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MATHEMATICS

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PAPER - VI

REAL AND COMPLEX ANALYSIS

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REAL AND COMPLEX ANALYSIS

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Unit - 2 Completeness - definition and example - Cantor's intersection theorem - Baire's category theorem

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Unit - 5 Compactness - definition and example - compact subsets of R-equivalent characterization for Compactness - Continuity and Compactness.

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Reference Books

1. Modern Analysis : Arumugam and Issac
2. Complex Analysis : S. Narayanan and T.K.Manickavasagam Pillai

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REAL ANALYSIS

Lesson Compiled by

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INTRODUCTION

The concepts of sets and functions are indispensable to almost all branches of pure mathematics. The usual material of elementary set theory is so current that we take it for granted.

i) A is a subset of B written as $A \subseteq B$

ii) Union of two sets A and B written as $A \cup B$

iii) Intersection of two sets A and B writtenas $A \cap B$

iv) Complement of subset of A of X written as A^C

v) Difference of two sets A and B written as $A - B$

vi) Cartesian product of two sets A and B written as $f : A \times B$

vii) A function f from a set A to a set B written as $f : A \rightarrow B$

viii) The empty set which contains no element is denoted by ϕ.

Certain letters are reserved to denote particular sets which occur often. They are:

N, the set of all natural numbers

Z, the set of all integers

Q, the set of al rational numbers

Q^+, the set of all positive rational numbers

R, the set of all real numbers

R^n, the set of all ordered n-tuples $(x_1, x_2, \ldots\ldots, x_n)$ of real numbers

C, the set of all complex numbers

C^n, the set of all ordered n-tuples $(z_1, z_2, \ldots\ldots, z_n)$ of complex numbers

The concept of union and intersection can be extended to any collection of sets.

Let I be a nonempty set. For each $i \in I$, let A_i be a set. Then we say that $\{A_i / i \in I\}$ is a family of sets indexed by the set I.

We define $\bigcup_{i \in I} A_i = \{x / x \in A_i \text{ for at least one } i \in I\}$

and $\bigcap_{i \in I} A_i = \{x / x \in A_i \text{ for all } i \in I\}$

Example :

For each $i \in N$, let $A_i = \{i, i+1, \ldots, i+n, \ldots\}$

$\therefore A_1 = \{1, 2, \ldots\}$; $A_2 = \{2, 3, \ldots\}$; .....

Then $\{A_i / i \in N\}$ is a family of sets indexed by N.

Here $\bigcup_{i \in N} A_i = \{1, 2, \ldots, n, \ldots\} = N$ and

$\bigcap_{i \in N} A_i = \phi$.

Note 1 :

$\bigcup_{i \in N} A_i$ is also written as $\bigcup_{i=1}^{\infty} A_i$ and

$\bigcap_{i \in N} A_i$ as $\bigcap_{i=1}^{\infty} A_i$.

Note 2 :

The distributive laws for union and intersection and De Morgan's laws for finite number of sets can be generalised to any collection of sets as follows.

i) $\left(\bigcup_{i \in I} A_i\right)^C = \bigcap_{i \in I} A_i^C$

ii) $\left(\bigcap_{i \in I} A_i\right)^C = \bigcup_{i \in I} A_i^C$

iii) $A \cap \left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} (A \cap B_i)$

iv) $A \cup \left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} (A \cup B_i)$

Intervals in R :

Let a, b $\in$ R and a<b. Then

i) (a, b) = $\{x/x \in R$ and $a<x<b\}$ is called the **open interval** with a and b as end points.

ii) [a, b] = $\{x/x \in R$ and $a \leq x \leq b\}$ is called the **closed interval** with a and b as end points.

iii) (a, b] = $\{x/x \in R$ and $a<x \leq b\}$ is called the **open-closed interval** with a and b as end pionts.

iv) [a, b) = $\{x/x \in R$ and $a \leq x<b\}$ is called the **closed-open interval** with a and b as end points.

v) $[a, \infty) = \{x/x \in R$ and $x \geq a\}$

vi) $(a, \infty) = \{x/x \in R$ and $x>a\}$

vii) $(-\infty, a] = \{x/x \in R$ and $x \leq a\}$

viii) $(-\infty, a) = \{x/x \in R$ and $x<a\}$

ix) $(-\infty, \infty) = R$

Any subset of R which is one of the above forms is called an **interval.** Any **interval** of the form (i), (ii), (iii) or (iv) is called a **finite interval** or **bounded interval** and any interval of the form (v), (vi), (vii), (viii) or (ix) is called an **infinite interval** or an **unbounded interval.**

The singleton set {a} is considered to be a degenerate closed interval [a, a].

COUNTABLE SETS

If a set A is finite then we can actually count the number of elements in this set. In other words we can label the elements of A by using the natural numbers 1, 2,....,n for some n and the number of elements in this set A is n.

In this case there exists a bijection f from A onto the set {1, 2,.....,n} and hence if A and B are two finite sets having the same number of elements, then there exists a bijection from A to B.

Definition :

Two sets A and B are said t be equivalent if there exists a bijection f from A to B.

Note :

Two finite sets A and B are equivalent iff they have the same number of elements. Hence a finite set cannot be equivalent to a proper subset of itself. However an infinite set can be equivalent to a proper subset.

Example 1 :

Let A = N and B = {2, 4, 6,.....,2n,.....}

Then $f : A \to B$ defined by $f(n) = 2n$ is a bijection. Hence A is equivalent to B eventhough A has actually `more' elements than B.

Example 2 :

N is equivalent to Z

The function $f : N \to Z$ defined by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if n is even} \\ \frac{1-n}{2} & \text{if n is odd is a bijection} \end{cases}$$

Hence N is equivalent to Z.

Definition :

A set A is said to be **countably infinite** if A is equivalent to the set of natural numbers N.

A is said to be **countable** if it is finite or countably infinite.

Note :

Let A be a countably infinite set. Then there is a bijection f from N to A.

Let $f(1) = a_1$, $f(2) = a_2$, ..., $f(n) = a_n$,....

Then $A = \{a_1, a_2, \ldots\ldots, a_n, \ldots\}$

Thus all the elements of A can be labelled by using the elements of N.

Example 1 :

{2, 4, 6,.....,2n,......} is a countable set.

Example 2 :

Z is countable.

Example 3 :

Let $A = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\ldots\ldots\right\}$

The function $f : N \rightarrow A$ defined by $f(n) = \frac{n}{n+1}$ is a bijection.

Hence A is countable.

Theorem 1 :

A subset of a countable set is countable.

Proof :

Let A be a countable set and let $B \subseteq A$. If A or B is finite, then obviously B is countable. Hence let A and B be both infinite.

Since A is countably infinite, we can write $A = \{a_1, a_2, \ldots\ldots, a_n, \ldots..\}$. Let a_{n_1} be the first element in A such that $a_{n_1} \in B$. Let a_{n_2} be the first element in A which follows a_{n_1} such that $a_{n_2} \in B$.

Proceeding like this we get $B = \{a_{n_1}, a_{n_2}, \ldots..\}$. Thus all the elements of B canbe labelled by using the elements of N. Hence B is countable.

Theorem 2 :

Q^+ is countable.

Proof :

Take all positive rational numbers whose numerator and denominator add up to 2. We have only one number namely $\frac{1}{1}$.

Next we take all positive rational numbers whose numerator and denominator add up to 3.

We have $\frac{1}{2}$ and $\frac{2}{1}$.

Next we take all positive rational numbers whose numerator and denominator add up to 4.

We have $\frac{3}{1}, \frac{2}{2}$ and $\frac{1}{3}$.

Proceeding like this, we can list all the positive rational numbers together from the beginning omitting those which are already listed.

Thus we obtain the set $\left\{1, \frac{1}{2}, 2, 3, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, 4, \ldots\ldots\right\}$. This let contains every positive rational number each occuring exactly once.

Thus Q^+ is countable.

Theorem 3 :

Q is countable.

Proof :

We know that Q^+ is countable.

Let $Q^+ = \{r_1, r_2, \ldots\ldots\ldots, r_n, \ldots\ldots\}$

$\therefore$ $Q = \{0, \pm r_1, \pm r_2, \ldots\ldots\ldots, \pm r_n, \ldots\ldots\ldots\}$

Let $f : N \to Q$ be defined by

$f(1) = 0$, $f(2n) = r_n$ and $f(2n+1) = -r_n$.

Clearly f is a bijection and hence Q is countable.

Theorem 4 :

$N \times N$ is countable.

Proof :

$N \times N = \{(a, b) / a, b \in N\}$

Take all ordered pairs (a, b) such taht a+b = 2. There is only one such pair namely (1, 1).

Next take all ordered pairs (a, b) such that a+b = 3. We have (1, 2) and (2, 1)

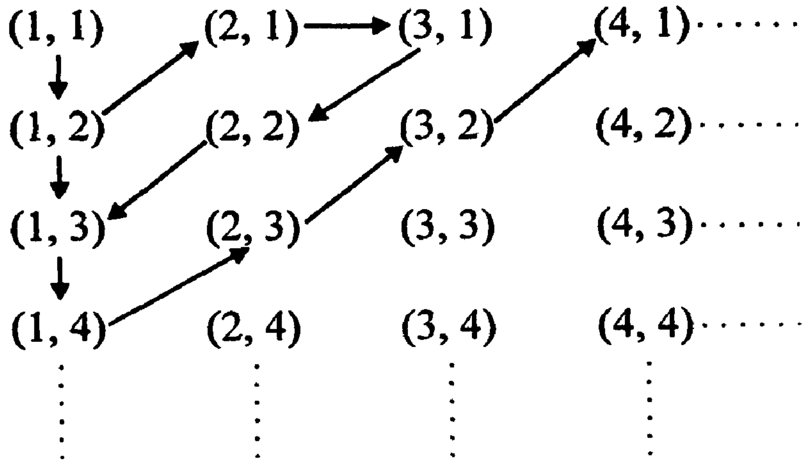
Next take all ordered pairs (a, b) such that a+b = 4. We have (3, 1), (2, 2) and (1, 3).

Proceeding like this and listing all the ordered pairs together from the beginning, we get the set {(1, 1), (1, 2), (2, 1), (3, 1), (2, 2), (1, 3),......}. This set contains every ordered pair belogning to N×N exactly once

Thus N×N is countable.

Note :

The above process of arranging the elements of N×N as a sequence can be represented by means of a diagram. This process is known as Cantor's diagonalisation process.



Theorem 5 :

If A and B are countable sets then A×B is also countable.

Proof :

We assume that A and B are countably infinite.

Let $A = \{a_1, a_2, \ldots\ldots, a_n, \ldots\ldots\}$

$B = \{b_1, b_2, \ldots\ldots, b_n, \ldots\ldots\}$

Define $f : N \times N \to A \times B$ by $f(i, j) = (a_i, b_j)$

We claim that f is a bijection.

Suppose $x = (p, q) \in N \times N$ and $y = (u, v) \in N \times N$

$$f(x) = f(y) \Rightarrow (a_p, b_q) = (a_u, b_v)$$
$$\Rightarrow a_p = a_u, b_q = b_v$$
$$\Rightarrow p = u \text{ and } q = v$$
$$\Rightarrow (p, q) = (u, v)$$
$$\Rightarrow x = y$$

∴ f is 1–1.

Now, suppoe $(am, an) \in A \times B$.

Then $(m, n) \in N \times N$ and $f(m, n) = (a_m, a_n)$.

∴ f is onto. Hence f is a bijection.

Hence A×B is equivalent to N×N which is countable.

Hence A×B is countable.

Theorem 6 :

Let A be a countably infinite set and f be a mapping of A onto a set B. Then B is countable.

Proof :

Let A be a countably infinite set and $f: A \to B$ be an onto map. Let $b \in B$. Since f is onto, there exists at least one pre-image for b. Choose one element $a \in A$ such that $f(a)=b$.

Define $g : B \to A$ by $g(b) = a$.

Clearly g is 1–1.

$\therefore$ B is equivalent to a subset of the countable set A.

$\therefore$ B is countable (by theorem 1)

Theorem 7 :

Countable union of countable sets is countable.

Proof :

Let $S = \{A_1, A_2, \ldots, A_n, \ldots\}$ be a countable family of countable sets.

Case (i) :

Let each A_i be countably infinite.

Let

$$
\begin{aligned}
A_1 &= \{a_{11}, a_{12}, \ldots, a_{1n}, \ldots\} \\
A_2 &= \{a_{21}, a_{22}, \ldots, a_{2n}, \ldots\} \\
&\vdots \\
A_n &= \{a_{n1}, a_{n2}, \ldots, a_{nn}, \ldots\} \\
&\vdots
\end{aligned}
$$

We define a map $f : N \times N \to \cup A_n$ by $f(i, j)=a_{ij}$. Clearly f is onto.

Also by theorem (4), $N \times N$ is countably infinite.

Hence by theorem (6), $\cup A_n$ is countably infinite.

Case (ii)

Let each A_i be countable.

For each i choose a set B_i such that B_i is a countably infinite set and $A_i \subseteq B_i$.

Then $\cup A_i \subseteq B_i$.

$\cup B_i$ is countable

$\therefore \cup A_i$ is countable (by theorem 1).

Worked Examples :

Example 1 :

Any countably infinite set is equivalent to a proper subset of itself.

Solution :

Let A be a countable infinite set.

Hence $A = \{a_1, a_2, \ldots\ldots, a_n, \ldots\}$

Let $B = \{a_2, a_3, \ldots\ldots, a_n, \ldots\}$

Clearly B is a proper subset of A.

Define a map $f: A \to B$ by $f(a_n) = a_{n+1}$

Clearly f is a bijection. Hence A is equivalent to B.

Example 2 :

Any infinite set is equivalent to a proper subset of itself.

Solution :

Let A be an infinite set.

Choose any element $a_1 \in A$.

Since A is infinite set, we can choose another element $a_2 \in A - \{a_1\}$

Suppose we have chosen $a_1, a_2, \ldots, a_n$ from A.

Since A is infinite, $A - \{a_1, a_2, \ldots, a_n\}$ is also infinite.

$\therefore$ We can choose a_{n+1} from $A - \{a_1, a_2, \ldots, a_n\}$

$B = \{a_1, a_2, \ldots\ldots, a_n, a_{n+1}, \ldots\}$ is countably infinite subset of A.

Clearly $A = (A-B) \cup B$.

Consider the following subset C of A given by

$$C = (A-B) \cup \{a_2, a_3, \ldots, a_n, \ldots\} = A - \{a_1\}$$

Clearly C is a proper subset of A.

Consider the function $f : A \to C$ defined by $f(x) = x$ if $x \in A-B$ and $f(a_n) = a_{n+1}$

Obviously f is a bijection.

Hence A is equivalent to C.

Exercise :

1. Show that N and A = {101, 102, 103,....} are equivalent.
2. Show that for any two sets A and B, the set $A \times B$ is equivalent to the set $B \times A$.
3. Prove that the set of all even integers is countably infinite.

UNCOUNTABLE SETS

Definition :

A set which is not countable is called uncountable.

Theorem 8 :

(0, 1] is uncountable.

Proof :

Every real number in (0, 1] can be written uniquely as a non-terminating decimal $0.a_1a_2,......a_n....$ where $0 \le a_i \le 9$ for each i subject to the following restriction that any terminating decimal $0.a_1a_2........a_n000.......$ is written as $0.a_1a_2....(a_{n-1})999....$

For example 0.54 = 0.53999......

1 = 0.999.....

Suppose (0, 1] is countable.

Then the elements of (0, 1] can be listed as

$\{x_1, x_2, ...x_n, ...\}$ where

$$\begin{aligned} x_1 &= 0.a_{11}a_{12} \ldots\ldots a_{1n} \ldots\ldots \\ x_2 &= 0.a_{21}a_{22} \ldots\ldots a_{2n} \ldots\ldots \\ &\vdots \\ x_n &= 0.a_{n1}a_{n2} \ldots\ldots a_{nn} \ldots\ldots \\ &\vdots \end{aligned}$$

For each positive integer n choose an integer b_n such that $0 \le b_n \le 9$ and $bn \ne 0$ and $b_n \ne a_{nn}$.

Let $y = 0.b_1\ b_2\ b_3$ ......

Clearly $y \in (0, 1]$

Also y is different from each x_i at least in the i^{th} place.

Hence $y \neq x_i$ for each i which is a contradiction.

Hence (0, 1] is uncountable.

Corollary 1 :

Any subset A of R which contains (0, 1] is uncountable.

Proof :

Suppose A is countable.

∴ By theorem (1) any subset of A is countable.

Hence we get (0, 1] is countable which is a contradiction.

∴ A is uncountable.

Corollary 2 :

R is uncountable.

The result follows directly by taking A = R.

Corollary 3 :

The set S of irrational numbers is uncountable.

Proof :

Suppose S is countable.

We know that Q is countable.

∴ $S \cup Q = R$ is countable which is a contradiction (by corollary 1)

∴ S is uncountable.

Exercise :

1. Prove that C is uncountable.
2. Prove that any interval in R which contains more than one point is uncountable.
3. Prove that the set of all irratational numbers lying in the interval (0, 1] is uncountable.

INEQUALITIES OF HOLDER AND MINKOWSKI THEOREM (HOLDER'S INEQUALITY)

If $p > 1$ and q is such that $\frac{1}{p}+\frac{1}{q}=1$ then $\sum_{i=1}^{n} |a_i b_i| \le \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |b_i|^q\right]^{1/q}$

where $a_1, a_2, \ldots\ldots, a_n$ and $b_1, b_2, \ldots\ldots, b_n$ are real numbers.

Proof :

First we shall prove the inequality.

$x^{1/p}\, y^{1/q} \le \frac{x}{p}+\frac{y}{q}$ where $x \ge 0$ and $y \ge 0$.

This inequality is trivial if $x = 0$ or $y = 0$.

Let $x, y > 0$.

$$\text{Consider } f(t) = t^{\lambda}-\lambda t+\lambda-1 \text{ where } \lambda = \frac{1}{p} \text{ and } t \ge 0$$

$$\text{Then } f'(t) = \lambda t^{\lambda-1}-\lambda = \lambda\,(t^{\lambda-1}-1)$$

$$\therefore\ f(t) = f'(1) = 0$$

Also $f'(t) > 0$ for $0 < t < 1$ and $f'(t) < 0$ for $t > 1$.

$\therefore$ $f(t) \le 0$ for all $t \ge 0$ and in particular $f\left(\frac{x}{y}\right) \le 0$.

$$\therefore\ \left(\frac{x}{y}\right)^{\lambda} - \lambda\left(\frac{x}{y}\right)+\lambda-1 \le 0$$

$$\therefore\ \left(\frac{x}{y}\right)^{1/p} - \frac{1}{p}\left(\frac{x}{y}\right)+\frac{1}{p}-1 \le 0$$

Multiplying by y we get $x^{1/p}y^{(1-1/p)} - \frac{x}{p} - \left(1-\frac{1}{p}\right)y \le 0$

$$\therefore\ x^{1/p}y^{(1-1/p)} - \frac{x}{p} - \frac{y}{q} \le 0 \quad \left(\because\ 1-\frac{1}{p}=\frac{1}{q}\right)$$

$$\therefore \quad x^{1/p}y^{1/q} \leq \frac{x}{p} + \frac{y}{q}$$

To prove Holder's inequality, we apply the above inequality to the numbers

$$x_j = \frac{|a_j|^p}{\sum_{i=1}^{n} |a_i|^p}; \; y_j = \frac{|b_j|^q}{\sum_{i=1}^{n} |b_i|^q} \text{ for each } j = 1, 2, \ldots\ldots, n.$$

We get $$\frac{|a_j|\,|b_j|}{\left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |b_i|^q\right]^{1/q}} \leq \frac{x_j}{p} + \frac{y_j}{q} \text{ for all } j = 1, 2, \ldots\ldots, n.$$

Adding these n inequalities we get

$$\frac{\sum_{i=1}^{n} |a_i||b_i|}{\left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |b_i|^q\right]^{1/q}} \leq \sum_{j=1}^{n} \left(\frac{x_j}{p} + \frac{y_j}{q}\right) \qquad \text{------(1)}$$

Now $$\sum_{j=1}^{n} \left(\frac{x_j}{p} + \frac{y_j}{q}\right) = \frac{1}{p}\sum_{j=1}^{n} x_j + \frac{1}{q}\sum_{j=1}^{n} y_j$$

$$= \frac{1}{p} + \frac{1}{q} \left(\text{since } \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j = 1\right)$$

$$= 1$$

Using this in (1) we get

$$\sum_{i=1}^{n} |a_i||b_i| \leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \cdot \left[\sum_{i=1}^{n} |b_i|^q\right]^{1/q}$$

$$\therefore \quad \sum_{i=1}^{n} |a_i b_i| \leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |b_i|^q\right]^{1/q}$$

Note :

If we put p = 2 = q in Holder's inequality we get the following inequality which is known as **Cauchy-Schurarz inequality.**

Theorem 10 : (Minkowski's nequality)

If $p \geq 1$, $\left[\sum_{i=1}^{n} |a_i + b_i|^p\right]^{1/p} \leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} + \left[\sum_{i=1}^{n} |b_i|^p\right]^{1/p}$ where $a_1, a_2, \ldots\ldots, a_n$ and $b_1, b_2, \ldots\ldots, b_n$ are real numbers.

Proof :

This inequality is trivial when p = 1. Let p>1.

Clearly, $$\left[\sum_{i=1}^{n} |a_i + b_i|^p\right]^{1/p} \leq \left[\sum_{i=1}^{n} (|a_i| + |b_i|)^p\right]^{1/p} \qquad \text{-------(1)}$$

$$\sum_{i=1}^{n} [|a_i| + |b_i|]^p = \sum_{i=1}^{n} [|a_i| + |b_i|]^{p-1} (|a_i| + |b_i|)$$

$$= \sum_{i=1}^{n} |a_i|(|a_i| + |b_i|)^{p-1} + \sum_{i=1}^{n} |b_i|(|a_i| + |b_i|)^{p-1}$$

$$\leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} (|a_i| + |b_i|)^{(p-1)q}\right]^{1/q}$$

$$+ \left[\sum_{i=1}^{n} |b_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} (|a_i| + |b_i|)^{(p-1)q}\right]^{1/q}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ (using Holder's inequality).

Since $\frac{1}{p} + \frac{1}{q} = 1$ we have p+q = pq

Hence (p–1)q = p

Dividing by $\left[\sum_{i=1}^{n} (|a_i| + |b_i|)^p\right]^{1/q}$ we get

$$\left[\sum_{i=1}^{n} (|a_i| + |b_i|)^p\right]^{1-1/q} \leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} + \left[\sum_{i=1}^{n} |b_i|^p\right]^{1/p}$$

$$\left[\sum_{i=1}^{n} (|a_i| + |b_i|)^p\right]^{1/p} \leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} + \left[\sum_{i=1}^{n} |b_i|^p\right]^{1/p} \qquad \text{------(2)}$$

From (1) and (2) we get the required inequality.

METRIC SPACE

The concept of convergence of sequences of real numbers depends on the absolute value of the difference between any two real numbers. We observe that this absolute value is nothing but the distance between the two numbers when they are considered as points on the real line. For the study of the concepts like continuity and convergence the algebraic properties of R are irrelevent. A set equipped with a reasonable concept of distance is called a **metric space.**

Definition :

A **metric space** is a non empty set M together with a function $d: M \times M \to R$ satisfying the following conditions.

i) $d(x, y) \geq 0$ for all $x, y \in M$

ii) $d(x, y) = 0$ iff $x = y$

iii) $d(x, y) = d(y, x)$ for all $x, y \in M$

iv) $d(x, z) \leq d(x, y)+d(y, z)$ for all $x, y, z \in M$ **(triangle inequality)**

d is called a **metric** or **distance function** and d(x, y) is called the **distance** between x and y.

Note :

The metric space M with the metric d is denoted by (M, d) or simply by M.

Example 1 :

In R we define $d(x, y) = |x-y|$. Then d is a metric on R. This is called the **usual metric** on R.

Proof :

Clearly $d(x, y) = |x-y| \geq 0$

Also $d(x, y) = 0 \Leftrightarrow |x-y| = 0$

$\Leftrightarrow x = y$

$$d(x, y) = |x-y|$$
$$= |y-x|$$
$$= d(y, x)$$

Let $x, y, z \in R$

Then
$$\begin{aligned} d(x, z) &= |x-z| = |x-y+y-z| \\ &\leq |x-y|+|y-z| \\ &\leq d(x, y)+d(y, z) \\ \therefore d(x, z) &\leq d(x , y)+d(y, z) \end{aligned}$$

Hence d is a metric on R.

Example 2 :

In C we define $d(z, w) = |z-w|$. Then d is a metric on C. This is called the **usual metric** on C.

Note :

If the complex number $z = x+iy$ is identified with the point (x, y) of the two dimensional Euclidean plane then the above distance formula takes the form

$d(z, w) = \sqrt{(x-u)^2+(y-v)^2}$ where $z = x+iy$ and $w = u+iv$

Example 3 :

On any non-empty set M we define d as follows.

$$d(x, y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

Then d is a metric on M. This is called the **discrete metric** on M.

Proof :

Clearly $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$

Also $d(x, y) = d(y, x) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$

$\therefore d(x, y) = d(y, x)$ for all $x, y \in M$. Let $x, y, z \in M$

Case (i) :

$$x = z$$

Then $d(x, z) = 0$

Also $d(x, y)+d(y, z) \geq 0$

$\therefore d(x, z) \leq d(x, y)+d(y, z)$

Case (ii) :

$$x \neq z$$

Then $d(x, z) = 1$

Also since x, z are distinct, y cannot be equal to both x and z.

Hence either $y \neq x$ or $y \neq z$.

$$d(x, y)+d(y, z) \geq 1$$

$$d(x, z) \leq d(x, y)+d(y, z)$$

Thus $d(x, z) \leq d(x, y)+d(y, z)$ for all $x, y, z \in M$

Hence d is a metric on M.

Example 4 :

In R^n we define $d(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}$ where $x = (x_1, x_2, \ldots\ldots, x_n)$ and $y = (y_1, y_2, \ldots\ldots, y_n)$. Then d is a metric on R^n. This is called the **usual metric** on R^n.

Proof :

$$d(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2} \geq 0$$

$$d(x, y) = 0 \Leftrightarrow \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2} = 0$$

$$\Leftrightarrow (x_i - y_i)^2 = 0 \text{ for all } i = 1, 2, \ldots\ldots, n$$

$$\Leftrightarrow x_i = y_i \text{ for all } i = 1, 2, \ldots.., n$$

$$\Leftrightarrow (x_1, x_2, \ldots., x_n) = (y_1, y_2, \ldots., y_n)$$

$$\Leftrightarrow x = y$$

Also, $d(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}$

$$= \left[\sum_{i=1}^{n} (y_i - x_i)^2\right]^{1/2}$$

$$= d(y, x)$$

To prove the triangle inequality, take $a_i = x_i - y_i$, $b_i = y_i - z_i$ and $p = 2$ in Minkowski's inequality.

We get, $$\left[\sum_{i=1}^{n}(x_i - z_i)^2\right]^{1/2} \leq \left[\sum_{i=1}^{n}(x_i - y_i)^2\right]^{1/2} + \left[\sum_{i=1}^{n}(y_i - z_i)^2\right]^{1/2}$$

i.e., $$d(x, z) \leq d(x, y) + d(y, z)$$

$\therefore$ d is a metric on R^n.

Note :

R^n with usual metric is called the **n-dimensional Euclidean space.**

Example 5 :

Consider R^n. Let $p > 1$.

We define $$d(x, y) = \left[\sum_{i=1}^{n}|x_i - y_i|^p\right]^{1/p}$$

$x = (x_1, x_2, \ldots\ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$

Then d is a metric on R^n.

The proof is similar to that of Example 4.

Example 6 :

Let $x, y \in R^2$. Then $x = (x_1, x_2)$ and $y = (y_1, y_2)$ where $x_1, x_2, y_1, y_2 \in R$. We define $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$. Then d is a metric on R^2.

Proof :

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2| \geq 0$$

$$\begin{aligned} d(x, y) = 0 &\Leftrightarrow |x_1 - y_1| + |x_2 - y_2| = 0 \\ &\Leftrightarrow |x_1 - y_1| = 0 \text{ and } |x_2 - y_2| = 0 \\ &\Leftrightarrow x_1 = y_1 \text{ and } x_2 = y_2 \\ &\Leftrightarrow (x_1, x_2) = (y_1, y_2) \\ &\Leftrightarrow x = y \end{aligned}$$

$$d(x, y) = |x_1-y_1| + |x_2-y_2|$$
$$= |y_1-x_1| + |y_2-x_2|$$
$$= d(y, x)$$

Let $x, y, z \in R^2$.

$$d(x, z) = |x_1-z_1| + |x_2-z_2|$$
$$= |x_1-y_1+y_1-z_1| + |x_2-y_2+y_2-z_2|$$
$$\leq \{|x_1-y_1|+|y_1-z_1|\} + \{|x_2-y_2|+|y_2-z_2|\}$$
$$= \{|x_1-y_1|+|x_2-y_2|\} + \{|y_1-z_1|+|y_2-z_2|\}$$
$$= d(x, y) + d(y, z)$$

Thus $d(x, z) \leq d(x, y) + d(y, z)$

Hence d is a metric on R^2.

Note :

More generally in R^n we define $d(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Then d is a metric on R^n.

Example 7 :

In R^n we define $d(x, y) = \max\{|xi-yi|, i = 1, 2, \ldots\ldots, n\}$

$$x = (x_1, x_2, \ldots\ldots, x_n)$$

and $y = (y_1, y_2, \ldots\ldots, y_n)$

Then d is a metric on R^n.

Proof :

$$d(x, y) = \max\{|x_1-y_1|, |x_2-y_2|, \ldots\ldots, |x_n-y_n|\} \geq 0$$
$$d(x, y) = 0 \Leftrightarrow \max\{|x_1-y_1|, \ldots\ldots, |x_n-y_n|\} = 0$$
$$\Leftrightarrow x_i-y_i = 0 \text{ for all } i = 1, 2, \ldots\ldots, n$$
$$\Leftrightarrow x_i = y_i \text{ for all } i = 1, 2, \ldots\ldots, n$$
$$\Leftrightarrow (x_1, x_2, \ldots\ldots, x_n) = (y_1, y_2, \ldots\ldots, y_n)$$
$$\Leftrightarrow x = y$$

$$\begin{aligned} d(x, y) &= \max\{|x_i-y_i|\} \\ &= \max\{|y_i-x_i|\} \\ &= d(y, x) \end{aligned}$$

Let x, y, z $\in R^n$. Since each $x_i, y_i, z_i \in R$

We have $|x_i-z_i| \leq |x_i-y_i|+|y_i-z_i|$ for all i = 1, 2,....,n.

$\therefore \max|x_i-z_i| \leq \max|x_i-y_i| + \max|y_i-z_i|$

$\therefore d(x, z) \leq d(x, y)+d(y, z)$

Hence d is a metric on R^n.

Example 8 :

Let $p \geq 1$. Let l_p denote the set of all sequences (x_n) such that $\sum_1^{\infty}|x_n|^p$ is convergent. Define $d(x, y) = \left[\sum_{n=1}^{\infty}|x_n - y_n|^p\right]^{1/p}$ where $x = (x_n)$ and $y = (y_n)$.

