or exponential etc.,

Ex: $x^5 + 2x^4 - 3x^3 + x - 9 = 0$ is an algebric equation, while $3x^2 + \log(x+1) + e^{-x} + \cos x = 0$ is a transcendental equation **solution** of an equation f(x) = 0 means to find its roots or zeros.

The problem is to find an approximate value of x which satisfies the equation f(x) = 0. One way is to draw the graph of y = f(x) and find the approximate values of the abscissa of the points where the graph crosses the x – axis. It is not always easy to draw the graph of y = f(x). In some cases the equation can be written as $f_1(x) = f_2(x)$

Then the abscissa of the intersecting points of the two graphs $y = f_1(x)$ and $y = f_2(x)$ will give the real roots of the equation f(x)=0. These two methods will give at the most an approximate values of the real roots correct to one place of decimal.

Before finding an approximate value of the root by drawing graphs, it is better to find the limits between which a real root lies. "**If f(x)** is a continuous functions function in the range (a,b) and if f(a) and f(b) have different signs, then there is at least one real root between a and b".

1. Find the positive root of the equation

$$x^3 - x - 1 = 0$$

Solution:

Given $f(x) = x^3 - x - 1 = 0$

We shall find the values of f(x) when x=0,1,2,3,......

$x = 0$; $f(0) = (0)^3 - 0 - 1 = -1$

$x = 1$; $f(1) = (1)^3 - 1 - 1 = -1$

$x = 2$; $f(2) = (2)^3 - 2 - 1 = 8 - 2 - 1 = 5$

$x = 3$; $f(3) = (3)^3 - 3 - 1 = 27 - 3 - 1 = 23$

$x = 4$; $f(4) = (4)^3 - 4 - 1 = 64 - 4 - 1 = 59$

Here f(1) is (–ve) and f(2) is (+ve)

∴ Hence a root lies between 1 and 2

To find a still closer approximation find the values of f(x) for intermediate values between 1 and 2.

Let x be 1.5 $\left[x = \frac{1+2}{2} = \frac{3}{2} = 1.5\right]$

$f(1.5) = (1.5)^3 - 1.5 - 1$

$= 0.815$

Here f(1) is (– ve) and f(1.5) is (+ve)

Hence the root lies between 1 and 1.5

To find still closer limits find f(x) for x =1.1,1.2,1.3,1.4 .

$x = 1.1$

$f(x) = (1.1)^3 - 1.1 - 1 = 1.331 - 1.1 - 1 = -0.769$

$x = 1.2$; $f(x) = (1.2)^3 - 1.2 - 1 = 1.728 - 1.2 - 1 = -0.372$

$x = 1.3$; $f(x) = (1.3)^3 - 1.3 - 1 = 2.197 - 1.3 - 1 = -0.103$

$x = 1.4$; $f(x) = (1.4)^3 - 1.4 - 1 = 2.744 - 1.4 - 1 = -0.344$

Here f(1.3) is (–ve) and f(1.4) are (+ve)

∴ The root lies between 1.3 and 1.4. $\left[x = \frac{1.3+1.4}{2} = \frac{2.7}{2} = 1.35\right]$

To find still closer limits, find f(x) = 1.35

$f(1.35) = (1.35)^3 - 1.35 \; - 1 = 2.460375 - 1.35 - 1$

$= 0.1104$ & is (+ve)

∴ The root lies between 1.3 and 1.35.

To find still closer limits find f(x) for

x = 1.31 , 1.32, 1.33, 1.34.

$x = 1.31 \;;\; f(1.31) = (1.31)^3 - 1.31 - 1 = 2.248091 - 1.31 - 1$
$= -0.619$

$x = 1.32 \;;\; f(1.32) = (1.32)^3 - 1.32 - 1 = 2.299968 - 1.32 - 1$
$= -0.02$

$x = 1.33 \;;\; f(1.33) = (1.33)^3 - 1.33 - 1 = 2.352637 - 1.33 - 1$
$= +0.0226$

$x = 1.34 \;;\; f(1.34) = (1.34)^3 - 1.34 - 1 = 2.406104 - 1 - 1.34 - 1$
$= +0.0661.$

Here f(1.32) is (–ve) and f(1.33) is (+ve)

Hence the root lies between 1.32 and 1.33.

To find still closer limits, find f(x) for x = 1.325 $\left[x = \frac{1.32 + 1.33}{2} = \frac{2.65}{2} = 1.325\right]$

$f(1.325) = (1.325)^3 - 1.325 - 1$

$= 0.0012.$

Here f(1.32) is (–ve) & f(1.325) is (+ve).

Hence the root lies between. 1.32 & 1.325

Find f(x) for x = 1.321, 1.322,1.323,1324.

$x = 1.321 \;;\; f(1.321) = (1.321)^3 - 1.321 - 1 = 2.305199161 - 1.321 - 1$
$= -0.0158$

$x = 1.322$; $f(1.322) = (1.322)^3 - 1.322 - 1 = 2.310438248 - 1.322 - 1$
$= -0.01156$

$x = 1.323$; $f(1.323) = (1.323)^3 - 1.323 - 1 = 2.315685267 - 1.323 - 1$
$= -0.0073$

$x = 1.324$; $f(1.324) = (1.324)^3 - 1.324 - 1 = 2.320940224 - 1.324 - 1$
$= -0.031.$

Here f(1.324) is (–ve) and f(1.325) is (+ve)

Hence the root lies between 1.324 and 1.325.

To find still closer limits, find f(x) for x = 1.3245 $\left[x = \frac{1.324 + 1.325}{2} = \frac{2.649}{2} = 1.3245\right]$

$f(1.3245) = (1.3245)^3 - 1.3245 - 1 = 2.323570681 - 1.3245 - 1$
$= -0.0009$

Here f(1.3245) is (–ve) and f(1.325) is (+ve)

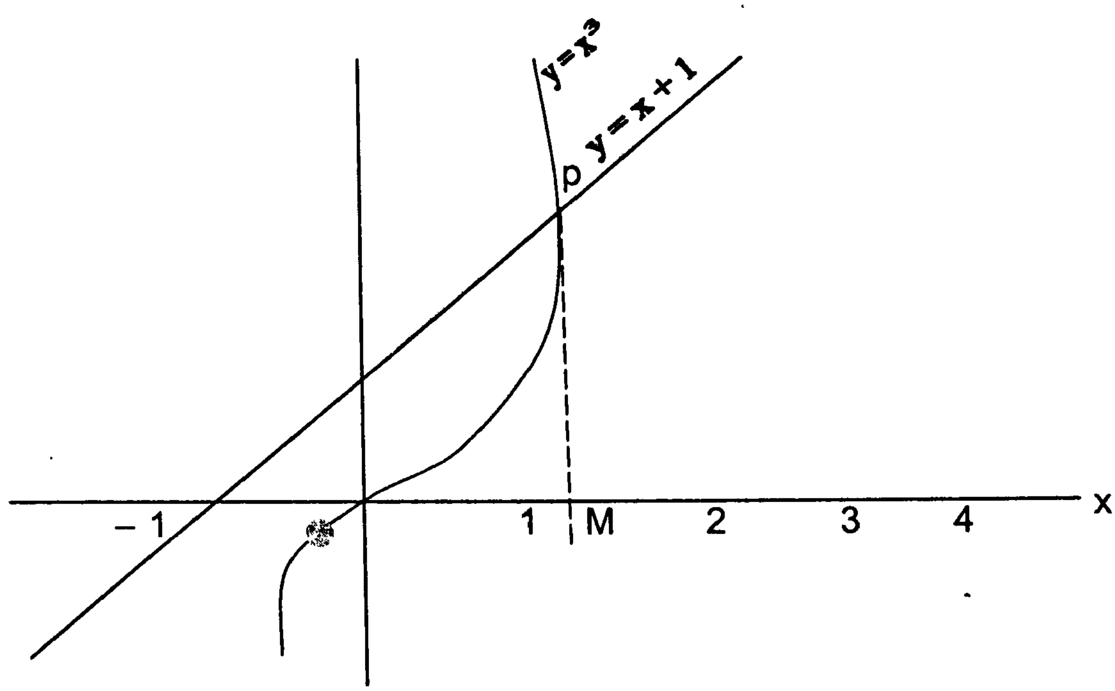
∴ The roots lies between 1.3245 & 1.325

To a third place of decimals.

We can take the value as 1.325.

Aliter:

The equation $x^3 - x - 1 = 0$ can be written in the form $x^3 = x + 1$.



Draw the graphs of the equation

$Y = x^3$ and $y = x+1$

And find the x coordinates of the points of intersection from the graphs it in seen that the two curves intersect at only one point an its abscissa lies between 1 and 2.

If we take a bigger scale, it can be shown that the root lies between 1.3 and 1.4.

4.2 INTERATION METHOD

Suppose $f(x) = 0$ can be written in the form

$$x = F(x) \qquad (1)$$

Let x_0 be an approximation to the root a

Then $x_1 = F(x_0)$ (2)

$x_2 = F(x_1)$

,

,

,

,

gives a future approximation.

In general $x_{k+1} = F(x_k)$, where $k = 0,1,2,\ldots\ldots$(3) will give further approximations.

This formula (3) is known as the Iteration formula.

This is valid if f

$x_1, x_2 \ldots x_k, x_{k+1} \ldots\ldots$ converge to 'a'

subtracting (2) from (1)

we get

$$x - x_1 = F(x) - F(x_0)$$

but $F(x) - F(x_0) = (x - x_0)\, F^1\left(\varepsilon_0\right)$ where $x \le \varepsilon_0 \le x_1$,

By mean value theorem.

$$\therefore\ x - x_1 = (x - x_1)\, F'\left(\varepsilon_0\right).$$

Similarly

$$x - x_2 = (x - x_1)\ F'(\varepsilon_1).$$

$$x - x_3 = (x - x_2)\ F'(\varepsilon_2).$$

....

....

$$x - x_n = (x - x_{n-1})\ F'(\varepsilon_{n-1}).$$

Multiplying all these equation & concelling the common factor on both sides.

$$(x - x_1)(x - x_2)(x - x_3).....(x - x_n)$$
$$= [(x - x_0)(x - x_1)(x - x_2).....(x - x_{n-1})]\ F'(\varepsilon_0)\ F'(\varepsilon_1)\ F'(\varepsilon_2).....F'(\varepsilon_{n-1})$$

We get

$$x - x_n = (x - x_0)\ F'(\varepsilon_0)\ F'(\varepsilon_1).....F'(\varepsilon_{n-1})$$

If the maximum value of $|F'(x)| < |$, then each of the quantities

$|F'(\varepsilon_0)|, |F'(\varepsilon_1)|.......|F'(\varepsilon_{n-1})| < |$

(ie) less than a proper fraction λ, then

$$|F'(\varepsilon_0)|\,|F'(\varepsilon_1)|.......|F'(\varepsilon_{n-1})| < \lambda^n$$

Hence $|x - x_n| < |x - x_0|\ \lambda^n$

Since λ is a proper fraction

$$\lim_{n \to \alpha} |x - x_0|\ \lambda^n = 0$$

$\therefore$ The condition for the convergence of the errors in x_0, x_1, x_2 ....is $|F'(x)| < |$.

Example: 1

Find the positive root of the equation

$$x^3 - x - 1 = 0.$$

Solution:

Given $f(x) = x^3 - x - 1 = 0$.

The curve $y = x^3$ and $y = x+1$

We shall find the values of f(x), when x = 0,1,2......

$x = 0$; $f(0) = (0)^3 - 0 - 1 = -1$

$x = 1$; $f(1) = (1)^3 - 1 - 1 = -1$

$x = 2$; $f(2) = (2)^3 - 2 - 1 = 8-2-1 = 5$

Here f(1) is '–ve' and f(2) is '+ve'.
Hence a root lies between 1 and 2.

To find a still closer approximation find the values of f(x) for intermediate values between 1 and 2.

$x = 1.5$ $\left[x = \frac{1+2}{2} = \frac{3}{2} = 1.5\right]$

$f(1.5) = (1.5)^3 - 1.5 - 1 = 0.815$

Hence the root lies between 1 and 1.5

To find still closer limits find f(x) for x = 1.1, 1.2 1.3, 1.4.

$x = 1.1$; $f(x) = (1.1)^3 - 1.1 - 1 = 1.331 - 1.1 - 1$
$= -0.769$

$x = 1.2$; $f(1.2) = (1.2)^3 - 1.2 - 1 = 1.728 - 1.2 - 1.$
$= -0.472$

$x = 1.3$; $f(1.3) = (1.3)^3 - 1.3 - 1 = 2.197 - 1.3 - 1$
$= -0.103$

$x = 1.4$; $f(1.4) = (1.4)^3 - 1.4 - 1 = 2.744 - 1.4 - 1$
$= -0.344$

∴ The root lies between 1.3 and 1.4

This equation can be put in the form

$x = x^3 - 1$

Hence $F(x) = x = x^3 - 1$

$F'(x) = 3x^2$

$F'(1.3) = 3(1.3)^2$
$= 5.07$

Hence $|F'(1.3)|$ is not less than one.

Hence this way of writing f(x), will not give any valid iteration process.

Another way of writing the equation to apply the process is

$$x^3 = 1 + x \Rightarrow x^2.x = 1+x$$

$$\text{(ie) } x = \frac{1+x}{x^2}$$

In this case $F(x) = \dfrac{1+x}{x^2} \Rightarrow$

$$F'(x) = -\frac{2}{x^3} - \frac{1}{x^2}$$

$$F'(1.3) = -\frac{2}{(1.3)^3} - \frac{1}{(1.3)^2}$$

$$= -1.5020$$

Here $|F'(1.3)| \not< 1$

Hence this iteration process is not valid.

A third way of writing the equation is

$$x^3 = 1 + x$$

$$x = (1+x)^{1/3}$$

In this case $F(x) = (1+x)^{1/3}$

$$F'(x) = \frac{1}{3}(1+x)^{-2/3}$$

$x = 1.3$

$$\text{Hence } F'(1.3) = \frac{1}{3(1+1.3)^{2/3}}$$

$$= 0.1913$$

$$\therefore |F'(1.3)| < 1.$$

Hence this process can start with 1.3

In general $x_{k+1} = F(x_k)$.

K=0;

$x_1 = F(x_0)=(1+1.3)^{1/3} = (2.3)^{1/3}$
$x_1 = 1.3200$

$x_2 = F(x_1) = (1+1.32)^{1/3} = (2.32)^{1/3}$
$= 1.3238$

$x_3 = F(x_2) = (1+1.3238)^{1/3} = (2.3238)^{1/3}$
$= 1.3245$

$x_4 = F(x_3) = (1+1.3245)^{1/3} = (2.3245)^{1/3}$
$= 1.3247$

$x_5 = F(x_4) = (1+1.3247)^{1/3} = (2.3247)^{1/3}$
$= 1.3247$

Since x_4 & x_5 give the same value.

$\therefore$ Hence the root is 1.3247 (correct to 4 places of decimals).

Example: 2

Find the positive root of the equation x = cos x correct to 4 places of decimals.

Solution:

In this case x = cos x.

$\therefore F(x) = \cos x$

$\therefore F'(x) = -\sin x$

x = 0

$F(0) = \cos 0 = 0$

x = 0

$F(1) = \cos 1 = 0.54030$

Hence the roots are 0 and 1

This equation can be put in the form

$F(x) = \cos x$

$F'(x) = \sin x$

$x = 0.5 \Rightarrow |F'(0.5)| = |0.5| = |\sin(0.5)| < |$

Hence this process can be start with 0.5 Formula: $x_{k+1} = F(x_k)$.

$K = 0,\ x_0 = 5 \Rightarrow x_1 = F(x_0) = \cos x_0 = \cos(0.5) = 0.8776$

$K = 1,\ x_1 = 0.8776;\ x_2 = F(x_1) = \cos x_1 = \cos(0.8776) = 0.6390$

$K = 2,\ x_2 = 0.6390 \Rightarrow x_3 = F(x_2) = \cos x_2 = \cos(0.639) = 0.8027$

$K = 3,\ x_3 = 0.8027 \Rightarrow x_4 = F(x_3) = \cos x_3 = \cos(0.8027) = 0.6948$

$K = 4,\ x_4 = 0.6948 \Rightarrow x_5 = F(x_4) = \cos x_4 = \cos(0.6948) = 0.7682$

$K = 5,\ x_5 = 0.7682 \Rightarrow x_6 = F(x_4) = \cos x_5 = \cos(0.7682) = 0.7192$

$K = 6,\ x_6 = 0.7192 \Rightarrow x_7 = F(x_6) = \cos x_6 = \cos(0.7192) = 0.7523$

$K = 7,\ x_7 = 0.7523 \Rightarrow x_8 = F(x_7) = \cos x_7 = \cos(0.7523) = 0.7301$

$K = 8,\ x_8 = 0.7301 \Rightarrow x_9 = F(x_8) = \cos x_8 = \cos(0.7301) = 0.7451$

$K = 9,\ x_9 = 0.7451 \Rightarrow x_{10} = F(x_9) = \cos x_9 = \cos(0.7451) = 0.7350$

$K = 10,\ x_{10} = 0.7350;\ x_1 = F(x_{10}) = \cos x_{10} = \cos(0.7350) = 0.7418$

$K = 11,\ x_{11} = 0.7418;\ x_{12} = F(x_{11}) = \cos x_{11} = \cos(0.7418) = 0.7373$

$K = 12\ x_{12} = 0.7373;\ x_{13} = F(x_{12}) = \cos x_{12} = \cos(0.7373) = 0.7403$

$K = 13,\ x_{13} = 0.7403;\ x_{14} = F(x_{13}) = \cos x_{13} = \cos(10.7403) = 0.7383$

$K = 14,\ x_{14} = 0.7383;\ x_{15} = F(x_{14}) = \cos x_{14} = \cos(0.7383) = 0.7396$

$K = 15,\ x_{15} = 0.7396;\ x_{16} = F(x_{15}) = \cos x_{15} = \cos(0.7396) = 0.7387$

$K = 16,\ x_{16} = 0.7387;\ x_{17} = F(x_{16}) = \cos x_{16} = \cos(0.7387) = 0.7393$

K = 17, $x_{17} = 0.7393$; $x_{18} = F(x_{17}) = \cos x_{17} = \cos(0.7393) = 0.7389$

K = 18, $x_{18} = 0.7389$; $x_{19} = F(x_{18}) = \cos x_{18} = \cos(0.7389) = 0.7392$

K = 19, $x_{19} = 0.7392$; $x_{20} = F(x_{19}) = \cos x_{19} = \cos(0.7392) = 0.7390$

K = 20, $x_{20} = 0.7390$; $x_{21} = F(x_{20}) = \cos(0.7390) = 0.7391$

K = 21, $x_{21} = 0.7391$; $x_{22} = F(x_{21}) = \cos(0.7391) = 0.7391$

Since x_{21} and x_{22} give the same value we take the root as 0.7391. Correct to 4 places of decimals.

Example: 3

The equation $4x = e^x$ has two roots one near 0.3 and the other near 2.1 Find them.

Solution:

Given the equation can be written in the form

$$x = \frac{1}{4}e^x$$

Formula: $x_{k+1} = F(x_k)$

$$\therefore \quad F(x) = \frac{1}{4}e^x$$

Hence $F'(x) = \frac{1}{4}e^x$

Given $x_o = 0.3$

$$F'(x_o) = \frac{1}{4}e^{0.3} = \frac{1.349858808}{4} = 0.3375$$

$$|F'(0.3)| = 0.3375 < |$$

Given x = 2.1

$$F'(2.1) = \frac{1}{4}e^{2.1} = \frac{8.16617}{4} = 2.0415$$

$|F'(2.1)| = 2.0415 \not< 1$

Hence this process can start with 0.3

$$x_1 = F(x_0) \Rightarrow x_1 = \frac{1}{4}e^{0.3} = \frac{1.34986}{4} = 0.3375$$

$$x_2 = F(x_1) \Rightarrow x_2 = \frac{1}{4}e^{0.3375} = \frac{1.40144}{4} = 0.3504$$

$$x_3 = F(x_2) \Rightarrow x_3 = \frac{1}{4}e^{0.3504} = \frac{1.41964}{4} = 0.3549$$

$$x_4 = F(x_3) \Rightarrow x_4 = \frac{1}{4}e^{0.3549} = \frac{1.42604}{4} = 0.3565$$

$$x_5 = F(x_4) \Rightarrow x_5 = \frac{1}{4}e^{0.3565} = \frac{1.42832}{4} = 0.3571$$

$$x_6 = F(x_5) \Rightarrow x_6 = \frac{1}{4}e^{0.3571} = \frac{1.42918}{4} = 0.3573$$

$$x_7 = F(x_6) \Rightarrow x_7 = \frac{1}{4}e^{0.3573} = \frac{1.42946}{4} = 0.3574$$

$$x_8 = F(x_8) \Rightarrow x_8 = \frac{1}{4}e^{0.3574} = \frac{1.42961}{4} = 0.3574$$

Here $x_7 = x_8$

Hence the root is 0.3574 correct to 4 places of decimals.

Here 2.1 can not be the starting point for this form $x = \frac{1}{4}e^x$

Since $F'(2.1) \not< 1$.

The equation is $e^x = 4x$

Taking log on both side.

$x = 10g_e 4x$

Here, $F(x) = 10g_e(4x) = 1og4 + logx.$

$$F'(x) = \frac{1}{x}$$

$$\therefore F'(2.1) = \frac{1}{2.1} = 0.47619 < 1$$

$$\therefore |F'(2.1)| < 1.$$

Hence we can start with 2.1 in this form for iteration process.

Let $x_0 = 2.1$.

$x_1 = F(x_0) \Rightarrow x_1 = \log 4\ (2.1) = 10g_e(8.4) = 2.1282$

$x_2 = F(x_1) \Rightarrow x_2 = \log 4\ (2.1282) = 10g_e(8.5128) = 2.1416$

$x_3 = F(x_2) \Rightarrow x_3 = \log 4\ (2.1416) = 10g_e(8.5664) = 2.1478$

$x_4 = F(x_3) \Rightarrow x_4 = \log 4\ (2.1478) = 10g_e(8.5912) = 2.1507$

$x_5 = F(x_4) \Rightarrow x_5 = \log 4\ (2.1507) = 10g_e(8.6028) = 2.1521$

$x_6 = F(x_5) \Rightarrow x_6 = \log 4\ (2.1521) = 10g_e(8.6084) = 2.1527$

$x_7 = F(x_6) \Rightarrow x_7 = \log 4\ (2.1527) = 10g_e(8.6108) = 2.1530$

$x_8 = F(x_7) \Rightarrow x_8 = \log 4\ (2.1530) = 10g_e(8.6120) = 2.1532$

$x_9 = F(x_8) \Rightarrow x_9 = \log 4\ (2.1532) = 10g_e(8.6128) = 2.1532$

Here $x_8 = x_9$.

Hence we take 2.1532 as the root correct four places of decimals.

Example:4

Find a real root of the equation $\cos x = 3x - 1$ correct to 4 decimal places by iteration method.

Solution:

Let $f(x) = \cos x - 3x+1=0$

$f(0) = \cos o - 3(0) +1 = 0 = +ve$

$f\left(\pi/2\right) = \cos \pi/2 - 3\ \pi/2+1 = 0\ -\frac{(3\pi - 2)}{2} = -ve$

$\therefore$ a root lies between 0 and $\pi/2$

The given equation may be written as

$x = 1/3(1+\cos x) = \phi\ (x)$

$\phi'(x) = -1/3 \sin x$

$|\phi'(x)| = \left|\frac{1}{3}\sin x\right| < 1\ \ \forall\ x$ & in particular in $\left(o, \pi/2\right)$ (ie) (0,1,5,708)

Hence the iteration method may be applied.

Let us take $x_0 = 0.6$.

$x_1 = \frac{1}{3}(1+\cos x_1) = \frac{1}{3}(1+\cos(0.6)_2$

$x_1 = 0.60845$

$x_2 = \frac{1}{3}[1+(0.60845)]$

$x_2 = 0.60684$

$x_3 = \frac{1}{3}[1+\cos(0.60684)] = 0.60715$

$x_4 = \frac{1}{3}[1+\cos(0.60715)] = 0.60709$

$x_5 = \frac{1}{3}[1+\cos(0.60709)] = 0.60710$

$$x_6 = \frac{1}{3}[1+\cos(0.60710)] = 0.60710$$

Due to repetition of x_5 and x_6, we step our work here.

Hence the roots 0.6071 correct to 4 decimal places.

Example: 5

Solve the equation $x_3 + x_2 - 1 = 0$ for the positive by iteration method.

Solution:

Let $f(x) = x_3 + x_2 - 1 = 0$.

$f(o) = -1 = -ve$: $f(1) = 1 = +ve$

The root lies between 1 and 1

The given equation as $x^{2(x+1)} = 1$.

$$\text{(ie) } x = \frac{1}{\sqrt{x+1}} = \phi\ (x).$$

$$\phi'(x) = -\frac{1}{2}\ \frac{1}{(x+1)^{3/2}}$$

$$\text{(ie) } |\phi'(0)| = \frac{1}{2} < 1 \text{ and } |\phi'(1)| < |$$

$$\text{(ie) } |\phi'(x)| < | \quad \forall x \in (0,1)$$

Hence, the iterative method can be applied.

Take $x_0 = 0.75$ as starting value.

$$x_1 = \frac{1}{\sqrt{1+x_0}} = \frac{1}{\sqrt{1+0.75}} = \frac{1}{\sqrt{1.75}} = 0.75593$$

$$x_2 = \frac{1}{\sqrt{1+0.75593}} = \frac{1}{\sqrt{1.75593}} = 0.75465$$

$$x_3 = \frac{1}{\sqrt{1+0.75465}} = \frac{1}{\sqrt{1.75465}} = 0.75493$$

$$x_4 = \frac{1}{\sqrt{1+0.75493}} = \frac{1}{\sqrt{1.75493}} = 0.75487$$

$$x_5 = \frac{1}{\sqrt{1+0.75487}} = \frac{1}{\sqrt{1.75487}} = 0.75488$$

$$x_6 = \frac{1}{\sqrt{1+0.75488}} = 0.75488$$

Due to repetition of x_5 and x_6. We stop out work here.

Hence the root is 0.75488.

Exercise

1. Solve the following by iteration method
   - i) $3x - \cos x - 2 = 0$
   - ii) $3x = 6 + \log_{10}x$
   - iii) $x^3 + x + 1 = 0$
   - iv) $2x - \log 10x = 7$
   - v) $3x + \sin x = e^x$.
   - vi) $x^3 + x^2 = 100$

4.3 Newton – Raphson Method

When an approximate value of a root of an equation is known, a closer approximation may be obtained by the following method commonly known as "Newton Raphson Method".

Let α be a root of the equation $f(x) = 0$ &

Let x_o be an approximation to α.

$\therefore \ \alpha = x_o + h$ where h is small, positive or negative.

$\therefore \ f(x_o + h) = o$

(ie) $f(x_0) + h\ f'(x_o) + \frac{h_2}{2'} f''(x_o) + \cdots = o$ [By Taylor's Theorm].

Neglecting higher powers of h.

We get,

$$f(x_o) + hf'(x_o) = o \text{ approximately}$$

$$\therefore h = \frac{-f(x_o)}{f'(x_o)} \text{ approximately}$$

$$\therefore \alpha = x_o - \frac{f(x_o)}{f'(x_o)} \text{ approximately} \quad \left[\begin{matrix}\alpha = x_o + h \\ h = \alpha - x_o\end{matrix}\right]$$

Let this approximation be x_1

Then $x_1 = x_o - \dfrac{f(x_o)}{f'(x_o)}$

Repeating this process, We get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

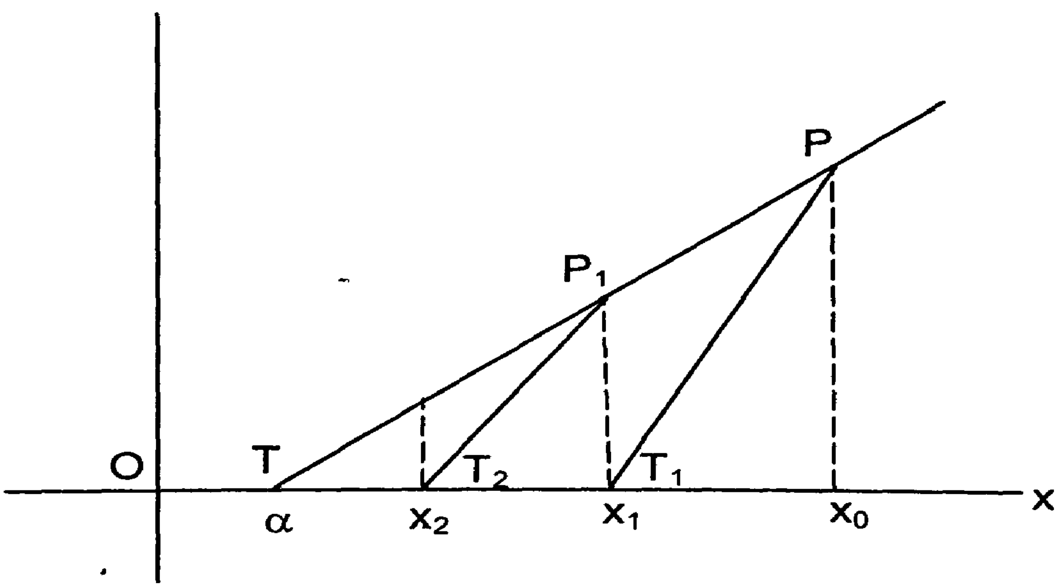
....

....

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

This formula is known as Newton's Raphson formula.

Geometrical Interpretation



Let the curve y = f(x) cuts the x – axis at $x = \alpha$.

Then α is a root of f(x) = 0

Let x_o be an approximate root of f(x) = 0

Then $x = x_o$ is a point near to $x = \alpha$.

If the ordinate at $x = x_o$, meets the curve y = f(x) at P, then the tangent PT_1 to the curve y = f(x) will in general meet the x– axis at the point x_1, which is nearer to the root α than the first approximation.

$$\frac{f(x_o)}{x_o - x_1} = f'(x_o)$$

$$\therefore\ x_o - x_1 = \frac{f(x_o)}{f'(x_o)}$$

$$\therefore\ x_1 = x_o - \frac{f(x_o)}{f'(x)}$$

Repeating this process,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Then T_2 is the point $x = x_2$ continuing the process,

We get points

$T_3(x = x_3)$, $T_4\ (x = x_4)$ .... $T_k(x = x_k)$ ....

Which will approach the point which is $x = \alpha$.

Error in Newton – Raphson Method:

By the Newton – Raphson Method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

This is really an Iteration Method, where

$$x_{k+1} = F(x_k) \; \& \; F(x_k) = x_k - \frac{f(x_k)}{f'(x_k)}$$

Hence the equation is of the form

$$x = F(x) \text{ where } F(x) = x - \frac{f(x)}{f'(x_k)}$$

$x_0, x_1, x_2 \ldots\ldots x_{k+1}$ converges to the true root

$$\text{if } |F'(x)| < 1$$

$$\text{(ie) } \left| 1 - \frac{[f'(x)]^2 - f(x) f''(x)}{[f'(x)]^2} \right| < 1$$

$$\left[F(x) = x - \frac{f(x)}{f'(x)} \Rightarrow F'(x) = 1 - \frac{f'(x) . f^1(x) - f(x) f''(x)}{[f'(x)^2]} \right]$$

$$\left| \frac{[f'(x)]^2 - [f'(x)]^2 + f(x) f''(x)}{[f'(x)]^2} \right| < 1$$

$$\left| \frac{f''(x) f(x)}{[f'(x)]^2} \right| < 1$$

$$\text{(ie) } |f(x) f''(x)| < |f'(x)|^2$$

This is the desired creteriam for convergence.

Order of Convergence of the error

Let $E_k = x_k - \alpha$ wher E_k is the error in the K^{th} stage.

If the sequence $\{x_k\}$ converges to α. Then the sequence $\{E_k\}$ converges to O.

If the error E_k is related to the error E_{k+1}.

(ie) $x_{k+1} - \alpha$ by the formula

$|E_{k+1}| \leq \lambda |E_k|^P$ where λ,p are constants and $\lambda > 0$, $P \gtrsim 0$

then we say that the convergence is of order p.

If P = 1, the convergence is said to be linear.

If P = 2, the convergence is said to be quadratic and so on.

Obviously the convergence is faster if p is larger & λ is smaller.

Case i) Let α is a root of the equation f(x) = 0 of order one.

If the equation is expressed in the form

$$X = F(x), \text{ then } x_{k+1} = F(x_k)$$

But $F(x_k) = F(\alpha + E_k)$ where E_k is the error in the k^{th} stage. $(\because E_k = x_k - \alpha)$

$$\therefore x_{k+1} = F(\alpha + E_k)$$

$$= F(\alpha) + \frac{E_k}{1!}F'(\alpha) + \frac{(E_k)^2}{2!}F''(\alpha) + \ldots\ldots \text{ By Taylor's Theorem.}$$

$$= \alpha + \frac{E_k}{1!}F'(\alpha) + \frac{(E_k)^2}{2!}F''(\alpha) + \ldots\ldots \quad [\because x\,F(x)]\alpha \quad = F(\alpha)$$

$$x_{k+1} - \alpha = \frac{E_k}{1!}F'(\alpha) + \frac{(E_k)^2}{2!}F''(\alpha) + \ldots\ldots$$

$$\text{(ie) } E_{k+1} = \frac{E_k}{1!}F'(\alpha) + \frac{(E_k)^2}{2!}F''(\alpha) + \ldots\ldots$$

In the Newton – Raphson formula

$$F(x) = x - \frac{f(x)}{f'(x)}$$

$$F(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2}$$

$$F'(x) = \frac{[f^1(x)]^2 - [f'(x)]^2 + f(x)f''(x)}{[f^1(x)]^2}$$

$$F'(x) = \frac{f(x)\, f''(x)}{[f'(x)]^2}$$

$$\therefore\ F'(\alpha) = \frac{f(\alpha)\, f''(\alpha)}{[f'(\alpha)]^2} = 0 \text{ since } f(\alpha) = 0$$

$$\text{Again } F''(x) = \frac{[f'(x)]^2\, [f(x)\, f'''(x) + f'(x)\, f''(x)] - f(x)\, f''(x)\, 2f'(x)\, f''(x)}{[f'(x)]^4}$$

$$\therefore F''(\alpha) = \frac{[f'(\alpha)]^2\, [f(\alpha)\, f''(\alpha) + f''(\alpha)\, f''(\alpha)] - f(\alpha)\, f''(\alpha)\, 2f'(\alpha)\, f''(\alpha)}{[f'(\alpha)]^4}$$

$$\therefore\ F''(\alpha) = \frac{[f'(\alpha)]^3\, f''(\alpha)}{[f'(\alpha)]^4} = \frac{f''(\alpha)}{f^2(\alpha)}$$

Hence $E_{k+1} = \dfrac{(E_k)^2}{2!}\dfrac{f''(\alpha)}{f^1(\alpha)}$ Omitting higher powers of E_k.

$\therefore$ The convergence is quadratic.

Case ii) If α is a multiple root of order p of f(x) = 0, then

$$f(x) = (x - \alpha)^P Q(x) \text{ where } Q(\alpha) \neq 0$$

$$f'(x) = P(x - \alpha)^{p-1} Q(x) + (x - \alpha)^P Q'(x)$$

$$\frac{f(x)}{f'(x)} = \frac{(x-\alpha)^P Q(x)}{P(x-\alpha)^{P-1} Q(x) + (x-\alpha)^P Q'(x)}$$

$$= \frac{(x-\alpha)^P Q(x)}{(x-\alpha)^P [P(x-\alpha)^{-1} + Q'(x)]}$$

$$= \frac{Q(x)(x-\alpha)}{PQ(x) + (x-\alpha)Q'(x)}$$

Let us take $F(x) = x - \dfrac{\lambda f(x)}{f'(x)}$

$$= x - \frac{\lambda (x-\alpha) Q(x)}{PQ(x) + (x-\alpha)Q'(x)}$$

Finding F' (x) & than substituting α for x, we get

$$F'(x) = 1 - \frac{[PQ(x) + (x-\alpha)Q'(x)][\lambda Q(x) + \lambda(x-\alpha)Q'(x)] + [\lambda(x-\alpha)Q(x)][PQ'(x) + Q'(x) + (x-\alpha)Q''(x)]}{[PQ(x) + (x-\alpha)Q'(x)]^2}$$

$$F'(\alpha) = 1 - \frac{PQ(\alpha)\,\lambda Q(x)}{[PQ(\alpha)]^2}$$

$$F'(\alpha) = 1 - \frac{P\lambda\, Q^2(\alpha)}{P^2\, Q^2(\alpha)} = 1 - \frac{\lambda}{p}$$

As proved in case (i)

$$E_{K+1} = F'(\alpha)E_k + F''(\alpha)\frac{(E_K)^2}{2!} + \ldots\ldots \text{ where } E_k = x_K - \alpha.$$

When $\lambda = 1$.

We get the Newton – Raphson formula.

In that case

$F'(\alpha) = 1 - \frac{1}{p}$ and $E_{k+1} = \left(1 - \frac{1}{p}\right)_{E_K}$ omitting higher powers of E_k.

$\therefore$ The convergence is linear.

Hence if f(x) = 0 has a multiple root, the convergence of the errors is linear. On the otherland if $\lambda = P$

$F'(\alpha) = 0$

Hence $E_{k+1} = F''(\alpha)\frac{(E_k)^2}{2}$ neglecting higher order terms.

This shows that the convergence is quadratic.

Hence in the case of multiple root of order λ of an equation, it is better to take the modified formula of Newton – Raphson.

$F(X) = x - \frac{\lambda f(x)}{f'(x)}$ and tabulate the iteration.

Example: 1

Find the positive root of $x^3 - x - 1 = 0$ correct to 4 places of decimals.

Solution:

Let $f(x) = x^3 - x - 1$

$f'(x) = 3x^2 - 1$

Take x = 1, 2 ....

$f(1) = (1)^3 - 1 - 1 \quad = 1 - 1 - 1 = -1$

$f(2) = (2)^3 - 1 - 1 \quad = 8 - 2 - 1 = 5$

$\therefore$ hence a root lies between 1 and 2.

$$x_0 = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

$f(1.5) = (1.5)^3 - 1.5 - 1 = 3.375 - 1.5 - 1$

$= 0.875$

$\therefore$ Hence the root lies between 1 and 1.5 x = 1.3

$f(1.3) = (1.3)^3 - 1.3 - 1 = 2.197 - 1.3 - 1$

$= -0.103$

$f(1.4) = (1.4)^3 - 1.4 - 1 = 2.744 - 1.4 - 1$

$= 0.344$

Hence the first approximation to the root 1.3 & 1.4

Newton Raphson Formula:

$$X_{K+1} = x_K - \frac{f(x_k)}{f'(x_k)}$$

$$\therefore \quad x_{K+1} = x_K - \frac{x_K^3 - x_K - 1}{3x_k^2 - 1}$$

$$= \frac{x_k\,(3x_k^2 - 1) - x_k^3 + x_k + 1}{3x_k^2 - 1}$$

$$= \frac{3x_k^3 - x_k - x_k^3\, x_k - 1}{3x_k^2 - 1}$$

$$x_{k+1} = \frac{2x_k^3 + 1}{3x_k^2 - 1}$$

$k = 0$, $x_0 = 1.3$

$$\therefore\ x_1 = \frac{2x_0^3 + 1}{3x_0^2 - 1} = \frac{2(1.3)^3 + 1}{3(1.3)^2 - 1} = \frac{2(2.197) + 1}{3(1.69) - 1}$$

$$= 5.394/4.07 = 1.3253$$

$k = 1$, $x_1 = 1.3253$

$$x_2 = \frac{2x_1^3 + 1}{3x_1^2 - 1} = \frac{2(1.3253)^3 + 1}{3(1.3253)^2 - 1} = \frac{2(2.32778) + 1}{3(1.7564) - 1}$$

$$= \frac{5.65556}{4.26926} = 1.3247$$

$k = 2$, $x_2 = 1.3247$,

$$x_3 = \frac{2x_2^3 + 1}{3x_2^3 - 1} = \frac{2(1.3247)^3 + 1}{3(1.3247)^2 - 1} = \frac{(2.3246)2 + 1}{3(1.7548) - 1}$$

$$= \frac{5.6492}{4.26449} = 1.3247$$

since $x_2 = x_3$

The root correct to 4 places of decimals 1.3247

The varies approximations converge to the true value if $|f(x)\,f''(x)| < [f'(x)]^2$

This can be verified by putting x = 1.3 in the formula. $f''(x) = 3x$

$$|(-0.10\,3)(3.9)| < [14.21]^2$$

Example: 2

Find the positive root of the equation x = cos x. Correct to 4 places of decimals.

Solution:

Let f(x) = x – Cos x

x = 0, f(0) = 0 – Cos 0 = – 1

x = 1, f(1) = 1 – Cos 1 = 1 – ·5403 = ·4597

Hence the root lies between o and 1

$$x_0 = \frac{0+1}{2} = \frac{1}{2} = 0.5$$

Newton – Raphson Formula:

$$X_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Here f(x) = x – Cos x

f′ (x) = 1 + Sin x

$$x_{k+1} = x_k - \frac{x_k - \cos x_k}{1+\sin x_k}$$

$$= \frac{x_k(1+\sin x_k) - x_k + \cos x_k}{1+\sin x_k}$$

$$x_{k+1} = \frac{x_k \sin x_k + \cos x_k}{1+\sin x_k}$$

k = 0, $x_0 = 0.5$

$$\text{Hence } x_1 = \frac{0.5\sin(0.5)+\cos(0.5)}{1+\sin(0.5)}$$

$$= \frac{0.5\,(47943)+0.87758}{1+0.47943}$$

$$= \frac{0.239172+0.87758}{1.47943} = 0.7552$$

$x_1 = 0.7552, \quad k = 1$

$$x_2 = \frac{0.7552 \sin(0.7552) + \cos(0.7552)}{1 + \sin(0.7552)}$$

$$= \frac{0.7552\,(0.68543) + 0.72813}{1 + 0.68543}$$

$$= \frac{0.51764 + 0.72813}{1.68543} = 0.7391$$

$x_2 = 0.7391, \quad k = 2$

$$x_3 = \frac{0.7391 \sin(0.7391) + \cos(0.7391)}{1 + \sin(0.7391)}$$

$$= \frac{0.7391\,(0.67362) + 0.73908}{1 + 0.67362} = \frac{0.49787 + 0.73908}{1.67362}$$

$x_3 = 0.7391, \quad \therefore x_2 = x_3.$

Further continuation is not necessary.

Hence the root is 0.7391 correct to 4 places of decimals.

Example: 3

The equation $4x = e^x$ has two roots one near 0.3 and the other near 2.1. Find them correct to 4 places of decimals.

Solution:

Let $f(x) = 4x - e^x$; $\quad f'(x) = 4 - e^x$

Given $x_0 = 0.3$

Newton' Raphson Formula is

$$X_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)_{xk}}$$

$$= x_k - \frac{4x_k - e^{x_k}}{4 - e^{x_k}}$$

$$= \frac{x_k(4 - e^{x_k}) - 4x_k + e^{x_k}}{4 - e^{x_k}}$$

$$= \frac{4x_k - x_k e^{x_k} - 4x_k + e^{x_k}}{4 - e^{x_k}}$$

$$x_{k+1} = \frac{e^{x_k} - x_k\, e^{x_k}}{4 - e^{xk}} = \frac{e^{x_k}\,(1 - x_k)}{4 - e^{x_k}}$$

$k = 0,\; x_0 = 0.3$

$$\therefore x_1 = \frac{e^{0.3}\,(1 - 0.3)}{4 - e^{0.3}} = \frac{1.349859\,(0.7)}{4 - 1.349859} = \frac{0.9449012}{2.650141}$$

$$= 0.35655$$

$k = 1,\; x_1 = 0.\ 35655$

$$\therefore x_2 = \frac{e^{0.35655}\,(1 - 0.35655)}{4 - e^{0.35655}} = \frac{1.4284\,(0.64345)}{4 - 1.4284}$$

$$= \frac{0.9191}{2.5716} = 0.3574$$

$k = 2,\; x_2 = 0.3574$

$$x_3 = \frac{e^{0.3574}\,(1 - 0.3574)}{4 - e^{0.3574}} = \frac{1.42961\,(0.6426)}{2.57039}$$

$$= 0.3574$$

$$\therefore x_2 = x_3$$

Hence the root near 0.3 is 0.3574

Given $x_0 = 2.1$

$$x_1 = \frac{e^{2.1}\,(1 - 2.1)}{4 - e^{2.1}} = \frac{8.1662\,(-1.1)}{-4.1662}$$

$$x_1 = \frac{-8.98282}{-4.1662} = 2.1562$$

$k = 1,\ x_1 = 2.1562$

$$x_2 = \frac{e^{2.1562}(1-2.1562)}{4-e^{2.1562}} = \frac{8.6375(-1.1562)}{-4.6375}$$

$x_2 = 2.1535$

$k = 2,\ x_2 = 2.1535$

$$x_3 = \frac{e^{2.1535}(1-2.1535)}{4-e^{2.1535}} = \frac{8.61496(-1.1535)}{-4.61496}$$

$x_3 = 2.1533$

$k = 3,\ x_3 = 2.1533$

$$x_4 = \frac{e^{2.1533}(1-2.1533)}{4-e^{2.1535}} = \frac{8.6132(-1.1533)}{-4.6132}$$

$x_4 = 2.1533,\quad x_3 = x_4$

Hence the root near 2.1 it is 2.1533

Example: 4

Find the double root of $x^3 - 5.4x^2 + 9.24x - 5.096 = 0$. Correct to first decimal place given that it is near 1.5.

Solution:

Let $f(x) = x^3 - 5.4x^2 + 9.24x - 5.096$

$f'(x) = 3x^2 - 10.8x + 9.24$

Formula:

$$F(x) = x - \frac{\lambda f(x)}{f'(x)}$$

$$\therefore F(x) = x - \frac{2f(x)}{f'(x)}$$

$$= x = \frac{2(x^3 - 5.4x^2 + 9.24x - 5.096)}{3x^2 - 10.8x + 9.24}$$

$$= \frac{3x^3 - 10.8x^2 + 9.24x - 2x^3 + 10.8x^2 - 18.48x + 10.192}{3x^2 - 10.8x + 9.24}$$

$$F(x) = \frac{x^3 - 9.24x + 10.192}{3x^2 - 10.8x + 9.24}$$

$$\therefore x_{k+1} = \frac{x_k^3 - 9.24x_k + 10.192}{3x_k^2 - 10.8x_k + 9.24}$$

Hence k = 0, x_0 = 1.5

$$x_1 = \frac{(1.5)^3 - 9.24\,(1.5) + 10.192}{3(1.5)^2 - 10.8\,(1.5) + 9.24}$$

$$= \frac{3.375 - 13.86 + 10.192}{6.75 - 16.2 + 9.24} = \frac{-0.293}{-0.21} = 1.3952$$

k = 1, x_1 = 1.3952

$$x_2 = \frac{(1.3952)^3\ 9.24(1.3952) + 10.192}{3(1.3952)^2 - 10.8\,(1.3952) + 9.24}$$

$$= \frac{2.7159 - 12.8916 + 10.192}{5.8399 - 15.06816 + 9.24} = \frac{-0.0163}{-0.01154}$$

$$= 1.4125$$

k = 2, x_2 = 1.4125

$$x_3 = \frac{(1.4125)^3 - 9.24\,(1.4125) + 10.192}{3(1.4125)^2 - 10.8\,(1.4125) + 9.24}$$

$$= \frac{2.81803 - 13.0515 + 10.192}{5.98547 - 15.255 + 9.24} = \frac{-0.04147}{-0.02953}$$

$$x_3 = 1.4043$$

k = 3, x_3 = 1.404

$$x_4 = \frac{(1.4043)^3 - 9.24\,(1.4043) + 10.192}{3\,(1.4043)^2 - 10.8\,(1.4043) + 9.24}$$

$$= \frac{2.76936 - 12.9757 + 10.192}{5.91618 - 15.166 + 9.24} = \frac{-0.01434}{-0.00982} = 1.4603$$

Example: 5

Find an iterative formula to find $\sqrt{N}$ (where N is a positive number) and hence find $\sqrt{5}$

Solution:

Let $x = \sqrt{N}$

$x^2 - N = 0$

Let $f(x) = x^2 - N \Rightarrow f'(x) = 2x$; $\alpha_{i+1} = \alpha_i - \dfrac{\alpha_i^2 - N}{2\alpha_i}$

$$\alpha_{i+1} = \alpha_i - \frac{\alpha_i}{2} + \frac{N}{2\alpha_i} = \frac{1}{2}\left(\alpha_i + \frac{N}{\alpha_i}\right)$$

$\alpha_{i+1} = \dfrac{1}{2}\left(\alpha_i + \dfrac{N}{\alpha i}\right)$ is the iterative formula to find $\sqrt{N}$

To find $\sqrt{5}$, put N = 5

Also $x = \sqrt{5}$ lies between 2 and 3.

Take $\alpha_0 = 2$

$$\therefore \alpha_1 = \frac{1}{2}\left(\alpha_0 + \frac{5}{\alpha_0}\right) = \frac{1}{2}\left(2 + \frac{5}{2}\right) = 2.25$$

$$\alpha_2 = \frac{1}{2}\left(2.25 + \frac{5}{2.25}\right) = 2.23611111$$

$$\alpha_3 = \frac{1}{2}\left(2.23611111 + \frac{5}{2.3611111}\right) = 2.23606798$$

Similarly, $\alpha_4 = \dfrac{1}{2}\left(\alpha_3 + \dfrac{5}{\alpha_3}\right) = \dfrac{1}{2}\left(2.23606798 + \dfrac{5}{2.23606798}\right)$

α_4 2.23606798

Hence the approximate value of $\sqrt{5}$ is 2.23606798

Exercise:

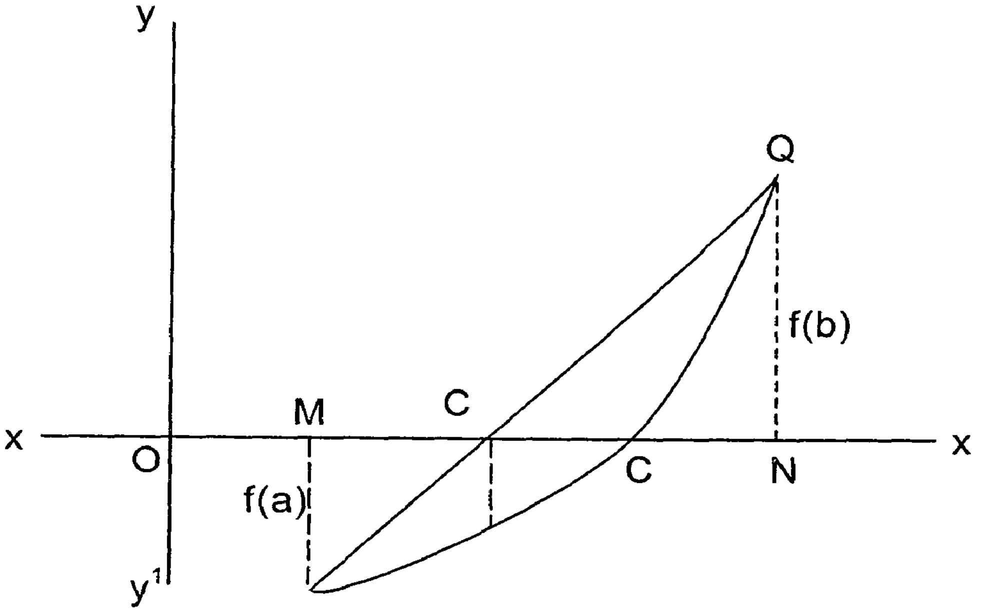
1. Find the positive root of $f(x) = 2x^3 - 3x - 6 = 0$ by Newton – Raphson method correct to five decimal places.
2. Using Newton Raphson method find the root between 0 and 1 of $x^3 = 6x - 4$ correct to 5 decimal places.
3. Find an iterative formula to find the reciprocal of a given number N and hence find the value of $\dfrac{1}{19}$.
4. Using Newton's method, find the root of the equation $x^3 - 3x - 5 = 0$
5. Find a real root of $x^3 + 2x^2 + 50x + 7 = 0$.

4.4 Method of False Position or REGULA FALSI

Suppose we have to solve the equation

$f(x) = 0$

Let f(a) and f(b) have different signs. Let M and N be respectively,. the points x= a x = b.



Let c be the point ($x = \alpha$) where the graph y = f(x) crosses the axis.

Then α is the true value of the root of the equation f(x) = 0.

Let P, Q be respectively the points whose abscissa are x = a and x = b

Let PQ intersect the x – axis at $C_1(x = x_1)$ Δ^{le} PMC_1 and c_1 NQ are similar.

$$\frac{PM}{NQ} = \frac{MC_1}{C_1N}$$

(ie) $$\frac{|f(a)|}{|f(b)|} = \frac{x_1 - a}{b - x_1}$$

(ie) $$\frac{|f(b)|}{|f(a)|} = \frac{b - x_1}{x_1 - a}$$

$$\frac{|f(b)|}{|f(a)|} + 1 = \frac{b - x_1}{x_1 - a} + 1$$

$$\frac{|f(b)| + |f(a)|}{|f(a)|} = \frac{b-a}{x_1 - a}$$

(ie) $x_1 = a + \dfrac{(b-a)|f(a)|}{|f(b)| + |f(a)|}$

Hence the first approximation to the root is x_1

where $x_1 = a + h$ & $h = \dfrac{|f(a)||b-a|}{|f(a)| + |f(b)|}$

Find the sign of $f(x_1)$ and if has a sign different from f(b) continue the process.

Then $x_2 = x_1 + h_1$ where $h_1 = \dfrac{|f(x_1)||b-x_1|}{|f(x_1)| + |f(b)|}$

By continuing this process,

We get x_3, x_4 .... which will converge to α.

This method is known as method of Regula Falsi or false position.

Example: 1

Solve the equation $x_3 - x - 1 = 0$ correct to 4 places of decimals

Solution:

Let $f(x) = x^3 - x - 1$

Take x = 1, $\Rightarrow f(1) = (-1)^3 - (-1) - 1 = -1$

x = 1.5 $\Rightarrow f(1.5) = (1.5)^3 - (1.5) - 1$

$= 0.875$

[$f(1) = -1, f(2) = 5$
$\therefore$ root 1 and 2
$x_0 = \frac{1+2}{2} = \frac{3}{2}$
$x_0 = 1.5$
$f(1.5) = 0.815$
$\therefore$ 1 & 1.5]

Hence the root lies between 1 and 1.5

Formula:

$$x_2 = a + \frac{(b-a)|f(a)|}{|f(b)| + |f(a)|}$$

$x_0 = a = 1,\quad x_1 = b = 1.5$

$$X^2 = 1 + \frac{(1.5-1)(-1)|}{|1|+10.875|} = 1 + \frac{(1.5-1)(1)}{1+0.875}$$

$= 1.2667$

$f(x_2) = (1.2667)^3 - 1.2667 - 1$

$= -0.2142$

$x_1 = 1.5,\ x_2 = 1.2667$

$$\therefore\ x_3 = 1.2667 + \frac{0.2142\,(1.5-0.2142)}{0.2142+0.875}$$

$= 1.3229$

$f\,(x_3) = (1.3229)^3 - 1.3229 - 1$

$= -0.0077$

Hence the root lies between 1.3229 & 1.5

$$x_4 = 1.3229 + \frac{0.0077\,(1.5-1.3229)}{0.0077+0.875}$$

$= 1.3245$

$f\,(x_4) = (1.3245)^3 - 1.3245 - 1$

$= -0.0009$

$x_1 = 1.5 = b,\ x_4 = 1.3245$

$$x_5 = 1.3245 + \frac{0.0009\,(1.5-1.3229)}{1.3245+0.875}$$

$= 1.3247$

$f\,(x_5) = (1.3247)^3 - 1.3247 - 1$

$= 2.32462342 - 1.3247 - 1$

$= -0.00007657978$

The root lies between 1.3247 & 1.5

$$x_6 = 1.3247 + \frac{.00007657978\,(1.5 - 1.3247)}{.00007657978 + 0.875}$$

$$x_6 = 1.3247 + \frac{.0000134242987}{0.875076579} = 1.32471534$$

Hence the root is 1.3247 correct to 4 places of decimals.

Example: 2

Find the positive root of the equation x = cos x correct to 4 places of decimals.