Then d is a metric on l_p.

Proof :

Let a, b $\in l_p$.

First we prove d(a, b) is a real number. By Minkowskis' inequality we have

$$\left[\sum_{i=1}^{n}|a_i + b_i|^p\right]^{1/p} \leq \left[\sum_{i=1}^{n}|a_i|^p\right]^{1/p} + \left[\sum_{i=1}^{n}|b_i|^p\right]^{1/p} \quad \text{------(1)}$$

Since a, b $\in l_p$ the right hand side of (1) has a finite limit as $n\to\infty$.

$\therefore \left[\sum_{i=1}^{n}|a_i + b_i|^p\right]^{1/p}$ is a convergent series. Similarly we can prove that $\left[\sum_{i=1}^{n}|a_i - b_i|^p\right]^{1/p}$ is also a convergent series and hence d(a, b) is a real number.

Taking limit as $n\to\infty$ in (1) we get

$$\left[\sum_{i=1}^{\infty}|a_i + b_i|^p\right]^{1/p} \leq \left[\sum_{i=1}^{\infty}|a_i|^p\right]^{1/p} + \left[\sum_{i=1}^{\infty}|b_i|^p\right]^{1/p} \quad \text{------(2)}$$

Obviously $d(x, y) \geq 0$

$$d(x, y) = 0 \text{ iff } x = y$$

and $$d(x, y) = d(y, x)$$

Let $x, y, z \in l_p$

Taking $a_i = x_i - y_i$ and $b_i = y_i - z_i$ in (2) we get

$$\left[\sum_{i=1}^{\infty} |x_i - z_i|^p\right]^{1/p} \leq \left[\sum_{i=1}^{\infty} |x_i - y_i|^p\right]^{1/p} + \left[\sum_{i=1}^{\infty} |y_i - z_i|^p\right]^{1/p}$$

$\therefore$ $$d(x, z) \leq d(x, y) + d(y, z)$$

Hence d is a metric on l_p.

Note :

In particular l_2 is a metric space with the metric defined by

$$d(x, y) = \sum_{n=1}^{\infty} \left[|x_n - y_n|^2\right]^{1/2}$$

Example 9 :

Let M be the set of all bounded real valued functions defined on a non-empty set E. Define $d(f, g) = \sup\{|f(x)-g(x)|\}/x \in E\}$ d is a metric on M.

Proof :

$$d(f, g) = \sup\{|f(x)-g(x)|\} \geq 0$$

Also, $$d(f, g) = 0 \Leftrightarrow \sup\{|f(x)-g(x)|\} \geq 0$$

$$\Leftrightarrow |f(x)-g(x)| = 0 \text{ for all } x \in E$$

$$\Leftrightarrow f(x) = g(x) \text{ for all } x \in E$$

$$\Leftrightarrow f = g$$

Also $$d(f, g) = \sup\{|f(x)-g(x)|\}$$

$$= \sup\{|g(x)-f(x)|\}$$

$$= d(g, f)$$

Let f, g, h ∈ M

We have $|f(x)-h(x)| \le |f(x)-g(x)|+|g(x)-h(x)|$

∴ $\sup\{|f(x)-h(x)|\} \le \sup\{|f(x)-g(x)|\}+\sup\{|g(x)-h(x)|\}$

∴ $d(f, h) \quad d(f, g)+d(g, h)$

Hence d is a metric on M.

Example 10 :

Let M be the set of all sequences in R. Let x, y ∈ M and let $x = (x_n)$ and $y=(y_n)$.

Define $$d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)}$$

Then d is a metric on M.

Proof :

Let x, y ∈ M. First we prove that d(x, y) is a real number ≥ 0.

We have $\frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)} \le \frac{1}{2^n}$ for all n.

Also $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent series.

∴ $\sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)}$ is a convergetn series.

∴ d(x, y) is a real number and d(x, y) ≥ 0

$$d(x, y) = 0 \Leftrightarrow \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)} = 0$$

$\Leftrightarrow |x_n - y_n| = 0$ for all n.

$\Leftrightarrow x_n = y_n$ for all n.

$\Leftrightarrow x = y$

Also $$d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)}$$

$$= \sum_{n=1}^{\infty} \frac{|y_n - x_n|}{2^n(1+|y_n - x_n|)}$$

$$= d(y, x)$$

Let x, y, z ∈ M. Then

$$\frac{|x_n - z_n|}{1+|x_n - z_n|} = 1 - \frac{1}{1+|x_n - z_n|}$$

$$\leq 1 - \frac{1}{(1+|x_n - y_n| + |y_n - z_n|)}$$

$$= \frac{|x_n - y_n| + |y_n - z_n|}{1+|x_n - y_n| + |y_n - z_n|}$$

$$= \frac{|x_n - y_n|}{1+|x_n - y_n| + |y_n - z_n|} + \frac{|y_n - z_n|}{1+|x_n - y_n| + |y_n - z_n|}$$

$$\leq \frac{|x_n - y_n|}{1+|x_n - y_n|} + \frac{|y_n - z_n|}{1+|y_n - z_n|}$$

Multiplying both sides of the inequailty by $\frac{1}{2^n}$ and taking the sum from n = 1 to ∞ we get $d(x, z) \leq d(x,y) + d(y, z)$

∴ d is a metric on M.

Example 11 :

Let l^∞ denote the set of all bounded sequences of real numbers. Let $x = (x_n)$ and $y = (y_n) \in l^\infty$ define d on l^∞ as $d(x, y) = \text{lub}|x_n - y_n|$.

Then d is a metric on l^∞.

Solution :

$$d(x, y) = \text{lub}|x_n - y_n| \geq 0$$

$$d(x, y) = 0 \Leftrightarrow \text{lub}|x_n - y_n| = 0$$

$$\Leftrightarrow |x_n - y_n| = 0 \text{ for } 1 \leq n < \infty$$

$$\Leftrightarrow x_n = y_n \text{ for } 1 \le n < \infty$$

$$\Leftrightarrow (x_n) = (y_n)$$

$$\Leftrightarrow x = y$$

$$d(x, y) = \text{lub}|x_n - y_n|$$

$$= \text{lub}|y_n - x_n|$$

$$= d(y, x)$$

Let $z = (z_n)$

$$|x_n - z_n| \le |x_n - y_n| + |y_n - z_n|$$

$$\le \text{lub}|x_n - y_n| + \text{lub}|y_n - z_n|$$

$$= d(x, y) + d(y, z)$$

$\therefore$ $\text{lub}|x_n - z_n| \le d(x, y) + d(y, z)$

$\therefore$ $d(x, z) \le d(x, y) + d(y, z)$

$\therefore$ d is a metric on l^{∞}.

Worked Examples :

Example 1 :

Let d_1 and d_2 be two metrics on M. Define $d(x, y) = d_1(x, y) + d_2(x, y)$. Prove that d is a metric on M.

Solution :

$$d(x, y) = d_1(x, y) + d_2(x, y) \ge 0$$

$$d(x, y) = 0 \Leftrightarrow d_1(x, y) + d_2(x, y) = 0$$

$$\Leftrightarrow d_1(x, y) = 0 \text{ and } d_2(x, y) = 0$$

$$\Leftrightarrow x = y$$

$$d(x, y) = d_1(x, y) + d_2(x, y)$$

$$= d_1(y, x) + d_2(y, x)$$

$$= d(y, x)$$

Let $x, y, z \in M$. Then we have

$$d_1(x, z) \le d_1(x, y) + d_1(y, z) \text{ and}$$

$$d_2(x, z) \le d_2(x, y) + d_2(y, z)$$

Adding, we get $d(x, z) \le d(x, y) + d(y, z)$

$\therefore$ d is a metric on M.

Example 2 :

Determine whether d(x, y) defined on R by $d(x, y) = (x-y)^2$ is a metric or not.

Solution :

Let $x, y \in R$

$$d(x, y) = (x-y)^2 \geq 0$$

$$d(x, y) = (x-y)^2 = (y-x)^2$$

$$= d(y, x)$$

But triangle inequality does not hold.

Take $x = -5$, $y = -4$, and $z = 4$

Then $d(x, y) = (-5+4)^2 = 1$

$$d(y, z) = (-4-4)^2 = 64$$

$$d(x, z) = (4+5)^2 = 81$$

$$d(x, z) > d(x, y)+d(y, z)$$

Hence triangle inequality does not hold.

∴ d is not a metric on R.

Example 3 :

If d is a metric on M, prove that $\sqrt{d}$ is a metric on M.

Solution :

Let $x, y, z \in M$

∴ We have $\sqrt{d(x,y)} \geq 0$

Also $\sqrt{d(x,y)} = \sqrt{d(y,x)}$

$$d(x, z) \leq d(x, y)+d(y, z)$$

∴ $$\sqrt{d(x,z)} \leq \sqrt{d(x,y)+d(y,z)}$$

$$\leq \sqrt{d(x,y)}+\sqrt{d(y,z)} \quad \left(\text{since } \sqrt{a+b} \leq \sqrt{a}+\sqrt{b}\right)$$

Hence $\sqrt{d}$ is a metric on M.

Example 4 :

Let (M, d) be a metric space. Define $d_1(x, y) = \frac{d(x,y)}{1+d(x,y)}$. Prove that d_1 is a metric on M.

Solution :

$$d_1(x, y) = \frac{d(x,y)}{1+d(x,y)} \geq 0 \text{ (since } d(x, y) \geq 0)$$

$$d_1(x, y) = 0 \Leftrightarrow \frac{d(x,y)}{1+d(x,y)} = 0$$

$$\Leftrightarrow d(x, y) = 0$$

$$\Leftrightarrow x = y \text{ (since d is a metric)}$$

Also

$$d_1(x, y) = \frac{d(x,y)}{1+d(x,y)}$$

$$= \frac{d(y,x)}{1+d(y,x)}$$

$$= d_1(y, x)$$

Let $x, y, z \in M$

Then

$$d_1(x, z) = \frac{d(x,z)}{1+d(x,z)}$$

$$d_1(x, z) = 1-\frac{1}{1+d(x,z)}$$

$$\leq 1-\left[\frac{1}{1+d(x,y)+d(y,z)}\right]$$

$$= \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)}$$

$$= \frac{d(x,y)}{1+d(x,y)+d(y,z)}+\frac{d(y,z)}{1+d(x,y)+d(y,z)}$$

$$\leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}$$

$$= d_1(x, y)+d_1(y, z)$$

Thus $\quad d_1(x, z) \leq d_1(x, y) + d_1(y, z)$

$\therefore$ d_1 is a metric on M.

Example 5 :

Let (M, d) be a metric space. Define $d_1(x, y) = \min\{1, d(x, y)\}$. Prove that d_1 is a metric on M.

Solution :

$$d_1(x, y) = \min\{1, d(x, y)\} \geq 0$$

$$\therefore d_1(x, y) \geq 0$$

$$d_1(x, y) = 0 \Leftrightarrow \min\{1, d(x, y)\} = 0$$

$$\Leftrightarrow d(x, y) = 0$$

$$\Leftrightarrow x = y$$

Also $d_1(x,y) = \min\{1, d(x, y)\}$

$$= \min\{1, d(y, x)\}$$

$$= d_1(y, x)$$

Let $x, y, z \in M$

Then $\quad d_1(x, z) = \min\{1, d(x, z)\} \leq 1$

To prove $\quad d_1(x, z) \leq d_1(x, y)+d_1(y, z)$

If $d_1(x, y) = 1$ or $d_1(y, z) = 1$ the inequality is obvious.

Let $d_1(x, y) < 1$ and $d_1(y, z) < 1$.

Then $\quad d_1(x, y)+d_1(y, z) = \min\{1, d(x, y)\}+\min\{1, d(y, z)\}$

$$= d(x, y)+d(y, z)$$

$$\geq d(x, z)$$

$$\geq \min\{1, d(x, z)\}$$

$$= d_1(x, z)$$

Thus $\quad d_1(x, y)+d_1(y, z) \geq d_1(x, z)$

$\therefore$ d_1 is a metric on M.

Example 6 :

Let M be a non-empty set.

Let $d: M \times M \to R$ be a function such that

(i) $d(x, y) = 0$ iff $x = y$

(ii) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in M$.

Prove that d is a metric on M.

Solution :

Put $y = x$ in (ii)

We have $d(x, x) \leq d(x, z) + d(x, z)$

$\therefore$ $0 \leq 2d(x, z)$ (by (i))

$\therefore$ $d(x, z) \geq 0$

To prove $d(x,y) = d(y, x)$

Putting $z = x$ in (ii)

we get $d(x, y) \leq d(x, x) + d(y, x)$

i.e., $d(x, y) \leq d(y, x)$ [using (i)]

Since this is true for all $x, y \in M$

we have $d(y, x) \leq d(x, y)$

Hence $d(x, y) = d(y, x)$

Now (ii) can be written as $d(x, y) \leq d(x, z)+d(z, y)$ which is the triangle inequality.

$\therefore$ d is a metric on M.

Example 7 :

If $(M_1, d_1), (M_2, d_2), \ldots\ldots, (M_n, d_n)$ are metric spaces then $M_1 \times M_2 \times \ldots\ldots \times M_n$ is a metric space with metric d defined by $d(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i)$. where $x = (x_1, x_2, \ldots\ldots, x_n)$, $y = (y_1, y_2, \ldots\ldots, y_n)$.

Solution :

$$d(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i) \geq 0$$

Also $d(x, y) = 0 \Leftrightarrow \sum_{i=1}^{n} d_i(x_i, y_i) = 0$

$\Leftrightarrow d_i(x_i, y_i) = 0$ for all $i = 1, 2, \ldots, n$

$\Leftrightarrow x_i = y_i$ for all $i = 1, 2, \ldots, n$

$\Leftrightarrow (x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$

$\Leftrightarrow x = y$

Also
$$d(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i)$$
$$= \sum_{i=1}^{n} d_i(y_i, x_i)$$
$$= d(y, x)$$

Let $x, y, z \in M$

Then
$$d(x, z) = \sum_{i=1}^{n} d_i(x_i, z_i)$$
$$\leq \sum_{i=1}^{n} \left[d_i(x_i, y_i) + d_i(y_i, z_i)\right]$$
$$= \sum_{i=1}^{n} d_i(x_i, y_i) + \sum_{i=1}^{n} d_i(y_i, z_i)$$
$$= d(x,y) + d(y, z)$$

$\therefore \quad d(x, z) \leq d(x, y) + d(y, z)$

Hence d is a metric on M.

Example 8 :

In a metric space (M, d) prove that $|d(x, z) - d(y,z)| \leq d(x, y)$ for all $x, y, z \in M$.

Solution :

Let $x, y, z \in M$

We have $d(x, z) \leq d(x, y) + d(y, z)$

$\therefore \quad d(x, z) - d(y, z) \leq d(x, y)$ ------(1)

Interchanging x and y in (1) we get

$$d(y, z) - d(x, z) \leq d(y, x) = d(x, y)$$

$$\therefore \quad d(y, z) - d(x, z) \leq d(x, y) \qquad \text{------}(2)$$

From (1) and (2) we get $|d(x, z) - d(y, z)| \leq d(x, y)$

Exercis' :

1. If d is a metric on M prove that

   (i) 2d is a metric on M

   (ii) nd is a metric on M where $n \in N$.

2. Let M denote the set of all sequences in R.

   Define $d(x, y) = \sum_{i=1}^{\infty} \frac{|x_n - y_n|}{n!(1+|x_n - y_n|)}$

   Prove that d is a metric on M.

3. Determine whether $d(x,y) = |x-2y|$ defined on R is a metric or not.

4. Let $(M_1, d_1), (M_2, d_2), \ldots, (M_n, d_n)$ be metric spaces.

   Let $M = M_1 \times M_2 \times \ldots \times M_n$. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be elements of M.

   Define $d(x, y) = \max d_i(x_i, y_i)$

   Prove that d is a metric on M.

BOUNDED SETS IN A METRIC SPACE

Definition :

Let (M, d) be a metric space. We say that a subset A of M is bounded if there exists a positive real number K such that $d(x, y) \leq K$ for all $x, y \in A$.

Example 1 :

Any finite subset A of a metric space (M, d) is bounded.

Proof :

Let A be any finite subset of M.

If $A = \phi$ then A is obviously bounded.

Let $A \neq \phi$. Then $\{d(x, y)/x, y \in A\}$ is a finite set of real numbers.

Let $K = \max\{d(x, y)/x, y \in A\}$

Clearly $d(x, y) \leq K$ for all $x, y \in A$

$\therefore$ A is bounded.

Example 2 :

[0, 1] is a bounded subset of R with usual metric since $d(x, y) \leq 1$ for all $x, y \in [0, 1]$.

More generally any finite interval and any subset of R which is contained in a finite interval are bounded subsets of R.

Example 3 :

$(0, \infty)$ is an unbounded subset of R.

Example 4 :

If we consider R with discrete metric then $(0, \infty)$ is a bounded subset of R, sicne $d(x, y) \leq 1$ for all $x, y \in (0, \infty)$.

More generally any subset of a discrete matric space M is a bounded subset of M.

Example 5 :

In l_2 let

$$e_1 = (1, 0, \ldots\ldots, 0, \ldots\ldots)$$
$$e_2 = (0, 1, 0, \ldots\ldots, 0, \ldots)$$
$$e_3 = (0, 0, 1, \ldots\ldots, 0, \ldots\ldots)$$
$$\ldots\ldots \quad \ldots \quad \ldots\ldots$$

Let $A = \{e_1, e_2, \ldots, e_n, \ldots\}$

Then A is a bounded subset of l_2.

Proof :

$$d(e_n, e_m) = \begin{cases} \sqrt{2} \text{ if } n \neq m \\ 0 \text{ if } n = m \end{cases}$$

$\therefore d(e_n, e_m) \leq \sqrt{2}$ for all $e_n, e_m \in A$

$\therefore$ A is a bounded set in l_2.

Example 6 :

Let (M, d) be a metric space.

Define $d_1(x, y) = \frac{d(x,y)}{1+d(x,y)}$

We know that (M, d_1) is also a metric space.

Also $d_1(x,y)<1$ for all $x, y \in M$

Hence (M, d_1) is a bounded metric space.

Definition :

Let (M, d) be a metric space. Let $A \subseteq M$. Then the **diameter** of A, denoted by d(A), is defined by $d(A) = \text{l.u.b.}\{d(x, y)/x, y \in A\}$.

Note 1 :

A non empty set A is a bounded set iff d(A) is finite.

Note 2 :

Let $A, B \subseteq M$. Then $A \subseteq B \Rightarrow d(A) \leq d(B)$

Example 1 :

The diameter of any non-empty subset in a discrete metric space is 1.

Example 2 :

In R the diameter of any interval is equal to the length of the interval. For example the diameter of [0, 1] is 1.

Example 3 :

In any metric space, $d(\phi) = -\infty$

Exercise :

1. Let (M, d) be a bounded metric space. Define $d_1(x, y) = 2d(x, y)$. Prove that (M, d_1) is a bounded metric space.
2. Prove that in a metric space any subset of a bounded set is bounded.
3. Find the diameter of the following subset of R with usual metric.

   (i) {1, 3, 5, 7, 9}, (ii) [−3, 5] (iii) $[1, 2] \cup [5, 6]$

Open ball (open sphere) in a metric space

Definition :

Let (M, d) be a metric space. Let $a \in M$ and r be a positive real number. Then the **open ball** or the **open sphere** with centre a and radius r denoted by $B_d(a, r)$ is the subset of M given by $B_d(a, r) = \{x \in M / d(a, x) < r\}$.

When the metric d under consideration is clear we write B(a, r) instead of $B_d(a, r)$.

Note 1 :

B(a, r) is always nonempty since it contains at least its centre a.

Note 2 :

B(a, r) is a bounded set.

For let $x, y \in B(a, r)$

$\therefore d(a, x) < r$ and $d(a, y) < r$

$\therefore d(x, y) \leq d(x, a)+d(a, y) \leq r+r = 2r$

Example 1 :

Consider R with usual metric.

Let $a \in R$

Then
$$\begin{aligned} B(a, r) &= \{x \in R / d(a,x) < r\} \\ &= \{x \in R / |a-x| < r\} \\ &= \{x \in R / a-r < x < a+r\} \\ &= (a-r, a+r) \end{aligned}$$

Example 2 :

Consider C with usual metric Let $a \in C$.

Then
$$\begin{aligned} B(a, r) &= \{z \in C \ / \ d(a, z) < r\} \\ &= \{z \in C \ / \ |z-a| < r\} \end{aligned}$$

This is the interior of the circle with centre a and radius r.

Example 3 :

In R^2 with usual metric B(a, r) is the interior of the circle with centre a and radius r.

Example 4 :

Let d be the discrete metric on M.

Then $$B(a, r) = \begin{cases} M \text{ if } r > 1 \\ \{a\} \text{ if } r \le 1 \end{cases}$$

Proof :

We have $$d(x, y) = \begin{cases} 1 \text{ if } x \ne y \\ 0 \text{ if } x = y \end{cases}$$

Let $a \in M$. Let r be any positive real number.

Case (i) :

Let r > 1.

Then $B(a, r) = \{x \in M / d(a, x) < r\}$

Clearly every point $x \in M$ is such that $d(a, x) < r$.

Hence B(a, r) = M

Case (ii) :

Let $r \le 1$. In this case for any point $x \ne a$, $d(a, x) = 1 \ge r$

Hence $x \notin B(a, r)$ so that $B(a, r) = \{a\}$

$$\therefore B(a, r) = \begin{cases} M \text{ if } r > 1 \\ \{a\} \text{ if } r \le 1 \end{cases}$$

Example 5 :

Consider M = [0, 1] with usual metric $d(x, y) = |x-y|$

Here $$B(0, \tfrac{1}{2}) = \{x \in [0, 1] / d(0, x) < \tfrac{1}{2}\}$$
$$= \{x \in [0, 1] / |x| < \tfrac{1}{2}\}$$
$$= [0, \tfrac{1}{2})$$

Example 6 :

Consider R^2 with the metric d given by

$$d((x_1, y_1), (x_2, y_2)) = |x_1-x_2|+|y_1-y_2|$$

Then $B((0,0), 1) = \{(x, y)\in R^2 \,/\, |x-0|+|y-0|<1\}$

$= \{(x, y)\in R^2 \,/\, |x|+|y|<1\}$

This is the interior of the square bounded by the four lines $x+y=1$, $-x+y=1$, $x+y=-1$ and $x-y = 1$.

Exercise :

1. In R with usual metric find (i) $B(-1, 1)$, (ii) $B(½, 1)$
2. In R^2 with usual metric find (i) $B((0, 0), ½)$, (ii) $B((1, 1), 1)$
3. In [0, 1] with usual metric find (i) $B(1, ½)$, (ii) $B(0, ¼)$

OPEN SETS

Definition :

Let (M, d) be a metric space. Let A be a subset of M. Then A is said to be **open** in M if for every $x\in A$ there exists a positive real number r such that $B(x, r)\subseteq A$.

Example 1 :

In R with usual metric (0, 1) is an open set.

Proof :

Let $x\in(0, 1)$

Choose $r = \min\{x-0, 1-x\} = \min\{x, 1-x\}$

Clearly $r>0$ and $B(x, r) = (x-r, x+r)\subseteq(0,1)$

$\therefore$ (0, 1) is open.

Example 2 :

In R with usual metric [0, 1) is not open since no open ball with centre 0 is contained [0, 1).

Example 3 :

Consider $M=[0, 2)$ with usual matric. Let $A = [0, 1)\subseteq M$. Then A is open in M.

Proof :

Let $x \in [0, 1)$

If $x = 0$, then $B(0, ½) = [0, ½) \subseteq A$

If $x \neq 0$ choose $r = \min\{x, 1-x\}$

$r>0$ and $B(x, r) = (x-r, x+r) \subseteq [0, 1)$

$\therefore$ A is open in M.

Example 4 :

Any open interval (a, b) is an open set in R with usual metric.

Proof :

Let $x \in (a, b)$

Let $r = \min\{x-a, b-x\}$

Then $B(x, r) \subseteq (a, b)$

Hence (a, b) is an open set.

Note : $(-\infty, a)$ and (a, ∞) are open sets.

Example 5 :

In R with usual metric the set {0} is not an open set since any open ball wtih centre 0 is not contained in {0}.

Example 6 :

In R with usual matric any finite non empty subset A of Ris not an open set.

Proof :

Any open ball in R is a bounded open interval which is an infinite subset of R. Hence it cannot be contained inthe finite subset A. Hence A is not open in R.

Example 7 :

Q is not open in R.

Proof :

Let $x \in Q$. Then for any $r>0$ the intervl $(x-r, x+r)$ contains both rational and irrational numbers.

$\therefore$ (x–r, x+r) is not a subset of Q.

$\therefore$ Q is not open in R.

Example 8 :

Z is not open in R.

Proof :

Let $x \in Z$. Then for any $r>0$, the interval $(x-r, x+r)$ is not a subset of Z. Hence Z is not open in R.

Note : The set of irrational numbers is not open in R.

Theorem :

In any metric space M, (i) ϕ is open, (ii) M is open.

Proof :

(i) Trivially ϕ is an open set.

(ii) Let $x \in M$. Clearly for any $r>0$, $B(x, r) \subseteq M$

Hence M is an open set.

Theorem :

In any metric spac (M, d) each open ball is an open set.

Proof :

Let B(a, r) be an open ball in M.

Let $x \in B(a, r)$

Then $d(a, x) < r$

$\therefore r-d(a, x)>0$

Let $r_1 = r-d(a, x)$

We claim that $B(x, r_1) \subseteq B(a, r)$

Let $y \in B(x, r_1)$

$\therefore d(x, y) < r_1 = r-d(a, x)$

$\therefore d(x, y)+d(a, x) < r$ ------(1)

$d(a, y) \leq d(a, x)+d(x, y) < r$ (by (1))

∴ $d(a, y) < r$

∴ $y \in B(a, r)$

Hence $B(x, r_1) \subseteq B(a, r)$

∴ $B(a, r)$ is an open set.

Theorem :

In any metric space the union of any family of open sets is open.

Proof :

Let (M, d) be a metric space.

Let $\{A_i / i \in I\}$ be a family of open set in M.

Let $A = \bigcup_{i \in I} A_i$

If $A = \phi$ then A is open

∴ Let $A \neq \phi$. Let $x \in A$

Then $x \in A_i$ for some $i \in I$

Since A_i is open there exists an open ball $B(x, r)$ such that $B(x, r) \subseteq A$

∴ $B(x, r) \subseteq A$

Hence A is open.

Theorem :

In any metric space the intersection of a finite number of open sets is open.

Proof :

Let (M, d) be a metric space.

Let $A_1, A_2, \ldots, A_n$ be open sets inM.

Let $A = A_1 \cap A_2 \cap \ldots \cap A_n$

If $A = \phi$ then A is open.

∴ Let $A \neq \phi$. Let $x \in A$

∴ $x \in A_i$ for each $i = 1, 2, \ldots, n$.

Since each A_i is an open set there is a positive real number r_i such that

$$B(x, r_i) \subseteq A_i \qquad \text{------(1)}$$

Let $r = \min\{r_1, r_2, \ldots r_n\}$

Obviously r is a positive real number and $B(x, r) \subseteq B(x, r_i)$ for all i=1,2,....,n.

Hence $B(x, r) \subseteq A_i$ for all i = 1,2,......,n (by (1))

$\therefore B(x, r) \subseteq \bigcap_{i=1}^{n} A_i$

$\therefore B(x, r) \subseteq A$

$\therefore$ A is open.

Note :

The intersection of an infinite number of open sets ina metric space need not be open. For example, consider R with usual metric.

Let $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$

Then A_n is open in R for all n.

But $\bigcap_{n=1}^{\infty} A_n = \{0\}$ which is not open in R.

Worked Examples :

Example 1 :

Let (M, d) be a metric space. Let $x \in M$ show that $\{x\}^c$ is open

Solution :

Let $y \in \{x\}^c$. Then $y \neq x$.

$\therefore d(x, y) = r > 0$

Clearly $B\left(y, \frac{1}{2}r\right) \subseteq \{x\}^c$

$\therefore \{x\}^c$ is open.

Example 2 :

Let (M, d) be a metric space show that every subset of M is open iff $\{x\}$ is open for all $x \in M$.

Solution :

Suppose every subset of M is open. Then obviously $\{x\}$ is open for all $x \in M$. Conversely let $\{x\}$ be open for all $x \in M$.

Let A be any subset of M.

If $A = \phi$ then A is open.

Let $A \neq \phi$. Then $A = \bigcup_{x \in A} \{x\}$

By hypothesis $\{x\}$ is open.

Hence A is open.

Example 3 :

Prove that any open subset of R can be expressed as the union of a countable number of mutually disjoint open intervals.

Solution :

Let A be an open subset of R. Let $x \in A$. Then there exists a positive real number r such that $B(x, r) = (x-r, x+r) \subseteq A$. Thus there exists an open interval I such that $x \in I$ and $I \subseteq A$.

Let I_x denote the largest open interval such that $x \in I$ and $I_x \subseteq A$.

Clearly $\bigcup_{x \in A} I_x = A$

Let $x, y \in A$

We claim that $I_x = I_y$ or $I_x \cap I_y = \phi$

Suppose $I_x \cap I_y \neq \phi$.

Then $I_x \cup I_y$ is an open interval contained in A. But I_x is the largest open interval such that $x \in Ix$ and $I_x \subseteq A$.

∴ $I_x \cup I_y = I_x$ so that $I_y \subseteq I_x$

Similarly $I_x \subseteq I_y$

∴ $I_x = I_y$. Thus the intervals I_x are mutually disjoint.

We claim that the set $F = \{I_x / x \in A\}$ is countable.

For each $I_x \in F$ choose a rational number $r_x \in I_x$.

Since the intervals Ix are mutually disjoint $I_x \neq Iy \Rightarrow r_x \neq r_y$

∴ $f:F \to Q$ defined by $f(I_x) = r_x$ is 1–1

∴ F is equivalent to a subset of Q which is countable.

∴ F is countable.

EQUIVALENT METRICS

Definition :

Let d and ρ be the two metrics on M. Then the metrics d and ρ are said to be **equivalent** if the open sets of (M, ρ) are the open sets of (M, d) and conversely.

Example :

Let (M, d) be a metric space. Define $\rho(x, y) = 2d(x, y)$. Then d and ρ are equivalent metrics.

Solution :

We know that ρ is a metric on M. We first prove that $B_d(a, r) = B_\rho(a, 2r)$.

Let $x \in B_d(a, r)$

∴ $d(a, x) < r$

∴ $2d(a, x) < 2r$

∴ $\rho(a, x) < 2r$

Hence $x \in B_\rho(a, 2r)$

∴ $B_d(a, r) \subseteq B_\rho(a, 2r)$ -------(1)

Let $x \in B_\rho(a, 2r)$

∴ $\rho(a, x) < 2r$

∴ $\frac{1}{2}\rho(a, x) < r$

∴ $d(a, x) < r.$

Hence $x \in B_d(a, r)$

∴ $B_\rho(a, 2r) \subseteq B_d(a, r)$ -------(2)

∴ By (1) and (2) we get $B_d(a, r) = B\rho(a, 2r)$ -------(3)

Let G be any open subset in (M, d).

Let $a \in G$. Hence there exists $r>0$ such that $B_d(a, r) \subseteq G$

∴ $B_\rho(a, 2r) \subseteq G$ (using (3))

∴ G is open in (M, ρ)

Conversely suppose G is open in (M, ρ)

Let $a \in G$. Hence there exists $r>0$ such that $B_\rho(a, r) \subseteq G$

Hence $B_d\left(a, \frac{1}{2}r\right) \subseteq G$. (using (3))

Hence G is open in (M, d)

∴ d and ρ are equivalent metrics.

Example :

Let (M, d) be a metric space. Define $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$. Prove that d and ρ are equivalent metrics on M.

Solution :

We know that ρ is a metric on M. We first prove $B_\rho(a, r) = B_d\left(a, \frac{r}{1-r}\right)$ provided $0<r<1$.

Let $x \in B_\rho(a, r)$. Hence $\rho(a, x)<r$.

$$\therefore \quad \frac{d(a,x)}{1+d(a,x)} < r$$

$$\therefore \quad d(a,x) < r[1+d(a, x)]$$

$$\therefore \quad d(a, x)[1-r] < r$$

$$\therefore \quad d(a, x) < \frac{r}{1-r} \text{ (since } 0<r<1)$$

$$\therefore \quad x \in B_d\left(a, \frac{r}{1-r}\right)$$

$\therefore \quad B_\rho(a, r) \subseteq B_d\left(a, \frac{r}{1-r}\right)$ -------(1)

Let $x \in B_d\left(a, \frac{r}{1-r}\right)$. Hence $d(a,x) < \frac{r}{1-r}$

$\therefore \quad d(a, x)(1-r) < r$

$\therefore \quad d(a, x) < r[1+d(a, x)]$

$\therefore \quad \frac{d(a,x)}{1+d(a,x)} < r$

$\therefore \quad \rho(a, x) < r$

$\therefore \quad x \in B_\rho(a, r)$

$\therefore \quad B_d\left(a, \frac{r}{1-r}\right) \subseteq B_\rho(a, r)$ -------(2)

$\therefore$ By (1) and (2) we get $B_d\left(a, \frac{r}{1-r}\right) = B_\rho(a, r)$ --------(3)

Let G be open in (M, ρ)

Let $a \in G$. Hence there exists $r>0$ such that $B_\rho(a, r) \subseteq G$.

Without loss of generality we may assume that $r<1$.

$\therefore \quad B_d\left(a, \frac{r}{1-r}\right) \subseteq G$ (by (3))

$\therefore$ G is open in (M, d)

Conversely let G be open in (M, d)

$\therefore$ There exists $r>0$ such that $B_d(a, r) \subseteq G$

$\therefore \quad B_\rho\left(a, \frac{r}{1-r}\right) \subseteq G$ (using (3))

$\therefore$ G is open in (M, ρ)

Hence d and ρ are equivalent metrics.

Example :

If d and ρ are metrics onM and if there exists $K>1$ such that $\frac{1}{K}\rho(x, y) \leq d(x, y) \leq K\rho(x,y)$ for all $x,y \in M$. Prove that d and ρ are equivalent metrics.

Solution :

Suppose there exists $K>1$ such that for all $x, y \in M$.

$$\frac{1}{K}\rho(x, y) \leq d(x, y) \leq K\rho(x,y) \qquad \text{-------(1)}$$

Let G be an open set in (M, d).

Let $a \in G$. Hence there exists $r>0$ such that $B_d(a, r) \subseteq G$ -------(2)

We now claim that $B_\rho\left(a, \frac{r}{K}\right) \subseteq G$

Let $x \in B_\rho\left(a, \frac{r}{K}\right)$

$\therefore \quad \rho(a, x) < \frac{r}{K}$

$\therefore \quad K\rho(a, x) < r$

$\therefore \quad d(a, x) < r$ (using (1))

$\therefore \quad x \in B_d(a, r) \subseteq G$ (by (2))

$\therefore \quad x \in G.$

Hence $B_\rho\left(a, \frac{r}{K}\right) \subseteq G$

$\therefore$ G is open in (M, ρ)

Conversely let G be open in (M, ρ). Let $a \in G$.

There exists $r>0$ such that $B_\rho(a, r) \subseteq G$ ------(3)

We claim $B_d\left(a, \frac{r}{K}\right) \subseteq G$

Let $x \in B_d\left(a, \frac{r}{K}\right)$

$d(a, x) < \frac{r}{K}$

$Kd(a, x) < r$

$\rho(a, x) < r$ (using (1))

$$x \in B_\rho(a, r) \subseteq G \qquad \text{(by (3))}$$

$$x \in G.$$

Hence $B_d\left(a, \frac{r}{K}\right) \subseteq G$

Hence G is open in (M, d)

$\therefore$ d and ρ are equivalent metrics.

Exercise :

1. Determine which of the following subsets of R are open in R with usual metric.

   (i) $(1, 2) \cup (3, 4)$ (ii) $\left(-\frac{1}{2}, \frac{1}{2}\right) \cup \{1\}$ (iii) $(-\infty, a)$

   (iv) $(-\infty, a]$ (v) (a, ∞)

2. Determine which of the following subsets of C are open in C with usual metric.

   (i) $\{z/1 < |z| \leq 2\}$ (ii) $\{z / |z| < 1\}$

   (iii) $\{x+iy/x \geq 0\}$ (iv) $\{x+iy/y>0\}$

3. Prove that any subset of R with usual metric is not an open set in R^2 with usual metric.

4. Prove that the complement of any finite subset of a metric space M is open.

SUBSPACE

Definition :

Let (M, d) be a metric space. Let M_1 be a non-empty subset of M. Then M_1 is also a metric space with the same metric d. We say that (M_1, d) is a **subspace** of (M, d).

Note :

If M_1 is a subspace of M a set which is open in M_1 need not be open in M.

For example, if M=R with usual metric and $M_1=[0,1]$ then $[0, ½)$ is open in M_1 but not open in M.

Theorem :

Let M be a metric space and M_1 a subspace of M. Let $A_1 \subseteq M_1$. Then A_1 is openin M_1 iff there exists an open set A in M such tht $A_1 = A \cap M_1$.

Proof :

Let M_1 be a subspace of M. Let $a \in M_1$. We denote $B_1(a, r)$ the open ball in M_1 with centre a, radius r.

Then $B_1(a, r) = \{x \in M_1 / d(a, x) < r\}$

Also $B(a, r) = \{x \in M / d(a, x) < r\}$

Hence $B_1(a, r) = B(a, r) \cap M_1$ -------(1)

Let A_1 be an open set in M_1.

$$A_1 = \bigcup_{x \in A_1} B_1(x, r(x))$$

$$= \bigcup_{x \in A_1} [B(x, r(x)) \cap M_1] \quad \text{(by (1))}$$

$$= \left[\bigcup_{x \in A_1} B(x, r(x))\right] \cap M_1$$

$= A \cap M_1$ where $A = \bigcup_{x \in A_1} B(x, r(x))$ which is open in M.

Conversely let $A_1 = A \cap M_1$ where A is open in M.

We claim that A_1 is open in M_1.

Let $x \in A_1$.

$\therefore$ $x \in A$ and $x \in M_1$.

Since A is open in M there exists a positive real number r such that $B(x, r) \subseteq A$.

$\therefore$ $M_1 \cap B(x, r) \subseteq M_1 \cap A$

i.e., $B_1(x, r) \subseteq A_1$ (using (1))

$\therefore$ A_1 is open in M_1.

Example :

Let M = R and $M_1 = [0, 1]$. Let $A_1 = \left[0, \frac{1}{2}\right)$. Now $A_1 = \left[0, \frac{1}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right) \cap [0,1]$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is open in R.

$\therefore$ $\left[0, \frac{1}{2}\right)$ is open in [0, 1].

INTERIOR OF A SET

Definition :

Let (M, d) be a metric space. Let $A \subseteq M$. Let $x \in A$. Then x is said to be an **interior point** of A if there exists a positive real number r such that $B(x, r) \subseteq A$.

The set of all interior points of A is called the **interior** of A and it is denoted by Int A.

Note : Int $A \subseteq A$.

Example :

Consider R with usual metric.

(a) Let $A=[0, 1]$. Clearly 0 and 1 are not interior points of A and any point $x \in (0, 1)$ is an interior point of A. Hence Int A = (0, 1)

(b) Let A be a finite subset of R. Then Int A = ϕ.

Theorem :

Let (M, d) be a metric space. Let $A, B \subseteq M$.

(i) A is open iff A = Int A

In particular Int $\phi = \phi$ and Int M = M

(ii) Int A = union of all open sets contained in A.

(iii) Int A is an open subset of A and if B is any other open set contained in A then $B \subseteq$ Int A. i.e., Int A is the largest open set contained in A.

(iv) $A \subseteq B \Rightarrow$ Int $A \subseteq$ Int B.

(v) Int $(A \cap B)$ = Int $A \cap$ Int B

(vi) Int$(A \cup B) \supseteq$ Int $A \cup$ Int B

Proof :

(i) Follows from the definition of open set.

(ii) Let $G = \cup\{B/B$ is an open subset of $A\}$

To prove that Int A = G

Let $x \in$ Int A

$\therefore$ There exists a positive real number r such that $B(x, r) \subseteq A$

Thus B(x, r) is an open set contained in A.

∴ $B(x, r) \subseteq G$

∴ $x \in G$

∴ Int $A \subseteq G$ ------(1)

Let $x \in G$

Then there exists an open set B such that $x \in B$ and $B \subseteq A$

Since B is open and $x \in B$ there exists a positive real number such that $B(x, r) \subseteq B \subseteq A$

∴ x is an interior point of A

Hence $G \subseteq$ Int A -------(2)

From (1) and (2), we get G = Int A.

(iii) Since union of any collection of open sets is open (ii) ⇒ Int A is an open set.

Trivially Int $A \subseteq A$

Let B be any open set contained in A.

Then $B \subseteq G$ = Int A (by (2))

∴ Int A is the largest open set contained in A.

(iv) Let $x \in$ Int A

∴ There exists a real number $r>0$ such that $B(x, r) \subseteq A$

But $A \subseteq B$. Hence $B(x, r) \subseteq B$

∴ $x \in$ Int B. Hence Int $A \subseteq$ Int B

(v) $A \cap B \subseteq A$

∴ Int$(A \cap B) \subseteq$ Int A (by (iv))

Similarly Int $(A \cap B) \subseteq$ Int B

∴ Int $(A \cap B) \subseteq$ Int $A \cap$ Int B --------(1)

Int $A \subseteq A$; Int $B \subseteq B$

Hence Int $A \cap$ Int $B \subseteq A \cap B$

Thus Int $A \cap$ Int B is an open set contained in $A \cap B$. But Int$(A \cap B)$ is the largest open set contained in $A \cap B$.

∴ Int $A \cap$ Int $B \subseteq$ Int$(A \cap B)$ -------(2)

From (1) and (2) we get Int $(A \cap B)$ = Int $A \cap$ Int B.

(vi) $A \subseteq A \cup B$

$\therefore$ Int $A \subseteq$ Int$(A \cup B)$ (by (iv))

Similarly Int $B \subseteq$ Int $(A \cup B)$

$\therefore$ Int $A \cup$ Int $B \subseteq$ Int $(A \cup B)$

Note :

Int $(A \cap B)$ need not be equal to Int $A \cup$ Int B. For example, in R with usual metric consider.

$A = (0, 2]$ and $B = (2, 3)$

Then $A \cup B = (0, B)$. Clearly Int $(A \cup B) = (0, 3)$

But Int $A \cup$ Int $B = (0, 2) \cup (2, 3) = (0, 3) - \{2\}$

$\therefore$ Int $(A \cup B) \neq$ Int $A \cup$ Int B

CLOSED SETS

Definition :

Let (M, d) be a metric space. Let $A \subseteq M$. Then A is said to be closed in M if the complement of A is open in M.

Example 1 :

In R with usual metric any closed interval [a, b] is closed set.

Proof :

$[a, b]^C = R - [a, b] = (-\infty, a) \cup (b, \infty)$

Also $(-\infty, a)$ and (b, ∞) are open in R.

i.e., $[a, b]^C$ is open in R.

$\therefore$ [a, b] is closed in R.

Example 2 :

In R with usual metric [a, b) is neither closed nor open.

Proof :

[a, b) is not open in R since a is not an interior point of [a, b).

$[a, b)^C = R - [a, b) = (-\infty, a) \cup [b, \infty)$ and this set is not open since b is not an interior point.

$\therefore$ [a, b) is not closed in R.

Hence [a, b) is neither open nor closed in R.

Example 3 :

Z is closed.

Proof :

$$Z^C = \bigcup_{n=-\infty}^{\infty} (n, n+1)$$

The open interval (n, n+1) is open and union of open sets is open.

Z^C is open. Hence Z is closed.

Example 4 :

Q is not closed in R.

Proof :

Q^C = the set of irrationals which is not open in R.

$\therefore$ Q is not closed in R.

Example 5 :

In R with usual metric every singleton set is closed.

Proof :

Let $a \in R$

Then $\{a\}^C = R - \{a\} = (-\infty, a) \cup (a, \infty)$

Since $(-\infty, a)$ and (a, ∞) are both open sets, $(-\infty, a) \cup (a, \infty)$ is open.

$\therefore$ $\{a\}^C$ is open in R. Hence $\{a\}$ is closed in R.

Definition :

Let (M, d) be a metric space. Let $a \in M$. Let r be any positive real number. Then the **closed ball** or the **closed sphere** with centre a and radius denoted by $B_d[a, r]$ is defined by $B_d[a, r] = \{x \in M / d(a, x) \leq r\}$.

When the metric d under consideration is clear we write B[a, r] instead of $B_d[a,r]$.

Example 1 :

In R with usual metric B[a, r]=[a–r. a+r]

Example 2 :

In R^2 with usual metric let.

$a = (a_1, a_2) \in R^2$

Then $B[a, r] = \{(x, y) \in R^2 \mid d((a_1,a_2), (x, y)) \leq r\}$

$= \{(x,y) \in R^2 \mid (x-a_1)^2+(y-a_2)^2 \leq r^2\}$

Hence B[a, r] is the set of all points which lie within and on the circumference of the circle with centre a and radius r.

Theorem :

In any metric space every closed ball is a closed set.

Proof :

Let (M, d) be a metric space.

Let B[a, r] be a closed ball in M.

Case (i) :

Suppose $B[a, r]^C = \phi$.

$\therefore$ $B[a, r]^C$ is open and hence B[a, r] is closed.

Case (ii) :

Suppose $B[a, r]^C \neq \phi$

Let $x \in B[a, r]^C$

$\therefore$ $x \notin B[a, r]$

$\therefore$ $d(a, x) > r$

$\therefore$ $d(a, x)-r > 0$

Let $r_1 = d(a, x)-r$

We claim that $B(x, r_1) \subseteq B[a, r]^C$

Let $y \in B(x, r_1)$

Then $d(x, y)<r_1 = d(a, x)-r$

$\therefore \quad d(a, x) > d(x, y)+r$ --------(1)

$d(a, x) \leq d(a, y)+d(y, x)$

$\therefore \quad d(a, y) \geq d(a, x)-d(y, x)$

$> d(x, y)+r-d(y, x)$ (by (1))

$= r$

Thus $d(a, y) > r$

$\therefore \quad y \notin B[a, r]$

Hence $y \in B[a, r]^C$

$\therefore \quad B(x, r_1) \subseteq B[a, r]^C$

$\therefore$ $B[a, r]^C$ is open in M

$\therefore$ $B[a, r]$ is closed in M.

Theorem :

In any metric space M, (i) ϕ is closed. (ii) M is closed.

Proof :

Since $M^C = \phi$ is open, M is closed. Similarly $\phi^C = M$ is open and hence is ϕ closed.

Note : In any metric space M, ϕ and M are both open and closed.

Theorem :

In any metric space **arbitrary intersection** of closed sets is closed.

Proof :

Let (M, d) be a metric space.

Let $\{A_i / i \in I\}$ be a collection of closed sets.

We claim that $\bigcap_{i \in I} A_i$ is closed.

We have $\left(\bigcap_{i \in I} A_i\right)^C = \bigcup_{i \in I} A_i^C$ (by De Morgan's law)

Since A_i is closed A_i^C is open.

Hence $\bigcup_{i \in I} A_i^C$ is open.

$\therefore \left(\bigcap_{i\in I} A_i\right)^C$ is open.

$\therefore \bigcap_{i\in I} A_i$ is closed.

Theorem :

In any metric space the union of a **finite number** of closed sets is closed.

Proof :

Let (M, d) be a metric space.

Let $A_1, A_2, \ldots, A_n$ be closed sets in M.

By De-Morgan's law $(A_1 \cup A_2 \cup \ldots\ldots \cup A_n)^C = A_1^C \cap A_2^C \cap \ldots\ldots \cap A_n^C$

Since each A_i is closed A_i^C is open

Hence $A_1^C \cap A_2^C \cap \ldots\ldots \cap A_n^C$ is open

$\therefore (A_1 \cup A_2 \cup A_3 \cup \ldots\ldots\ldots \cup A_n)^C$ is open.

Hence $A_1 \cup A_2 \cup \ldots\ldots\ldots \cup A_n$ is closed.

Note : The union of an infinite collection of closed sets need not be closed.

For example, consider R with usual metric.

Let $A_n = \left[\frac{1}{n}, 1\right]$ where n = 1, 2, 3,.......

Then $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right] = \{1\} \cup \left[\frac{1}{2}, 1\right] \cup \left[\frac{1}{3}, 1\right] \cup \ldots\ldots$

$= (0, 1]$ which is not closed in R.

$\therefore \bigcup_{n=1}^{\infty} A_n$ is not closed

Theorem :

Let M be a metric space and M_1 be a subspace of M. Let $F_1 \subseteq M_1$. Then F_1 is closed in M_1 iff there exists a set F which is closed in M such that $F_1 = F \cap M_1$.

Proof :

Let F_1 be closed in M_1.

∴ M_1-F_1 is open in M_1.

∴ $M_1-F_1 = A \cap M_1$ where A is open in M.

$$F = M_1-(A \cap M_1)$$

$$= M_1-A = A^C \cap M_1$$

Also since A is open in M, A^C is closed in M.

∴ $F_1 = F \cap M_1$ where $F = A^C$ is closed in M.

Conversely, there exists a set F which is closed in M such that $F_1 = F \cap M_1$

To prove That F_1 is closed in M_1.

$$F_1 = F \cap M_1 \text{ where F is closed in M}$$

Let $F = A^C$

∴
$$F_1 = F \cap M_1$$
$$= A^C \cap M_1$$
$$= M_1 \cap A^C$$
$$= M_1-A = M_1-(A \cap M_1)$$

Since A^C is closed in M, A is open in M.

So, $F_1 = M_1-(A \cap M_1) \Rightarrow M_1-F_1 = A \cap M_1$ which is open in M_1.

M_1-F_1 is open in $M_1 \Rightarrow F_1$ is closed in M_1.

Exercise :

1. Prove that any finite subset of a metric space is closed.
2. Let M_1 be a subspace of a metric space M. Prove that every closed set A_1 of M_1 is closed in M iff M_1 itself is closed in M.
3. Let M_1 be a subspace of a metric space M. Prove that every closed set A_1 of M_1 is closed in M iff M_1 itself is closed in M.

Closure :

Let (M, d) be a metric space. Let $A \subseteq M$. Consider the collection of all closed sets which contain A. This collection is non empty since at least M is a member of this collection.

Definition :

Let A be a subset of metric space (M, d). The **closure** of A denoted by $\overline{A}$ is defined to be the intersection of all closed sets which contain A. Thus $\overline{A} = \cup\{B/B$ is closed in M and $A \subseteq B\}$.

Note : Since intersection of any collection of closed sets is closed $\overline{A}$ is a **closed set.** Further $\overline{A} \supseteq A$. Also if B is any clsoed set containing A then $\overline{A} \subseteq B$. Thus $\overline{A}$ is the **smallest closed set containing A.**

Theorem :

A is closed iff $A = \overline{A}$

Proof :

Suppose $A = \overline{A}$

Since $\overline{A}$ is closed A is closed. Conversely, suppose A is closed. Then the smallest closed set contains A is A itself.

∴ $A = \overline{A}$

Note :

In particular (i) $\phi = \overline{\phi}$ (ii) $M = \overline{M}$ (iii) $A = \overline{A}$

Theorem :

Let (M, d) be a metric space. Let $A, B \subseteq M$

Then (i) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$

(ii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(iii) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

Proof :

(i) Let $A \subseteq B$

$\overline{B} \supseteq B \supseteq A$

∴ $\overline{B}$ is a closed set containing A.

But $\overline{A}$ is the smallest closed set containing A.

∴ $\overline{A} \subseteq \overline{B}$

(ii) We have $A \subseteq A \cup B$

$\therefore \overline{A} \subseteq \overline{A \cup B}$ (by (i))

Similarly $\overline{B} \subseteq \overline{A \cup B}$

$\therefore \overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$ --------(1)

$\overline{A}$ is a closed set containing A and $\overline{B}$ is a closed set containing B.

$\therefore \overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$

But $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$

$\therefore \overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$ -------- (2)

From (1) and (2) we get $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(iii) We have $A \cap B \subseteq A$

$\overline{A \cup B} \subseteq \overline{A}$ (by (i))

Similarly, $\overline{A \cap B} \subseteq \overline{B}$

$\therefore \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

Note :

$\overline{A \cap B}$ need not be equal to $\overline{A} \cap \overline{B}$.

For example in R with usual matric, take A=(0, 1) and B=(1, 2)

Then $A \cap B = \phi$

$\therefore \overline{A \cap B} = \overline{\phi} = \phi$

But $\overline{A} \cup \overline{B} = [0, 1] \cap [1, 2] = \{1\}$

$\therefore \overline{A \cap B} \neq \overline{A} \cap \overline{B}$

LIMIT POINT

Definition :

Let (M, d) be a metric space. Let $A \subseteq M$. Let $x \in M$. Then x is called a **limit point** or a **cluster point** or an **accumulation point** of A if every open ball with centre x contains at least one point of A differetn from x.

i.e., $B(x, r) \cap (A - \{x\}) \neq \phi$ for all $r > 0$.

The set of all limit points of A is called the **derived set** of A and is denoted by D(A).

Note : x is not a limit point of A iff there exists an open ball B(x, r) such that $B(x, r) \cap (A - \{x\}) = \phi$.

Example 1 :

Consider R with usual metric.

Let A = [0, 1)

Any open ball with centre 0 is of the form (–r, r) which contains a point of [0, 1) other than 0.

Hence 0 is a limit point of [0,1)

Similarly 1 is a limit point of [0, 1)

2 is not a limit point of A, since $\left(2-\frac{1}{2}, 2+\frac{1}{2}\right) \cap [0,1) = \left(\frac{3}{2}, \frac{5}{2}\right) \cap [0,1) = \phi$

In this case all points of [0, 1] are limit points of [0, 1) and no other point is a limit point. Hence D[0, 1) = [0, 1]

Example 2 :

Z has no limit point.

Let x be any real number.

If x is an integer, then $B\left(x, \frac{1}{2}\right) = \left(x-\frac{1}{2}, x+\frac{1}{2}\right)$ does not contain any integer other than x. Hence x is not a limit point of Z.

If x is not an integer, let n be the integer which is closert to x.

Choose r such that $0 < r < |x-n|$

Then B(x, r) = (x–r, x+r) contains no integer.

Hence x is not a limit point of Z.

Since x is arbitrary Z has no limit point.

∴ D(Z) = ϕ.

Example 3 :

Let (M, d) be a discrete metric space. Let $A \subseteq M$. Let $x \in M$

Then $B\left(x, \frac{1}{2}\right) \cap (A - \{x\}) = \{x\} \cap (A - \{x\}) = \phi$

$\therefore$ x is not a limit point of A

Since $x \in M$ is arbitrary A has no limit point.

$\therefore D(A) = \phi$

Thus any subset of a discrete metric space has no limit point.

Theorem :

Let (M, d) be a metric space. Let $A \subseteq M$. Then x is a limit point of A iff each open ball with centre x contains an infinite number of points of A.

Proof :

Let x be a limit piont of A.

Suppose an open ball B(x, r) contains only a finite number of points of A.

Let $B(x, r) \cap (A - \{x\}) = \{x_1, x_2, \ldots\ldots, x_n\}$

Let $r_1 = \min\{d(x, x_i)/i = 1, 2, \ldots\ldots, n\}$

Since $x \neq x_i$, $d(x, x_i) > 0$ for all $i = 1, 2, \ldots., n$ and hence $r_1 > 0$

Also $B(x, r_1) \cap (A - \{x\}) = \phi$

$\therefore$ x is not a limit point of A which is a contradiction.

Hence every open ball with centre x contains infinite number of points of A.

The converse is obvious.

Corollary :

Any finite subset of a metric space has no limit point.

Proof :

Let A be a finite subset of M.

Suppose A has limit point say x. Then B(x, r) contains infinite number of points of A. This is a contradiction since A is finite.

Theorem :

Let M be a metric space and $A \subseteq M$. Then $\overline{A} = A \cup D(A)$

Proof :

Let $x \in A \cup D(A)$. We shall prove that $x \in \overline{A}$

Suppose $x \notin \overline{A}$

$\therefore$ $x \in M - \overline{A}$ and since $\overline{A}$ is closed $M - \overline{A}$ is open.

$\therefore$ There exists an open ball $B(x, r) \subseteq M - \overline{A}$

$\therefore$ $B(x, r) \cap \overline{A} = \phi$

$\therefore$ $B(x, r) \cap A = \phi$ (since $A \subseteq \overline{A}$)

$\therefore$ $x \notin A \cup D(A)$ which is a contradiction

$\therefore$ $x \in \overline{A}$

$\therefore$ $A \cup D(A) \subseteq \overline{A}$ --------(1)

Now let $x \in \overline{A}$. To prove $x \in A \cup D(A)$

If $x \in A$, Clearly $x \in A \cup D(A)$

Suppose $x \notin A$. We claim that $x \in D(A)$

Suppose $x \notin D(A)$. Then there exists an open ball $B(x, r)$ such that $B(x, r) \cap A = \phi$.

$\therefore$ $B(x, r)^C \supseteq A$ and $B(x, r)^C$ is closed.

But $\overline{A}$ is the smallest closed set containing A.

$\therefore$ $\overline{A} \subseteq B(x, r)^C$

But $x \in \overline{A}$ and $x \notin B(x, r)^C$ which is a countradiction

Hence $x \in D(A)$

$\therefore$ $x \in A \cup D(A)$

$\therefore$ $\overline{A} \subseteq A \cup D(A)$ --------(2)

From (1) and (2) we get $\overline{A} = A \cup D(A)$

Corollary 1 :

A is closed iff A contains all its limit points.

Corollary 2 :

$x \in \overline{A} \Leftrightarrow B(x, r) \cap A \neq \phi$ for all $r>0$.

DENSE SETS

Definition :

1) A subset A of a metric space M is said to be **dense** in M or **everywhere dense** if $\overline{A} = M$.

2) A metric space M is said to be separable if there exists a countable dense subset in M.

Example 1 :

Let M be a metric space. Trivially, M is dense in M.

Hence any countable metric space is separable.

Example 2 :

Let M be a discrete metric space.

Let $A \subset M$ and $A \neq M$

Since A is closed, $\overline{A} = A$

A is not dense.

Hence any uncountable discrete metric space is not separable.

Theorem :

Let M be a metric space and $A \subseteq M$. Then the following are equivalent.

(i) A is dense in M

(ii) The only closed set which contains A is M

(iii) The only open set disjoint from A is ϕ

(iv) A intersects every non-empty open set

(v) A intersects every open ball

Proof :

(i) $\Rightarrow$ (ii)

Suppose A is dense in M

Then $\overline{A} = M$ -------(1)

Let $F \subseteq M$ be any closed set containing A.

Since $\overline{A}$ is the smallest closed set containing A, we have $\overline{A} \subseteq F$

Hence $M \subseteq F$ (by (1))

∴ $M = F$

∴ The only closed set which contain A is M.

(ii) ⇒ (iii)

Suppose (iii) is not true

Then there exists a non empty open set B such that $B \cap A = \phi$

∴ B^C is a closed set and $B^C \supseteq A$

Further, since $B \neq \phi$ we have $B^C \neq M$ which is a contradiction to (ii)

Hence (ii) ⇒ (iii)

Obviously (iii) ⇒ (iv)

(iv) = (v), since every open ball is an open set.

(v) ⇒ (i)

Let $x \in M$. Suppose every open ball $B(x, r)$ intersects A.

Then by corolary (2) $x \in \overline{A}$.

∴ $M \subseteq \overline{A}$

But trivially $\overline{A} \subseteq M$

∴ $\overline{A} = M$

∴ A is dense in M.

Exercise :

1. Prove that any finite subset of a metric space is closed.
2. Prove that the set of all limit points of a subset of a metric space is closed.
3. If G is an open set and $G \cap A = \phi$ prove that $G \cap \overline{A} = \phi$
4. Prove that in a metric space M, the only set which is both closed and dense is M.

COMPLETE METRIC SPACE

COMPLETENESS

Definition :

Let (M, d) be a metric space. Let $(x_n) = x_1, x_2, \ldots, x_n \ldots$ be a sequence of points in M. Let $x \in M$. We say that (x_n) **converges** to x if given $\in>0$ there exists a positive integer n_0 such that $d(x_n, x)<\in$ for all $n \geq n_0$. Also x is called a **limit** of (x_n). If (x_n) converges to x we write $\underset{n\to\infty}{\text{Lt}}\ x_n = x$ or $(x_n)\to x$.

Note 1 : $(x_n)\to x$ iff for each open ball $B(x, \in)$ with centre x there exists a positive integer n_0 such that $x_n \in B(x, \in)$ for all $n \geq n_0$. Thus the open ball $B(x, \in)$ contains all but a finite number of terms of the sequence.

Note 2 : $(x_n)\to x$ iff the sequence of real numbers $(d(x_n, x))\to 0$.

Theorem 1 :

For a convergent sequences (x_n) the limit is unique.

Proof :

Suppose $(x_n)\to x$ and $(x_n)\to y$.

Let $\in>0$ be given. Then there exist positive integers n_1 and n_2 such that $d(x_n, x)<\frac{1}{2}\in$ for all $n \geq n_1$ and $d(x_n, y)<\frac{1}{2}\in$ for all $n \geq n_2$. Let m be a positive integer such that $m \geq n_1, n_2$.

Then
$$d(x, y) \leq d(x, x_m)+d(x_m, y)$$
$$< \frac{1}{2}\in+\frac{1}{2}\in = \in$$

$\therefore \quad d(x, y) < \in$

Since $\in>0$ is arbitrary, $d(x, y) = 0$

$\therefore x = y$

Theorem 2 :

(i) Let M be a metric space and $A \subseteq M$. Then (i) $x \in \overline{A}$ iff there exists a sequence (x_n) in A such that $(x_n) \to x$.

(ii) x is a limit point of A iff there exists a sequence (x_n) of distinct points in A such that $(x_n) \to x$.

Proof :

Let $x \in \overline{A}$

Then $x \in A \cup D(A)$

∴ $x \in A$ or $x \in D(A)$

If $x \in A$, then the constant sequence x, x, ..., is a sequence in A converging to x.