Solution:

Let $f(x) = x - \cos x$

Take $x = 0$

$f(0) = 0 - \cos 0 = -1$

$x = 1,\ f(1) = 1 - \cos 1 = 1 - 0.54030 = 0.4597$

$$x_0 = \frac{0+1}{2} = 0.5$$

Hence the root lies between 0.5 and 1

$f(0.5) = +(0.5) - \cos(0.5)$

$= -0.3776$

$f(1) = 0.4597$

$x_0 = a = 0.5,\ x_1 = b = 1$

$$x_2 = 0.5 + \frac{0.3776\ (1 - 0.5)}{0.3776 + 0.4597}$$

$= 0.7255$

$f(x_2) = f(0.7255) = (0.7255) - \cos(0.7255)$

$= -0.0227$

$x_2 = a = 0.7255,\ x_1 = b = 1$

$$x_3 = 0.7255 + \frac{0.0227\,(1-0.7255)}{0.0227+0.4597}$$

$= 0.738945$

$f(0.738945) = 0.738945 - \cos(0.738945)$

$= 0.000235$

The root lies between 0.738945 and 1.

$$x_4 = \frac{0.738945 \times 0.459698 + 1 \times 0.000235}{0.459698 + 0.000235}$$

$$= \frac{0.339927}{0.459933} = 0.739079$$

$\therefore$ The root is 0.7391 correct to 4 decimal places.

Example: 3

Solve for a positive root of $x^3 - 4x + 1 = 0$ by regular Falsi method.

Solution:

Let $f(x) = x^3 - 4x + 1 = 0$

$f(1) = -2 = -\text{ve},\ f(2) = 1 = +\text{ve},\ f(0) = 1 = +\text{ve}$

$\therefore$ A root lies between 0 and 1.

Another root lies between 1 and 2.

We shall find the root that lies between 0 and 1.

Here a = 0, b = 1.

$$x_1 = \frac{a\,f(a) - b\,(f)\,(a)}{f(b) - f(a)} = \frac{0 \times f(1) - 1 \times f(0)}{f(1) - f(0)} = \frac{-1}{-2-1} = 0.333333$$

$f(x_1) = f\left(\frac{1}{3}\right) = \frac{1}{27} - \frac{4}{3} + 1 = -0.2963$

Now f (0) and f $\left(\frac{1}{3}\right)$ are opposite in sign.

Hence the root lies between 0 and $\frac{1}{3}$.

$$\text{Hence } x_2 = \frac{0.f\left(\frac{1}{3}\right) - \frac{1}{3}f(0)}{f\left(\frac{1}{3}\right) - f(0)} = \frac{-\frac{1}{3}}{-1.2963} = 0.25714$$

Now $f(x_2) = f(0.25714) = -0.011558 = -ve$.

$\therefore$ The root lies between 0 and 0.25714

$$x^3 = \frac{0 \times f(0.25714) - 0.25714\, f(0)}{f(0.25714) - f(0)} = \frac{-0.25714}{-1.011558} = 0.25420$$

$f(x_3) = f(0.25420) = (0.25420)^3 - 4(0.25420) + 1 = 0.0003742$

$\therefore$ The root lies between 0 and 0.25420

$$\therefore x_4 = \frac{0 \times f(0.25420) - 0.25420 \times f(0)}{f(0.25420) - f(0)}$$

$$= \frac{-0.25420}{-1.0003742} = 0.25410$$

$f(x_4) = f(0.25410) = (0.25410)^3 - 4(0.25410) + 1$

$= -0.000012936.$

$\therefore$ The root lies between 0 and 0.25410

$$x_5 = \frac{0 \times f(0.25410) - 0.25410 \times f(0)}{f(0.25410) - f(0)}$$

$$= \frac{-0.25410}{-1.000012936} = 0.25410$$

$\therefore$ Hence root is 0.25410

Example: 4

Find an approximate root of $x \log_{10} x - 1.2 = 0$ by regula – False method

Solution:

Let $f(x) = x \log_{10} x - 1.2$

$f(1) = 1 \log 1 - 1.2 = -1.2 = -$ ve.

$f(2) = 2 \times 0.30103 - 1.2 = -0.597940$

$f(3) = 3 \times 0.47712 - 1.2 = 0.231364 = +$ve.

Hence a root lies between 2 and 3.

$$x_1 = \frac{2(f(3)) - 3f(2)}{f(3) - f(2)} = \frac{2 \times 0.23136 - 3 \times (-0.59794)}{0.23136 + 0.59794}$$

$$= 2.721014$$

$$f(x_1) = f(2.72104) = (2.721014) \log (2.721014)$$

$$= -0.017104$$

The root lies between 2.721014 & 3

$$x_2 = \frac{x_1 \text{ x } f(3) - 3 \text{ x } f(x_1)}{f(3) - f(x_1)} = \frac{2.721014 \times 0.231364 - 3 \times (-0.017104)}{0.23136 + 0.017104}$$

$$x_2 = \frac{0.68084}{0.24846} = 2.74021 \mid$$

$$f(x^2) = f(2.7402) = 2.7402 \times \log (2.7402) - 1.2$$

$$= -0.00038905$$

$\therefore$ The root lies between 2.740211 and 3.

$$x_3 = \frac{2.7402 \times f(3) - 3 \times f(2.7402)}{f(3) - f(2.7402)} = \frac{2.7402 \times 0.23136 + 3 \times (0.00038905)}{0.23136 + 0.00038905}$$

$$= \frac{0.63514}{0.23175} = 2.740627$$

$f(2.740627) = (2.740627) \log 2.740627 - 1.2$

$= 0.00011998.$

The root lies between 2.740211 & 2.740627

$$x_4 = \frac{2.7402 \times f(2.7406) - 2.7406 \times f(2.7402)}{f(2.7406) - f(2.7402)}$$

$$= \frac{2.7402 \times 0.00011998 + 2.7406 \times 0.00038905}{0.00011998 + 0.00038905}$$

$$x_4 = \frac{0.0013950}{0.00050903} = 2.7405$$

Hence the root is 2.7405.

4.5 BISECTION METHOD

To solve the equation $f(x) = 0$,

First, we find two numbers $a \times b$ such that $f(a)$ and $f(b)$ have opposite signs.

Then the root lies between a and b.

First, we shall take $\frac{a+b}{2}$ as the first approximation, let $x_1 = \frac{a+b}{2}$.

Find $f(x_1)$ and if it is not equal to zero. Then the root lies either between a and x_1 or between x_1 or between x_1 and b.

From the sign of $f(x_1)$,

We can locate the position of the root.

Suppose $f(x_1)$ and $f(a)$ have different signs then the root lies between x_1 and a.

Hence the second approximation is $\frac{x_1 + a}{2}$

Repeating the process, approximations $x_1, x_2, x_3 \ldots\ldots$ are obtained.

After k bisections, the length of the sub-interval which contains x_K is $\frac{b-a}{2^k}$

Hence $| x_K - \alpha | \leq \frac{b-a}{2^k}$

$\therefore$ As $k \to \infty$, $x_k \to \alpha$

If the error is to be made less than a small quantity (say) δ, then

$$\frac{b-a}{2^k} < \delta$$

(i.e.) $2^k > \frac{b-a}{\delta}$

Taking logarithms $k > \dfrac{\log\left(\frac{b-a}{\delta}\right)}{\log 2}$

This formula is useful in the determination of the number of Bisectors required to achieve a desired accuracy.

Example: 1

Find the positive root of the equation $x^3 - x - 1 = 0$. Correct to 4 places of decimals.

Solution:

Let $f(x) = x^3 - x - 1$

Take $x = 1 \Rightarrow f(1) = (1)^3 - 1 - 1 = -1$

$x = 2 \Rightarrow f(2) = (2)^3 - 2 - 1 = 8 - 2 - 1 = 5$

Hence the root lies between 1 and 2.

$$x_0 = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

$f(1.5) = (1.5)^3 - 1.5 - 1$

$= 0.875$

Hence the root lies between 1 and 1.5

$$x_1 = \frac{1+1.5}{2} = 1.25$$

$$f(1.25) = (1.25)^3 - 1.25 - 1 = -0.2969$$

Hence the root lies between 1.25 & 1.5

$$x_2 = \frac{1.25+1.5}{2} = \frac{2.75}{2} = 1.375$$

$$f(1.375) = (1.375)^3 - 1.375 - 1$$
$$= 0.2246$$

Hence the root lies between 1.25 and 1.375

$$x_3 = \frac{1.25+1.375}{2} = 1.3125$$

$$f(1.3125) = (1.3125)^3 - 1.3125 - 1$$

$$= -0.515$$

Hence the root lies between 1.3125 and 1.375

$$x_4 = \frac{1.3125+1.375}{2} = \frac{2.6875}{2} = 1.3438$$

$$f\ (1.3438) = (1.3438)^3 - 1.3438 - 1$$

$$= \quad 0.1838$$

Hence the root lies between 1.3125 and 1.3438

$$x_5 = \frac{1.3125+1.3438}{2} = \frac{2.6564}{2} = 1.3282$$

$$f\ (1.3287) = (1.3287)^3 - (1.3287) - 1$$

$$= 0.0170$$

Hence the root lies between 1.3125 & 1.3287

$$x_6 = \frac{1.3125+1.3282}{2} = \frac{2.6412}{2} = 1.3204$$

$$f\ (1.3204) = (1.3206)^3 - 1.3206 - 1 = -0.018340 - 1.3206 - 1$$
$$= 1.3206$$

$f(4) = 4^3 - 9(4) + 1 = 64 - 36 + 1 = 28$

$\therefore$ A root lies between 2 & 4.

Let $x_0 = \frac{2+4}{2} = 3$

Now, $f(3) = 3^3 - 9(3) + 1 = 27 - 27 + 1 = 1$

Hence the root lies between 2 and 3.

$$x_1 = \frac{2+3}{2} = 2.5$$

$$f(2.5) = (2.5)^3 - 9(2.5) + 1$$

$$= -\text{ve}.$$

The root lies between 2.5 and 3.

$$x_2 = \frac{2.5+3}{2} = 2.75$$

$$f(2.75) = (2.75)^3 - 9(2.75) + 1$$

$$= -\text{ve}.$$

The root lies between 2.75 & 3

$$x_3 = \frac{1}{2}(2.75 + 3) = \frac{5.75}{2} = 2.875$$

$$f(x_3) = f(2.875) = (2.875)^3 - 9(2.875) + 1$$

$$= -\text{ve}.$$

$\therefore$ The root lies between 2.875 and 3

$$x_4 = \frac{1}{2}(2.875 + 3) = \frac{5.875}{2} = 2.9375$$

$$f(2.9375) = (2.9375)^3 - 9(2.9375) + 1$$

$$= -\text{ve}.$$

The roots lies between 2.9375 and 3

$$x_5 = \frac{1}{2}(2.9375 + 3) = \frac{5.9375}{2} = 2.9688$$

f (2.9688) = $(2.9688)^3 - 9(2.9688) + 1$

= + ve.

The roots lies between 2.9688 and 2.9375

$$x_6 = \frac{1}{2}(2.9375 + 2.9688) = \frac{5.9063}{2} = 2.9532$$

f (2.9532) = $(2.9532)^3 - 9(2.9532) + 1$

= + ve.

Hence the root lies between 1.3206 and 1.3287

$$\therefore\ x_7 = \frac{1.3204 + 1.3282}{2} = \frac{2.6486}{2} = 1.3243$$

f (1.3243) = $(1.3243)^3 - 1.3243 + 1$

= – ve.

Hence the root 1.3243 & 1.3282

$$\therefore\ x_8 = \frac{1.3243 + 1.3282}{2} = \frac{2.6525}{2} = 1.3263$$

= + ve.

f (1.3263) = $(1.3263)^3 - 1.3263 + 1$

= + ve.

$$x_9 = \tfrac{1}{2}(1.3243 + 1.3263) = 1.3253)$$

f (1.3253) = + ve.

The root lies between 1.3243 & 1.3253

$$x_{10} = \frac{1}{2}\,(1.3243 + 1.3263) = \frac{2.6496}{2} = 1.3248$$

$f\,(1.3248) = (1.3248)^3 - 1.3248 + 1$

$= +\,ve$

The root lies between 1.3243 & 1.3248

$$x_{11} = \frac{1}{2}\,(1.3243 + 1.3248) = \frac{2.6491}{2} = 1.32455$$

$f\,(1.32455) = (1.32455)^3 - 1.32455 + 1$

$= -\,ve.$

The root lies between 1.3248 & 1.32455

$$x_{12} = \frac{1}{2}\,(1.3248 + 1.32455) = \frac{2.6503}{2} = 1.3247$$

$f\,(1.3247) = (1.3247)^3 - 1.3247 + 1$

$= -\,ve.$

The root lies between 1.3247 & 1.3248

$$\therefore\ x_{13} = \frac{1}{2}\,(1.3247 + 1.3248) = \frac{2.6495}{2} = 1.32475$$

$\therefore$ The approximate root is 1.32475

Example: 2

Assuming that a root of $x^3 - 9x + 1 = 0$ lies in the interval (2,4), find that root by bisection method.

Solution:

Let $f\,(x) = x^3 - 9\,x + 1$

$f\,(2) = 2^3 - 9\,(2) + 1 = 8 - 18 + 1 = -11$

The root lies between 2.9735 and 2.9532

$$\therefore\ x_7 = \frac{2.9375 + 2.9532}{2} = \frac{5.8907}{2} = 2.9454$$

f (2.9454) = $(2.9454)^3 - 9\ (2.9454) + 1$

= + ve.

The root lies between 2.9375 and 2.9454

$$x_8 = \frac{2.9375 + 2.9454}{2} = \frac{5.8829}{2} = 2.9415$$

f (2.9415) = $(2.9415)^3 - 9\ (2.9415) + 1$

= – ve.

The root lies between 2.9415 and 2.9454

$$x_9 = \frac{2.9415 + 2.9454}{2} = \frac{5.8869}{2} = 2.9435$$

f (2.9435) = $(2.9435)^3 - 9\ (2.9435) + 1$

= + ve.

The root lies between 2.9415 & 2.9435

$$x_{10} = \frac{2.9415 + 2.9435}{2} = \frac{5.8850}{2} = 2.9425$$

f (2.9425) = $(2.9425)^3 - 9\ (2.9425) + 1$

= – ve.

The root lies between 2.9425 & 2.9435

$$x_{11} = \frac{2.9425 + 2.9435}{2} = \frac{5.8860}{2} = 2.9430$$

f (2.9430) = $(2.9430)^3 - 9\ (2.9430) + 1$

= + ve.

The root lies between 2.9425 & 2.9430

$$x_{12} = \frac{2.9425 + 2.9430}{2} = \frac{5.8855}{2} = 2.94275$$

$f(2.94275) = (2.94275)^3 - 9(2.94275) + 1$

$= 25.48356042 - 26.48475 + 1$

$= .00118958$

Approximate root is 2.9429

Example: 3

Find the positive root of $x - \cos x = 0$ by bisection method.

Solution:

Let $f(x) = x - \cos x$

$f(0) = 0 - \cos 0 = -1$

$f(0.5) = 0.5 - \cos(0.5)$

$f(0.5) = -0.37758$

$f(1) = 1 - \cos 1$

$f(1) = 0.45970$

Hence, the root lies between 0.5 and 1.

$$x_0 = \frac{0.5 + 1}{2} = 0.75$$

$f(0.75) = 0.75 - \cos(0.75)$

$f(0.75) = 0.018311$

The root lies between 0.5 and 0.75

$$x_1 = \frac{0.5 + 0.75}{2} = 0.625$$

$f(0.625) = 0.625 - \cos(0.625)$

$= 0.18596$

$\therefore$ The root lies between 0.625 and 0.750

$$x_2 = \frac{1}{2}(0.625 + 0.750) = \frac{1.375}{2} = 0.6875$$

$$f(0.6875) = 0.6875 - \cos(0.6875)$$

$$= -\text{ ve.}$$

The root lies between 0.6875 and 0.75

$$x_3 = \frac{0.6875 + 0.75}{2} = \frac{1.4375}{2} = 0.71875$$

$$f(0.71875) = 0.71875 - \cos(0.71875)$$

$$= -0.033879$$

The root lies between 0.71875 and 0.75

$$x_4 = \frac{0.71875 + 0.75}{2} = \frac{1.46875}{2} = 0.73438$$

$$\therefore \quad f(0.73438) = 0.734338 - \cos(0.73438)$$
$$= -0.0078664 = -\text{ ve.}$$

$\therefore$ The root lies between 0.73438 and 0.75

$$x_5 = \frac{0.73438 + 0.75}{2} = \frac{1.48438}{2} = 0.742190$$

$$f(0.742190) = 0.742190 - \cos(0.742190)$$
$$= 0.0051999$$

$$= +\text{ ve.}$$

The root lies between 0.73438 & 0.742190

$$x_6 = \frac{1}{2}(0.73438 + 0.742190)$$
$$= 0.73829$$

$$f(0.73829) = 0.73829 - \cos(0.73829)$$

$= -0.0013305$

The root lies between 0.73829 and 0.74219

$$x_7 = \frac{1}{2}(0.73829 + 0.74219) = 0.7402$$

$$f(0.7402) = 0.7402 - \cos(0.7402)$$
$$= 0.0018663$$

The root lies between 0.73829 & 0.7402

$$x_8 = \frac{0.73829 + 0.7402}{2} = = 0.73925$$

$f(0.73925) = 0.00027593$

The roots lies between 0.73829 & 0.73925

$$x_9 = \frac{0.73829 + 0.73925}{2}$$
$$= 0.7388$$

$\therefore x_8 = x_9$

$\therefore$ The root is 0.739.

Example:

Using bisection method, find the negative root of $x_3 - 4x + 9 = 0$ by bisection method.

Solution:

Let $f(x) = x_3 - 4x + 9$

$$f(-x) = x^3 + 4x + 9$$

The negative root of $f(x) = 0$ is the positive root of $f(-x) = 0$.

We will find, the positive root of $f(-x) = 0$

(i.e.) $\phi(x) = x^3 - 4x + 9 = 0$

ϕ (2) = – ve and ϕ (3) = + ve

∴ The root lies between 2 and 3

Hence $x_0 = \frac{2+3}{5} = 2.5$

ϕ (2.5) = $(2.5)^3$ – 4 (2.5) – 9
= – ve

The root lies between 2.5 and 3

Hence $x_1 = \frac{1}{2}$ (2.5 + 3) = 2.75

ϕ (2.75) = $(2.75)^3$ – 4 (2.75) + 9
= + ve.

The root lies between 2.5 & 2.75

$x_2 = \frac{1}{2}$ (2.5 + 2.75)
= 2.625

ϕ (2.625) = $(2.625)^3$ – 4 (2.625) – 9

= 1.4121 = – ve.

The root lies between 2.625 and 2.75

$x_3 = \frac{1}{2}$ (2.625 + 2.75) = 2.6875

ϕ (2.6875) = $(2.6875)^3$ – 4 (2.6875)

= – ve.

The root lies between 2.6875 & 2.71875

$x_5 = \frac{1}{2}$ (2.6875 + 2.71875)
= 2.703125

ϕ (2.703125) = $(2.703125)^3 - 4\,(2.703125) - 9$

= – ve.

$\therefore$ The root lies between 2.703125 & 2.71875

$$x_6 = \frac{1}{2}\,(2.703125 + 2.71875)$$

= 2.710938

ϕ (2.703125 = $(2.703125)^3 - 4\,(2.703125) - 9$

= 19.92318445 – 10.8437529
= 0.07943245

The root lies between 2.703125 and 2.710938

$$x_7 = \frac{2.703125 + 2.710938}{2} = 2.7070315$$

ϕ (2.7070315) = $(2.7070315)^3 - 4\,(2.7070315) - 9$

= 19.83717973 – 10.828126 – 9
= 0.00905373.
The root lies between 2.703125 and 2.7070315

$$x_8 = \frac{2.703125 + 2.7070315}{2} = 2.70507825$$

ϕ (2.70507825) = $(2.70507825)^3 - (2.70507825) - 9$

= 19.79427035 – 10.820313 + 9

= – 0.02604265

The root lies between 2.70507825 & 2.7070315

$$x_9 = \frac{2.70507825 + 2.7070315}{2} = 2.706054875$$

ϕ (2.706054875) = $(2.706054875)^3 - 4\ (2.706054875) - 9$

$= 19.8157173 - 10.8242195 - 9 = .0085022$

The root lies between 2.7070315 and 2.706054875

$$x_{10} = \frac{2.707031 + 2.706054}{2} = 2.70654$$

ϕ (2.70654) = $(2.70654)^3 - 4\ (2.70654) - 9$

$= 19.82637653 - 10.826169$

$= .00021653$

The root lies between 2.706054 and 2.70654

$$x_{10} = \frac{2.706054 + 2.70654}{2} = 2.706297$$

ϕ (2.706297) = $(2.706297)^3 - 4\ (2.706297) - 9$

$= 19.82103682 - 10.825188 - 9 = -\ .00415118$

The root lies between 2.706297 and 2.70654

$$x_{12} = \frac{2.706297 + 2.70654}{2} = 2.7064185$$

ϕ (2.7064185) = $(2.7064185)^3 - 4\ (2.70654) - 9$

$= 19.82370656 - 10.82616 - 9$

$= -\ .00245344$

The root lies between 2.7064185 and 2.70654

$$x_{13} = \frac{2.7064185 + 2.70654}{2} = 2.70647925$$

$\phi\ (2.70647925) = (2.70647925)^3 - 4\ (2.70647925 - 9)$

$$= 19.82504151 - 10.825917 - 9$$

$$= .0008755$$

The root lies between 2.70647925 & 2.70654

$$x_{14} = \frac{2.70647925 + 2.70654}{2} = 2.706509625$$

$\therefore$ Hence the negative root of the given equation is $-$ 2.7065

Exercise:

1. Find a positive root of the following equation by bisection method.
   i) $e^x = 3x$
   ii) $x^3 + x^2 - 1 = 0$
   iii) $3x = \sqrt{1 + \sin x}$
   iv) $x^3 + 3x - 1 = 0$

UNIT – V

SOLUTIONS OF SIMULTANEOUS LINEAR EQUATIONS

5.1 Let the system of n simultaneous linear equations be

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

This system of equations can be put in the form

A X= B (2) where A is matrix

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdot & \ldots & \ldots & \ldots \\ \cdot & \ldots & \ldots & \ldots \\ \cdot & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

and X is the column matrix $\{x_1\ x_2 \ldots x_n\}$
and B is the column matrix $\{b_1\ b_2 \ldots b_n\}$

If A is a non–singular matrix, its inverse A^{-1} can be found.

Multiplying equation (2) by A^{-1}

We get $A^{-1}AX = A^{-1}B$
(i.e.) $X = A^{-1}B$

$A^{-1}B$ is a column matrix containing n elements. Equating the elements of A with that of $A^{-1}B$. We get the solution of the system.

Another method of solving the system of equations (1) of article 1 is by that is known as Cramer's Rule.

$$x_k = \frac{1}{|A|} \begin{vmatrix} a_{11} & a_{12} & \ldots\ldots & b_1 & \ldots\ldots & a_{1n} \\ a_{11} & a_{12} & \ldots\ldots & b_1 & \ldots\ldots & a_{1n} \\ \ldots & \ldots & & \ldots & & \ldots \\ \ldots & & & \ldots & & \ldots \\ \ldots & & & \ldots & & \ldots \\ a_{n1} & a_{n2} & & b_n & & a_{nn} \end{vmatrix}$$

where the k_{th} column of $|A|$ is *replaced* by the column $\{b_1, b_2, \ldots\ldots b_n\}$

Thus

$$x_1 = \frac{1}{|A|} = \begin{vmatrix} b_1 & a_{12} & \ldots\ldots & a_{1n} \\ b_2 & a_{12} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & & \ldots \\ \ldots & & & \ldots \\ b_n & a_{n2} & & a_{nn} \end{vmatrix}$$

$$x_2 = \frac{1}{|A|} = \begin{vmatrix} a_{11} & b_1 & a_{13} & \ldots\ldots & a_{1n} \\ a_{21} & b_2 & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & & & \ldots \\ \ldots & & & & \ldots \\ a_{n1} & b_n & a_{n2} & & a_{nn} \end{vmatrix}$$ and

so on.

In practice these methods are useful when the number of equations in the system is at the most four.

For larger number of equations is practice these methods are laborious. We shall give below some methods which are useful solutions of linear simultaneous equations with larger number of equations.

5.2 GAUSS' METHOD

Let the system of n simultaneous linear equations be

$$\left.\begin{aligned} &a_{11} x_1 + a_{12} x_2 + \ldots\ldots + a_{1n} x_n = b_1 \quad (1)\\ &a_{21} x_1 + a_{22} x_2 + \ldots\ldots + a_{2n} x_n = b_2 \quad (2)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &a_{n1} x_1 + a_{n2} x_2 + \ldots\ldots\ a_{nn} x_n = b_n \quad (5.) \end{aligned}\right\}$$

Retain the first equation as it is Multiply equation (1) by + $\frac{a_{21}}{a_{11}}$ and subtract its from equation (2)

$$a_{11} \times \frac{a_{21}}{a_{11}} x_1 + \frac{a_{21}}{a_{11}} \times a_{12} x_2 + \ldots\ldots + a_{1n} \times \frac{a_{21}}{a_{11}} x_n = b_1 \times \frac{a_{21}}{a_{11}} \quad (A)$$

Subtract (A) from (2)

$$(a_{21} - a_{21})x_1 + \left(\left(\frac{a_{21}}{a_{11}} \times a_{12}\right) - a_{22}\right) x_2 + \ldots\ldots + \left(a_{1n} \times \frac{a_{21}}{a_{11}} - a_{2n}\right) = b \times \frac{a_{21}}{a_{11}} - b_2$$

$$0.x_1 + b_{22} x_2 + b_{23} x_3 + \ldots\ldots + b_{2n} x_n = C_1$$

where $b_{22} = \frac{a_{21}}{a_{11}} \times a_{12} - a_{22}, \ldots\ldots\ b_{2n} = a_{1n} \times \frac{a_{21}}{a_{11}} - a_{2n}$

$$C_1 = b_1 \times \frac{a_{21}}{a_{11}} - b_2$$

Then the equation (2) will reduce to an equation with no x_1 term.

Similarly reduce equation (3), (4) ..... (n) with no x_1 term.

These equation will reduce to

$$a_{11}\, x_1 + a_{12}\, x_2 + \ldots\ldots + a_{1n}\, x_n = b_1 \qquad (a_1)$$

$$0 + b_{22}\, x_2 + \ldots\ldots\ldots + b_{2n}\, x_n = C_1 \qquad (a_2)$$

$$0 + b_{32}\, x_2 + \ldots\ldots\ldots + b_{3n}\, x_n = C_2 \qquad (a_3)$$

. . . .

. . . .

$$0 + b_{n2}\, x_2 + \ldots\ldots\ldots b_{nn}\, x_n = \ell_1 \qquad (a_n)$$

Retaining the first two equations and Eliminating x_2 from the remaining equations, Multiply equation a_2 by $\frac{b_{32}}{b_{22}}$ and substract it from equation a_3.

$$0 + b_{22} \times \frac{b_{32}}{b_{22}}\, x_2 + b_{23} \times \frac{b_{32}}{b_{22}}\, x_3 + \ldots\ldots + b_{2n} \times \frac{b_{32}}{b_{22}} = C_1 \times \frac{b_{32}}{b_{22}} \qquad (B)$$

Substract (B) from (a_3)

$$(b_{32} - b_{32})\, x_2 + \left(b_{23} \times \frac{b_{32}}{b_{22}} - b_{33}\right) x_3 + \ldots\ldots + \left(b_{2n} \times \frac{b_{32}}{b_{22}} - b_{3n}\right) x_n = \left(C_1 \times \frac{b_{32}}{b_{22}} - c_2\right)$$

$$0.\, x_2 + C_{33}\, x_3 + \ldots\ldots\ldots + C_{3n}\, x_n = d_1$$

where $C_{33} = b_{23} \times \frac{b_{32}}{b_{22}} - b_{33}, \ldots\ldots C_{3n} = b_{2n} \times \frac{b_{32}}{b_{22}} - b_{3n}$

$$d_1 = C_1 \times \frac{b_{32}}{b_{22}} - C_2$$

Similarly

$$0 + C_{43}\, x_3 + \ldots\ldots + C_{3n}\, x_n = d_2$$

....

....

$$0 + C_{43}\, x_3 + \ldots\ldots + C_{nn}\, x_n = e$$

These equations will reduce to

$$a_{11}\, x_1 + a_{12}\, x_2 + \ldots\ldots + a_{1n}\, x_n = b_1$$

$$0 + b_{22}\, x_2 + \ldots\ldots\ldots + b_{2n}\, x_n = C_1$$

$$0 + 0 + C_{33}\, x_3 + \ldots + C_{3n}\, x_n = d_1$$

....

....

$$0 + 0 + C_{n3}\, x_3 + \ldots + C_{nn}\, x_n = e$$

By continuing this process

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1$$

$$b_{22}x_2 + \ldots\ldots\ldots + b_{2n}x_n = C_1$$

$$C_{33}x_3 + \ldots\ldots + C_{3n}x_n = d_1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots + K_{nn}x_n = k_1$$

From the last equation x_n can be found. Substituting the value of x_n is the last but one equation.

We get x_{n-1}

Similarly by means of back substations $x_1, x_2, x_3 \ldots\ldots x_{n-2}$ can be determined.

This method is known as Gauss' Elimination Method.

Example: 1

Solve the equations

$$4x - 3y = 11$$

$$3x + 2y = 4$$

Solution:

Eliminating x from the 2nd equation by subtracting $\frac{3}{4}$ of the equation (1)

Multiply equation (1) by $\frac{3}{4}$.

$$4 \times \frac{3}{4}x - 3 \times \frac{3}{4}y = 11 \times \frac{3}{4} \qquad \text{(A)}$$

$$\left[\begin{array}{l} 2 + \frac{9}{4} \\ \frac{8+9}{4} = \frac{-17}{4} \end{array}\right]$$

Subtract (A) from (2)

$$(3x - 3x) + (2 - (-3) \times \frac{3}{4})\, y = 4 - 11 \times \frac{3}{4}$$

$$\frac{16-33}{4} = \frac{17}{4}$$

$$0 \quad + \frac{17}{4}y = -\frac{17}{4}$$

Hence the system of equations become

$$4x - 3y = 1$$

$$\frac{17}{4}\,y = -\,\frac{17}{4}$$

$$\text{Hence } y = -\,\frac{17}{4} \times \frac{4}{17}$$

$$\therefore y = -1$$

Substituting this value in equation (1)

we get

$$4x - 3(-1) = 11$$

$$4x + 3 = 11$$

$$4x = 11 - 3$$

$$4x = 8 \Rightarrow x = \frac{8}{4}$$

$$\therefore x = 2$$

Another Method,

$$\left|\begin{array}{cc|c} 4 & -3 & 11 \\ 3 & 2 & 4 \end{array}\right|$$

$$\left|\begin{array}{cc|c} 4 & -3 & 11 \\ 0 & \frac{17}{4} & -\frac{17}{4} \end{array}\right| \qquad R_2 \to R_2 - \frac{3}{4} R_1$$

Hence $y = -1$, $\quad x = \dfrac{11 + 3(-1)}{4} = 2$

Example: 2

Solve the equations.

$$3x + 4y + 5z = 18$$

$$2x - y + 8z = 13$$

$$5x - 2y + 7z = 20$$

Solution:

The working can be exhibited as follows:

$$\Rightarrow \left|\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{array}\right|$$

$$\Rightarrow \left|\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ & -\frac{11}{3} & \frac{14}{3} & 1 \\ & \frac{-26}{3} & \frac{-4}{3} & -10 \end{array}\right| \qquad \begin{array}{l} R_2 \to R_2 - \frac{2}{3} R_1 \\ R_3 \to R_3 - \frac{5}{3} R_1 \end{array}$$

$a_{21}\ R_2 \to 2 - \frac{2}{3} \times 3 = 0$

$a_{22} = -1 - \frac{2}{3} \times 4 = \frac{-11}{3}$

$a_{23} = -8 - \frac{2}{3} \times 5 = \frac{14}{10}$

$C_1 = 1$

$a_{31} = 5 - \frac{5}{3} \times 3 = 0$

$a_{32} = -2 - \frac{5}{3} \times 4 = \frac{-26}{3}$

$a_{33} = 7 - \frac{5}{3} \times 5 = \frac{4}{3}$

$d_1 = 20 - \frac{5}{3} \times 18 = -10$

$$\Rightarrow \left|\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ & -11 & 14 & 3 \\ & & \frac{204}{11} & \frac{204}{11} \end{array}\right| \qquad R_3 \to R_3 - \frac{13}{11} R_2$$

$a_{32} = R_3 \to 13 + \frac{13}{11}(-11) = 0$

$a_{33} \to 2 + \frac{13}{11}(14)$

$\to \frac{204}{11}$

$$\frac{204}{11} z = \frac{204}{11} \Rightarrow z = \frac{204}{11} \times \frac{11}{204}$$

$\therefore\ z = 1$

$$y = \frac{3 - 14(1)}{-11} \Rightarrow \frac{-11}{-11} = 1$$

$$x = \frac{18 - 5(1) - 4\ (1)}{3} \Rightarrow \frac{18 - 9}{3} = \frac{9}{3} = 3$$

$\therefore$ Hence the solution: x = 3, y = 1, z = 1.

Example: 3

Solve the equations

$$x + y + z + w = 2$$

$$2x - y + 2z - w = -5$$

$$3x + 2y + 3z + 4w = 7$$

$$x - 2y - 3z + 2w = 5$$

Solution:

The working is given below.

$$\Rightarrow \left|\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 2 & -1 & 2 & -1 & -5 \\ 3 & 2 & 3 & 4 & 7 \\ 1 & -2 & -3 & 2 & 5 \end{array}\right|$$

$$\Rightarrow \left|\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 2 & -3 & 0 & -3 & -9 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -3 & -4 & 1 & 3 \end{array}\right|$$

$R_2 - 2R_1 \rightarrow R_2$

$R_3 - 3R_1 \rightarrow R_3$

$R_4 - R_1 \rightarrow R_4$

$a_{21} = 2 - 2(1) = 0$
$a_{22} = -1 - 2(1) = -3$
$a_{23} = 2 - 2(1) = 0$
$a_{24} = -1 - 2(1) = -3$
$C_1 = -5 - 2(2) = -9$
$a_{31} = 3 - 3(1) = 0$
$a_{32} = 2 - 3(1) = 1$
$a_{33} = 3 - 3(1) = 0$
$a_{34} = 4 - 3(1) = 1$
$d_1 = 7 - 3(2) = 1$
$a_{41} = 1 - 1 = 0$, $a_{42} = -2 - 1 = -3$
$a_{43} = -3 - 1 = -4$, $a_{44} = 2 - 1 = 1$

$$\Rightarrow \left|\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -3 & -4 & 1 & 3 \end{array}\right|$$

$R_2 \rightarrow \frac{-1}{3} R_2$

$$\Rightarrow \left|\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -2 & 0 \end{array}\right|$$

$R_3 \rightarrow R_3 + R_2$

$a_{31} = 0$
$a_{32} = -1 + 1 = 0$
$a_{33} = 0$, $a_{34} = 1 + 1 = 2$

$$\Rightarrow \left|\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & -4 & -2 & 0 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right| \qquad \text{Interchanging } R_3 \text{ and } R_4$$

$\therefore$ $2w = 4$

$\therefore$ $w = 2$

$z = \dfrac{2(2)}{-4} = 1$

$y = 3 - 1(2) = 1$

$x = 2 - 2 + 1 - 1 = 0$

$\therefore$ The solution is $x = 0$, $y = 1$, $z = -1$, $w = 2$.

Example: 4

Solve the system by Gauss – Elimination method $2x + 3y - z = 5$; $4x + 4y - 3z = 3$ and $2x - 3y + 2z = 2$.

Solution:

The system is equivalent to

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$$

$$A \quad x \quad = \quad B$$

$$\therefore (A, B) = \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ 2 & -3 & 2 & 2 \end{array}\right)$$

Taking $a_{11} = 2$ as the Pivot reduce all elements below that to zero.

$$(A, B) \sim \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & -6 & 3 & -3 \end{array}\right) \qquad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

Taking the element – 2 in the position (2, 2) as pivot, reduce all elements below that to zero.

$$(A, B) \sim \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & -6 & 18 \end{array}\right) \qquad R_3 \to R_3 - 3R_2$$

Hence $2x + 3y - z = 5$

$-2y - z = -7$

$6z = 18 \Rightarrow z = 3$

$-2y - 3 = -7$

$-2y = -7 + 3 \Rightarrow -2y = -4$

$y = 2$

$\therefore\ 2x + 3y - z = 5$

$2x + 3(2) - 3 = 5$

$2x = 5 - 6 + 3 \Rightarrow 2x = 2$

$x = 1$

$\therefore$ The solution $x = 1$, $y = 2$, $z = 3$.

Example: 5

Using Gauss – Elimination method, solve the system

$3.15x - 1.96y + 3.85z = 12.95$
$2.13x + 5.12y - 2.89z = 8.61$
$5.92x + 3.05y + 2.15z = 6.68$

Solution:

$$\begin{pmatrix} 3.15 & -1.96 & 3.85 \\ 2.13 & 5.12 & -2.89 \\ 5.92 & 3.05 & 2.15 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12.95 \\ -8.61 \\ 6.68 \end{pmatrix}$$

$$A \qquad \times \qquad = \qquad B$$

$$(A, B) = \left(\begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 2.13 & 5.12 & -2.89 & -8.61 \\ 5.92 & 3.05 & 2.15 & 6.88 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 6.4453 & -5.4933 & -17.3666 \\ 0 & 6.7335 & -5.0855 & -17.4578 \end{array}\right) \quad \begin{array}{l} R_2 \to R_2 \left(-\frac{2.13}{3.15}\right) R_1 \\ R_3 \to R_3 \left(-\frac{5.92}{3.15}\right) R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 6.4453 & -5.4933 & -17.3666 \\ 0 & 0 & 0.6534 & 0.6853 \end{array}\right)$$

$$\therefore \quad 3.15\,x - 1.96\,y + 3.85\,z = 12.95$$

$$6.4453\,y - 5.4933\,z = -17.3666$$

$$0.6534\,z = 0.6853$$

$$\therefore \quad z = \frac{0.6853}{0.6534} = 1.0488$$

$$\therefore \quad y = \frac{5.4933\ (1.0488) - 17.3666}{6.4453} = 1.8005$$

$$\therefore \quad x = \frac{12.95 + 1.96\,(1.8005) + 3.85\ (1.0488)}{3.15}$$

$$x = 1.7089$$

$\therefore$ The solution is

$x = 1.7089, \quad y = 1.8005 \quad z = 1.0488$

Exercise

1. Solve the following simultaneous equation using Gauss – Elimination method.

i) $x + 2y + 5z = 23$

$3x + y + 4z = 26$

$6x + y + 7z = 47$

ii) $x + y + 3z = -6$

$2x + 4y + z = 7$

$3x + 2y + 9z = 14$

iii) $x + 2y - z = -1$

$3x - y - 2z = -5$

$x - y - 3z = 0$

iv) $2x - y + 3z + w = 9$

$-x + 2y + z - 2w = 2$

$3x + y - 4z + 3w = 3$

$5x - 4y + 3z - 6w = 2$

v) $4.12x - 9.68y + 2.01z = 4.93$

$1.88x - 4.62y + 5.50z = 3.11$

$1.10x - 0.96y + 2.72z = 4.02$

Answer:

i) $x = 4$, $y = 2$, $z = 3$

ii) $x = 1$, $y = 1$, $z = 1$

iii) $x = 2$, $y = -1$, $z = 1$

iv) $x = 2$, $y = 2$, $z = 2$, $w = 1$.

5.3 Gauss Jordan Methods

This method is a modification of the above Gauss elimination method. In this method, the coefficient matrix A of the system A x = B in brought to a diagonal matrix or unit matrix by making the matrix A not only upper triangular but also lower triangular by making all elements above the leading diagonal of A also as zeros. By this way, the system Ax = B will reduce to the form.

$$\left(\begin{array}{ccccc|c} a_{11} & 0 & 0 & 0 & 0 & b_1 \\ 0 & b_{22} & 0 & 0 & 0 & c_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & d_3 \\ 0 & 0 & 0 & 0 & \alpha_{nn} & d_3 \end{array}\right) \qquad (1)$$

From (1)

$$x_n = \frac{k_n}{\alpha_{nn}} \ldots\ldots\ldots x_2 = \frac{c_2}{b_{22}}, \; x_1 = \frac{b_1}{a_{11}}$$

Note:

By this method, the values of $x_1, x_2, \ldots\ldots x_n$ are got immediately without using the process of back substitutions.

Example:1

Solve the equations

$4x - 3y = 11$

$3x + 2y = 4$

Solution:

The working in given below

$$\Rightarrow \left|\begin{array}{cc|c} 4 & -3 & 11 \\ 3 & 2 & 4 \end{array}\right|$$

$$a_{11} = \frac{1}{4} \times 4 = 1$$

$$a_{12} = \frac{1}{4} \times (-3) = -3/4$$

$$c_1 = 11/4$$

$$\Rightarrow \left|\begin{array}{cc|c} 1 & \frac{-3}{4} & \frac{11}{4} \\ 1 & \frac{2}{3} & \frac{4}{3} \end{array}\right| \quad \begin{array}{l} R_1 \to \frac{1}{4} R_1 \\ R_2 \to \frac{1}{3} R_2 \end{array}$$

$$a_{21} = \frac{1}{3} \times 3 = 1$$

$$a_{22} = \frac{1}{3} \times 2 = \frac{2}{3}$$

$$d_1 = 4/3$$

$$\Rightarrow\left|\begin{array}{cc|c} 1 & \frac{-3}{4} & \frac{11}{4} \\ 0 & \frac{17}{12} & \frac{-17}{12} \end{array}\right| \quad R_2 \rightarrow R_2 - R_1$$

$a_{21} = 1 - 1 = 0$

$a_{22} = \frac{2}{3} + \frac{3}{4} = \frac{17}{12}$

$c_1 = \frac{4}{3} - \frac{11}{4} = \frac{17}{12}$

$$\Rightarrow\left|\begin{array}{cc|c} 1 & \frac{-3}{4} & \frac{11}{4} \\ 0 & 1 & -1 \end{array}\right| \quad R_2 \rightarrow \frac{12}{17} R_2$$

$a_{21} = 0$

$a_{22} = \frac{12}{17} x \frac{17}{12} = 1$

$c_1 = \frac{12}{17} x \frac{-17}{12}$

$$\Rightarrow\left|\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right| \quad R_1 \rightarrow R_1 + \frac{3}{4} R_2$$

$a_{11} = 1 + 0 = 1$

$a_{12} = -\frac{3}{4} + \frac{3}{4}(1) = 0$

$c_1 = \frac{11}{4} + \frac{3}{4}(-1)$

$= \frac{18}{4} = +2$

$\therefore\ x = 2\ ,\ y = -1$

Hence the solution: $x = 2$, $y = -1$.

Example: 2

Solve the equation

$$3x + 4y + 5z = 18$$
$$2x - y + 8z = 13$$
$$5x - 2y + 7z = 20$$

Solution:

$$\left|\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{array}\right|$$

$$\Rightarrow\left|\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 0 & \frac{-11}{3} & \frac{14}{3} & 1 \\ 5 & \frac{-26}{3} & \frac{-4}{3} & -10 \end{array}\right| \quad \begin{array}{l} R_2 \to R_2 - \frac{2}{3} R_1 \\ R_3 \to R_3 - \frac{5}{3} R_1 \end{array}$$

$a_{21} = 2 - \frac{2}{3}(3) = 0$

$a_{22} = -1 - \frac{2}{3}(4) = \frac{-11}{3}$

$a_{23} = 8 - \frac{2}{3}(5) = \frac{+14}{3}$

$a_{31} = 5 - \frac{5}{3}(3) = 0$

$a_{32} = -2 - \frac{5}{3}(4) = \frac{-26}{3}$

$a_{33} = 7 - \frac{5}{3}(5) = \frac{-4}{3}$

$$\Rightarrow\left|\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 0 & 11 & -14 & -3 \\ 5 & 13 & 2 & 5 \end{array}\right| \quad \begin{array}{l} R_2 \to -3R_2 \\ R_3 \to \frac{3}{2} R_3 \end{array}$$

$a_{21} = 0,\ a_{22} = +11,\ a_{23} = 14$

$a_{31} = 0,\ a_{32} = +13,\ a_{33} = 2$

$$\Rightarrow\left|\begin{array}{ccc|c} 3 & 0 & \frac{111}{11} & \frac{210}{11} \\ 0 & 11 & -14 & -3 \\ 0 & 0 & \frac{204}{11} & \frac{204}{11} \end{array}\right| \quad \begin{array}{l} R_1 \to R_1 - \frac{4}{11} R_2 \\ R_3 \to R_3 - \frac{13}{11} R_2 \end{array}$$

$a_{11} = 3 - \frac{4}{11}(0) = 3$

$a_{22} = 4 - \frac{4}{11}(11) = 0$

$a_{23} = 5 - \frac{4}{11}(-14) = \frac{-111}{11}$

$a_{31} = 0 - \frac{13}{11}(0) = 0$

$a_{32} = 13 - \frac{13}{11}(1) = 0$

$a_{33} = 2 - \frac{13}{11}(-14) = \frac{204}{11}$

$$\Rightarrow\left|\begin{array}{ccc|c} 33 & 0 & 111 & 210 \\ 0 & 11 & -14 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right| \quad \begin{array}{l} R_2 \to 11 R_1 \\ R_3 \to \frac{11}{204} R_3 \end{array}$$

$a_{11} = 11(3) = 33,\ a12 = 11(0) = 0$

$a_{13} = 11\left(\frac{111}{11}\right) = 111$

$a_{31} = \frac{11}{204}(0) = 0,\ a_{32} = \frac{11}{204}(0) = 0$

$a_{33} = \frac{11}{204} \times \frac{204}{1} = 1$

$$\Rightarrow\left|\begin{array}{ccc|c} 33 & 0 & 0 & 99 \\ 0 & 11 & 0 & 11 \\ 0 & 0 & 1 & 1 \end{array}\right| \quad \begin{array}{l} R_1 \to R_1 - 111R_3 \\ R_2 \to R_2 + 14 R_3 \end{array}$$

$a_{11} = 33 - 11(0) = 33$
$a_{12} = 0 - 111\ (0) = 0$
$a_{13} = 111 - 111(1) = 0$
$a_{21} = 0 + 14(0) = 0$
$a_{22} = 11 + 14(0) = 11$
$a_{33} = -14 + 14(1) = 0$

$$\Rightarrow \left|\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right| \quad R_1 \to \frac{1}{33}R_1 \quad R_2 \to \frac{1}{11}R_2$$

$a_{11} = \frac{1}{33}(33) = 1$
$a_{12} = 0$ $a_{13} = 0$
$a_{21} = 0$ $a_{22} = \frac{1}{11}(11) = 1$
$a_{23} = 0$

$\therefore$ The solution x = 3, y = 1, z = 1.

Example :3

$X + y + z + w = 2$
$2x - y + 2z - w = -5$
$3x + 2y + 3z + 4w = 7$
$x - 2y - 3z + 2w = 5$

Solution:

The working is given below:

$$\Rightarrow \left|\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 2 & -1 & 2 & -1 & -5 \\ 3 & 2 & 3 & 4 & 7 \\ 1 & -2 & -3 & 2 & 5 \end{array}\right|$$

$$\Rightarrow \left|\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & 0 & -3 & -9 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -3 & -4 & 1 & 3 \end{array}\right| \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 3R_1 \\ R_4 \to R_4 + R_1 \end{array}$$

$a_{21} = 2 - 2(1) = 0$
$a_{22} = -1 - 2(1) = -3$
$a_{23} = 2 - 2(1) = 0$
$a_{24} = -1 - 2(1) = -3$
$a_{31} = 3-3(1)=0$
$a_{32} = 2-3(1)= -1$
$a_{33} = 3-3(1)= 0$
$a_{34} = 4-3(1)=1$
$a_{41} = 1 - 1 = 0$
$a_{42} = -2-1= -3$
$a_{43} = -3-1= -4,$
$a_{44} = 2-1= 1,$

$$\Rightarrow \left|\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & 0 & -3 & -9 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & 4 & 12 \end{array}\right| \quad \begin{array}{l} R_3 \to R_3 - \frac{1}{3}R_2 \\ R_4 \to R_4 - R_2 \end{array}$$

$a_{31} = 0$
$a_{32} = -1\frac{1}{3}(-3)= 0$
$a_{33} = 0$
$a_{34} = 1-\frac{1}{3}(-3)= 2$
$a_{41} = 0$
$a_{42} = -3 + 3= 0$
$a_{43} = -4-0= -4,$
$a_{44} = 1+3 = 4$

$$\Rightarrow \left|\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & -3 \end{array}\right| \quad R_2 \to -\frac{1}{3}R_2, \quad R_3 \to \frac{1}{2}R_3, \quad R_4 \to -\frac{1}{4}R_4$$

$$\Rightarrow \left|\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right| \quad \text{Interchanging } R_3 \text{ and } R_4$$

$$\Rightarrow \left|\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right| \quad R_2 \to R_1 - R_2 \qquad \begin{array}{ll} a_{11}=1, & a_{12}=0 \\ a_{13}=1, & a_{14}=0 \end{array}$$

$$\Rightarrow \left|\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right| \quad \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_4 \\ R_3 \to R_3 + R_4 \end{array} \qquad \begin{array}{llll} a_{11}=1, & a_{12}=0, & a_{13}=0, & a_{14}=1 \\ a_{21}=0, & a_{22}=1, & a_{23}=0, & a_{24}=0 \\ a_{31}=0, & a_{32}=0, & a_{33}=1, & a_{34}=0 \end{array}$$

$$\Rightarrow \left|\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right| \quad R_1 \to R_1 - R_4$$

The solution: $x = 0$, $y = 1$, $x = -1$, $w = 2$.

Example: 4

Solve the following system by Gauss – Jordan method.

$5x_1 + x_2 + x_3 + x_4 = 4$; $\qquad x_1 + 7x_2 + x_3 + x_4 = 12.$

$x_1 + x_2 + 6x_3\ x_4 = -5$; $\qquad x_1 + x_2 + x_3 + 4x_4 = -6$

Solution:

Interchange the first and last equation, So that the coefficient of x1 in the first equation is,1. Then we have

$$(A,B) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 1 & 7 & 1 & 1 & 12 \\ 1 & 1 & 6 & 1 & -5 \\ 5 & 1 & 1 & 1 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & -4 & -4 & -19 & 34 \end{array}\right) \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 + 5R_1 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & -4 & -4 & -19 & 34 \end{array}\right) R_2 \to R_2/6$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 4.5 & -9 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & 0 & -4 & -21 & 46 \end{array}\right) \quad R_{42} \to R_{42}(4)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 4.5 & -9 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & 1 & -0.6 & -0.6 \\ 0 & 0 & -4 & -21 & -21 \end{array}\right) \quad R_3 \to R_3/5$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 5.1 & -9.2 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & 1 & -0.6 & 0.2 \\ 0 & 0 & 0 & -23.4 & 46.8 \end{array}\right) \quad R_4 \to R_{43}(14)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 5.1 & -9.2 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & 1 & -0.6 & 0.2 \\ 0 & 0 & 0 & -23.4 & 2 \end{array}\right) R_4 \to \frac{R_4}{23.4}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{array}\right) \qquad \begin{array}{l} R_3 \to R_3\left(-\frac{3}{5}\right) \\ R_2 \to R_2\left(-\frac{1}{2}\right) \\ R_1 \to R_1(5.1) \end{array}$$

$\therefore$ The solution is

$x_1 = 1,\ x_2 = 2,\ x_3 = -1,\ x_4 = -2.$

Example: 5

Apply Gauss – Jordan method to find the solution of the following system $10x + y + z = 12,\ 2x + 10y + z = 13,\ x + y + 5z = 7.$

Solution:

Since the coefficient of x in the last equation is unity, we rewrite the equations interchanging the first and last. Hence the argument matrix is

$$(A,B) = \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{array}\right) \qquad \begin{array}{l} R_2 \to R_2 + (-2)R_1 \\ R_3 \to R_{3+}\ (-10)R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & \frac{-9}{8} & -\frac{1}{8} \\ 0 & -9 & -49 & -58 \end{array}\right) \qquad R_2 \to \frac{R_2}{8}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & \frac{-9}{8} & \frac{-1}{8} \\ 0 & 0 & \frac{-473}{8} & \frac{-473}{8} \end{array}\right) \qquad R_3 \to R_3\left(\frac{-8}{473}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{49}{8} & \frac{57}{8} \\ 0 & 1 & \frac{-9}{8} & \frac{-1}{8} \\ 0 & 0 & 1 & 1 \end{array}\right) \quad R_1 \rightarrow R_1 - R_2$$

$$R_2 \rightarrow R_2 + \frac{9}{8} R_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad R_1 \rightarrow R_1 + \left(\frac{-49}{8}\right) R_3$$

$\therefore$ The solution is

$x = 1, y = 1, z = 1.$

Exercise

1. Solve the following simultaneous equations using Gauss – Jordan method.

i) $x + 2y + 5z = 23$
$3x + y + 4z = 26$
$6x + y + 7z = 47$

ii) $x + 2y + 3z = 6$
$2x + 4y + z = 7$
$3x + 2y + 9z = 14$

iii) $x + 2y - z = -1$
$3x - y - 2z = 5$
$x - y - 3z = 0$

iv) $2x - y + 3z + w = 9$
$-x + 2y + z - 2w = 2$
$3x + y - 4z + 3w = 3$
$5x - 4y + 3z - 6w = 2$

v) $6x - y + z = 13$
$x + y + z = 9$
$10x + y - z = 19$

5.4 Iteration methods

Suppose we have to solve the system of equations

$a_1x + b_1y + c_1z = d_1$ ....(1)

$a_2x + b_2y + c_2z = d_2$ ....(2)

$a3x + b3y + c3z = d_3$ ....(3)

Express x from equation (1), in terms of the other two variables

$$a_1x + b_1y + c_1z = d_1$$

$$a_1x = d_1y - b_1z - c_1z$$

$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z) \quad \text{...........} (A_1)$$

Express y from equation (2), in terms of the other two variables

$$a_2x + b_2y + c_2z = d_2$$

$$b_2y = d_2 - a_2x - c_2z$$

$$y = \frac{1}{b_2}(d_2 - c_2z - a_2x) \quad \text{...........} (A_2)$$

Express z form equation (3), in terms of the other two variables

$$a_3x + b_3y + c_3z = d_3$$

$$c_3z = d_3 - a_3x - b_3y$$

$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y) \quad \text{...........} (A_3)$$

If $x^{(0)}$, $y^{(0)}$, $z^{(0)}$ are the initial values of x,y,z respectively then $x^{(1)} = \frac{1}{a_1}(d_1 - b_1\ y^{(0)} - c_1z^{(0)})$, $y^{(1)}) = \frac{1}{b_2}(d_2 - a_2\ x^{(0)} - c_2z^{(0)})$, $z^{(1)} = \frac{1}{c_3}(d_3 - a_3\ x^{(0)} - b_3\ y^{(0)})$. We take the iteration scheme as $[\because x_{k+1} = F(x_k)]$ proceeding in this some way, If the r^{th} iterates $x^{(r)}$, $y^{(r)}$, $z^{(r)}$ the iteration scheme reduces to

$$\left.\begin{aligned} x_{k+1} &= \frac{1}{a_1}(d_1 - b_1\ y_k - c_1z_k) \\ y_{k+1} &= \frac{1}{b_2}(d_2 - c_2\ z_k - a_2x_k) \\ z_{k+1} & \frac{1}{c_3}(d_3 - c_3\ x_k - b_3y_k) \end{aligned}\right\} -B$$

To start with, assume $K = 0$ & $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ substituting these values in the equations of (B). We get x_1, y_1, z_1 the first approximations. Then put $k = 1$ and the values of x_1, y_1, z_1 in the system of equations (B). And get x_2, y_2, z_2 the second approximations continuing this process.

We can get subsequent approximations of x , y , z.

This method is known as Jacobi method.