If $x \in D(A)$ then the open ball $B\left(x, \frac{1}{n}\right)$ contains infinite number of points of A.

∴ We can choose $x_n \in B\left(x, \frac{1}{n}\right) \cap A$ such that $x_n \neq x_1, x_2, \ldots . . x_{n-1}$ for each n.

∴ (x_n) is a sequence of distinct points in A.

Also $d(x_n, x) < \frac{1}{n}$ for all n.

∴ $\underset{n \to \infty}{\text{Lt}} d(x_n, x) = 0$

∴ $(x_n) \to x$

Conversely, suppose there exists a sequence (x_n) in A such that $(x_n) \to x$. Then for any $r>0$ there exists a positive integer n_0 such that $d(x_n, x)<r$ for all $n \geq n_0$.

∴ $x_n \in B(x, r)$ for all $n \geq n_0$ -------(1)

∴ $B(x, r) \cap A \neq \phi$

∴ $x \in \overline{A}$

Further if (x_n) is a sequence of distinct points $B(x,r) \cap A$ is infinite.

∴ $x \in D(A)$

∴ x is a limit point of A.

Definition :

Let (M, d) be metric space. Let (x_n) be a sequence of points inM. (x_n) is said to be a **cauchy sequence** in M if given $\in>0$ there exists a positive integer n_0 such that $d(x_m, x_n)<\in$ for all $m, n\geq n_0$.

Theorem 3 :

Let (M, d) be a metric space. Then any convergent sequence in M is a Cauchy sequence.

Proof :

Let (x_n) be a convergent sequence in M converging to $x\in M$.

Let $\in>0$ be given.

Then there exists a positive integer n_0 such that $d(x_n, x)<\frac{1}{2}\in$ for all $n\geq n_0$.

$$\therefore \quad d(xn, xm) \leq d(x_n, x)+d(x, x_m)$$
$$< \frac{1}{2}\in+\frac{1}{2}\in \text{ for all } n, m \geq n_0$$
$$= \in \text{ for all } m, n \geq n_0$$

Thus $\quad d(x_n, x_m) < \quad$ for all $m, n \geq n_0$

$\therefore (x_n)$ is a Cauchy sequence.

Note :

The converse of the above theorem is not true.

For example, consider the metric space (0, 1] with usual metric.

$\left(\frac{1}{n}\right)$ is a Cauchy sequence in (0, 1]. But this sequence does not converge to any point in (0, 1].

Definition :

A metric space M is said to be **complete** if every Cauchy sequence in M converges to a point in M.

Example 1 :

R with usual metric is complete.

Example 2 :

C with usual metric is complete.

Proof :

Let (Z_n) be a Cauchy sequence in C.

Let $Z_n = x_n+iy_n$ where $x_n, y_n \in R$

We claim that (x_n) and (y_n) are Cauchy sequences in R.

Let $\epsilon>0$ be given.

Since (Z_n) is a Cauchy sequence, there exists a positive integer n_0 such that $|Z_n-Z_m|<\epsilon$ for all n, m $\geq n_0$.

Now $|x_n-x_m| \leq |Z_n-Z_m|$ and $|y_n-y_m| \leq |Z_n-Z_m|$

Hence $|x_n-x_m|<\epsilon$ for all n, m $\geq n_0$ and

$|y_n-y_m|<\epsilon$ for all n, m $\geq n_0$

$\therefore$ (x_n) and (y_n) are Cauchy sequences in R.

Since R is complete, there exist x, y $\in$ R such that $(x_n)\to x$ and $(y_n)\to y$

Let $Z = x+iy$. We claim that $(Z_n)\to Z$

We have
$$|Z_n-Z| = |(x_n+iy_n)-(x+iy)|$$
$$= |(x_n-x)+i(y_n-y)|$$
$$\leq |x_n-x|+|y_n-y| \quad \text{--------(1)}$$

Let $\epsilon>0$ be given.

Since $(x_n)\to x$ and $(y_n)\to y$ there exist positive integers n_1 and n_2 such that

$$|x_n-x| < \frac{1}{2}\epsilon \text{ for all } n\geq n_1 \text{ and}$$

$$|y_n-y| < \frac{1}{2}\epsilon \text{ for all } n\geq n_2$$

Let $$n_3 = \max\{n_1, n_2\}$$

From (1) we get $$|Z_n-Z| < \frac{1}{2}\epsilon+\frac{1}{2}\epsilon = \epsilon \text{ for all } n\geq n_3$$

$\therefore$ $$(Z_n) \to Z$$

$\therefore$ C is complete

Example 3 :

Any discrete metric space is complete.

Proof :

Let (M, d) be a discrete metric space. Let (x_n) be a Cauchy sequence in M. Then there exists a positive integer n_0 such that $d(x_n, x_m)<\frac{1}{2}$ for all $n,m\geq n_0$

Since d is the discrete metric distance between any two points is either 0 or 1.

∴ $d(x_n, x_m) = 0$ for all $n, m\geq n_0$

∴ $x_n = x_{n_0} = x$(say) for all $n\geq n_0$

∴ $d(x_n, x) = 0$ for all $n\geq n_0$

∴ $(x_n)\to x$.

Hence M is complete.

Example 4 :

R^n with usual metric is complete.

Proof :

Let (x_p) be a Cauchy sequence in R^n.

Let $x_p = (x_{p_1}, x_{p_2}, \ldots\ldots, x_{p_n})$ Let $\epsilon>0$ be given.

Then there exists a positive integer n_0 such that $d(x_p, x_q)<\epsilon$ for all $p, q \geq n_0$

∴ $\left[\sum_{k=1}^{n}\left(x_{p_k} - x_{q_k}\right)^2\right]^{1/2} <\epsilon$ for all $p, q \geq n_0$

∴ $\sum_{k=1}^{n}\left(x_{p_k} - x_{q_k}\right)^2 <\epsilon^2$ for all $p, q \geq n_0$

∴ For each $k = 1,2,\ldots\ldots,n$ we have $|x_{p_k} - x_{q_k}|<\epsilon$ for all $p, q \geq n_0$

∴ (x_{p_k}) is a Cauchy sequence in R for each $k = 1,2,\ldots\ldots,n$.

Since R is complete, there exists $y_k\in R$ such that $(x_{p_k})\to y_k$.

Let $y = (y_1, y_2,\ldots\ldots,y_n)$. We claim that $(x_p)\to y$.

Since $(x_{p_k}) \to y_k$ there exists a positive integer m_k such that

$$|x_{p_k} - y_k| < \frac{\epsilon}{\sqrt{n}} \text{ for all } p \geq m_k.$$

Let $m_0 = \max\{m_1, m_2, \ldots\ldots, m_n\}$

Then
$$d(x_p, y) = \left[\sum_{k=1}^{n} \left(x_{p_k} - y_k\right)^2\right]^{1/2}$$

$$< \left[n\left(\frac{\epsilon}{\sqrt{n}}\right)^2\right]^{1/2} \quad \text{for all } p \geq m_0$$

$$= \epsilon \text{ for all } p \geq m_0$$

Thus $d(x_p, y) < \epsilon$ for all $p \geq m_0$.

$\therefore (x_p) \to y$. Hence R^n is complete.

Example 5 :

l_2 is complete.

Proof :

Let (x_p) be a cauchy sequence in l_2.

Let $x_p = \{x_{p_1}, x_{p_2}, \ldots\ldots, x_{p_n}, \ldots\ldots\}$

Let $\epsilon > 0$ be given. Then there exists a positive integer n_0 such that $d(x_p, x_q) < \epsilon$ for all $p, q \geq n_0$.

i.e., $\left[\sum_{n=1}^{\infty} \left(x_{p_n} - x_{q_n}\right)^2\right]^{1/2} < \epsilon$ for all $p, q \geq n_0$

$\therefore \sum_{n=1}^{\infty} \left(x_{p_n} - x_{q_n}\right)^2 < \epsilon^2$ for all $p, q \geq n_0$ --------(1)

For each $n = 1, 2, 3, \ldots\ldots$ We have

$|x_{p_n} - x_{q_n}| < \epsilon$ for all $p, q \geq n_0$.

$\therefore (x_{p_n})$ is a Cauchy sequence in R for each n. Since R is complete, there exists $y_n \in R$ such that $(x_{p_n}) \to y_n$ --------(2)

Let $y = (y_1, y_2, \ldots\ldots, y_n, \ldots\ldots)$

We claim that $y \in l_2$ and $(x_p) \to y$

For any fixed positive integer m, we have

$$\sum_{n=1}^{m} \left(x_{p_n} - x_{q_n}\right)^2 < \in^2 \text{ for all } p, q \geq n_0 \qquad \text{(using (1))}$$

Fixing q and allowing $p \to \infty$ in this finite sum we get

$$\sum_{n=1}^{m} \left(y_n - x_{q_n}\right)^2 \leq \in^2 \text{ for all } q \geq n_0 \qquad \text{(using (2))}$$

Since this is true for every positive integer m

$$\sum_{n=1}^{\infty} \left(y_n - x_{q_n}\right)^2 \leq \in^2 \text{ for all } q \geq n_0 \qquad \text{---------(3)}$$

Now

$$\left[\sum_{n=1}^{\infty} |y_n|^2\right]^{1/2} = \left[\sum_{n=1}^{\infty} \left|y_n - x_{q_n} + x_{q_n}\right|^2\right]^{1/2}$$

$$\leq \left[\sum_{n=1}^{\infty} \left|y_n - x_{q_n}\right|^2\right]^{1/2} + \left[\sum_{n=1}^{\infty} \left|x_{q_n}\right|^2\right]^{1/2}$$

(by Minkowski's inequailty)

$$\leq \in + \left[\sum_{n=1}^{\infty} \left|x_{q_n}\right|^2\right]^{1/2} \text{ for all } q \geq n_0 \quad \text{(by (3))}$$

∴ Since $x_q \in l_2$ we have $\left[\sum_{n=1}^{\infty} \left|x_{q_n}\right|^2\right]^{1/2}$ converges.

∴ $y \in l_2$

Also (3) gives $d(y, x_q) \leq \in$ for all $q \geq n_0$

∴ $(x_p) \to y$

∴ l_2 is complete.

Note :

A subspace of a complete metric space need not be complete.

For example R with usual metric is complete.

But the subspace (0,1] is not complete.

Theorem 4 :

A subset A of a complete metric space M is complete iff A is closed.

Proof :

Suppose A is complete.

To prove that A is closed, we shall prove that A contains all its limit points.

Let x be a limit point of A.

Then there exists a sequence (x_n) in A such that $(x_n) \to x$ (by theorem (2))

Since A is complete, $x \in A$

$\therefore$ A contains all its limit points.

Hence A is closed.

Conversely, let A be a closed subset of M. Let (x_n) be a Cauchy sequence in A. Then (x_n) is a Cauchy sequence in M also and since M is complete there exists $x \in M$ such that $(x_n) \to x$. Thus (x_n) is a sequence in A converging to x.

$\therefore x \in \overline{A}$ (by theorem (2))

Since A is closed $A = \overline{A}$

$\therefore x \in A$

Thus every Cauchy sequence (x_n) in A

converges to a point in A.

$\therefore$ A is complete.

Note 1 :

[0, 1] with usual metric is complete since it is a closed subset of the complete metric space R.

Note 2 :

Consider Q.

Since $\overline{Q} = R$, Q is not a closed subset of R. Hence Q is not complete.

Problem :

Let A, B be subsets of R. Prove that $\overline{A \times B} = \overline{A} \times \overline{B}$

Solution :

Let $(x, y) \in \overline{A \times B}$

$\therefore$ There exists a sequence $((x_n, y_n)) \in A \times B$ such that $((x_n, y_n)) \to (x, y)$ (by theorem (2))

$\therefore$ $(x_n) \to x$ and $(y_n) \to y$

Also (x_n) is a sequence in A and (y_n) is a sequence in B.

$\therefore$ $x \in \overline{A}$ and $y \in \overline{B}$ (by theorem (2))

$\therefore$ $(x, y) \in \overline{A} \times \overline{B}$

$\therefore$ $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$ ---------(1)

Now let $(x, y) \in \overline{A} \times \overline{B}$

$\therefore$ $x \in \overline{A}$ and $y \in \overline{B}$

$\therefore$ There exists a sequence (x_n) in A and a sequence (y_n) in B such that $(x_n) \to x$ and $(y_n) \to y$

$\therefore$ $((x_n, y_n))$ is a sequence in $A \times B$ which converges to (x, y)

$\therefore$ $(x, y) \in \overline{A \times B}$

$\therefore$ $\overline{A} \times \overline{B} \subseteq \overline{A \times B}$ --------(2)

$\therefore$ By (1) and (2) we get $\overline{A} \times \overline{B} = \overline{A \times B}$.

Exercise :

1. Prove that lp is a complete metric space for any $p \geq 1$.
2. Determine which of the following subsets of R are complete.

   (i) (a, b), (ii) (a, b], (iii) $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\right)$ (iv) $[0, 1] \cup [2, 3]$
3. Prove that R^n with $d_1(x,y) = \max\{|x_i - y_i| / i = 1, 2, \ldots. n\}$ is complete.

CANTOR'S INTERSECTION THEOREM

Theorem 5 :

Let M be a metric space. M is complete iff for every sequence (F_n) of non-empty closed subsets of M such that

$F_1 \supseteq F_2 \supseteq \ldots\ldots\ldots\ldots \supseteq F_n \supseteq \ldots\ldots\ldots$ and $(d(F_n)) \to 0$

$\bigcap_{n=1}^{\infty} F_n$ is non empty.

Proof :

Let M be a complete metric space. Let (F_n) be a sequence of closed subsets of M such that

$F_1 \supseteq F_2 \supseteq \ldots\ldots\ldots \supseteq F_n \supseteq \ldots\ldots\ldots$ --------(1)

and $(d(F_n)) \to 0$ --------(2)

We claim that $\bigcap_{n=1}^{\infty} F_n \neq \phi$

For each positive integer n, choose a point $x_n \in F_n$.

By (1), $x_n, x_{n+1}, x_{n+2}, \ldots\ldots$ all lie in F_n

i.e., $x_m \in F_n$ for all $m \geq n$ --------(3)

Since $(d(F_n)) \to 0$, given $\epsilon > 0$, there exists a positive integer n_0, such $d(F_n) < \epsilon$ for all $n \geq n_0$.

In particular $d(Fn) < \epsilon$ --------(4)

∴ $d(x,y) < \epsilon$ for all $x, y \in F_n$

$x_m \in F_{n_0}$ for all $m \geq n_0$ (by (3))

∴ $m, n \geq n_0 \Rightarrow x_m, x_n \in F_{n_0}$

$\Rightarrow d(x_m, x_n) < \epsilon$ (by (4))

∴ (x_n) is a Cauchy sequence in M.

Since M is complete there exists a point $x \in M$ such that $(x_n) \to x$

We claim that $x \in \bigcap_{n=1} F_n$

Now for any positive integer n, $x_n, x_{n+1}, \ldots\ldots$ is a sequence in F_n and this sequence converges to x.

∴ $x \in \overline{F_n}$ (by theorem (2))

But $\overline{F_n}$ is closed and hence $\overline{F_n} = F_n$

∴ $x \in F_n$

$\therefore\ x \in \bigcap_{n=1}^{\infty} F_n$

Hence $\bigcap_{n=1}^{\infty} F_n \neq \phi$.

To prove the converse let, (x_n) be any Cauchy sequence in M.

Let
$$F_1 = \{x_1, x_2, \ldots, x_n, \ldots\}$$
$$F_2 = \{x_2, x_3, \ldots, x_n, \ldots\}$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$F_n = \{x_n, x_{n+1}, \ldots\}$$

Clearly $F_1 \supseteq F_2 \supseteq \ldots \supseteq F_n \supseteq \ldots$

$\therefore\ \overline{F_1} \supseteq \overline{F_2} \supseteq \ldots \supseteq \overline{F_n} \ldots$

$\therefore\ (\overline{F_n})$ is a decreasing sequence of closed sets. Since (x_n) is a Cauchy sequence given $\epsilon>0$ there exists a positive integer n_0, such that $d(x_n, x_m)<\epsilon$ for all $n, m \geq n_0$.

$\therefore$ For any integer $n \geq n_0$, the distance between any two points of F_n is less than ϵ.

$\therefore\ d(F_n) < \epsilon$ for all $n \geq n_0$

But $d(F_n) = d(\overline{F_n})$

$\therefore\ d(\overline{F_n}) < \epsilon$ for all $n \geq n_0$ ------(5)

$\therefore\ (d(\overline{F_n})) \to 0$

Hence $\bigcap_{n=1}^{\infty} \overline{F_n} \neq \phi$

Let $x \in \bigcap_{n=1}^{\infty} \overline{F_n}$. Then x and $x_n \in \overline{F_n}$

$\therefore\ d(x_n, x) \leq d(\overline{F_n})$

$\therefore\ d(x_n, x) < \epsilon$ for all $n \geq n_0$ (by (5))

$\therefore\ (x_n) \to x$

$\therefore$ M is complete.

Note 1 :

In the above theorem $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

For suppose $\bigcap_{n=1}^{\infty} F_n$ contains two distint points x and y.

Then $d(F_n) \geq d(x, y)$ for all n.

$\therefore$ $(d(F_n))$ does not tend to zero which is a contradiction.

$\therefore$ $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

Note 2 :

In the above theorem $\bigcap_{n=1}^{\infty} F_n$ may be empty if each F_n is not closed.

For example, consider $F_n = \left(0, \frac{1}{n}\right)$ in R.

Clearly $F_1 \supseteq F_2 \supseteq \ldots\ldots \supseteq F_n \supseteq \ldots\ldots$ and

$(d(F_n)) = \left(\frac{1}{n}\right) \to 0$ as $n \to \infty$

But $\bigcap_{n=1}^{\infty} F_n = \phi$.

Note 3 :

In this theorem $\bigcap_{n=1}^{\infty} F_n$ may be empty if the hypothesis $(d(F_n)) \to 0$ is omitted.

For example, consider $F_n = [n, \infty)$ in R. Clearly (F_n) is a sequence of closed sets and

$F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots\ldots \supseteq F_n \supseteq \ldots\ldots$

Also $\bigcap_{n=1}^{\infty} F_n = \phi$

Here $d(F_n) = \infty$ for all n and hence the hypothesis $(d(F_n)) \to 0$ is not true.

BAIRE'S CATEGORY THEOREM

Definition :

A subset A of a metric space is said to be **nowhere dense** in M if Int $\overline{A} = \phi$.

Definition :

A subset A of a metric space M is said to be of **first category** in M if A can be expressed as a countable union of nowhere dense sets.

A set which is not of first category is said to be of **second category.**

Note : If A is of first category then $A = \bigcup_{n=1}^{\infty} E_n$ where E_n is nowhere dense subsets in M.

Example 1 :

In R with usual metric.

$A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{n}, \ldots\ldots\right\}$ is nowhere dense.

For $\overline{A} = A \cup D(A) = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{n}, \ldots\ldots\right\}$

Clearly Int $\overline{A} = \phi$

Example 2 :

In any discrete metric space M, any non-empty subset A is not nowhere dense.

For, in a discrete metric space every subset is both open and closed.

∴ $\overline{A}$ = Int $\overline{A}$ = Int A = A

∴ Int $\overline{A} \neq \phi$

∴ A is not nowhere dense.

Example 3 :

In R with usual metric any finite subset A is nowhere dense.

For, let A be any finite subset of R.

ThenA is closed and hence $A = \overline{A}$

Also since A is finite, no point of A is an interior piont of A.

$\therefore$ Int $\overline{A}$ = Int A = ϕ

$\therefore$ A is nowhere dense.

Example 4 :

Consider R with usual metric. Any singleton set $\{x\}$ is nowhere dense.

Any countable subset of R being a countable union of singleton sets is of first category.

In particular Q is of first category.

Note :

If A and B are sets of first category in a metric space M then $A \cup B$ is also of first category.

For, since A and B are of first category in M we have $A = \bigcup_{n=1}^{\infty} E_n$ and $B = \bigcup_{n=1}^{\infty} H_n$ where E_n and H_n are nowhere dense subsets in M.

$\therefore$ $A \cup B$ is a countable union of nowhere dense subsets of M.

Hence $A \cup B$ is of first category.

Theorem 6 : (Baire's Category Theorem) :

Any complete metric space is of second category.

Proof :

Let M be a complete metric space. We claim that M is not of first category. Let (A_n) be a sequence of nowhere dense sets in M.

We claim that $\bigcup_{n=1}^{\infty} A_n \neq M$

Since M is open and A_1 is nowhere dense, there exists an open ball say B_1 of radius less than 1 such that B_1 is disjoint from A_1.

Let F_1 denote the concentric closed ball whose radius is $\frac{1}{2}$ times that of B_1.

Now Int F_1 is open and A_2 is nowhere dense.

$\therefore$ Int F_1 contains an open ball B_2 of radius less than $\frac{1}{2}$ such that B_2 is disjoint from A_2.

Let F_2 be the concentric closed ball whose radius is $\frac{1}{2}$ times that of B_2. Now Int F_2 is open and A_2 is nowhere dense.

∴ Int F_2 contains an open ball B_3 of radius less than $\frac{1}{4}$ such that B_2 is disjoint from A_3.

Proceeding like this we get a sequence of non-empty closed balls F_n such that

$F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots\ldots\ldots \supseteq F_n \supseteq \ldots\ldots$ and $d(F_n) < \frac{1}{2^n}$

Hence $(d(F_n)) \to 0$ as $n \to \infty$

Since M is complete, by Cantor's intersection theorem there exists a point x in M such that $x \in \bigcap_{n=1}^{\infty} F_n$

Also each F_n is disjoint from A_n.

Hence $x \notin A_n$ for all n.

∴ $x \notin \bigcup_{n=1}^{\infty} A_n$

∴ $\bigcup_{n=1}^{\infty} A_n \neq M$. Hence M is of second category.

Corollary :

R is of second category.

Note :

The converse of the above theorem is not true. i.e., A metric space which is of second category need not be complete.

For example, consider $M = R - Q$, the space of irrational numbers.

We know that Q is of first category.

Suppose M is of first category. Then $M \cup Q = R$ is also of first category which is a contradiction.

∴ M is of second category.

Also M is not a closed subspace of R and hence M is not complete.

SOLVED PROBLEMS :

Problem 1 :

Prove that any nonempty open interval (a, b) in R is of second category.

Solution :

Let (a, b) be a nonempty open interval in R.

Suppose (a, b) is of first category.

Now, $[a, b] = (a, b) \cup \{a\} \cup \{b\}$

$\therefore$ [a, b] is of first category.

But [a, b] is a complete metric space and hence is of second category which is a contradiction.

$\therefore$ (a, b) is of second category.

Problem 2 :

Prove that a closed set A in a metric space M is nowhere dense iff A^C is everywhere dense.

Solution :

Let A be a closed set in M.

$\therefore A = \overline{A}$ --------(1)

Suppose A is nowhere dense in M

$\therefore \text{Int } \overline{A} = \phi$

$\therefore \text{Int } A = \phi$ (by (1)) --------(2)

Now we claim that $\overline{A^C} = M$

Obviously $\overline{A^C} \subseteq M$ --------(3)

Let $x \in M$. Let G be any open set such that $x \in G$

Since $\text{Int } A = \phi$ we have $G \not\subseteq A$

$\therefore G \cap A^C \neq \phi$

$\therefore x \in \overline{A^C}$

$\therefore M \subseteq \overline{A^C}$ --------(4)

$\therefore$ By (3) and (4) we have $M = \overline{A^C}$

$\therefore$ A^C is everywhere dense in M.

Conversely let A^C be everywhere dense in M.

$\therefore$ $\overline{A^C} = M$

We claim that Int $A = \phi$

Let G be any nonempty open set in M.

Since $\overline{A} = M$ we have $G \cap A^C \neq \phi$

$\therefore$ $G \not\subset A$

$\therefore$ The only open set which is contained in A is the empty set.

$\therefore$ Int $A = \phi$. $\therefore$ Int $\overline{A} = \phi$ (by (1))

$\therefore$ A is nowhere dense in M.

Exercise :

1. Prove that a subset of a nowhere dense set is a nowhere dense set.

2. Prove that the union of a countable number of sets which are of first category is again of first category.

3. Prove that any complete metric space M in which every finite subset is nowhere dense is uncountable. Deduce that R is uncountable.

   (Hint. If M is countable then M is of first category).

CONTINUITY

The definition of continuity for real valued functions depends on the usual metric of the real line. Hence the concept of continuity can be extended for functions defined from one metric space to another in a natural way.

Definition :

Let (M_1, d_1) and (M_2, d_2) be metric spaces. Let $f:M_1 \to M_2$ be a function. Let $a \in M_1$ and $l \in M_2$. The function f is said to have **limit** as $x \to a$ if given $\epsilon > 0$, there exists $\delta > 0$ such that $0 < d_1(x, a) < \delta \Rightarrow d_2(f(x), l) < \epsilon$.

We write $\underset{x \to a}{\text{Lt}} f(x) = l$

Definition :

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Let $a \in M_1$. A function $f: M_1 \to M_2$ is said to be **continuous** at a if given $\epsilon > 0$, there exists $\delta > 0$ such that $d_1(x, a) < \delta \Rightarrow d_2(f(x)), f(a)) < \epsilon$.

f is said to be **continuous** if it is continuous at every point of M_1.

Note 1 : f is continuous at a iff $\underset{x \to a}{\text{Lt}} f(x) = f(a)$.

Note 2 : The condition $d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \epsilon$ can be rewritten as (i) $x \in B(a, \delta) \Rightarrow f(x) \in B(f(a), \epsilon)$ or (ii) $f(B(a, \delta)) \subseteq B(f(a), \epsilon)$.

Example 1 :

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Then any constant function. $f:M_1 \to M_2$ is continuous.

Proof :

Let $f:M_1 \to M_2$ be given by $f(x) = a$ where $a \in M_2$ is a fixed element.

Let $x \in M_1$ and $\epsilon > 0$ be given.

Then for any $\delta > 0$, $f(B(x, \delta)) = \{a\} \subseteq B(a, \epsilon)$

$\therefore$ f is continuous at x.

Since $x \in M_1$ is arbitrary, f is continuous.

Example 2 :

Let (M_1, d_1) be a discrete metric space and let (M_2, d_2) be any metric space. Then any function $f:M_1 \to M_2$ is continuous. i.e., any function whose domain is a discrete metric space is continuous.

Proof :

Let $x \in M_1$. Let $\epsilon > 0$ be given.

Since M_1 is discrete for any $\delta < 1$, $B(x, \delta) = \{x\}$

$\therefore f(B(x, \delta)) = \{f(x)\} \subseteq B(f(x), \epsilon)$

$\therefore$ f is continuous at x.

Theorem 1 :

Let (M_1, d_1) and (M_2, d_2) be two metric spaces.

Let $a \in M_1$. A function $f:M_1 \to M_2$ is continuous at a iff $(x_n) \to a \Rightarrow (f(x_n)) \to f(a)$.

Proof :

Suppose f is continuous at a.

Let (x_n) be a sequence in M_1 such that $(x_n) \to a$

We claim that $(f(x_n)) \to f(a)$.

Let $\epsilon > 0$ be given. By definition of continuity there exists $\delta > 0$ such that

$$d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \epsilon \qquad \text{--------(1)}$$

Since $(x_n) \to a$ there exists a positive integer n_0 such that $d_1(x_n, a) < \delta$ for all $n \geq n_0$.

$\therefore d_2(f(x_n), f(a)) < \epsilon$ for all $n \geq n_0$ (by (1))

$\therefore (f(xn)) \to f(a)$

Conversely, suppose $(x_n) \to a \Rightarrow (f(x_n)) \to f(a)$

We claim that f is continuous at a.

Suppose f is not continuous at a.

Then there exists an $\epsilon > 0$ such that for all $\delta > 0$, $f(B(a, \delta)) \not\subseteq B(f(a), \epsilon)$

In particular $f(B(a, 1/n)) \not\subseteq B(f(a), \epsilon)$.

Choose x_n such that $x_n \in B(a, 1/n)$ and $f(x_n) \not\subset B(f(a), \in)$.

∴ $d_1(x_n, a) < 1/n$ and $d_2(f(x_n), f(a)) \geq \in$.

∴ $(x_n) \to a$ and $(f(x_n))$ does not converge to $f(a)$ which is a contradiction to the hypothesis.

∴ f is continuosu at a.

Corollary :

A function $f : M_1 \to M_2$ is continuous iff $(x_n) \to x \Rightarrow (f(x_n)) \to f(x)$.

Theorem 2 :

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. $f: M_1 \to M_2$ is continuous iff $f^{-1}(G)$ is open in M_1 whenever G is open in M_2. i.e., f is continuous iff inverse image of every open set is open.

Proof :

Suppose f is continuous.

Let G be an open set in M_2.

We claim that $f^{-1}(G)$ is open in M_1.

If $f^{-1}(G)$ is empty, then it is open.

Let $f^{-1}(G) \neq \phi$

Let $x \in f^{-1}(G)$. Hence $f(x) \in G$

Since G is open, there exists an open ball $B(f(x), \in)$ such that

$$B(f(x), \in) \subseteq G \quad \text{-------(1)}$$

By definition of continuity, there exists an open ball $B(x, \delta)$ such that

$$f(B(x, \delta)) \subseteq B(f(x), \in)$$

∴ $f(B(x, \delta)) \subseteq G$ (by (1))

∴ $B(x, \delta) \subseteq f^{-1}(G)$

Since $x \in f^{-1}(G)$ is arbitrary, $f^{-1}(G)$ is open.

Conversely, suppose $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

We claim that f is continuous.

Let $x \in M_1$.

Now $B(f(x), \epsilon)$ is an open set in M_2.

∴ $f^{-1}(B(f(x), \epsilon)$ is open in M_1 and $x \in f^{-1}(B(f(x), \epsilon))$

∴ There exists $\delta > 0$ such that $B(x, \delta) \subseteq f^{-1}(f(x), \epsilon))$

∴ $f(B(x, \delta)) \subseteq B(f(x), \epsilon)$

∴ f is continuous at x.

Since $x \in M_1$ is arbitrary f is continuous.

Note 1 :

If $f : M_1 \to M_2$ is continuous and G is open in M_1, then it is not necessary that f(G) is open in M_2.

i.e., under a continuous map the image of an open set need not be an open set.

For example let $M_1 = R$ with discrete metric and let $M_2 = R$ with usual metric.

Let $f : M_1 \to M_2$ be defined by $f(x) = x$. Since M_1 is discrete every subset of M_1 is open.

Hence for any open subset G of M_2, $f^{-1}(G)$ is open in M_1.

∴ f is continuous.

$A = \{x\}$ is open in M_1.

But $f(A) = \{x\}$ is not open in M_2.

Note 2 :

In the above example f is a continuous bijection whereas $f^{-1} : M_2 \to M_1$ is not continuous.

For, $\{x\}$ is an open set in M_1.

$(f^{-1})^{-1} (\{x\}) = \{x\}$ which is not open in M_2.

∴ f^{-1} is not continuous.

Thus if f is a continuous bijection, f^{-1} need not be continuous.

Theorem 3 :

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. A function $f:M_1 \rightarrow M_2$ is continuous iff $f^{-1}(F)$ is closed in M_1 whenever F is closed in M_2.

Proof :

Suppose $f:M_1 \rightarrow M_2$ is continuous.

Let $F \subseteq M_2$ be closed in M_2.

$\therefore$ F^C is open in M_2.

$\therefore$ $f^{-1}(F^C)$ is open in M_1.

But $f^{-1}(F^C) = [f^{-1}(F)]^C$

$f^{-1}(F)$ is closed in M_1.

Conversely, suppose $f^{-1}(F)$ is closed in M_1 whenever F is closed in M_2.

We claim that f is continuous.

Let G be an open set in M_2.

$\therefore$ G^C is closed in M_2.

$\therefore$ $f^{-1}(G^C)$ is closed in M_1.

$\therefore$ $[f^{-1}(G)]^C$ is closed in M_1.

$\therefore$ $f^{-1}(G)$ is open in M_1.

$\therefore$ f is continuous.

Theorem 4 :

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Then $f:M_1 \rightarrow M_2$ is continuous iff

$$f(\overline{A}) \subseteq \overline{f(A)} \text{ for all } A \subseteq M_1.$$

Proof :

Suppose f is continuous.

Let $A \subseteq M_1$. Then $f(A) \subseteq M_2$

Since f is continuous, $f^{-1}\left(\overline{f(A)}\right)$ is closed in M_1.

Also $f^{-1}(\overline{f(A)}) \supseteq A$ (since $\overline{f(A)} \supseteq f(A)$)

But $\overline{A}$ is the smallest closed set containing A.

$\therefore \overline{A} \subseteq f^{-1}(\overline{f(A)})$

$\therefore f(\overline{A}) \subseteq \overline{f(A)}$

Conversely, let $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$.

To prove that f is continuous, we shall show that if F is a closed set in M_2, then $f^{-1}(F)$ is closed in M_1.

$$\begin{aligned} \text{By hypothesis, } f(\overline{f^{-1}(F)}) &\subseteq \overline{f\, f^{-1}(F)} \\ &\subseteq \overline{F} \\ &= F \text{ (since F is closed)} \end{aligned}$$

$$\begin{aligned} \text{Thus} \quad f(\overline{f^{-1}(F)}) &\subseteq F \\ \therefore \quad \overline{f^{-1}(F)} &\subseteq f^{-1}(F) \\ \text{Also} \quad f^{-1}(F) &\subseteq \overline{f^{-1}(F)} \\ \therefore \quad f^{-1}(F) &= \overline{f^{-1}(F)} \end{aligned}$$

Hence $f^{-1}(F)$ is closed.

$\therefore$ f is continuous.

Solved Problems :

Problem 1 :

Let f be a continuous real valued function defined on a metric space M.

Let $A = \{x \in M\}/f(x) \geq 0\}$. Prove that A is closed.

Solution :

$$\begin{aligned} A &= \{x \in M/f(x) \geq 0\} \\ &= \{x \in M/f(x) \in [0, \infty)\} \\ &= f{-}1([0, \infty)) \end{aligned}$$

Also $[0, \infty)$ is a closed subset of R.

Since f is continuous, $f^{-1}([0, \infty))$ is closed in M.

∴ A is closed.

Problem 2 :

Show that the function f:R→R defined by

$$f(x) = \begin{cases} 0 \text{ if x is irrational} \\ 1 \text{ if x is rational} \end{cases}$$

is not continuous by each of the following methods.

(i) By the usual $\in$, δ method.

(ii) By exhibiting a sequence (x_n) such that $(x_n) \to x$ and $(f(x_n))$ does not converge to f(x).

(iii) By exhibiting an open set G such that $f^{-1}(G)$ is not open.

(iv) By exhibiting a closed subset F such that $f^{-1}(F)$ is not closed.

(v) By exhibiting a subset of A of R such that $f(\overline{A}) \not\subset \overline{f(A)}$

Solution :

(i) To prove that f is not continuous at x we have to show that there exists an $\in>0$ such that for all $\delta>0$,

$f(B(x, \delta)) \not\subset B(f(x), \in)$

Let $\in = 1/2$

For any $\delta>0$, $B(x, \delta) = (x-\delta, x+\delta)$ contains both rational and irrational numbers.

If x is rational, choose $y \in B(x, \delta)$ such that y is irrational and if x is irrational choose $y \in B(x, \delta)$ such that y is rational.

Then $|f(x)-f(y)| = 1$ (by definition of y)

i.e., $d(f(x), f(y)) = 1$

∴ $f(y) \notin B(f(x), 1/2)$

Thus $y \in B(x, \delta)$ and $f(y) \notin B(f(x), 1/2)$

∴ $f(B(x, \delta))$ $B(f(x), \delta)$

Hence f is not continuous at x.

(ii) Let $x \in R$. Suppose x is rational. Then $f(x) = 1$

Let (x_n) be a sequence of irrational numbers such that $(x_n) \to x$.

Then $(f(x_n)) \to 0$ and $f(x) = 1$

$\therefore$ $(f(x_n))$ does not converge to $f(x)$.

Proof is similar if x is irrational.

(iii) Let

$$G = \left(\frac{1}{2}, \frac{3}{2}\right). \text{ Clearly G is open in R.}$$

$$\begin{aligned} f^{-1}(G) &= \{x \in R / f(x) \in G\} \\ &= \left\{x \in R / f(x) \in \left(\frac{1}{2}, \frac{3}{2}\right)\right\} \\ &= Q \end{aligned}$$

But Q is not open in R.

Thus $f^{-1}(G)$ is not open in R.

$\therefore$ f is not continuous.

(iv) Choose $F = \left[\frac{1}{2}, \frac{3}{2}\right]$

Then $f^{-1}(F) = Q$ which is not closed in R.

$\therefore$ f is not continuous.

(v) Let $A = Q$. Then $\overline{A} = R$

$\therefore$ $f(\overline{A}) = f(R) = \{0, 1\}$ (by definition of y)

Also $f(A) = f(Q) = \{1\}$

$\therefore$ $\overline{f(A)} = \overline{\{1\}} = \{1\}$

$\therefore$ $\overline{f(A)} \not\subset f(\overline{A})$

$\therefore$ f is not continuous.

Problem 3 :

Let M_1, M_2, M_3 be metric spaces. If $f: M_1 \to M_2$ and $g: M_2 \to M_3$ are continuous functions, prove that $gof: M_1 \to M_3$ is also continuous.

i.e., composition of two continuous functions is continuous.

Solution :

Let G be open in M_3.

Since g is continuous, $g^{-1}(G)$ is open in M_2.

Now, since f is continuous, $f^{-1}(g^{-1}(G))$ is open in M_1.

i.e., $(gof)^{-1}(G)$ is open in M_1.

∴ gof is continuous.

Problem 4 :

Let M be a metric space. Let $f:M \to R$ and $g:M \to R$ be two continuous functions. Prove that $f+g:M \to R$ is continuous.

Solution :

Let (x_n) be a sequence converging to x in M.

Since f and g are continuous functions,

$(f(x_n)) \to f(x)$ and $(g(x_n)) \to g(x)$

∴ $(f(x_n)+g(x_n)) \to f(x)+g(x)$

i.e., $((f+g)(x_n)) \to (f+g)(x)$

∴ f+g is continuous.

Problem 5 :

If $f:R \to R$ and $g:R \to R$ are both continuous functions on R and if $h:R^2 \to R^2$ is defined by $h(x, y)=(f(x), g(y))$ prove that h is continuous on R^2.

Solution :

Let (x_n, y_n) be a sequence in R^2 converging to (x, y).

We claim that $(h(x_n, y_n))$ converges to h(x, y).

Since $((x_n, y_n)) \to (x, y)$ in R^2, $(x_n) \to x$ and $(y_n) \to y$ in R.

Also f and g are continuous.

∴ $(f(x_n)) \to f(x)$ and $g(y_n) \to g(y)$

∴ $(f(x_n), g(y_n)) \to (f(x), g(y))$

∴ $(h(x_n, y_n)) \to h(x, y)$

∴ h is continuous on R^2.

Problem 6 :

Let (M, d) be a metric space. Let $a \in M$ show that the function $f: M \rightarrow R$ defined by $f(x) = d(x, a)$ is continuous.

Solution :

Let $x \in M$

Let (x_n) be a sequence in M such that $(x_n) \rightarrow x$

We claim that $(f(x_n)) \rightarrow f(x)$

Let $\epsilon > 0$ be given.

Now, $|f(x_n) - f(x)| = |d(x_n, a) - d(x, a)| \leq d(x_n, x)$

Since $(x_n) \rightarrow x$, there exists a positive integer n_1 such that $d(x_n, x) < \epsilon$ for all $n \geq n_1$.

∴ $|f(xn) - f(x)| < \epsilon$ for all $n \geq n_1$.

∴ $(f(x_n)) \rightarrow f(x)$

∴ f is continuous.

Problem 7 :

Define $f: l_2 \rightarrow l_2$ as follows. If $s \in l_2$ is the sequence $s_1, s_2, \ldots$, let f(s), be the sequence $0, s_1, s_2, \ldots\ldots$ show that f is continuous on l_2.

Solution :

Let $y = (y_1, y_2, \ldots\ldots, y_n, \ldots\ldots) \in l_2$

Let (x_n) be a sequence in l_2 converging to y.

Let $x_n = (x_{n_1}, x_{n_2}, \ldots\ldots, x_{n_k}, \ldots)$

Then $(x_{n_1}) \rightarrow y_1$; $(x_{n_2}) \rightarrow y_2, \ldots\ldots, (x_{n_k}) \rightarrow y_k, \ldots\ldots$

∴ $(f(x_n)) = ((0, x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots\ldots)) \rightarrow (0, y_1, y_2, \ldots y_n, \ldots) = f(y)$

∴ $(f(x_n)) \rightarrow f(y)$

∴ f is continuous.

Problem 8 :

Let G be an open subset of R. Prove that the characteristic function on G defined by $\chi_G(x) = \begin{cases} 1 & \text{if } x \in G \\ 0 & \text{if } x \notin G \end{cases}$ is continuous at every point of G.

Solution :

Let $x \in G$ so that $\chi_G(x) = 1$

Let $\in > 0$ be given.

Since G is open and $x \in G$, we can find a $\delta > 0$ such that

$$B(x, \delta) \subseteq G$$

$$\therefore \quad \chi G(B(x, \delta)) \subseteq \chi_G(G)$$

$$= \{1\}$$

$$\subseteq B(1, \in)$$

Thus $\quad \chi_G(B(x, \delta)) \subseteq B(\chi_G(x), \in)$.

$\therefore \chi_G$ is continuous at x.

Since $x \in G$ is arbitrary, χ_G is continuous on G.

Exercise :

1. Prove that that map $f: R^n \to R$ defined by $f(x_1, x_2, \ldots, x_n) = x_1$ is continuous.

2. If f and g are two continuous functions from a metric spae M_1 into another metric space M_2 and if $f(x) = g(x)$ for all $x \in A$, then prove that $f(x) = g(x)$ for all $x \in \overline{A}$ where $A \subseteq M_1$.

3. Let $f: R \to R$ be defined by $f(x) = \begin{cases} -2 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases}$

   Prove that f is not continuous.

HOMEOMORPHISM

Definition :

Let (M_1, d_1) and (M_2, d_2) be metric spaces. A function $f:M_1 \to M_2$ is called a **homeomorphism** if

(i) f is 1–1 and onto

(ii) f is continuosu

(iii) f^{-1} is continuous

M_1 and M_2 are said to be homeomorphic if there exists a homeomorphism $f:M_1 \to M_2$.

Definition :

A function $f:M_1 \to M_2$ is said to be **an open map** if f(G) is open in M_2 for every open set G in M_1.

i.e., f is an open map if the image of an open set in M_1 is an open set in M_2.

f is called a **closed map** if f(F) is closed in M_2 for every closed set F in M_1.

Note 1 :

Let $f:M_1 \to M_2$ be a 1–1 onto function.

Then f^{-1} is continuous iff f is an open map.

For, f^{-1} is continuous iff for any open set G in M_1, $(f^{-1})^{-1}$ (G) is open in M_2.

But, $(f^{-1})^{-1}(G) = f(G)$

∴ f^{-1} is continuous iff for every open set G in M_1, f(G) is open in M_2.

∴ f^{-1} is continuous iff f is an open map.

Note 2 :

f^{-1} is continuous iff f is a closed map.

Note 3 :

Let $f:M_1 \to M_2$ be a 1–1 onto map. Then the following are equivalent.

(i) f is a homeomorphism

(ii) f is a continuous open map

(iii) f is a continuous closed map.

Note 4 :

Let $f:M_1 \to M_2$ be a homeomorphism. $G \subseteq M_1$ is open in M_1 iff f(G) is open in M_2.

For, since f is an open map G is open in $M_1 \Rightarrow f(G)$ is open in M_2.

Also, since f is continuous, f(G) is open in $M_2 \Rightarrow f^{-1}(f(G))=G$ is open in M_1.

∴ G is open in M_1 iff f(G) is open in M_2 -------(1)

Conversely, if $f:M_1 \to M_2$ is a 1–1 onto map satisfying (1) then f is a homeomorphism.

Thus a homeomorphism $f:M_1 \to M_2$ is simply a 1–1 onto map between the pionts of the two spaces such that their open sets are also in 1–1 correspondence with each otehr.

Note 5 :

Let $f:M_1 \to M_2$ be a 1–1 onto map. Then f is a homeomorphism iff it satisfies the following condition.

F is closed in M_1 iff f(F) is closed in M_2.

Example 1 :

The metric spaces [0,1] and [0, 2] with usual metric are homeomorphic.

Proof :

Define $f:[0, 1] \to [0, 2]$ by $f(x) = 2x$.

Clearly f is 1–1 and onto

Also $f^{-1}(x) = \frac{1}{2}x$

f and f^{-1} are both continuous

∴ f is a homeomorphism.

Example 2 :

The metric spaces $(0, \infty)$ and R with usual metrics are homeomorphic.

Proof :

$f:(0, \infty)\to R$ defined by $f(x)=\log_e x$ is the required homeomorphism. Here $f^{-1}(x) = e^x$.

Example 3 :

The metric spaces $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and R with usual metrics are homeomorphic and $f:\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\to R$ defined by f(x) = tax x is the required homeomorphism.

In this example, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is not a complete metric space whereas R is complete.

This shows that completeness of metric spaces is not preserved under homeomorphism.

Example 4 :

The metric spaces (0, 1) and $(0, \infty)$ with usual metrics are homeomorphic.

Proof :

Define $f:(0, 1)\to(0, \infty)$ by $f(x) = \frac{x}{1-x}$

We claim that f is 1–1 and onto.

Let $f(x) = f(y)$

$\therefore \quad \frac{x}{1-x} = \frac{y}{1-y}$

$\therefore \quad x-xy = y-xy$

$\therefore \quad x = y$

Hence f is 1–1.

Let $y\in(0, \infty)$

$\therefore \quad f(x) = y \Rightarrow \frac{x}{1-x} = y$

$$\Rightarrow y - xy = x$$

$$\Rightarrow x(1+y) = y$$

$$\Rightarrow x = \frac{y}{1+y}$$

$\therefore \frac{y}{1+y} \in (0, 1)$ is the preimage of y under f. Clearly f and f^{-1} are continuous.

$\therefore$ f is a homeomorphism.

Example 5 :

R with usual metric is not homeomorphic to R with discrete metric.

Proof :

Let M_1 = R with usual metric.

Let M_2 = R with discrete metric

Let $f: M_1 \to M_2$ be any 1–1 onto map.

Now, $\{a\}$ is open in M_2.

But $f^{-1}(\{a\}) = \{f^{-1}(a)\}$ is not open in M_1.

Hence f is not continuous.

Thus any bijection $f: M_1 \to M_2$ is not a homeomorphism. Hence M_1 is not homeomorphic to M_2.

Definition :

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Let $f: M_1 \to M_2$ be a 1–1 onto map. f is said to be an **isometry** if $d_1(x, y) = d_2(f(x), f(y))$ for all $x, y \in M_1$. In other words, an isometry is a distance preserving map.

M_1 and M_2 are said to be isometric if there exists an isometry f from M_1 onto M_2.

Example 6 :

R^2 with usual metric and C with usual metric are isometric and $f: R^2 \to C$ defined by $f(x, y) = x + iy$ is the required isometry.

Proof :

Let d_1 denote the usual metric on R^2 and d_2 denote the usual metric on C.

Let $a = (x_1, y_1)$ and $b = (x_2, y_2) \in R^2$.

Then
$$\begin{aligned} d_1(a, b) &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= |(x_1 - x_2) + i(y_1 - y_2)| \\ &= |(x_1 + iy_1) - (x_2 + iy_2)| \\ &= d_2(f(a), f(b)) \end{aligned}$$

$\therefore$ f is an isometry.

Example 7 :

Let d_1 be the usual metric on [0, 1] and d_2 be the usual metric on [0, 2]. The map $f:[0, 1] \rightarrow [0, 2]$ defined by $f(x) = 2x$ is not an isometry.

Proof :

Let $x, y \in [0, 1]$

Then
$$\begin{aligned} d_2(f(x), f(y)) &= |f(x) - f(y)| \\ &= |2x - 2y| \\ &= 2|x - y| \\ &= 2d_1(x, y) \end{aligned}$$

$$\therefore \quad d_1(x, y) \neq d_2(f(x), f(y))$$

Hence f is not an isometry.

Note :

Since an isometry f preserves distances, the image of an open ball B(x, r) is the open ball B(f(x), r)

Hence it follows that under an isometry the image of an open set is also an open set. Also if f is an isometry f^{-1} is also an isometry.

Hence under an isometry the inverse image of an open set is open.

Hence an isometry is a homeomorphism.

But a homeomorphism from one metric space to another need not be an isometry.

For example $f:[0, 1]\to[0, 2]$ defined by $f(x) = 2x$ is a homeomorphism.

But f is not an isometry.

Exercise :

1. Prove that any two open intervals are homeomorphic.

2. Show that $f:(0, 1)\to R$ defined by $f(x) = \frac{1}{1-x} - \frac{1}{x}$ is a homeomorphism.

3. Prove that (2, 5) and (8, 11) are isometric

4. Prove that homeomorphism is an equivalence relation among metric spaces.

5. Prove that isometry is an equivalance relation among metric spaces.

UNIFORM CONTINUITY

We introduce the concept of uniform continuity.

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Let $f:M_1\to M_2$ be a continuous function. For each $a\in M_1$ the following is true.

Given $\epsilon>0$, there exists a $\delta>0$ such that $d_1(x, a)<\delta \Rightarrow d_2(f(x), f(a))<\epsilon$.

In general the number δ depends on ϵ and the point a under consideration.

For example, consider $f:R\to R$ given by $f(x) = x^2$

Let $a\in R$. Let $\epsilon>0$ be given.

We want to find $\delta>0$ such that

$$|x-a|<\delta \Rightarrow |f(x)-f(a)|<\epsilon \qquad \text{-------(1)}$$

Clearly if $\delta>0$ satisfies (1) then any δ_1 where $0<\delta_1<\delta$ also satisfies (1).

Hence if there exists a $\delta>0$ satisfying (1) then we can find another δ_1 such that $0<\delta_1<1$ and δ_1 also satisfies (1).

Hence we may restrict x such that $|x-a|<1$.

$a-1 < x < a+1$

$x+a < 2a+1$

$$|f(x)-f(a)| = |x^2-a^2| = |x+a|\,|x-a|$$
$$< |2a+1|\,|x-a| \text{ if } |x-a| < 1$$

Hence if we choose $\delta = \min\left\{1, \dfrac{\epsilon}{|2a+1|}\right\}$ then we have $|x-a|<\delta \Rightarrow |f(x)-f(a)|<\epsilon$.

Thus we see that the number δ depends on both ϵ and the point a under consideration and if a becomes large, δ has to be chosen correspondingly small. In fact, there is no $\delta>0$ such that (1) holds for all a.

For, suppose there exists $\delta>0$ such that

$$|x-a|<\delta \Rightarrow |f(x)-f(a)|<\epsilon \text{ for all } a\in R$$

Take $$x = a+\frac{1}{2}\delta$$

Clearly, $$|x-a| = \frac{1}{2}\delta<\delta$$

$$\therefore \quad |f(x)-f(a)| < \epsilon$$

$$\therefore \quad \left|\left(a+\frac{1}{2}\delta\right)^2-a^2\right| < \epsilon$$

$$\therefore \quad \frac{1}{2}\delta\left|\frac{1}{2}\delta+2a\right| < \epsilon$$

This inequality cannot be true for all $a\in R$, since by taking a sufficiently large, we can make $\frac{1}{2}\delta\left|\frac{1}{2}\delta+2a\right|>\epsilon$

Thus there is no $\delta>0$ such that (1) holds for all $a\in R$

Let $f:R\to R$ be given by $f(x) = 2x$

Let $a\in R$. Let $\epsilon>0$ be given.

Then $|f(x)-f(a)| = |2x-2a| = 2|x-a|$

∴ If we choose $\delta = \frac{1}{2}\in$ then we have

$$|x-a|<\delta \Rightarrow |f(x)-f(a)|<\in$$

Here δ depends on ∈ and not an a.

i.e., for a given $\in>0$ we are able to find $\delta>0$ such that δ works uniformly for all $a\in R$.

Definition :

Let (M_1, d_1) and (M_2, d_2) be metric spaces. A function $f:M_1\to M_2$ is said to be **uniformly continuous** on M_1 if given $\in>0$ there exists a $\delta>0$ such that $d_1(x, y)<\delta \Rightarrow d_2(f(x), f(y))<\in$.

Note 1 :

If $f:M_1\to M_2$ is uniformly continuous on M_1, then f is continuous at every point of M_1.

Moreover for a given $\in>0$ there exists a $\delta>0$ such that $x, y\in M_1$ and $d_1(x, y)<\delta \Rightarrow d_2(f(x), f(y))<\in$.

Thus uniform continuity is continuity plus the added condition that for a given $\in>0$ we can find $\delta>0$ which works uniformly for all points of M_1.

Note 2 :

A continuous function $f:M_1\to M_2$ need not be uniformly continuous on M_1.

For example $f:R\to R$ defined by $f(x) = x^2$ is continuous but not uniformly continuous on R.

Solved Problems :

Problem 1 :

Prove that $f:[0, 1]\to R$ defined by $f(x) = x^2$ is uniformly continuous on [0, 1]

Solution :

Let $\in>0$ be given. Let $x, y\in[0, 1]$.

Then $|f(x)-f(y)| = |x^2-y^2| = |x+y||x-y|$

$\leq 2|x-y|$ (since $x\leq 1$ and $y\leq 1$)

∴ $|x-y|<\frac{1}{2}\in \Rightarrow |f(x)-f(y)|<\in$

∴ f is uniformly continuous on [0, 1]

Problem 2 :

Prove that the function $f:(0, 1)\rightarrow R$ defined by $f(x)=\frac{1}{x}$ is not uniformly continuous.

Solution :

Let $\in>0$ be given. Suppose there exists a $\delta>0$ such that

$$|x-y|<\delta \Rightarrow |f(x)-f(y)| < \in$$

Take $x = y+\frac{1}{2}\delta$

Clearly $|x-y| = \frac{1}{2}\delta<\delta$

∴ $|f(x)-f(y)| < \in$

∴ $\left|\frac{1}{x}-\frac{1}{y}\right| < \in.$

∴ $\left|\frac{1}{y+\frac{1}{2}\delta}-\frac{1}{y}\right| < \in$

∴ $\left|\frac{\delta}{2\left(y+\frac{1}{2}\delta\right)y}\right| < \in$

∴ $\frac{\delta}{(2y+\delta)y} < \in$

This inequality cannot be true for all $y \in (0, 1)$ since $\frac{\delta}{(2y+\delta)y}$ becomes arbitrarily large as y approaches zero.

∴ f is not uniformly continuous.

Problem 3 :

Prove that the function $f:R \to R$ defined by $f(x) = \sin x$ is uniformly continuous on R.

Solution :

Let $x, y \in R$ nd $x>y$

$$\sin x - \sin y = (x-y)\cos z \text{ where } x>z>y \text{ (by mean value theorem)}$$

$$\therefore \quad |\sin x - \sin y| = |x-y|\,|\cos z|$$

$$\leq |x-y| \quad (\text{since } |\cos z| \leq 1)$$

Hence for a given $\epsilon>0$, if we choose $\delta=\epsilon$,

we have $|x-y|<\delta \Rightarrow |f(x)-f(y)| = |\sin x - \sin y| < \epsilon$.

∴ $f(x) = \sin x$ is uniformly continuous on R.

Exercise :

1. Let $f:R \to R$ and $g:R \to R$ be two functions uniformly continuous on R. Prove that f+g is also uniformly continuous on R.
2. Is the product of uniformly continuous real valued functions again uniformly continuous?
3. Determine whether $f:[0, 1] \to R$ defined by $f(x)=\sqrt{x}$ is uniformly continuous.

DISCONTINUOUS FUNCTIONS ON R

Definition :

A function $f:R \to R$ is said to approach to a limit l as x tends to a if given $\epsilon>0$ there exists a $\delta>0$ such that $0<|x-a|<\delta \Rightarrow |f(x)-l|<\epsilon$ and we write $\underset{x \to a}{\text{Lt}}\, f(x) = l$.

A function f is said to have l as the **right limit** at x=a if given $\epsilon>0$ there exists a $\delta>0$ such that $a<x<a+\delta \Rightarrow |f(x)-l|<\epsilon$ and we write $\underset{x\to a+}{\text{Lt}} f(x) = l$.

Also we denote the right limit l by f(a+).

A function f is said to have l as the **left limit** at x=a if given $\epsilon>0$ there exists a $\delta>0$ such that $a-\delta<x<a \Rightarrow |f(x)-l|<\epsilon$ and we write $\underset{x\to a-}{\text{Lt}} f(x) = l$.

Also we denote the left limit l by f(a–).

Note 1 :

$\underset{x\to a}{\text{Lt}} f(x) = l$ iff $\underset{x\to a+}{\text{Lt}} f(x) = \underset{x\to a-}{\text{Lt}} f(x) = l$

Note 2 :

f is continuous at a iff f(a+) = f(a–) = f(a)

Note 3 :

If $\underset{x\to a}{\text{Lt}} f(x)$ does not exist then one of the following happens.

(i) $\underset{x\to a+}{\text{Lt}} f(x)$ does not exist.

(ii) $\underset{x\to a-}{\text{Lt}} f(x)$ does not exist.

(iii) $\underset{x\to a+}{\text{Lt}} f(x)$ and $\underset{x\to a-}{\text{Lt}} f(x)$ exist and are unequal.

Definition :

If a function f is discontinuous at a then a is called a **point of discontinuity** for the function.

If a is a point of discontinuity of a function then any one of the following cases arises.

(i) $\underset{x\to a}{\text{Lt}} f(x)$ exists but is not equal to f(a).

(ii) $\underset{x\to a+}{\text{Lt}} f(x)$ and $\underset{x\to a-}{\text{Lt}} f(x)$ exist and are not equal.

(iii) Either $\underset{x\to a-}{\text{Lt}} f(x)$ or $\underset{x\to a+}{\text{Lt}} f(x)$ does not exist.

Definition :

Let a be a point of discontinuity for f(x) a is said to be a point of **discontinuity of the first kind** if $\underset{x \to a+}{\text{Lt}} f(x)$ and $\underset{x \to a-}{\text{Lt}} f(x)$ exist and both of them are finite and unequal.

a is said to be a point of **discontinuity of the second kind** if either $\underset{x \to a+}{\text{Lt}} f(x)$ or $\underset{x \to a-}{\text{Lt}} f(x)$ does not exist.

Definition :

Let $A \subseteq R$. A function $f:A \to R$ is called **monotonic increasing** if x, $y \in A$ and $x<y \Rightarrow f(x) \le f(y)$.

f is called monotonic decreasing if x, $y \in A$ and $x>y \Rightarrow f(x) \ge f(y)$.

f is called **monotonic** if it is either monotonic increasing or monotonic decreasing.

Theorem 5 :

Let $f:[a, b] \to R$ be a monotonic increasing function. Then f has a left limit and a right limit at every point of (a,b). Also f has a right limit at a and f has a left limit at b.

Further $x<y \Rightarrow f(x+) \le f(y-)$

Similar result is true for monotonic decreasing functions.

Proof :

Let $f:[a, b] \to R$ be monotonic increasing.

Let $x \in [a, b]$. Then $\{f(t) / a \le t < x\}$ is bounded above by f(x).

Let $l = l.u.b. \{f(t) / a \le t < x\}$

We claim that $f(x-) = l$

Let $\epsilon > 0$ be given. By definition of l.u.b. there exists t such that $a \le t < x$ and $l - \epsilon < f(t) \le l$

$\therefore \quad t<u<x \Rightarrow l-\epsilon<f(t) \le f(u) \le l$ (since f is monotonic increasing)

$\Rightarrow l-\epsilon<f(u) \le l$

$\therefore \quad x-\delta<u<x \Rightarrow l-\epsilon<f(u) \le l$ where $\delta = x-t$

$\therefore \quad f(x-) = l$

Similar we can prove that

$$f(x+) = g.l.b.\{f(t)/x<t\leq b\}$$

Now we shall prove that

$$x<y \Rightarrow f(x+)\leq f(y-)$$

Let $x<y$

Now, $f(x+) = g.l.b.\ \{f(t)/x<t\leq b\}$

$= g.l.b.\ \{f(t)/x<t\leq y\}$ --------(1)

(since f is monotonic increasing)

Also $f(y-) = l.u.b.\ \{f(t)/a\leq t<y\}$

$= l.u.b.\ \{f(t)/x\leq t<y\}$ --------(2)

∴ $f(x+) \leq f(y-)$ (by (1) and (2))

Theorem 6 :

Let $f : [a, b] \rightarrow R$ be a monotonic function. Then the set of points of [a, b] at which f is discontinuous is countable.

Proof :

We shall prove the theorem for a monotonic increasing function.

Let $E = \{x/x\in[a, b]$ and f is discontinuous at $x\}$

Let $x\in E$. Then $f(x+)$ and $f(x-)$ exist and $f(x-)\leq f(x)\leq f(x+)$

If $f(x-) = f(x+)$ then $f(x-) = f(x) = f(x+)$

∴ f is continuous at x which is a contradiction.

∴ $f(x-) \neq f(x+)$

∴ $f(x-) < f(x+)$

Choose a rational number $r(x)$ such that $f(x-)<r(x)<f(x+)$

This defines a map r from E to Q which maps x to r(x).

We claim that r is 1–1.

Let $x_1<x_2$

∴ $f(x_1+)<f(x_2-)$

Also $f(x_1-)<r(x_1)<f(x_1+)$

and $f(x_2-)<r(x_2)<f(x_2+)$

$\therefore$ $r(x_1)<f(x_1+)<f(x_2-)<r(x_2)$

Thus $x_1<x_2 \Rightarrow r(x_1)<r(x_2)$

$\therefore$ $r:E\to Q$ is 1–1.

Hence E is countable.

Thus we have proved that the set of discontinuties of a monotonic function is countable.

Definition :

A subset D of R is said to be of type F_σ if D can be expressed as a countable union of closed sets.

i.e., $D=\cup F_n$ where every F_n is a closed subset of R.