Convergence of the iteration:

System of the equations (A) are

$$x = F_1(y, z) \text{ where } F_1 = \frac{1}{a_1}(d_1 - b_1y - c_1z)$$

$$y = F_2(z, x) \text{ where } F_2 = \frac{1}{b_2}(d_2 - c_2z - a_2x)$$

$$z = F_3(x, y) \text{ where } F_3 = \frac{1}{c_3}(d_3 - a_3x - b_3y)$$

The conditions of convergence are

$$\left|\frac{\partial F_1}{\partial x}\right| + \left|\frac{\partial F_2}{\partial x}\right| + \left|\frac{\partial F_3}{\partial x}\right| < 1$$

$$\left|\frac{\partial F_1}{\partial y}\right| + \left|\frac{\partial F_2}{\partial y}\right| + \left|\frac{\partial F_3}{\partial y}\right| < 1$$

$$\left|\frac{\partial F_1}{\partial z}\right| + \left|\frac{\partial F_2}{\partial z}\right| + \left|\frac{\partial F_3}{\partial z}\right| < 1$$

These in equalities are satisfied if

$$\left|\frac{\partial F_1}{\partial x}\right| + \left|\frac{\partial F_1}{\partial y}\right| + \left|\frac{\partial F_1}{\partial z}\right| < 1$$

$$\left|\frac{\partial F_2}{\partial x}\right| + \left|\frac{\partial F_2}{\partial y}\right| + \left|\frac{\partial F_2}{\partial z}\right| < 1$$

$$\left|\frac{\partial F_3}{\partial x}\right| + \left|\frac{\partial F_3}{\partial y}\right| + \left|\frac{\partial F_3}{\partial z}\right| < 1$$

Here $\left|\frac{\partial F_1}{\partial x}\right| = 0, \left|\frac{\partial F_1}{\partial y}\right| = -\frac{b_1}{a_1}, \left|\frac{\partial F_1}{\partial z}\right| = -\frac{c_1}{a_1}$

$$\left|\frac{\partial F_2}{\partial x}\right| = -\frac{a_2}{b_2}, \left|\frac{\partial F_2}{\partial y}\right| = 0, \left|\frac{\partial F_2}{\partial z}\right| = -\frac{c_2}{b_2}$$

$$\left|\frac{\partial F_3}{\partial x}\right| = -\frac{a_3}{c_3}, \left|\frac{\partial F_3}{\partial y}\right| = -\frac{b_3}{c_3}, \left|\frac{\partial F_3}{\partial z}\right| = 0.$$

$$\therefore \left|-\frac{b_1}{a_1}\right| + \left|\frac{c_1}{a_1}\right| < 1$$

(ie) $|b_1| + |c_1| < |a_1|$

$$\left|-\frac{a_2}{b_2}\right| + \left|-\frac{c_2}{b_2}\right| < 1$$

(ie) $|a_2| + |c_2| < |b_2|$

$$\left|-\frac{a_3}{c_3}\right| + \left|-\frac{b_3}{c_3}\right| < 1$$

(ie) $|a_3| + |b_3| < |c_3|$

This condition is satisfied if the absolute values of the diagonal elements of the matrix $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ are greater than the sum of the absolute values of the other two elements in the row. Since we can arrange the equations in such a way that the elements along the diagonal are dominant, we can take the rule as if in each equation the absolute value of the largest coefficient is greater than the sum of the absolute values of all the remaining coefficients in that equation, the approximations converge to the true values. This condition is sufficient but not necessary.

Example:1

Solve the equation $14x - 5y = 5.5$; $2x + 7y = 19.3$ by Jacobi method.

Solution:

$$14x - 5y = 5.5$$

$$14x = 5.5 + 5y$$

$$x = \frac{1}{14}(5.5 + 5y)$$

$$2x + 7y = 19.3$$

$$7y = 19.3 - 2x$$

$$y = \frac{1}{7}(19.3 - 2x).$$

Hence we shall take the iteration scheme as

$$x_{k+1} = \frac{1}{14}(5.5 + 5y_k)$$

$$y_{k+1} = \frac{1}{7}(19.3 + 2x_k)$$

First Iteration:

Let the initial values be (0, 0)

$x = 0,\ y = 0$

$$x_1 = \frac{1}{14}(5.5 + 5y_0) = \frac{1}{14}(5.5)$$

$$x_1 = 0.3929,\ y1 = \frac{1}{7}(19.3 - 2x_0) = \frac{1}{7}(19.3) = 2.7571$$

Second iteration:

Using these values in $x_1 = 0.3929$, $y_1 = 2.7571$

$$x_2 = \frac{1}{14}(5.5 + 5(2.7571))$$

$$= 1.3758$$

$$y_2 = \frac{1}{7}(19.3 - 2(0.3929))$$

$$= 2.6449$$

Third iteration:

Using the value of $x_2 = 1.3758$, $y_2 = 2.6449$

$$x_3 = \frac{1}{14}(5.5 + 5(2.6449))$$

$= 1.3374$

$$y_3 = \frac{1}{7}(19.3 - 2(1.3758))$$

$= 2.3641$

Fourth iteration:

Using the value of $x_3 = 1.3374$, $y_3 = 2.3641$

$$x_4 = \frac{1}{14}(5.5 + 5(2.3641)) = 1.2372.$$

$$y_4 = \frac{1}{7}(19.3 - 2(1.3374))$$

$= 2.3750$

Fifth iteration:

Using the value of $x_4 = 1.2372$, $y_4 = 2.3750$

$$x_5 = \frac{1}{14}(5.5 + 5(2.3750))$$

$= 1.2410.$

$$y_5 = \frac{1}{7}(19.3 - 2(1.2372))$$

$= 2.4037$

Sixth iteration:

Using the value of $x_5 = 1.2410$, $y_5 = 2.4037$

$$x_6 = \frac{1}{14}(5.5 + 5(2.4037))$$

$= 1.2513$

$$y_6 = \frac{1}{7}(19.3 - 2(1.2410))$$

$= 2.4026$

The iteration is shown below

x :	0.3929	1.3758	1.3374	1.2372	1.2410
y :	2.7571	2.6449	2.3641	2.3750	2.4037

x: 1.2513

y: 2.4026.

∴ The actual value are x = 1.25, y = 2.4

Example: 2

Using the Jacobi method solve the equation.

$3x + y + z = 9.3$, $2x + 5y - z = 10.5$

$x - 2y + 10z = 30.6$.

Solution:

$3x + y + z = 9.3$

$$x = \frac{1}{3}(9.3 - y - z)$$

$2x + 5y - z = 10.5$

$$y = \frac{1}{5}(10.5 + z - 2x)$$

$x - 2y + 10z = 30.6$

$$z = \frac{1}{10}(30.6 - x + 2y)$$

Hence the iteration scheme is

$$x_{k+1} = \frac{1}{3}(9.3 - y_k - z_k)$$

$$y_{k+1} = \frac{1}{5}(10.5 + z_k - 2x_k)$$

$$z_{k+1} = \frac{1}{10}(30.6 - x_k - 2y_k)$$

Starting the process with x = 0, y = 0, z = 0,

$$x_1 = \frac{1}{3}(9.3) = 3.1$$

$$y_1 = \frac{1}{5}(10.5) = 2.1$$

$$z_1 = \frac{1}{10}(30.6) = 3.06$$

Hence $x_2 = \frac{1}{3}(9.3 - 2.1 - 3.06) = 1.38$

$$y_2 = \frac{1}{5}(10.5 + 3.06 - 2(3.1)) = 1.472$$

$$z_2 = \frac{1}{10}(30.6 - 3.1 + 2(2.1)) = 3.17$$

$$x_3 = \frac{1}{3}(9.3 - 1.472 - 3.17) = 1.5227$$

$$y_3 = \frac{1}{5}(10.5 + 3.17 - 2(1.38)) = 2.182$$

$$z_3 = \frac{1}{10}(30.6 - 1.38 + 2(1.472) = 3.2168$$

$$x_4 = \frac{1}{3}(9.3 - 2.182 - 3.2164) = 1.3005$$

$$y_4 = \frac{1}{5}(10.5 + 3.2164 - 2(1.5227)) = 2.1342$$

$$z_4 = \frac{1}{10}(30.6 - 1.3005 + 2(2.182)) = 3.3441$$

$$x_5 = \frac{1}{3}(9.3 - 2.1342 - 3.3441) = 1.2739$$

$$y_5 = \frac{1}{5}(10.5 + 3.3441 - 2(1.3005) = 2.2486$$

$$z_5 = \frac{1}{10}(30.6 -.1.3005 + 2(2.1342) = 3.3568$$

$$x_6 = \frac{1}{3}(9.3 - 2.2486 - 3.356) = 1.2315$$

$$y_6 = \frac{1}{5}(10.5 + 3.3568 - 2(1.2739)) = 2.2618$$

$$z_6 = \frac{1}{10}(30.6 - 1.2739 + 2(2.486)) = 3.3823.$$

The iteration is given below:

x :	3.1	1.38	1.5227	1.3005	1.2739	1.2315
y :	2.1	1.472	2.182	2.1342	2.2486	2.2618
z :	3.06	3.17	3.2164	3.3441	3.3568	3.3823

Hence x = 1.2315, y = 2.2618, z= 3.3823

∴ The Actual values aer x = 1.2, y = 2.3, z = 3.4

Example: 3

Solve the following system by Gauss – Jacobi methods.

$$10x + 5y - 2z = 3; \quad 4x - 10y + 3z = -3, \ x + 6y + 10z = -3$$

Solution:

Here, we see that the diagonal elements are dominant.

Hence, the iteration process can be applied (ie) the coefficient matrix $\begin{pmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{pmatrix}$ is diagonally dominant, since

$$|10| < |-5| + |-2|, |-10| > |4| + |3| \ \& \ |10| > |1| + |6|$$

Solving for x , y, z,

We have

$$x = \frac{1}{10}(3 + 5y + 2z) \quad (1)$$

$$y = \frac{1}{10}(3 + 4x + 3z) \quad (2)$$

$$z = \frac{1}{10}(-3 - x - 6y) \quad (3)$$

First iteration:

Let the initial values be (0, 0, 0) using these initial values in (1) (2) (3)

$$x^{(1)} = \frac{1}{10}[3+5(0) + 2(0)] = 0.3$$

$$y^{(1)} = \frac{1}{10}[3+4(0) + 3(0)] = 0.3$$

$$z^{(1)} = \frac{1}{10}[3 - 0 - 6(0)] = 0.3$$

Second Iteration:

Using these values in (1), (2),(3) We get

$$x^{(2)} = \frac{1}{10}[3+5(0.3) + 2(-0.3)] = 0.39$$

$$y^{(2)} = \frac{1}{10}[3+4(0.3) + 3(-0.3)] = 0.33$$

$$z^{(2)} = \frac{1}{10}[-3 - (0.3) - 6(0.3)] = 0.51.$$

Third iteration:

Using the values of $x^{(2)}$, $y^{(2)}$, $z^{(2)}$ in (1), (2), (3) we get

$$x^{(3)} = \frac{1}{10}[3 + 5(0.33) + 2(-0.51)] = 0.363$$

$$y^{(3)} = \frac{1}{10}[3 + 4(0.39) + 3(-0.51)] = 0.303$$

$$z^{(3)} = \frac{1}{10}[-3 - 4(0.39) - 6(-0.33)] = -0.537$$

Fourth iteration:

$$x^{(4)} = \frac{1}{10}[3 + 5(0.303) + 2(-0.537)] = 0.3441$$

$$y^{(4)} = \frac{1}{10}[3 + 4(0.363) + 3(-0.537)] = 0.2841$$

$$z^{(4)} = \frac{1}{10}[-3 - 0.363) - 6(0.303)] = -0.5181$$

Fifth iteration:

$$x^{(5)} = \frac{1}{10}[3 + 5(0.2841) + 2(-0.5181)] = 0.33843$$

$$y^{(5)} = \frac{1}{10}[3 + 4(0.3441) + 3(-0.5181)] = 0.2822$$

$$z^{(5)} = \frac{1}{10}[-3 - (0.3441) - 6(0.2841)] = -0.50487$$

Sixth iteration:

$$x^{(6)} = \frac{1}{10}[3 + 5(0.2822) + 2(-0.50487)] = 0.340126$$

$$y^{(6)} = \frac{1}{10}\left[3 + 4(0.33843) + 3(-0.50487)\right] = 0.283911$$

$$z^{(6)} = \frac{1}{10}\left[-3 - (0.33843) - 6(0.2822)\right] = -0.503163$$

Seventh iteration:

$$x^{(7)} = \frac{1}{10}\left[3 + 5(0.283911) + 2(-0.503163)\right] = 0.3413229$$

$$y^{(7)} = \frac{1}{10}\left[3 + 4(0.340126) + 3(-0.503163)\right] = 0.2851015$$

$$z^{(7)} = \frac{1}{10}\left[-3 - (0.340126) - 6(0.283911)\right] = -0.5043592$$

Eighth iteration:

$$x^{(8)} = \frac{1}{10}\left[3 + 5(0.2851015) + 2(-0.5043592)\right] = 0.34167891$$

$$x^{(9)} = \frac{1}{10}\left[3 + 5(0.2852214) + 2(-0.50519319)\right] = 0.341572062$$

$$y^{(9)} = \frac{1}{10}\left[3 + 4(0.34167891) + 3(-0.50519319)\right] = 0.285113607$$

$$z^{(9)} = \frac{1}{10}\left[-3 - (0.34167891) - 6(0.2852214)\right] = -0.505300731$$

Hence correct to 3 decimal places, the values are

$x = 0.342, \quad y = 0.285, \quad z = -0.505$

x	:	0.3	0.39	0.363	0.3441	0.33843	0.340126
y	:	0.3	0.33	0.303	0.2841	0.2822	0.283911
z	:	– 0.3	– 0.51	– 0.537	– 0.5181	– 0.50487	– 0.503163

x	:	0.3413229	0.34167891	0.341572062
y	:	0.2851015	0.2852214	0.2811360 7
z	:	– 0.5043592	– 0.50519319	– 0.505300731

The solution is

$$x = 0.342, \quad y = 0.285, \quad z = -0.505$$

Examples:

Solve the following system of equations by

Using Gauss Jacobi Method

$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33$$
$$6x + 3y + 12z = 35$$

Solution:

Since the diagonal elements are dominant; the coefficient matrix,

$$x = \frac{1}{8}[20 + 3y - 2z] \quad (1)$$

$$y = \frac{1}{11}[33 - 4x + z] \quad (2)$$

$$z = \frac{1}{12}[35 - 6x - 3y] \quad (3)$$

First iteration:

Let the initial values be x = 0, y= 0, z = 0

Using the values, x = 0, y = 0, z = 0 in (1), (2), (3)

We get

$$x^{(1)} = \frac{1}{8}[20 + 3(0) - 2(0)] = 2.5$$

$$y^{(1)} = \frac{1}{11}[33 + 4(0) + 0)] = 3.0$$

$$z^{(1)} = \frac{1}{12}[35 - 6(0) - 3(0)] = 2.916666$$

Second iteration:

Using these values $x^{(1)}$, $y^{(1)}$, $z^{(1)}$ again in (1), (2), (3)

We get

$$X^{(2)} = \frac{1}{8}[20 + 3(3.0) - 2(2.916666)] = 2.895833$$

$$Y^{(2)} = \frac{1}{11}[33 - 4(2.5) + (2.916666)] = 2.356060$$

$$Z^{(2)} = \frac{1}{12}[35 - 6(2.5) - 3(3.0)] = 0.916666$$

Third iteration:

$$x^{(3)} = \frac{1}{8}[20 + 3(2.356060) - 2(0.916666)] = 3.154356$$

$$y^{(3)} = \frac{1}{11}[33 - 4(2.895833) + 0.916666] = 2.030303$$

$$z^{(3)} = \frac{1}{12}[35 - 6(2.895833) - 3(2.356060)] = 0.879735$$

Fourth iteration:

$$x^{(4)} = \frac{1}{8}[20 + 3(2.030303) - 2(0.879735)] = 3.041430$$

$$y^{(4)} = \frac{1}{11}[33 - 4(3.154356) + (0.879735)] = 1.932937$$

$$z^{(4)} = \frac{1}{12}[35 - 6(3.154356) - 3(2.030303)] = 0.831913$$

Fifth iteration:

$$x^{(5)} = \frac{1}{8}[20 + 3(1.932937) - 2(0.831913)] = 3.016873$$

$$y^{(5)} = \frac{1}{11}[33 - 4(3.041430) + (0.831913)] = 1.969654$$

$$z^{(5)} = \frac{1}{12}[35 - 6(3.041430 + 3(1.932937)] = 0.912717$$

Sixth iteration:

$$x^{(6)} = \frac{1}{8}[20 + 3(1.969654) - 2(0.912717)] = 3.010441$$

$$y^{(6)} = \frac{1}{11}[33 - 4(3.016873 + (0.912717)] = 1.985930$$

$$z^{(6)} = \frac{1}{12}[35 - 6(3.016873 - 3(1.969654)] = 0.915817$$

Seventh iteration:

$$x^{(7)} = \frac{1}{8}[20 + 3(1.985930) - 2(0.915817)] = 3.015770$$

$$y^{(7)} = \frac{1}{11}[33 - 4(3.010441) + (0.915817)] = 1.988550$$

$$z^{(6)} = \frac{1}{12}[35 - 6(3.010441) - 3(1.985930)] = 0.914964$$

Eighth iteration:

$$x^{(8)} = \frac{1}{8}\left[20 + 3\,(1.988550) - 2\,(0.914964)\right] = 3.016946$$

$$y^{(8)} = \frac{1}{11}\left[33 - 4\,(3.015770 + (0.914964)\right] = 1.986535$$

$$z^{(8)} = \frac{1}{12}\left[35 - 6\,(3.015770 - 3\,(1.988550)\right] = 0.911644$$

Ninth iteration:

$$x^{(9)} = \frac{1}{8}\left[20 + 3\,(1.986535) - 2\,(0.911644)\right] = 3.017039$$

$$y^{(9)} = \frac{1}{11}\left[33 - 4\,(3.016946 + (0.911644)\right] = 1.985805$$

$$z^{(9)} = \frac{1}{12}\left[35 - 6\,(3.016946) - 3\,(1.986535)\right] = 0.911560$$

Tenth iteration:

$$x^{(10)} = \frac{1}{8}\left[20 + 3\,(1.985805) - 2\,(0.911560)\right] = 3.016786$$

$$y^{(10)} = \frac{1}{11}\left[33 - 4\,(3.017039) + (0.911560)\right] = 1.985764$$

$$z^{(10)} = \frac{1}{12}\left[35 - 6\,(3.017039) - 3\,(1.985805)\right] = 0.911696$$

x	:	2.5	2.895833	3.154356	3.041436	3.016873
y	:	3.0	2.356060	2.030303	1.932937	1.969654
z	:	2.916666	0.916666	0.879735	0.831913	0.912717
x	:	3.010441	3.015770	3.016946	3.017039	3.016786
y	:	1.985930	1.988550	1.986535	1.985805	1.985764
z	:	0.915817	0.914964	0.911644	0.911560	0.911696

∴ The actual values are

$x = 3.0168, \quad y = 1.9858, \quad z = 0.9117$

Exercise:

1. Solve the following system of equation by Jacobi.

i) $8x - 6y + z = 13.67$

$3x + 11y - 2z = 17.59$

$2x - 6y + 9z = 29.29$

ii) $9x + 2y - z = 21.1$

$4x + 6y + z = 33.4$

$5x - 3y + 11z = 50.9$

iii) $7.6x - 2.4y + 1.3z = 20.396$

$3.7x - 7.9y - 2.5z = 35.866$

$1.9x - 4.3y + 8.2z = 32.514$

iv) $9.862x - 5.821y + 1.231z = 4.3135$

$2.431x - 6.375y - 3.042z = 24.8298$

$3.754x - 4.872y + 9.635z = 3.9959$

5.5 GAUSS – SEIDAL METHOD OF ITERATION

This is only a refinement of Jacobi Method as before,

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \qquad (1)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 z)$$

We start with the initial values y_0, z_0 for y and z and get x_1 from the first equation.

$$\text{(ie) } x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

While using the second equation, we use z_0 for z and x_1 for x instead of x_0 as in the Jacobi's Method.

$$\text{We get } y_1 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_0)$$

Now, having known x_1 and y_1

Use x_1 for x and y_1 for y in the third equation.

We get

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

In finding the values of the unknowns, we use the latest available values on the R.H.S.

If x_k, y_k, z_k are the iterates, then the iteration scheme will be

$$x_{k+1} = \frac{1}{a_1}(d_1 - b_1 y_k - c_1 z_k)$$

$$y_{k+1} = \frac{1}{b_2}(d_2 - a_2 x_{k+1} - c_2 z_k)$$

$$z_{k+1} = \frac{1}{c_3}(d_3 - a_3 x_{k+1} - b_3 y_{k+1})$$

This process of iteration is continued until the convergence is assured. As the current values of the unknowns at each stage of iteration are used in getting the values of unknowns, the convergence in Gauss – Seidel method is very fast when compared to Gauss–Jacobi method. The rate of convergence is Gauss-Seidel method is roughly two times than that of Gauss–Jacobi method. As we saw the sufficient conditions already, the sufficient condition for the convergence of this method is also the same as we stated earlier. That is the method of iteration will converge if in each equation of the given system, the absolute values of all the remaining co–efficients. (The largest coefficients must be the coefficients for different unknowns).

Note:

1. For all systems of equation, this method will not work. (Since convergence is not assured). It converges only for special systems of equations.

2. Iteration method is self–correcting method that is any error made in computation is corrected in the subsequent iterations.

3. The iteration is stopped when the values of x, y, z start repeating with the required degree of accuracy.

Example:

Solve the equations.

$$14x - 5y = 5.5$$

$$2x + 7y = 19.3$$ by Gauss Seidel method

Solution:

$$14x = 5.5 + 5y$$

$$x = \frac{1}{14}(5.5 + 5y)$$

$$7y = 19.3 - 2x$$

$$y = \frac{1}{7}(19.3 - 2x)$$

Put y = 0, we get $x = \frac{5.5}{14} = 0.3929$

Putting x = 0.3929 in equation(2)

$$y = \frac{1}{7}\{19.3 - 2(0.3929)\} = 2.6445$$

The iteration scheme is

$$x_{k+1} = \frac{1}{14}(5.5 + 5y_k)$$

$$y_{k+1} = \frac{1}{7}(19.3 - 2x_{k+1})$$

Starting the iteration with the base

$$x_1 = 0.3929, \quad y_1 = 2.6445$$

We get $x_2 = \frac{1}{14}\{5.5 + 5(2.6445)\} = 1.3373$

$$y_2 = \frac{1}{7}\{19.3 - 2(1.3373)\} = 2.3751$$

$$x_3 = \frac{1}{14}\{5.5 + 5(2.3751)\} = 1.2411$$

$$y_3 = \frac{1}{7}\{19.3 - 2(1.2411)\} = 2.4025$$

$$x_4 = \frac{1}{14}\{5.5 + 5\,(2.4025)\} = 1.2509$$

$$y_4 = \frac{1}{7}\{19.3 - 2\,(1.2509)\} = 2.3997$$

$$x_5 = \frac{1}{14}\{5.5 + 5\,(2.3997)\} = 1.2499$$

$$y_5 = \frac{1}{7}\{19.3 - 2\,(1.2499)\} = 2.4000$$

The iteration is shown below.

x:	0.3929	1.3373	1.2411	1.2509	1.2499
y:	2.6445	2.3751	2.4025	2.3997	2.4000

Hence x = 1.2499, y = 2.4000.

The actual values are x = 1.25, y=2.4.

Example: 2

Using the Gauss– Seidel Method Solve the equation

$$3x + y + z = 9.3$$
$$2x + 5y - z = 10.5$$
$$x - 2y + 10z = 30.6$$

Solution:

$$3x + y + z = 9.3$$

$$3x = 9.3 - y - z$$

$$x = \frac{1}{3}(9.3 - y - z)$$

$$2x + 5y + z = 10.5$$

$$5y = 10.5 - 2x + z$$

$$y = \frac{1}{5}(10.5 - 2x + z)$$

$$x - 2y + 10z = 30.6$$

$$10z = 30.6 - x + 2y$$

$$z = \frac{1}{10}(30.6 - x + 2y)$$

First iteration:

Put y = 0, z = 0

$$x_1 = \frac{1}{3}(9.3 - y_0 - z_0)$$

$$x_1 = \frac{1}{3}(9.3 - 0 - 0) \Rightarrow x_1 = \frac{9.3}{3}$$

$$x_1 = 3.1$$

$$y_1 = \frac{1}{5}(10.5 - 2(3.1) + 0) = 0.86$$

$$z_1 = \frac{1}{10}(30.6 + 2(0.86) - 3.1$$

$$= 2.922$$

Hence the starting points for the iteration are

$$x_1 = 3.1, \qquad y_1 = 0.86, \qquad z_1 = 2.922$$

The iteration scheme is

$$x_{k+1} = \frac{1}{3}(9.3 - y_k - z_k)$$

$$y_{k+1} = \frac{1}{5}(10.5 - 2x_{k+1} + z_k)$$

$$z_{k+1} = \frac{1}{10}\left(30.6 + 2y_{k+1} - x_{k+1}\right)$$

In computing the successive approximation.

We use the latest approximate values.

Second iteration:

$$x_2 = \frac{1}{3}(9.3 - 0.86 - 2.922) = 1.8159$$

$$y_2 = \frac{1}{5}(10.5 - 2(1.8159) + 2.922) = 1.9580$$

$$z_2 = \frac{1}{10}(30.6 - 2(1.9580) - 1.8159) = 3.2694$$

Third iteration:

$$x_3 = \frac{1}{3}(9.3 - 1.9580 - 3.2694) = 1.3575$$

$$y_3 = \frac{1}{10}(10.5 - 2(1.3575) + 3.2694) = 2.2109$$

$$z_3 = \frac{1}{10}(30.6 - 2(2.2109) - 1.3575) = 3.3664$$

Fourth Iteration

$$x_4 = \frac{1}{3}(9.3 - 2.2109 - 3.3664) = 1.2409$$

$$y_4 = \frac{1}{5}(10.5 - 2(1.2409) + 3.3664) = 2.2769$$

$$z_4 = \frac{1}{10}(30.6 - 2(2.2769) - 1.2409) = 3.3913$$

Fifth Iteration:

$$x_5 = \frac{1}{3}(9.3 - 2.2769 - 3.3913) = 1.2106$$

$$y_5 = \frac{1}{5}(10.5 - 2(1.2106) + \ 3.3913) = 2.2940$$

$$z_5 = \frac{1}{10}(30.6 + 2(2.2940) + \ 1.2409) = 3.3977$$

Sixth Iteration:

$$X_6 = \frac{1}{3}(9.3 - 2.2940 - 1.2106) = 1.2028$$

$$Y_6 = \frac{1}{5}(10.5 - 2(1.2028) + \ 3.3977) = 2.2984$$

$$Z_6 = \frac{1}{10}(30.6 + 2(2.2984) - 1.2028) = 3.3994$$

The iteration is shown below.

x:	3.1	1.8159	1.3575	1.2409	1.2106	1.2028
y :	0.86	1.9580	2.2109	2.2769	2.2940	2.2984
z:	2.922	3.2694	3.3664	3.3913	3.3977	3.3994

Hence the sixth approximation is

$$X = 1.2028, \quad y = 2.2984, \quad z = 3.399$$

∴ The actual value are $x = 1.2$, $y = 2.3$, $z = 3.4$

Example: 3

Solve, by Gauss–Seidal method, the following system

$$28x + 4y - z = 32$$
$$x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35$$

Solution:

Since the diagonal elements in the coefficient matrix are not dominant, we rearrange the equation as follows, such that the elements in the coefficient matrix are dominant.

$$28x + 4y - z = 32$$
$$2x + 17y + 4z = 35$$
$$x + 3y + 10z = 24$$

Hence, $x = \frac{1}{28}(32 - 4y + z)$

$$y = \frac{1}{17}(35 - 2x - 4z)$$

$$z = \frac{1}{10}(24 - x - 3y)$$

Setting y = 0, z = 0, we get

First iteration:

$$x^{(1)} = \frac{1}{28}[32 - 4(0) + 0] = 1.1429$$

$$y^{(1)} = \frac{1}{17}[35 - 2(1.1429) - 4(0)] = 1.9244$$

$$z^{(1)} = \frac{1}{10}[24 - 1.1429 - 3(1.9244)] = 1.8084$$

Second iteration:

$$x^{(2)} = \frac{1}{28}[32 - 4(1.9244) + 1.8084] = 0.9325$$

$$y^{(2)} = \frac{1}{17}[35 - 2(0.9325) - 4(1.8084)] = 1.5236$$

$$z^{(2)} = \frac{1}{10}[24 - 0.9325 - 3(1.5236)] = 1.8497$$

Third iteration:

$$x^{(3)} = \frac{1}{28}[32 - 4(1.5236) + 1.8497] = 0.9913$$

$$y^{(3)} = \frac{1}{17}[35 - 2(0.9913) - 4(1.8497)] = 1.5070$$

$$z^{(3)} = \frac{1}{10}[24 - 0.9913 - 3(1.5070)] = 1.8488$$

Fourth iteration:

$$x^{(4)} = \frac{1}{28}\left[32 - 4(1.5070) + 1.8488\right] = 0.9936$$

$$y^{(4)} = \frac{1}{17}\left[35 - 2(0.9936) - 4(1.8488)\right] = 1.5069$$

$$z^{(4)} = \frac{1}{10}\left[24 - 0.9936 - 3(1.5069)\right] = 1.8486$$

Fifth iteration:

$$x^{(5)} = \frac{1}{28}\left[32 - 4(1.5069) + 1.8486\right] = 0.9936$$

$$y^{(5)} = \frac{1}{17}\left[35 - 2(0.9936) - 4(1.8486)\right] = 1.5069$$

$$z^{(5)} = \frac{1}{10}\left[24 - 0.9936 - 3(1.5069)\right] = 1.8486$$

since the values of x, y, z in the 4^{th} & 5^{th} iterations are same, we stop the process here

$\therefore$ x = 0.9936, y = 1.5069, z= 1.8486

Example:

Solve the following system of equations by using Gauss Seidel methods
$8x - 3y + 2z = 20$; $4x + 11y - z = 33$; $6x + 3y + 12z = 35$

Solutions:

Since the diagonal elements are dominant in the coefficient matrix.

$$x = \frac{1}{8}\left[20 + 3y - 2z\right] \qquad (1)$$

$$y = \frac{1}{11}\left[33 - 4x + z\right] \qquad (2)$$

$$z = \frac{1}{12}\left[35 - 6x - 3y\right] \qquad (3)$$

Take the initial values as y = 0, z = 0 & use

First iteration:

$$x^{(1)} = \frac{1}{8}\left[20 + 3(0) - 2(0)\right] = 2.5$$

$$y^{(1)} = \frac{1}{11}\left[33 - 4(2.5) + 0\right] = 2.090909$$

$$z^{(1)} = \frac{1}{12}\left[35 - 6(2.5) - 3(2.090909)\right] = 1.143939$$

Second iteration:

$$x^{(2)} = \frac{1}{8}[20 + 3(2.090909) - 2(1.143939)] = 2.998106$$

$$y^{(2)} = \frac{1}{11}[33 - 4(2.998106) + (1.143939)] = 2.013774$$

$$z^{(2)} = \frac{1}{12}[35 - 6(2.998106) - 3(2.013774)] = 0.914170$$

Third iteration:

$$x^{(3)} = \frac{1}{8}[20 + 3(2.013774) - 2(0.914170)] = 3.026623$$

$$y^{(3)} = \frac{1}{11}[33 - 4(3.026623) + 0.914170] = 1.982516$$

$$z^{(3)} = \frac{1}{12}[35 - 6(3.026623) - 3(1.982516)] = 0.907726$$

Fourth iteration:

$$x^{(4)} = \frac{1}{8}[20 + 3(1.982516) - 2(0.907726)] = 3.041430$$

$$y^{(4)} = \frac{1}{11}[33 - 4(3.016512) + 0.907726] = 1.985607$$

$$z^{(4)} = \frac{1}{12}[35 - 6(3.016512) - 3(1.985607)] = 0.912009$$

Fifth iteration:

$$x^{(5)} = \frac{1}{8}[20 + 3(1.985607) - 2(0.912009)] = 3.01660$$

$$y^{(5)} = \frac{1}{11}[33 - 4(3.016600) + 0.912009] = 1.985964$$

$$z^{(5)} = \frac{1}{12}[35 - 6(3.016600) - 3(1.985964)] = 0.911876$$

Sixth iteration:

$$x^{(6)} = \frac{1}{8}[20 + 3(1.985964) - 2(0.911876)] = 3.016767$$

$$y^{(6)} = \frac{1}{11}[33 - 4(3.016767) + 0.911876] = 1.985892$$

$$z^{(6)} = \frac{1}{12}[35 - 6(3.016767) - 3(1.985892)] = 0.911810$$

Seventh iteration:

$$x^{(7)} = \frac{1}{8}[20 + 3(1.985892) - 2(0.911810)] = 3.016751$$

$$y^{(7)} = \frac{1}{11}[33 - 4(3.016757) + 0.911810] = 1.985889$$

$$z^{(7)} = \frac{1}{12}[35 - 6(3.016757) - 3(1.985889)] = 0.911816$$

Since the sixth and seventh iterations give the same values for x, y, z correct to 4 decimal places.

$\therefore$ x = 3.0168, y = 1.9859, z = 0.9118

x	2.5	2.998106	3.026623	3.016512	3.016600
y	2.090909	2.013774	1.982516	1.985607	1.985964
z	1.143939	0.914170	0.907726	0.912009	0.911876
x	3.016767	3.016757			
y	1.985892	1.985889			
z	0.911810	0.911816			

Exercise:

Solve the following systems of equation Gauss–Seidal method.

i) $8x - 6y + z = 13.67$
$3x + 11y - 2z = 17.59$
$2x - 6y + 9z = 29.29$

ii) $9x + 2y - z = 21.1$
$4x + 6y + z = 33.4$
$5x - 3y + 11z = 50.9$

iii) $7.6x - 2.4y + 1.3z = 20.396$
$3.7x + 7.9y - 2.5z = 35.866$
$1.9x - 4.3y + 8.2z = 32.514$

iv) $9.862x - 5.821y + 1.231z = 4.3135$
$2.431x + 6.375y - 3.042z = 24.8298$
$3.754x - 4.812y + 9.635z = 3.9959$

v) $8.7x - 2.3y + 4.1z + 1.7w = 18.23$
$3.4x + 12.4y - 4.5z - 3.6w = -0.34$
$4.3x - 3.2y + 5.6z - 13.4w = -22$
$2.8x - 4.7y + 20.4z - 7.8w = 34.33$

UNIT – VI
INTERPOLATION

6.1 Introduction:

Interpolation has been described as the of reading between the line of a table and in elementary mathematics, it mean the process of computing intermediate value of a function from a given set of tabulator value of the function. Suppose the following table represents a set of corresponding value of x and y

x: x_0 x_1 x_2 x_3.........., x_n

Y: y_0 y_1 y_2 y_3..........., y_n

Now, we require the value of $y = y_i$ corresponding to a value $x = x_i$ where $x_0 < x_i < x_n$.

Extrapolation is used to denote the process of finding the values out side the interval (x_0, x_n). But, in general the word interpolation is used in both processes.

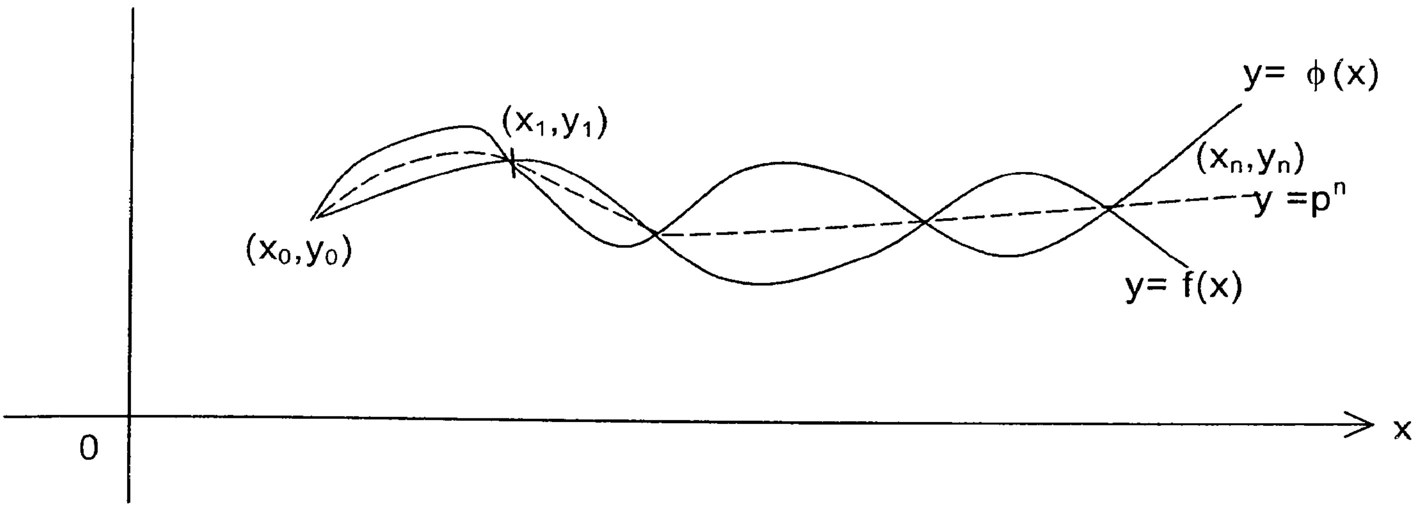
Let y = F (x) be the function taking the values $y_0, y_1, \ldots\ldots y_n$ corresponding to $x = x_0, x_1 \ldots\ldots x_n$. In other words, $y_1 = F(x_1)$,I = 0, 1,2,.....n. If f(x) is known the value of y can be calculated for any x. But in many cases we have to find y = f(x) such that $y_i = F(x_i)$ from the given table. This is not easy because there are infinity of function $y = \phi(x)$ such that $y_1 = \phi(x_i)$. Hence, from the table we cannot find a unique $\phi(x)$ such that $y = \phi(x)$ satisfier the set of table we cannot find a unique $\phi(x)$ such that $y = \phi(x)$ satisfies the set of values given in the table above of the sequences of function $\{\phi(x)\}$, there is a unique n^{th} degree polynomial $p_n(x)$ such that $y_i = p_n(x_i)$, I = 0, 1, 2,..... n (Ref fig.1)

The function $\phi(x)$ is called interpolating function or smoothing function or interpolating formula.

The polynomial function $p_n(x)$ may be taken as an interpolating polynomial or collocation polynomial where.

$y_i = F(x_i) = p_n(x_i)$, i = 0,1,2,......,n.

Other types or approximating function may be taken suitable for different purpose. In this chapter, we will be mostly concerned with the polynomial interpolation only.



Polynomial interpolation is mostly function preferred because of the following reasons:

1. They are simple forms of functions which can be easily manipulated.
2. Computations for define values of the argument, integration and differentiation of such function, are easy.
3. Polynomials are free feam singularities whereas rational functions or other types, do have singularities

The basis of finding such collocation polynomial is the fact that there is exactly only one collocation polynomial $p_n(x)$of degree n such that the values of $p_n(x)$ at $x_0, x_1, x_2, \ldots x_n$ coincide with the given functionnal values $y_0, y_1, y_2, \ldots y_n$. Here, $p_n(x)$ is called polynomial approximation to f(x). We shall see below a few of the method of finding such interpolating polynomials. The simplest of all interpolations in which the interpolating polynomial is linear. Let us assume that the set of values of x & y are given below. Linear interpolation or methods of proportional parts.

x: x_0 x_1 x_2 $x_3 \ldots\ldots , x_n$

Y: y_0 y_1 y_2 $y_3 \ldots\ldots , y_n$

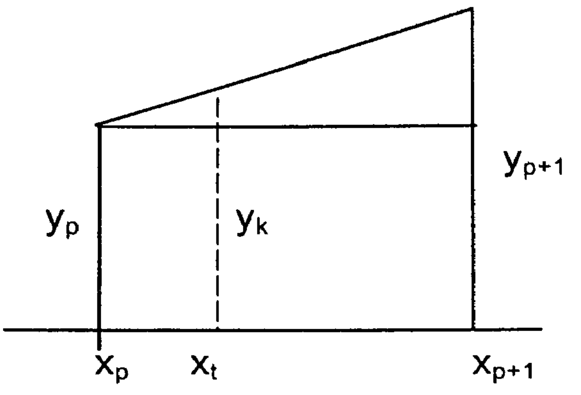
Now we require the value of y corresponding to x_k which lies between x_r and x_{r+1}.

We will assume the polynomial to be linear (ie.st.line)

The line equation is $\frac{y - y_r}{x - x_r} = \frac{y_{r+1} - y_r}{x_{r+1} - x_r}$

$\therefore y_k = y_r + \left[\frac{y_{r+1} - y_r}{x_{r+1} - x_r}\right] (x_k - x_r)$ gives the value of y at x_r x_k, $x_r < x_k < x_{r+1}$.

This methods may be successful in the difference between succeeding pairs of value of the variable are small and regular. But, if the intervals between the two pairs of value are large, and irregular, this method of simple proportion cannot be used without large error.



Example:

The following are the measurements t made on a curve recorded by an oscillograph representing a change of current I due to a change in the condition of an electric current:

T:	1.2	2.0	2.5	3.0
I:	1.36	0.58	0.34	0.20.

Find the value of i and t = 1.6.

Solution:

Let $i_{(1.6)}$ be the required value

$$\text{Then } y_k = y_p + \frac{x_k - x_p}{x_{p+1} - x_p}\,(y_{p+1} - yp)$$

$$i_{(1.6)} = 1.36 + \frac{1.6 - 1.2}{2.0 - 1.2}\,(0.58 - 1.36)$$

$$= 1.36 - \frac{0.4}{0.8}(0.78)$$

$$= 0.97$$

Example:

Using the method of proportional parts, find. y at x =0.5, x = 0.75, given the following table

x:	0	1	2	5
y:	2	3	12	147

Solution:

$$y_k = y_r + \left(\frac{y_{r+1} - y_r}{x_{r+1} - x_r}\right)(x_k - x_r)$$

$$y_{(0.5)} = 2 + \frac{(3-2)}{(1-0)}(0.5 - 0)$$

$$= 2.5$$

$$y_{(0.75)} = 2 + \frac{(3-2)}{(1-0)}(0.75 - 0)$$

$$= 2.75$$

6.2 Newton's Interpolation Formula

Let y = f(x) denote a function which takes a set of corresponding values of two quantities x and y.

x: x_0 x_1 x_2 $x_3 \ldots\ldots, x_n$

y: y_0 y_1 y_2 $y_3 \ldots\ldots, y_n$

Let us suppose that the values of x viz . x_0 , $x_1 \ldots$,x_n are equidistant.

(ie) $x_i - x_{i-1} = h$ for i = 1, 2,....n

$\therefore x_k = x_0 + kh,\ x_1 = x_0 + 1h$ etc.

$\therefore x_2 = x_0 + 2h, \ldots.. i = 1, 2 \ldots.. n.$

$$k = \frac{x_k - x_0}{h}$$

Let us assume that $P_n(x)$ be a polynomial of the n^{th} degree in x, such that $y_i = f(x_i) = P_n(x_i)$ i = 0,1,....n.

Let us assume $P_n(x)$ in the form given below $P_n(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots a_nx^n$ satisfies the (n+1) pairs of tabulated values (x_0,y_0), (x_1,y_1),$(x_2,y_2)\ldots\ldots(x_n,y_n)$.

Then Δy_i where (i = 0,1....n)are constants and the subsequent differences are zero.

We have

$$y_k = (1+\Delta)^k y_0$$

$$= y_0 + k\Delta y_0 + \frac{k(k-1)}{2!}\Delta^2 y_0 + \ldots\ldots + \Delta^k y_0$$

Finite Differences

$$\Delta y_k = y_{k+1} - y_k$$
$$\Delta^2 y_k = \Delta y_{k+1} - \Delta y_k$$
$$\vdots$$
$$\Delta^r y_k = \Delta^{r-1} y_{k+1} - \Delta^{r-1} y_k$$
$$\Delta y_0 = y_1 - y_0$$
$$y_1 = y_0 + \Delta y_0 = (1+\Delta) y_0$$
$$y_2 = y_0 + 2\Delta y_0 = \Delta^2 y_0$$
$$\vdots \quad = (1+\Delta)^2 y_0$$
$$y_k = (1+\Delta)^k y_0$$

Substituting the value of k in the above equation.

We have

$$y_k = y_0 + \left(\frac{x_k - x_0}{h}\right)\Delta y_0 + \frac{\left(\frac{x_k - x_0}{h}\right)\left(\frac{x_k - x_0}{h} - 1\right)}{2!}\Delta^2 y_0 + \ldots\ldots + \Delta^k y_0$$

If we assume that the value of y corresponding to an arbitrary x can be obtained from the above formula by replacing x_k by x, then

We have

$$y = y_0 + \frac{x - x_0}{h}\Delta y_0 + \frac{\frac{x - x_0}{h}\left(\frac{x - x_0}{h} - 1\right)}{2!}\Delta^2 y_0 + \ldots\ldots$$

Put $\frac{x - x_0}{h} = x$

Then we get

$$y = y_0 + x\Delta y_0 \frac{x(x-1)}{2i}\Delta^2 y_0 + \ldots\ldots + \frac{x(x-1)(x-2)\ldots(x-n+1)}{ni}\Delta^n y_0.$$

This is known as Newton's forward differences Interpolation formula.

In this formula for the computation of y, we have to take (P+1) terms if the P^{th} order of differences are constant or the P^{th} order of differences become very small.

Aliter:

Let the polynomial

$y = a_0 + a_1x + a_2x^2 + \ldots\ldots a_nx^n$ satisfies the (n+1) pairs of tabulated values $(x_0, y_0), (x_1,y_1), (x_2,y_2)\ldots\ldots (x_n, y_n)$.

Let us assume that the polynomial can be put in the form.

$$y = A_0 + A_1(x-x_0) + A_2 (x - x_0) (x - x_1) + A_3 (x - x_0) (x - x_1) (x - x_2)+ \ldots\ldots + A_n(x - x_0)(x - x_1) \ldots\ldots (x - x_{n-1})$$

We have to determine the constants A_0 , A_1, ... A_n

When $x = x_0$, $y = y_0$.

$\therefore y_0 = A$ (1)

When $x = x_1$, $y = y_1$

$\therefore y_1 = A_0 + A_1 (x_1 - x_0)$

$= y_0 + A_1 h$ since $x_1 - x_0 = h$

$\therefore A_1 = \dfrac{y_1 - y_0}{h}$

$= \dfrac{\Delta y_0}{h}$ (2)

When $x = x_2$, $y = y_2$.

$\therefore y_2 = A_0 + A_1 (x_2 - x_0) A_2(x_2 - x_0) (x_2 - x_1)$

$= y_0 + \dfrac{\Delta y_0}{h}(2h) + A_2 (2h) (h)$

$\therefore 2h^2 A_2 = y_2 - y_0 - 2\Delta y_0$

but we have shown that

$y_2 = y_0 + 2\Delta y_0 + \Delta^2 y_0$,

$\therefore 2h^2 A_2 = \Delta^2 y_0$

Hence $A_2 = \dfrac{\Delta^2 y_0}{2h^2}$

Continuing the calculation of the coefficients in this manner we shall find

$$A_3 = \frac{\Delta^3 y_0}{3!h^3} \ldots\ldots A_n = \frac{\Delta^n y_0}{n!h^n}$$

Hence

$$y = y_0 + (x - x_0)\frac{\Delta y_0}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 y_0}{2!h^2}$$

$$+ (x - x_0)(x - x_1)(x - x_2)\frac{\Delta^3 y_0}{3!h^3} + \ldots\ldots$$

$$+ (x - x_0)(x - x_1)\ldots\ldots(x - x_{n-1})\frac{\Delta^n y_0}{n!h^n}$$

If $X = \frac{x - x_0}{h}$, then $x = x_0 + x_h$

$$\begin{aligned}(x - x_1) &= (x - x_0) - (x_1 - x_0)\\ &= Xh - h\\ &= (X - 1)h\end{aligned}$$

$$\begin{aligned}(x - x_2) &= (x - x_0) - (x_2 - x_0)\\ &= Xh - 2h\\ &= (X - 2)h\end{aligned}$$

....

....

$$\begin{aligned}x - x_{n-1} &= (X - \overline{n-1}\ h)\\ &= (X - n+1)h.\end{aligned}$$

$$\therefore\ y = y_0 + X\Delta y_0 + \frac{X(X-1)}{2!}\Delta^2 y_0 + \ldots\ldots + \frac{X(X-1)(X-2)\ldots\ldots(X_n+1)}{n!}\Delta^n y_0$$

Newton's Back ward Differences Interpolation
Formula:

Newton's forward interpolation formula cannot be used for interpolating a value of y nearer to the end of the table of values.

For this purpose, we get another backward interpolation formula,

Suppose $y = f(x)$ takes the values $y_0, y_1 \ldots\ldots y_n$ corresponding to the values $x_0, x_1 \ldots\ldots x_n$ of x.

Let $x_k - x_{k-1} = h$ for $k = 1,2....n$ (equal intervals)

$$\therefore\ x_k = x_0 + kh,\ i = 0,1,2\ldots\ldots$$

Now, we want to find a collocation polynomial $P_n(x)$ of degree in x such that

$$P_n(x_i) = y_i,\ I = 0,1,2,\ldots..n$$

Let

$$\begin{aligned}P_n(x) = a_0 + a_1 (x - x_n) + a_2 (x - x_n)(x - x_{n-1}) + \ldots\ldots \\ + a_r (x - x_n)(x - x_{n-1})\ldots\ldots (x - x_{n-r+1}) + \ldots\ldots \\ + a_n (x - x_n)(x - x_{n-1}) \ldots..(x - x_1)\end{aligned}$$

Since $x_{n-1} = x_n - h$

$x_{n-2} = x_n - 2h\ldots\ldots\ x_{n-r+1} = x_n - (r-1)$

$x_1 = x_n - (n-1)h.$

We have

$$\begin{aligned}P_n(x) = a_0 + a_1 (x - x_n) + a_2 (x - x_n)(x - x_n + h) \\ + a_3 (x - x_n)(x - x_n + h)(x - x_n + 2h) + \ldots\ldots \\ + a_n (x - x_n)(x - x_n + h) \ldots..(x - x_n + (n-1)h)\end{aligned}$$

We shall find $a_0\ a_1\ldots..a_n$ such that $p_n(x_i) = y_i$

Since $\nabla = E^{-1}\Delta$.

$$\nabla y_x = y_x - y_{x-h}$$

$$\vdots$$

$$\nabla^r y_k = \nabla^{r-1} y_k - \nabla^{r-1} y_{k-1}$$

$$y_{n-1} = (1 - \nabla) y_n$$

$$y_{n-2} = (1 - \nabla^2) y_n$$

$$\vdots$$

$$y_{n-k} = (1 - \nabla)^k y_n$$

$$Y_{n-k} = (1 - \nabla)^k y_n$$

$$= Y_n - k\nabla y_n + \frac{k(k-1)}{2!}\nabla^2 y_n - \ldots\ldots..\ (1)$$

If the quantities $x_0, x_1, x_2 \ldots\ldots. x_n$ are equally spaced, then

$x_{n-k} = x_n - kh$

$$\therefore k = \frac{x_n - x_{n-k}}{h}$$

Substituting this values k in (10)

$$Y_{n-k} = y_n - \frac{(x_n - x_{n-k})}{h}\nabla y_n + \frac{\left(\frac{(x_n - x_{n-k})}{h}\right)\left(\frac{(x_n - x_{n-k})}{h} - 1\right)}{2!}\nabla^2 y_n \ldots\ldots\ldots..$$

If we assume that the value of y corresponding to an arbitrary x can be obtained from the above formula by replacing x_{n-k} by x, then we have

$$y = y_n - \frac{x_n - x}{h}\nabla y_n + \frac{\left(\frac{x_n - x}{h}\right)\left(\frac{x_n - x}{h} - 1\right)}{2!}\nabla^2 y_n - \ldots\ldots\ldots$$

Putting $\frac{x_n - x}{h} = -X$

We have

$$y = y_n + x \,.\, \nabla y_n + \frac{X(X+1)}{2!}\ \nabla^2 y_n + \ldots\ldots$$

This is known as Backward differences Newton's Interpolation formula.

The number of terms taken on the left side is P + 1 if the p^{th} order differences are constants they are very small.

Note 1:

Newton's formulae with forward and backward differences are most appropriate for calculation near the beginning and end respectively of a tabulation and their use is mainly restricted to such situations.

Note 2:

The process of computing the value of a function outside the range of given values is called extrapolation. It should be used with caution, but if the function is known to run smoothly near the ends of the range of given values and if h is taken as small as it should be, we are usually safe in extrapolating for a distance h outside the range of given values.

Aliter:-

Let the polynomial be expressed as

$$Y = B_0 + B_1(x - x_n) + B_2(x - x_n)(x - x_{n-1}) + \ldots\ldots. + B_n(x - x_n)(x - x_{n-1}) \ldots\ldots.(x - x_n)$$

When $x = x_n$, $y = y_n$,

$\therefore\ y_n = B_0$.

When $x = x_{n-1}$, $y = y_{n-1}$

$$y_{n-1} = B_0 + B_1\,(x_{n-1} - x_n)$$
$$= y_n + B_1(-h)$$
$$B_1 = \frac{y_n - y_{n-1}}{h} = \frac{\Delta y_n}{h}$$

When $x = x_{n-2}$, $y = y_{n-2}$,

$$y_{n-2} = B_0 + B_1(x_{n-2} - x_n) + B_2\,(x_{n-2} - x_n)\,(x_{n-2} - x_{n-1})$$
$$= B_0 + B_1\,(-2h) + B_2(-2h)\,(-h)$$
$$= y_n - 2(y_n - y_{n-1}) + 2h^2.B_2$$

$$\therefore 2h^2B_2 = y_n - 2(y_n) - 2y_{n-1}) + y_{n-2}$$
$$= y_n - 2y_{n-1} + y_{n-2}$$
$$= (y_n - y_{n-1}) - (y_{n-1} - y_{n-2})$$
$$= \Delta y_n - \Delta y_{n-1}$$
$$= \Delta^2 y_n$$
$$B^2 = \frac{\nabla^2 y_n}{2h^2}$$

Similarly, we get $B_3 = \dfrac{\nabla^3 y_n}{3!h^3} \ldots\ldots\ B_n = \dfrac{\nabla^n y_n}{n!h^n}$

$$\therefore\ y = y_n + (x - x_n)\frac{\nabla y_n}{n} + (x - x_n)(x - x_{n-1})\frac{\nabla^2 y_n}{2!h^2} + \ldots\ldots + (x - x_n)\ldots\ldots(x - x_1)\frac{\nabla^n y_n}{n!h^n}$$

Put $X = \dfrac{x - x_n}{h}$ (ie) $X = x_n + hx$

$$\frac{x - x_{n-1}}{h} = \frac{(x - x_n) + (x_n - x_{n-1})}{h} = \frac{hX + h}{h} = \frac{h(X+1)}{h} = X+1$$

$$\frac{x - x_{n-2}}{h} = \frac{(x - x_n) + (x_n - x_{n-2})}{h} = \frac{hX + 2h}{h} = \frac{h(X+2)}{h} = X+2$$

$$\frac{x - x_1}{h} = \frac{(x - x_n) + (x_n - x_1)}{h} = \frac{hX + (n-1)h}{h} = \frac{h(X + (n-1))}{h} = X + (n-1)$$

$$\therefore\ y = y_n + x \cdot \nabla y_n + \frac{x(x+1)}{2!}\nabla^2 y_n + \ldots\ldots + \frac{x(x+1)\ldots(x+(n-1))}{n!}\nabla^n y_n$$

Example: 1

The following data given the melting point of an alloy of lead and zinc; θ is the temperature is degrees centigrade; x is percent of lead:-

x :	40	50	60	70	80	90
θ :	184	204	226	250	276	304

Find θ when x =43 and when x = 84.

Solution:

Since x = 43 is nearer to the beginning of the table. We use Newton's forward formula.

We from the difference table.

Also h = constant =10.

x	θ	Δθ	$\Delta^2\theta$	$\Delta^3\theta$
40	184	20	2	0
50	204	22	2	0
60	226	24	2	0
70	250	26	2	
80	276	28		
90	304			

The topmost diagonal gives the forward differences of y_0 while the lowermost diagonal gives the backward differences of y_n.