Definition :

Consider any function $f:R\to R$. Let I be a bounded open interval in R. Then the **oscillation** of f over I denoted by $\omega(f, I)$ is defined by

$$\omega(f, I) \quad = \quad l.u.b.\{f(x)/x\in I\}$$

If $a\in R$ the oscillation of f at a denoted by $\omega(f, a)$ is defined by $\omega(f, a) = g.l.b.\ \omega(f, I)$ where g.*l*.b. is taken over all bounded open intervals containing a.

Example :

Consider the function $f:R\to R$ defined by $f(x) = [x]$

Let a=4. Let I be any bounded open interval containing 4.

Suppose I does not contain any integer other than 4. Then $\omega(f, I) = 4-3 = 1$.

For any other open interval I containing 4,

$$\omega(f, I) \geq 1$$

$$\therefore \qquad \omega(f, 4) = 1$$

In general, for any $n\in z$, $\omega(f, n) = 1$

Theorem 7 :

$f:R\to R$ is continuous at $a\in R$ iff $\omega(f, a) = 0$.

Proof :

Suppose f is continuous at a.

Let $\in > 0$ be given. Then there exists $\delta > 0$ such that $|x-a| < \delta \Rightarrow |f(x)-f(a)| < \frac{1}{2}\in$.

Let $I = (a-\delta, a+\delta)$

$\therefore$ For any $x \in I$, $|f(x)-f(a)| < \frac{1}{2}\in$.

$\therefore$ For any $x, y \in I$, $|f(x)-f(y)| \leq |f(x)-f(a)| + |f(a)-f(y)| < \frac{1}{2}\in + \frac{1}{2}\in = \in$.

$\therefore$ $\omega(f, I) < \in$.

Since $\in > 0$ is arbitary $\omega(f, a) = 0$

Conversely, let $\omega(f, a) = 0$. We claim that f is continuous at a.

Let $\in > 0$ be given.

Since $\omega(f, a) = \text{g.}l\text{.b. } \omega(f, I) = 0$, there exists a bounded open interval I containing a such that

$$\omega(f, I) < \in \qquad \text{--------(1)}$$

Let $x_1, x_2 \in I$

Then $f(x_1) \leq \text{l.u.b. } \{f(x)/x \in I\}$

and $f(x_2) \geq \text{g.l.b.} \{f(x)/x \in I\}$

$\therefore$ $|f(x_1)-f(x_2)| \leq \omega(f, I) < \in$ by (1)

Thus for any two points $x_1, x_2 \in I$, $|f(x_1)-f(x_2)| < \in$.

In particular $|f(x)-f(a)| < \in$ for all $x \in I$.

Since I is a bounded open interval containing a we can choose $\delta > 0$ such that $(a-\delta, a+\delta) \subseteq I$.

$\therefore$ $|f(x)-f(a)| < \in$ for all $x \in (a-\delta, a+\delta)$

$\therefore$ $|x-a| < \delta \Rightarrow |f(x)-f(a)| < \in$.

$\therefore$ f is continuous at a.

Theorem 8 :

Let f:R→R be any function. Let r>0. Then $E_r = \left\{a \in R / \omega(f,a) \geq \frac{1}{r}\right\}$ is a closed set.

Proof :

Let x be any limit point of E_r.

We claim that $x \in E_r$.

For this wemust prove that $\omega(f, x) \geq \frac{1}{r}$.

Let I be any bounded open interval containing x. Since x is a limit point of E_r, I contains a piont y of E_r.

Hence I is a bounded open interval containing y.

∴ $\omega(f, y) \leq \omega(f, I)$

But $\omega(f, y) \geq \frac{1}{r}$ (since $y \in E_r$)

∴ $\omega(f, I) \geq \frac{1}{r}$ and this is true for any bounded open interval I containing x.

∴ $\omega(f, x) \geq \frac{1}{r}$.

∴ $x \in E_r$

∴ E_r contains all its limit points.

∴ E_r is closed.

Theorem 9 :

Let D be the set of points of discontinuities of a function f:R→R. Then D is of type F_σ.

Proof :

Let $x \in D$. Then f is discontinuous at x.

∴ $\omega(f, x) > 0$ (by theorem 7)

∴ $$\omega(f, x) \geq \frac{1}{n} \text{ for some positive integer } n.$$

∴ $x \in E_n$ for some positive integer n where E_n is defined as in theorem 8.

∴ $$x \in \bigcap_{n=1}^{\infty} E_n$$

∴ $$D \subseteq \bigcup_{n=1}^{\infty} E_n \qquad \text{--------(1)}$$

Let $$x \in \bigcup_{n=1}^{\infty} E_n$$

Then $x \in E_n$ for some positive integer n.

∴ $$\omega(f, x) \geq \frac{1}{n}.$$

Hence $\omega(f, x) > 0$.

∴ f is discontinuous at x. Hence $x \in D$.

∴ $$\bigcup_{n=1}^{\infty} E_n \subseteq D \qquad \text{-------(2)}$$

Thus $$D = \bigcup_{n=1}^{\infty} E_n \quad \text{(by (1) and (2))}.$$

Also each E_n is closed (by theorem 8)

Thus D is a countable union of closed sets.

∴ D is of type F_σ.

Theorem 10 :

There is no function $f:R \rightarrow R$ such that f is continuous at each rational number and discontinuous at each irrational number.

Proof :

Because of theorem 9 it is enough to prove that the set A of all irrational numbers is not of type F_σ.

Suppose A is of type F_σ.

Then $A = \bigcup_{n=1}^{\infty} F_n$ where each F_n is closed.

Since F_n contains only irrational numbers, F_n cannot contain any open interval.

$\therefore$ $\text{Int } F_n = \phi$

$\therefore$ $\text{Int } \overline{F_n} = \phi$ (since F_n is closed)

$\therefore$ F_n is nowhere dense.

$\therefore$ A is of first category which is a contradiction.

$\therefore$ A is not of type F_σ.

Hence the theorem.

Exercise :

1. If $f : M_1 \to M_2$ is a continuous bijection then $f^{-1} : M_2 \to M_1$ is also continuous.
2. If M_1 is homeomorphic to M_2 and M_1 is complete then M_2 is complete.
3. If $f : M_1 \to M_2$ is continuous at every point of M_1 then f is uniformly continuous on M_1.
4. If $f : M_1 \to M_2$ is uniformly continuous on M_1 then f is continuous at every point of M_1.
5. $f : [0, 1] \to R$ defined by $f(x) = x^2$ is uniformly continuous on [0, 1].

CONNECTEDNESS

In R consider the subsets A = [1, 2] and B = [1, 2]∪[3, 4]. The set A consists of a single 'piece' whereas B consists of 'two pieces'. We say that A is a connected set and B is not a connected set.

Definition and Examples :

Definition :

Let (M, d) be a metric space. M is said to be **connected** if M cannot be represented as the union of two disjoint non-empty open sets.

If M is not connected it is said to be **disconnected.**

Example 1 :

Let M = [1, 2]∪[3, 4] with usual metric then M is disconnected.

Proof :

[1, 2] and [3, 4] are open in M.

Thus M is the union of two disjoint non empty open sets namely [1, 2] and [3, 4].

Hence M is disconnected.

Example 2 :

Any discrete metric space M with more than one point is disconnected.

Proof :

Let A be a proper non-empty subset of M. Since M has more than one point such a set exists.

Then A^C is also non-empty.

Since M is discrete every subset of M is open.

∴ A and A^C are open.

Thus $M = A \cup A^C$ where A and A^C are two disjoint non-empty open sets.

∴ M is not connected.

Theorem 1 :

Let (M, d) be a metric space. Then the following are equivalent.

(i) M is connected.

(ii) M cannot be written as the union of two disjoint non-empty closed sets.

(iii) M cannot be written as the union of two non-empty sets A and B such that $A \cap \overline{B} = \overline{A} \cap B = \phi$.

(iv) M and ϕ are the only sets which are both open and closed in M.

Proof :

(i) ⇒ (ii)

Suppose (ii) is not true

∴ $M = A \cup B$ where A and B are closed $A \neq \phi$, $B \neq \phi$ and $A \cap B = \phi$

∴ $A^C = B$ and $B^C = A$

Since A and B are closed, A^C and B^C are open.

∴ B and A are open.

Thus M is the union of two disjoint non-empty open sets.

∴ M is not connected which is a contradiction.

∴ (i) ⇒ (ii)

(ii) ⇒ (iii)

Suppose (iii) is not true.

Then $M = A \cup B$ where $A \neq \phi$, $B \neq \phi$ and $A \cap \overline{B} = \overline{A} \cap B = \phi$.

We claim that A and B are closed.

Let $x \in \overline{A}$

∴ $x \notin B$ (since $\overline{A} \cap B = \phi$)

∴ $x \in A$ (since $A \cup B = M$)

∴ $\overline{A} \subseteq A$

But $A \subseteq \overline{A}$

∴ $A = \overline{A}$ and hence A is closed.

Similarly B is closed.

Now $A \cap B = \overline{A} \cap B$ (since $A = \overline{A}$)

$= \phi$

Thus $M = A \cup B$ where $A \neq \phi$, $B \neq \phi$, A and B are closed and $A \cap B = \phi$ which is a contradiction to (ii)

∴ (ii) ⇒ (iii)

(iii) ⇒ (iv)

Suppose (iv) is not true.

Then there exists $A \subseteq M$ such that $A \neq M$ and $A \neq \phi$ and A is both open and closed.

Let $B = A^C$

Then B is also both open and closed and $B \neq \phi$.

Also $M = A \cup B$

Further $\overline{A} \cap B = A \cap A^C$ (since $\overline{A} = A$ and $B = A^C$)

$= \phi$

Similarly $A \cap \overline{B} = \phi$

∴ $M = A \cup B$ where $A \cap \overline{B} = \phi = \overline{A} \cap B$ which is a contradiction to (iii).

∴ (iii) ⇒ (iv)

(iv) ⇒ (i)

Suppose M is not connected.

∴ $M = A \cup B$ where $A \neq \phi$, $B \neq \phi$, A and B are open and $A \cap B = \phi$.

Then $B^C = A$

Now since B is open A is closed.

Also $A \neq \phi$ and $A \neq M$ (sicne $B \neq \phi$)

∴ A is a proper non-empty subset of M which is both open and closed which is a contradiction to (iv).

∴ (iv) ⇒ (i).

Theorem 2 :

A metric space M is connected iff there does not exist a continuous function f from M onto the discrete metric space {0, 1}.

Proof :

Suppose there exists a continuous function f from M onto $\{0, 1\}$

Since $\{0, 1\}$ is discrete, $\{0\}$ and $\{1\}$ are open.

$A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$ are open in M.

Since f is onto, A and B are non-empty.

Clearly $A \cap B = \phi$ and $A \cup B = M$.

Thus $M = A \cup B$ where A and B are disjoint non-empty open sets.

$\therefore$ M is not connected which is a contradiction.

Hence there does not exist a continuous function from onto the discrete metric space $\{0, 1\}$

Conversely, suppose M is not connected.

Then there exist disjoint non-empty open sets A and B in M such that $M = A \cup B$

Now define $f: M \to \{0, 1\}$ by $f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$

Clearly f is onto.

Also $f^{-1}(\phi) = \phi$, $f^{-1}(\{0\}) = A$, $f^{-1}(\{1\}) = B$ and $f^{-1}(\{0,1\}) = M$

Thus the inverse image of every open set in $\{0, 1\}$ is open in M.

Hence f is continuous.

Thus there exists a continuous function f from M onto $\{0, 1\}$ which is a contradiction. Hence M is connected.

Note : The above theorem can be restated as follows. M is connected iff every continuous function $f: M \to \{0, 1\}$ is not onto.

Solved Problems :

Problem 1 :

Let M be a metric space. Let A be a connected subset of M. If B is a subset of M such that $A \subseteq B \subseteq \overline{A}$ then B is connected. In particular $\overline{A}$ is connected.

Solution :

Suppose B is not connected.

Then $B = B_1 \cup B_2$ where $B_1 \neq \phi$, $B_2 \neq \phi$, $B_1 \cap B_2 = \phi$ and B_1 and B_2 are open in B.

Now since B_1 and B_2 are open in B there exist open sets G_1 and G_2 in M such that

$$B_1 = G_1 \cap B \text{ and } B_2 = G_2 \cap B$$

$$\therefore \quad B = B_1 \cup B_2 = (G_1 \cap B) \cup (G_2 \cap B) = (G_1 \cup G_2) \cap B$$

$$\therefore \quad B \subseteq G_1 \cup G_2$$

$$\therefore \quad A \subseteq G_1 \cup G_2 \quad (\text{since } A \subseteq B)$$

$$\therefore \quad A = (G_1 \cup G_2) \cap A$$

$$= (G_1 \cap A) \cup (G2 \cap A)$$

Now, $G_1 \cap A$ and $G_2 \cap A$ are open in A.

$$\text{Further, } (G_1 \cap A) \cap (G_2 \cap A) = (G_1 \cap G_2) \cap A$$

$$= (G_1 \cap G_2) \cap B \quad (\text{since } A \subseteq B)$$

$$= (G_1 \cap B) \cap (G_2 \cap B)$$

$$= B_1 \cap B_2$$

$$= \phi.$$

$$\therefore \quad (G_1 \cap A) \cap (G_2 \cap A) = \phi.$$

Now, since A is connected, either $G_1 \cap A = \phi$ or $G_2 \cap A = \phi$.

Without lossof generality let us assume that $G_1 \cap A = \phi$.

Since G_1 is open in M, we have $G_1 \cap \overline{A} = \phi$.

$\therefore$ $G_1 \cap B = \phi$ (since $B \subseteq \overline{A}$)

$\therefore$ $B_1 = \phi$ which is a contradiction.

$\therefore$ B is connected.

Problem 2 :

If A and B are connected subsets of a metric space M and if $A \cap B \neq \phi$, prove that $A \cup B$ is connected.

Solution :

Let $f: A \cup B \rightarrow \{0,1\}$ be a continuous function.

Since $A \cap B \neq \phi$ we can choose $x_0 \in A \cap B$

Let $f(x_0) = 0$

Since $f: A\cup B \rightarrow \{0, 1\}$ is continuous $f/A : A \rightarrow \{0, 1\}$ is also continuous.

But A is connected.

Hence f/A is not onto.

$\therefore$ $f(x) = 0$ for all $x \in A$ or $f(x) = 1$ for all $x \in A$.

But $f(x_0) = 0$ and $x_0 \in A$

$\therefore$ $f(x) = 0$ for all $x \in A$

Similarly $f(x) = 0$ for all $x \in B$

$\therefore$ $f(x) = 0$ for all $x \in A\cup B$

Thus any continuous function $f: A\cup B \rightarrow \{0, 1\}$ is not onto.

$\therefore$ $A\cup B$ is connected.

Exercise :

1. Prove that $\{0, 1\}$ is not a connected subset of R with discrete metric.
2. Let $A_1, A_2, \ldots, A_n, \ldots$ be connected subsets of a metric space M each of which intersects its successor. Prove that $\bigcup_{n=1}^{\infty} A_n$ is connected.
3. Let $\{A_\alpha\}$ be a family of connected subsets of a metric space M such that $\cap A_\alpha \neq \phi$. Then prov that $A = \cup A_\alpha$ is a connected subset of M.
4. Prove that the set of all components of a metric space M forms a partition of M.
5. Prove that in a discrete metric space each component consists of a single piont.

CONNECTED SUBSETS OF R

Theorem 3 :

A subspace of R is connected iff it is an interval.

Proof :

Let A be a connected subset of R. Suppose A is not an interval.

Then there exist $a, b, c \in R$ such that $a<b<c$ and $a, c \in A$ but $b \notin A$.

Let $A_1 = (-\infty, b) \cap A$ and $A_2 = (b, \infty) \cap A$

Since $(-\infty, b)$ and (b, ∞) are open in R, A_1 and A_2 are open sets in A.

Also $A_1 \cap A_2 = \phi$ and $A_1 \cup A_2 = A$

Further $a \in A_1$ and $c \in A_2$

Hence $A_1 \neq \phi$ and $A_2 \neq \phi$

Thus A is the union of two disjoint non-empty open sets A_1 and A_2.

Hence A is not connected which is a contradiction.

Hence A is an interval.

Conversely, let A be an interval.

We claim that A is connected.

Suppose A is not connected.

Let $A = A_1 \cup A_2$ where $A_1 \neq \phi$, $A_2 \neq \phi$, $A_1 \cap A_2 = \phi$ and A_1 and A_2 are closed sets in A.

Choose $x \in A_1$ and $z \in A_2$.

Since $A_1 \cap A_2 = \phi$ we have $x \neq z$

Without loss of generality we assume that $x < z$.

Now, since A is an interval we have $[x, z] \subseteq A$.

i.e., $[x, z] \subseteq A_1 \cup A_2$

$\therefore$ Every element of $[x, z]$ is either in A_1 or in A_2.

Now let $y = l.u.b. \{[x, z] \cap A_1\}$

Clearly $x \leq y \leq z$

Hence $y \in A$

Let $\in > 0$ be given. Then by the definition of l.u.b. there exists $t \in [x, z] \cap A_1$ such that $y - \in < t \leq y$

$(y-\in, y+\in) \cap ([x, z] \cap A_1) \neq \phi$.

$\therefore y \in \overline{[x,z] \cap A_1}$

$\therefore y \in [x, z] \cap A_1$ (since $[x, z] \cap A_1$ is closed in A)

$\therefore y \in A_1$ ---------(1)

Again by the definition of y, $y+\in \in A_2$ for all $\in > 0$ such that $y+\in \leq z$.

$\therefore y \in \overline{A_2}$.

$\therefore y \in A_2$ (since A_2 is closed) --------(2)

$\therefore y \in A_1 \cap A_2$ [by (1) and (2)] which is a contradiction sicne $A_1 \cap A_2 = \phi$.

Hence A is connected.

Theorem 4 :

R is connected.

Proof :

$R = (-\infty, \infty)$ is an interval.

$\therefore$ R is connected.

Solved Problems :

Problem 1 :

Give an example to show that a subspace of a connected metric space need not be connected.

Solution :

We know that R is connected.

$A = [1, 2] \cup [3, 4]$ is a subspace of R which is not connected.

Problem 2 :

Prove or disprove if A and C are connected subsets of a metric space M and if $A \subseteq B \subseteq C$, then B is connected.

Solution :

We disprove this statement by giving a counter example.

Let $A = [1, 2]$; $B = [1, 2] \cup [3, 4]$; $C = R$.

Clearly $A \subset B \subset C$

Here A and C are connected. But B is not connected.

Exercise :

Determine which of the following are connected subsets of R.

1. $[4, 6] \cup [8, 10]$
2. $[4, 6] \cup [5, 7]$
3. z
4. $R - \{0\}$
5. $(-\infty, 0)$

CONNECTEDNESS AND CONTINUITY

Theorem 5 :

Let M_1 be connected metric space. Let M_2 be any metric space. Let $f:M_1 \rightarrow M_2$ be a continuous function. Then $f(M_1)$ is a connected subset of M_2.

i.e., Any continuous image of a connected set is connected.

Proof :

Let $f(M_1) = A$ so that f is a function from M_1 onto A.

We claim that A is connected.

Suppose A is not connected. Then there exists a proper non-empty subset B of A which is both open and closed in A.

$\therefore$ $f^{-1}(B)$ is a proper non-empty subset of M_1 which is both open and closed in M_1. Hence M_1 is not connected which is a contradiction.

Hence A is connected.

Theorem 6 :

Let f be a real valued continuous function defined on an interval I. Then f takes every value between any two values it assumes.

(This is known as **the intermediate value theorem**)

Proof :

Let a, $b \in I$ and let $f(a) \neq f(b)$. Without loss of generality we assume that $f(a) < f(b)$.

Let c be such that $f(a) < c < f(b)$.

The interval I is a connected subset of R.

$\therefore$ f(I) is a connected subset of R.

$\therefore$ f(I) is an interval.

Also $f(a), f(b) \in f(I)$. Hence $[f(a), f(b)] \subseteq f(c)$

$\therefore$ $c \in f(I)$ (since $f(a) < c < f(b)$)

$\therefore$ $c = f(c)$ for some $x \in I$.

Solved Problem :

Prove that if f is a non-constant real valued continuous function on R then the range of f is uncountable.

Solution :

We know that R is connected.

Since f is a continuous function on R, f(R) is a connected subset of R.

∴ f(R) is an interval in R.

Also, since f is a non-constant function the interval, f(R) contains more than one point.

∴ f(R) is uncountable. i.e., The range of f is uncountable.

Exercise :

1. Prove that if $f: R \to R$ is a continuous function which assumes only rational values then f is a constant function.
2. Prove that $A = \{(x, y) / x^2+y^2=1\}$ is a connected subset of R^2.

   [Hint : Consider $f : [0, 2\pi] \to A$ given by $f(x) = (\cos x, \sin x)$]
3. Determine whether Q is connected or not.

COMPACTNESS

COMPACT METRIC SPACES

Definition :

Let M be a metric space. A family of open sets $\{G_\alpha\}$ in M is called an **open cover** for M if $\cup G_\alpha = M$.

A subfamily of $\{G_\alpha\}$ which itself is an open cover is called a **subcover.**

A metric space M is said to be **compact** if every open cover for M has finite subcover.

i.e., for each family of open sets $\{G_\alpha\}$ such that $\cup G_\alpha = M$, there exist a finite subfamily $\{G_{\alpha_1}, G_{\alpha_2}, \ldots\ldots, G_{\alpha_n}\}$ such that $\bigcup_{i=1}^{n} G_{\alpha_i} = M$.

Example 1 :

R with usual metric is not compact.

Proof :

Consider the family of open intervals $\{(-n, n)/n \in N\}$.

This is a family of open sets in R. Clearly $\bigcup_{n=1}^{\infty}(-n, n) = R$

$\therefore$ $\{(-n, n)/n \in N\}$ is an open cover for R and this open cover has no finite subcover.

$\therefore$ R is not compact.

Example 2 :

(0, 1) with usual metric is not compact.

Proof :

Consider the family of open intervals $\left\{\left(\frac{1}{n}, 1\right) \Big/ n = 2, 3, \ldots\ldots\right\}$

Clearly $\bigcup_{n=2}^{\infty}\left(\frac{1}{n},1\right) = (0, 1)$

$\therefore$ $\left\{\left(\frac{1}{n},1\right)/n = 2,3,\ldots\ldots\right\}$ is an open cover for (0, 1) and this open cover has no finite subcover. Hence (0, 1) is not compact.

Example 3 :

[0, ∞) with usual metric is not compact.

Proof :

Consider the family of intervals $\{[0, n)/n \in N\}$

Also $\bigcup_{n=1}^{\infty}[0,n) = [0, \infty)$

$\therefore$ $\{[0, n)/n \in N\}$ is an open cover for [0, ∞) and this open cover has no finite subcover. Hence [0, ∞) is not compact.

Example 4 :

Let M be an infinite set with discrete metric. Then M is not compact.

Proof :

Let $x \in M$. Since M is a discrete metric space $\{x\}$ is open in M.

Also $\bigcup_{x=M}\{x\} = M.$

Hence $\{\{x\}/x \in M\}$ is an open cover for M and since M is infinite, this open cover has no finite subcover.

Hence M is not compact.

Theorem 1 :

Let M be a metric space. Let $A \subseteq M$. A is compact iff given a family of open sets $\{G_\alpha\}$ in M such that $\cup G_{\in} \supseteq A$ thre exists a subfamily $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}$ such that $\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$.

Proof :

Let A be a compact subset of M. Let $\{G_\alpha\}$ be a family of open sets in M such that $\cup G_\alpha \supseteq A$.

Then $(\cup G_\alpha) \cap A = A$

$\therefore \cup(G_\alpha \cap A) = A$

Also $G_\alpha \cap A$ is open in A.

$\therefore$ The family $\{G_\alpha \cap A\}$ is an open cover for A.

Since A is compact this open cover has a finite subcover, say $G_{\alpha_1} \cap A$, $G_{\alpha_2} \cap A, \ldots, G_{\alpha_n} \cap A$

$$\therefore \bigcup_{i=1}^{n} \left(G_{\alpha_i} \cap A\right) = A$$

$$\therefore \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) \cap A = A$$

$$\therefore \bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$$

Conversely let $\{H_\alpha\}$ be an open cover for A.

$\therefore$ Each H_α is open in A.

$H_\alpha = G_\alpha \cap A$ where G_α is open in M.

Now, $\cup H_\alpha = A$

$\therefore \cup(G_\alpha \cap A) = A$

$\therefore (\cup G_\alpha) \cap A = A$

$\therefore \cup G_\alpha \supseteq A$

Hence by hypothesis there exists a finite subfamily $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}$ such that $\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$

$$\therefore \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) \cap A = A$$

$$\therefore \bigcup_{i=1}^{n} \left(G_{\alpha_i} \cap A\right) = A$$

$$\therefore \quad \bigcup_{i=1}^{n} H_{\alpha_i} = A$$

Thus $\{H_{\alpha_1}, H_{\alpha_2}, \ldots\ldots, H_{\alpha_n}\}$ is a finite subcover of the open cover $\{H_\alpha\}$.

$\therefore$ A is compact.

Theroem 2 :

Any compact subset A of a metric space M is bounded.

Proof :

Let $x_0 \in A$

Consider $\{B(x_0, n)/n \in N\}$

Cleraly $\bigcup_{n=1}^{\infty} B(x_0, n) = M$

$$\therefore \quad \bigcup_{n=1}^{\infty} B(x_0, n) \supseteq A$$

Since A is compact there exists a finite subfamily say, $B(x_0, n_1)$, $B(x_0, n_2), \ldots, B(x_0, n_k)$ such that $\bigcup_{i=1}^{k} B(x_0, n_i) \supseteq A$

Let $n_0 = \max\{n_1, n_2, \ldots\ldots, n_k\}$

Then $\bigcup_{i=1}^{k} B(x_0, n_i) = B(x_0, n_0)$

$$\therefore \quad B(x_0, n_0) \supseteq A$$

We know that $B(x_0, n_0)$ is a bounded set and a subset of a bounded set is bounded. Hence A is bounded.

Note :

The converse of the above theorem is not true.

For example, (0, 1) is a bounded subset of R.

But it is not compact.

Theorem 3 :

Any compact subset A of a metric space (M, d) is closed.

Proof :

To prove that A is closed we shall prove that A^C is open.

Let $y \in A^C$ and let $x \in A$. Then $x \neq y$

∴ $d(x, y) = r_x > 0$

It can be easily verified that $B\left(x, \frac{1}{2} r_x\right) \cap B\left(y, \frac{1}{2} r_x\right) = \phi$.

Now consider the collection $\left\{B\left(x, \frac{1}{2} r_x\right) / x \in A\right\}$

Cleary $\bigcup_{x \in A} B\left(x, \frac{1}{2} r_x\right) \supseteq A$.

Since A is compact there exists a finite number of such open balls say, $B\left(x_1, \frac{1}{2} r_{x_1}\right), \ldots\ldots, B\left(x_n, \frac{1}{2} r_{x_n}\right)$ such that

$$\bigcup_{i=1}^{n} B\left(x_i, \frac{1}{2} r_{x_i}\right) \supseteq A \qquad \text{-------(1)}$$

Let $$V_y = \bigcap_{i=1}^{n} B\left(y, \frac{1}{2} r_{x_i}\right)$$

Clearly, V_y is an open set containing y.

Since $B\left(y, \frac{1}{2} r_y\right) \cap B\left(x, \frac{1}{2} r_x\right) = \phi$ we have $V_y \cap B\left(x, \frac{1}{2} r_{x_i}\right) = \phi$ for each i=1,2,....,n.

$$\therefore \quad V_y \cap \left[\bigcup_{i=1}^{n} B\left(x, \frac{1}{2} r_{x_i}\right)\right] = \phi$$

$$\therefore \quad V_y \cap A = \phi \qquad \text{(by (1))}$$

$$\therefore \quad V_y \subseteq A^C$$

$$\therefore \quad \bigcup_{y \in A^C} V_y = A^C \text{ and each } V_y \text{ is open.}$$

∴ A^C is open. Hence A is closed.

Note 1 :

The converse of the above theorem is not true.

For example, $[0, \infty)$ is a closed subset of R. But it is not compact.

Note 2 :

Any compact subset of a metric space is closed and bounded.

Theorem 4 :

A closed subspace of a compact metric space is compact.

Proof :

Let M be a compact metric space. Let A be a non-empty closed subset of M.

We claim that A is compact.

Let $\{G_\alpha / \alpha \in I\}$ be a family of open sets in M such that $\bigcup_{\alpha \in I} G_\alpha \supseteq A$

$$\therefore A^C \cup \left[\bigcup_{\alpha \in I} G_\alpha \right] = M$$

Also A^C is open. (since A is closed)

$\therefore$ $\{G_\alpha / \alpha \in I\} \cup \{A^C\}$ is an open cover for M. Since M is compact it has a finite subcover say $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}, A^C$.

$$\therefore \quad \left(\bigcup_{i=1}^{n} G_{\alpha_i} \right) \cup A^C = M$$

$$\therefore \quad \bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$$

$\therefore$ A is compact.

Exercise :

1. Show that every finite metric space is compact.
2. A and B are two compact subsets of a metric space M. Prove that $A \cup B$ is also compact.
3. Give an example of a connected subset of R which is not compact.
4. Give an example of an open cover which has no finite subcover for the following subsets of R.

   (i) $(5, 6)$ (ii) $(5, \infty)$ (iii) $[5, \infty)$ (iv) $[7, 9)$.

COMPACT SUBSETS OF R

We know that every compact subset of a metric space is closed and bounded. However the converse is not true. For example, consider an infinite discrete metric space (M, d).

Let A be an infinite subset of M.

Then A is bounded since $d(x, y) \leq 1$ for all $x, y \in A$

Also A is closed since any subset of a discrete metric space is closed.

Hence A is closed and bounded.

However A is not compact.

Theorem 5 : (Heine Borel Theorem)

Any closed interval [a, b] is a compact subset of R.

Proof :

Let $\{G_\alpha / \alpha \in I\}$ be a family of open sets in R such that $\bigcup_{\alpha \in I} G_\alpha \supseteq [a, b]$.

Let $S = \{x / x \in [a,b]$ and $[a, x]\}$ can be covered by a finite number of G_α's.

Clearly $a \in S$ and hence $S \neq \phi$.

Also S is bounded above by b.

Let c denote the *l*.u.b. of S.

Clearly $c \in [a, b]$.

∴ $c \in G_{\alpha_1}$ for some $\alpha_1 \in I$.

Since G_{α_1} is open, there exists $\epsilon > 0$ such that $(c-\epsilon, C+\epsilon) \subseteq G_{\alpha_1}$.

Choose $x_1 \in [a, b]$ such that $x_1 < C$ and $[x_1, c] \subseteq G_{\alpha_1}$.

Now, since $x_1 < C$, $[a, x_1]$ can be covered by a finite number of G_α' s.

These finite number of G_α' s together with G_{α_1} cover [a, c]

∴ By definition of S, $c \in S$.

Now, we claim that c=b.

Suppose $c \neq b$.

Then choose $x_2 \in [a, b]$ such that $x_2 > c$ and $[c, x_2] \subseteq G_{\alpha_1}$.

As before, $[a, x_2]$ can be covered by a finite number of G_α, s.

Hence $x_2 \in S$.

But $x_2 > c$ which is a contradiction, sicne C is the l.u.b. of S.

∴ $c = b$

∴ [a, b] can be covered by a finite number of G_α, s.

∴ [a, b] is a compact subset of R.

Theorem 6 :

A subset A of R is compact iff A is closed and bounded.

Proof :

If A is compact then A is closed and bounded.

Conversely, let A be subset of R which is closed and bounded.

Since A is bounded we can find a closed interval [a, b] such that $A \subseteq [a, b]$

Since A is closed in R, A is closed in [a, b] also.

Thus A is a closed subset of the compact space [a, b].

Hence A is compact (by theorem (4)).

Exercise :

1. Determine which of the following subset of R are compact.

   (i) Z (ii) Q (iii) [1, 2]

   (iv) (3, 4) (v) $[1, 2] \cup [3, 4]$ (vi) $[1, 3] \cap [3, 4]$

   (vii) $\left\{1, \frac{1}{2}, \frac{1}{3}, \dots\dots\dots, \frac{1}{n}, \dots\dots\right\}$

2. If A and B are compact subsets of R prove that $A \cap B$ is also a compact subset of R.

EQUIVALENT CHARACTERISATIONS FOR COMPACTNESS

Definition :

A family $\mathfrak{F}$ of subsets of a set M is said to have the **finite intersection property** if any finite members of $\mathfrak{F}$ have non-empty intersection.

Example :

In R the family of closed intervals $\mathfrak{F}=\{[-n, n]/n\in N\}$ has finite intersection property.

Theorem 7 :

A metric space M is compact iff any family of closed sets with finite intersection property has non-empty intersection.

Proof :

Suppose M is compact.

Let $\{A_\alpha\}$ be a family of closed subsets of M with finite intersection property.

We claim that $\cap A_\alpha \neq \phi$

Suppose $\cap A_\alpha = \phi$ then $(\cap A_\alpha)^C = \phi^C$.

$\therefore \quad \cap A_\alpha{}^C = M$

Also, since each A_α is closed, $A_\alpha{}^C$ is open.

$\therefore$ $\{A_\alpha{}^C\}$ is an open cover for M.

Since M is compact this open cover has a finite subcover say, $A_1{}^C, A_2{}^C, \ldots\ldots, A_n{}^C$.

$$\therefore \quad \bigcup_{i=1}^{n} A_i{}^C = M$$

$$\therefore \quad \left(\bigcap_{i=1}^{n} A_i\right)^C = M$$

$$\therefore \quad \bigcap_{i=1}^{n} A_i = \phi$$ which is a contradiction to the finite intersection property.

$$\therefore \quad \cap A_\alpha \neq \phi$$

Conversely, suppose that each family of closed sets in M with finite intersection property has non empty intersection.

To prove that M is compact, let $\{G_\alpha / \alpha \in I\}$ be an open cover for M.

$$\therefore \quad \bigcup_{\alpha \in I} G_\alpha = M$$

$$\therefore \quad \left(\bigcup_{\alpha \in I} G_\alpha\right)^C = M^C$$

$$\therefore \quad \bigcap_{\alpha \in I} G_\alpha^{\ C} = \phi.$$

Since G_α is open, $G_\alpha^{\ C}$ is closed for each α.

$\therefore$ $\mathfrak{F} = \{G_\alpha^{\ C} / \alpha \in I\}$ is a family of closed sets whose intersection is empty.

Hence by hypothesis this family of closed sets does not have the finite intersection property.

Hence there exists a finite sub-collection of $\mathfrak{F}$ say $\{G_1^{\ C}, G_2^{\ C}, \ldots, G_n^{\ C}\}$ such that $\bigcap_{i=1}^{n} G_i^{\ c} = \phi$.

$$\therefore \quad \left(\bigcup_{i=1}^{n} G_i\right)^C = \phi$$

$$\therefore \quad \bigcup_{i=1}^{n} G_i = M$$

$\therefore$ $\{G_1, G_2, \ldots, G_n\}$ is a finite subcover of the given open cover.

Hence M is compact.

Definition :

A metric space M is said to be **totally bounded** if for every $\epsilon > 0$ there exists a finite number of elements $x_1, x_2, \ldots, x_n \in M$ such that $B(x_1, \epsilon) \cup B(x_2, \epsilon) \cup \ldots \cup B(x_n, \epsilon) = M$.

A non-empty subset A of a metric space M is said to be totally bounded if the subspace A is a totally bounded metric space.

Theorem 8 :

Any compact metric space is totally bounded.

Proof :

Let M be a compact metric space. Then $\{B(x, \in)/x \in M\}$ is an open cover for M.

Since M is compact this open cover has a finite subcover say $B(x_1, \in)$, $B(x_2, \in), \ldots\ldots, B(x_n, \in)$

∴ $M = B(x_1, \in) \cup B(x_2, \in) \cup \ldots\ldots \cup B(x_n, \in)$

∴ M is totally bounded.

Theorem 9 :

Let A be a subset of a metric space M. If A is totally bounded then A is bounded.

Proof :

Let A be a totally bounded subset of M. Let $\in > 0$ be given.

Then there exists a finite number of points $x_1, x_2, \ldots\ldots, x_n \in A$ such that $B_1(x_1, \in) \cup B_1(x_2, \in) \cup \ldots\ldots \cup B_1(x_n, \in) = A$, where $B_1(x_i, \in)$ is an open ball in A.

Further we know that an open ball is a bounded set.

Thus A is the union of a finite number of bounded sets and hence A is bounded.

Note :

The converse of the above theorem is not true.

For let M be an infinite set with discrete metric.

Clearly M is bounded.

Now $B(x, ½) = \{x\}$

Since M is infinite, M cannot be written as the union of a finite number of open balls $B(x, ½)$

∴ M is not toally bounded.

Definition :

Let (x_n) be a sequence in a metric space M. Let $n_1 < n_2 < \ldots\ldots < n_k < \ldots\ldots$ be an increasing sequence of positive integers. Then (x_{n_k}) is called a subsequence of (x_n).

Theorem 10 :

A metric space (M, d) is totally bounded iff every sequence in M has a Cauchy subsequence.

Proof :

Suppose every sequence in M has a Cauchy subsequence.

We claim that M is totally bounded.

Let $\in>0$ be given. Choose $x_1 \in M$.

If $B(x_1, \in) = M$ then obviously M is totally bounded.

If $B(x_1, \in) \neq M$, choose $x_2 \in M-B(x_1, \in)$ so that $d(x_1, x_2) \geq \in$.

Now, if $B(x_1, \in) \cup B(x_2, \in) = M$ the proof is complete.

If not choose $x_3 = M-[B(x_1, \in) \cup B(x_2, \in)]$ and so on.

Suppose this process does not stop at a finite stage.

Then we obtain a sequence $x_1, x_2, \ldots, x_n, \ldots$ such that $d(x_n, x_m) \geq \in$ if $n \neq m$.

Clearly this sequence (x_n) cannot have a Cauchy subsequence which is a contradiction.

Hence the above process stops at a finite stage and we get a finite set of points $\{x_1, x_2, \ldots, x_n\}$ such that $M = B(x_1, \in) \cup B(x_2, \in) \cup \ldots \cup B(x_n, \in)$

$\therefore$ M is totally bounded.

Conversely suppose M is totally bounded.

Let $S_1 = \{x_{i_1}, x_{i_2}, \ldots, x_{i_n}, \ldots\}$ be a sequence in M.

If one term of the sequence is infinitely repeated then S_1 contains a constant subsequence which is obviously a Cauchy subsequence.

Hence we assume that no term of S_1 is infinitely repeated so that the range of S is infinite.

Now since M is totally bounded M can be covered by a finite number of open balls of radius $\frac{1}{2}$.

Hence atleast one of these balls must contain an infinite number of terms of the sequence S_1.

∴ S_1 contains a subsequence $S_2 = (x_{2_1}, x_{2_2}, \ldots\ldots, x_{2_n}, \ldots\ldots)$ all terms of which lie within an openball of radius $\frac{1}{2}$.

Similarly S_2 contains a subsequence $S_3 = (x_{3_1}, x_{3_2}, \ldots\ldots, x_{3_n}, \ldots\ldots)$ all terms of which lie within an openball of radius $\frac{1}{3}$.

We repeat this process of forming successive subsequence and finally we take the diagonal sequence.

$S = (x_{1_1}, x_{2_2}, \ldots\ldots, x_{n_n}, \ldots\ldots)$

We claim that S is a Cauchy subsequence of S_1.

If m>n, both x_{m_m} and x_{n_n} lie within an open ball of radius $\frac{1}{n}$.

∴ $d(x_{m_m}, x_{n_n}) < \frac{2}{n}$

Hence $d(x_{m_m}, x_{n_n}) < \epsilon$ if $n, m > \frac{2}{\epsilon}$

This shows that S is a Cauchy subsequence of S_1.

Thus every sequence in M contains a Cauchy subsequence.

Corollarly :

A non-empty subset of a totally bounded set is totally bounded.

Proof :

Let A be a totally bounded subset of a metric space M.

Let B be a non-empty subset of A.

Let (x_n) be a sequence in B.

∴ (x_n) is a sequence in A.

Since A is totally bounded (x_n) has a Cauchy subsequence.

Thus every sequence in B has a Cauchy subsequence.

∴ B is totally bounded.

Definition :

A metric space M is said to be **sequentially compact** if every sequence in M has a convergent subsequence.

Theorem 11 :

Let (x_n) be a Cauchy sequence in a metric space M. If (x_n) has a subsequence (x_{n_k}) converging to x, then (x_n) converges to x.

Proof :

Let $\epsilon>0$ be given. Since (x_n) is a Cauchy sequence, there exists a positive integer m_1, such that $d(x_n, x_m) < \frac{1}{2}\epsilon$ for all $n, m \geq m_1$ -------(1)

Also since $(x_{n_k}) \to x$, there exists a positive integer m_2 such that

$$d(x_{n_k}, x) < \frac{1}{2}\epsilon \text{ for all } n_k \geq m_2 \qquad \text{-------(2)}$$

Let $m_0 = \max\{m_1, m_2\}$ and fix $n_k \geq m_0$.

Then
$$\begin{aligned} d(x_n, x) &\leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2} \text{ for all } n \geq m_0 \text{ (by (1) and (2))} \\ &= \epsilon \text{ for all } n \geq m_0. \end{aligned}$$

Hence $(x_n) \to x$.

Theorem 12 :

In a metric space M the following are equivalent.

(i) M is compact.

(ii) Any infinite subset of M has a limit point.

(iii) M is sequentially compact.

(iv) M is totally bounded and complete.

Proof :

(i) $\Rightarrow$ (ii)

Let A be an infinite subset of M.

Suppose A has no limit point in M.

Let $x \in M$

Since x is not a limit point of A there exists an open ball $B(x, r_x)$ such that

$$B(x, r_x) \cap (A - \{x\}) = \phi$$

$$\therefore \quad B(x, r_x) \cap A = \begin{cases} \{x\} & \text{if } x \in A \\ \phi & \text{if } x \notin A \end{cases}$$

Now, $\{B(x, r_x)/x \in M\}$ is open cover for M. Also each $B(x, r_x)$ covers atmost one point of the infinite set A.

Hence this open cover cannot have a finite subcover which is a contradiction to (i). Hence A has atleast one limit point.

(ii) $\Rightarrow$ (iii). Let (x_n) be a sequence in M. If one term of the sequence is infinitely repeated then (x_n) contains a constant subsequence which is convergent.

Otherwise (x_n) has an infinite number of terms. By hypothesis this infinite set has a limit point, say x.

For any $r>0$, the open ball $B(x, r)$ contains infinite number of terms of the sequence (x_n).

Choose a positive integer n, such that $x_{n_1} \in B(x, 1)$. Then choose $n_2 > n_1$ such that $x_{n_2} \in B\left(x, \frac{1}{2}\right)$.

In general for each positive integer K choose n_k such that $n_k > n_{k-1}$ and $x_{n_k} \in B\left(x, \frac{1}{K}\right)$

Clearly (x_{n_K}) is a subsequence of (x_n).

Also $d(x_{n_K}, x) < \frac{1}{K}$

$\therefore (x_{n_K}) \to x$

Thus (x_{n_K}) is a convergent subsequence of (x_n). Hence M is sequentially compact.

(iii) $\Rightarrow$ (iv)

By hypothesis every sequence in M has a convergent subsequence. But every convergent sequence is a Cauchy sequence.

Thus every sequence in M has a Cauchy subsequence.

By theorem (10), M is totally bounded.

Now we prove that M is complete.

Let (x_n) be a Cauchy sequence in M.

∴ By hypothesis (x_n) contains a convergent subsequence (x_{n_K})

Let $(x_{n_K}) \to x$ (say)

Then $(x_n) \to x$ (by theorem (11))

∴ M is complete.

(iv) ⇒ (i)

Suppose M is not compact.

Then there exists an open cover $\{G_\alpha\}$ for M which has no finite subcover.

Let $r_n = \frac{1}{2^n}$

Since M is totally bounded, M can be covered by a finite number of open balls of radius r_1.

Since M cannot be covered by a finite of G_α's atleast one of these open balls, say $B(x_1, r_1)$ cannot be covered by a finite number of G_α's.

Now $B(x_1, r_1)$ is totally bounded.

Hence we can find $x_2 \in B(x_1, r_1)$ such that $B(x_2, r_2)$ cannot be covered by a finite number of G_α's.

Proceeding like this we obtain a sequence (x_n) in M such that $B(x_n, r_n)$ cannot be covered by a finite number of G_α's and $x_{n+1} \in B(x_n, r_n)$ for all n.

Now,
$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1})+d(x_{n+1}, x_{n+2})+\ldots\ldots d(x_{n+p-1}, x_{n+p}) \\ &< r_n+r_{n+1}+\ldots\ldots+r_{n+p-1} \\ &= \frac{1}{2^n}+\frac{1}{2^{n+1}}+\ldots+\frac{1}{2^{n+p-1}} \end{aligned}$$

$$= \frac{1}{2^{n-1}}\left(\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^p}\right)$$

$$< \frac{1}{2^{n-1}}$$

$\therefore$ (x_n) is a Cauchy sequence in M.

Since M is complete there exists $x \in M$ such that $(x_n) \to x$.

Now, $x \in G_\alpha$ for some α.

Since G_α is open we can find $\in > 0$ such that $B(x, \in) \subseteq G_\alpha$ -------(1)

We have $(x_n) \to x$ and $(r_n) = \left(\frac{1}{2^n}\right) \to 0$.

Hence we can find a positive integer n_1 such that

$$d(x_n, x) < \frac{1}{2}\in$$

and $$r_n < \frac{1}{2}\in \quad \text{for all } n \geq n_1.$$

Now fix $n \geq n_1$

We claim that $B(x_n, r_n) \subseteq B(x, \in)$

Let $y \in B(x_n, r_n)$

$$\therefore \quad d(y, x_n) < r_n < \frac{1}{2}\in \text{ (since } n \geq n_1)$$

Now $$d(y, x) \leq d(y, x_n)+d(x_n, x)$$

$$< \frac{1}{2}\in+\frac{1}{2}\in = \in$$

$$\therefore \quad y \in B(x, \in)$$

$$\therefore \quad B(x_n, r_n) \subseteq B(x, \in) \subseteq G_\alpha \quad \text{(by (1))}$$

Thus $B(x_n, r_n)$ is covered by the single set G_α which is a contradiction since $B(x_n, r_n)$ cannot be covered by a finite number of G_α's.

Hence M is compact.

Theorem 13 :

R with usual metric is complete.

Proof :

Let (x_n) be a Cauchy sequence in R. Then (x_n) is a bounded sequence and hence is contained in a closed interval [a, b].

Now [a, b] is compact and hence is complete.

Hence (x_n) converges to some point $x \in [a, b]$. Thus every Cauchy sequence (x_n) in R converges to some point x in R and hence R is complete.

Solved Problems :

Problem 1 :

Give an example of a closed and bounded subset of l_2 which is not compact.

Solution :

Consider $0 = (0, 0, 0, \ldots\ldots\ldots) \in l_2$

Consider the closed ball B[0, 1]

Clearly B[0, 1] is a closed set.

We claim that B[0, 1] is not compact.

Consider $e_1 = (1, 0, 0, \ldots.)$; $e_2 = (0, 1, 0, \ldots.)$; ....$e_n = (0, 0, 0, \ldots., 1, 0, \ldots..)$

Now, $d(0, e_n) = 1$ and hence $e_n \in B[0, 1]$ for all n.

Thus (e_n) is a sequence in B[0, 1]

Also $d(e_n, e_m) = \sqrt{2}$ if $n \neq m$

Hence the sequence (e_n) does not contain a Cauchy subsequence.

∴ B[0, 1] is not totally bounded.

∴ B[0, 1] is not compact.

Problem 2 :

Prove that any totally bounded metric space is separable.

Solution :

Let M be a totally bounded metric space.

For each natural number n let $A_n = \{x_{n_1}, x_{n_2}, \ldots\ldots x_{n_K}\}$ be a subset of M such that

$$\bigcup_{i=1}^{K} B\left(x_{n_i}, \frac{1}{n}\right) = M \qquad \text{-------(1)}$$

Let $$A = \bigcup_{n=1}^{\infty} A_n$$

Since each A_n is finite, A is a countable subset of M.

We claim that A is dense in M.

Let $B(x, \in)$ be any open ball.

Choose a natural number n such that $\frac{1}{n} < \in$.

Now, $$x \in B\left(x_{n_i}, \frac{1}{n}\right) \text{ for some } i \text{ (by (1)).}$$

$$\therefore \quad d(x_{n_i}, x) < \frac{1}{n} < \in$$

$$\therefore \quad (x_{n_i}) \in B(x, \in)$$

$$\therefore \quad B(x, \in) \cap A \neq \phi.$$

Thus every open ball in M has non-empty intersection with A. Hence A is dense in M. Thus A is a countable dense subset of M. Hence M is separable.

Problem 3 :

Prove that any bounded sequence in R has a convergent subsequence.

Solution :

Let (x_n) be a bounded sequence in R. Then there exists a closed interval [a, b] such that $x_n \in [a, b]$ for all n.

Thus (x_n) is a sequence in the compact metric space [a,b].

Hence by theorem (12), (x_n) has a convergent sub-sequence.

Problem 4 :

Prove that the closure of a totally bounded set is totally bounded.

Solution :

Let A be a totally bounded subset of a metric space M.

We claim that $\overline{A}$ is totally bounded.

We show that every sequence in $\overline{A}$ contains a Cauchy subsequence.

Let (x_n) be a sequence in $\overline{A}$.

Let $\in>0$ be given.

Then since $x_n \in \overline{A}$, $B\left(x_n, \frac{1}{3}\in\right) \cap A \neq \phi$.

Choose $y_n \in B\left(x_n, \frac{1}{3}\in\right) \cap A$

$$\therefore \quad d(y_n, x_n) < \frac{1}{3}\in \quad \text{-------(1)}$$

(y_n) is a sequence in A. Since A is totally bounded (y_n) contains a Cauchy sequence say (y_{n_K}).

Hence there exists a natural number m such that

$$d(y_{n_i}, y_{n_j}) < \frac{1}{3}\in \text{ for all } n_i, n_j \geq m \quad \text{------(2)}$$

$$\therefore \quad d(x_{n_i}, x_{n_j}) \leq d(x_{n_i}, y_{n_i})+d(y_{n_i}, y_{n_j})+d(y_{n_j}, x_{n_j})$$

$$< \frac{1}{3}\in+\frac{1}{3}\in+\frac{1}{3}\in = \in \text{ for all ni, nj} \geq m \text{ by (1) and (2).}$$

Hence (x_{n_K}) is a Cauchy subsequence of (x_n).

$\therefore$ $\overline{A}$ is totally bounded.

Problem 5 :

Let A be a totally bounded subset of R. Prove that $\overline{A}$ is compact.

Solution :

Since A is totally bounded $\overline{A}$ is also totally bounded.

Also since $\overline{A}$ is a closed subset of R and R is complete $\overline{A}$ is complete.

Hence $\overline{A}$ is totally bounded and complete.

$\therefore$ $\overline{A}$ is compact.

Exercise :

1. Let M be a complete metric space. Prove that a closed subset A of M is compact iff A is totally bounded.
2. Prove that a compact metric space is separable.
3. Prove that a connected subset of a discrete metric space M is compact.
4. Give an example of a complete metric space which is not compact.
5. Prove that any Cauchy sequence in a metric space is totally bounded.

COMPACTNESS AND CONTINUITY

Theorem 14 :

Let f be a continuous mapping from a compact metric space M_1 to any metric space M_2. Then $f(M_1)$ is compact. i.e., continuous image of a compact metric space is compact.

Proof :

Without loss of generality we assume that $f(M_1) = M_2$

Let $\{G_\alpha\}$ be a family of open sets in M_2 such that $\cap G_\alpha = M_2$.

$$\therefore \quad \cup G_\alpha = f(M_1)$$

$$\therefore \quad f^{-1}(\cup G_\alpha) = M_1$$

$$\therefore \quad \cup f^{-1}(G_\alpha) = M_1$$

Also since f is continuous $f^{-1}(G_\alpha)$ is open in M_1 for each α.

$\therefore$ $\{f^{-1}(G_\alpha)\}$ is an open cover for M_1.

Since M_1 is compact this open cover has a finite subcover, say, $f^{-1}(G_\alpha),\ldots\ldots,f^{-1}(G_{\alpha_n})$.

$\therefore$ $f^{-1}(G_{\alpha_1}) \cup f^{-1}(G_{\alpha_2}) \cup \ldots\ldots \cup f^{-1}(G_{\alpha_n}) = M_1$

$$\therefore \ f^{-1}\left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) = M_1$$

$\therefore \bigcup_{i=1}^{n} G_{\alpha_i} = f(M_1) = M_2$

$\therefore G_{\alpha_1}, G_{\alpha_2}, \ldots G_{\alpha_n}$ is a cover for M_2. Thus the given open cover $\{G_\alpha\}$ for M_2 has a finite subcover.

$\therefore M_2$ is compact.

Corolarly 1 :

Let f be a continuous map from a compact metric space M_1 into any metric M_2. Then $f(M_1)$ is closed and bounded.

Proof :

$f(M_1)$ is compact and hence is closed and bounded.

Corollary 2 :

Any continuous real valued function f defined on a compact metric space is bounded and attains its bounds.

Proof :

Let M be a compact metric space.

Let $f: M \rightarrow R$ be a continuous real valued function. Then f(M) is a compact subset of R.

$\therefore$ f(M) is a closed and bounded subset of R.

Since f(M) is bounded f is a bounded function.

Let a = l.u.b of f(M) and b = g.l.b. of f(M).

By definition of l.u.b. and g.l.b. a, b $\in \overline{f(M)}$

But f(M) is closed. Hence $f(M) = \overline{f(M)}$

$\therefore$ a, b $\in$ f(M)

$\therefore$ There exist x, y $\in$ M such that f(x) = a and f(y) = b.

Hence f attains its bounds.

Note :

Corollary (2) is not true if M is not compact.

The function $f:(0, 1)\rightarrow R$ defined by $f(x) = \frac{1}{x}$ is continuous but not bounded.

The function $g : (0, 1)\rightarrow R$ defined by $g(x) = x$ is bounded having $l.u.b = 1$ and $g.l.b. = 0$. However this function never attains these bounds at any point in $(0, 1)$.

Theorem 15 :

Any continuous mapping f defined on a compact metric space (M_1, d_1) into any other metric space (M_2, d_2) is uniformly continuous on M_1.

Proof :

Let $\in>0$ be given. Let $x\in M_1$.

Since f is continuuos at x there exists $\delta_x > 0$ such that

$$d_1(y, x)<\delta_x \Rightarrow d_2(f(y), f(x))<\frac{1}{2}\in \qquad \text{--------(1)}$$

The family of open balls $\left\{B\left(x,\frac{1}{2}\delta_x\right)\big/ x\in M_1\right\}$ is an open cover for M_1.

Since M_1 is compact this open cover has a finite subcover say $B\left(x_1,\frac{1}{2}\delta_{x_1}\right),\ldots,B\left(x_n,\frac{1}{2}\delta_{x_n}\right)$

Let $\delta = \min\left\{\frac{1}{2}\delta_{x_1},\ldots,\frac{1}{2}\delta_{x_n}\right\}$

We claim that $d_1(p, q)<\delta \Rightarrow d_2(f(p), f(q))<\in$.

Let $p\in B\left(x_i,\frac{1}{2}\delta_{x_i}\right)$ for some i where $1\leq i\leq n$.

$$\therefore \quad d_1(p, x_i) < \frac{1}{2}\delta_{x_i}$$

$$\therefore \quad d_2(f(p), f(x_i)) < \frac{1}{2}\in \quad \text{(by (1))} \qquad \text{--------(2)}$$

Now,
$$\begin{aligned} d_1(q, x_i) &\leq d_1(q, p)+d_1(p, x_i) \\ &\leq \delta+\frac{1}{2}\delta_{x_i} \end{aligned}$$

$$\leq \frac{1}{2}\delta_{x_i} + \frac{1}{2}\delta_{x_i} = \delta_{x_i}$$

Thus $\quad d_1(q, x_i) < \delta_{x_i}$

$\therefore \quad d_2(f(q), f(x_i)) < \frac{1}{2}\in \quad$ (by (1)) -------(3)

$$d_2(f(p), f(q)) \leq d_2(f(p), f(x_i)) + d_2(f(x_i), f(q))$$

$$< \frac{1}{2}\in + \frac{1}{2}\in = \in \quad \text{(by (2) and (3))}$$

Thus $\quad d_1(p, q) < \delta \Rightarrow d_2(f(p), f(q)) < \in$.

This proves that f is uniformly continuous on M_1.

Note : The above theorem is not true if M_1 is not compact.

Theorem 16 :

Let f be a 1–1 continuous function from a compact metric space M_1 onto any metric space M_2. Then f^{-1} is continuous on M_2. Hence f is a homeomorphism from M_1 onto M_2.

Proof :

We shall show that f^{-1} is continuous by proving that F is a closed set in $M_1 \Rightarrow (f^{-1})^{-1}(F) = f(F)$is a closed set in M_2.

Let F be a closed set in M_1.

Since M_1 is compact F is compact.

Since f is continuous f(F) is a compact subset of M_2.

$\therefore$ f(F) is a closed subset of M_2.

$\therefore$ f^{-1} is continuous on M_2.

Solved Problems :

Problems 1 :

Prove that the range of a continuous real valued function f on a compact connected metric space M must be either a single point or a closed and bounded interval.

Solution :

Let $f : M \to R$ be a continuous function.

Case (i) :

Suppose f is a constant function. Then the range of f is a single point.

Case (ii) :

Suppose f is not a constant function. Then the range of f contains more than one point. Since M is connected f(M) is a connected subset of R.

∴ f(M) is an interval in R.

Also since M is compact and f is continuous

f(M) is a compact subset of R.

∴ f(M) is a closed and bounded subset of R.

Thus f(M) is a closed and bounded interval of R.

Problem 2 :

Prove that any continuous function $f:[a, b] \to R$ is not onto.

Solution :

Suppose f is onto. Then f([a, b]) = R. Since [a, b] is compact and f is continuous, f([a, b])=R is compact which is a contradiction.

∴ f is not onto.

Exercise :

1. Prove that any continuous function from a compact metric space to any other metric space is a closed map.
2. Does there exist a continuous function f from [a, b] onto (a, b)?
3. Prove that any continuous function defined on a closed interval [a, b] is bounded and attains its bounds and also prove that f is uniformly continuous.

COMPLEX NUMBERS

We observe that in the real number system the equation $x^2+1 = 0$ has no solution. This leads to the definition of complex numbers in which equations of the form $x^2+a=0$ where $a>0$, have solutions.

Definition :

A complex number Z is of the form $x+iy$ where x and y are real numbers and i an imaginary unit with the property $i^2 = -1$. x and y are called the **real** and **imaginary** parts of z and we write $x = \text{Re } z$ and $y = \text{Im } z$.

If $x = 0$ the complex number z is called purely imaginary. If $y = 0$ then z is real.

Two complex numbers are said to be equal if and only if they have the same real parts and the same imaginary parts.

Let C denote the set of all complex numbers. Thus $C = \{x+iy/x,y \in R\}$

The complex number $x-iy$ is said to be the conjugate of $x+iy$.

ALGEBRAIC OPERATIONS

Complex numbers are assumed to obey the following laws of Algebra.

1. Addition :

$$z_1+z_2 = (x_1+iy_1)+(x_2+iy_2) = (x_1+x_2)+i(y_1+y_2)$$

2. Subtraction :

$$z_1-z_2 = (x_1+iy_1)-(x_2+iy_2) = (x_2-x_2)+i(y_1-y_2)$$

We note that

$$z+\bar{z} = (x+iy) + (x-iy) = 2x$$

$$z-\bar{z} = (x+iy)-(x-iy) = 2iy$$

so that $x = \text{Re } z = \dfrac{z+\bar{z}}{2}$

$$y = \text{Im } z = \frac{z-\bar{z}}{2i}$$

3. Multiplication :

$$
\begin{aligned}
z_1 z_2 &= (x_1+iy_1)(x_2+iy_2) \\
&= x_1x_2+ix_1y_2+ix_2y_1+i^2y_1y_2 \\
&= (x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1) \ (\because i^2=-1) \\
z\bar{z} &= (x+iy)(x-iy) = x^2+y^2
\end{aligned}
$$

4. Division

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{x_1+iy_1}{x_2+iy_2} = (x_1+iy_1)\frac{(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)} \\
&= \frac{(x_1+iy_1)(x_2-iy_2)}{x_2^2+y_2^2} \\
&= \frac{(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)}{x_2^2+y_2^2}
\end{aligned}
$$

Results :

1. $i^2=-1,\quad i^3=-i,\quad i^4=1,\quad i^{4n}=1,\quad (i)^{4n+1}=i,\quad (i)^{4n+2}=-1,\quad (i)^{4n+3}=-i.$

2. If z_1 and z_2 are two complex numbers, then

   (i) $\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$

   (ii) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

3. $\bar{z} = z$ iff z is real and $\bar{z} = -z$ iff z is imaginary.

4. If the coefficients in a polynomial equation are real, then its complex roots form pairs of complex conjugate.

Worked Examples :

Example 1 :

Express the following in the form (a+ib)

(a) $\frac{1}{2+2i}$, (b) $\frac{-1}{1-2i}$ (c) $\frac{2-i}{3+2i}$

(a) $$\frac{1}{2+2i} = \frac{2-2i}{4+4} = \frac{1-i}{4} = \frac{1}{4}-\frac{i}{4}$$

(b) $$\frac{-1}{1-2i} = \frac{-(1+2i)}{1+4} = \frac{-(1+2i)}{5} = -\frac{1}{5}-\frac{2i}{5}$$

(c) $$\frac{2-i}{3+2i} = \frac{(2-i)(3-2i)}{9+4} = \frac{(6-2)-i(3+4)}{13}$$

$$= \frac{4-7i}{13} = \frac{4}{13}-\frac{7}{13}i$$

Example 2 :

Show that if the equation $z^2+\alpha z+\beta = 0$ has a pair of complex conjugate roots, then α and β are both real and $\alpha^2<4\beta$.