By Newton's Forward interpolation formula

$$y = y_0 + X \, . \, \Delta y_0 + X(X-1)/2! \; \Delta^2 y_0 + \ldots\ldots + \frac{X(X-1)(X-2)\ldots.(X-n+1)}{n!} \Delta^n y_0.$$

$$\therefore \; \theta = \theta_0 + X \, . \, \Delta y_0 + \frac{X(X-1)}{2!} \; \Delta^2 \theta_0$$

Here $\theta_0 = 184$, $\Delta \theta_0 = 20$, $\Delta^2\theta_0 = 2$

Let $\theta_{(43)}$ be the value of x and x = 43

$$X = \frac{43-40}{10} = \frac{+3}{10} = +0.3$$

$$\theta_{(43)} = 184 + (0.3)\,20 + \frac{(0.3)\,(0.3-1)}{2\,!} \,.2$$

$$= 184 + 6 + (-0.105)\,(2) = 184 + 6 - 0.21$$

$\theta_{(43)} = 189.79$

θ_{43} can also be calculated from the value of $\theta = 204$.

In that case $X = \dfrac{43-50}{10} \qquad = -0.7$

$$\theta_{(43)} = 204 + (-0.7)\ 22 + \frac{(-0.7)(-0.7-1)}{2!} \qquad (2)$$

$$= 204 - 15.4 + 1.19$$

$$= 189.79$$

Since x = 84 is near the end of the table. We have to use Newton's Backwards Interpolation.

Formula:

$$y = y_o + X.\ \Delta y_0 \frac{X(X+1)}{2!}\ \nabla^2 Y_n +,,,,,- + \frac{X(X+1)(X+n-1)}{n!} \nabla^n Y_n$$

In that case $\theta = \theta_n + x\ \nabla \theta_n + \frac{X(X+1)}{2!}\ \nabla^2 \theta_n$

$$-X = \frac{x_n - x}{h} = \frac{90-84}{10} = 0.6$$

$$\therefore X = -0.6$$

$$\therefore\ \theta_{84} = 304 + (-0.6)\ 28 + \frac{(-0.6)(-0.6+1)}{2}.2$$

$$= 304 - 16.8 - 0.24$$

$$= 286.96$$

θ_{84} can also be calculated from $\theta = 276$.

In this case $-X = \frac{80-84}{10} = \frac{4}{10}$

$$\therefore X = 0.4$$

$$\theta_{84} = 276 + (0.4)\ (26) + \frac{(0.4)(0.4+1)}{2}\ .2$$

$$= 276 + 10.4 + 0.56$$

$\theta_{84} = 286.96,$

Suppose we have to determine the relation between x and θ.

We have $\theta = \theta_0 + X\ \Delta\ \theta_0 + \frac{X(X-1)}{2!}\ \Delta^2\ \theta_0$

Here $\theta_0 = 184$, $\Delta\theta_0 = 20$, $\Delta^2\theta_0 = 2$, $x = \frac{x-40}{10}$

$$\therefore \theta_0 = 184 + \frac{x-40}{10}(20) + \frac{\left(\frac{x-40}{10}\right)\left(\frac{x-40}{10}-1\right)}{2!}$$

$$= 184 + 2x - 80 + \frac{(x-40)(x-50)}{100}$$

$$= 184 + 2x - 80 + \frac{(x-40)(x-50)}{100}$$

$$= 184 + 2x - 80 + \frac{x^2}{100} - \frac{50x}{100} - \frac{40x}{100}x + \frac{2000}{100}$$

$$= 124 + 0.01x^2 - 0.9x + 2x$$

$$= 124 + 1.1x + 0.01x^2$$

Example:2

Calculate the value of y when x = 0.47 from following data:-

x:	0	0.1	0.2	0.3	0.4	0.5
y:	1.0000	1.1103	1.2428	1.3997	1.5836	1.7974

Solution:

Since we have to find the value of y corresponding to a value near the end of the table.

Using Newton's Backward formula.

The backward differences are calculated and tabulated below:-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1.0000				
		0.1103			
0.1	1.1103		0.0222		
		0.1325		0.0022	
0.2	1.2428		0.0244		0.0004
		0.1569		0.0026	
0.3	1.3997		0.0270		0.0003
		0.1839		0.0029	
0.4	1.5836		0.0299		
		0.2138			
0.5	1.7974				

In this case h = 0.1 and $-X = \dfrac{x_n - x}{h} = \dfrac{0.03}{0.1}$

$\therefore X = -0.3.$

Newton's Backward difference formula is

$$y = y_n + X\Delta y_n + \frac{X(X+1)}{2!}\nabla^2 y_n + \frac{X(X+1)(X+2)}{3!}\nabla^3 y_n + \frac{X(X+1)(X+2)(X+3)}{4!}\nabla^4 y_n$$

We shall take 5 terms in the left side.

$$y = 1.7974 + (-.3)(0.2138) + \frac{(-0.3)(-0.3+1)}{2!}(0.0299)\frac{(-0.3)(-0.3+1)(-0.3+2)}{3!}$$

$$(0.0099)\frac{(-0.3)(-0.3+1)(-0.3+2)(-0.3+3)}{4!}(0.0003)$$

$= 1.7974 - 0.06414 - .0031395 - .00058905 - 00001204875$

$y. = 1.7295$

Example:3

From the following table find the value of tab $45°15^1$.

$x°$:	45	46	47	48	49	50
tan $x°$:	1.00000	1.03553	1.07237	1.11061	1.5037	1.19175

Solution:

We can use forward interpolation formula:

Also h = 1

$$X = \frac{x - x_0}{h} = \frac{45°15^1 - 45°}{1°}$$

x	y = tan x°	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
45°	1.00000	0.03553	0.00131	0.00009	0.00003	−0.00005
46°	1.03553	0.03684	0.00140	0.00012	−0.00002	
47°	1.07237	0.03824	0.00152	0.00010		
48°	1.11061	0.03976	0.00162			
49°	1.15037	0.04138				
50°	1.19175					

$$\therefore y = y_0 + \Delta y_0 + \frac{X(X-1)}{2}\Delta^2 y_0 + \ldots\ldots$$

$$=1.0000+(0.25)(0.03553)+\frac{(0.25)(0.25-1)}{2}(0.00131)+\frac{(0.25)(0.25-1)(0.25-2)}{3!}(0.00009)$$

$$+\frac{(0.25)(0.25-1)(0.25-2)(0.25-3)}{4!}(0.00003)$$

$$+\frac{(0.25)(0.25-1)(0.25-2)(0.25-3)(0.25-4)}{5!}(0.00005)$$

$$= 1.0000 + .0088825 - .0001228125 + .000004921875 - 0.037597656$$
$$+0.000001409912109$$

y. = 1.00876.

Example: 4

The population of a town is as follows.

Year x :	1941	1951	1961	1971	1981	1991
Population in lakhs :	20	24	29	36	46	51

Estimate the population increase during the period 1946 to 1976.

Solution:

Let us find the population at x = 1946 and x = 1976.

Since, six data are given p(x) is of degree 5.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1941	20					
		4				
1951	24		1			
		5		1		
1961	29		2		0	
		7		1		−9
1971	36		3		9	
		10		8		
1981	46		5			
		5				
1991	51					

$$X = \frac{x-x_0}{h} = \frac{1946-1941}{10} = \frac{5}{10} = \frac{1}{2}$$

Using forward Formula:-

$$y = y_0 + X\Delta y_0 + \frac{X(X-1)}{2!}\Delta^2 y_0 + \frac{X(X-1)(X-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots$$

$$\therefore y = 20 + \frac{1}{2}(4) + \frac{\frac{1}{2}\left(+\frac{1}{2}-1\right)}{2!}(1) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(1) +$$

$$\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!}(0) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-4\right)}{5!}(-1)$$

$$= 20+2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{243}(0)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{120}(-9)$$

$$= 20 + 2 - 0.125 + 0.0625 - 0.24609375$$

$y = 21.69.$

Using Backward Formula:

$$X = \frac{1976-1991}{10} = \frac{-15}{10} = -\frac{3}{2}$$

$$y = y_n + x\,\nabla y_n + \frac{x(x+1)}{2!}\nabla^2 y_n + \frac{x(x+1)(x+2)}{3!}\nabla^3 y_n + \ldots\ldots\ldots$$

$$= 51 - \frac{3}{2}(5) + \frac{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)}{2}(-5) + \frac{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}{6}(-8) + \frac{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{24}$$

$$(-9) + \frac{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{120}(-9)$$

$$= 51 - 7.5 - 1.875 - 0.5\ 0.2109375 - 0.10546875$$

$y = 40.8085938$

$\therefore$ Increase in population during the period.

$= 41.809 - 21.69$

$= 20.119$ lakhs

Exercise

1. Find an approximate value of θ when t = 3.5 given

t:	0	1	2	3	4	5	6	7	8
θ:	50	41.66	34.46	28.28	22.94	18.32	14.42	11.06	8.06

2.

x:	1.0	1.1	1.2	1.3	1.4
f(x):	0.84147	0.89121	0.93204	0.96356	0.98545

x:	1.5	1.6	1.7	1.8
f(x):	0.99749	0.99957	0.99166	0.97358

Calculate f(1.02) and /f(1.75) correct to five decimal places.

3) From the following data determine an approximate value for y corresponding to x = 2.2 correct to 3 places of decimals:

x:	1	2	3	4	5	6	7	8
y:	1.105	1.808	2.614	3.604	4.857	6.451	8.467	10.985

4. Given the data

x:	19	20	21	22	23	24	25
y:	91.00	100.25	110.00	120.25	131.00	142.25	154.00

find an approximate value of y when x = 23.6 & x = 25.5

5. The following table gives the value of the elliptic integral $F(\phi) = \int_0^{\phi} \frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}$ for certain in equidistant values of ϕ. Find the value $F(23.5°)$.

ϕ :	21°	22°	23°	24°	25°	26°
$F\phi$:	0.3706	0.3887	0.4068	0.4250	0.4433	0.4616

6.3 Divided Differences

Let the function y = f(x)

Assume the values $f(x_0)$, $f(x_1)$.....$f(x_n)$ corresponding to the arguments x_0, x_1, x_2. . . . x_n respectively where the intervals $x_1 - x_0$, $x_2 - x_1$,.... $x_n - x_{n-1}$ need not be equal.

The divided differences of y are defined as follows:

First order divided difference:

$$\frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{y_1-y_0}{x_1-x_0} = y\,(x_1, x_0) = \underset{x_1}{\Delta} f(x_0)$$

In the same notation

$$f(x_1,x_2) = \underset{x_2}{\Delta} f(x_1) = \frac{y_2-y_1}{x_2-x_1} = y(x_2, x_1)$$

$$f\,(x_3,x_2) = \underset{x_3}{\Delta}\ f(x_2) = \frac{y_3-y_2}{x_3-x_2} = y(x_3, x_2)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$f(x_{n-1}, x_n) = \underset{x_n}{\Delta}\ f(x_{n-1}) = \frac{y_n-y_{n-1}}{x_n-x_{n-1}} y(x_n, x_{n-1})$$

Second divided Difference:

The second divided difference of f(x) for three arguments x_0, x_1 ,x_2 is defined as

$$f(x_0, x_1 ,x_2) = \underset{x_1,x_2}{\Delta^2}\ f(x_0) = \frac{y(x_2,x_1)-y(x_1,x_0)}{x_2-x_1} = y(x_2,x_1,x_0).$$

$$f(x_1, x_2 ,x_3) = \underset{x_2,x_3}{\Delta^2}\ f(x_1) = \frac{y(x_3,x_2)-y(x_2,x_1)}{x_3-x_1} = y(x_3,x_2,x_1)$$

$$\vdots$$

Third order Differences:

$$f(x_0, x_1 ,x_2,x_3) = \underset{x_1,x_2,x_3}{\Delta^3}\ f(x_0) = \frac{y(x_3,x_2,x_1)-y(x_2,x_1,x_0)}{x_3-x_0} = y(x_3,x_2,x_1,x_0)$$

$$f(x_0, x_1 ,x_2,x_3 ,x_4) = \underset{x_2,x_3,x_4}{\Delta^3}\ f(x_1) = \frac{y(x_4,x_3,x_2)-y(x_3,x_2,x_1)}{x_4-x_1} = y(x_3,x_2,x_1,x_0)$$

And so on.

$$\therefore \Delta^r\ y_k = \Delta^{r-1} y_{k+1} - \Delta^{r-1} y_k\,.\ s$$

1. Properties of Divided Differences:

The value of any divided difference is independent of the order of the arguments.(ie) the divided differences are

It is seen that $y(x_1,x_0) = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_0 - y_1}{x_0 - x_1} = y(x_0, x_1)$

Also $y(x_1,x_0) = \frac{y_1}{x_1 - x_0} - \frac{y_0}{x_1 - x_0} = \frac{y_1}{x_1 - x_0} + \frac{y_0}{x_0 - x_1}$

$$Y(x_2,x_1,x_0) = \frac{y(x_2,x_1) - y(x_1,x_0)}{x_2 - x_0}$$

$$= \frac{1}{x_2 - x_0}\left[\frac{y_2}{x_2 - x_1} - \frac{y_1}{x_2 - x_1} - \frac{y_1}{x_1 - x_0} - \frac{y_0}{x_0 - x_1}\right]$$

$$= \frac{1}{x_2 - x_0}\left[\frac{y_2}{x_2 - x_1} + y_1\left(\frac{1}{x_1 - x_2} - \frac{1}{x_1 - x_0}\right) - \frac{y_0}{x_0 - x_1}\right]$$

$$= \frac{1}{x_2 - x_0}\left[\frac{y_2}{x_2 - x_1} + \frac{(x_2 - x_0)y_1}{(x_1 - x_2)(x_1 - x_0)} + \frac{y_0}{x_1 - x_0}\right]$$

$$= \frac{y_2}{(x_2 - x_0)(x_2 - x_1)} + \frac{y_1}{(x_1 - x_0)(x_2 - x_0)} + \frac{y_0}{(x_0 - x_1)(x_0 - x_2)}$$

Similarly

$$y(x_3,x_2,x_1,x_0) = \frac{y_3}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}$$
$$+ \frac{y_1}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} + \frac{y_0}{(x_0 - x_1)(x_0 - x_1)(x_0 - x_2)}$$

Continuing this process,

We get

$$y(x_n,x_{n-1},\ldots\ldots x_0) = \frac{y_n}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})} + \frac{y_{n-1}}{(x_{n-1} - x_0)(x_{n-1} - x_1)\ldots(x_{n-1} - x_n)}$$
$$+\ldots + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} + \frac{y_0}{(x_0 - x_1)\ldots(x_0 - x_n)}$$

This is symmetrical w.r. to any two arguments.

$\therefore$ The divided differences are symmetrical w.r.t any two argument.

2. The operator $\Delta\!\!\!\!|\;$ is linear

If f(x) and g(x) are two function and α and β are constant. Then

$$\Delta\!\!\!\!|\;[\alpha f(x) + \beta g(x) = \frac{[\alpha f(x_1) + \beta g(x_1) - \alpha f(x_0) + \beta g(x_0)]}{x_1 - x_0}$$

$$= \alpha \frac{f(x_1) - f(x_0)}{x_1 - x_0} + \beta \frac{g(x_1) - g(x_0)}{x_1 - x_0}$$

$$= \alpha \Delta\!\!\!\!|\; f(x) + \beta \Delta\!\!\!\!|\; g(x).$$

Remark:

i) Setting $\alpha = \beta = 1$

$$\Delta\!\!\!\!|\;[f(x) + g(x)] = \Delta\!\!\!\!|\; f(x) + \Delta\!\!\!\!|\; g(x)$$

ii) Setting $\beta = 0$

$$\Delta\!\!\!\!|\;[\alpha f(x)] = \alpha \Delta\!\!\!\!|\; f(x)$$

3. The n^{th} divided difference of a polynomial of degree n are constants.

Proof:

Taking $f(x) = x^n$ where n is a positive integer

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1^n - x_0^n}{x_1 - x_0} = \left(x_1^n - x_0^n\right)\left(x_1 - x_0\right)^{-1}$$

$$= \left(x_1^n - x_0^n\right)\left[x_1^{-1} + x_1^{-2} x_0 + x_1^{-3} x_0^2 + \ldots\ldots + x_0^{-1}\right]$$

$$= x^{n-1} + x_0\ x_1^{n-2} + x_0^2\ x_1^{n-3} + \ldots\ldots + x_0^{n-1}$$

= a polynomial function of degree (n – 1) and symmetrical in x_0, x_1 with leading coefficient 1.

Again,

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$= \frac{\left(x_2^{n-1} + x_1\ x_2^{n-2} + \ldots\ldots + x_1^{n-1}\right) - \left(x_0^{n-1} + x_1\ x_0^{n-2} + \ldots\ldots + x_1^{n-1}\right)}{x_2 - x}$$

$$= \frac{x_2^{n-1} - x_0^{n-1}}{x^2 - x_0} + \frac{x_1\left(x_2^{n-2} - x_0^{n-2}\right)}{x_2 - x_0} + \ldots\ldots\ldots + \frac{x_1^{n-2}\,(x_2 - x_0)}{x_2 - x_0}$$

$$= \left(x_2^{n-2} + x_0\, x_2^{n-3} + \ldots\ldots + x_0^{n-2}\right) + x_1 \left[x_2^{n-3} + x_0\, x_2^{n-4} + \ldots\ldots + x_0^{n-3}\right] + \ldots\ldots\ldots\ldots + x_1^{n-2}$$

= a polynomial of degree (n – 2) and symmetrical x_0, x_1, x_2 with leading coefficient 1.

Proceeding in this way, the r^{th} divided differences of x^n will be a polynomial of degree (n – r) and symmetrical in x_0, x_1, x_2 ....x_r with leading coefficient 1.

Hence n^{th} order divided differences of x^n will be a polynomial of degree $n - n = 0$,

(ie) $\Delta^n x^n = 1$

$\Delta^{n+1} x^n = 0$ for I = 1,2 ......

$\Delta^n \left[a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_n\right]$

$= a_0 \Delta^n x^n + a_1 \Delta^n x^{n-1} + \ldots\ldots\ldots + \Delta^n a_n$

$= a_0 . 1 + 0 + 0 + \ldots\ldots\ldots + 0 = a_0$

The converse is also true. (ie) if the n^{th} divided difference of a polynomial is constant, the polynomial is of degree n.

Example: 1

Find the divided differences of y given the following table:-

x :	2	3	5	8	13
y :	24	32	84	108	208

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	24				
3	32	$\frac{32-34}{3-2} = 8$	$\frac{26-8}{5-2} = 6$		
5	84	$\frac{84-32}{5-3} = 26$	$\frac{8-26}{8-3} = -3.6$	$\frac{-3.6-6}{8-2} = 1.6$	$\frac{0.51+1.6}{13-2} = -.192$
8	108	$\frac{108-84}{8-5} = 8$	$\frac{20-8}{13-5} = 1.5$	$\frac{1.5+3.6}{13-3} = 0.51$	
13	208	$\frac{208-108}{13-8} = 20$			

Example: 2

Find the divided differences of y given the following table:-

x :	1	2	7	8
f(x) :	1	5	5	4

Solution:

X	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1	$\frac{5-1}{2-1}=4$		
			$\frac{0-4}{7-1}=\frac{2}{3}$	
2	5	$\frac{5-5}{7-2}=0$		$\frac{-\frac{1}{6}+\frac{2}{3}}{8-1}=\frac{1}{14}$
			$\frac{-1-0}{8-2}=-\frac{1}{6}$	
7	5	$\frac{4-5}{8-7}=1$		—
			—	
8	4	–		

Example: 3

Find the differences of y given the following table:

x :	4	5	7	10	11	13
f(x) :	48	100	294	900	1210	2028

Solution:

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48	$\frac{100-48}{5-4}=52$			
			$\frac{97-52}{7-4}=15$		
5	100	$\frac{294-100}{7-5}=97$		$\frac{27-21}{11-5}=1$	
			$\frac{202-97}{10-5}=21$		0
7	294	$\frac{900-294}{10-7}=202$		$\frac{21-15}{10-4}=1$	
			$\frac{310-202}{11-7}=27$		0
10	900	$\frac{1210-900}{11-10}=310$		$\frac{33-27}{13-7}=1$	
			$\frac{409-310}{13-10}=33$		
11	1210	$\frac{2028-1210}{13-11}=409$			
13	2028	—			

Exercise:

1. Find the divided differences of y given 1ı following table.

i)

x :	3	4	6	9	10
y :	50	102	296	902	1212

ii)

x :	– 2	– 1	1	2	5
f(x) :	– 83	– 3	– 5	– 15	351

6.4 NEWTONS DIVIDED DIFFERENCES FORMULA

Let y = f(x) take values $f(x_0)$, $f(x_1)$, $f(x_2)$..........$f(x_n)$ corresponding to the arguments $x_0, x_n, \ldots\ldots. x_n$.

$$f(x, x_0) = y(x, x_0) = \frac{y - y_0}{x - x_0}$$

$$y(x, x_0)(x - x_0) = y - y_0$$

$$y = y_0 + y(x_1\ x_0)(x - x_0) \qquad (1)$$

Similarly $y(x, x_0, x_1) = \dfrac{y(x_1, x_0) - y(x_0, x_1)}{x - x_1}$

$$y(x, x_0, x_1)(x - x_1) = y(x, x_0) - y(x_0, x_1)$$

$$\therefore\ y(x, x_0) = y(x_0, x_1) + (x - x_1)\, y(x, x_0, x_1) \qquad (2)$$

Similarly

$$y(x, x_0, x_1, x_2) \equiv \frac{y(x, x_0, x_1) - y(x_0, x_1, x_2)}{x - x_2}$$

$$y(x, x_0, x_1, x_2)(x - x_2) = y(x, x_0, x_1) - y(x_0, x_1, x_2)$$

$$\therefore\ y(x, x_0, x_1) = y(x_0, x_1, x_2) + (x - x_2)\, y(x, x_0, x_1, x_2) \qquad (3)$$

....

....

$$y(x_1, x_0, x_1, \ldots\ldots\ldots x_{n-1}) = y(x_0, x_1, \ldots\ldots\ldots x_n) + (x - x_n)\, y(x, x_0, \ldots\ldots x_n)$$

Multiplying the equation (2) by $x - x_0$

$$y(x, x_0)(x - x_0) = y(x_0, x_1)(x - x_0) + (x - x_1)(x - x_0)\, y(x, x_0, x_1)$$

Multiplying equation (3) by $(x - x_0)(x - x_1)$

$$(x - x_0)(x - x_1)\, y(x_1, x_0, x_1) = (x - x_0)(x - x_1)\, y(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)\, y(x, x_0, x_1, x_2)$$

and so on an

Adding, we get

$$y = y_0 + (x - x_0)\, y(x_0, x_1) + (x - x_0)(x - x_1)\, y(x_0, x_1, x_2) \ldots\ldots\ldots + (x - x_0)(x - x_1) \ldots\ldots (x - x_{n-1})\, y(x_0, x_1, \ldots\ldots x_n) + R_n$$

Where $R_n = (x - x_0)(x - x_1) \ldots\ldots\ldots (x - x_n)\, y(x, x_0, x_1, \ldots\ldots x_n)$

If y is a polynomial of degree n is x,

We get $y(x, x_0, x_1 \ldots\ldots x_n) = 0$.

$$\therefore\ y = y_0 + (x - x_0)\, y(x_0, x_1) + (x - x_0)(x - x_0)(x - x_1)\, y(x_0, x_1, x_2) \ldots\ldots + \ldots\ldots + (x - x_0)(x - x_1) \ldots\ldots (x - x_{n-1})\, y(x_0, x_1, \ldots\ldots x_n)$$

This equation is known as Newtons divided differences formula.

This can be written as

$$y = y_0 + (x - x_0) \Delta y_0 + (x - x_0)(x - x_1) \Delta^2 y_0 + \ldots\ldots\ldots + (x - x_0)(x - x_1) \ldots\ldots (x - x)_{n-1} \Delta^n y_0$$

If y is polynomial of degree higher than n,

$$R_n(x) \neq 0$$

Hence the error term is

$$R_n(x) = (x - x_0)(x - x_1) \ldots\ldots\ldots (x - x_n)\, y(x, x_0, x_1, \ldots\ldots x_n).$$

Relation between Divided Differences and forward Differences:

If the arguments $x_0, x_1, x_2 \ldots\ldots$ are equalty spaced then
We have

$$x - x_0 = x_2 - x_1 = \ldots\ldots = x_n - x_{n-1} = h$$

$$\underset{|}{\Delta} y_0 = \frac{y_1 - y_0}{y_1 - y_0} = \frac{1}{h}\Delta y_0$$

$$\underset{|}{\Delta}^2 y_0 = \frac{\underset{|}{\Delta} Y_1 - \underset{|}{\Delta} Y_0}{x_2 - x_1} = \frac{\frac{1}{h}\Delta y_1 - \frac{1}{h}\Delta y_0}{2h} = \frac{1}{2h^2}(\Delta y_1 - \Delta y_0)$$

$$\underset{|}{\Delta}^2 y_0 = \frac{\Delta^2 y_0}{2!\,h^2} \ldots\ldots\ldots\ldots\ldots\ldots$$

Similarly $\underset{|}{\Delta}^3 y_0 = \dfrac{\Delta^3 y_0}{3!\,h^3}$

In general $\underset{|}{\Delta}^n y_0 = \dfrac{\Delta^3 y_0}{n!\,h^n}$

Substituting these values in the divided difference formula, we get

$$y = y_0 + (x - x_0)\frac{\Delta y_0}{h} + \frac{(x - x_0)(x - x_1)}{2!\,h^2}\Delta^2 y_0 + \ldots\ldots + \ldots\ldots + \frac{(x - x_0)(x - x_1)\ldots\ldots(x - x_{n-1})}{n!\,h^n}\Delta^n y_0$$

If $x = x_0 + xh$ & $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \ldots\ldots = x_n - x_{n-1} = h$

Where n need not be an integer.

$$y = y_0 + Xh\,\frac{\Delta y_0}{h} + \frac{Xh(Xh - h)}{2!\,h^2}\Delta^2 y_0 + \ldots\ldots$$

$$y = y_0 + X\,\Delta y_0 + \frac{X(X-1)}{2!}\Delta^2 y_0 + \ldots\ldots$$

If x is not an integer, it is an infinite series, whereas if x is an integer, it is a finite series.

Example:

The value of f(x) for values of x are given as f(1) = 1, f(2) = 5, f(7) = 5, f(8) = 4, Find f(x).

Solution:

The divided difference table is given below:

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1			
		$\frac{5-1}{2-1}=4$		
2	5		$\frac{0-4}{7-1} = -4/6 = -\frac{2}{3}$	
		$\frac{5-5}{7-2}=0$		$\frac{-1/6+\frac{2}{3}}{8-1} = -1/14$
			$\frac{-1-0}{8-2} = -1/6$	
7	5	$\frac{4-5}{8-7}=1$		
8	4			

Newton Divided Formula is

$y = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0 + \ldots..$

In this case $x_0 = 1$, $x_1 = 2$, $x_2 = 7$

$$y_0 = 1, \qquad \Delta y_0 = 4, \qquad \Delta^2 y_0 = -\frac{2}{3}, \qquad \Delta^3 Y_0 = \frac{1}{14}$$

$$\therefore f(x) = 1 + (x-1)(4) + (x-1)(x-2)(-2/3) + (x-1)(x-2)(x-7)\frac{1}{14}$$

$$= 1 + 4x - 4 - 2/3\,x^2 + 2x - 4/3 + \frac{1}{14}x^3 - \frac{10}{14}x^2 + \frac{23}{14}x - 1$$

$$= \frac{1}{42}\left(3x^3 - 58x^2 + 321x - 224\right)$$

Example: 2

A certain biquadratic polynomial passes through the points (2,3) (4,43) (5, 138) (7, 778) and (8, 1515). Find its equation.

Solution:

The given points are at unequal intervals and the table of divided difference in

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	3				
		$\frac{43-3}{42-2} = \frac{40}{2} = 20$			
			$\frac{95-20}{5-2} = \frac{75}{3} = 25$		
4	43				
		$\frac{138-43}{5-4} = \frac{95}{1}$		$\frac{75-25}{7-2} = \frac{50}{5} = 10$	
			$\frac{320-95}{7-4} = \frac{225}{3} = 75$		$\frac{16-10}{8-2} = \frac{6}{6} = 1$
5	138	$\frac{778-138}{7-5} = \frac{640}{2} = 320$			
				$\frac{139-75}{8-4} = \frac{64}{4} = 16$	
7	778		$\frac{737-320}{8-5} = \frac{417}{3} = 139$		
		$\frac{1515-778}{8-7} = 737$			
8	1515	–	–	–	–

Newton's Formula for divided differences is

$$y = y_0 + (x - x_0)\ \underset{|}{\Delta} y_0 + (x - x_0)(x - x_1)\ \underset{|}{\Delta}^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\ \underset{|}{\Delta}^3 y_0$$
$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)\ \underset{|}{\Delta}^4 y_0$$

In this case $x_0 = 2,\ x_1 = 4,\ x_2 = 5,\ x_3 = 7,\ x_4 = 8.$

$y_0 = 3,\ \underset{|}{\Delta} y_0 = 20,\ \underset{|}{\Delta}^2 y_0 = 25,\ \underset{|}{\Delta}^3 y_0 = 10,\ \underset{|}{\Delta}^4 y_0 = 1$

$$\therefore\ Y = 3 + (x - 2)(20) + (x - 2)(x - 4)(25) + (x - 2)(x - 4)(x - 5)(10)$$
$$+ (x - 2)(x - 4)(x - 5)(x - 7)(1)$$

$$= 3 + 20x - 40 + 25x^2 - 150x + 200 + 10x^3 - 10x^2 + 380x - 400 + x^4 - 18x^3 - 39x^2 + 226x - 280$$

$$= x^4 - 10x^3 + 36x^2 - 36x - 5.$$

Example: 3

By means of divided differences, find the value of y_{20} from the following table:-

x :	12	18	22	24	32
y :	146	836	1948	2796	9236

Solution:

The divided difference table in this case

x	y_x	$\underset{|}{\Delta} y_x$	$\underset{|}{\Delta}^2 y_x$	$\underset{|}{\Delta}^3 y_x$	$\underset{|}{\Delta}^4 y_x$
12	146				
		115			
18	836		16.3		
		278		0.59	
22	1948		24.33		.02
		424		0.98	
24	2796		38.10		
		805			
32	9236				

By Newton's Formula for dividend differences

$$y_x = y_0 + (x - x_0)\underset{|}{\Delta} y_0 + (x - x_0)(x - x_1)\underset{|}{\Delta}^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\underset{|}{\Delta} y_0$$

$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)\ \underset{|}{\Delta}^4 y_0$$

In this case $x = 20,\ x_0 = 12,\ x_1 = 18,\ x_2 = 32,\ x_3 = 24$

$y_0 = 146,\ \underset{|}{\Delta} y_0 = 115,\ \underset{|}{\Delta}^2 y_0 = 16.3,\ \underset{|}{\Delta}^3 y_0 = 0.59,\ \underset{|}{\Delta}^4 y_0 = 0.2$

y_{20}=146+(20 −12)115+(20 −12)(20 − 18)(16.3)+(20 − 12)(20 − 18)(20 − 24) (0.59) + (20 − 12) (20 − 18) (20 − 32) (20 − 24) (0.02)

= 146 + 920 + 260.8 − 37.76 + 15.36

Y_{20} = 1305.36

Example: 4

Find y_x given

$y_0 = -9,\ y_1 = 0,\ y_3 = 0,\ y_5 = -124,\ y_6 = 0,\ y_9 = 6552$

Since $y_1 = 0$, $y_3 = 0$, $y_6 = 0$, (x − 1) (x − 3) (x − 6) must be factor of y_x.

Solution:

Since 6 entries are given, y_x may be taken as a polynomial of second degree.

Hence $y_x = (x-1)(x-3)(x-6)\, f(x)$

Where f(x) is a polynomial of second degree.

$$f(x) = \frac{y_x}{(x-1)(x-3))(x-6)}$$

$$f(0) = \frac{y_0}{(-1)(-3)(-6)} = \frac{-9}{-18} = 0.5$$

$$f(5) = \frac{y_5}{(4)(2)(-1)} = \frac{-124}{-8} = 15.5$$

$$f(9) = \frac{y_9}{(8)(6)(3)} = \frac{6552}{144} = 45.5$$

We can find f(x) from their divided differences.

x	f(x)	Δf(x)	Δ^2f(x)
0	0.5		
		3	
5	15.5		0.5
		7.5	
9	45.5		

Hence $f(x) = f(0) + (x - 0)\Delta f(x) + (x - 0)(x - 5)\Delta^2 f(x)$

$$= 0.5 + x(3) + x(x - 5)(0.5)$$

$$= 0.5 + 3x + 0.5x^2 - 2.5x$$

$$f(x) = 0.5 + 0.5x + 0.5x^2 = \frac{x^2 + x + 1}{2}$$

Hence $y_x = \frac{1}{2}(x - 1)(x - 3)(x - 6)(x^2 + x + 1)$

Exercise:

1. Using the method of divided differences calculate

i) f(8) to the nearest integer from the following data:

x :	1	2	4	7	12
f(x) :	23	31	83	107	207

ii) f(3.1) from the following table:

x :	− 2	− 1	1	2	5
f(x) :	− 83	− 3	− 5	− 15	351

iii) y_T and y_{14} from the following table:

x :	3	4	6	9	10	12
y :	50	102	296	902	1212	2030

iv) log(1.45) from the following table:

x :	143	147	148	150
log x :	2.1553	2.1673	2.1703	2.1761

2. Using the method of divided differences find f(x) from the following table.

x :	0	1	4	5
f(x) :	9	12	69	124

3. Find the four the divided differences with arguments x_0, x_1, x_2, x_3, x_4 of the function $\frac{1}{x}$.

4. If $f(x) = x^4 - x^2 + 1$, find $\Delta^4 f(x)$. [Divided Difference]

Answer:

1. i) 94 ii)23.896 iii) $y_7 = 450$, $y_{14} = 3152$ iv) 2.1614

2. $x^3 - x^2 + 3x + 9$

3. $\dfrac{1}{x_0\ x_1\ x_2\ x_3\ x_4}$

4. 19.

6.5 GAUSS'S FORMULA

Central Difference Interpolation Formula:

Newton's forward and backward formulae are best suited for interpolation near the beginning and end of a table of differences. For interpolation near the middle of a difference table, a centre difference interpolation is required. For this origin is shifted to some convenient point in the middle so that the arguments are

.... $x_0 - 3h$, $x_0 - 2h$, $x_0 - h$, $x_0 + h$, $x_0 + 2h$, $x_0 + 3h$, ...and their corresponding entires are

$y_{-3}, y_{-2}, y_{-1}, y_0, y_1, y_3,$ .....

Newton's Divided Differences Formula is

$$y = y_0 + (x - x_0)\, y(x_0, x_1) + (x - x_0)(x - x_1)\, y(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)\, y(x_0, x_1, x_2, x_3) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\, y(x_0, x_1, x_2, x_3, x_4) + \ldots\ldots\ldots$$

In this formula,

Put $x_1 = x_0 + h$, $x_2 = x_0 - h$, $x_3 = x + 2h$, $x_4 = x - 2h$, $x_5 = x + 3h$, $x_6 = x - 3h$ and so on.

We get

$$y = y_0 + (x - x_0)\, y(x_0, x_0 + h) + (x - x_0)(x - x_0 - h)\, y(x_0, x_0 + h_1\, x_0 - h) + (x - x_0)(x - x_0 - h)(x - x_0 + h)\, y(x_0, x_0 + h, x_0 - h, x_0 + 2h) + (x - x_0)(x - x_0 - h)(x - x_0 + h)(x - x_0 - 2h)\, y(x_0, x_0 + h, x_0 - h, x_0 + 2h, x_0 - 2h) + \ldots\ldots\ldots$$

Since in the divided difference, the arguments may be interchanged provided their corresponding values of y are interchanged.

Put $X = \frac{x - x_0}{h}$, (ie) $x - x_0 = xh$.

$$\therefore\ y = y_0 + Xh\, y(x_0, x_0 + h) + Xh(xh - h)\, y(x_0, x_0 + h, x_0 - h) + Xh\,(Xh - h)\,(Xh + h)\, y(x_0, x_0 + h, x_0 - h, x_0 + 2h) + Xh\,(Xh - h)\,(Xh + h)\,(Xh - 2h)\, y(x_0, x_0 + h, x_0 - h, x_0 + 2h, x_0 - 2h) + \ldots\ldots$$

$$y = y_0 + Xhy(x_0, x_0 + h) + X(X - 1)h^2\, y(x_0, x_0 + h, x_0 - h) + X(X - 1)\,(X + 1)\,h^3\, y(x_0, x_0 + h, x_0 - h, x_0 + 2h) + X(X - 1)\,(X + 1)\,(X - 2)h^4\, y(x_0, x_0 + h, x_0 - h, x_0 + 2h, x_0 - 2h) + \ldots\ldots$$

We have show that

$$y(x_0, x_0 + h) = \frac{\Delta y_0}{h}$$

$$y\,(x_0, x_0 + h, x_0 - h) = y\,(x_0 - h, x_0, x_0 + h) = \frac{\Delta^2 y_{-1}}{2!\,h^2}$$

Since the order of arguments can be changed

$$y(x_0, x_0 + h, x_0 - h, x_0 + 2h) = y(x_0 - h, x_0, x_0 + h, x_0 + 2h) = \frac{\Delta^3\, y_{-1}}{3!\,h^3}$$

and $y(x_0, x_0 + h, x_0 - h, x_0 + 2h, x_2 - 2h) = y(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h)$

$$= \frac{\Delta^4\, y_{-2}}{4!\,b^4} \text{ and so on.}$$

Substituting these values in (1)
The equation becomes

$$y = y_0 + X\,\Delta y_0 + \frac{X(X-1)}{2!}\,\Delta^2 y_{-1} + \frac{X(X^2-1)^2}{3!}\,\Delta^3 y_{-1} + \frac{X\,(X^2-1^2)\,(X-2)}{4!} + \Delta^4 y_{-2} + \frac{X(X^2-1^2)\,(X^2-2^2)}{5!}\,\Delta^5 y_{-2} + \ldots\ldots$$

This is known as Gauss Forward Formula.

Gauss's Backward Formula:

In the Newton's Divided differences formula

Put $x_1 = x_0 - h$, $x_2 = x_0 + h$, $x_3 = x_0 - 2h$, $x_4 = x_0 + 2h$, $x_5 = x_0 - 3h$, $x_6 = x_0 + 3h$ and so on.

It will become

$$y = y_0 + (x - x_0)\, y(x_0, x_0 - h) + (x - x_0)(x - x_0 + h)\, y(x_0, x_0 - h, x_0 + h)$$
$$+ (x - x_0)(x - x_0 + h)(x - x_0 - h)\, y(x_0, x_0 - h, x_0 + h, x_0 - 2h)$$
$$+ (x - x_0)(x - x_0 + h)(x - x_0 - h)(x - x_0 + 2h)$$
$$y(x_0, x_0 - h, x_0 + h, x_0 - 2h, x_0 + 2h) + \ldots\ldots.$$

put $X = \dfrac{x - x_0}{h}$ (ie) $x = x_0 + Xh$

Then the equation reduces to

$$y = y_0 + Xh(x_0, x_0 - h) + Xh(Xh + h)\, y(x_0, x_0 - h, x_0 + 2h) + Xh(Xh + h)$$
$$(Xh - h)\, y(x_0, x_0 - h, x_0 + h, x_0 - 2h) + Xh(Xh + h)(Xh - h)(Xh + 2h)$$
$$y(x_0, x_0 - h, x_0 - 2h, x_0 + 2h) + \ldots\ldots\ldots\ldots$$

$$y = y_0 + Xhy(x_0, x_0 - h) + X(x+1)\, h^2 y\, (x_0, x_0 - h, x_0 + h) + X(X + 1)(X - 1)$$
$$h^3 y\, (x_0, x_0 - h, x_0 + h, x_0 - 2h) + X(X + 1)(X - 1)(X + 2)$$
$$h^4 y\, (x_0, x_0 - h, x_0 + h, x_0 - 2h, x_0 + 2h) + \ldots\ldots\ldots\ldots \quad \textbf{(1)}$$

We know that $y(x_0, x_0 - h) = y(x_0 - h, x_0) = \dfrac{\Delta y_{-1}}{h}$

$$y(x_0, x_0 - h, x_0 + h) = y(x_0 - h, x_0, x_0 + h) = \frac{\Delta^2 y_{-1}}{2!\, h^2}$$

$$y(x_0, x_0 - h, x_0 - 2h, x_0 - 2h) = y(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h)$$
$$= \frac{\Delta^4 y_{-2}}{4!\, h^1} \text{ and so on.}$$

Substituting these values in (1)

We get

$$y = y_0 + X\Delta y_{-1} + \frac{X(X+1)}{2!}\Delta^2 y_{-1} + \frac{X(X^2 - 1^2)}{3!}\Delta^3 y_{-1} +$$

$$\frac{X(X^2 - 1^2)(X+2)}{4!}\Delta^4 y_{-2} + \frac{X(X^2 - 1^2)(X^2 - 2^2)}{5!}\Delta^5 y_{-3} + \ldots\ldots.$$

This is known as Gauss's backwards formula.

Note: 1

1. This formula is known as Gauss's forward interpolation formula.
2. This formula involves odd differences below the central line ($x = a$) and even differences on the line.
3. Taking the central line and the next line from the table, we have the differences occurring in the formula.

$$\text{Central line } y_0 \;\cdots\cdots\; \Delta^2 y_{-1} \;\cdots\cdots\; \Delta^4 y_{-2} \;\cdots\cdots\; \Delta^6 y_{-3}$$

$$\Delta y_0 \qquad \Delta^3 y_{-1} \qquad \Delta^5 y_{-2}$$

Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
$a-3h$	y_{-3}					
		Δy_{-3}				
$a-2h$	y_{-2}		$\Delta^2 y_{-3}$			
		Δy_{-2}		$\Delta^3 y_{-3}$		
$a-h$	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$	
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$
a ……	y_0 ……		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$	
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$
$a+h$	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$	
		Δy_1		$\Delta^3 y_0$		
$a+2h$	y_2		$\Delta^2 y_1$			
		Δy_2				
$a+3h$	y_3					

4. The formula can be written easily with the help of the following table:

Coefficients	1	$\binom{x}{1}$	$\binom{x}{2}$	$\binom{x+1}{2}$	$\binom{x+1}{4}$	$\binom{x+1}{5}$	…….
Differences	y_0	Δy_0	$\Delta^2 y_{-1}$	$\Delta^3 y_{-1}$	$\Delta^4 y_{-2}$	$\Delta^5 y_{-2}$	…….

5. The formula is useful when u lies between 0 and 1.

Gauss's Backward Formula

Note:

1. Gauss's Backward formula involves odd differences above the central line and even differences on the central line.

2. Taking the central line and the previous line of the Table 1, we have the difference occurring in the formula.

Previous line : .... Δy_{-1} .... $\Delta^3 y_{-2}$ ...... $\Delta^5 y_{-3}$

Central line : ...y_0 $\Delta^2 y_{-1}$ .... $\Delta^4 y_{-2}$ .... $\Delta^6 y_{-3}$

3. This backward formula is useful when u lies between –1 and 0

4. The formula can be easily written with the help of the following table:

Coefficients	1	$\binom{X}{1}$	$\binom{X+1}{2}$	$\binom{X+1}{4}$	$\binom{X+2}{4}$	....
Differences	y_0	Δy_{-1}	$\Delta^2 y_{-1}$	$\Delta^3 y_{-2}$	$\Delta^4 y_{-2}$	....

Example: 1

Apply Gauss's forward central difference formula and estimate f(32) from the following table.

x:	25	30	35	40
y= f(x)	0.2707	0.3027	0.3386	0.3794

Solution:

Given x = 32 lies between 30 and 35

Let us take 30 as the origin; here h = 5

$$X = \frac{x - x_0}{h} = \frac{32-30}{5} = \frac{2}{5} = 0.4$$

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
y_{-1}	25	0.2707			
			.0320		
y_0	30	0.3027		.0039	
			.0359		.0010
y_1	35	0.3386		.0049	
			.0408		
y_2	40	0.3794			

Since we apply forward formula of Gauss, we enclose differences occurring in the terms by rectangle.

By Gauss's Forward Formula,

We have

$$Y(x) = y(x_0 + xh) = y_0 + X\ \Delta y_0 + \frac{x(x-1)}{2!}\ \Delta^2 y_{-1} + \frac{X(X^2 - 1^2)}{3!}\ \Delta^3 y_{-1} + \ldots..$$

$$y(x = 32) = y(x = 0.4) = 0.3027 + 0.4(0.0359) + \frac{0.4\,(0.4 - 1)}{2!}\ (0.0039) + \frac{0.4\,(0.4^2 - 1^2)}{3!}\ (0.0010)$$

$$= 0.3027 + (0.4)\ (0.0359) + \frac{(0.4)\,(-0.6)}{2}\ (0.0039) + \frac{(1.4)\,(0.4)\,(-0.6)}{6}\ (0.0010)$$

$$= 0.3027 + 0.01436 - 0.000468 - 0.00006$$

$$y = 0.31653$$

Example: 2

Using Gauss's backward interpolation formula, find the population for the year 1936 given that

Year	x :	1901	1911	1921	1931	1941	1951
Population In thousand	y :	12	15	20	27	39	52

Solution:

Since we require at x = 1936

Take 1941 are the origin h = 10,

$$X = \frac{x - 1941}{10} = \frac{1936 - 1941}{10} - 0.5$$

x		y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^3 y$	$\Delta^3 y$
1901	y_{-4}	12					
			3				
1911	y_{-3}	15		2			
			5		0		
1921	y_{-2}	20		2		3	−10
			7		3		
1931	y_{-1}	27		5		−7	
			12		−4		
1941	y_{-0}	39		1			
			13				
1951	y_{-1}	52					

We enclose those values required in the formula by rectangles.

By Gauss's backward formula,
We have

$$y(x) = y(x_0 + uh) = y(x = 0.5) = y_0 + x\Delta y_{-1} + \frac{x(x+1)}{2!}\Delta^2 y_{-1} + \frac{x(x^2-1^2)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{x(x^2-1^2)(x+2)}{4!}\Delta^3 y_{-2} + \ldots\ldots$$

$$y = 39 + (0 - .5)(12) + \frac{-0.5(-0.5+1)}{2!}0 - (1) + \frac{-0.5\left((-0.5)^2-1^2\right)}{3!}(-4)$$

$$= 39 - 6 + \frac{(0.5)(-0.5)}{2} + \frac{(0.5)(-0.5)(-1.5)}{6}(4)$$

$$= 33 - \frac{1}{8} - \frac{1}{4} = 32.625$$

$\therefore$ y = 32.625

Example: 3

The following table gives the value of the probability integral $f(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2/2}\,dx$ for certain equidistant values of x. Using a) Gauss's forward formula. b) Gauss's backward formula. Find the value of the integral when x = 0.68.

x :	0.50	0.55	0.60	0.65	0.70	0.75	0.80
f(x) :	0.1915	0.2088	0.2258	0.24422	0.2580	0.2734	0.2881

Solution:

Since x = 0.68 lies between 0.65 and 0.70.

We shall take the origin (ie) $x = x_0$ at 0.65.

We have $X = \frac{x - x_0}{h} = \frac{0.68 - 0.65}{0.05} = \frac{.03}{.05} = 0.6$

The difference table is given below:

x	y	f(x) (10^4)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.50	y_{-3}	1915					
			173				
0.55	y_{-2}	2088		– 3			
			170		– 3		
0.60	y_{-1}	2258		– 6		3	
			164		0		– 1
0.65	y_0	2422		– 6		2	
			158		2		3
0.70	y_1	2580		– 4		5	
			154		– 3		
0.75	y_2	2734		– 7			
			147				
0.80	y_3	2881					

a) By Gauss's forward formula:

$$y = y_0 + X\,\Delta y_0 + \frac{X(X-1)}{2!}\,\Delta^2 y_{-1} + \frac{X(X^2-1^2)}{3!}\,\Delta^3 y_{-1} + \frac{X(X^2-1^2)(x-2)}{4!}\,\Delta^4 y_{-2}$$

$$+ \frac{X(X^2-1^2)(X^2-2^2)}{5!}\,\Delta^5 y_{-2} + \ldots\ldots$$

$$= 2422 + (0.6)(158) + \frac{.6(.6-1)}{2}(-6) + \frac{.6(.6^2-1^2)}{6}(0)$$

$$+ \frac{.6(.6^2-2)(.6-2)}{24}(2) + \frac{.6(.6^2-1)(.6^2-4)}{120}(3)$$

$$= 2422 + 94.8 - 0.72 + 0 + 0.0448 + 0.034944$$

$$y = 2517.598$$

$$\therefore\ f(x) = 0.25176 \qquad \left[\because f(x)\,10^4 = 2517.598 \Rightarrow f(x) = \frac{2517.598}{10^4}\right]$$

b) By Gauss's Backward formula:

$$Y = y_0 + X\,\Delta y_{-1} + \frac{X(X+1)}{2!}\,\Delta^2 y_{-1} + \frac{X(X^2-1^2)}{3!}\,\Delta^3 y_{-2} + \frac{X(X^2-1^2)(X+2)}{4!}\,\Delta^4 y_{-2}$$

$$+ \frac{X(X^2-1^2)(X^2-2^2)}{5!}\,\Delta^5 y_{-3} + \ldots..$$

$$= 2422 + (0.6)(164) + \frac{0.6(1.6)}{2}(-6) + \frac{.6(.6^2-1)}{6}(0)$$

$$+ \frac{.6(.6^2-1)(.6+2)}{24}(2) + \frac{.6(.6^2-1)(.6^2-4)}{120}(-.$$

$$= 2422 + 98.4 - 2.88 + 0 - 0.0832 - 0.011648$$

$y = 2517.7$

$f(x) = 0.25177$

$$\left[\begin{array}{l} f(x) = \dfrac{2517.7}{10^4} \\ f(x) = .25177 \end{array}\right]$$

Exercise:

1. Interpolate by Gauss's formula the value of y when x = 17 from the following table:

x :	5	10	15	20	25	30
y :	0.3797	2.4622	4.0939	5.3725	6.3742	7.1591

2. Apply Gauss's forward formula to get y_{30} given that

$y_{21} = 18.4708$, $y_{25} = 17.8144$, $y_{29} = 17.1070$, $y_{33} = 16.3432$, $y_{37} = 15.5154$

3. Use Gauss's backward formula to obtain sin 45° given the table below:

x^o :	20	30	40	50	60	70
sin x^o :	0.34202	0.50200	0.64279	0.76604	0.86603	0.93969

4. Use Gauss forward formula. The value of e^{-x} for various values of x are given below. Find $e^{-1.7425}$

x :	1.72	1.73	1.74	1.75	1.76
e – x :	0.17907	0.17728	0.17552	0.17377	0.17204

6.6 STIRLINGS FORMULA

By Gauss's forward formula,

We have

$$y = y_0 + X\,\Delta y_0 + \frac{X(X-1)}{2!}\Delta^2 y_{-1} + \frac{X(X^2-1^2)}{3!}\Delta^3 y_{-1} + \frac{X(X^2-1^2)(X-2)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{X(X^2-1^2)(X^2-2^2)}{5!}\Delta^5 y_{-2} + \ldots\ldots \quad (1)$$

By Gauss's backward formula,

We have

$$y = y_0 + X\ \Delta y_{-1} + \frac{X(X+1)}{2!} + \frac{X(X^2-1^2)}{3!}\ \Delta^3 y_{-2} + \frac{X(X^2-1^2)(X+2)}{4!}\ \Delta^4 y_{-2}$$

$$+ \frac{X(X^2-1^2)(X^2-2^2)}{5!}\ \Delta^5 y_{-3} + \ldots\ldots \qquad (2)$$

Adding (1) and (2) and finding the average, We get

$$2y = 2y_0 + X(\Delta y_0 + \Delta y_{-1}) + \frac{X^2 - X + X^2 + X}{2!}\ \Delta^2 y_{-1} + \frac{X(X^2-1^2)}{3!}$$

$$\left(\Delta^3 y_{-1} + \Delta^3 y_{-2}\right) + \frac{X(X^2-1^2)}{4!}\left[X-2+X+2\right]\Delta^4 y_{-2}$$

$$+ \frac{X(X^2-1^2)(X^2-2^2)}{5!}\left[\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right] + \ldots\ldots\ldots\ldots \qquad (3)$$

$(\div 2)$

$$y = y_0 + X\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{X^2}{2!}\ \Delta^2 y_{-1} + \frac{X(X^2-1^2)}{3!}\ \left[\Delta^3 y_{-1} + \Delta^3 y_{-2}\right]$$

$$+ \frac{X^2(X^2-1^2)}{4!}\ \Delta^4 y_{-2} + \frac{X(X^2-1^2)(X^2-2^2)}{5!}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \ldots\ldots$$

Note:

The formation of even terms

$$X\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right)\quad \frac{X(X^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$\frac{X(X^2-1^2)(X^2-2^2)}{5!}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right)$$

From this the next even term n

$$\frac{X(X^2-1^2)(X^2-2^2)(X^2-3^2)}{7!}\left(\frac{\Delta^7 y_{-3} + \Delta^4 y_{-4}}{2}\right)$$

The odd terms are $\frac{X^2}{2!}\Delta^2 y_{-1}$, $\frac{X^2(X^2-1^2)}{4!}\Delta^4 Y_{-2}$

From this the next odd term is

$$\frac{X^2(X^2-1^2)(X^2-2^{2)}}{6!}\ \Delta^6 y_{-3}.$$

Similarly we can form all the terms.

Example: 1

Using Stirlings formula, find y(1.22) from the following table.

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	0.84147	0.89121	0.93204	0.96356	0.98545	0.99749	0.99957
x	1.7	1.8					
y	0.99385	0.97385					

Solution:

Since we require y at x = 1.22

Take the origin at x = 1.2 and h = 0.1

$$X = \frac{x - x_0}{h} = \frac{1.22 - 1.2}{0.1} = \frac{0.02}{0.1} = 0.2$$

We form the central difference table below.

Since x= 1.2 is the origin,

We take values on both sides of 1.2 to the required stage.

By Stirlings formula

We have

$$y = y_0 + X\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{X^2}{2}\Delta^2 y_{-1} + \frac{X(X^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{X^2(X^2-1^2)}{4!}\Delta^4 y_{-2} + \ldots\ldots\ldots$$

Difference Table

x		y	Δy	$\Delta^2 y$	$\Delta^2 y$	$\Delta^4 y$
1.0	y_{-2}	0.84147				
			0.04974			
1.1	y_{-1}	0.89121		–0.00891		
			0.04083		–0.00040	
1.2	y_0	0.93204		–0.00931		– 0.00008
			0.03152		–0.00032	
1.3	y_1	0.96356		–0.00963		
			0.02189			
1.4	y_2	0.98545				

$$y_{1.2} = 0.93204 + (0.2)\left[\frac{0.04083 + 0.03152}{2}\right] + \frac{(0.2)^2}{2}(-0.00931)$$

$$+ \frac{(0.2)(0.04-1)}{6}\left[\frac{-0.00040-0.00032}{2}\right]$$

$$+ \frac{(0.04)(0.04-1)}{24}(0.00008) + \ldots\ldots\ldots\ldots$$

$$= 0.93204 + 0.007235 - 0.0001862 + 0.00001152 - 0.000000128$$

y_{12} = 0.939100192.

Example: 2

From the following table estimate $e^{0.644}$ correct to five decimals using Stirling's formula

x :	0.61	0.62	0.63	0.64	0.65	0.66	0.67
e^x :	1.840431	1.858928	1.877610	1.896481	1.915541	1.934792	1.95421

Solution:

Take x = 0.64 as the origin

$$X = \frac{x - x_0}{h} = \frac{0.644 - 0.64}{0.01} = 0.4$$

Formula:

$$y = y_0 + X\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{X^2}{2}\Delta^2 y_{-1} + \frac{X(X^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \frac{X^2(X^2-1^2)}{4!}\Delta y_{-2} + \ldots\ldots\ldots$$

x		y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.61	y_{-3}	1.840431				
			0.018497			
0.62	y_{-2}	1.858928		0.000185		
			0.018682		.000004	
0.63	y_{-1}	1.877610		0.000189		−0.000004
			0.018871		.0.0	
0.64	y_0	1.896481		0.000189		0.000002
			0.019060		.000002	
0.65	y_1	1.915541		0.000191		0.000001
			0.019251		0.000003	
0.66	y_2	1.934792		0.000194		
			0.019445			
0.67	y_3	1.954237				

$$\therefore\ y = 1.896481 + (0.4)\left(\frac{0.01887 + 0.019060}{2}\right) + \frac{0.16}{2}(0.000189)$$

$$+ \frac{(0.4)(0.16-1)}{6}\left(\frac{0 + 0.000002}{2}\right) + \frac{(0.4)^2(0.16-1)}{4!}(0.000002)$$

$$= 1.896481 + 0.0075862 + 0.00001512 - 0.000000056 - 0.000000269$$

$$y = 1.904081996$$

Example: 3

The following table gives the value of the probability integral $f(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2/2}\,dx$ for certain equidistant values of x. Using Stirling's formula.

Find the value of the integral when x = 0.68

x	0.50	0.55	0.60	0.65	0.70	0.75	0.80
f(x)	0.1915	0.2088	0.2258	0.2422	0.2580	0.2734	0.2881

Solution:

Since x = 0.68 lies between 0.65 and 0.70, We shall take the origin. (ie) $x = x_0$ at 0.65.

$$X = \frac{x - x_0}{h} = \frac{0.68 - 0.65}{0.05} = 0.6.$$

The difference table is given below

X		f (x)(10^4)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.50	y_{-3}	1915	173	−3	−3	3	−1
0.55	y_{-2}	2088	170	−6	0	2	3
0.60	y_{-1}	2258	164	−6	2	5	
0.65	y_0	2422	158	−4	−3		
0.70	y_1	2580	154	−7			
0.75	y_2	2734	147				
0.80	y_3	2881					

Stirling's Formula:

$$y = y_0 + X\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{X^2}{2!}\Delta^2 y_{-1} + \frac{X(X^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \frac{X(X^2-1^2)}{4!}\ \Delta^4 y_{-2} + \frac{X(X^2-1^2)(X^2-2^2)}{5!}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \ldots$$

$$= 2422 + \frac{0.6\,(158+164)}{2} + \frac{(0.6)^2(-4)}{2} + \frac{(0.6)\,(0.6^2-1)}{6}\left(\frac{2+0}{2}\right)$$

$$+ \frac{(0.6)^2\,(0.6^2-1)}{24}\,(2) + \frac{0.6\,((0.6)^2-1)\,(.6^2-2)}{120}\left(\frac{3-1}{2}\right)$$

$$= 2422 + 96.6 - 0.72 - 0.064 - 0.0192 - .005248$$

$y = 2517.8$

$f(x)\,(10^4) = 2517.8$

$\therefore f(x) = 0.25178$

Example: 4

Given the following table, find y (35), by using Stirling's formula.

x :	20	30	40	50
y :	512	439	346	243

Solution:

We will take $x_0 = 30$ as the origin.

$$X = \frac{x - x_0}{6} = \frac{35-30}{10} = 0.5$$

$$y(35) = y_0 + \frac{X}{2}\,(\Delta y_0 + \Delta\, y_{-1}) + \frac{X^2}{2}\,\Delta^2 y_{-1} + \frac{X(X^2-1)}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \ldots\ldots$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	512 y_{-1}			
		−73 Δy_{-1}	−20 $\Delta^2 y_{-1}$	
30	439 y_0	−93 Δy_0		10 $\Delta^3 y_{-1}$
			−10 $\Delta^2 y_0$	
40	346 y_1	−103		
50	243			

$$y_{35} = 439 + \frac{0.5}{2}\,(-93-73) + \frac{0.25}{2}\,(-20) + \frac{(0.5)\,(0.25-1)}{6} \qquad (10)$$

$$= 439 - 41.50 - 2.50$$

$y_{35} = 395.$

Exercise:

1. Use Stirling's formula to find Y_x, given

x :	10	20	30	40
y_x :	51.2	43.9	34.6	24.3

2. Use Stirling's formula, estimate f (1.22) from the following table:

x:	20	25	30	35	40
f (x):	49225	48316	47236	45926	44306

3. Estimate $\sqrt{1.12}$ using Stirling's formula from the following table.

x:	1.0	1.05	1.10	1.15	1.20	1.25	1.30
f(x):	1.00000	1.02470	1.04881	1.07238	1.09544	1.11803	1.14017

4. Use Stirling's formula to get tem 89° 26' from the table.

X:	89° 21'	89° 23'	89° 25'	89° 27'	89° 29'
tan x :	88.14	92.91	98.22	104.17	110.90

5. Use Stirling formula to get the value of Y (45) given.

x:	40	44	48	52
y:	51.08	63.24	70.88	79.84

6.7 BESSEL'S FORMULA

In the Gauss's Backward formula, instead of x_0 take x_1 (ie.) we have to advance the subscripts of x and y by one unit.

$$X = \frac{x - x_1}{h}, \text{ (i.e.) } X = \frac{x - x_0 - h}{h}$$

$$\text{(i.e.) } X = \frac{x - x_0}{h} - 1$$

Hence X is to be replaced by X – 1.

Similarly (X – k) is to be replaced by (X – k – 1)

Then the

Gauss's Backward Formula reduces to

$$y = y_0 + (X - 1)\ \Delta y_0 + \frac{(X-1)x}{2!}\ \Delta^2 y_0 + \frac{X(X-1)(X-2)}{3!}\ \Delta^2 y_{-1}$$

$$+ \frac{X(X^2-1^2)(X-2)}{4!}\ \frac{\Delta^4 y_{-1}}{4!} + \frac{X(X^2-1^2)(X-2)(X-3)}{5!}\ \Delta^5 y_2 + \ldots\ldots (1)$$

Gauss's forward formula is

$$y = y_0 + X\ \Delta\ y_0\ \frac{X(X-1)}{2!}\ \Delta y_{-1} + \frac{X(X^2-1^2)}{3!}\ \Delta^3 y_{-1}$$

$$+ \frac{X(X^2-1^2)(X-2)}{4!}\ \Delta^4 y_{-2} + \frac{X(X^2-1^2)(X^2-2^2)}{5!}\ \Delta^5 y_{-2} + \ldots\ldots (2)$$

Finding the mean of (1) and (2), we get

$$\frac{2y}{2} = \frac{y_0+y_1}{2} + \frac{(X-1)\Delta y_0 + X\Delta y_0}{2} + \frac{(X-1)X\Delta^2 y_0 + X(X-1)\Delta y_{-1}}{(2!)}$$

$$+ \frac{(X-1)x(X-2)\Delta^2 y_{-1} + X(X^2-1^2)\Delta^3 y_{-1}}{2\ \ (3!)} +$$

$$\frac{x(x^2-1^2)(x-2)\Delta^4 y_{-1} + x(x^2-1^2)(x-2)\Delta^4 y_2 +}{4!\ (2)}\ \ldots\ldots\ldots$$

$$y = \frac{y_0+y_1}{2} + \left(X - \frac{1}{2}\right)\ \Delta y_0 + \frac{(X-1)x}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+ \frac{X(X-\frac{1}{2})(X-1)}{3!}\ \Delta^3 y_{-1} + \frac{X(X^2-1^2)(X-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-2}}{2}\right)$$

$$+ \frac{X(X-\frac{1}{2})(X^2-1^2)(X^2-1^2)(X^2-1^2)}{5!}\ \Delta^5 y_{-2} + \ldots\ldots$$

This is known as Bessel's formula of interpolation.

Example: 1

1. Given the following table, find y (35) by using Bessel's formula.

x:	20	30	40	50
y:	512	439	346	243

Solution:

Take $x_0 = 30$ as the origin.

$$X = \frac{x - x_0}{h} = \frac{35 - 30}{10} = 0.5$$

The difference table is

x		y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	y_{-1}	512			
			Δy_{-1} −73		
30	y_0	439		$\Delta^2 y_{-1}$ −20	
			Δy_0 −93		$\Delta^3 y_{-1}$ 10
40	y_1	346		$\Delta^2 y_0$ −10	
			−103		
50	y_2	243			

Using Bessel's formula:

$$y = \frac{y_0 + y_1}{2} + \left(X - \frac{1}{2}\right)\Delta y_0 + \frac{X(X-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{X\left(X - \frac{1}{2}\right)(X-1)}{3!}\Delta^3 y_{-1} + \ldots\ldots$$

$$= \frac{439 + 346}{2} + (0.5 - \tfrac{1}{2})(-93) + \frac{0.5(0.5-1)}{2!}\left(\frac{-20-10}{2}\right)$$

$$+ \frac{(0.5)(0.5 - \tfrac{1}{2})(0.5-1)}{3!}(10)$$

$$= \frac{439 + 346}{2} + 0 + 0.25\left(\frac{-30}{2}\right)$$

$$= 392.5 + 1.875$$

$$y = 394.375$$

Example: 2

From the following table, estimate $e^{0.644}$ correct to five decimals using Bessel's formula.

x	0.61	0.62	0.63	0.64	0.65	0.66	0.67
e^X	1.840431	1.858928	1.877610	1.896481	1.915541	1.934792	1.954237

Solution:

Take x = 0.64 as the origin.

$$X = \frac{x - x_0}{h} = \frac{0.644 - 0.64}{0.01} = 0.4$$

Difference Table

x		$y=e^x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^3 y$
0.61	y_{-3}	1.840431				
			0.018497			
0.62	y_{-2}	1.858928		0.000185		
			0.018682		0.000004	
0.63	y_{-1}	1.877610		0.000189		–0.000004
			0.018871		0.0	
0.64	y_0	1.896481		0.000189		0.000002
			0.019060		0.000002	
0.65	y_1	1.915541		0.000191		0.000001
			0.019251		0.000002	
0.66	y_2	1.934792		0.000194		
			0.019445			
0.67	y_3	1.954237				

Using Bessel's formula

$$y = \frac{y_0 + y_1}{2} + \left(X - \tfrac{1}{2}\right) \Delta y_0 + \frac{X(X-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+ \frac{X\left(X - \frac{1}{2}\right)(X-1)}{3!} \Delta^3 y_{-1} + \left(\frac{X(X^2 - 1^2)(X-2)}{4!}\right)\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \ldots\ldots$$

We get

$$y = \frac{1.896481 + 1.915541}{2} + \left(0.4 - \frac{1}{2}\right)(0.019060)$$

$$+ \frac{(0.4)(-0.6)}{2}\left(\frac{0.000189 + 0.000191}{2}\right)$$

$$+ \frac{(0.4)(-0.6)(0.4 - 0.5)}{6}(0.000002) + \ldots \frac{(0.4)\left[(0.4)^2 - 1\right)(0.4-2)}{41}\left(\frac{0.000002 + .000001}{2}\right)$$

$= 1.906011 - 0.001906 - 0.0000228$

$= 1.904082$

Example: 3

The following table gives the value of the probability integral $f(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2/2}\, dx$ for certain equidistant values of x. Using Bessel's formula.

Find the value of the integral when x = 0.68.

x:	0.50	0.55	0.60	0.65	0.70	0.75	0.80
f (x):	0.1915	0.2088	0.2258	0.2422	0.2580	0.2734	0.2881

Solution:

Since x =0.68 lies between 0.65 and 0.70. We shall take the origin, (i.e.) x = x_0 at 0.65.

We have $X = \frac{x - x_0}{h} = \frac{0.648 - 0.65}{0.05} = 0.6$

By Bessel's formula

$$y = y_0 + X\, \Delta y_0 + \frac{X(X-1)}{2!}\left(\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right) + \frac{X\left(X - \frac{1}{2}\right)(X-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{X(X^2 - 1^2)(X-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \frac{X\left(X - \frac{1}{2}\right)(X^2 - 1^2)(X^2 - 2^2)}{5!}\Delta^5 y_2 + \ldots$$

x		f(x) (10^4)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.50	y_{-3}	1915					
			173				
0.55	y_{-2}	2088		−3			
			170		−3		
0.60	y_{-1}	2258		−6		3	
			164		0		−1
0.65	y_0	2422		−6		2	
			158		2		3
0.70	y_1	2580		−4		5	
			154		−3		
0.75	y_2	2734		−7			
			147				
0.80	y_3	2881					

we get

$$y = 2422 + (0.6)(158) + \frac{0.6(0.6-1)}{2}\left(\frac{-4-6}{2}\right) + \frac{(0.6)(.6-.5)(.6-1)}{6} \qquad (2)$$

$$+ \frac{.6\,(0.6^2-1)(0.6-2)}{24}\left(\frac{2+5}{2}\right)+\frac{.6\,(.6-.5)(-6^2-1)(.6^2-4)}{120} \quad (3)$$

$= 2422 + 94.8 + 0.6 - .008 + 0.00784 + .0034944$

$y = 2517.55$

$f(x)\,(10^4) = 2517.55$

$\therefore f(x) = 0.25176$

Exercise:

1. From the following table using Bessel's formula find y (5).

X:	0	4	8	12
Y:	14.27	15.81	17.72	19.96

2. Apply Bessel's formula to obtain.

i) y_{45} given

x :	40	44	48	52
y:	51.08	63.24	70.88	79.84

ii) the value of y when x = 5 from the following table:

x:	0	4	8	12
y:	14.27	15.81	17.72	19.96

3. Using Bessel's formula obtain the value of y (5) given

x:	0	4	8	12
y:	14.27	15.81	17.72	19.96

6.8 LAPLACE – EVERETT FORMULA

Gauss's forward formula is

$$y_{(X)} = y\,(x_0 + X\,h) = y_0 + \Delta\, y_0 + \frac{x\,(x-1)}{2!}\,\Delta^2 y_{-1} + \frac{x\,(x^2-1^2)}{3!}\,\Delta^3 y_{-1}$$

$$+ \frac{x(x^2-1^2)(x-2)}{4!}\Delta^4 y_{-2} + \frac{x(x^2-1^2)(x^2-2^2)}{5!}\Delta^5 y_{-2} + \ldots \quad (1)$$

we have $\Delta y_0 = y_1 - y_0$; $\Delta^3 y_{-1} = -\Delta^2 y_0 - \Delta^2 y_{-1}$;

$$\Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2} \text{ etc.}$$

Substituting these in (1)
we have

$$y(x_0 + Xh) = y_0 + X(y_1 - y_0) + \frac{X(X-1)}{2!}\Delta^2 y_{-1} + \frac{X(X^2-1^2)}{3!}(\Delta^2 y_0 - \Delta^2 y_{-1})$$

$$+ \frac{X(X^2-1^2)(X-2)}{4!}\Delta^4 y_{-2} + \frac{X(X^2-1^2)(X^2-2^2)}{5!}\left(\Delta^4 y_{-1} - \Delta^4 y_{-2}\right) + \ldots$$

$$y = y_0 + Xy_1 - Xy_0 + \frac{X(X-1)}{2!}\Delta^2 y_{-1} + \frac{X(X^2-1^2)}{3!}\Delta^2 y_0 - \frac{X(X^2-1^2)}{3!}\Delta^2 y_0$$

$$+ \frac{X(X^2-1^2)(X-2)}{4!}\Delta^4 y_{-2} + \frac{X(X^2-1^2)(X^2-2^2)}{5!}\Delta^4 y_{-1}$$

$$- \frac{X(X^2-1^2)(X^2-2^2)}{5!}\Delta^4 y_{-2} + \ldots \ldots$$

$$= (1-X)y_0 + Xy_1 + \left(\frac{X(X-1)}{2!}\left(\frac{X(X^2-1^2}{3!}\right)\right)\Delta^2 y_{-1} + \frac{X(X^2-1^2)}{3!}\Delta^2 y_0$$

$$+\left(\frac{X(X^2-1^2)(X-2)}{4!} - \frac{X(X^2-1^2)(X^2-2^2)}{5!}\right)\Delta^4 y_{-2} + \frac{X(X^2-1^2)(X^2-2^2)}{5!}\Delta^2 y_{-1} \quad (2)$$

Using this result $\binom{X}{r} + \binom{X}{r+1} = \binom{X+1}{r+1}$

Changing $1 - X = v = Y$ or $X = 1 - v$ in equation (2)
we get

$$\binom{X}{3} = \frac{X(X-1)(X-2)}{3!} = \frac{(1-v)(-v)(-v-1)}{3!} = \frac{(v+1)(v)(v-1)}{} = 0$$

$$= \binom{v+1}{3}$$

Similarly,

$$\binom{X+1}{5} = -\binom{v+2}{5} \text{ etc.}$$

Hence, equation (2) reduces to

$$y(x) = \left[V y_0 + \binom{v+1}{3} \Delta^2 y_{-1} + \binom{v+2}{5} \Delta^4 y_{-2} + \dots \right]$$

$$+ \left[X y_1 + \binom{X+1}{3} \Delta^2 y_0 + \binom{X+2}{5} \Delta^4 y_{-1} + \dots \right] \quad \text{——— (3)}$$

$$y = \left[V y_0 + \frac{V(v^2-1^2)}{3!} \Delta^2 y_{-1} + \frac{V(V^2-1^2)(V^2-2^2)}{5!} \Delta^4 y_{-2} + \dots \right]$$

$$+ \left[X y_1 + \frac{X(X^2-1^2)}{3!} \Delta^2 y_0 + \frac{X(X^2-1^2)(X^2-2^2)}{5!} \Delta^4 y_{-1} + \dots \right] \quad (4)$$

This formula is known as Laplace – Everett formula.

Note: 1. This formula involves even differences on and below the central line.

2. It involves only even order differences.

3. This can be used if $0 < u < 1$.

Example: 1

From the following table, estimate $e^{0.644}$ correct to five decimals using Laplace – Everett's formula. Also find e^x

X:	0.61	0.62	0.63	0.64	0.65	0.66	0.67
e^x:	1.840431	1.858928	1.877610	1.896481	1.915541	1.934792	1.954237

Solution:

Take x = 0.64 as the origin

$$X = \frac{x - x_0}{h} = \frac{0.644 - 0.64}{0.01} = 0.4$$

Difference Table

x		$y=e^x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.61	y_{-3}	1.840431				
			+0.018497			
0.62	y_{-2}	1.858928		0.000185		
			0.018682		.000004	
0.63	y_{-1}	1.877610		0.000189		–0.000004
			0.018871		0.0	
0.64	y_0	1.896481		0.000189		0.000002
			0.019060		0.000002	
0.65	y_1	1.915541		0.000191		0.000001
			0.019251		0.000003	
0.66	y_2	1.934792		0.000194		
			0.019445			
0.67	y_3	1.954237				

Using Laplace Everett's Formula, we get

$$y = (0.644) = \left[V\,Y_0 + \frac{V(V^2-1^2)}{3!}\,\Delta^2 y_{-1} + \frac{V(V^2-1^2)(v^2-2^2)}{5!}\Delta^4 y_{-2} + \ldots\ldots \right]$$

$$+ \left[X\,y_1 + \frac{x(x^2-1^2)}{3!}\,\Delta^2 y_0\ \frac{x(x^2-1^2)(x^2-2^2)}{5!}\ \Delta^4 y_{-1} + \ldots\ldots \right]$$

$$V = 1 - X \Rightarrow V = 1 - 0.4 = 0.6$$

$$Y = \left[(0.6)(1.896481) + \frac{(0.6)(0.36-1)}{6}\ (0.000189)\ \frac{(0.6)(0.6^2-1^2)(0.6^2-2^2)}{5!} / \ 0.000002) \right]$$

$$+ (0.4)\ (1.915541) + \frac{(0.4)(0.16-1)}{6}\ (0.000191)$$

$$+ \frac{(0.6)(0.6^2-1^2)(0.6^2-2^2)}{5!}\ (0.0000.01) \Big]$$

$$= [1.1378886 - 0.000012096 + 0.00000002396] + [0.7662164$$

$$- 0.00001069 + .000000011648]$$

$$= 1.137876527 + 0.766205722$$

$$y = 1.904082.$$

Example: 2

The following table gives the values of the probability integral

$f(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$ for certain values of x. Find the value of this integral when x = 0.5437 using Laplace Everett's formula,

x:	0.51	0.52	0.53	0.54	0.55	0.56
Y=f(x)	0.5292437	0.5378987	0.5464641	0.5549392	0.5633233	0.5716157

x:	0.57
f(x):	0.5798158

Solution:

We take the origin $x_0 = 0.54$ and x = 0.5437, h = 0.01

$$X = \frac{x - x_0}{h} = \frac{0.5437 - 0.54}{0.01} = 0.37$$

$$V = 1 - X = 1 - 0.37 = 0.63$$

Difference table:

x		f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.51	y_{-3}	0.5292437				
			0.0086550			
0.52	y_{-2}	0.5378987		−0.0000896		
			0.0084751		−0.0000007	
0.53	y_{-1}	0.5464641		−0.0000903		0.0
			0.0084751		−0.0000007	
0.54	y_0	0.5549392		−0.0000910		0.0
			0.0083841		−0.0000007	
0.55	y_1	0.5633232		−0.0000917		0.0000001
			0.0082924		−0.0000006	
0.56	y_2	0.5716157		−0.0000923		
			0.0082001			
0.57	y_3	0.5798158				

By Laplace Everett's Formula

$$y = (x = 0.5437) = \left[X y_1 + \frac{X(X^2-1)}{6} \Delta^2 y_0 + \frac{X(X^2-1)(X^2-4)}{12} \Delta^4 y_{-1} + \ldots\ldots \right]$$

$$+ \left[V y_0 + \frac{v(v^2-1)}{6} \Delta^2 y_{-1} + \frac{v(v^2-1)(v^2-1)(v^2-4)}{120} \Delta^4 y_{-2} + \ldots\ldots \right]$$

$$y = \left[(0.37)\,(0.5633233) + \frac{(0.37)\left((.37)^2 - 1 \right)}{6} (-\,0.0000917) \right.$$

$$\left. + \frac{(.37)\,(0.37^2 - 1)\,(0.37^2 - 4)}{120} - (0) \right] + \left[(0.63)\,(0.5549392) \right.$$

$$+ (0.63)\,\frac{\left((0.63\;)^2 - 1\;\right)}{6}\;(-0.0000910) + \left[\frac{(0.63)\,(0.63^2 - 1)\,(0.63^2 - 4)}{120} \begin{pmatrix} \\ 0 \end{pmatrix}\right]$$

$$= [0.208429621 + 0.00000488 + 0\,] + [0.34961196$$

$$+ 0.00000576262 + 0]$$

$y = 0.55805195.$

Example: 3

From the following table,

x:	20	25	30	35	40
f (x)	11.4699	12.7834	13.7648	14.4982	15.0463

find f (34) using Laplace Everett's formula.

Solution:

Take the origin x_0 as 30; h = 5

$$X = \frac{x - x_0}{h} = \frac{34 - 30}{5} = 0.8$$

$$V = 1 - X = -\,0.8 = 0.2$$

Difference Table

x		f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
20	y_{-2}	11.4699				
			1.3135			
25	y_{-1}	12.7834		−0.3321		
			0.9814		0.0841	
30	y_0	13.7648		−0.2480		−0.0214
			0.7334		0.0627	
35	y_1	14.4982		−0.1853		
			0.5481			
40	y_2	15.0463				

By Laplace Everett's Formula

$$y(x) = \left[X y_1 + \frac{X(X^2-1^2)}{3!} \Delta^2 y_0 + \frac{X(X^2-1^2)(X^2-2^2)}{5!} \Delta^4 y_{-1} + \ldots \right]$$

$$+ \left[V y_0 + \frac{v(v^2-1^2)}{3!} \Delta^2 y_{-1} + \frac{v(v^2-1^2)(v^2-2^2)}{5!} \Delta^4 y_{-2} + \ldots \right]$$

y (x = 34 = y (x = 0.8)

$$= \left[(0.8)(14.4982) + \frac{(0.8)(0.64-1)}{6} (0.1853) \right] + [(0.2)(13.7648)$$

$$+ \frac{(0.2)(0.04-1)}{6}(-0.2480) + \frac{(0.2)(.04-1)(0.04-4)}{120}(-.0214)$$

= 11.59856 + 0.0088944 + 2.75296 + .007936 – .0001355904

Y = 14.368214

Example: 4

From the following the table, estimate f (337.5) by proper interpolation formula.

x:	310	320	330	340	350	360
f(x) = y = log x	2.4913617	2.5051500	2.5185139	2.5314789	2.5440680	2.5563025

Solution:

Take x_0 =330 as the origin

$$h = 10, X = \frac{x - x_6}{h} = \frac{337.5 - 330}{10} = 0.75$$

Since $X = 0.75 > \frac{1}{2}$

We can use Everett's formula for better result v = 1 – x = 0.25.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
310	2.4913617				
		0.0137883			
320	2.5051500		–0.0004244		
		0.0133639		0.0000255	
330	2.5185139		–0.0003989		–0.0000025
		0.0129650		0.0000230	
340	2.5314789		–0.0003757		–0.0000017
		0.0125891		0.0000213	
350	2.5440680		–0.0003546		
		0.0122345			
360	2.5563025				

By Everett's formula,

$$y(0.75) = \left[X y_1 + \frac{X(X^2-1)}{3!} \Delta^2 y_0 + \frac{X(X^2-1)(X^2-4)}{5!} \Delta^4 y_{-1} + \ldots \right]$$

$$+ \left[V y_0 + \frac{v(v^2-1)}{3!} \Delta^2 y_{-1} + \frac{v(v^2-1)(v^2-4)}{5!} \Delta^4 y_{-2} + \ldots \right]$$

$$= \left[(0.75)(2.5314789) + \frac{(0.75)(0.5625-1)}{6}(-0.0003759) \right.$$

$$+ \frac{(0.75)\ (0.5625-1)(0.5625-4)}{120}\ (-0.0000017)$$

$$+ \left[(0.25)\ (2.585139) + \frac{(0.25)(0.0625-1)}{6}(-0.0003989) \right.$$

$$+ \frac{(0.25)\ (0.0625-1)(0.0625-4)}{120}\ (-0.0000025)$$

Y (0.75) = 2.5282736

Exercise:

1. Using Everett's formula, find log 2375 given

x:	21	22	23	24	25	26
log x :	1.3222	1.3424	1.3617	1.3802	1.3979	1.4150

2. Find y (12) if y (0) = 0, y (10) = 43214, y (20) = 86002.

Y (30) = 128372 using Everett's formula,

3. Using Everett's formula, estimate y (30) given.

x:	20	28	36	44
y (x) :	2854	3162	7088	7984

4. Apply Everett's formula to evaluate y (26) and y (27) given.

x:	15	20	25	30	35	40
y(x) :	12.849	16.351	19.524	22.396	24.999	27.356

6.9 LAGRANGE'S INTERPOLATION FORMULA [unequal intervals]

The forward and backward interpolation formula of Newton can be used only when the values of independent variable x are equally spaced. Further, the differences must become ultimately small. In cases, where the values of independent variable are not equally spaced and in cases when the differences of dependent variable are not small, ultimately, we will use Lagrange's interpolation formula.

Let $y = f(x)$ be a function.

Let f(x) takes the values $y_0, y_1, y_2 \ldots\ldots y_n$ corresponding to $x = x_0, x_1, x_2, \ldots\ldots x_n$, (i.e.) $y_i = f(x)$ $i = 0, 1, 2, \ldots.n$.

Let $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3) \ldots\ldots (x_n, y_n)$ denote $(n + 1)$ corresponding pairs of values of any two variables x and y.

Let $y = a_0 + a_1x\ a_2x^2 + \ldots\ldots + a_nx^n$ be a polynomial fitting this data well.

Let the polynomial be written in the following from:-

$$y = A_0 (x - x_1)(x-x_2) \ldots\ldots (x-x_n) + A_1 (x-x_1)(x-x_2) \ldots\ldots (x-x_n)$$

$$+ A_2 (x - x_0)(x-x_1) \ldots\ldots (x-x_n) + \ldots\ldots\ldots\ldots$$

$$+ A_n (x - x_0)(x-x_1) \ldots\ldots\ldots\ldots (x-x_{n-1}) \qquad \text{———— (1)}$$

When $x = x_0$, $y = y_0$. Substituting the value in the equation (1).

We get $y_0 = A_0 (x-x_1)(x_0-x_1) \ldots\ldots (x_0-x_n)$

$$\therefore\ A_0 = \frac{Y_0}{(x_0 - x_1)(x_0 - x_2)\ldots\ldots(x_0 - x_n)}$$

When $x = x_1$, $y = y_1$. Substituting the value in the equation (1).

We get $y_1 = A_{\cdot} (x_1-x_0)(x_1-x_2) \ldots\ldots (x_1-x_n)$

$$A_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)\ldots\ldots(x_1 - x_n)}$$

When $x = x_2$, $y = y_2$. Substituting the value in the equation (1).

$$Y_2 = A_2 (x_2-x_0)(x_2-x_1) \ldots .(x_2-x_n)$$

$$A_2 = \frac{y_2}{(x_2 - x_0)(x_2 - x_1).......(x_2 - x_n)}$$

. . . .

Similarly, ...

$$A_n = \frac{y_n}{(x_n - x_0)(x_n - x_1).......(x_n - x_{n-1})}$$

Substituting the values of A_0, A_1, ... A_n in the equation (1)
We have

$$y = \frac{(x - x_1)(x - x_2).....(x - x_n)}{(x_0 - x_1)(x_0 - x_2).....(x_0 - x_n)} y_0$$

$$+ \frac{(x - x_0)(x - x_2).....(x - x_n)}{(x_1 - x_0)(x_1 - x_2).....(x_1 - x_n)} y_1$$

$+$ . . . .

. . . .

$$+ \frac{(x - x_0)(x - x_1).....(x - x_{n-1})}{(x_n - x_0)(x_n - x_1).....(x_n - x_{n-1})} y_n \qquad (2)$$

This equation is known as **Lagrange's interpolation formula**.
Cor: Dividing both sides of equation (2) by

$(x - x_0)\ (x - x_1)\ ....\ (x - x_n)$
we get

$$\frac{f(x)}{(x - x_0)(x - x_1).....(x - x_n)} = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)....(x_0 - x_n)} \cdot \frac{1}{x - x_0}$$

$$+ \frac{y_1}{(x_1 - x_0)(x_1 - x_2)....(x_1 - x_n)} \cdot \frac{1}{x - x_1}$$

$+$ . . . .

. . . .

$$+ \frac{y_n}{(x_n - x_0)(x_n - x_1)....(x_n - x_{n-1})} \cdot \frac{1}{x - x_n}$$

Example:1

Using Lagrange's interpolation formula, find y (10) from the following table

	x_0	x_1	x_2	x_3
x:	5	6	9	11
	y_0	y_1	y_2	y_3
y:	12	13	14	16

Solution:

By Lagrange's interpolation formula, we have

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\,y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\,y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\,y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\,y_3$$

$\therefore \quad x_0 = 5, \quad x_1 = 6, \quad x_2 = 4, \quad x_3 = 11$

$y_0 = 12 \quad y_1 = 13 \quad y_2 = 14, \quad y_3 = 16$

$$\therefore\ y = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}.12 + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}.13$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}.14 + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}.16$$

Put x = 10

$$y = \frac{(10-6)(10-9)(10-11)}{(-1)(-4)(-6)}.12 + \frac{(10-5)(10-9)(10-11)}{(1)(-3)(-5)}.13$$

$$+ \frac{(10-5)(10-6)(10-11)}{(4)(3)(-2)}.14 + \frac{(10-5)(10-6)(10-9)}{(6)\ (5)\ (2)}.16$$

$$= \frac{(4)(1)(-1)}{-(24)}.12 + \frac{(5)(1)(-1)}{+15}.13$$

$$+ \frac{(5)(4)(-1)}{(-24)}.14 + \frac{(5)(4)(1)}{(60)}.16$$

$= 24.333333 + 11.666667 + 5.333$

Y (10) = 14.666666

Example: 2

The following are the measurements t made on a curve recorded by an oscillograph representing a change of current i due to a change in the conditions of an electric current.

t:	1.2	2.0	2.5	3.0
i:	1.36	0.58	0.34	0.20

Find the value of i when t = 1.6.

Solution:

Since there are only four corresponding pairs of values given.

The polynomial representing the data is

$$i_t = \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)}\, i_0 + \frac{(t-t_0)(t-t_2)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)}\, i_1$$

$$+ \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_2-t_0)(t_2-t_1)(t_2-t_3)}\, i_2 + \frac{(t-t_0)(t-t_1)(t-t_2)}{(t_3-t_0)(t_3-t_1)(t_3-t_2)}\, i_3$$

Here $t_0 = 1.2$, $t_1 = 2.0$, $t_2 = 2.5$, $t_3 = 3.0$

$i_0 = 1.36$, $i_1 = 0.58$, $i_2 = 0.34$, $i_3 = 0.20$.

$$\therefore\ i = \frac{(t-2)(t-2.5)(t-3)}{(1.2-2)(1.2-2.5)(1.2-3)}\ 1.36 + \frac{(t-t_0)(t-t_2)(t-t_3)}{(2-1.2)(2-2.5)(2-3)}\ (.58)$$

$$+ \frac{(t-1.2)(t-2)(t-3)}{(2.5-1.2)(2.5-2)(2.5-3)}\ (0.34) + \frac{(t-1.2)(t-2)(t-2.5)}{(3-1.2)(3-2)(3-2.5)}\ (0.20)$$

Let $i_{(1\ 6)}$ be the value corresponding to t = 1.6

Then

$$i_{(16)} = \frac{(1.6-2)(1.6-2.5)(1.6-3)}{(-.8)(-1.3)(-1.8)}\ (1.36) + \frac{(1.6-1.2)(1.6-2.5)(1.6-3)}{(-.8)(-.5)(-1)}\ (.58)$$

$$+ \frac{(1.6-1.2)(1.6-2)(1.6-3)}{(1.3)(.5)(-0.5)}\ (0.34) + \frac{(1.6-1.2)(1.6-2.5)(1.6-3)}{(1.8)(1)(0.5)}\ (.20)$$

$$= \frac{(-.4)(-.9)(-1.4)}{(-.8)(-1.3)(-1.8)}(1.36) + \frac{(-.4)(-.9)(-1.4)}{(-.8)(-.5)(-1)}(.58)$$

$$+ \frac{(.4)(-.4)(-1.4)}{(1.3)(.5)(-0.5)}(0.34) + \frac{(0.4)(-0.4)(-0.9)}{(1.8)(0.5)}(0.20)$$

$$= \frac{-0.68544}{-1.872} + \frac{-0.29232}{-.40} + \frac{+.07616}{-0.325} + \frac{.0288}{.9}$$

$$= 0.366153846 + 0.7308 - .23434 + .032 = 0.8932$$

Example: 3

Using Lagrange's interpolation formula, find polynomial y (9.5) given

x:	7	8	9	10
y:	3	1	1	9

Solution:

By Lagrange's Formula.

$$y = f(x) = \frac{(x-8)(x-9)(x-10)}{(7-8)(7-9)(7-10)} \times 3 + \frac{(x-7)(x-9)(x-10)}{(8-7)(8-9)(8-10)} \times 1$$

$$+ \frac{(x-7)(x-8)(x-10)}{(9-7)(9-8)(9-10)}(1) + \frac{(x-7)(x-8)(x-9)}{(10-7)(10-8)(10-9)} \times 9$$

$$y_{9.5} = f(9.5) = \frac{(1.5)(0.5)(-0.5)}{(-1)(-2)(-3)}(3) + \frac{(2.5)(0.5)(-0.5)}{(1)(-1)(-2)}$$

$$+ \frac{(2.5)(1.5)(-0.5)}{(2)(1)(-1)} + \frac{(2.5)(1.5)(0.5)}{(3)(2)(1)} \times 9$$

$$f(9.5) = 0.1875 - 0.3125 + 0.9375 + 2.8125$$

$$y = 3.625$$

Example: 4

Use Lagrange's formula to fit a polynomial to the data.

x:	–1	0	2	3
y:	–8	3	1	12 and hence find y (x = 1).

Solution:

By Lagrange's Formula:

$$y = f(x) = \frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(-8) + \frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)} \times 3$$

$$+ \frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}(1) + \frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)} \times 12$$

$$y = \frac{x^3-5x^2+6x}{(-12)}(-8) + \frac{x^3-4x^2+x+6}{6}(3) + \frac{x^3-2x^2-3x}{-6} + \frac{x^3-x^2-3x}{12}(12)$$

$$= \frac{2}{3}(x^3-5x^2+6x) + \frac{1}{2}(x^3-4x^2+x+6) - \frac{1}{6}(x^3-2x^2-3x) + (x^3-x^2-3x)$$

$$= \frac{4x^3-20x^2+24x+3x^3-12x^2+3x+18-x^3+2x^2+3x+6x^3-6x^2-18x}{6}$$

$$= \frac{1}{6}\left[12x^3-36x^2+12x+18\right]$$

$$= 2x^3 - 6x^2 + 3x + 3$$

$$f(1) = 2 - 6 + 3 + 3 = 2$$

$$\therefore f(1) = 2$$

6.10 INVERSE INTERPOLATION

So far, given a set of values of x and y we were finding the values of y corresponding to some $x = x_k$ (which is not given in the table). Here we treat y as a function of x. Now the problem is given some $y = y_0$, we should find the corresponding x. This process of finding x given y is called Inverse interpolation.

In such a case, we will take y as independent variable and x as dependent variable and use Lagrange's interpolation formula.

Taking y as independent variable.

$$x = \frac{(y-y_1)(y-y_2)\ldots\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots\ldots(y_0-y_n)} x_0 + \frac{(y-y_0)(y-y_2)\ldots\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots\ldots(y_1-y_0)} x_1$$

$$+\ldots\ldots+ \frac{(y-y_0)(y-y_1)\ldots\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots\ldots(y_n-y_{n-1})} x_n$$

This formula is called formula of inverse interpolation.

Example: 1

From the data given below, find the value of x when y = 13.5

x:	93.0	96.2	100.0	104.2	108.7
y:	11.38	12.80	14.70	17.07	19.91

Solution:

By Lagrange's formula for inverse interpolation.

$$x = \frac{(y-12.80)(y-14.70)(y-17.07)(y-19.91)}{(11.38-12.80)(11.38-14.70)(11.38-17.07)(11.38-19.91)} (93.0)$$

$$+ \frac{(y-11.38)(y-14.70)(y-17.07)(y-19.91)}{(12.80-11.38)(12.80-14.70)(12.80-17.07)(12.80-19.91)} (96.2)$$

$$+ \frac{(y-11.38)(y-12.80)(y-17.07)(y-19.91)}{(14.70-11.38)(14.70-12.80)(14.70-17.07)(14.70-19.91)} \times (100.0)$$

$$+ \frac{(y-11.38)(y-12.80)(y-14.70)(y-19.91)}{(17.07-11.38)(17.07-12.80)(17.07-14.70)(17.07-19.91)} (104.2)$$

$$+ \frac{(y-11.38)(y-12.80)(y-14.70)(y-17.07)}{(19.91-11.38)(19.91-12.80)(19.91-14.70)(19.91-17.07)} \times (108.7)$$

Putting y = 13.5.

$$x = \frac{(0.7)(-1.2)(-3.57)(-6.41)(93)}{(-1.42)(-3.32)(5.69)(-8.53)} + \frac{(2.12)(-1.2)(-3.57)(-6.4)}{(1.42)(-1.9)(-4.27)(-7.11)} (96.2)$$

$$+ \frac{(2.12)(0.7)\ (3.57)(-6.4)(100)}{(13.32)(1.9)(-2.37)(-2.84)} + \frac{(2.12)(0.7)\ (-1.2)(-6.47)(-108.7)}{(5.69)(4.27)(2.37)(2.84)}$$

$$\frac{(2.12)(0.7)(-1.2)(-3.57)(96.2)}{(18.53)(7.11)(5.21)(2.84)}$$

= 7.8126929 + 68.3721132 + 43.595887 – 7.2733428 + 0.770048198

x = 97.6557503.

Example: 2

The following table gives the values of the probability integral $\frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$ corresponding to certain values of x. For what value of x is the integral equal to 0.37.

x:	0.4	0.6	0.8
$y = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$:	0.3683	0.3332	0.2897

Solution:

Since Lagrange's formula is merely a relation between two variables either of which may be taken as the independent variable. It is evident that by considering y as the independent variable we can write a formula giving x as a function of y.

Hence

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)} x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)} x_1$$

$$+ \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)} x_2$$

In this case

$x_0 = 0.4$, $x_1 = 0.6$ $x_2 = 0.8$

$y_0 = 0.3683$ $y_1 = 0.3332$ $y_2 = 0.2897$

Let $x_{(3)}$ be the value corresponding to y = 0.3

Then $x_{(0.3)} = \frac{(0.3-0.3332)(.3-0.2897)}{(0.3683-0.3332)(0.3682-0.2897)} (0.4)$

$$+ \frac{(0.3-0.3683)(0.3-0.2897)}{(0.3332-0.3683)(0.3-0.3332)} (0.6)$$

$$+ \frac{(0.3-0.3683)(0.3-0.3332)}{(0.28977-0.3683)(0.2897-0.3332)} (0.8)$$

$$= \frac{(-0.0332)(0.0103)}{(0.0351)(0.0785)} (0.4) + \frac{(-0.0683)(0.0103)}{(-0.0351)(0.0435)} (0.6)$$

$$+ \frac{(-0.0683)(-0.0332)}{(-0.0786)(-0.0435)} (0.8)$$

$$= \frac{(-.00034196)}{.00275535} (0.4) + \frac{(-.00070349)}{(-.00152685)} (.6) + \frac{-.00226756}{.0034191} (0.8)$$

$$= -0.049643958 + 0.276447588 + 0.530563013$$

X = 0.75739505

Example: 3

Using Lagrange's formula, prove

$y_1 = y_3 - 0.3 (y_5 - y_3) + 0.2 (y_{-3} - y_{-5})$ nearly.

Solution:

y_{-5}, y_{-3}, y_3, y_5 occur in the answers. So we can have the table.

x:	−5	−3	3	5
y:	y_{-5}	y_{-3}	y_3	y_5

By Lagrange's Formula: $x_0 = -5$, $x_1 = -3$, $x_2 = 3$, $x_3 = 5$

$$y(x) = \frac{(x+3)(x-3)(x-5)}{(-5+3)(-5-3)(-5-5)} y_{-5}$$

$$+ \frac{(x+5)(x-3)(x-5)}{(-3+5)(-3-3)(-3-5)} \; y_{-3}$$

$$+ \frac{(x+5)(x+3)(x-5)}{(3+5)(3+3)(3-5)} \; y_3 + \frac{(x+5)(x+3)(x-3)}{(5+5)(5+3)(5-3)} \; y_5$$

$$y_{(1)} = \frac{(4)(-2)(-4)}{(-2)(-8)(-10)} \; y_{-5} + \frac{(6)(-2)(-4)}{(2)(-6)(-8)} \; y_{-3}$$

$$+ \frac{(6)(4)(-4)}{(8)(6)(-2)} \; y_3 + \frac{(6)(4)(-2)}{(10)(8)(2)} \; y_5$$

$$= (0.2)\, y_{-5} + (0.5)\, y_{-3} + y_3 - (0.3)\, y_5$$

$$y_1 = y_3 - 0.3\,(y_5 - y_{-3}) + (0.2)\,(y_{-3} - y_{-5})$$

Example: 4

The mode of a certain frequency curve $y = f(x)$ is very nearer to $x = 9$ and the values of the frequency density $f(x)$ for $x = 8.9, 9, 9.3$ are respectively 0.30, 0.35 and 0.25. Calculate the approximate value of the mode.

Solution:

Given that

x:	8.9	9.0	9.3
f(x):	0.30	0.35	0.25

By Lagrange's interpolation Formula,

$$f(x) = \frac{(x-9)(x-9.3)}{(8.9-9)(8.9-9.3)}(0.30) + \frac{(x-8.9)(x-9.3)}{(9-8.9)(9-9.3)}(0.35) + \frac{(x-8.9)(x-9)}{(9.3-8.9)(9.3-9)}(0.25)$$

$$= \frac{(x-9)(x-9.3)}{(-0.1)(-0.4)}(0.30) + \frac{(x-8.9)(x-9.3)}{(0.1)(-0.3)} + \frac{(x-8.9)(x-9)}{(0.4)\;(0.3)}(0.25)$$

$$= (x^2 - 18.3x + 83.7)(7.5) - (x^2 - 18.2x + 82.77)(11.67) + (x^2 - 17.9x + 80.1)\;(2.083)$$

$$= 7.5x^2 - 137.25x + 627.75 - 11.67x^2 + 212.394x - 965.9259 + 2.083x^2 - 37.2857x + 166.8483$$

$$f(x) = 2.087x^2 + 7.8583 - 171.3276$$

To get the mode, $f^1(x) = 0$ and $f^{11}(x) = -ve$.

$$\therefore f^1(x) = 0 \Rightarrow$$

$f^1(x) = -4.174x + 37.8583 = 0$

$$\therefore x = \frac{+37.8583}{+4.174} = 9.07$$

$f^{11}(x) = -4.174 = (-ve)$.

$\therefore$ f(x) is maximum at x = 9.07

$\therefore$ Mode is 9.07

Exercise:

1. Use Lagrange's interpolation formula to fit a polynomial to the data.

x:	−1	0	2	3
y:	−8	3	1	2

2. Given $U_1 = 22$. $U_2 = 30$, $U_4 = 82$, $U_7 = 106$, $U_8 = 206$ find U_6. Using Lagrange's interpolation formula.

3. Using Lagrange's formula find f (6) given

x:	2	5	7	10	12
f(x):	18	180	448	1210	2028

4. If $y_0 = 1$, $y_3 = 19$, $y_4 = 49$ & $y_6 = 181$ find y_5.

Inverse Interpolation:

1. Find x, given y = 0.3 from the data

x:	0.4	0.6	0.8
y:	0.3683	0.3332	0.2897

2. Find the value of x when y(x) = 19 given .

x:	0	1	2
y:	0	1	20

3. If cos hx = 1.285 find x given.

x:	0.735	0.736	0.737	0.738
cos hx::	1.2824937	1.2832974	1.2841023	1.2849085
x:	0.739	0.740	0.741	0.742
cos hx::	1.2857159	1.2865247	1.2873348	1.2881461

4. Given f (30) = −30, f(34) = −13, f(38) = 3 and f(42) = 18 find x so that f(x) = 0.

UNIT – VII

FINITE DIFFERENCES

7.1: FORWARD DIFFERENCES

Let $y = f(x)$ be a given function of x and let $y_0, y_1, y_2, \dots y_n$ be the values of y corresponding to $x_0, x_1, x_2, \dots x_n$ the values of x.

The independent variable x is called the argument and the corresponding dependent value y is called the entry.

In general, the difference between any two-consective values of need not be same or equal.

We can write the arguments and entires as below.

$x: x_0 \quad x_1 \quad x_2 \dots\dots\dots, x_{n-1} \quad x_n$

$y: y_0 \quad y_1 \quad y_2 \dots\dots\dots, y_{n-1} \quad y_n$

If we subtract from each value of y(except y_0) the preceeding value of y,

We get

$y_1 - y_0, y_2 - y_1, y_3 - y_2, \dots y_n - y_{n-1}$

These results are called the first differences of y.

The first differences of y are denoted by Δy.

(ie) $\Delta y_0 = y_1 - y_0$

$\Delta y_1 = y_2 - y_1$

$\Delta y_2 = y_3 - y_2$

$\vdots$

$\Delta y_{n-1} = y_n - y_{n-1}$

$\therefore \Delta y_k = y_{k+1} - y_k$

Here, the symbol Δ denotes an operation called forward differences operator.

Higher Differences:

The second and higher differences are defined as below:

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$$

$$\Delta^2 y_1 = \Delta(\Delta y_1) = \Delta(y_2 - y_1) = \Delta y_2 - \Delta y_1$$

$$\vdots$$

$$\Delta^2 y_{n-1} = \Delta(\Delta y_{n-1}) = \Delta(y_n - y_{n-1}) = \Delta y_n - y_{n-1}$$

Here, Δ^2 is an operator called second order forward difference operator.

In the same way,

The third order forward difference operator Δ^3 is as follows:

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$

$$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$$

....

....

In general

$$\therefore \Delta^r y_k = \Delta^{r-1} y_{K+1} - \Delta^{r-1} y_k$$

There differences are called forward differences and these differences are usually represented in ta+bular form

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		$\Delta^6 y_1$
		Δy_4		$\Delta^3 y_3$		$\Delta^5 y_2$	
x_5	y_5		$\Delta^2 y_4$		$\Delta^4 y_3$		
		Δy_5		$\Delta^3 y_4$			
x_6	y_6		$\Delta^2 y_5$				
		Δy_6					
x_7	y_7						

The quantities in each column represent the difference between the quantities in the proceeding column. They are equally placed midway between the quantities being subtracted.

Result:

$\Delta y_0 = y_1 - y_0$

$\therefore\ y_1 = y_0 + \Delta y_0$ (1)

we have $y_2 = y_1 + \Delta y_1$

but $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$

$\therefore\ \Delta y_1 = \Delta y_0 + \Delta^2 y_0$

Hence $y_2 = y_0 + \Delta y_0 + \Delta y_0 + \Delta^2 y_0 = y_0 + 2\Delta y_0 + \Delta^2 y_0$

$y_2 = (1 + 2\Delta + \Delta^2)y_0 = (1+\Delta)^2 y_0$ (2)

We have $y_3 = y_2 + \Delta y_2$

but $\Delta^2 y_1 = \Delta y_2 - \Delta y_1$

$\therefore \Delta y_2 = \Delta y_1 + \Delta^2 y_1$

Also $\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$

$\therefore \Delta^2 y_1 = \Delta^2 y_0 + \Delta^3 y_0$

Hence $y_3 = y_0 + 2\Delta y_0 + \Delta^2 y_0 + \Delta y_0 + \Delta^2 y_0 + \Delta^2 y_0 + \Delta^3 y_0$

$= y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0$ (3)

$= y_0 (1 + 3\Delta + 3\Delta^2 + \Delta^3) = (1+\Delta)^2 y_0$

The results (1), (2) and (3) can be written symbolically as

$y_1 = (1+\Delta)y_0$

$y_2 = (1+\Delta)^2 y_0$

$y_3 = (1+\Delta)^3 y_0$

In which $(1+\Delta)^r$ is an operator in y with the exponent on the Δ indicating the order of the difference.

From the expressions for y_1, y_2, y_3 ....

We get $y_k = (1+\Delta)^k y_0$

$= (1 + kc_1\Delta + kc_2\Delta^2 + \ldots\ldots + \Delta^k) y_0$

$y_k = y_0 + kc_1 \Delta y_0 + kc_2 \Delta^2 y_0 + \ldots\ldots + \Delta^k y_k$

This formula enables us to represent every value of y_k in term of y_0 and the forward differences Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$,........

Example: 1

Find difference table of the numbers is given below:

x:	– 1	0	1	2	3	4	5
y:	–17	–6	23	76	159	278	439

verified y_5

Solution:

Difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-1	$-17 y_{-1}$				
		$11 = \Delta y_{-1}$			
0	$-6 y_0$		$18 = \Delta^2 y_{-1}$		
		$29 = \Delta y_0$		$6 = \Delta^3 y_{-1}$	
1	$23 y_1$		$24 = \Delta^2 y_0$		0
		$53 = \Delta y_1$		$6 = \Delta^3 y_0$	
2	$76 y_2$		$30 = \Delta^2 y_1$		0
		$83 = \Delta y_2$		$6 = \Delta^3 y_1$	
3	$159 y_3$		$36 = \Delta^2 y_2$		0
		$119 = \Delta y_3$		$6 = \Delta^3 y_2$	
4	$278 y_4$		$42 = \Delta^2 y_3$		
		$161 = \Delta y_4$			
5	$439 y_5$				

$y_k = (1 + \Delta)^k y_0$

$k=5$

$y_5 = (1+ \Delta)^5 y_0$

$= y_0 + 5\Delta y_0 + 10\Delta^2 y_0 + 10\Delta^3 y_0 + 5\Delta^4 y_0 + \Delta^5 y_0$

$= -6 + 5(29) + 10(24) + 10(6) + 5(0) + 0$

$y_5 = 439$

Example:2

Find the 8^{th} term and the general term of the series 3,3,5,9,15,23......

Solution:

Difference Table

y	Δy	$\Delta^2 y$	$\Delta^3 y$
3			
	0		
3		2	
	2		0
5		2	
	4		0
9		2	
	6		0
15		2	
	8		
23			

Taking 3 as y_0, the eight term of the series in y_7.

$y_7 = (1+\Delta)^7 y_0$

$= (1 + 7\Delta + 7c_2\Delta^2 + 7c_3\Delta^3 + 7c_4\Delta^4 + 7c_5\Delta^5 + 7c_6\Delta^6 + \Delta^7)\, y_0$

$= (1 + 7\Delta + 21\Delta^2 + 35\Delta^3 + 35\Delta^4 + 21\Delta^5 + 7\Delta^6 + \Delta^7)\, y_0$

$y_0 = 3,$

$\Delta y_0 = 0$, $\Delta^2 y_0 = 2$, $\Delta^3 y_0 = 0$

$\therefore\ y_7 = y_0 + 7\Delta y_0 + 21\Delta^2 y_0 + 35\Delta^3 y_0 + \ldots\ldots.$

$\therefore\ y_7 = 3 + 7(0) + 21(2) + 35(0)$

$\therefore\ y_7 = 45$

Since the subsequent differences are zeros.

$$\therefore y_k = (1+\Delta)^k y_0$$

$$= \left[1 + k\Delta + \frac{k(k-1)}{2}\Delta^2 + \frac{k(k-1)(k-2)}{6}\Delta^3\right] y_0$$

$$y_k = y_0 + k\Delta y_0 + \frac{k(k-1)}{2}\Delta^2 y_0 + \frac{k(k-1)(k-2)}{6}\Delta^3 y_0$$

$$= 3 + k(0) + \frac{k(k-1)}{2}(2) + \frac{k(k-1)(k-2)}{6}(0)$$

$$y_k = 3 + k^2 - k + 0$$

$$\therefore y_k = k^2 - k + 3$$

Example: 3

Find the sixth term of the sequence 8, 12, 19, 29, 42

Solution:

Difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	8			
		4		
1	12		3	
		7		0
2	19		3	
		10		0
3	29		3	
		13		
4	42			

$y_0 = 8$, $\Delta y_0 = 4$, $\Delta^2 y_0 = 3$, $\Delta^3 y_0 = 0$

Taking 8 as y_0, the sixth term of the series in y_5

$y_6 = (1 + \Delta)^5 y_0$

$= y_0 + 5\Delta y_0 + 10\Delta^2 y_0 + 10\Delta^3 y_0 + 5\Delta^4 y_0 + \Delta^5 y_0$

$= 8 + 5(4) + 10(3) + 10(0) + 0$

$y_6 = 58$

Example: 4

Find the 7^{th} term of the sequence 2,9,28,65,126,217 and also find the general term.

Solution:

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	2				
		7			
1	9		12		
		19		6	
2	28		18		0
		37		6	
3	65		24		0
		61		6	
4	126		30		
		91			
5	217				

Taking 2 as y_0, the 7^{th} term of the series is y_6

$$y_7 = (1+\Delta)^6 y_0 = y_0 + 6c_1 \Delta^2 y_0 + 6c_3 \Delta^3 y_0 + 6c_4 \Delta^4 y_0 + 6c_5 \Delta^5 y_0 + \Delta^6 y_0$$

$$y_0 = 2,\ \Delta y_0 = 7,\ \Delta^2 y_0 = 12,\ \Delta^3 y_0 = 6,\ \Delta^4 y_0 = 0.$$

$$\therefore\ y_7 = 2 + 6(7) + 15 + 20\,(6) + 15(0)$$

$$y_7 = 2 + 42 + 180 + 120$$

$$\therefore\ y_7 = 344$$

$$y_n = (1+\Delta)^n\, y_0$$

$$= y_0 + nc_1 \Delta y_0 + nc_2 \Delta^2 y_0 + nc_3 \Delta^3 y_0 + nc_4 \Delta^4 y_0 + \ldots\ldots.$$

$$= 2 + n(7) + \frac{n(n-1)}{2}\,(12) + \frac{n(n-1)(n-2)}{6}\,(6) + 0$$

$$y_n = n^3 + 3n^2 + 3n + 2$$

$$= (n+1)^3 + 1$$

$$\therefore y_6 = (6+1)^3 + 1 = 7^3 + 1 = 344.$$

Example: 5

Find f(x) from the table below. Also find f(7)

x :	0	1	2	3	4	5	6
f(x):	–1	3	19	53	111	199	323

Solution:

Difference Table

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	–1				
		4			
1	3		12		
		16		6	
2	19		18		0
		34		6	
3	53		24		0
		58		6	
4	111		30		0
		88		6	
5	199		36		
		124			
6	323				

$\Delta^4 f(x)$, $\Delta^5 f(x)$ ...... are all zero

$y_x = (1 + \Delta)^x y_0$

$= y_0 + xc_1 \Delta y_1 + xc_2 \Delta^2 y_0 + xc_3 \Delta^3 + xc_4 \Delta^4 y_0 + \ldots\ldots$

$$= (-1) + x\,(4) + \frac{x(x-1)}{2}(12) + \frac{x(x-1)(x-2)}{16}(16) + 0$$

$= -1 + 4x + 6x^2 - 6x + x^3 - 3x^2 + 2x$

$f(x) = x^3 + 3x^2 - 1$

$f(7) = 7^3 + 3(49) - 1$

$= 489$

Exercise

1. Compute the third difference of f(32) by the formula from the table of entries.

   x : 32 33 34 35

   f(x): 539 8568 8765 24364
2. Find y_3 given $y_5 = 4$, $y_6 = 3$, $y_7 = 4$, $y_8 = 10$, $y_9 = 24$ $y_{10} = 49$, the third differences being constant.
3. Show that $\Delta^2 y_0 = y_3 . - 3y_2 + 3y_1 - y_0$
4. For the function y = sinhx, write down the table by taking x = 1.5, 1.6, 1.7, 1.8......2.1.
5. Find the 5th term of the sequence 3, 611,18.
6. Obtain the 6th 7th terms of the sequence 0,4,16,42,88.
7. If the third differences are constants, find u_6 if $u_0 = 9$, $u_1 = 18$, $u_2 = 20$, $u_3 = 24$.
8. Calculate $\Delta^4 u_6$ if $u_6 = 2$. $u_7 = -6$, $u_8 = 8$, $u_9 = 9$ and u10 = 17.

Answer :

1. 23234
2. 0
5. 27
6. 160,264
7. 138
8. 55

7.2 Backward Differences

If y_0, y_1, y_2.....y_n denote a set of values of any function $y = f(x)$ for $x_0, x_1, x_2.....x_n$ respectively then

$y_1 - y_0$, $y_2 - y_1$, $y_3 - y_2$, ... $y_n - y_{n-1}$ are the first differences and they are denoted by

$\nabla^2 y_2$, $\nabla^2 y_3$, $\nabla^2 y_n$

$\therefore \ \nabla^2 y_2 = \nabla y_2 - \nabla y_1$

$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$

....

....

$\nabla^2 y_k = \nabla y_k - \nabla y_{k-1}$

....

In general $\nabla^r y_k = \nabla^{r-1} y_k - \nabla^{r-1} y_{k-1}$

These differences when this notation is used are called backward differences.

A table of backward differences is indicated in the following table where the differences $\nabla^k y_i$ with fixed subscript i is along the diagonal starting up as shown by arrows.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		$\nabla^5 y_6$
x_4	y_4		$\nabla^2 y_5$		$\nabla^4 y_6$	
		∇y_5		$\nabla^3 y_6$		$\nabla^5 y_7$
x_5	y_5		$\nabla^2 y_6$		$\nabla^4 y_7$	
		∇y_6		$\nabla^3 y_7$		
x_6	y_6		$\nabla^2 y_7$			
		∇y_7				
x_7	y_7					

Result:

$\nabla y_n = y_n - y_{n-1}$

$\therefore\ y_{n-1} = y_n - \nabla y_n = (1-\nabla)y_n$

$y_{n-2} = y_{n-1} - \nabla y_{n-1}$

but $\nabla^2 y_n = y_n - \nabla y_{n-1}$

$\therefore\ \nabla y_{n-1} = \nabla y_n - \nabla^2 y_n$

Hence $y_{n-2} = y_n - \nabla y_n - (\nabla y_n - \nabla^2 y_n)$

$= y_n - 2\nabla y_n + \nabla^2 y_n$

$y_{n-2} = (1-\nabla)^2\, y_n$

Similarly,

$y_{n-3} = (1-\nabla)^3 y_n$

In general $y_{n-k} = (1-\nabla)^k y_n$

$= (1 - k\,c_1\nabla + kc_2\,\nabla^2 \ldots\ldots)y_n$

$y_{n-k} = y_n - k\,c_1\nabla y_n + kc_2\,\nabla^2\, y_n \ldots\ldots\ldots$

It shows that any value of y in the above table can expressed in terms of y_n and the back differences of y_n.

Example:1

Find y(−1) if y(0) = 2, y(1) = 9, y(2) = 28, y(3) = 65 y(4) = 126, y(5) = 127

Solution:

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	2				
		7			
1	9		12		
		19		6	
2	28		18		0
		37		6	
3	65		24		0
		61		6	
4	126		30		
		91			
5	217				

$\nabla y_5 = 91,\quad \nabla^2 y_5 = 30,\quad \nabla^3 y_5 = 6,\quad \nabla^4 y_5 = 0$

$y(-1) = y_{-1} = y_{5-6}$

$y(-1) = y_5 - 6c_1\nabla y_5 + 6c_2\nabla^2 y_5 + 6c_3\nabla^3 y_5 + 6c_4\nabla^4\, y_5 + \ldots\ldots$

$= 217 - 546 + 450 - 120$

$y(-1) = 667 - 666 = 1$

We can verify the value of y(0)

$y(0) = y_0 = y_{5-5}$

$y_0 = y_5 - 5c_1 \Delta y_5 + 5c_2 \Delta^2 y_5 - 5c_3 \Delta^3 y_5 + \ldots\ldots$

$= 217 - 5(91) + 10(30) - 10(6)$

$= 217 - 455 + 300 - 60 = 2$

This is exactly the same given value $y_0 = 2$.

Example: 2

Find the first term of the series whose second and subsequent terms are 8,3,0, –1, 0........

Solution:

Let y_0 be the first term.

$\therefore\ y_1 = 8,\ y_2 = 3,\ y_3 = 0,\ y_4 = -1$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	8			
		–5		
2	3		2	
		–3		0
3	0		2	
		1		0
4	–1		2	
		1		
5	0			

The differences of y_1 are $y_1 = 8$, $\Delta y_1 = -5$, $\Delta^2 y_1 = 2$, $\Delta^3 y_1 = 0$, $\Delta^4 y_1 = 0\ldots\ldots$

$y_0 = (1+\Delta)^{-1} y_1$

$=(1 - \Delta + \Delta^2 - \Delta^3 + \Delta^4 \ldots\ldots) y_1$

$= y_1 - \Delta y_1 + \Delta^2 y_1 - \Delta^3 y_1 + \ldots\ldots$

$= 8 - (-5) + (-2) - 0 + \ldots\ldots$

$= 15$

Example: 3

Obtain backward difference table for $y = f(x)$ $f(x) = x^3 - 3x^2 - 5x - 7$ for $x = -1, 0, 1, 2, 3, 4, 5$.

Solution:

x :	–1	0	1	2	3	4	5
f(x):	–6	–7	–14	–21	–22	–11	18

Backward Difference table

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
–1	–6				
		–1			
0	–7		–6		
		–7		6	
1	–14		0		0
		–7		6	
2	–21		6		0
		–1		6	
3	–22		12		0
		11		6	
4	–11		18		
		29			
5	18				

Example: 4

Find first term of the series whose second and subsequent terms are 46, 66, 81, 93, 101

Solution:

Let y_0 be the first term.

$y_1 = 46$, $y_2 = 66$, $y_3 = 81$, $y_4 = 93$, $y_5 = 101$

Difference Table

x		y	∇y	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
0	y_0					
		46				
1	y_1		20			
		66		–5		
2	y_2		15		2	
		81		–3		–3
3	y_3		12		–1	
		93		–4		
4	y_4		8			
		101				
5	y_5					

The difference of y_1 are $y_1 = 46$, $\Delta y_1 = 20$ $\Delta^2 y_1 = -5$, $\Delta^3 y_1 = 2$ $\Delta^4 y_1 = -3$

$$y_1 = (1+\Delta)^{-1}$$
$$= (1 - \Delta + \Delta^2 - \Delta^3 - \Delta^4 \ldots\ldots.)y_1$$
$$= y_1 - \Delta y_1 + \Delta^2 y_1 - \Delta^3 y_1 + \ldots\ldots$$
$$= 46 - 20 + (-5) - (2) + (-3) \ldots\ldots$$
$$y_0 = 16$$

Exercise

1. Obtain backward difference table $y = f(x)$, $f(x) = x^3 + 3x^2 - 5x + 8$ for $x = -1, 0, 1, 2, 3, 4, 5, 6$.
2. Constant a backward difference table, given Sin 30° = 0.5000, Sin 35° = 0.5736, Sin 40° = 0.6428 and Sin 45° = 0.7071, assuming third differences are constants, find Sin 25°.
3. Find $y(-1)$, if $y(0) = 3$, $y(1) = 11$, $y(2) = 30$, $y(3) = 67$, $y(4) = 128$, $y(5) = 217$.
4. Find the first term of the series whose second & subsequent terms 3, 19, 53,111, 199,323.

7.3 Operators

We have already defined the forward and Backward difference operator (ie) Δ and ∇.

Central Difference operator (δ):

The central difference operator δ is defined by

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

(or)

$$\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}$$

Shifting or displacement or translation operator E.

We define the shifting operator E such that

$Ef(x) = f(x+h)$

(or)

$Ey_x = y_{x+h}$

Hence $Ey_1 = y_2$, $E(y_2) = y_3$ etc.

$E^2 y_x = E(Y_{x+h}) = Y_{x+2h}$

$E^n y_x = EY_{x+nh}$ and $E^n f(x) = f(x+nh)$

Averaging operator μ:

The averaging operator μ is defined by

$$\mu y_x = \frac{1}{2}\left(y_{x+\frac{h}{2}} + y_{x-\frac{h}{2}}\right)$$

(ie)

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)\right]$$

Result:

$$Y_{x+rh} = E^r y_x$$
$$= (1+\Delta)^r y_x$$
$$= (1+ rc_1\Delta + rc_2\Delta^2 + \ldots\ldots)y_x$$
$$= y_x + rc_1\Delta y_x + rc_2\Delta^2 y_x + \ldots\ldots \Delta^r y_x$$

Relation:

Relation between E and Δ .

We know $\Delta(f(x)) = f(x + h) - f(x)$

$$= Ef(x) - 1.\ f(x)$$

$$\Delta(f(x)) = (E - 1)\ f(x)$$

$$\therefore\ \Delta = (E - 1)$$

This is called separation of symbols

$$\Delta = E - 1$$
$$\therefore E = 1 + \Delta$$

Relation between E and ∇:

$$\nabla f(x) = f(x) - f(x - h)$$
$$= 1.f(x) - E^{-1} f(x)$$
$$\nabla f(x) = (1 - E^{-1})\ f(x)$$
$$\therefore \nabla = 1 - E^{-1}$$

$E^{-1} = (1 - \nabla)^{-1}$ Since $\left(E^{-1}\right)^{-1} = E$

Relation between E and δ.

$$\delta\ f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

$$= E^{1/2} f(x) - E^{-1/2} f(x) = \left(E^{1/2} - E^{-1/2}\right) f(x)$$

$$\therefore \delta = E^{1/2} - E^{-1/2} = E^{-1/2}(E-1) = E^{-1/2}\,\Delta$$

Also $\delta = E^{1/2}(1 - E^{-1}) = E^{1/2}\,\nabla$

$$\therefore \delta = E^{-1/2}\,\Delta = E^{1/2}\,\nabla$$

Relation between E and μ:

$$\mu\, f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

$$= \frac{1}{2}\left[E^{1/2} f(x) + E^{-1/2} f(x)\right]$$

$$\mu\, f(x) = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] f(x)$$

(1) $$\therefore \mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$$

Relation between D and Δ:

$$D\, f(x) = \frac{d}{dx} f(x)$$

(2) **By Taylor's theorem:**

$$f(x+h) = f(x) + \frac{h}{1!} f^{1}(x) + \frac{h^2}{2!} f^{11}(x) + \ldots\ldots\ldots$$

$$E\, f(x) = f(x) + \frac{h}{1!} D\, f(x) + \frac{h^2}{2!} D^2 f(x) + \ldots\ldots\ldots$$

$$E\, f(x) = \left[1 + \frac{hD}{1!} + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} + \ldots\ldots\ldots\right] f(x)$$

$$E\, f(x) = e^{hD} f(x)$$

$$\therefore E = e^{hD}$$

$$\therefore E = 1 + \Delta = e^{hD}$$

Taking log on both side.

$$h\,D = \log E = \log(1+\Delta).$$

$$h\,D = \Delta - \frac{\Delta^2}{2!} + \frac{\Delta^3}{3!} \ldots\ldots\ldots\ldots$$

$$D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2!} + \frac{\Delta^3}{3!}\ldots\ldots\ldots\ldots\right]$$

Example:

$$\text{Prove } \left(\frac{\Delta^2}{E}\right)u_x \neq \frac{\Delta^2 u_x}{\Delta E U_x}$$

Solution:

$$\text{Now } \frac{\Delta^2 U_x}{E} = \left(\Delta^2 E^{-1}\right) u_x$$

$$= \left[(E-1)^2 E^{-1}\right] U_x$$

$$= [\,E - 2 + E^{-1}\,]\, u_x$$

$$= u_{x+h} - 2\,u_x + u_{x-h}$$

$$\frac{\Delta^2 u_x}{E U_x} = \frac{(E-1)^2 u_x}{u_{x+h}}$$

$$= \frac{(E^2 - 2E + 1) U_x}{u_{x+h}}$$

$$= \frac{u_{x+2h} - 2u_{x+h} + u_x}{u_{x+h}}$$

From (1) and (2)

The R.H.S. of 1 and 2 are not equal.

$$\therefore \quad \left(\frac{\Delta^2}{E}\right)U_x \neq \frac{\Delta^2 u_x}{E u_x}$$

Example: 1

Show that $\Delta\nabla = \nabla\Delta$

$$\nabla\Delta\, y_x = \nabla\,(y_{x+h} - y_x)$$

$$= \nabla\, y_{x+h} - \nabla\, y_x$$

Also we have

$$\Delta \ \nabla \ y_x = \Delta (y_x - y_{x-h})$$

$$= \Delta \ y_x - \Delta \ y_{x-h}$$

$$= (y_{x+h} - y_x) - (y_x - y_{x-h})$$

$$\Delta \ \nabla \ y_x = Y_{x+h} - 2y_x + y_{x-h}$$

$$\therefore \ \nabla \ \Delta \ y_x = \Delta \nabla \ y_x$$

$$\therefore \quad \nabla \ \Delta = \Delta \ \nabla$$

Example:

Show that $\delta = 2 \sin h \left(\frac{hD}{2}\right)$

Solution:

We have shown that

$$\delta = E^{1/2} - E^{-1/2} \text{ and } E = e^{hD}$$

$$\therefore \quad \delta = e^{1/2 hD} - e^{-1/2 hD}$$

$$\left[\because \sinh \theta = \frac{e^{\theta} - e^{-\theta}}{2}\right]$$

$$\delta = 2 \sin h \left(\frac{hD}{2}\right)$$

Example:

Show that $\mu^2 = 1 + \frac{1}{4} \delta^2$

Solution:

We have defined $\mu y_x = \frac{1}{2}\left(y_{x+h/2} + y_{x-h/2}\right)$

$$\mu y_x = \frac{1}{2}\left(E^{1/2} y_x + E^{-1/2} y_x\right)$$

$$\mu y_x = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right) y_x$$

$$\therefore \quad \mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$$

Hence $\mu^2 = \frac{1}{4}\left(E^{1/2} + E^{-1/2}\right)^2$ $\quad [\because (a+b)^2 = (a-b)^2 + 4ab]$

$$= \frac{1}{4}\left[\left(E^{1/2} - E^{-1/2}\right)^2 + 4\right]$$

$$= \frac{1}{4}\left[\delta^2 + 4\right] \qquad \left[\because \delta = E^{1/2} - E^{-1/2}\right]$$

$$\mu^2 = 1 + \frac{1}{4}\delta^2$$

Example:

Use the method of separation of symbols to prove the identity.

$y_x = y_{x-1} + \Delta\, y_{x-2} + \ldots\ldots + \Delta^{n-1}\, y_{x-n} + \Delta^n\, y_{x-n}$

Solution:

We have learnt that $y_{x-r} = E^{-r}\, y_x$

$\therefore Y_{x-1} = E^{-1}\, y_x$; $y_{x-2} = E^{-2}\, y_x \ldots\ldots y_{x-n} = E^{-n}\, y_x$.

Hence the R.H.S.

$$= E^{-1}\, y_x + \Delta\, E^{-2}\, y_x - \ldots\ldots + \Delta^{n-1}\, E^{-n}\, y_x + \Delta^n E^{-n}\, y_x$$

$$= \left(\frac{1}{E} + \frac{\Delta}{E^2} + \ldots\ldots\ldots + \frac{\Delta^{n-1}}{E^n} + \frac{\Delta^n}{E^n}\right) y_x$$

$$= \frac{1}{E}\left(1 + \frac{\Delta}{E} + \ldots\ldots + \frac{\Delta^{n-1}}{E^{n-1}}\right) y_x + \frac{\Delta^n}{E^n}\, y_x$$

$$= \left[\frac{1}{E}\left(\frac{\left(\frac{\Delta}{E}\right)^n - 1}{\frac{\Delta}{E} - 1}\right) + \frac{\Delta^n}{E_n}\right] y_x$$

$$= \left(\frac{\Delta^n - E^n}{\Delta - E} \cdot \frac{1}{E_n} + \frac{\Delta^n}{E^n}\right) y_x$$

$$= \left(\frac{\Delta^n - E^n}{-E^n} + \frac{\Delta^n}{E^n}\right) y_x \text{ since } \Delta + 1 = E$$

$$= y_x$$

Example:

Show that $a_0 + \frac{a_1 x}{1!} + \frac{a_2 x^2}{2!} + \dots\dots\dots = e^x \left(a_0 + \frac{x \Delta a_0}{1!} + \frac{x^2 \Delta^2 a_0}{2!} + \dots\dots\right)$ and hence sum the series to infinity

$$5 + \frac{4x}{1!} + \frac{5x^2}{2!} + \frac{14x^3}{3!} + \frac{37x^4}{4!} + \frac{80x^5}{5!} + \dots\dots\dots$$

Solution:

Denoting a_r by $E^r a_0$

We have $a_r = E^r a_0 = (1 + \Delta)^r a_0$

$$\therefore a_0 + \frac{a_1 x}{1!} + \frac{a_2 x^2}{2!} + \dots\dots\dots\dots\dots\dots\dots$$

$$= a_0 + \frac{x(1+\Delta) a_0}{1!} + \frac{\left(x^2 (1+\Delta)^2 a_0\right)}{2!} + \dots\dots\dots$$

$$= \left[1 + \frac{x(1+\Delta)}{1!} + \frac{(x(1+\Delta))^2}{2!} + \frac{(x(1+\Delta))^3}{3!} + \dots\dots\dots\dots\dots\right] a_0$$

$$= \left[e^{x(1+\Delta)}\right] a_0$$

$$= \left(e^{x + x\Delta}\right) a_0$$

$$= e^x \left(e^{x\Delta}\right) a_0$$

$$= e^x \left(1 + x\Delta + \frac{x^2 \Delta^2}{2!} \dots\dots\dots\right) a_0$$

$$= e^x \left(a_0 + x \Delta a_0 + \frac{x^2}{2!} \Delta^2 a_0 + \dots\dots\dots\right)$$

The following table gives the difference table for the coefficients a_0, a_1 .... of the given series.

5				
	−1			
4		2		
	1		6	
5		8		0
	9		6	
14		14		0
	23		6	
37		20		
	43			
80				

Hence the sum of the series is

$$e^x\left(5-x+\frac{2x^2}{2!}+\frac{6x^3}{3!}\right)=e^x\left(5-x+x^2+x^3\right)$$

Example:

Prove that $\sum_{x=1}^{n} y_x = n\,C_1\, y_1 + n\,C_2\,\Delta\, y_1 + n\,C_3\,\Delta^2 y_1 + \ldots\ldots + \Delta^{n-1}\, y_1$ and use it to determine.

Solution:

We have $\sum_{x=1}^{n} y_x = y_1 + y_2 + y_3 + \ldots. + y_n.$

$$= y_1 + E\,y_1 + E^2\,y_1 + \ldots\ldots + E^{n-1}\,y_1$$

$[E\,y_x = y_{x+h}]$

$$= (1 + E + E^2 + \ldots\ldots E^{n-1})\,y_1$$

$$= \frac{E^n-1}{E-1}\,y_1$$

$$= \left\{\frac{(1+\Delta)^n-1}{(1+\Delta)-1}\right\} y_1 \quad \text{since } E = 1+\Delta$$

$$= \frac{n\,C_1\,\Delta + n\,C_2\,\Delta^2 + \ldots\ldots + \Delta^n}{\Delta}\,y_1$$

$$= (n\,C_1 + n\,C_2\,\Delta + \ldots\ldots + \Delta^{n-1})\,y_1$$

$$\sum_{x=1}^{n} y_x = n\,C_1\,y_1 + n\,C_2\,\Delta\,y_1 + \ldots.+ \Delta^{n-1}\,y_1$$

We have $y_x = x^3$

	$\Delta\ y_1$	$\Delta^2 y_1$	$\Delta^3 y_1$	$\Delta^4 y_1$
$y_1 = 1$	7	12	6	0
$y_2 = 8$	19	18	6	0
$y_3 = 27$	37	24	6	
$y_4 = 64$	61	30		
$y_5 = 125$	91			
$y_6 = 216$				

$\therefore\ y_1 = 1,\ \Delta\ y_1 = 7,\quad \Delta^2 y1 = 12,\quad \Delta^3 y_1 = 6,\quad \Delta^4\quad y_1 = 0$

$$\therefore\ \sum_1^3 x^3 = n\,C_1\, y_1 + n\,C_2\ \Delta\ y_1 + n\,C_3\ \Delta^2\, y_1 + n\,C_4\ \Delta^3\, y_1$$

$$= n\,C_1\,(1) + n\,C_2\,(7) + n\,C_3\,(12) + n\,C_4\,(6)$$

$$= n + \frac{n(n-1)(7)}{2} + \frac{n(n-1)(n-2)}{3!} + \frac{n(n-1)(n-2)(n-3)}{4!} \qquad (6)$$

$$= n + \frac{7n^2}{2} - \frac{7n}{2} + \frac{n^3 - 3n^2 + 2n}{6}\ (12) +$$

$$\frac{n^4 - 3n^3 - 3n^3 + 9n^2 + 2n^2 - 6n}{24}\ (6)$$

$$= \frac{4n + 14n^2 - 14n + 4n^3 - 24n^2 + 16n + n^4 - 6n^3 + 11n^2 - 6n}{4}$$

$$= \frac{(0)\,n + n^2 + 2n^3 + n^4}{4}$$

$$\sum_1^3 x^3 = \frac{n^2(n^2 + 2n + 1)}{4} = \frac{n^2\,(n+1)^2}{4}$$

Example:

Prove the results.

i) $E\ \nabla = \Delta = \nabla\ E$

Proof:

$(E\ \nabla)\ y_x = E\ (\nabla\ y_x) = E\ (y_x - y_{x-h})$

$= E\ y_x - E\ y_{x-h}$

$= y_{x+h} - y_x = \Delta\ y_x$

$\therefore E\nabla = \Delta$

Also $(\nabla E)\, y_x = \nabla (E y_x) = \nabla y_{x+h} = y_{x+h} - y_x = \Delta y_x$

$\therefore \nabla E = \Delta$

Hence $E\nabla = \Delta = \nabla E$.

ii) $\delta E^{1/2} = \Delta$

Proof:

$$\delta E^{1/2} y_x = \delta\, y_{x+\frac{h}{2}} = (E^{1/2} - E^{-1/2})\, y_{x+\frac{h}{2}} \qquad [\, \delta = E^{1/2} - E^{-1/2} \,]$$

$$= E^{1/2} y_{x+\frac{h}{2}} - E^{-1/2} y_{x+\frac{h}{2}} = y_{x+h} - y_x = \Delta y_x$$

$$\therefore \delta E^{1/2} = \Delta$$

iii) $h D = \log(1+\Delta) = -\log(1-\nabla) = \sin h^{-1}(\mu\delta)$

Proof:

$$E = e^{hD}$$

$$\therefore e^{hD} = E = 1 + \Delta$$

Taking logarithm,

$$h D = \log(1+\Delta)$$

Also $\nabla = 1 - E^{-1}$; $\qquad \therefore E^{-1} = 1 - \nabla$

(i.e) $e^{-hD} = 1 - \nabla$

Taking logarithm,

$$-h D = \log(1-\nabla)$$

$$\therefore hD = -\log(1-\nabla)$$

$$\sin(h D) = \frac{e^{hD} - e^{-hD}}{2} = \frac{E - E^{-1}}{2} = \left(\frac{E^{1/2} + E^{-1/2}}{2}\right)\left(E^{1/2} - E^{-1/2}\right)$$

$$\sin(hD) = \mu\delta$$

$$\therefore \sinh hD = \mu\delta$$

$$\therefore \; hD = \sinh^{-1}(\mu\delta).$$

iv) $1 + \mu^2 \delta^2 = (1 + \frac{1}{2}\delta^2)^2$

Proof:

$$1 + \mu^2\delta^2 = 1 + \left(\frac{E^{1/2} + E^{-1/2}}{2}\right)^2 \left(E^{1/2} - E^{-1/2}\right)^2$$

$$= 1 + \left(\frac{E - E^{-1}}{2}\right)^2 = \frac{4 + (E - E^{-1})^2}{4} = \left(\frac{E + E^{-1}}{2}\right)^2 \qquad (1)$$

$$\left(1 + \frac{1}{2}\delta^2\right)^2 = \left[1 + \frac{1}{2}(E^{1/2} - E^{-1/2})^2\right]^2$$

$$= \left[1 + \frac{1}{2}(E + E^{-1} - 2)\right]^2$$

$$= \left[\frac{E + E^{-1}}{2}\right]^2 \qquad (2)$$

From (1) and (2)

$$1 + \mu^2\delta^2 = \left(1 + \frac{1}{2}\delta^2\right)^2$$

v) $E^{1/2} = \mu + \frac{1}{2}\delta$

Proof:

$$\mu + \frac{1}{2}\delta = \frac{E^{1/2} + E^{-1/2}}{2} + \frac{E^{1/2} - E^{-1/2}}{2} = \frac{E^{1/2} + E^{-1/2} + E^{1/2} - E^{-1/2}}{2}$$

$$\mu + \frac{1}{2}\delta = \frac{2E^{1/2}}{2} = E^{1/2}$$

vi) $E^{-1/2} = \mu - \frac{1}{2}\delta$

Proof:

$$\mu - \frac{1}{2}\delta = \frac{E^{1/2} + E^{-1/2}}{2} - \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)$$

$$= \frac{E^{1/2} + E^{-1/2} - E^{1/2} + E^{-1/2}}{2} = \frac{2E^{-1/2}}{2}$$

$$\mu - \frac{1}{2}\delta = E^{-1/2}$$

vii) $\mu\delta = \frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta$

Proof:

$$\frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta = \frac{1}{2}\Delta\,(E^{-1} + 1)$$

$$= \frac{1}{2}(E - 1)(E^{-1} + 1) = \frac{1}{2}(E.E^{-1} + E - E^{-1} - 1)$$

$$\frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta = \frac{1}{2}(E - E^{-1}) = \mu\delta$$

viii) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}}$

Proof:

$$\frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}} = \frac{1}{2}\delta\left[\delta + 2\sqrt{1 + \frac{\delta^2}{4}}\right]$$

$$= \frac{1}{2}\delta\left[\delta + \sqrt{4 + \delta^2}\right]$$

$$= \frac{1}{2}\delta\left[\left(E^{1/2} - E^{-1/2}\right) + \sqrt{4 + \left(E^{1/2} - E^{-1/2}\right)^2}\right]$$

$$= \frac{1}{2} \quad \delta \left[\quad \left(E^{1/2} - E^{-1/2}\right) + \sqrt{\left(E^{1/2} + E^{-1/2}\right)^2} \right.$$

$$= \frac{1}{2} \quad \left(E^{1/2} - E^{-1/2}\right) + \left[E^{1/2} - E^{-1/2} + E^{1/2} + E^{-1/2} \right]$$

$$= \frac{1}{2} \times 2 \quad \left(E^{1/2} - E^{-1/2}\right) + \ E^{1/2}$$

$$\frac{1}{2} \ \delta^2 + \delta \ \sqrt{1 + \frac{\delta^2}{4}} \qquad = E^{1/2} . E^{1/2} - E^{-1/2} . E^{1/2} = E - 1 = \Delta$$

ix) $\nabla \ \Delta = \Delta - \nabla = \delta^2$

Proof:

$$\nabla \ \Delta = (1 - E^{-1})\ (E - 1) = E - 1 - E^{-1} \ \ E + E^{-1} = E + E^{-1} - 2 = \delta^2$$

$$\Delta - \nabla \ = (E - 1) - (1 - E^{-1}) = E - 1 - 1 + E^{-1} = E + E^{-1} - 2 = \delta^2$$

(x) $(1 + \Delta)\ (1 - \nabla) = 1$

Proof:

$$(1 + \Delta)\ (1 - \nabla) = E \ . \ E^{-1} = 1$$

Exercise:

1. Prove the results

i) $\Delta^3 y_2 = \nabla^3 y_3$

ii) $\Delta = \mu \ \ \delta + \frac{1}{2} \ \delta^2$

iii) $\nabla = -\frac{1}{2} \ \delta^2 + \delta \ \sqrt{1 + \frac{1}{4} \ \delta^2}$

iv) $\delta = \Delta \ (1 + \Delta)^{-1/2} = \nabla \ (1 - \nabla)^{-1/2}$

v) $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

vi) $\Delta^2 = (1 + \Delta) \ \delta^2$

2. Prove that

$$u_0 + u_1 + u_2 + \ldots\ldots + u_n = (n+1)\ C_1\, u_0 + (n+1)\ C_2\, \Delta u_0 + \ldots\ldots + (n+1)\, C_{n+1}\, \Delta^n u_0$$

3. Prove $\Delta^n e^x = e^{x+n} - n\,C_1\, e^{x+n-1} + n\,C_2\, e^{x+n-2} + \ldots\ldots + (-1)^n e^x$

4. Prove that

$$u_0 - u_1 + u_2 - U_3 + \ldots\ldots\ldots = \tfrac{1}{2}\, u_0 - \frac{1}{4}\ \Delta u_0 + \frac{1}{8}\ \Delta^2 u_0 - \frac{1}{16}\Delta^3 u_0 + \ldots\ldots\ldots$$

PROPERTIES

7.4 Properties of operators

The operators $\Delta, \nabla, E, \delta, \mu$ and D are all linear operators.

i) $\Delta(a\, f(x) + b\ \phi\ (x)) = [a\,f(x+h) + b\,\phi(x+h)] - [a\,f(x) + b\,\phi(x)]$

$$= a\ [f(x+h) - f(x)] + b\ [\phi(x+h) - \phi(x)]$$

$$= a\ \Delta\ f(x) + b\ \Delta\ \phi\ (x)$$

Hence Δ is a linear operator.

Putting a = b = 1.

$$\Delta\ [f(x) + g(x)] = \Delta\ f(x) + \Delta\ g(x)$$

and by putting b = 0,

$$\Delta\ [a\,f(x)] = a\ \Delta\ f(x)$$

ii) The operators is distributive over addition.

$$\Delta^m\ \Delta^n\ f(x) = \Delta^{m+n}\ f(x) = \Delta^n\ \Delta^m\ f(x)$$

$$\Delta^m\ \Delta^n\ f(x) = (\Delta . \Delta\ \ldots\ldots m \text{ factors})\ (\Delta \ldots \Delta\ n \text{ factors})\ f(x)$$

$$= \Delta^{m+n}\ f(x)$$

iii) Also $\Delta\ [\ f(x) + g(x)\ ] = \Delta\ [g(x) + f(x)\ ]$

We known that D, is differentiation operate obeys many laws of algebra, such as

$$D\ (U_x + V_x) = D\ (U_x) + D\ (V_x)$$

$$D\ (C\ U_x) = C\ D\ (U_x)$$

$D^m (D^n U_X) = D^n (D^m U_X) = D^{m+n} U_X$.

If $y = x^n$, when n is a positive integer

$\Delta y = (x + h)^n - x^n$

$$= nx^{n-1} h + \frac{n(n-1)}{2} x^{n-2} h^2 + \ldots\ldots + h^n$$

= a polynomial of degree n – 1.

Similarly it can be show that

$\Delta^2 y$ = polynomial of degree n – 2 in x.

$\Delta^3 y$ = polynomial of degree n – 3 in x.

.. . .

.. . .

$\Delta^n y$ = polynomial of degree n – n (i.e.) 0 in x.

Hence the n^{th} difference of x^n where n is a positive integers are constants and so the n^{th} difference of any polynomial of n^{th} degree.

$$a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_n,$$

where n is a positive integer, are constants. If in forming the difference of a function, some order of differences (say n^{th}) becomes constant, the function is a polynomial of degree n,

Even if the n^{th} order differences become approximately as a polynomial of degree n,

$\Delta^n y_x = a_0 n!$ where $y_x = a_0 x^n + \ldots. a_n$.

Example: 1

Find the cubic polynomial in x which takes on the values –3, 3, 11, 27, 57, 107 when x = 0, 1, 2, 3, 4, 5 respectively.

Let the function be y_x.

Since the function is cubic polynomial, the fourth order differences. (i.e) $\Delta^4 y_x$ should be zeros.

x	y	Δ y	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	–3				
		6			
1	3		2		
		8		6	
2	11		8		0
		16		6	
3	27		14		0
		30		6	
4	57		20		
		50			
5	107				

We have $y_x = E^x \, y_0$

$$= (1 + \Delta)^x \, y_0$$

$$= \left(1 + x\,\Delta + \frac{x(x-1)}{2!}\,\Delta^2 + \frac{x(x-1)(x-2)}{3!}\Delta^3 \right) y_0$$

$$= y_0 + x\,\Delta\, y_0 + \frac{x(x-1)}{2!}\,\Delta^2 y_0 + \frac{x(x-1)(x-2)}{6} \Bigg) \Delta^3 y_0$$

$$= -3 + x\,(6) + \frac{x(x-1)}{2}\,(2) + \frac{x(x-1)(x-2)}{6} \qquad (6)$$

$$= -3 + 6x + x^2 - x + x^3 - 2x^2 - x^2 + 2x$$
$$= -3 + 7x - 2x^2 + x^3$$

$\therefore y_x = x^3 - 2x^2 + 7x - 3$

Example: 2

Find y_6, if $y_0 = 9$, $y_1 = 18$, $y_2 = 20$, $y_3 = 24$ given that the third differences are constants.

Solution:

Since third differences are constants.

$$\Delta^4 y_0 = 0, \quad \Delta^5 y_0 = \Delta^6 y_0 = 0$$

$y_6 = E^6\, y_0 = (1 + \Delta)^6\, y_0$

$$= (1 + 6\,C_1\,\Delta + 6\,C_2\,\Delta^2 + 6\,C_3\,\Delta^3 + 6\,C_4\,\Delta^4 + 6\,C_5\,\Delta^5 + \Delta^6)\, y_0$$

$= (1 + 6\,\Delta + 15\,\Delta^2 + 20\,\Delta^3)\, y_0$ since other terms vanish

$$= [1 + 6\,(E - 1) + 15\,(E - 1)^2 + 20\,(E - 1)^3] y_0$$

$= [1 + 6E - 6 + 15E^2 - 30E + 15 + 20E^3 - 60E^2 + 60E - 20]\, y_0$

$= (-10 + 36E - 45E^2 + 20E^3)\, y_0$

$= -10\, y_0 + 36E\, y_0 - 45E^2\, y_0 + 20E^3\, y_0$

$= -10\, y_0 + 36\, y_1 - 45 y_2 + 20\, y_3$

$= -10\,(9) + 36\,(18) - 45\,(20) + 20\,(24)$

$= -90 + 648 - 900 + 480 = 1128 - 990 = 1384.$

7.5 FINDING MMISSING TERMS

1. Find the missing term in the following table:

x	:	7	9	11	13	15	17
y	:	32	78	–	144	257	381

Solution:

Since 5 values of y are given, we assume that the 5th differences zeros.

$\Delta^5\, y_1 = 0$

(i.e.) $(E - 1)^5\; y_1 = 0$

(i.e.) $(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)\, y_1 = 0$

(i.e.) $E^5\, y_1 - 5E^4\, y_1 + 10E^3\, y_1 - 10E^2\, y_1 + 5E\, y_1 - y_1 = 0$

(i.e.) $y_6 - 5\, y_5 + 10\, y_4 - 10\, y_3 + 5\, y_2 - y_1 = 0$

Substituting the values for y_1, y_2, y_4, y_5, y_6

We get

$381 - 5\,(257) + 10\,(144) - 10\, y_3 + 5\,(78) - 32 = 0$

(i.e.) $894 - 10\, y_3 = 0$

$y_3 = 89.4$

Alites:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
7	32					
		46				
9	78		a – 124			
		a – 78		346 – 3a		
11	a		222 – 2a		–193 + 6a	
		144 – a		–253 + 3a		98 – 10 a
13	144		–31 + a		–95 – 4 a	
		113		+ 42 – a		
15	257		11			
		124				
17	381					

Example: 2

Find the missing terms in the following table:

x	:	10	15	20	25	30	35	40
y	:	270	–	222	200	–	164	148

Solution:

Since 5 values of y are given, we assume that the fifth differences are zeros. Hence $\Delta^5 y_k = 0$.

$$(i.e)\ (E-1)^5 y_K = 0 \Rightarrow (E^5 - 5E^4 + 5C_2 E^3 - 5C_3 E^2 + 5C_2 E + 1)\, y_K = 0.$$

$$(i.e.)\ y_K - 5E^4 y_K + 10E^3 y_K - 10E^2 y_K + 5E y_K - y_K = 0$$

$$(i.e)\ y_{K+5} - 5y_{K+4} + 10y_{K+3} - 10y_{K+2} + 5y_{K+1} - y_K = 0$$

K = 1 and 2,

We get $y_6 - 5y_5 + 10y_4 - 10y_3 + 5y_2 - y_1 = 0$

$$y_7 - 5y_6 + 10y_5 - 10y_4 + 5y_3 - y_2 = 0$$

Substituting the values.

$y_1 = 270,\ y_3 = 222,\ y_4 = 200,\ y_6 = 164,\ y_7 = 145$

In these equations, we get

$$164 - 5y_5 + 2000 - 2220 + 5y_2 - 270 = 0$$

$$148 - 820 + 10y_5 - 2000 + 1100 - y_2 = 0$$

(i.e) $5y_5 - 5y_2 = -326$

$$10y_5 - y_2 = 1562$$

Solving these equations,

(1) $\times$ 2 $\Rightarrow 10y_5 - 10y_2 = -652$

(2) $\times$ 1 $\Rightarrow 10y_5 - y_2 = 1562$

(–) (+) (–)

$$-9y_2 = -2214$$

$$y_2 = \frac{2214}{9} = 246$$

$$y_2 = 246$$

$$10\ y_5 - 246 = 1562$$

$$10\ y_5 = 1562 + 246$$

$$y_5 = \frac{1808}{10} = 180.8$$

Hence the missing terms are 246 and 180.8

Example: 3

From the following table, find the missing value.

x :	2	3	4	5	6
f(x):	45.0	49.2	54.1	–	67.4

Solution:

Since only four values of f (x) are given we assume that the polynomial which fits the data, (i.e) collection polynomial is of degree three.

Hence fourth polynomial differences are zeros.

(i.e) $\Delta^4 y_0 = 0$

$$\therefore\ (E - 1)^4\ y_0 = 0$$

(i.e) $(E^4 - 4E^3 + 6E^2 - 4E + 1)\ y_0 = 0$

$E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4E y_0 + y_0 = 0$ where $y_0 = 45.0$

$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0,$

$67.4 - 4y_3 + 6\,(54.1) - 4\,(49.2) + 45.0 = 0$

$240.2 - 4y_3 = 0$

$$y_3 = \frac{240.2}{4}$$

$$\therefore y_3 = 60.05$$

Missing term is 60.05

Example:

Estimate the production for 1964 and 1966 from the following data.

Year:	1961	1962	1963	1964	1965	1966	1967
Production:	200	220	260	–	350	–	430

Solution:

Since five values are given, Collaction polynomial is of degree four.

Hence $\Delta^5 y_k = 0$

(ie) $(E-1)^5 y_k = 0$

(ie) $(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)\, y_k = 0$

$E^5 y_0 - 5E^4 y_0 + 10E^3 y_0 - 10E^3 y_0 + 5Ey_0 - y_0 = 0$ (Take k = 0)

$$y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$$

$$y_5 - 5(350) + 10y_3 - 10(260) + 5(220) - 200 = 0$$

$$y_5 + 10y_3 = 3450 \qquad (1)$$

Taking k = 1

$$y_6 - 5y_5 + 10y_4 - 10y_3 + 5y_2 - y_1 = 0$$

$$430 - 5y_5 + 10(350) - 10y_3 + 5(260) - 220 = 0$$

$$5y_5 + 10y_3 = 5010 \qquad (2)$$

Solving for y_3, y_5 from (1) and (2)

(1) x 5 $\Rightarrow$ $5y_5 + 50y_3 = 17250$

(2) x 1 $\Rightarrow$ $5y_5 + 10y_3 = 5010$

(–)

$$40y_3 = 12240$$

$$y_3 = \frac{12240}{40} = 306$$

Substituting in(1) $y_5 + 10(306) = 3450$

$$y_5 + 3060 = 3450$$

$$y_5 = 390$$

Hence missing values are 306 and 390.

Exercise:

1. Find the cubic polynomial from the data

x	0	1	2	3	4
y	– 5	– 10	– 9	4	35

2.

x	4	6	8	10	12
f(x)	– 43	15	185	515	1053

3. From the following data, find the missing term.

x	2	3	4	5	6
f(x)	45.0	49.2	54.1	–	67.4

4. From the following data, find the value of f(31)

x	30	32	33	34
f(x)	8.84	33.56	45.13	56.20

5. From the following data, find the missing term

x	2	3	4	5	6
f(x)	45.0	49.2	54.1	–	67.4

6. Estimate the production in the year 1966 from the following data:–

Year	1962	1964	1968	1970
Production	100	112	136	180

9. The following table gives the quantity of cement in thousands of tons manufactured in India in the year x. Find the probable production in the year 1970:–

year x	1966	1968	1970	1972	1974	1976
Quantity y	39	85	–	151	264	388

8. Estimate the production for 1974 and 1976 from the following data:

Year	1971	1972	1973	1974	1975	1976	1977
Production In 1000tons	200	220	260	–	350	–	430

9. If y_x is a polynomial of fifth degree in x & $y_1 + y_7 = -786$, $y_2 + y_6 = 686$, $y_3 + y_5 = 1088$ find y_4.

10. Find the cubic polynomial y(x) such that
$y(0) = -5$, $y(1) = 1$, $y(2) = 9$, $y(3) = 25$, $y(4) = 55$, $y(5) = 105$

The generality of the results will not be affected by taking the interval of difference in the independent variable as unity.

$$\Delta y_x = y_{x+h} - y_x$$

Suppose we change the independent variable x to t such that x = th, then

$$\Delta y_{tn} = y_{h(x+1)} - y_{xh}$$

ie, $\Delta y_t = y_{t+1} - y_t$, where $y_{th} = y_t$.

Hence we have successive difference whose common difference is unity.

In this case $\Delta y_x = y_{x+1} - y_n$ with this notation,
We get

i) $\Delta^n (x^n) = n!$

ii) $\Delta^n y_x = a_6\, n!$ where $y_x = a_0 x^n + \dots\dots + a_n$

iii) $(a + bx)^{|n|}$ is defined as

$(a + bx)(a + \overline{bx+1})(a + \overline{bx+2}) \dots\dots (a + \overline{bx+n-1})$ and $(a + bx)^{(n)}$ is defined as

$$(a + bx)(a + \overline{bx-1})(a + \overline{bx-2}) \dots\dots (a + \overline{bx-n+1})$$

Hence $x^{|n|} = x(x + 1)(x + 2) \dots\dots\dots (x + n - 1)$

$$x^{(n)} = x(x - 1)(x - 2) \dots\dots\dots (x - n + 1)$$

$$\therefore \Delta (a + bx)^{(n)} = (a + \overline{bx-1})(a + bx) \dots\dots\dots (a + \overline{bx-n+2})$$
$$- (a + bx)(a + \overline{bx-1}) \dots\dots (a + \overline{bx-n+1})$$
$$= (a + bx) \dots\dots (a + bx - n + 2) \times \{a + bx + a - a - bx - n + 1\}$$

$$= bn(a + bx)^{(n-1)}$$

Hence $\Delta\ x^{(n)} = nx^{(n-1)}$

$$\therefore\ \Delta^n x^{(n)} = n\,!$$

iv) $$\Delta \frac{1}{(a+bx)^{|n|}} = -\frac{b_n}{(a+bx)^{|n+1|}}$$

$$\therefore\ \Delta \frac{1}{x^{|n|}} = -\frac{n}{x^{|n+|}}$$

Note: The corresponding results in differential calculate are

$$D(a + bx)^n = n6(a + bx)^{n-1}$$

$$D\frac{1}{(a+bx)^n} = \frac{-nb}{(a+bx)^{n+1}}$$

Example: 1

Prove that $y_x = 2^x(A + Bx)$ where A and B are constants satisfy the equation $y_{x+2} - 4y_{x+1} + 4y_x = 0$

$$y_x = 2^x (A + Bx)$$

$$y_{x+1} = 2^{x+1} \{A + B(x+1)\}$$

$$= 2^x \{2A + 2B(x + 1)\}$$

$$y_{x+2} = 2^{x+2} \{A + B(x + 2)\}$$

$$= 2^x \{4A + 4B(x + 2)\}$$

$$\therefore\ y_{x+2} - 4y_{x+1} + 4y_x$$

$$= 2^x\{4A + 4B(x + 2)\} - 4.2^x \{2A + 2B(x + 1)\} + 4.2^x (A + Bx)$$

= 0 on simplifications.

7.6 INVERSE OPERATORS

If $\Delta y_x = u_x$ Then $y_x = \Delta^{-1} u_x$ Here Δ^{-1} is called finite integration operator or inverse of operator Δ.

If C(x) is a periodic function of period h which is equal to the interval of differencing,

$$\Delta C(x) = C(x + h) - C(x) \text{ by definition of } \Delta$$
$$= C(x) - C(x). \qquad \because C(x) \text{ is periodic}$$
$$= 0$$

This shows that, if C(x) is periodic function whose period and interval of differencing is same h, then $\Delta C(x) = 0$.

Hence if $\Delta y(x) = u(x)$

then $\Delta\ (y(x) + C(x)) = \Delta y(x) + \Delta C(x)$

$$= \Delta y(x) + 0 = u(x)$$

$$\therefore\ \Delta^{-1} u(x) = y(x) + C(x)$$

Where c(x) is the periodic function of period h (similar to constant of integration in integration).

The following inverse operator results can be remembered from the corresponding forward operator results.

1. $\Delta^{-1} (e^{ax+h}) = \dfrac{e^{ax+b}}{e^{ah-1}}$

   Hence, $\Delta^{-1} e^x = \dfrac{e^x}{e^h - 1}$

2. $\Delta^{-1} (a^x) = \dfrac{a^x}{a^h - 1}$, $a \neq 1$

3. $\Delta^{-1} (u_x + v_x) = \Delta^{-1} \mu_x + \Delta^{-1} v_x$
4. $\Delta^{-1} (cu_x) = c\Delta^{-1} u_x$

5. $\Delta^{-1}(a+bx)^{(n)} = \dfrac{(a+bx)^{(n+1)}}{(n+1)hb}$, $n \neq -1$

6. $\Delta^{-1}x^{(n)} = \dfrac{x^{(n+1)}}{n+1}$, $n \neq -1$ and $h = 1$.

Summation of series

An important application of finite calculus is finding the sum of series. Let us find the sum of the series

$$u_1 + u_2 + u_3 + \ldots\ldots + u_n$$

Let the x^{th} term u_x be such that $u_x = \Delta y_x$

$$\therefore\ u_x = \Delta y_x = y_{x+1} - y_x \text{ (here } h = 1)$$

Hence $u_1 = y_2 - y_1$

$u_2 = y_3 - y_2$

$u_3 = y_4 - y_3$

....

....

$u_n = y_{n+1} - y_n$

Adding vertically,

$$S_n = u_1 + u_2 + \ldots\ldots\ \mu_n = y_{n+1} - y_1 = (y_x)_1^{n+1} = \left[\Delta^{-1}(u_x)\right]_1^{n+1}$$

$$\text{Hence } \sum_{x=1}^{n} u_x = \left[\Delta^{-1}u_x\right]_1^{n+1}$$

Example: 1

Find $\Delta^{-1}x(x+1)(x+2)$

Solution:

Here $(x+2)(x+1)x = (x+2)^{(3)}$, if $h = 1$

Hence, $\Delta^{-1}(x+2)(x+1)x = \Delta^{-1}(x+2)^{(3)}$

$$= \frac{(x+2)^{(4)}}{4} + c(x)$$

$$= \frac{(x+2)(x+1)(x)}{4} + c(x)$$

Where c(x) is a periodic function of period 1.

Example: 2

Find $\Delta^{-1} \frac{1}{x(x+1)(x+2)}$

Solution:

$$\frac{1}{x(x+1)(x+2)} = (x-1)^{(-3)}$$

$$\Delta^{-1} \frac{1}{x(x+1)(x+2)} = \Delta^{-1} \left[(x-1)^{(-3)}\right]$$

$$= \frac{(x-1)^{(-2)}}{(-2)} + c(x)$$

$$= -\frac{1}{2} \frac{1}{x(x+1)}$$

Example: 3

Sum the series to n terms of

1.2.3 + 2.3.4 + 3.4.5 + ...

Solution:

n^{th} term = μ_n = n(n+1) (n+2)

Sum of series to n terms = $\sum_{x=1}^{n} u_x$

$$= \left[\Delta^{-1} u_x\right]_1^{n+1}$$

$$= \left[\Delta^{-1} (x+2)^{(3)}\right]_1^{n+1}$$

$$= \left[\frac{(x+2)^{(4)}}{4} \right]_1^{n+1}$$

$$= \frac{1}{4} \left[(n+3)^{(4)} - 3^{(4)} \right]$$

$$= \frac{1}{4} \left[(n+3)(n+2)(n+1)n - 3.2.1.0 \right]$$

$$= \frac{1}{4} \left[(n+3)(n+2)(n+1)n \right]$$

Example: 4

Sum to n terms of the series

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \text{.........}$$

Solution:

$$u_x = \frac{1}{x(x+1)(x+2)} = (x-1)^{(-3)}$$

$$\text{Sum to n terms} = \sum_{x=1}^{n} u_x$$

$$= \left(\Delta^{-1} u_x \right)_1^{n+1}$$

$$= \left[\Delta^{-1} . (x-1)^{(-3)} \right]_1^{n+1}$$

$$= \left[\frac{(x-1)^{(-2)}}{-2} \right]_1^{n+1}$$

$$= -\frac{1}{2} \left[n^{(-2)} - 0^{(-2)} \right]$$

$$= -\frac{1}{2} \left[\frac{1}{(n+1)(n+2)} - \frac{1}{1.2} \right]$$

$$= \frac{1}{2} \left[\frac{1}{2} - \frac{1}{(n+1)(n+2)} \right]$$

Example: 5

Find Δ^{-1} {x(x+1)}

$$X(x+1) = (x+1)^{(2)}$$

We have Δ^{-1} $(a + bx)^{(a)} = \dfrac{(a+bx)^{n+1}}{b\,(n+1)}$

$$\therefore\ \Delta^{-1}\,(x + 1)^{(2)} = \frac{(x+1)^3}{3}$$

$$= \frac{(x+1)\,x\,(x-1)}{3}$$

Example: 6

Sum the series

$$2.3 + 3.4 + 4.5 + \ldots. + (n+1)\,(n+2)$$

This series = $\sum\limits_{x=1}^{n} (x+1)\,(x+2)$

$$\therefore\ u_x = (x + 1)\,(x + 2)$$

Hence $\sum\limits_{x=1}^{n} u_x = \left[\Delta^{-1} u_X\right]_1^{n+1}$

$$= \left[\Delta^{-1}\{(x+1)\,(x+2)\}\right]_1^{n+1}$$

$$= \left[\frac{(x+2)^{(3)}}{3}\right]_1^{n+1}$$

$$= \left[\frac{(x+2)\,(x+1)\,x}{3}\right]_1^{n+1}$$

$$= \frac{(n+3)\,(n+2)\,(n+1)}{3} - \frac{3.2.1}{3}$$

$$= \frac{n\,(n^2 + 6n + 11)}{3}$$

Exercise:

1. Sum the series of n terms.

 i) $\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+......$ ii)3.537 + 5.7.9 +7.9.11 + ...

2. Find i) $\Delta^{-1}\ [(x+1)(x+2)(x+3)]$ ii) $\Delta^{-1}\left[\frac{1}{(2x-1)(2x+1)(2x+3)}\right]$

3. Find the sum to n terms of the series

 i) n(n – 1) (n – 2) ii) (n+1) (n+2) (n+3) (n+4) iii) n^4

7.7 FACTORIAL NOTATION

Factorial Polynomial

A factorial polynomial $x^{(n)}$ is defined as

$$x^{(n)} = x(x - h)(x - 2h) \ldots\ldots (x -(n - 1)h)$$

where n is a positive integer.

(Read $x^{(n)}$ = as x raised to the power n factorial. Thus $x^{(1)} = x$, $x^{(2)} = x(x - h)$, $x^{(3)} = x(x - h)(x - 2h)$ .... etc.

Differences of $x^{(n)}$.

i) $\Delta x^{(n)} = (x + h)^{(n)} - x^{(n)}$

$$= (x + h)(x)(x - h) \ldots\ldots [x - (n -2) h] - x(x - h)(x - 2) \ldots\ldots [x - (n - 1)h]$$

$$= x(x - h)(x - 2h) \ldots\ldots [x - (n - 2) h] \{(x+h) - (x - (n - 1) h\}$$

$$= x^{(n-1)}.nh$$

$$= nhx^{(n-1)}$$

Similarly $\Delta^2 x^{(n)} = \Delta [nhx^{(n-1)}] = (nh)(n - 1) hx^{(n-2)} = n(n - 1)h^2 x^{n-2}$

Proceeding like this,

$$\Delta^r x^{(n)} = n(n - 1)(n - 2) \ldots\ldots (n - r + 1) h^r x^{(n-r)}$$

Where r is a positive integer and r < n

Note:

i) In particular $\Delta^n x^{(n)} = n!\, h^n$.

ii) If h = 1 (ie) the interval of differencing is units, then $\Delta^r x^{(n)} = n(n-1)(n-2)\ldots(n-r+1)\, x^{(n-r)}$ which is analogus to the differentiation of x^n

iii) If h = 1, $\Delta^n x^{(n)} = n!$ & $\Delta^r x^{(n)} = 0$ if r > n.

iv) Whenever we require $\Delta^r x^{(n)}$, it is difficult to find $\Delta^r x^{(n)}$ and hence we express x^n in terms of factorial polynomial and hence we calculate $\Delta^r x^{(n)}$.

Reciprocal factorial:

The reciprocal factorial function $x^{(-n)}$ is defined as

$$x^{(-n)} = \frac{1}{(x+h)(x+2h)\ldots(x+n)}$$ where n is a '+ve' integer.

Differences of a reciprocal factorial function

i) $\Delta x^{(-n)} = (x+h)^{(-n)} - x^{(-n)}$

$$= \frac{1}{(x+2h)(x+3h)\ldots[x+(n+1)h]} - \frac{1}{(x+h)(x+2h)\ldots(x+nh)}$$

$$= \frac{1}{(x+h)(x+2h)\ldots[x+(n+1)h]} = \left(-n)\, h\, x^{-(n+1)}\right)$$

ii) $\Delta^2 x^{(-n)} = \Delta(\Delta x^{-n})$

$= \Delta\left(-nh\, x^{-(n+1)}\right)$

$= (-nh)\,[-(n+1)h]\, x^{-(n+2)}$

$= (-1)^2 h^2 n(n+1)\, x^{-(n+2)}$

Similarly

$\Delta^r x^{-n} = (-1)^r n(n+1)(n+2)\ldots(n+r-1)\, x^{(n+r)} h^r$.

Polynomial in factorial notation:

Any polynomial $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n$ can be expressed in the factorial polynomial form as.

$A_0 x^{(n)} + A_1 x^{(n-1)} + A_2 x^{(n-2)} + \ldots + A_n$.

Since, $f(x) = A_0 x^{(n)} + A_1 x^{(n-1)} + \ldots + A_n$.

$$= A_0 x(x-h)\ldots(x-(n-1)h) + A_1 x(x-h)\ldots(x-(n-2h) + A_2 x(x-h)\ldots(x-(n-3)h) + \ldots + A_{n-1}x + A_n \quad (1)$$

Dividing the R.H.S of (1) by x, the remainder is A_n & dividing the quotient again by $x - h$, the remainder is A_{n-1} and then dividing the quotient again by $x - 2h$, the remainder is A_{n-2} etc.

Thus, dividing f(x) successively by x, $x - h$, $x - 2h$, ... The coefficients A_n, A_{n-1}, A_{n-2} , ... are got which are nothing but the remainders of f(x) in that order.

Note: If h = 1, divide f(x) successively by x, $x - 1$, $x - 2$, ... to get A_n, A_{n-1}......

Example:

Express $x^4 + 3x^3 - 5x^2 + 6x - 7$ in factorial polynomials and get their successive forward differences taking h = 1.

Solution:

First divide $x^4 + 3x^3 - 5x^2 + 6x - 7$ successively by x, $x - 1$, $x - 2$...... by synthetic division method.

0	1	3	– 5	6	– 7
	0	0	0	0	0
1	1	3	– 5	+6	– 7
	0	1	4	–1	
2	1	4	– 1	5	
	0	2	12		
3	1	6	11		
	0	3			
	1	9			

∴ Factorial polynomial is

$$f(x) = 1.x^{(4)} + 9x^{(3)} + 11x^{(2)} + 5x^{(1)} - 7$$

$$\Delta f(x) = 4x^{(3)} + 27x^{(2)} + 22x^{(1)} + 5$$

$$\Delta^2 f(x) = 12x^{(12)} + 54x^{(1)} + 22$$

$$\Delta^3 f(x) = 24x^{(1)} + 54$$

$$\Delta^4 f(x) = 24$$

$$\Delta^r f(x) = 0 \qquad \text{if } r>4$$

Aliter:

$y = x^4 + 3x^3 - 5x^2 + 6x - 7$ can be written as

$$y = x^4 + 3x^3 - 5x^2 + 6x - 7 = Ax(x-1)(x-2)(x-3) + Bx(x-1)(x-2) + Cx(x-1) + Dx + E.$$

$$= Ax^{(4)} + Bx^{(3)} + cx^{(2)} + Dx^{(1)} + E.$$

Put $x = 0$; $E = -7$

$x = 1$; $D + E = -2 \quad \therefore D = 5$

$x = 2$; $2C+2D+2E = 16 + 24 - 20 + 12 - 7; \therefore C = 11$

Put $x = 3$; $6B + 6C + 3D + E = 81 + 81 - 45 + 18 - 7$

$6B = 54; \qquad B = 9.$

Equate coefficient of x^3 on both sides;

$A = 1.$

$$\therefore y = x^{(4)} + 9x^{(3)} + 11x^{(2)} + 5x^{(1)} - 7$$

Example: 2

Express $3x^3 - 2x^2 + 7x - 6$. In factorial polynomials and get their successive forward differences taking h = !

Solution:

Now, express $3x^3 - 2x^2 + 7x - 6$ is factorial polynomial,
Using synthetic division process,

0	3	– 2	7	– 6
		0	0	0
1	3	– 2	7	– 6
		3	1	
2	3	1	8	
		6		
	3	7		

Hence $\phi(x) = 3x^{(3)} + 7x^{(2)} + 8x^{(1)} - 6$ (here h = 1)

$$\Delta\,\phi(x) = 9x^{(2)} + 14x^{(1)} + 8$$

$\Delta^2 \phi(x) = 18x^{(1)} + 14$

$\Delta^3 \phi(x) = 18$

$\Delta^r \phi(x) = 0$ for $r > 3$.

Example: 3

Express $x^3 + x^2 + x + 1$ is factorial polynomials and get their successive forward differences, taking $h = 1$.

Solution:

Now, express $\phi(x) = x^3 + x^2 + x + 1$

$$\begin{array}{c|cccc}
0 & 1 & 1 & 1 & 1 \\
 & & 0 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & \boxed{1} \\
 & & 1 & 2 & \\
\hline
2 & 1 & 2 & \boxed{3} & \\
 & & 3 & & \\
\hline
 & 1 & \boxed{4} & &
\end{array}$$

$\phi(x) = x^{(3)} + 4x^{(2)} + 3x^{(1)} + 1$

$\Delta \phi(x) = 3x^{(2)} + 8x^{(1)} + 3$

$\Delta^2 \phi(x) = 6x^{(1)} + 8$

$\Delta^3 \phi(x) = 6$

$\Delta^r \phi(x) = 0$ for $r > 3$.

Example: 4

Represent the function $f(x) = 2x^3 - 3x^2 + 4x - 8$ and its differences in the factorial notation.

Solution:

Let $2x^3 - 3x^2 + 4x - 8 = 2x(x - 1)(x - 2) + ax(x - 1) + bx + c$

put $x = 0$, then $c = -8$ $\quad [\because -8 = 0 + 0 + 0 + c]$

put $x = 1$, then $-5 = b - 8$

$\therefore b = 3$ [$\because 2 - 3 + 4 - 8 = 0 + 0 + b + c - 5 = b - 8$]

put $x = 2$, then $4 = 2a + 2b - 8$

$\therefore 4 = 2a + 6 - 8$ [$16 - 12 + 8 - 8 = 0 + 2a + 2b + c$

$4 = 2a + 2(3) - 8$]

$\therefore a = 3$

Hence $2x^3 - 3x^2 + 4x - 8 = 2(x - 1)(x - 2) + 3x(x - 1) + 3x$
$- 8 - 2x^{(3)} + 3x^{(2)} + 3x^{(1)} - 8$

$$\therefore \Delta f(x) = 2\Delta x^{(3)} + 3\Delta x^{(2)} + 3\Delta x^{(1)} - \Delta(8)$$

$$= 6x^{(2)} + 6x^{(1)} + 3(1)$$

$$= 6x(x - 1) + 6x + 3$$

$$= 6x^2 - 6x + 6x + 3$$

$$= 6x^3 + 3$$

$$\Delta^2 f(x) = 6\ \Delta(x^{(2)}) + 6\Delta\ (x^{(1)}) + \Delta(3)$$
$$= 12x^{(1)} + 6$$

$$\Delta^3 f(x) = 12$$

$$\Delta^4 f(x) = 0$$

Example: 5

Find the function whose first difference is $5x^2 - 6x + 7$.

Solution:

$$\Delta\ \{f(x)\} = 5x^2 - 6x + 7$$

$$= 5x(x - 1) + ax + b.$$

putting $x = 0$, $b = 7$ [$\because 5x^2 - 6x + 7 = 5x(x - 1) + ax + b$]
$7 = b$

putting $x = 1$, $6 = a + 7$ [$\because 5(1) - 6(1) + 7 = 0 + a + b$]
$6 = a + 7$

$\therefore\ \Delta\ \{ f(x) \} = 5x\ (x - 1) - x + 7$

$= 5x^{(2)} - x^{(1)} + 7$

$f(x) = 5\ \Delta^{-1}\{x^{-2}\} - \Delta^{-1}\{x^{(1)}\} + \Delta^{-1}\ [7]$

$= 5\frac{x^{(3)}}{3} - \frac{x^{(2)}}{2} + 7x^{(1)} + k$ (where k is an arbitrary constant).

$$= \frac{5x\,(x-1)\,(x-2)}{3} - \frac{x(x-1)}{2} + 7x + k$$

$$= \frac{5\,(x^3 - 3x^2 + 2x)}{3} - \frac{x^2 - x}{2} + 7x + k$$

$$= \frac{1}{6}\left[10x^3 - 30x^2 + 20x - 3x^2 + 3x + 42x + 6k\right]$$

$$= \frac{1}{6}\left[10x^3 - 33x^2 + 65x + 6k\right]$$

Exercise:

1. Obtain the function whose first differences in $8x^2 + 5$.
2. Represent the function.
   $f(x) = x^4 - 12x^3 + 24x^2 - 30x + 14$ & its successive differences in factorial notation.
3. Find second difference of $f(x) = 7x^4 + 12x^3 - 6x^2 + 5x - 3$, if $h = 2$.
4. Express the following functions in terms of factorial polynomials and find their differences:
   i) $3x^4 + 8x^3 + 3x^2 - 27x + 9$ ii) $2x^3 - 3x^2 + 3x + 10$.

UNIT – VIII
NUMERICAL DIFFERENTIATION

8.1 Introduction:

So far, we were finding the polynomial curve y = f(x) passing through the (n+1) ordered pairs (x_i, y_i) i = 0,1,....n now we are trying to find the derivative value of such curves at a given $x = x_k$ (say)whose $x_0 < x_k < x_n$ (or even outside the range but closer to starting or end values). To get the derivative, we first find the curve y = f(x) through the points and then differentiate and get its value at the required point.

If the values of x are equally spaced, we get the interpolating polynomial due to Newton's – Gregory. If the derivate is required at a point nearer to the starting value in the table.

We use Newton's forward interpolation formula. If we require the derivative at the end of the table, we use Newton's backward interpolation formula. If the value of derivative is required near the middle of the table value we use of the central difference interpolation formulae. In the case of unequal intervals, we can use Newton's divided difference formula or Lagrange's interpolation formula to get the derivative value.

8.2 First and Second Derivative:

Newton's forward difference formula to get the derivative:

We are given (n+1) ordered pairs (x_i; y_i;) i = 0 to n.

We went to find the derivative of y = f(x) passing through the (n+1)points, at a point nearer to the starting value $x = x_0$.

Newton's forward difference interpolation formula is

$$y(x_0 + x_h) = y_0 = y_0 + x\,\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 y_0 + \ldots\ldots..(1)$$

where y(x) is a polynomial of degree n is x & $X = \dfrac{x - x_0}{h}$

Differentiating y(x) with respect to x,

$$\frac{dy}{dx} = \Delta y_0 + \frac{2(x-1)}{2!}\Delta^2 y_0 + \frac{3x^2 - 6x + 2}{3!}\Delta^3 y_0 + \ldots\ldots.$$

but $\frac{dy}{dX} = \frac{dy}{dx} \cdot \frac{dx}{dX} = h \cdot \frac{dy}{dx}$ since $\frac{dX}{dx} = h.$

$$\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{2x-1}{2!}\Delta^2 y_0 + \frac{3x^2-6x+2}{3!}\Delta^3 y_0 + \ldots\ldots\right\} \qquad (2)$$

The series on the right side gives the value of $\frac{dy}{dx}$ at any x,

If we put X = 0, then $x = x_0$,

If we put X= 0, then $x = x_0$,

Hence in the series on the right if we put x = 0, it gives the values of $\frac{dy}{dx}$ at $x = x_0$.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left\{\Delta y_0 \mp \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\ldots\right\}$$

If we differentiate equation (2)

We get $\frac{d^2y}{dx^2} = \frac{1}{h}\left[\frac{2\Delta^2 y_0}{2!} + \frac{6x-6}{3!}\Delta^3 y_0 + \ldots\ldots\right] \frac{dX}{dx}.$

$$\left[\frac{d^2y}{dx^2} = \frac{d}{dX}\left(\frac{dy}{dx}\right)\cdot\frac{dX}{dx} = \frac{d}{dX}\left(\frac{dy}{dx}\right)\cdot\frac{1}{h}\right]$$

$$= \frac{1}{h^2}\left\{\Delta^2 y_0 + (x-1)\Delta^3 y_0 + \frac{6x^2-18x+11}{12}\Delta^4 y_0 + \ldots\ldots\right\}$$

Putting x = 0, $x = x_0$.

Then $\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 \ldots\ldots\right\}$

This gives the value of $\frac{d^2y}{dx^2}$ at $x = x_0$,

Hence, $\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{12x-18}{12}\Delta^4 y_0 + \ldots\ldots\right]$

Aliter:

We have shown that $1 + \Delta = e^{hD}$

$hD = \log_e(1+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 \ldots\ldots$

Applying this identity to y_0,

We get

$$D(y_0) = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3.....\right]y_5$$

$$\frac{dy}{dx} \text{ at } x_0 = \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0....\right)$$

Again $\frac{d^2y}{dx^2}$ at $x_0 = D^2y_0 = \frac{1}{h^2}[\log(1+\Delta]^2 y_0$

$$= \frac{1}{h^2}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3....\right]^2 y_0$$

$$= \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4....\right]y_6$$

$$\frac{d^2y}{dx^2} \text{ at } x_0 = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0....\right]$$

Newton's Backward Difference Formula to compute the derivative.

Now, Consider Newton's Backward difference interpolation formula:

$$y(x) = y(x_n+vh) = y_n + X\nabla y_n + \frac{X(X+1)}{2!}\nabla^2 y_n + \frac{X(X+1)(X+2)}{3!}\nabla^3 y_n + ...+ \frac{X(X+1)(X+2)...(X+n-1)}{n!}\nabla^n y_n \text{ .... (1)}$$

where

$$-X = \frac{x_n - x}{h}$$

Differentiate with respect to x

$$\left[\frac{dy}{dx} = \frac{dy}{dx}.\frac{dx}{dx} \Rightarrow \frac{dy}{dx}.h\right] \because \frac{x_n - x}{h} = -X;\ \frac{1}{h} = \frac{dX}{dx}$$

$$\frac{dy}{dX} = \nabla y_n + \frac{2x+1}{2!}\nabla^2 y_n + \frac{3x^2+6x+2}{3!}\nabla^3 y_n + ...$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2x+1}{2!}\nabla^2 y_n + \frac{3x^2+6x+2}{3!}\nabla^3 y_n + ...\right] \quad (2)$$

$\therefore$ Putting x = 0, (ie) when x= x_n,

$$\frac{dy}{dx} \text{ at } x^n = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n ....\right)$$

If we differentiating equation (2) once again with respect to x.

We get

$$\frac{d^2y}{dx^2} = \frac{1}{h}\left[\nabla^2 y_n + \frac{6X+6}{3!}\nabla^3 y + \frac{12X^2+36X+22}{24}\nabla^4 y_n + ...\right] \frac{dx}{dX}$$

which on simplication gives,

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (X+1)\nabla^3 y_n + \frac{6X^2+18X+11}{12}\nabla^4 y_n + ...\right]$$

Putting X = 0,when x = x_n,

$$\frac{d^2y}{dx^2} \text{ at } x_n = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + ...\right]$$

$$\frac{d^3y}{dx^3} \text{ at } x_n = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + ...\right]$$

Aliter:

We can easily show that $\nabla = 1 - E^{-1}$

$$\therefore \frac{1}{E} = 1 - \nabla$$

$$\therefore e^{hD} = E = (1 - \nabla)^{-1}$$

Hence $hD = -\log_e(1 - \nabla)$.

Applying this identity to y_n,

We get

$$D(y_n) = -\frac{1}{h}\log(1 - \nabla)y_n$$

$$= \frac{1}{h}\left(\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 ......\right) y_n$$

$$\frac{dy}{dx} \text{ at } x_n = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n......\right]$$

Again

$$\frac{d^2y}{dx^2} \text{ at } x_n = D^2 y_n$$

$$= \frac{1}{h^2}[\log(1-\nabla)]^2 \; y_n$$

$$= \frac{1}{h^2}\left(\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3......\right)^2 y_n$$

$$= \frac{1}{h^2}\left(\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4......\right) y_n$$

$$= \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + ...\right)$$

Example: 1

Find the first and second derivative of $\sqrt{x}$ at x = 15 and x = 23 from the table.

x :	15	17	19	21	23	25
(x)= $\sqrt{x}$:	3.873	4.123	4.359	4.583	4.796	5.000

Solution:

The forward differences are tabulated below:

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
15	3.873			
		0.250		
17	4.123		−0.014	
		0.236		0.002
19	4.359		−0.012	
		0.224		0.001
21	4.583		−0.011	
		0.213		0.002
23	4.796		−0.009	
		0.204		
25	5.000			

$$X = \frac{x - x_0}{h} = \frac{15-15}{2} = +0$$

$$\therefore\ f'(x) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 \ldots\ldots\right]$$

$$= \frac{1}{2}\left[0.250 - \frac{1}{2}(-0.014) + \frac{1}{3}(0.002)\right]$$

$$= \frac{1}{2}\left[0.250 + .007 + .0006667\right] = \frac{0.2576667}{2}$$

$$f'(x) = 0.1289$$

$$f''(x) = \frac{1}{h^2}\left[\Delta^2 y_0 + (x-1)\Delta^3 y_0 + \frac{6x^2 - 18x + 11}{12}\Delta^4 y_0 + \ldots\right]$$

$$f''(x) = \frac{1}{2^2}\,[\,[-0.014 + (0-1)(0.002)]$$

$$= \frac{1}{4}\left[-0.014 - 0.002\right] = \frac{-0.016}{4} = -0.004$$

$$f''(x) = -0.004$$

The Back Differences table in the same table with a different notation.

$$X = \frac{x_n - x}{h}$$

In this case $-X = \dfrac{25-23}{2} = \dfrac{2}{2} = 1$

$$\therefore\ X = -1\ \&\ h = 2$$

Hence $f'(x)$ at $x = 23$,

We have to substitute $X = -1$. In the formula

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2X+1}{2!}\nabla^2 y_n + \frac{3X^2 + 6X + 2}{3!}\nabla^3 y_n + \ldots\ldots\right]$$

$$\therefore\ \frac{dy}{dx} \text{ at } X = -1 = \frac{1}{2}\left[\nabla y_n - \frac{1}{2}\nabla^2 y_n - \frac{1}{6}\nabla^3 y_n\right]$$

$$=\frac{1}{2}\left[0.204-\frac{1}{2}(-0.009)-\frac{1}{6}(0.002)\right]$$

$$=\frac{1}{2}\left[0.204+.0045-.000333\right]$$

$$\frac{dy}{dx} \text{ at } X=-1=\frac{0.208167}{2}=0.1041$$

$$\frac{d^2y}{d^2x} \text{ at } x=23, \text{ (ie) } -X=\frac{25-23}{2}=1 \Rightarrow X=-1$$

$$\frac{d^2y}{d^2x} \text{ at } x=23=\frac{1}{h}\left[\nabla^2 y_n+(X+1)\nabla^3 y_n+\frac{6X^2+18X+11}{12}\nabla^4 y_n+\ldots\ldots\right]$$

$$\frac{d^2y}{d^2x} \text{ at } x=23=\frac{1}{4}\left[\nabla^2 y_n\ldots\right]$$

$$=\frac{1}{4}[-0.009]=-0.0023.$$

The correct values to four decimal places are

$$f'(15)=\frac{1}{2\sqrt{15}}=0.1291$$

$$f''(15)=\frac{-1}{4(15)^{3/2}}=-0.0043$$

$$f'(23)=\frac{1}{2\sqrt{23}}=0.1042$$

Example:2

Find the first two derivatives of $(x)^{1/3}$ at x = 50 and x = 56 given the table below:

x :	50	51	52	53	54	55	56
$y=x^{1/3}$:	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

Solution:

Since we require $f'(x)$ at $x = 50$.

We use Newton's forward formula and to get $f'(x)$ at $x= 56$, we use Newton's Backward formula.

Difference Table

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
50	3.6840			
		0.0244		
51	3.7084		–0.0003	
		0.0241		0
52	3.7325		–0.0003	
		0.0238		0
53	3.7563		–0.0003	
		0.0235		0
54	3.7798		–0.0003	
		0.0232		0
55	3.8030		–0.0003	
		0.0229		
56	3.8259			

By Newton;s forward Formula.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{dy}{dx}\right)_{x=0}$$

$$= \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 \ldots\ldots\right]$$

$$= \frac{1}{1}\left[0.0244 - \frac{1}{2}(-0.0003) + \frac{1}{3}(0)\right]$$

$\frac{dy}{dx}$ at $x = x_0 = 0.024\ 55$

$$\left(\frac{d^2y}{dx^2}\right)_{x=50} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \ldots\right]$$

$$= \frac{1}{1}[-0.0003] = -0.0003.$$

By Newton's backward difference formula

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \left(\frac{dy}{dx}\right)_{x=0} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \ldots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=56} = \frac{1}{1}\left[0.0229 + \frac{1}{2}(-0.0003) + 0\right] = 0.02275$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=56} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + ...\right]$$

$$= \frac{1}{1}[-0.0003] = -0.0003$$

Example:3

The population of a certain town is given below. Find the rate of growth of the population in 1931, 1941, 1961 and 1971.

Year x:	1931	1941	1951	1961	1971
Population in thousands y:	40.62	60.80	79.95	103.56	132.65

Solution:

We from the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1931	40.62				
		20.18			
1941	60.80		−1.03		
		19.15		5.49	
1951	79.95		+4.46		−4.47
		23.61		1.02	
1961	103.56		5.48		
		29.09			
1971	132.65				

We use the same table for backward and Forward differences

i) To get $f'(1931)$and $f'(1941)$

We use forward formula,

$x_0 = 1931$, $x_1 = 1941$.....

$X = \frac{x - x_0}{h}$, $x_0 = 1931$corresponds x = 0.

$$\left(\frac{dy}{dx}\right)_{x=1931} = \left(\frac{dy}{dx}\right)_{x=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + ...\right]$$

$$= \frac{1}{10}\left[20.18 - \frac{1}{2}(-1.03) + \frac{1}{3}(5.49) - \frac{1}{4}(-4.47)\right]$$

$$= \frac{1}{10}[20.18 + 0.515 + 1.83 + 1.1175]$$

$$= 2.36425. \qquad (1)$$

ii) If x = 1941, $X = \frac{x - x_0}{h} = \frac{1941 - 1931}{h} = 1$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2x-1}{2}\Delta^2 y_0 + \frac{3x^2 - 6x + 2}{6}\Delta^3 y_0 + \frac{4x^3 - 18x^2 + 22x - 6}{24}\Delta^4 y_0 .....\right]$$

We get

$$\left(\frac{dy}{dx}\right)_{x=1} = \frac{1}{10}\left[20.18 + \frac{1}{2}(-1.03) - \frac{1}{6}(5.49) + \frac{1}{12}(-4.47)\right]$$

$$= \frac{1}{10}[20.18 - 0.515 - 0.915 - 0.3725]$$

$$= 1.83775 \qquad (2)$$

Note: If we neglect the data against and take 1941 as x_0, we have $\Delta y_0 = 19.51$, $\Delta^2 y_0 = 4.46$, $\Delta^3 y_0 = 1.02$.

Now using,

$$\left(\frac{dy}{dx}\right)_{x=1941} = \frac{1}{4}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 .....\right]$$

$$= \frac{1}{10}\left[19.15 - \frac{1}{2}(4.46) + \frac{1}{3}(1.02)\right]$$

$$= 1.7260 \qquad (3)$$

Evidently the values given by (2) and (3) are not same. In getting the answer given by (2), we have assumed a polynomial of degree 4 whereas in getting the answer given by (3), we have assumed the interpolating polynomials assumed are different. Hence we see the difference in answers,

iii) To get f′(1971) use the formula,

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n - \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + ...\right]$$

$$= \frac{1}{10}\left[29.09 + \frac{1}{2}(5.48) + \frac{1}{3}(1.02) + \frac{1}{4}(-4.47)\right]$$

$$\left(\frac{dy}{dx}\right)_{1971} = \frac{1}{10}[31.0525] = 3.10525$$

iv) To get f′(1961), we use $X = \frac{x - x_n}{h} = \frac{1961-1971}{10} = -1$

$$\left(\frac{dy}{dx}\right)_{x=1961} = \left(\frac{dy}{dx}\right)_{X=-1} = \frac{1}{h}\left[\nabla y_n + \frac{2X+1}{2}\nabla^2 y_n + \frac{3X^2+6X+2}{6}\nabla^3 y_n + ...\right] X = -1$$

$$= \frac{1}{10}\left[29.09 - \frac{1}{2}(5.48) - \frac{1}{6}(1.02) - \frac{1}{12}(-4.471)\right]$$

$$= \frac{1}{10}[29.09 - 2.74 - 0.17 + 0.3725] = 2.65525.76$$

Example: 4

Find the first and second derivative of the function tabulated below at x = 0.6.

x :	0.4	0.5	0.6	0.7	0.8
y :	1.5836	1.7974	2.0442	2.3275	2.6511

Solution:

Since x = 0.6 is in the middle of the table, we will use stirling's formula

Difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.4	1.5836				
		0.2138			
0.5	1.7974		0.0330		
		0.2468 Δy_{-1}		0.0035	
0.6	2.0442 y_0		0.0365 $\Delta^2 y_{-1}$	$\Delta^3 y_{-2}$	0.0003 $\Delta^4 y_{-2}$
		0.2833 Δy_0		0.0038 $\Delta^3 y_{-1}$	
0.7	2.3275		0.0403 $\Delta^2 y_0$		
		0.3236			
0.8	2.6511				

By stirling's, formula

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + ...\right]$$

$$= \frac{1}{0.1}\left[\frac{1}{2}[0.2833 - 0.2468] - \frac{1}{2}(0.0038 + 0.0035)\right]$$

$$= 10[0.26505 - 0.0006083]$$

$$= 2.64442$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + ...\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_{0.6}} = \frac{1}{(0.01)}\left[0.0365 - \frac{1}{12}(0.0003)\right]$$

$$= 3.6475$$

Exercise:

1. Obtain the derivative at x = 7 of the function tabulated below:

x :	5	6	7	8	9	10
f(x) :	196	394	686	1090	1624	2306

2. Determine at x = 2.5 from the following data1

x :	2.3	2.5	2.7	2.9	3.1	3.3
y :	3617	3979	4317	4633	4929	5206

5. The following table gives corresponding values of pressure and specific value of superheated steam:-

v :	2	4	6	8	10
p :	105	42.7	25.3	16.7	13

4. The specific heat of silica glass at various temperature are as follows:

C° :	100	200	300	400	500
Specific heat in calories per degree centigrade per gram :	0.2372	0.2416	0.2460	0.2504	0.2545

Find the rate of change of specific heat with respect to temperature at 100°c.

5. The amount A of a substance remaining in a reaching system after an interval of time t in a certain chemical experiment is given below.

t :	2	5	8	11	14
A :	94.8	87.9	81.3	75.1	68.7

Find $\frac{dA}{dt}$ when t = 8.

8.3 MAXIMUM AND MINIMUM VALUE OF A FUNCTION FOR THE GIVEN DATA:

Given the ordered pairs (x_i, y_i) i = 0, 1,2,......n, we can get the interpolating polynomial of degree n, Now we went to find the value of x at which the curve is maximum or minimum.

Now, using Newton's forward interpolation formula and getting its derivative and equating it to zero, we get an equation from which the extremum values of y can be got.

From equation $\left(\frac{dy}{dx}\right)$,

We get

$$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left[\Delta y_0 + \frac{2X-1}{2}\Delta^2 y_0 + \frac{3X^2-6X+2}{6}\Delta^3 y_0 + \ldots\ldots\right]$$

$$\left(\frac{dy}{dx}\right) = 0 \Rightarrow \Delta y_0 + \frac{2X-1}{2}\Delta^2 y_0 + \frac{3X^2-6X+2}{6}\Delta^3 y_0 + \ldots\ldots = 0 \qquad \text{(A)}$$

If higher differences are small, we can take only the first three terms of (A) and solving it for x,5 (since it is a quadratic in x). We get x.

Using $x = x_0 + Xh$,
We can get the values x at which y is an extremum.

Note:

If the interval of differencing is not constant (ie) x's are not equally spaced)

We get Newton's divided difference formula or Lagranges interpolation formula for general x, and then differenting it w.r. to x. we can get the differenting at any x in the range.

Setting the particular value for x, say x_k we get the derivating value at x_k.

Example:

Find approximately the minimum value of f(x) from the following table:–

x	:	0	1	2	3	4	5	6	7	8	9
f(x)	:	890	844	769	668	541	389	401	462	495	530

Solution:

The minimum value appears to be in the neighbourhood of x = 5

∴ Tabulate the differences of f(x) in the neighbourhood of x = 5.

x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
5	389				
		12			
6	401		49		
		61		–77	
7	462		–28		107
		33		30	
8	495		2		
		35			
9	530				

For the values of x between x = 5 and x = 6, we have

$$\frac{df(x)}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2X-1}{2!}\Delta^2 y_0 + \frac{3X^2-6X+2}{6}\Delta^3 y_0 + ...\right]$$

In this case h = 1, $\Delta y_0 = 12$, $\Delta^2 y_0 = 49$, $\Delta^3 y_0 = -77$

$$\therefore \frac{df(x)}{dx} = 12 + \frac{(2X-1)}{2}(49) + \frac{3X^2-6X+2}{6}(-77)$$

$$= \frac{72 + 294X - 147\ 231X^2 + 462X - 154}{6}$$

$$= -\frac{(231X^2 - 756X + 229)}{6}$$

For a minimum $\frac{df}{dx} = 0$

Hence $231X^2 - 756X + 229 = 0$

Solving we get x = 3 or 0.3 approximately.

Since the function attains a minimum at a value between 5 and 6, x must be necessarily a fraction

Hence $X = 0.3$

$$\therefore\ x = x_0 + Xh$$

$$X = 5 + (0.3)\,(1) = 5.3$$

We find the value of f(x)
When x = 5.3 by Newton's interpolation formula

$$f(5.3) = 389 + (0.3)\,(12) + \frac{(0.3)\,(0.3-1)}{2\,!}(49) + \frac{(0.3)\,(0.3-1)\,(0.3-2)}{3\,!}(-7.7)$$

$$= 389 + 3{\cdot}6 - 5{\cdot}145 - {\cdot}45815$$

$$f(5.3) = 386.99685$$

Example:
Given the following data, find y'(6) and the maximum value of y.

x	:	0	2	3	4	7	9
y	:	4	26	58	112	466	922

Solution:
Since the arguments are not equally spaced, we will use Newton's divided difference formula (or even Lagrange's formula)

Divided Difference Table

x	y = f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	4				
		11			
2	26		7		
		32		1	
3	58		11		0
		54		1	
4	112		16		0
		118		1	
7	466		22		
		228			
9	922				

By Newton's Divided difference formula

$$y = f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0)\,(x - x_1)\, f(x_0, x_1, x_2) + \ldots\ldots$$

$$= 4 + (x - 0)\,(11) + (x - 0)\,(x - 2)\,(7) + (x - 0)\,(x - 2)\,(x - 3).$$

$= 4 + 11x + 7x^2 - 14x + x^3 - 5x^2 + 6x$

$= x^3 + 2x^2 + 3x + 4$

$\therefore\ y'(x) = 3x^2 + 4x + 3$

$y'(6) = 3(6)^2 + 4(6) + 3 = 135$

y(x) is maximum if $y'(x) = 0$

$\therefore\ 3x^2 + 4x + 3 = 0$

But the roots are imaginary

$\therefore$ There is no extremum value in the range. In fact, it is an increasing curve.

Example:

From the following table, find the value of x for which f(x) is a maximum. Also find the maximum value of f(x) from the table of values given below.

x	:	60	75	90	105	120
f(x)	:	28.2	38.2	43.2	40.9	37.7

Solution:

The maximum value appears to be in the neighbourhood of $x = 90^\circ$

Hence, we will use Stirling's formula. h = 15 (x's are equally spaced)

x	y = f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
60	28.2				
		10 (Δy_{-1})			
75	38.2		-5 ($\Delta^2 y_{-2}$)		
		5 (Δy_0)		-2.3 ($\Delta^3 y_{-1}$)	
90	43.2 (y_0)		-7.3 ($\Delta^2 y_{-1}$)		8.7
		-2.3		$+0.4$ ($\Delta^3 y_{-2}$)	
105	40.9		-0.9 ($\Delta^2 y_0$)		
		-3.2			
120	37.7				

By Stirling's formula.

$$y(x) = y(x_0 + Xh) = y_0 + \frac{X}{2}\left(\Delta y_0 + \Delta y_{-1}\right) + \frac{X^2}{2}\Delta^2 y_{-1} +$$

$$\frac{X(X^2 - 1^2)}{12}\left(\Delta^3 y_{-1} + \Delta^3 y_{-2}\right) + \frac{X^2(X^2 - 1^2)}{24}\Delta^2 y_{-2} + \ldots\ldots$$

Here $x_0 = 90$, $y_0 = 43.2$, $\Delta y_0 = -2.3$, $\Delta^2 y_0 = -0.9$, $\Delta y_{-1} = 5$,
$\Delta^2 y_{-1} = 7.3$, $\Delta^2 y_{-1} = -5$

$$\therefore\ y = 43.2 + \frac{X}{2}(-2.3 + 5) + \frac{X^2}{2}(-7.3) + \frac{(X^3 - X)}{12}(-2.3 + 6.4)$$

$$= 43.2 + 1.35X - 3.65X^2 + 0.3417(X^3 - X)$$

$$= 0.3417x^3 - 3.65x^2 + 1.0083x + 43.2$$

If y is maximum, $\frac{dy}{dx} = 0$

$$\therefore\ 3 \times 0.3417X^2 - 2 \times 3.65X + 1.0083 = 0$$

$$1.0251X^2 - 7.30X + 1.0083 = 0$$

$$X = \frac{7.30 \pm \sqrt{(7.30)^2 - 4(1.0251)(1.0083)}}{2 \times 1.0251}$$

$$= \frac{7.3 \pm 7.0111}{2.0502} = 6.9803 \text{ or or } 0.1409$$

X = 6.9803 goes beyond the range.

$\therefore$ Take x = 0.1409

$$x = x_0 + Xh = 90 + 15\,(0.1409) = 92.1135$$

Maximum $y = 0.3417\,(-0\,.1409)^3 - 3.65\,(0.1409)^2 + 1.0083\,(0.1409) + 43.2$

$= 43.27$

f(x) is maximum at x = 92.1135 & the maximum value is 43.27

Example:

The following table gives the results of an observation; θ is the observed temperature in degrees centigrade of a vessel of cooling water, t is the time in minutes from the beginning of observation: –

t	:	1	3	5	7	9
θ	:	85.3	74.5	67.0	60.5	54.3

Find the approximate rate of cooling when t = 3 & t = 3.5

Solution:

The rate of cooling is given by the expression $\frac{d\theta}{dt}$. Hence we have to find its value t = 3 and t = 3.5

$$\frac{d\theta}{dt} = \frac{1}{h}\left[\Delta\theta - \frac{1}{2}\Delta^2\theta + \frac{1}{3}\Delta^3\theta + \ldots\ldots\right]$$

The table of differences is calculated

t	θ	Δθ	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$
1	85.3				
		−10.8			
3	74.5		3.3		
		− 7.5		−2.3	
5	67.0		1.0		1.6
		− 6.5		−0.7	
7	60.5		0.3		
		− 6.2			
9	54.3				

At t = 3, the value of $\frac{d\theta}{dt}$ is given by substituting the value of $\Delta\theta$, $\Delta^2\theta$, $\Delta^3\theta$ starting at t = 3

∴ At t = 3,

$$\frac{d\theta}{dt} = \frac{1}{2}\left[-7.5 - \frac{1}{2}(1.0) + \frac{1}{3}(-0.7)\right]$$

$$= -4.12$$

Using the equation (2) t = 3.5

We nave $\frac{d\theta}{dt} = \frac{1}{h}\left[\Delta\theta + \frac{2x-1}{2!}\Delta^2\theta + \frac{3x^2-6x+2}{6}\Delta^3\theta\ \ldots\ldots\right]$

Where $X = \frac{x-3}{h}$ & h = 2

We have to find the value of $\frac{d\theta}{dt}$ at t = 3.5

$$\therefore X = \frac{0.5}{2} = 0.25$$

At t = 3.5

$$\frac{d\theta}{dt} = \frac{1}{2}\left[-7.5 + \frac{2(0.25)-1}{2}(1.0) + \frac{3(0.25)^2 - 6(0.25) + 2}{6}(-0.7)\right]$$

$$= \frac{1}{2}\left[-7.5 - 0.25 - 0.080208\right]$$

$$\frac{d\theta}{dt} = \frac{-7.83028}{2} = -3.915104$$

Exercise:

1. The following table gives the values of x and f(x). Find the maximum value of f(x).

x :	9	10	11	12	13	14	15
f(x) :	1330	1340	1320	1250	1120	930	725

2. Find approximately the minimum value of f(x) from the following table

x :	0	1	2	3	4	5	6	7	8	9
f(x) :	890	844	769	668	541	389	401	462	495	530

3. Find the minimum value of the polynomial f(x) which has the values

x :	0	2	4	6
f(x) :	3	3	11	27

4. Find maximum and minimum values of y from the table

x :	0	1	2	3	4	5
y :	0	0.25	0	2.25	16	56.25

5. Find the maximum value of f(x) given the table

x :	1.2	1.3	1.4	1.5	1.6
f(x) :	0.9320	0.9636	0.9855	0.9975	0.9996

Unit – IX
NUMERICAL INTEGRATION

9.1 Introduction:

We know that $\int_a^b f(x)\,dx$ represents the area between $y = f(x)$, x – axis and the ordinates $x = a$ and $x = b$.

This integration is possible only if the f(x) is explicitly given and if it is integrable. The problem of numerical integration can be stated as follows:

Given a set of (n+1) paired values (x_i, y_i) I = 0,1,2,.....n of the function $y = f(x)$ is not known explicity, it is required to compute $\int_{x_0}^{x_n} y\,dx$.

As we did in the case of interpolation or numerical differentiation, we replace f(x) by an interpolating polynomial $p_n(x)$ and obtain $\int_{x_0}^{x_n} p_n(x)\,dx$ which is approximately taken as the value for $\int_{x_0}^{x_n} f(x)\,dx$.

9.2 Newton's Cote's Formula:
A General Quadrature Formula:

Suppose we have to evaluate the define integral $\int_a^b f(x)\,dx$. Let the range (a,b) be divided into n equal parts (say) at $x_1, x_2 \ldots\ldots x_{n-1}$ and 'a' be x_0 and 'b' be x_n.

Let the values of f(x) at $x_0, x_1, x_2 \ldots\ldots x_n$ be $y_0, y_1, y_2 \ldots\ldots y_n$.

Suppose a curve passes through these points (x_0, y_0), (x_1, y_1), (x_2, y_2) .... (x_n, y_n), then

$\int_a^b f(x)\,dx$ = area between the curve $y = f(x)$, the x – axis and the ordinates at $x = x_0$ & $x = x_n$.

$= \int_{x_0}^{x_0+nh} y_x\,dx$ where $y = f(x)$ and h = length of the subinterval

$$= \int_0^n y_{x_0+kh}\, hdk \qquad \text{Putting } x = x_0 + kh$$

$$dx = hdk.$$

$$x = x_0;\ k = 0;$$

$$= h \int_0^n y_k\, dk$$

$$= h \int_0^n E^k y_0\, dk \qquad \because y_{x+h} = Ey_x$$

$$= h \int_0^n (1+\Delta)^k\, y_0\, dk \qquad E = 1 + \Delta$$

$$\int_a^b f(x)\,dx = h \int_0^n \left[1 + kc_1\Delta + kc_2\,\Delta^2 + \ldots\ldots\right] y_0\, dk$$

$$= h \int_0^n \left(y_0 + kc_1\,\Delta y_0 + kc_2\,\Delta^2 y_0 + \ldots\ldots\ldots\right)dk$$

$$\int_a^b f(x)dx = h \int_0^n \left(y_0 + kc_1\,\Delta y_0 + \frac{k(k-1)}{2!}\Delta^2 y_0 + \frac{k(k-1)(k-2)}{3!}\Delta^3 y_0 \ldots\ldots\ldots\right)dk$$

This is a basis integration formula and it is known as Newton's Cote's formula.

9.3 Trapezoidal Rule

Putting n = 1, in the quadrature formula (ie) there are only two paired values and interpolating polynomial is linear)

$$\int_{x_0}^{x_0+h} f(x)\,dx = h \int_0^1 \left(y_0 + k\,\Delta y_0\right)dk \quad \text{neglecting higher differences}$$

$$= h\left[y_0 k + \frac{k^2}{2}\Delta y_0\right]_0^1$$

$$= h\left(y_0 + \frac{1}{2}\Delta y_0\right)$$

$$= h\left(y_0 + \frac{y_1 - y_0}{2}\right)$$

$$= \frac{h}{2}(2y_0 + y_1 - y_0) = \frac{h}{2}(y_0 + y_1)$$

Similarly $\int_{x_0+h}^{x_0+2h} y\,dx = \frac{h}{2}(y_1 + y_2)$

....

....

$$\int_{x_0+(n-1)h}^{x_0+nh} y\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding these n integrals, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+h} f(x)\,dx + \int_{x_0+h}^{x_0+2h} f(x)\,dx + \ldots\ldots + \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx$$

$$= \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1+y_2) + \ldots\ldots + \frac{h}{2}(y_{n-1} + y_n)$$

$$= \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots.. + y_{n-1})\right]$$

$$= \frac{h}{2}\left[\text{(sum of the first and the last ordinates)} + 2\text{(sum of the remaining ordinates)}\right]$$

∴ This is known as Trapezoidal Rule.

Truncation error in Trapezoidal Rule:

In the neighbourhood of $x = x_0$, we can expand $y = f(x)$ by Taylor series in powers of $x - x_0$,

$$\text{(ie) } y(x) = y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \ldots\ldots \quad (1)$$

Where $y_0' = [y'(x)]_{x=x_0}$

$$\int_{x_0}^{x_1} y\,dx = \int_{x_0}^{x_1}\left[y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \ldots\ldots\right]dx$$

$$= \left[y_0x + \frac{(x-x_0)^2}{2!}y_0' + \frac{(x-x_0)^3}{3!}y_0'' + \ldots\ldots\right]_{x_0}^{x_1}$$

$$= y_0(x_1 - x_0) + \frac{(x_1-x_0)^2}{2!}\ y_0' + \frac{(x_1-x_0)^3}{3!}\ y_0'' + \ldots\ldots$$

$$= hy_0 + \frac{h^2}{2!}\ y_0' + \frac{h^3}{3!}\ y_0'' + \ldots\ldots \qquad (2)$$

If h is the equal interval length

Also $\int_{x_0}^{x_1} y\,dx = \frac{h}{2}\ (y_0 + y_1)$ (3)

= area of the first trapezoium = A

putting $x = x_1$ in (1)

$$y(x_1) = y_1 = y_0 + \frac{(x_1 - x_0)}{1!}\ y_0' + \frac{(x_1-x_0)^2}{2!}\ y_0'' + \ldots\ldots$$

$$\text{(ie) } y_1 = y_0 + \frac{h}{1!}\ y_0' + \frac{h^2}{2!}\ y_0'' + \ldots.. \qquad (4)$$

$$A_0 = \frac{h}{2}\left[y_0 + y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \ldots\ldots\right] \text{ using (4) in (3)}$$

$$= hy_0 + \frac{h^2}{2}\ y_0' + \frac{h^3}{2\times 2!}\ y_0'' + \ldots\ldots$$

Substracting A_0 value from (2)

$$\int_{x_0}^{x_1} y\,dx - A_0 = h^3 y_0''\left(\frac{1}{3!} - \frac{1}{2\times 2!}\right) + \ldots\ldots\ldots$$

$$= -\frac{1}{12} h^3 y_0'' + \ldots\ldots.$$

$\therefore$ The error in the first interval (x_0, x_1) is $-\frac{1}{12} h^3 y_0''$ (neglect other terms)

Similarly the error in the i^{th} interval = $-\frac{1}{12} h^3 y_{i-1}''$

$\therefore$ The total cumulative error (approx)

$$E = -\frac{1}{12} h^3 (y_0'' + y_1'' + y_2'' + \ldots\ldots.. + y_{n-1}'')$$

$$|E| < \frac{nh^3}{12}.M \text{ where M is the maximum value of}$$

$$|y_0''|, |y_1''|, |y_2''| \ldots\ldots\ldots$$

$$< \frac{(b-a)h^2}{12}.M \text{ if the interval is (a,b) \& } h = \frac{b-a}{n}$$

Hence, the error in the trapezoidal rule is of the order h^2.

Example: 1

Calculate $\int_0^1 \frac{dx}{1+x}$ using Trapezoidal Rule.

Solution:

For trapezoidal rule, the range is to be divided into any number of equal parts. All these are satisfied if the range is divided into six equal parts

Hence $h = \frac{1}{6}$.

We shall find the value of $\frac{1}{1+x}$ at the points of division. The table given below, given those values $x = \frac{1}{6}$; $\left[\frac{1}{1+\frac{1}{6}} = \frac{6}{7}\right]$

x	:	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$\frac{1}{1+x}$	:	1	$\frac{6}{7}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{6}{11}$	$\frac{1}{2}$
y	:	1	0.8571	0.75	0.6667	0.6	0.5455	0.5

By Trapezoidal Rule:

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \dots\dots\dots + y_n)\right]$$

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{1}{12}\left[(1+0.5) + 2\,(0.8571 + 0.7500 + 0.6667\ 0.6000)\right]$$

$$= \frac{1}{12}\left[1.5 + 5.7476\right] = \frac{7.2476}{12}$$

$$\int_0^1 \frac{1}{1+x}\,dx = 0.60397.$$

$\therefore$ The Error in this case is $\frac{(x_n - x_0)}{12}\ h^2 f'(\varepsilon)$

$$= \frac{1}{12}\ (1-0)\left(\frac{1}{6}\right)^2 \frac{2}{(1+\varepsilon)^2}$$

$$\left[\ f(x) = \frac{1}{1+x}\right.$$

$$f'(x) = -(1+x)^{-2}$$

$$f''(x) = +2\,(1+x)^{-3}$$

$$\left. f''(\varepsilon) = \frac{2}{(1+\varepsilon)^3}\ \right]$$

$$= \frac{1}{12}\left(\frac{1}{36}\right)\cdot\frac{2}{(1+\varepsilon)^2}$$

$$= 0.0046\ \ (\varepsilon = 0)$$

$$= 0.0005\ \ (\varepsilon = 1)$$

Hence the error lies between 0.0005 to 0.0046

Example: 2

Evaluate $\int_{-3}^{3} x^4\,dx$ by using Trapezoidal Rule.

Solution:

Here $y(x) = x^4$. Interval length $(b - a) = 6$ so, we divide 6 equal intervals with $h = \frac{6}{6} = 1$

x	:	– 3	– 2	–1	0	1	2	3
y	:	81	16	1	0	1	16	81

By Trapezoidal Rule

$$\int_{-3}^{3} y\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots\ldots\ldots y_{n-1})\right]$$

$$\int_{-3}^{3} x^4 dx = \frac{1}{2}\left[(81 + 81) + 2(16 + 1 + 0 + 16)\right]$$

$$= \frac{1}{2}\left[162 + 66\right]$$

$$\int_{-3}^{3} x^4 dx = 115$$

Example: 3

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule with h = 0.2. Hence obtain an approximate value of π. Can you use other formulae in this case.

Solution:

Let $y(x) = \frac{1}{1+x^2}$

Interval is (1 – 0) = 1

$\therefore$ The value of y are calculated as points taking h = 0.2

x	:	0	0.2	0.4	0.6	0.8	1.0
$y = \frac{1}{1+x^2}$	:	1	0.96154	0.86207	0.73529	0.60976	0.50000

By trapezoidal Rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots\ldots + y_{n-1})\right]$$

$$= \frac{0.2}{2}\left[(1 + 0.5) + 2(0.96154 + 0.86207 + 0.73529 + 0.60976\right]$$

$$= (0.1)\,[1.5 + 6.33732]$$

$$= 0.783732$$

By actual integration,

$$\int_0^1 \frac{dx}{1+x^2} = \left(\tan^{-1} x\right)_0^1 = \frac{\pi}{4}$$

$$\therefore\ \pi/4 \approx 0.783732$$

$\pi \approx 3.13493$ (approximately)

Example: 4

Evaluate the integral $I = \int_4^{5.2} \log_e x \, dx$ using Trapezoidal rule.

Solution:

Here $b - a = 5.2 - 4 = 1.2$

We shall divided the interval into 6 equal parts.

Hence, $h = \frac{1.2}{6} = 0.2$

x	4	4.2	4.4	4.6
$f(x) = \log_e x$	1.3862944	1.4350845	1.4816045	1.5260563
x :	4.8	5.0	5.2	
f(x) :	1.5686159	1.6094379	1.6486586	

By Trapezoidal Rule,

$$\int_4^{5.2} \log x \, dx = \frac{0.2}{2}\left[(1.3862944 + 1.6486586) + 2\,(1.4350845 + 1.4816045 + 1.5260563 + 1.5686159 + 1.6094379)\right]$$

$$= \frac{0.2}{2}\left[3.034953 + 15.2415982\right]$$

$$\int_4^{5.2} \log x \, dx = 1.82765512$$

Example: 5

From the following table find the area bounded by the course and the x - axis from x = 7.47 to x = 7.52

x :	7.47	7.48	7.49	7.50	7.51	7.52
y = f(x) :	1.93	1.95	1.98	2.01	2.03	2.06

Solution:

Since only 6 ordinates (n = 5) are given we will use Trapezoidal Rule.

$$\text{Area} = \int_{7.47}^{7.52} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2\,(y_1 + y_2 + \ldots\ldots + y_{n-1})\right]$$

$$= \frac{0.01}{2}\left[(1.93 + 2.06) + 2(1.95 + 1.98 + 2.01 + 2.03)\right]$$

$$= \frac{0.01}{2}[3.99 + 15.94]$$

$$\int_{7.47}^{7.52} f(x)\,dx = 0.09965$$

Exercise:

1. Evaluate $\int_1^{10} \frac{dx}{x}$ by trapezoidal rule, dividing the range into nine equal parts.

2. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ with $h = \frac{1}{6}$. Using Trapezoidal Rule.

3. Find $\int_{0.6}^{1.8} f(x)\,dx$ from the data:

x :	0.6	0.8	1.0	1.2	1.4	1.6	1.8
f(x) :	4.95	6.05	7.39	9.02	11.02	13.46	16.42

by Trapezoidal rule

4. Find $\int_5^{11} f(x)\,dx$ from the data.

x :	5	6	7	8	9	10	11
f(x) :	95.90	96.85	97.77	98.68	99.56	100.41	101.24

using Trapezoidal rule.

5. Evaluate $\int_0^2 \frac{dx}{x^2 + x + 1}$ to three decimals, dividing the range of integration into 8 equal parts. Using Trapezoidal rule. **Answer:** 0.8145

6. Compute the value of $\int_{0.2}^{1.4} (\operatorname{Sin} x - \log x + e^x)\,dx$ taking h = 0.2 & using Trapezoidal rule. **Answer:** 4.0715.

7. Calculate $\int_0^{\pi/2} \operatorname{Sin} x\,dx$ by dividing the interval into ten equal parts. Using Trapezoidal rule.

9.4 SIMPSON'S ONE THIRD RULE

Using Newton Cote's Formula

$$\int_{x_0}^{x_0+nh} f(x)dx = h\int_0^n\left[y_0 + k\Delta y_0 + \frac{k(k-1)}{2!}\Delta^2 y_0 + \frac{k(k-1)(k-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\right]dk$$

putting n = 2.

$$\int_{x_0}^{x_0+2h} y\,dx = h\int_0^2 \left[y_0 + k\Delta y_0 + \frac{k(k-1)}{2!}\Delta^2 y_0\right] dk$$

$$= h\left(2y_0 + 2\,\Delta y_0 + \frac{1}{3}\Delta^2 y_0\right)$$

$$= h\left[2y_0 + 2\,(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}[6y_0 + 6y_1 - 6y_0 + y_2 - 2y_1 + y_0]$$

$$= \frac{h}{3}\,[y_0 + 4y_1 + y_2]$$

Similarly

$$\int_{x_0+2h}^{x_0+4h} y\,dx = \frac{h}{3}\,(y_2 + 4y_3 + y_4)$$

....

....

$$\int_{x_0+(n-2)h}^{x_0+nh} y\,dx = \frac{h}{3}\,(y_{n-2} + 4y_{n-1} + y_n)$$

Adding all these integrals, we have

$$\int_{x_0+(n-2)h}^{x_0+nh} y\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4\,(y_1 + y_3 + y_{n-1}) + 2\,(y_2 + y_4 + \ldots\ldots + y_{n-2})\right]$$

Here n is ever since we neglect all differences above second, y is a polynomial of second degree. (ie) of the form $y = lx^2 + mx + n$, (ie) a parabola. Here we assume that the curve passing through the extremities of 3 consecutive ordinates is a parabola. This is known as Simpson's one third Rule.

Errors in the different Rules:

The leading term dropped in the expansion of $(1 + \Delta)^k$ gives the error.

In the case of Simpson's one – third rule, the leading term dropped is

$$h\int_0^2 \frac{k\,(k-1)\,(k-2)}{6}\,\Delta^3 y_0\,dk$$

$$\text{(ie)}\ \frac{h}{6}\left[\Delta^3 y_0 \left(\frac{k^4}{4} - k^3 + k^2\right)_0^2\right] = 0$$

Hence the next term dropped can be taken as error, which is equal

$$h\int_0^3 \frac{k(k-1)(k-2)(k-3)}{24}\Delta^4 y_0\, dk$$

$$= -\frac{h}{90}\Delta^4 y_0$$

$$= -\frac{h^5}{90} f^{(4)}(\varepsilon)$$

This error is for an interval of 2h. Hence the error in the interval of $h = -\frac{h^5}{180} f^{(4)}(\varepsilon)$.

Hence the error for the whole length $nh = -\frac{nh^5}{180} f^{(4)}(\varepsilon)$

$$= -\frac{(x_n - x_0)}{180} h^4 f^{(4)}(\varepsilon)$$

where $x_0 < \varepsilon < x_n$.

Example: 1

Calculate $\int_0^1 \frac{dx}{1+x}$ using one–third Simpson rule.

Solution:

For Simpson's $\frac{1^{rd}}{3}$ rule the range is to be divided respectively into even, multiple of three and multiple of six equal parts. All these are satisfied if the range is divided into six equal parts.

Hence $h = \frac{1}{6}$.

We shall find the value of $\frac{1}{1+x}$ at the points of division. The table given below, gives those values: $\left[0 + \frac{1}{6} - \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{2}{6}, \frac{1}{3} + \frac{1}{6} = \frac{3}{6} \ldots\ldots\ldots\ldots \right]$

x:	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$\frac{1}{1+x}$:	1	$\frac{6}{7}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{6}{11}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$\left[\frac{1}{1+0}=1,\ \frac{1}{1+\frac{1}{6}}=\frac{6}{7};\ \frac{1}{1+\frac{1}{3}}=\frac{3}{4}\ldots\ldots\ldots\right]$$

y: 1 0.8571 0.75 0.6667 0.6 0.5455 0.5

By Simpson's third rule.

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+\ldots\ldots y_{n-1})+2(y_2+y_4+\ldots\ldots y_{n-2})\right]$$

$$\int_0^1 \frac{dx}{1+x} = \frac{1}{6\times 3}\left[(1+.5)+4(0.8571+0.6667+0.5455)+2(0.7500+0.6000)\right]$$

$$= 0.6931$$

In this case the error is

$$-\frac{(x_n-x_0)}{180}\left(\frac{1}{6}\right)^4 f^{(4)}\ (\varepsilon)$$

$f(x) = (1+x)^{-1}$
$f^1(a) = -(1+x)^{-2}$
$f^{ii}(x) = 2(1+x)^{-3}$
$f^{iii}(x) = 6(1+x)^{-4}$
$f^{iv}(x) = 24(1+x)^{-5}$

(i.e.) $-\dfrac{1}{180\ \times 1296}\times\dfrac{24}{(1+\varepsilon)^5}$

(i.e.) $-\dfrac{1}{180\ \times 54}\times\dfrac{1}{(1+\varepsilon)^5}$

(i.e.) lies between $+\dfrac{1}{180\times 54\times 32}$ & $\dfrac{1}{180\times 54}$

$[\varepsilon = 2,0]$

(i.e.) 0.00003 and 0.0010.

Example: 2

Evaluate $\int_{-3}^{3} x^4\ dx$ by using Simpson's one third rule.

Solution:

Here $y(x) = x^4$

Interval length $(b-a) = 6$

So, we divide 6 equal intervals with $h = \frac{6}{6} = 1$.

x:	−3	−2	−1	0	1	2	3
$y = x^4$:	81	16	1	0	1	16	81

By Simpson's one-third rule

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \ldots\ldots y_{n-1}) + 2(y_2 + y_4 + \ldots\ldots y_{n-2})\right]$$

$$\int_{-3}^{3} x^4\,dx = \frac{1}{3}\left[(81+81) + 2(1+1) + 4(16+0+16)\right]$$

$$= \frac{1}{3}[162 + 4 + 128]$$

$$\int_{-3}^{3} x^4\,dx = 98$$

Example: 3

By dividing the range into ten equal parts. Evaluate $\int_0^{\pi} \sin x\,dx$ by Simpson's one–third rule.

Solution:

Range = $\pi - 0 = \pi$

Hence $h = \frac{\pi}{10}$

We tabulate below the values of y at difference x is

x :	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{5\pi}{10}$	$\frac{6\pi}{10}$
Y = sin x:	0.0	0.3090	0.5878	0.8090	0.9511	1.0	0.9511

x :	$\frac{7\pi}{10}$	$\frac{8\pi}{10}$	$\frac{9\pi}{10}$	π
Y = sin x:	0.8090	0.5878	0.3090	0

Use Simpson's one third rule

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+.......y_{n-1})+2(y_2+y_4+.......y_{n-2})\right]$$

$$\int_0^{\pi} \sin x\,dx = \frac{1}{3}\left(\frac{\pi}{10}\right)[(0+0) + 2(0.5878 + 0.9511 + 0.9511 + 0.5878) + 4(0.3090 + 0.8090 + 1 + 0.8090 + 0.3090)]$$

$$= \frac{1}{3}\left(\frac{\pi}{10}\right)[6.1556 + 12.944] = \frac{1}{3}\left(\frac{\pi}{10}\right)(19.0996)$$

$$\int_0^{\pi} \sin x\,dx = 1.99909$$

Example: 4

Evaluate $\int_0^1 e^x\,dx$ Simpson's one third rule correct to five decimal places, by proper choice of h.

Solution:

Here, interval length = b – a = 1

$y = e^x;\ y^1 = e^x;\ y^{11} = e^x;\ yi^{111} = e^x;\ y^{iv} = e^x;$

$$\text{Error} = |E| < \frac{(b-a)}{180} h^4 .\ M \text{ where } M = \text{Max}(e^x) \text{ in the range.}$$

$$< \frac{1}{180} h^4 . e$$

we require $(E) < 10^{-6}$

$$\frac{h^4 e}{180} < 10^{-6} \qquad e = 2.718282$$

$$h < \left(\frac{180 \times 10^{-6}}{e}\right)^{1/4} = 0.148$$

Hence we take h = 0.1 to have the accuracy required.

$$\therefore \int_0^1 e^x\,dx = \frac{0.1}{3}\left[(1+e)+2(e^{0.2}+e^{0.4}+e^{0.6}+e^{0.8})+4(e^{0.1}+e^{0.3}+e^{0.5}+e^{0.7}+e^{0.9})\right]$$

$$= \frac{0.1}{3} \; [(1 + 2.718282) + 2\,(1.22140) + 1.49182 + 1.8221188 + 2.22554) + 4\,(1.10517 + 1.34986 + 1.6487 + 2.01375 + 2.4596]$$

$$= \frac{0.1}{3}\,[3.718282 + 13.5217576 + 34.30832]$$

$$= \frac{0.1}{3}\,[51.5483596]$$

$$= 1.718278$$

Exercise:

1. Evaluate $\int_0^1 e^{-x^2}\,d\,x$ by Simpson's one third rule. (10 strips)
2. Compute the value of $\int_1^2 \frac{d\,x}{x}$ using Simpson on third rule. H = 025.
3. Evaluate $\int_1^2 \sin^3 x\,d\,x$ taking h = $\pi/6$ using Simpson's Rule.

9.5 SIMPSON'S THREE ENGHTH'S RULE

Using Newton Cote's formula

$$\int_{x_0}^{x_0+nh} f(x)\,d\,x = h\int_0^n \left[y_0 + k\,\Delta y_0 + \frac{k\,(k-1)}{2!}\Delta^2\,y_0 + \frac{k(k-1)(k-2)}{3!}\Delta^3\,y_0 + ...\right]$$

$$[(k^2 - k)\,(k - 2)\,k^3 - 2k^2 - k_2 + 2K]$$

x_0 + 3 h Putting n = 3

$$\int_{x_0}^{x_0+3h} f(x)\,d\,x = h\int_0^3 \left[\,y_0 + k\,\Delta\,y_0 + \frac{k\,(k-1)}{2!}\Delta^2\,y_0 + \frac{k(k-1)(k-2)}{3!}\Delta^3\,y_0\right]\,d\,k$$

$$= h\left[y_0\,k + \frac{k^2}{2}\Delta\,y_0 + \left(\frac{k^3}{6} - \frac{k^2}{4}\right)\Delta^2\,y_0 + \left(\frac{k^4}{24} - \frac{3k^3}{18} + \frac{2k^2}{12}\right)\Delta^3\,y_0\right]_0^3$$

$$= h\left[3\,y_0 + \frac{9}{2}\,\Delta\,y_0 + \frac{27}{12}\Delta^2\,y_0 + \frac{9}{24}\Delta^3\,y_0\right]$$

$$= h\left[3\,y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_0 - 2y_1 + y_2 + \frac{3}{8}[-y_0 + 3y_1 - 3y_2 + y_3]\right]$$

$$= h\left[3y_0 + \frac{9}{2}y_1 - \frac{9}{2}y_0 + \frac{9}{4}y_0 - \frac{18}{4}y_1 + \frac{9}{4}y_2 - \frac{3}{8}y_0 + \frac{3}{8}(3y_1) - \frac{3}{8}(3y_2) + \frac{3}{8}y_3\right]$$

$$= \frac{h}{24}[72y_0 + 108y_1 - 108y_0 + 54y_0 - 108y_1 + 54y_2 - 9y_0 + 27y_1 - 27y_2 + 9Y_3]$$

$$= \frac{9h}{24}[9y_0 + 27y_1 - 27y_2 + 9y_3]$$

$$= \frac{9h}{24}[y_0 + 3y_1 - 3y_2 + y_3]$$

$$\int_{x_0}^{x_0+3h} = \frac{3h}{8}[y_0 + 3y_1 - 3y_2 + y_3]$$

Similarly $\int_{x_0}^{x_0+6h} y\,dx = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$

. . . .

. . . .

$$\int_{x_0+(n-3)h}^{x_0+nh} y\,dx = \frac{3h}{8}[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

Here n is a multiple of 3 and y is a polynomial of degree 3, (i.e.) $y = lx^3 + mx^2 + px + q$. Since higher powers above the third differences are neglected. Hence in this case we assume a cubic curve passes through 4 consecutive ordinates extremities.

Adding all these we get

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\left[\begin{array}{l}(y_0 + y_n) + 3((y_1 + y_2 + y_4 + y_5 + y_7 + \dots\dots + y_{n-1}) \\ + 2(y_3 + y_6 + \dots\dots\dots + y_{n-3})\end{array}\right]$$

This is known as Simpson's three eighth's rule.

Errors in the Difference rules:

The leading term dropped in the expansion of $(1 + \Delta)^k$ gives the error.

In the case of Simpson's three eighth's rule the leading term dropped is

$$h \int_0^3 \frac{k(k-1)(k-2)(k-3)}{4!} \Delta^4 y_0 \, dk.$$

$$= \frac{h}{4!} \left[\frac{k^5}{5} - \frac{6k^4}{4} + \frac{11k^3}{3} - \frac{6k^2}{2} \right] \Delta^4 y_0$$

$$= \frac{h}{4!} \left[\frac{243}{5} - \frac{486}{4} + \frac{297}{3} - \frac{54}{2} \right] \Delta^4 y_0$$

$$= \frac{h}{24} \left[\frac{9}{10} \right] \Delta^4 y_0 = \frac{3h}{80} \Delta^4 y_0$$

$$= + \frac{3h^5}{80} f^{(4)}(\varepsilon) \text{ where } x_0 < \varepsilon < x_3.$$

This error is for the length of interval 3 h.

$\therefore$ The error for the length of interval

$$nh = -\frac{nh^5}{80} f^{(4)}(\varepsilon)$$

$$= -\frac{(x_n - x_0)}{80} h^4 f^{(4)}(\varepsilon)$$

where $x_0 < \varepsilon < x_n$

Example: 1

Calculate $\int_0^1 \frac{dx}{1+x}$ using three – eighths Simpson's rule.

Solution:

For Simpson's $\frac{3^{th}}{8}$ rule the range is to be divided respectively into even multiple of three and multiple of six equal parts. All these are satisfied if the range is divided into six equal parts.

Hence $h = \frac{1}{6}$

X	:	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$\frac{1}{1+x}$	:	1	$\frac{6}{7}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{6}{11}$	$\frac{1}{2}$
		1	0.8571	0.75	0.6667	0.6	0.5455	0.5

By Simpson's three – eighth rule

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\begin{bmatrix}(y_0+y_n)+3\left((y_1+y_2+y_4+y_5+y_7+\ldots\ldots+y_{n-1})\right)\\ +2\left(y_3+y_6+\ldots\ldots\ldots+y_{n-3}\right)\end{bmatrix}$$

$$= \frac{3}{8}\cdot\frac{1}{6}\left[(1+0.5)+3(0.8571+0.7500+0.6000+0.5455+2(0.6667)\right]$$

$$= 0.0625\,[1.5 + 8.2578 + 1.3334]$$

$$= 0.6932$$

In this case the error is $-\dfrac{(x_n - x_0)}{80}\; h^4 f^{(4)}(\varepsilon)$

(i.e.) $-\dfrac{1}{80}\left(\dfrac{1}{6}\right)^4 \dfrac{24}{(1+\varepsilon)^5}$ $\quad\quad \varepsilon = 1, 2$

(i.e.) lies between $\dfrac{1}{80\times 54\times 32}$ & $\dfrac{1}{80\times 54}$

$\therefore$ 0.000006 and .0002

Example: 2

Evaluate $\int_{-3}^{3} x^4\,dx$ by using Simpson's three eighths rule.

Solution:

Using Simpson's three – eighths rule.

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\begin{bmatrix}(y_0+y_n)+3\left((y_1+y_2+y_4+y_5+\ldots\ldots+y_{n-1})\right)\\ +2\left(y_3+y_6+y_9\ldots\ldots\ldots+y_n\right)\end{bmatrix}$$

Here $y = x^4$. Interval length b – a = 6. So we divide 6 equal intervals with $h = \dfrac{6}{6} = 1$.

x	:	–3	–2	–1	0	1	2	3
y	:	81	16	1	0	1	16	81

$$\int_{-3}^{3} y\,d\,x = \frac{3}{8}\ [(81+81)+3(16+1+1+16)+2(0)]$$

$$= \frac{3}{8}\ [162 + 99 + 0] = 97.875$$

Example: 3

Evaluate $\int_{4}^{5.2} \log_e x$ using Simpson's three –eighth's rule.

Solution:

Here b – a = 5. 2 – 4 = 1.2

We shall divide the interval into 6 equal parts.

Hence, $h = \frac{1.2}{6} = 0.2$

x:	4	4.2	4.4	4.6	4.8	5.0
$\log_e x$	1.3862944 y_0	1.4350845 y_1	1.4816045 y_2	1.5260563 y_3	1.568659 y_4	1.6094379 y_5

x:	5.2
$\log_e x$	1.6486586 y_6

By Simpson's three eighth rule,

$$\int_{x_0}^{x_0+nh} y\,d\,x = \frac{3h}{8}\left[\begin{array}{l}(y_0+y_n)+3\left((y_1+y_2+y_4+y_5+\ldots\ldots+y_{n-1}\right) \\ +2\left(y_3+y_6+y_9\ldots\ldots\ldots\ldots+y_n\right)\end{array}\right]$$

$$I = \frac{3\,(0.2)}{8}\left[\begin{array}{l}(1.3862944+1.6486586)+3(1.4350845+1.4816045+ \\ 1.5686159+1.6094379)+2(1.5260563)\end{array}\right]$$

$$= \frac{0.6}{8}\ [3.034953 + 18.2842284 + 3.0521126]$$

$$I = 1.82784705$$

Example: 4

Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by Simpson's three – eighth's rule; Also check up the results by actual integration.

Solution:

Here b – a = 6 – 0 = 6

Divide into 6 equal parts $h = \frac{6}{6} = 1$

x	:	0	1	2	3	4	5	6
$\frac{1}{1+x^2}$	:	1.00	0.500	0.200	0.100	0.58824	0.038462	0.027027

By Simpson's three eighth's rule.

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\left[\begin{array}{l}(y_0+y_n)+3((y_1+y_2+y_4+y_5+y_7........+y_{n-1})\\+2(y_3+y_6+..........+y_{n-3})\end{array}\right]$$

$$= \frac{3\,x1}{8}\,[(1+0.027027)+3(0.5+0.2+0.058824+0.038462)+2(0.1)]$$

$$= \frac{3}{8}\,[1.027027 + 2.391858 + 0.2]$$

$$I = 1.35708188$$

By actual integration,

$$I = \int_0^6 \frac{dx}{1+x^2} = (\tan^{-1} x)_0^6 = \tan^{-1} 6 = 1.40564765$$

Exercise:

1. Find $\int_{0.6}^{1.8} f(x)\,dx$ from the data.

x:	0.6	0.8	1.0	1.2	1.4	1.6	1.8
f(x):	4.95	6.05	7.39	9.02	11.02	13.46	16.42

Using Simpson's $\frac{3}{8}$ Rule.

2. Evaluate $\int_0^{\pi} \frac{dx}{x+\cos x}$ by Simpson's $\frac{3}{8}$ Rule with $h = \frac{\pi}{8}$.

3. Calculate $\int_{0.5}^{0.7} e^{-x} x^{1/2}\, dx$ taking 5 ordinates by Simpson's $\frac{3}{8}$ Rule

4. Evaluate $\int_1^{1.4} e^{-x^2}\, dx$ by taking h = 0.1 using Simpson's $\frac{3}{8}$ Rule.

5. Evaluate $\int_1^2 \frac{\sin x}{x}$ taking 6 intervals.

6. The speed of an electric train at various times after leaving one station unit it stops at the next station are given table.

Speed in kmph:	0	13	33	$39\frac{1}{2}$	40	40	36	15	0
e in minutes:	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{4}$	$3\frac{1}{2}$

9.6 WEDDLE'S RULE

By Newton's Cote's formula

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\left[\begin{array}{l}(y_0+y_n)+4((y_1+y_3+\ldots+y_{n-1}))\\ +2(y_2+y_4+\ldots\ldots+y_{n-2})\end{array}\right]$$

Putting n = 6 and neglecting all differences above the sixth, the Newton's cote's formula reduces to

$$\int_{x_0}^{x_0+6h} y\,dx = h\int_0^6 \left[y_0 + k\Delta y_0 + \frac{k(k-1)}{2!}\Delta^2 y_0 + \frac{k(k-1)(k-2)}{3!}\Delta^3 y_0 + \frac{k(k-1)(k-2)(k-3)\ldots\ldots(k-5)}{6!}\Delta^6 y_0\right]dk$$

$$= h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_6 + \frac{33}{10}\Delta^5 y_0 + \frac{42}{140}\Delta^6 y_0\right]$$

Replace the last term by $\frac{42}{140}\ \Delta^6 y_0$

(i.e) $\frac{3}{10}\ \Delta^6 y_0$ which is negligible when h and $\Delta^6 y_0$ are small.

$$\int_{x_0}^{x_0+6h} y\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_4 + y_6\right]$$

Similarly

$$\int_{x_0+6h}^{x_0+12h} y\,dx = \frac{3h}{10}\left[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}\right]$$

. . . .

. . . .

$$\int_{x_0+(n-6)h}^{x_0+nh} y\,dx = \frac{3h}{10}\left[y_{n-6} + 5y_{n-5} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n\right]$$

Adding all these integrals,

We have

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{10}\begin{bmatrix} y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 \\ + y_8 + 6y_9 + 5y_{11} + 2y_{12} + 5y_{n-1} + y_n \end{bmatrix}$$

Here n is a multiple of 6. This is known as Weddle's rule.

Errors in the Different Rules:

The leading term dropped in the expansion of $(1 + \Delta)^k$ gives the error.

In the Weddle's rule,

The last term is $\frac{41h}{140}\Delta^6 y_0$ but we have taken the last term as $-\ \frac{42h}{140}\ \Delta^6 y_0$

$$\text{Hence the error} = -\ \frac{h}{140}\ \Delta^6 y_0$$

$$= -\ \frac{1}{140}\ h^7\ f^{(6)}\ (\varepsilon)$$

where $x_0 < \varepsilon < x_6$.

Example: 1

Calculate $\int_0^1 \frac{dx}{1+x}$ using Weddle's rule.

Solution:

For Weddle's rule the range is to be divided respectively into even, multiple of three and multiple of six equal parts. All these we satisfied if the range is divided into six equal parts.

Hence $h = \frac{1}{6}$

x	:	0	$1/6$	$1/3$	$1/2$	$2/3$	$5/6$	1
$\frac{1}{1+x}$	:	1	$\frac{6}{7}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{6}{11}$	$\frac{1}{2}$
		1	0.8571	0.75	0.6667	0.6	0.5455	0.5

$$\int_0^1 \frac{dx}{1+x} = \frac{3h}{10}\begin{bmatrix} y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_4 + 2y_6 + 5y_7 + y_8 + 6y_9 \\ + y_{10} + 5y_{11} + 2y_{12} + \dots\dots + 5y_{n-1} + y_n \end{bmatrix}$$

$$= \frac{3}{10} \cdot \frac{1}{6}\left[1 + 5(0.8571) + 0.7500 + 6(0.6667) + 0.6000 + 5(0.5455) + 0.5000\right]$$

$$= \frac{3}{60}\,[1.4.2855 + 0.7500 + 4.0002 + 0.6000 + 2.7275 + 09.5000]$$

$= 0.6932$

In this case the error is $-\frac{(x_n - x_0)}{180} h^4 f^{(4)}(\varepsilon)$ and its lies between 0.00003 & .0010.

Actual value of $\int_0^1 \frac{1}{1+x}\,dx = \log(1+x)_0^1$

$= 108.2$

$= 0.69315$

Example: 2

Evaluate the integral $I = \int_{4}^{5.2} \log_e x \; d\,x$ using Weddle's rule.

Solution:

$$\int_{x_0}^{x_0+nh} y\,d\,x = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_4 + 2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + 2y_{12} + \ldots\ldots + 5y_{n-1} + y_n \right]$$

Here b – a = 5.2 – 4 = 1.2

We shall divide the interval into 6 equal parts.

Hence $h = \frac{1.2}{6} = 0.2$

x	:	4	4.2	4.4	4.6	4.8	5.0
$\log_e x$	:	1.3862944	1.4350845	1.4816045	1.5260563	1.5668159	1.6094379

X	:	5.2
$\log_e x$	:	1.6486586

$$I = \frac{3\,(0.2)}{10}\,[1.3862944 + 5\,(1.4350845) + 1.4816045 + 6\,(1.5260563) + 1.5686159 + 5\,(1.6094379) + 1.6486586]$$

$$= \frac{0.6}{10}\,[1.3862944 + 7.1754225 + 1.4816045 + 9.1563378 + 1.5686159 + 8.0471895 + 1.648658]$$

I = 1.82784734.

Example: 3

Evaluate $= I = \int_{0}^{6} \frac{1}{1+x}\,d\,x$ using Weddle's rule. Also check up by direct integration.

Solution:

Take the number of intervals as ơ.

$\therefore\ h = \frac{6-0}{6} = 1$

x	0	1	2	3	4	5	6
$\frac{1}{1+x}$:	1	0.5	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$

By Weddle's rule

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{10}\begin{bmatrix} y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + 6y_9 \\ + y_{10} + 5y_{11} + 2y_{12} + \dots\dots + 5y_{n-1} + y_n \end{bmatrix}$$

$$= \frac{3\times 1}{10}\left[1 + 5(0.5) + \frac{1}{3} + 6\left(\frac{1}{4}\right) + \frac{1}{5} + 5\left(\frac{1}{6}\right) + \frac{1}{7}\right]$$

$$= \frac{3}{10}\left[1 + 2.5 + 0.3333 + 1.5 + \frac{1}{5} + 0.8333 + 0.14286\right]$$

$= 1.95285$

By actual integration,

$$\int_0^6 \frac{1}{1+x}\,dx = [\log(1+x)]_0^6 = \log_e 7 = 1.949591015$$

Example: 4

Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by Weddle's Rule. Also check up the results by actual integration.

Solution:

Here b – a = 6 – 0 = 6

Divide into 6 equal parts $h = \frac{6}{6} = 1$.

x:	0	1	2	3	4	5	6
$\frac{1}{1+x^2}$:	1.00	0.500	0.200	0.100	0.058824	0.038462	0.027027

By Weddle's rule.

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{10}\begin{bmatrix} y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + 6y_9 \\ + y_{10} + 5y_{11} + 2y_{12} + \dots\dots + 5y_{n-1} + y_n \end{bmatrix}$$

$$= 3 \times \frac{1}{10}\,[1 + 5\,(0.5) + 0.2 + 6\,(0.1) + 0.058824 + 5\,(0.038462) + 0.027027]$$

$$= \frac{3}{10}[1 + 2.5 + 0.2 + 0.6 + 0.058824 + 0.19231 + 0.027027]$$

$$I = 1.3734483$$

By actual integration,

$$I = \int_0^6 \frac{dx}{1+x^2} = \left(\tan^{-1} x\right)_0^6 = \tan^{-1} 6 = 1.405648$$

Exercise

1. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ with $h = \frac{1}{6}$ by Weddle's rule.

2. Evaluate $\int_{0.2}^{1.4} (\sin x - \log x + e^x)\, dx$ with h = 0.1 by Weddle's rule

3.Find by Weddle's rule.

(a) $\int_{0.6}^{1.8} f(x)\, dx$ from the data.

x:	0.6	0.8	1.0	1.2	1.4	1.6	1.8
f(x):	4.95	6.05	7.39	9.02	11.02	13.46	16.42

b) $\int_3^{11} f(x)\, dx$ from the data

x:	5	6	7	8	9	10	11
f(x):	95.90	96.85	97.77	98.68	99.56	100.41	101.24

4. Show that the difference between the values of $\int_{x_0}^{x_6} f(x)\, dx$ obtained by Simpson's one third rule & Weddle's rule with six sub-intervals is $\frac{h}{30}\left(\Delta^4 + \Delta^5 + \Delta^6\right) f(x_0)$ where $h = \frac{x_6 - x_0}{6}$

5. Compute: $\int_5^{12} \frac{dx}{x}$

6. Evaluate $\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$ by Weddle's rule, taking h = 6.

UNIT – X

DIFFERENCE EQUATIONS

10.1 In the following article we take the common difference of the successive differences of the independent variable as unity.

An equation which express a relation between an independent variable x and successive differences or successive values of a dependent variable y_x is.

Difference equation:

Thus, examples of difference equation are

i) $\Delta^2 y_x - 3\Delta y_x = 0$

ii) $\Delta^3 y_x - 2\Delta^2 y_x - 3\Delta y_x + y_x = 0$

iii) $\Delta^3 y_x - 3\Delta y_x - 2y_x = x + 2$

By means of the relationship

$$\Delta^n y_x = y_{x+n} - nc_1 y_{x+n-1} + \ldots + (-1)^n y_x$$

We can express the difference equations in forms involving successive values of y_x instead of successive differences of y_x. Thus the above three equations may be written respectively.

$$y_{x+2} - 5y_{x+1} + y_x = 0 \quad (1)$$

$$y_{x+3} - y_{x+2} - 4y_{x+1} + 5y_x = 0 \quad (2)$$

$$y_{x+3} - 3y_{x+2} = x + 2 \quad (3)$$

Of the two forms, for purposes of solution, that involving successive values is usually preferable. The order of a difference equation written in this form is the difference between the highest and the lowest subscript of the y's.

Hence the order of the equation (1), (2) and (3) are respectively 2,3,and 1

The degree of a differential equation of this form is the highest power of the y's. These for a differential equation of the form $(y_x)^2 (y_{x+1})^3 - 2y_x y_{x+2} + 3y^2x+1 = 2x^2 - 5$ the order is 2 and the degree is 3.

General linear difference equation:

The most important type of difference equation is the linear differenc equation it has the general form.

$$y_{x+n} + A_{n-1} y_{x+n-1} + A_{n-2} y_{x+n-2} + \ldots + \Delta_1 y_{x+1} + A_0 y_x = f(x) \quad (1)$$

where $A_0, A_1 \ldots A_{n-1}$ f(x) are known functions of x.

If the right hand member of the equation (1)

If f(x) is zero, the equation is called homogenous. Equation (1) is known as complete equation.

With regard to homogeneous equation of the form.

$$y_{x+n} + A_{n-1}y_{x+n-1} + \ldots\ldots A_1 y_{x+1} + A_0 y_x = 0 \qquad (2)$$

The following results are easily established:-

i. $\phi_1(x)$ is a solution of (2), so is $c_1\phi_1(x)$

ii) If $\varphi_1(x)$, $\phi_2(x)$, $\phi_3(x)$...... $\phi_n(x)$ are n functions of x, which are independent solution of 2. Then

$$y_x = c_1\phi_1(x) + c_2\phi_2(x) + \ldots\ldots + c_n\phi_n(x) \qquad (3)$$

is a general solution of (2).

iii) If $c_1\phi_1(x) + c_2\phi_2(x) + \ldots\ldots + c_n\phi_n(x)$ is the general solution of (2), then the general solution of (1) is

$$y_x = c_1\phi_1(x) + c_2\phi_2(x) + \ldots\ldots + c_n\phi_n(x) + F(x).$$

We shall (3) the complementary function and F(x) the particular integral and hence the general solution of (1) is the sum of the complementary function and the particular integral. We thus see some analogy between this and the linear differential equation with constant coefficients.

Solution of first and second order equation with constant coefficient.

10.2 Linear Difference Equation of order one:

The general form of this equation is

$$y_{x+1} - f(x)y_x = \phi(x)$$

We shall first solve the equation when the right side is zero

$$y_{x+1} - f(x)y_x = 0$$

$$(ie)\ y_{x+1} - f(x)y_x$$

$$\therefore y_x = f(x-1)y_{x-1}$$

$$y_{x-1} = f(x-2)y_{x-2}$$

....

....

$$y_{r+1} = f(r)y_r$$

Hence $y_x = f(x-1)(x-2)\ldots\ldots f(r) y_r$.

If y_r has arbitrary value C. the solution becomes $yx = C \overset{x-1}{\underset{r}{\pi}} f(x)$. Where

$\overset{x-1}{\underset{r}{\pi}} f(x)$ stands for the continued product.

$f(r)\, f(r+1) \ldots\ldots f(x-2)\, f(x-1)$

We can continue to the process upto r = 0.

In that case the solution is

$$y_x = C \overset{x-1}{\underset{0}{\pi}} f(x)$$

Now, let us consider the equation

$$y_{x+1} - f(x)\, y_x = \phi(x) \qquad (1)$$

Let V_x be the solution of the equation

$$y_{x+1} - f(x)\, y_x = 0 \qquad (2)$$

Let us assume that $u_x\, v_x$ be a solution of the equation (1)

$$\therefore\ y_x = u_x\, v_x \qquad (3)$$

Hence $y_{x+1} = u_{x+1}\, v_{x+1}$

$$\Delta u_x = u_{x+1} - u_x$$

$$\therefore\ y_{x+1} = (\Delta u_x + u_x) v_{x+1} \qquad (4)$$

$\therefore$ Substituting (3) and (4) in (1)

we get

$$(\Delta u_x + u_x) v_{x+1} - f(x)\, u_x\, V_x = \phi(x)$$

(ie) $(\Delta u_x) v_{x+1} + u_x \{v_{x+1} - f(x)\, v_x\} = \phi(x)$

(ie) $(\Delta u_x) v_{x+1} = \phi(x)$

(ie) $\Delta u_x = \dfrac{\phi(x)}{v_{x+1}}$

(ie) $u_x = \Delta^{-1} \left\{ \dfrac{\phi(x)}{v_{x+1}} \right\}$

Hence the general solution of equation (1) is

$$y_x = cv_x + v_x\ \Delta^{-1}\left\{\frac{\phi(x)}{v_{x+1}}\right\}$$

Example: 1

Solve the equation $y_{x+1} - 2y_x = 0$

Solution:

The general form of this equation in $y_{r+1} - f(x)y_x = \phi(x)$

In this equation $f(x) = 2$.

$\therefore$ f(0), f(1),f(2), ... f(x − 1) are all equal to 2.

$$\therefore \underset{x=0}{\overset{x-1}{\pi}} f(x) = 2^x$$

$$[\because f(0).\ f(1).\ f(2).....f(x-1)$$
$$2\ .\ 2\ .\ 2......2\ = 2^x]$$

Hence the general equation

$$y_x = C\underset{0}{\overset{x-1}{\pi}} f(x)$$

$$\therefore y_x = c\ .\ 2^x$$

Example: 2

Solve the equation $y_{x+1} - 3y_x = 2$

Solution:

The solution of the equation

$y_{x+1} - 3y_x = 0$

In this equation $f(x) = 3\ .\ \phi(x) = 2$

$\therefore$ f(0),f (1),f(2)....f(x − 1) are equal to 3.

$$\therefore \underset{x=0}{\overset{x-1}{\pi}} f(x) = f(0), f(1), f(2)....f(x-1)$$

$$= 3\ .\ 3\ ....3 = 3^x$$

$$y_x = \overset{x-1}{\underset{x=0}{\pi}} f(x) = c.3^x$$

Hence the general solution of the equation is

Formula:

$$y_x = CV_x + v_x \Delta^{-1}\left[\frac{\phi(x)}{v_{x+1}}\right]$$

$$y_x = C.3^x + 3^x \Delta^{-1}\left[\frac{2}{3^{x+1}}\right]$$

$$y_x = C.3^x + 2.\frac{3x}{3} \Delta^{-1}\left(\frac{1}{3}\right)^x \left[\Delta^{-1}\left(m^x\right) = \frac{m^x}{m-1} m \neq 1\right]$$

$$= C.3^x + 2.3^{x-1} \Delta^{-1} \frac{\left(\frac{1}{3}\right)^x}{\frac{1}{x^3} - 1}$$

$$= C.3^x + 3^x \frac{\left(\frac{1}{3}\right)}{\left(-\frac{2}{3}\right)} \bullet 2$$

$$= c.3^x - 3^x \left(\frac{1}{3}\right)^x$$

$$y_x = c.3^x - 1$$

Aliter:

Let us find the general solution of the equation

$y_{x+1} - Ay_x = B$

$y_{x+1} = Ay_x + B$

Hence $y_x = Ay_{x-1} + B$

$$= A(Ay_{x-2} + B) + B$$

$$= A^2 y_{x-2} + B(1+A)$$

$$= A^2(y_{x-3} + B) + B(1+A)$$

$$= A^3 y_{x-3} + B(1+A+A^2)$$

Continuing this process, we get

$$y_x = A^x y_0 + B\,(1 + A + A^2 + \ldots\ldots\ldots + A^{x-1})$$

$$y_x = A^x y_0 + B\frac{1-A^x}{1-A} \quad \text{when } A \neq 1$$

$$y_x = y_0 + B\,(1+1+1+\ldots\ldots.+1) \text{ when } A=1$$

$$y_x = y_0 + xB$$

Hence we get that the solution of the equation $y_{x+1} - Ay_x = B$

$$y_x = A^x C + B.\frac{1-A^x}{1-A} \quad \text{when } A \neq 1$$

$$= C + x\,B \text{ when } A = 1$$

taking y_0 as the arbitrary constant.

Hence in the case of the equation

$$y_{x+1} - 3y_x = 2$$

The general solution is

$$y_x = c.3^x + 2\frac{3^x - 1}{3-1}$$

$$= c.3^x + 3^x - 1$$

$$y_x = c(3^x - 1\,)$$

Example: 3

Solve the equation $y_{x+1} - ay_x = 4^x$

Solution:-

The solution of the equation

$y_{x+1} - ay_x = 0$

In this equation $f(x) = a$

$\therefore$ f(0), f(1), f(2).....f(x−1) are all equal to a.

$\therefore \quad \overset{x-1}{\underset{x=0}{\pi}} f(x) = f(0) \,.\, f(1) \,.f(2) \,.\, \ldots. f_{(x-1)}$

$= a \,.\, a \,.\, a \ldots\ldots a = a^x.$

$\therefore$ The general equation is

$$y_x = C \overset{x-1}{\underset{x=0}{\pi}} f(x) = c.a^x$$

$$y_{x+1} - ay_x = 4^x$$

is $y_x = c.a^x + a^x \; \Delta^{-1} \left(\frac{4^x}{a^{x+1}}\right) \left[\because y_x = cv_x + v_x \Delta^{-1}\left\{\frac{\phi(x)}{v_{x+1}}\right\}\right]$

$$y_x = c.a^x + a^x. \; \frac{1}{a} \; \Delta^{-1}\left(\frac{4}{a}\right)^x$$

$$= c.a^x + a^{x-1} \; \frac{\left(\frac{4}{a}\right)^x}{\frac{4}{a}-1} \; \left[\because \Delta^{-1}(m^x) = \frac{m^x}{m-1}\right]$$

$$y_x = c.a^x + \frac{4^x}{4-a}$$

Example:

Solve the equation $y_{x+1} - \frac{x}{x+1} y_x = 0$

Solution:

In this equation $f(x) = \frac{x}{x+1}$

$\therefore$ f(0), f(1), f(2) ....f(x – 1)

$\therefore 0, \frac{1}{2} \frac{2}{3} \frac{3}{4}, \ldots\ldots\ldots \frac{x-1}{x}$

$$\therefore y_x = C \underset{x=1}{\overset{x-1}{\pi}} f(x) = c\, f(0)\,.\, f(1)\,.\, f(2) \,.....f(x-1)$$

$$\therefore y_x = C \left(\frac{1}{2}.\frac{2}{3}.\frac{3}{4}...\frac{x}{x-1}, \frac{x-1}{x}\right)$$

$$\therefore y_x = C\left(\frac{1}{x}\right) = cx^{-1}$$

Exercise

1.Solve the equation:

i) $y_{x+1} - y_x = 3^x$

ii) $2y_{x+1} - y_x = x$

iii) $y_{x+1} - y_x = 0$

iv) $3y_{x+1} + 2y_x = 0$ for which $y_1 = 0$,

v) $y_{x+1} - 56y_x = 2^x \,.\, x^2$

10.3 The solution of linear equation with constant coefficients of order more than one:

Let us consider an equation of the order n which of the form.

$$y_{x+n} + a_{n-1}\ y_{x+n-1} + a_{n-2}\, y_{x+n-2} + \,......\, + a_0 y_x = x$$

where a_0, a_1...... a_{n-1} are constants and x, a function of x.

From the definition of the operator E, we get $E(y_x) = y_{x+1}$

$$E^2(y_x) = y_{x+2}$$
$$\vdots$$
$$E^n y_x = y_{x+n}$$

Hence the equation (1) becomes

$$(E^n y_x + a_{n-1}\ E^{n-1}\, y_x + \,.......+ a_1 E + a_0\,)y_x = x$$

$$\text{(ie) } f(E) = x$$

where f(E) is a rational integral function of E.

We shall first discuss the methods of determining the complementary function and then the particular integral. The complementary function of (1) is the general solution of the homogeneous function

$$f(E)\,y_x = 0$$

$$\text{(ie) } (E^n + a_{n-1} E^{n-1} + \ldots\ldots + a_1 E + a_0)\, y_x = 0 \qquad (2)$$

Let m^x is a trial solution of this equation

We have

$$E(m^x) = m^{x+1},\ E^2(m^x) = m^{x+2}, \ldots\ldots\ E^n(m^x) = m^{x+n}$$

Hence

$$m^{x+n} + a_{n-1}\, m^{x+n-1} + \ldots\ldots\ldots\ a_1\, m^{x+n} + a_0 m^x = 0.$$

$$\text{(ie) } m^x\,(m^x + a_{n-1}\, m^{n-1} + \ldots\ldots\ldots\ a_1 m + a_0) = 0.$$

$\therefore$ If m^x is a solution of the equation it is necessary that

$$m^n + a_{n-1}\, m^{n-1} + \ldots\ldots\ldots + a_1 m + a_0 = 0.$$

Since m^x is not zero for any finite values of x.

If $m_1, m_2, m_3 \ldots\ldots m_n$ are n distinct roots of (3) then $m_1^x,\ m_2^x,\ m_3^x \ldots\ldots\ m_n^x$ are particular solutions of (2) and hence the general solution of (2) is

$$y_x = c_1 m_1^x + c_2 m_2^x + \ldots\ldots\ldots + c_n m_n^x$$

Equation (3) is called, the auxiliary equations are distinct and if they are $m_1, m_2 \ldots\ldots m_n$, then equation (2) can be written as

$$(E - m_1)\ (E - m_2) \ldots\ldots\ ((E - m_n)\, y_x = 0$$

If the auxiliary equation has multiple roots say $m_1 = m_2$, then the general solution becomes,

$$y_x = C_1 m_1{}^x + C_2 m_1{}^x + \ldots\ldots + C_n m_n{}^x$$

$$= (c_1 + c_2)\, m_1^x + .\ \ldots\ldots + c_n m_n^x$$

$$y_x = A_1 m_1^x + c_3 m_3^x + \ldots\ldots\ldots + c_n m_n^x$$

The solution contains only (n – 1) arbitrary constant and hence is solution not general.

In this case the difference equation becomes

$$(E - m_1)^2\, (E - m_3) \ldots\ldots (E - m_n)\, y_x = 0$$

Hence we have to find the particular solution corresponding to

$(E - m_1)^2 y_x = 0$

(ie) $(E - m_1)(E - m_1)y_x = 0$

Let $(E - m_1)y_x$ be U_x

Then $(E - m_1)u_x = 0$

$$\therefore u_x = c m_1^x$$

Hence $(E - m_1)y_x = c m_1^x$

$$y_x = c_1 m_1^x + m_1^x \Delta^{-1}\left(\frac{cm_1^x}{m_1^{x+1}}\right)$$

$$y_x = c_1 m_1^x + m_1^x \Delta^{-1}\left(\frac{c}{m_1}\right)$$

$$= c_1 m_1^x + m_1^x \frac{c}{m_1}\Delta^{-1} \quad (1)$$

$$= c_1 m_1^x + m_1^x \frac{c}{m_1}x.$$

$$= \left(c_1 + \frac{cx}{m1}\right) m_1^x$$

Hence the general solution of the equation is

$y_x = (c_1 + c_2x)\, m_1^x + c_3 m_3^x + \ldots\ldots.\ c_n m_n^x$

Similarly it can be shown that if the auxiliary equation has k equal roots m_1, then the general solution of the equation is

$$y_x = (c_1 + c_2x + c_3x^2 + \ldots..+c_kx^{k-1})\, m_1^x + c_{k+1}\, m^x{}_{k+1} + \ldots..+ c_n m_n^x$$

Suppose the auxiliary equation has imaginary and real roots. Imaginary roots occur in pairs. Suppose one pairs of imaginary roots is $A \pm iB$. Corresponding to these roots the terms in the complementary function are

$c_1(A+iB)^x + c_2(A - iB)^x$

A + iB can be expressed in the modulus amplitude form as $r(\cos\theta + i\sin\theta)$ where $r^2 = A^2 + B^2$ & $\theta = \tan^{-1}\left(\frac{B}{A}\right)$

$$\therefore A - iB = r(\cos\theta - i\sin\theta)$$

Hence the expression (5) becomes

$$c_1\{r(\cos\theta + i\sin\theta)\}^x + C_2\{r(\cos\theta - I\sin\theta)\}^x$$

(ie) $c_1 r^x(\cos x\theta + i\sin x\theta) + c_2 r^x(\cos x\theta - i\sin x\theta)$

(ie) $r^x\{(c_1 + c_2)\cos x\theta + i(c_1 - c_2)\sin x\theta\}$

(ie) of the form $r^x(A\cos x\theta + B\sin x\theta)$

where A and B are arbitrary constant.

Example:1

Solve the equation $y_{x+2} - 3y_{x+1} + 2y_x = 0$

Solution:

The equation can be written in the form

$(E^2 - 3E + 2)y_x = 0$

The auxiliary equation is

$m^2 - 3m + 2 = 0$

(ie) $(m - 1)(m - 2) = 0$

(ie) $m = 1$ or 2

Hence the general solution of the equation is

$y_x = c_1 1^x + c_2 2^x$

$\therefore y_x = c_1 + c_2 2^x$

Example: 2

Solve the equation $y_{x+2} - 4y_{x+1} + 4y_x = 0$

Solution:

The equation can be written in the form

$(E^2 - 4E + 4)y_x = 0$

(ie) $(E - 2)^2 y_x = 0$

Hence the auxiliary equation is $(m - 2)^2 = 0$

The roots are 2, 2,

∴ The general solution of the equation is

$$y_x = (A+Bx)2^x$$

Example: 3

Solve the equation $y_{x+2} - 2y_{x+1} + 2y_x = 0$.

Solution:

This equation can be put in the form

$$(E^2 - 2E + 2)\, y_x = 0$$

Hence the auxiliary equation is

$$m^2 - 2m + 2 = 0 \qquad \left[m = \frac{+2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} \right]$$

The roots of this equation are 1 + i and 1 − i.

∴ We have $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Hence the general solution of the equation is

$$y_x = \left(\sqrt{2}\right)^x \left(A\cos\frac{\pi x}{4} + B\sin\frac{\pi x}{4}\right)$$

Particular Solutions of the complete equations:

Let us find the general solution of the complete equation.

$$y_{x+n} + a_1\, y_{x+n-} + \ldots\ldots\ldots a_n\, y_x = x. \qquad (1)$$

Having already found the general solution of the corresponding homogeneous equation, it we add to it any particular solution of the equation (1) the sum will be the general solution of the complete equation.

A number of special techniques exist for finding particular solution of the equation. (1)

We have written the general linear difference equation with constant coefficients in the form $f(E)\, y_x = x$.

The particular integral to this equation is

$y_x = \frac{1}{f(E)} X$, where we define the right-hand member to the expression which when operated upon by f (E), produces x.

Some Useful Results:

1. If f (E) is a polynomial in E, then $f(E)\, m^x = m^x f(m)$.

Let $f(E) = a_0 + a_1 E + a_2 E^2 + \ldots + a_n E^n$

$$f(E)\, m^x = (a_0 + a_1 E + a_2 E^2 + \ldots + a_n E^n)\, m^x$$

$$= a_0 m^x + a_1 m^{x+1} + a_2 m^{x+2} + \ldots + a_n m^{x+n}$$

$$= m^x (a_0 + a_1 m + a_2 m^2 + \ldots + a_n m^n)$$

$$f(E)\, m^x = m^x f(m)$$

Using this result we can show that

$$\frac{1}{f(E)} m^x = \frac{1}{f(m)} m^x \text{ if } f(m) \neq 0.$$

If f (m) = 0, Let m, be a repeated root of order k.

Then $f(m) = (m - m_1)^k \phi(m)$.

In that case $f(E) = (E - m_1)^k \phi(E)$

$$\text{Hence } \frac{1}{f(E)} m^x = \frac{1}{(E - m_1)^k \phi(E)} m_1^x .= \frac{1}{\phi(m_1)(E - m_1)^k} m_1^x$$

Let us first find $\frac{1}{(E - m_1)} m_1^x$.

$$(E - m_1)\, x\, m_1^{x-1} = E(x\, m_1^{x-1}) - m_1\, x\, m_1^{x-1}$$

$$= E(x\, m_1^{x-1}) - x\, m_1^x$$

$$= (x + 1)\, m_1^x - x\, m_1^x$$

$$= m_1^x$$

$$\therefore \frac{1}{E - m_1} m_1^x = x\, m_1^x$$

We have $(E - m_1)^2 x (x - 1) m_1^{x-2}$

$$= (E - m_1)(E - m_1) x (x - 1) m_1^{x-2}$$
$$= (E - m_1) \left[(x+1) x m_1^{x-1} - x (x-1) m_1^{x-1}\right]$$

$$= (E - m_1) \left[2 x m_1^{x-1}\right]$$

$$= 2 \left[(x+1)\left(m_1^x - m_1 x m_1^{x-1}\right)\right] = 2 \left[(x+1) m_1^x - x m_1^x\right]$$

$$= 2 m_1^x.$$

Hence $\frac{1}{(E - m_1)^2} m_1^x = \frac{x(x-1)}{2} m_1^{x-2}$

$$= \frac{x^{(2)}}{2!} m_1^{x-2}$$

Similarly we can extend this result and show that

$$\frac{1}{(E - m_1)^k} m_1^x = \frac{x^{(k)}}{k!} m_1^{x-k}$$

$$\therefore \frac{1}{f(E)} m_1^x = \frac{1}{(E - m_1)^x \phi(m)} m_1^x$$

$$= \frac{1}{\phi(m_1)} \cdot \frac{x^{(k)}}{k!} m_1^{x-k}$$

2. If f (E) is a polynomial in E and F (x) is a function of x, then

$$f(E) m^x F(x) = m^x f(mE) F(x)$$
$$f(E) [m^x F(x)] = (a_0 + a_1 E + a_2 E^2 + \ldots + a_n E^n) \left[m^x F(x)\right]$$

$$= a_0 m^x F(x) + a_1 E [m^x F(x)] + a_2 E^2 \left[m^x F(x)\right] + \ldots\ldots + a_n E^n \left[m^x F(x)\right]$$

$$= a_0 m^x F(x) + a_1 m^{x+1} F(x+1) + a_2 m^{x+2} F(x+2) + \ldots + a_n m^{x+n} F(x+n)$$

$$= m^x\ [a_0 F(x) + a_1 m\ F(x+1) + a_2 m^2 F(x+2) + \ldots\ldots + a_n\ m^n\ F(x+n)]$$

$$= m^x\ [a_0 F(x) + a_1 m\ E\ F(x) + a_2 m^2 E^2 F(x) + \ldots\ldots + a_n\ m^n\ E^n\ F(x)]$$

$$f(E)\ [m^x F(x)] = m^x\ [a_0 + a_1\ m E + a_2\ m^2 E^2 + \ldots\ldots + a_n m^n E^n\ ] F(x)$$

$$= m^x\ f(m\ E)\ F(x).$$

Hence if f (x) is a polynomial in x,

$$\frac{1}{f(E)}\ m^x\ F(x) = m^x\ \frac{1}{f(mE)}\ F(x)$$

3. When x is a polynomial in x to find $\frac{1}{f(E)}$ x, $\frac{1}{f(E)}\ x = \frac{1}{f(1+\Delta)}$ x. $\frac{1}{f(1+\Delta)}$ can be expanded in ascending powers of Δ in the form $b_0 + b_1\Delta + b_2\Delta^2 + \ldots\ldots$

Then $\frac{1}{f(1+\Delta)}\ x = (b_0 + b_1\Delta + b_2\Delta^2 + \ldots\ldots)\ x$

$$= b_0\ x + b_1\Delta\ x + b_2\Delta^2 x + \ldots\ldots$$

Example: 1

Solve the equation $y_{x+2} - 7\ y_{x+1} + 12\ y_x = 2^x$.

Solution:

The equation can be written in the form

$(E^2 - 7\ E + 12)\ y_x = 2^x$.

The auxiliary equation $m^2 - 7\ m + 12 = 0$

(i.e.) $(m - 3)\ (m - 4) = 0$

The roots of the equation are 3 and 4.

Hence the complementary function is

$y_x = C_1\ 3^x + C_2\ 4^x$.

$$\text{Particular integral} = \frac{1}{E^2 - 7E + 12} 2^x$$

$$= \frac{1}{2^2 - 7(2) + 12} 2^x$$

$$= \frac{2^x}{2} = 2^{x-1}$$

∴ Hence the general solution is

$$y_x = C_1 3^x + C_2 4^x + 2^{x-1}$$

Example: 2

Solve the equation $y_{x+2} - 6 y_{x+1} + 8 y_x = 2^x$.

Solution:

The equation can be written in the form

$$(E^2 - 6E + 8) y_x = 2^x.$$

The auxiliary equation is $m^2 - 6m + 8 = 0$

The roots of this auxiliary equation are 2 and 4. Hence the complementary function is $C_1 2^x + C_2 4^x$.

$$\text{Particular Integral} = \frac{1}{E^2 - 6E + 8} 2^x$$

$$= \frac{1}{(E-2)(E-4)} 2^x$$

$$= \frac{-1}{2} \cdot \frac{1}{E-2} 2^x$$

$$= \frac{-1}{2} \cdot x_2^{x-1}$$

$$= -x. 2^{x-2}$$

Hence the general solution is

$$y_x = C_1 2^x + C_2 4^x - x_2^{x-2}$$

Example: 3

Solve the equation $y_{x+2} - 6\,y_{x+1} + 9\,y_x = 3^x$

Solution:

The equation can be written in the form of $(E^2 - 6E + 9)\,y_x = 3^x$.

The auxiliary equation is $m^2 - 6m + 9 = 0$

The roots of this equation are 3, 3.

Hence $C.F = (C_1 + C_2 x)\,3^x$

$$P.I = \frac{1}{(E^2 - 6E + 9)}\,3^x$$

$$= \frac{1}{(E-3)^2}\,3^x$$

$$= \frac{x^{(2)}\,3^{x-2}}{2!} = \frac{x(x-1)}{2}\,3^{x-2}$$

Hence the general solution is

$$y_x = (C_1 + C_2 x)\,3^x + \frac{x(x-1)}{2}\,3^{x-2}$$

Example: 4

Solve the equation $y_{x+2} - 5\,y_{x+1} + 6\,y_x = x^2 + x + 1$

Solution:

The equation can be written in the form.

$$(E^2 - 5E + 6)\,y_x = x^2 + x + 1$$

Its auxiliary equation is $m^2 - 5m + 6 = 0$.

The roots are 3, 2

Hence $C.F. = C_1\,3^x + C_2\,2^x$

$$P.I. = \frac{1}{(E^2 - 5E + 6)} \; (x^2 + x + 1)$$

$$= \frac{1}{(\Delta + 1)^2 - 5(\Delta + 1) + 6} \; (x^2 + x + 1)$$

$$= \frac{1}{\Delta^2 + 2\Delta + 1 - 5\Delta - 5 + 6} \; (x^2 + x + 1)$$

$$= \frac{1}{\Delta^2 - 3D + 2} \; (x^2 + x + 1)$$

$$= \frac{1}{(1 - \Delta)(2 - \Delta)} \; (x^2 + x + 1)$$

$$= \frac{(1 - \Delta)^{-1} \left(1 - \frac{\Delta}{2}\right)^{-1}}{2} \; (x^2 + x + 1)$$

$$P.I = \frac{(1 + \Delta + \Delta^2 + \ldots)\left(1 + \frac{\Delta}{2} + \frac{\Delta^2}{4} + \ldots\right)(x^2 + x + 1)}{2}$$

$$= \frac{1}{2}\left(1 + \frac{\Delta}{2} + \frac{\Delta^2}{4} + \ldots + \Delta + \frac{\Delta^2}{2} + \frac{\Delta^3}{4} + \ldots\ldots + \Delta^2 + \frac{\Delta^3}{2} + \frac{\Delta^4}{4} + \ldots\right) x^2 + x + 1$$

$$= \left(\frac{1}{2} + \frac{3\Delta}{4} + \frac{7\Delta^2}{8}\right)(x^2 + x + 1)$$

$$P.I = \frac{1}{2}\;(x^2 + x + 1) + \frac{3}{4}\,\Delta\;(x^2 + x + 1) + \frac{7}{8}\;\Delta^2\;(x^2 + x + 1)$$

We have, $\Delta\; x = (x + 1) - x = 1$

$$\Delta\;(x^2) = (x+1)^2 - x^2 = x^2 + 2x + 1 - x^2$$

$$= 2x + 1$$

$$\Delta^2\,(x) = 0$$

$$\Delta^2 (x^2) = 2 (x + 1) + 1 - (2 x + 1)$$

$$= 2 x + 2 + 1 - 2 x - 1 = 2$$

Hence $\quad P.I = \frac{1}{2} (x^2 + x + 1) + \frac{3}{4} (2x + 1 + 1) + \frac{7}{8} \qquad (2)$

$$= \frac{1}{2} x^2 + \frac{1}{2} x + \frac{1}{2} + \frac{6}{4} x + \frac{6}{4} + \frac{14}{8}$$

$$P.I = \frac{1}{2} x^2 + 2 x + \frac{15}{4}$$

Hence the general solution is

$$\therefore \qquad y_x = C_1 3^x + C_2 2^x + \frac{1}{2} x^2 + 2 x + \frac{15}{4}$$

Aliter:

By the method of undetermined coefficients also, the particular integral can be determined.

Let $a x^2 + b x + c$ be a particular solution of the equation

$(E^2 - 5 E + 6) y_x = x^2 + x + 1.$

Then $(E^2 - 5 E + 6) (a x^2 + 6 x + c) = x^2 + x + 1$

(i.e) $a (x + 2)^2 + b (x+2) + c - 5 [a (x+1)^2 + b (x+1) + c] + b (ax^2 + bx + c) = x^2 + x + 1$

Equating the coefficients of x^2, x and constant terms on both side.

$$a (x^2 + 2x + 4) + b (x+2) + c - 5 (a (x^2 + 2x + 1) + b (x+1) + c) + 6 (ax^2 + bx + c)$$
$$= x^2 + x + 1$$

$$ax^2 + 2xa + 4a + bx + 2b + c - 5 ax^2 - 10 ax - 5a - 5b x - 5b - 5c + 6 ax^2 + 6bx + 6c$$
$$= x^2 + x + 1$$

$$x^2 (a - 5a + 6a) + x (2a + b - 10 a - 5b + 6b) + 4 + 2b + c - 5a - 5c + 6$$
$$= x^2 + x + 1$$

Equating the coefficients x^2, x & constant.

$$2a = 1 \quad \Rightarrow a = \tfrac{1}{2}$$

$$-8a + 2b = 1 \quad \Rightarrow -8\left(\tfrac{1}{2}\right) + 2b = 1$$

$$-4 + 2b = 1$$

$$2b = 1 + 4$$

$$b = \frac{5}{2}$$

$$-5a + 2b - 4c + 10 = 1$$

$$-5a + 2b - 4c = -9$$

$$-5\left(\tfrac{1}{2}\right) + 2\left(\frac{5}{2}\right) - 4c = -9$$

$$-\frac{5}{2} + \frac{10}{2} - 4c = -9$$

$$\frac{5}{2} - 4c = -9$$

$$-4c = -9 - \frac{5}{2}$$

$$-4c = \frac{-18-5}{2}$$

$$-4c = \frac{-23}{2} = \frac{-23}{2}$$

Example: 5

Show that n straight lines, no two which are parallel and no three of which meet in a point, divide a plane into $\tfrac{1}{2}$ (n^2+n+2) parts.

Under these conditions, let the number of compartments formed by the n straight lines by y_n.

Draw the $(n+1)^{th}$ line in the plane.

It will meet each of the n lines once in `n' points and dissect (n+1) previously existing compartments and thus add (n+1) compartments more.

Hence $y_{n+1} = y_n + n + 1$

When n = 1, the number of compartments is 2.

Hence $y_1 = 2$.

So we have to solve the equation

$$y_{n+1} = y_n + n + 1$$

Subject to $y_1 = 2$.

The equation can be written in the form.

$$(E - 1)\ y_n = n + 1$$

Hence its C.F = $A\ (1)^n$

$$P.I = \frac{1}{E-1}\ (n+1)$$

$$= \frac{1}{\Delta}\ (n+1) = \Delta^{-1}\ (n+1)$$

$$= \frac{n\,(2)}{2} + n = \frac{n(n-1)}{2} + n$$

$$y_n = A + \frac{n(n-1)}{2} + n$$

when $n = 1$, $y_1 = 2$

$$\therefore A = 1$$

$$\therefore y_n = \frac{1}{2}\ (n^2 + n + 2)$$

Exercise:

1. $y_{x+2} - 2\,y_{x+1} + 4y_x = 0$
2. $y_{x+2} + y_x = 0$
3. $y_{x+2} - 3\,y_{x+1} - 2\,y_x = 1$
4. $y_{x+2} + 5\,y_{x+1} + 6\,y_x = 4^x$
5. $6y_{x+2} + 5\,y_{x+1} - 6\,y_x = 2^x$
6. $y_{x+2} + 2\,y_{x+1} + y_x = (9)\ 2^x$, given that $y_0 = 2 = \frac{1}{2}\ y_1$
7. $y_{x+2} - 7\,y_{x+1} - 8\,y_x = x\ (x - 1)\ 2^x$.