Solution :

Let $z_1 = xi+iy_1$ be a root.

Then $\overline{z_1} = x_1-iy_1$ is also a root.

So, $z_1^2+\alpha z_1+\beta = 0$ --------(1)

and $\overline{z_1}^2+\alpha\overline{z_1}+\beta = 0$ --------(2)

$(1)-(2) \Rightarrow z_1^2-\overline{z_1}^2+\alpha(z_1-\overline{z_1}) = 0$

$$\alpha = -\frac{z_1^2-\overline{z_1}^2}{z_1-\overline{z_1}} = -\frac{(z_1+\overline{z_1})(z_1-\overline{z_1})}{z_1-\overline{z_1}}$$

$$= -(z_1+\overline{z_1}) = -2x_1.$$

$(1)+(2) \Rightarrow z_1^2+\overline{z_1}^2+\alpha(z_1+\overline{z_1})+2\beta = 0$

i.e., $(x_1+iy_1)^2+(x_1-iy_1)^2-2x_1(2x_1)+2\beta = 0$

i.e., $2x_1^2-2y_1^2-4x_1^2+2\beta = 0$

i.e., $\beta = x_1^2+y_1^2$

$\alpha = -2x_1,\ \beta = x_1^2+y_1^2 \Rightarrow \alpha$ and β are real.

$$\alpha^2-4\beta = 4x_1^2-(4x_1^2+4y_1^2)$$

$$= -4y_1^2 < 0$$

i.e., $\alpha^2 < 4\beta.$

Example 3 :

The sum and the product of two complex numbers are real. Show that the two numbers are either both real or complex conjugates.

Solution :

Let $z_1 = a+ib$, $z_2 = c+id$ be the two complex numbers.

$$z_1+z_2 = (a+c)+i(b+d)$$

$$z_1 z_2 = (ac-bd)+i(ad+bc)$$

$$z_1+z_2 = \text{real} \Rightarrow b+d = 0, \text{ i.e., } b = -d$$

$$z_1 z_2 = \text{real} \Rightarrow ad+bc = 0$$

i.e., $ad-cd = 0 \quad (\because b = -d)$

i.e., $(a-c)d = 0$

$\therefore d = 0$ or $a-c = 0$ i.e., $a = c$

If $d = 0$, $b = 0$ and so, the two numbers are real

If $a = c$ then $b = -d$ gives

$z_1 = a+ib$, $z_2 = a-ib$

i.e., z_1 and z_2 are complex conjugates.

MODULUS AND AMPLITUDE OF A COMPLEX NUMBER

If $z = x+iy$, then $z\bar{z}=x^2+y^2$ is a positive real number.

We define the modulus or the absolute value of a complex number z to be the non negative real number $\left(\sqrt{x^2+y^2}\right)$ and denote it as $|z|$.

i.e., $|z| = |x+iy| = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$

Note :

If z is real, i.e., if Im $z = y = 0$ so that $z = x$, $|z| = \sqrt{x^2}$ non negative square root of x^2.

$$= x \text{ if } x \geq 0$$

$$= -x \text{ if } x < 0$$

The angle θ satisfying the equation $\cos\theta = \frac{X}{|z|}$, $\sin\theta = \frac{Y}{|z|}$ -------(A)

is defined as the argument or the amplitude of the complex number z.

We know that if θ is a solution $(2n\pi+\theta)$ is also a solution for every integer n. Thus there are infinite number of solutions to the above equation (A). The value of θ satisfying the inequality $-\pi \le \theta \le \pi$ is called the principal value of the argument and $(2n\pi+\theta)$ its general value.

Note 1 :

If z is real, i.e., if $z = x$ and $y = 0$ then the principal value of the argument of z is 0 or π according as X is positive or negative.

If z is imaginary, i.e., if $z = iy$ and $x = 0$ then the principal value of the argument of z is $(\pm \pi/2)$ according as y is positive or negative.

Note 2 :

If for $z = x+iy$, $|z| = r$ and θ is teh argument, then $\frac{x}{r} = \cos\theta$, $\frac{y}{r} = \sin\theta$

So
$$z = x+iy = r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

called the polar form of the complex number.

$$= r \text{ cis } \theta \quad \text{– a convenient notation}$$
$$= r e^{i\theta} \quad \text{– Euler's notation.}$$

Example :

Find the moduli and principal values of arguments of the following complex numbers.

(i)
$$z = 1+\sqrt{3}i$$
$$|z| = \sqrt{1+3} = 2$$
$$\text{Cos } \theta = \frac{1}{2},$$
$$\text{Sin } \theta = \frac{\sqrt{3}}{2}$$
$$\therefore \quad \theta = \frac{\pi}{3}$$
$$\therefore \quad |z| = 2, \ \arg z = \frac{\pi}{3}$$

(ii) $z = 1$

$\therefore \quad |z| = 1$

$\text{Cos } \theta = 1, \quad \sin\theta = 0$

$\therefore \quad \theta = 0.$

$\therefore \quad |z| = 1, \arg z = 0.$

(iii) $z = -i$

$\therefore \quad |z| = 1$

$\text{Cos } \theta = 0$

$\sin\theta = -1$

$\therefore \quad \theta = -\pi/2$

$|z| = 1$

$\arg z = -\pi/2$

(iv) $z = 2\sqrt{3}\,i - 2$

$|z| = \sqrt{12+4} = 4$

$$\cos\theta = -\frac{2}{4} = -\frac{1}{2}, \ \sin\theta = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

$$\therefore \quad \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$|z| = 4,$

$$\arg z = \frac{2\pi}{3}$$

PROPERTIES OF MODULUS OF A COMPLEX NUMBER

Result 1 :

If z, z_1, z_2 are any three complex numbers then

(i) $|-z| = |z|$

(ii) $|z| = |\bar{z}|$

(iii) $z\bar{z} = |z|^2$

(iv) $|z_1 z_2| = |z_1|\,|z_2|$

Proof of (iv) $|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2})$

$= (z_1 z_2)(\overline{z_1}\ \overline{z_2})$

$= (z_1 \overline{z_1})(z_2 \overline{z_2})$

$= |z_1|^2 \ |z_2|^2$

$\therefore \quad |z_1 z_2| = |z_1| \ |z_2|$

Corollary :

(i) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(ii) $|z_1 z_2 \ldots\ldots z_n| = |z_1| \ |z_2| \ldots\ldots |z_n|$

$|z^n| = |z|^n$

Result 2 :

If z, z_1, z_2 are any three complex numbers

(i) $-|z| \le \text{Re } z \le |z|$

(ii) $-|z| \le \text{Im } z \le |z|$

(iii) $|z| \le |\text{Re } z| + |\text{Im } z|$

(iv) $|z_1+z_2| \le |z_1|+|z_2|$

(v) $|z_1-z_2| \ge \big||z_1|-|z_2|\big|$

Proof :

Let $z = x+iy$,, $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$

(i) $|z| = \sqrt{x^2+y^2}$ Re $z = x$, Im $z = y$

Clearly $-\sqrt{x^2+y^2} \le x \le \sqrt{x^2+y^2}$

(ii) Also $-\sqrt{x^2+y^2} \le y \le \sqrt{x^2+y^2}$

(iii)

$$(|x|+|y|)^2 = |x|^2+|y|^2+2|x||y|$$
$$= x^2+y^2+2|x||y|$$
$$\geq x^2+y^2$$

$\therefore$ $$\sqrt{x^2+y^2} \leq |x|+|y|$$

i.e., $$|z| \leq |\text{Re } z|+|\text{Im } z|$$

(iv)

$$|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2})$$
$$= (z_1+z_2)(\overline{z_1}+\overline{z_2})$$
$$= (z_1\overline{z_1}+z_2\overline{z_2})+(z_1\overline{z_2}+\overline{z_1}z_2)$$
$$= \left(|z_1|^2+|z_2|^2\right)+2\,\text{Re}(z_1\overline{z_2})$$
$$\leq |z_1|^2+|z_2|^2+2|z_1\overline{z_2}| \quad \text{by (i)}$$
$$\leq |z_1|^2+|z_2|^2+2|z_1||z_2|$$
$$\leq (|z_1|+|z_2|)^2$$

$\therefore$ $$|z_1+z_2| \leq |z_1|+|z_2|$$

(v)

$$|z_1| = |(z_1-z_2)+z_2|$$
$$\leq |z_1-z_2|+|z_2|$$

i.e., $$|z_1|-|z_2| \leq |z_1-z_2|$$

$$|z_2| = |z_2-z_1+z_1|$$
$$\leq |z_2-z_1|+|z_1|$$
$$\leq |z_1-z_2|+|z_1|$$

i.e., $$|z_2|-|z_1| \leq |z_1-z_2|$$

Hence $$\big||z_1|-|z_2|\big| \leq |z_1-z_2|$$

Corollary :

$$|z_1+z_2+\ldots+z_n| \leq |z_1|+|z_2|+\ldots\ldots+|z_n|$$

PROPERTIES OF ARGUMENTS OF A COMPLEX NUMBER

Result 1 :

(i) $\arg \bar{z} = -\arg z$

(ii) $\arg z_1 z_2 = \arg z_1 + \arg z_2$

Proof :

(i) Let $z = r(\cos\theta + i \sin\theta)$

Then
$$\bar{z} = r(\cos\theta - i\sin\theta)$$
$$= r(\cos(-\theta) + i \sin(-\theta))$$

$\therefore$ $\arg \bar{z} = -\theta = -\arg z.$

(ii) Let
$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$
$$z_1z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)] + i(\cos\theta_1 + \sin\theta_2 + \sin\theta_1\cos\theta_2)]$$
$$= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Hence $\arg z_1z_2 = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$

We note that we may have to add 2π to L.H.S. if necessary.

Corollary :

$$\arg(z_1 z_2 \ldots\ldots z_n) + 2n\pi = \arg z_1 + \arg z_2 + \ldots\ldots + \arg z_n$$ for a suitable n.

Result 2 :

If z, z_1, z_2 are complex numbers, then

(i) $\arg \dfrac{1}{z} = -\arg z$

(ii) $\arg \dfrac{z_1}{z_2} = \arg z_1 - \arg z_2$

Proof :

(i) Let

$$z = r(\cos\theta + i\sin\theta)$$

$$\frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)} = \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= \frac{1}{r}[\cos(-\theta) + i\sin(-\theta)]$$

$$\therefore \quad \arg\frac{1}{z} = (-\theta) = -\arg z.$$

(ii)

$$\arg\frac{z_1}{z_2} = \arg z_1 + \arg\left(\frac{1}{z_2}\right)$$

$$= \arg z_1 - \arg z_2$$

Note :

$$\frac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$$

Result 3 : De Moiore's Theorem

(i) For any integer n, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(ii) $\text{Cos } n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$ for a rational number n.

Note :

$1, \omega, \omega^2, \ldots\ldots, \omega^{n-1}$ are n roots of 1 where

$$\omega = \text{cis}\frac{2\pi}{n}$$

$$\omega^n = \left(\text{cis}\frac{2\pi}{n}\right)^n = \text{cis } 2\pi = 1$$

$$1, \omega, \omega^2, \ldots\ldots, \omega^{n-1} = \frac{1-\omega^n}{1-\omega} = 0$$

Example 1 :

Evaluate $(i)^{1/7}$

$$i = 1\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = \text{cis}\frac{\pi}{2}$$

$$(i)^{1/7} = \text{cis}\frac{2K\pi + \frac{\pi}{2}}{7} \quad \text{for } K = 0, 1, 2, 3, 4, 5, 6.$$

$$\text{cis}\frac{\pi}{14}, \text{cis}\frac{5\pi}{14}, \text{cis}\frac{9\pi}{14}, \text{cis}\frac{13\pi}{14}, \text{cis}\frac{17\pi}{14}, \text{cis}\frac{2\pi}{14}, \text{cis}\frac{25\pi}{14}$$

Example 2 :

Show that

$$|z_1+z_2|^2+|z_1-z_2|^2 = 2\left(|z_1|^2+|z_2|^2\right)$$

Proof :

$$\begin{aligned}
|z_1+z_2|^2+|z_1-z_2|^2 &= (z_1+z_2)(\overline{z_1+z_2})+(z_1-z_2)(\overline{z_1-z_2}) \\
&= (z_1+z_2)(\overline{z_1}+\overline{z_2})+(z_1-z_2)(\overline{z_1}-\overline{z_2}) \\
&= (z_1\overline{z_1}+z_2\overline{z_2})+(z_1\overline{z_2}+\overline{z_1}z_2)+(z_1\overline{z_1}+z_2\overline{z_2}) \\
&\qquad -(z_1\overline{z_2}+\overline{z_1}z_2) \\
&= 2\left(|z_1|^2+|z_2|^2\right)
\end{aligned}$$

Example 3 :

If one of $|a|$ and $|b|$ is equal to 1, show that $\left|\frac{a-b}{1-\bar{a}b}\right| = 1$.

Proof :

$$\begin{aligned}
\left|\frac{a-b}{1-\bar{a}b}\right|^2 &= \frac{|a-b|^2}{|1-\bar{a}b|^2} = \frac{(a-b)(\bar{a}-\bar{b})}{(1-\bar{a}b)(1-a\bar{b})} \\
&= \frac{a\bar{a}+b\bar{b}-a\bar{b}-\bar{a}b}{1+a\bar{a}b\bar{b}-\bar{a}b-a\bar{b}} \\
&= \frac{|a|^2+|b|^2-(a\bar{b}+\bar{a}b)}{1+|a|^2|b|^2-(\bar{a}b+a\bar{b})}
\end{aligned}$$

If $|a|$ or $|b|$ is equal to 1,

$$\left|\frac{a-b}{1-\bar{a}b}\right|^2 = 1$$

Example 4 :

Show that

$$\left|1-\overline{z_1}z_2\right|^2-\left|z_1-z_2\right|^2 = \left(1-|z_1|^2\right)\left(1-|z_2|^2\right)$$

$$\begin{aligned}\left|1-\overline{z_1}z_2\right|^2-\left|z_1-z_2\right|^2 &= (1-\overline{z_1}z_2)(1-z_1\overline{z_2})-(z_1-z_2)(\overline{z_1}-\overline{z_2}) \\ &= (1-z_1\overline{z_2}-\overline{z_1}z_2+z_1\overline{z_1}z_2\overline{z_2}) \\ &\qquad -(z_1\overline{z_1}-z_1\overline{z_2}-\overline{z_1}z_2+z_2\overline{z_2}) \\ &= 1-|z_1|^2-|z_2|^2+|z_1|^2|z_2|^2 \\ &= (1-|z_1|^2)(1-|z_2|^2)\end{aligned}$$

EQUATION OF STRAIGHT LINE

Let A, B, P be the points in the complex plane representing the complex numbers a, b, z respectively. Then the complex number z–a, z–b are represented by the vectors $\overline{AP}$ and $\overline{BP}$ respectively. ∴ The principal value of $\arg\left(\frac{z-a}{z-b}\right)$ gives the angle between the line segment AP and BP taken in the appropriate sense.

If z, a, b are collinear then $\arg\left(\frac{z-a}{z-b}\right) = 0$ or π.

∴ $\frac{z-a}{z-b}$ is real.

$$\therefore \quad \frac{z-a}{z-b} = \overline{\left(\frac{z-a}{z-b}\right)}$$

$$\therefore \quad \frac{z-a}{z-b} = \frac{\overline{z}-\overline{a}}{\overline{z}-\overline{b}}$$

$$\therefore \quad (z-a)(\overline{z}-\overline{b})-(z-b)(\overline{z}-\overline{a}) = 0$$

$$(\overline{a}-\overline{b})z-(a-b)\overline{z}+(a\overline{b}-\overline{a}b) = 0$$

$$\therefore \quad (\overline{a}-\overline{b})z-(a-b)\overline{z}+2i\,\mathrm{Im}(a\overline{b}) = 0$$

$$\therefore \quad i(\overline{a}-\overline{b})z-i(a-b)\overline{z}-2\,\mathrm{Im}(a\overline{b}) = 0$$

This equation is of the form $\overline{\alpha}z+\alpha\overline{z}+\beta = 0$ where $\alpha \neq 0$ and β is real.

Any equation of the above form represents a straight line. This can be easily seen by changing the above equation into cartesian form. ∴ The general equationof a straight line is given by $\bar{\alpha}z+\alpha\bar{z}+\beta = 0$ where $\alpha \neq 0$ and β is real.

Theorem 1 :

Equation of the line joining a and b is $(\bar{a}-\bar{b})z+(b-a)\bar{z}+(a\bar{b}-\bar{a}b) = 0$.

Theorem 2 :

If a and b are two distinct complex numbers where $b\neq0$, then the equation $z = a+tb$ where t is a real parameter represents a straight line passing through a point a and parallel to b.

Proof :

Let z be any point on the line passing through a and parallel to b.

The vectors represented by $z-a$ and b are parallel.

Hence $z-a = tb$ for some real number t.

∴ $z = a+tb$, which is the equation of the required straight line.

Definition :

Two points P and Q are called **reflection points** for a given straight line l iff l is the perpendicular bisector of the segment PQ.

Theorem 3 :

The points z_1 and z_2 are reflection points for the line $\bar{\alpha}z+\alpha\bar{z}+\beta = 0$ iff $\bar{\alpha}z_1+\alpha\overline{z_2}+\beta = 0$.

Proof :

Let z_1 and z_2 be reflection points for the straight line $\bar{\alpha}z+\alpha\bar{z}+\beta = 0$ --------(1)

∴ For any point z on the line we have

$$|z-z_1| = |z-z_2|$$

$$∴ \quad |z-z_1|^2 = |z-z_2|^2$$

$$∴ \quad (z-z_1)(\bar{z}-\overline{z_1}) = (z-z_2)(\bar{z}-\overline{z_2})$$

$$∴ \quad z(\overline{z_2}-\overline{z_1})+\bar{z}(z_2-z_1)+z_1\overline{z_1}-z_2\overline{z_2} = 0 \quad \text{--------(2)}$$

Since the equation is true for any point z on the given line it must be regarded as the equation of the given line.

∴ From (1) and (2) we get

$$\frac{\overline{\alpha}}{\overline{z_2}-\overline{z_1}} = \frac{\alpha}{z_2-z_1} = \frac{\beta}{z_1\overline{z_1}-z_2\overline{z_2}} = K \text{ (say)}$$

∴ $\alpha = K(z_2-z_1)$; $\overline{\alpha} = K(\overline{z_2}-\overline{z_1})$ and $\beta = K(z_1\overline{z_1}-z_2\overline{z_2})$

$$\therefore \quad \overline{\alpha}z_1+\alpha\overline{z_2}+\beta = K\left[z_1(\overline{z_2}-\overline{z_1})+\overline{z_2}(z_2-z_1)+(z_1\overline{z_1}-z_2\overline{z_2})\right]$$

$$= 0$$

Conversely, suppose $\overline{\alpha}z_1+\alpha\overline{z_2}+\beta = 0$ -------(3)

Subtracting (3) from (1) we get $\overline{\alpha}(z-z_1)+\alpha(\overline{z}-\overline{z_2}) = 0$

i.e., $$\overline{\alpha}(z-z_1) = -\alpha(\overline{z}-\overline{z_2})$$

$$\therefore \quad |\overline{\alpha}|\,|z-z_1| = |\alpha|\,|\overline{z}-\overline{z_2}|$$

$$\therefore \quad |z-z_1| = |\overline{z}-\overline{z_2}|$$

$= |\overline{z-z_2}| = |z-z_2|$ for any point z on the given line.

∴ z_1 and z_2 are reflection points for the line (1).

GENERAL EQUATION OF CIRCLES

Equation of the circle with centre a and radius r is given by $|z-a| = r$.

i.e., $(z-a)(\overline{z}-\overline{a}) = r^2$

i.e., $z\overline{z}-a\overline{z}-\overline{a}z+a\overline{a}-r^2 = 0$

This equation is of the form $z\overline{z}+\overline{\alpha}z+\alpha\overline{z}+\beta = 0$ where β is a real number.

Any equation of the above form can be rewritten as $|z+a|^2 = \alpha\overline{\alpha}-\beta$ and hence represent a circle provided $\alpha\overline{\alpha}-\beta>0$.

Thus the general equation of a circle is given by $z\overline{z}+\overline{\alpha}z+\alpha\overline{z}+\beta = 0$ where β is real and $\alpha\overline{\alpha}-\beta>0$.

Definition :

Two points P and Q are said to be inverse points with respect to a circle with centre O and radius r if Q lies on the ray OP and OP. OQ = r^2.

Theorem 4 :

z_1 and z_2 are inverse pionts with respect to a circle $z\bar{z}+\bar{\alpha}z+\alpha\bar{z}+\beta = 0$ iff $z_1\overline{z_2}+\bar{\alpha}z_1+\alpha\overline{z_2}+\beta = 0$.

Proof :

Suppose z_1 and z_2 are inverse points with respect to the circle

$$z\bar{z}+\bar{\alpha}z+\alpha\bar{z}+\beta = 0 \qquad \text{--------(1)}$$

(1) can be rewritten as $|z+a|^2 = \alpha\bar{\alpha}-\beta$

∴ The centre of the circle is $-\alpha$ and radius is $\sqrt{(\alpha\bar{\alpha}-\beta)}$.

Since z_1 and z_2 are inverse points with respect to (1) we have

$$\arg(z_1+\alpha) = \arg(z_2+\alpha) \qquad \text{-------(2)}$$

and $$|z_1+\alpha|\,|z_2+\alpha| = \alpha\bar{\alpha}-\beta \qquad \text{-------(3)}$$

$$\therefore \quad \arg(z_1+\alpha)(\overline{z_2+\alpha}) = \arg(z_1+\alpha)+\arg(\overline{z_2+\alpha})$$

$$= \arg(z_1+\alpha)-\arg(z_2+\alpha)$$

$$= 0 \quad \text{by (2)}$$

∴ $(z_1+\alpha)(\overline{z_2+\alpha})$ is a positive real number

Hence using (3) we get $(z_1+\alpha)(\overline{z_2+\alpha}) = \alpha\bar{\alpha}-\beta$.

∴ $z_1\overline{z_2}+\bar{\alpha}z_1+\alpha\overline{z_2}+\beta = 0$

Converse can be similarly proved.

Note 1 :

Let z_1, z_2, z_3 and z_4 be four distinct points which are either concyclic or collinear.

Then $\arg\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$ is either 0 or π depending on the relative positions of the points.

Hence $\dfrac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$ is purely real.

Note 2 :

The equation $pz\bar{z}+\bar{\alpha}z+\alpha\bar{z}+\beta=0$ -------(1)

where p and β are real and $\alpha\bar{\alpha}-p\beta \geq 0$ can be taken as the joint equation of the family of circles and straight lines.

When $p \neq 0$ it represents a circle.

When $p = 0$ it represents a straight line.

Further z_1 and z_2 are inverse points or reflection points w.r.t. (1) iff $pz_1\overline{z_2}+\bar{\alpha}z_1+\alpha\overline{z_2}+\beta=0$.

Worked Examples :

Example 1 :

Prove that the equation $\left|\dfrac{z-z_1}{z-z_2}\right| = \lambda$ where λ is a non negative parameter represents a family of circles such that z_1 and z_2 are inverse points for every memberof the family.

Solution :

$$\left|\frac{z-z_1}{z-z_2}\right| = \lambda$$

$$\Rightarrow \left(\frac{z-z_1}{z-z_2}\right)\left(\frac{\bar{z}-\overline{z_1}}{\bar{z}-\overline{z_2}}\right) = \lambda^2$$

$$\Rightarrow \left(1-\lambda^2\right)z\bar{z}+\left(\overline{z_2}\lambda^2-\overline{z_1}\right)z+\left(z_2\lambda^2-z_1\right)\bar{z}+\left(z_1\overline{z_1}-\lambda^2 z_2\overline{z_2}\right) = 0 \qquad \text{------(1)}$$

∴ (1) represents a circle when $\lambda \neq 1$.

Using theorem (4) it can be verified that z_1 and z_2 are inverse points w.r.t. (1).

When $\lambda = 1$, the given equation represents a straight line which is the perpendicular bisector of the line segment joining z_1 and z_2.

Clearly z_1 and z_2 are reflections points for this line.

Example 2 :

Prove that $\arg\left(\frac{z-a}{z-b}\right) = \mu$ where μ is a real parameter, represents a familyof circles every member of which passes thruogh a and b.

Solution :

For any fixed value μ, $\arg\left(\frac{z-a}{z-b}\right) = \mu$ is the locus of a point z such that the angle between the lines joining a to z and b to z is μ.

Clearly this locus is the arc of a circle passing through a and b. The remaining part of the circle is represented by the equation $\arg\left(\frac{z-a}{z-b}\right) = \mu+\pi$. Hence the result follows.

Exercise :

1. Find one value of arg z where $z = \frac{-2}{1+i\sqrt{3}}$

2. Show that the inverse point of any point α with respect to the unit circle $|z| = 1$ is $1/\alpha$.

3. Find the inverse point of $-i$ with respect to the circle $2z\bar{z}+(i-1)z-(i+1)\bar{z} = 0$.

4. Find the equation of the circle passing through the points 1, i, 1+i.

5. Prove that the equation of the circle passing through three points z_1, z_2, z_3 is given by

$$\frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)} = \frac{(\bar{z}-\overline{z_1})(\overline{z_3}-\overline{z_2})}{(\bar{z}-\overline{z_2})(\overline{z_3}-\overline{z_1})}$$

Hint : If z is any point on the circle then $\frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)}$ is purely real.

ANALYTIC FUNCTIONS

We know that the distance between two points z_1 and z_2 in the complex plane is $|z_1-z_2|$. Hence the set C of complex numbers becomes a metric space with the metric d defined by $d(z_1, z_2) = |z_1-z_2|$. So we can talk about neighbourhood, interior point, open set, closed set, limit point, connected set etc. in the complex plane.

Definition :

Let z_0 be any complex number. Let $\in$ be a positive real number. Then the set of all points z satisfying $|z-z_0|<\in$ is called a **neighbourhood** of z_0 and is represented by $N_\in(z_0)$ or $S(z_0, \in)$.

$\therefore N_\in(z_0) = \{z/|z-z_0|<\in\}$

Note :

$|z-z_0| \leq \in$ represents the set of points on and inside the circle with centre z_0 and radius $\in$ and is called the **closed circular disc** with centre z_0 and radius $\in$.

Definition :

Let $S \subseteq C$. Let $z_0 \in S$. Then z_0 is said to be an **interior point** of S if there exists a neighbourhood $N_\in(z_0)$ such that $N_\in(z_0) \subseteq S$.

S is called an **open set** if every point of S is an interior point of S.

Definition :

Let $S \subseteq C$. Let $z_0 \in C$. Then z_0 is called a limit point of S if every neighbourhood of z_0 contain infinitely many points of S.

S is called a **closed set** if it contains all its limit points.

Definition :

Let $S \subseteq C$. Let $z_0 \in C$. Then z_0 is called a **boundary point** of S if z_0 is a limit point of both S and C–S. Thus z_0 is a boundary point of S iff evry neighbourhood of z_0 contains infinitely many pionts of S and infinitely many points of C–S.

Definition :

Let $S \subseteq C$. Then S is called a bounded set if there exists a real number K such that $|z| \leq K$ for all $z \in S$.

Definition :

Let $S \subseteq C$. Then S is called a **connected set** if every pair of points in S can be joined by a polygon which lies in S.

Definition :

A non empty open connected subset of C is called a **region** in C.

Functions of a complex variable :

We use the letters z and ω to denote complex variables. Thus to denote a complex valued function of a complex variable we use the notation $w = f(z)$.

The function $w = iz+3$ is defined in the entire complex plane.

The function $w = \frac{1}{z^2+1}$ is defined at all points of the complex plane except at $z = \pm i$.

The function $w = |z|$ is defined in the entire complex plane and this is a real valued function of the complex variable z.

If $a_0, a_1, a_2, \ldots, a_n$ are complex constants the function $P(z) = a_0+a_1z+\ldots+a_nz^n$ is defined in the entire complex plane and is called a **polynomial** in z. If P(z) and Q(z) are polynomials the quotient $\frac{P(z)}{Q(z)}$ is called a **rational function** and it is defined for all z wtih $Q(z) \neq 0$.

The function $f(z) = x^4+y^4+i(x^2+y^2)$ is defined over the entire complex plane.

In general if u(x, y) and v(x, y) are real valued functions of two variables both defined on a region S of the complex plane then

$f(z) = u(x, y)+iv(x,y)$ is a **complex valued function** defined on S.

Conversely each complex function $w=f(z)$ can be put in the form $w=f(z) = u(x,y)+iv(x,y)$ where u and v are real valued functions of the real variables x and y. u(x,y) is called the **real part** and v(x,y) is called the **imaginary part** of the function f(z).

For example $f(z) = z^2 = (x+iy)^2$

$= (x^2-y^2)+i(2xy)$ so that

$u(x, y) = x^2-y^2$

and $v(x, y) = 2xy$

Limits

Let $w = f(z)$ be a function defined in some region containing a point z_0 except at the point z_0. As z approaches z_0 the value f(z) of the function is arbitrarily close to a complex number l. Then we say that the limit of the function f(z) as z approaches z_0 is l.

Definition :

A function $w=f(z)$ is said to have the **limit** l as z tends to z_0 if given $\epsilon>0$ there exists $\delta>0$ such that $0<|z-z_0|<\delta \Rightarrow |f(z)-l|<\epsilon$. We write $\underset{z\to z_0}{\text{Lt}} f(z) = l$.

Lemma :

When the limit of a function f(z) exists as z tends to z_0 then the limit has a unique value.

Proof :

Suppose that $\underset{z\to z_0}{\text{Lt}} f(z)$ has two values l_1 and l_2. Then given $\epsilon>0$ there exists δ_1 and $\delta_2>0$ such that $0<|z-z_0|<\delta_1 \Rightarrow |f(z)-l_1| < \frac{\epsilon}{2}$ and $0<|z-z_0|<\delta_2 \Rightarrow |f(z)-l_2| < \frac{\epsilon}{2}$.

Let $\delta = \min\{\delta_1, \delta_2\}$

If $0<|z-z_0|<\delta$ we have

$$\begin{aligned} |l_1-l_2| &= |l_1-f(z)+f(z)-l_2| \\ &\leq |f(z)-l_1|+|f(z)-l_2| \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon \end{aligned}$$

Since $\epsilon>0$ is arbitrary $|l_1-l_2| = 0$ so that $l_1 = l_2$.

Example 1 :

$$\underset{z\to 2}{\mathrm{Lt}} \frac{z^2-4}{z-2} = 4$$

Solution :

Let $f(z) = \frac{z^2-4}{z-2}$. f(z) is not defined at z = 2 and when $z \neq 2$ we have

$$f(z) = \frac{(z+2)(z-2)}{z-2} = z+2$$

$\therefore \quad |f(z)-4| = |z+2-4| = |z-2|$ when $z \neq 2$

Given $\epsilon>0$, we choose $\delta = \epsilon$

Then $0<|z-2|<\delta \Rightarrow |f(z)-4|<\epsilon$.

$\therefore \underset{z\to 2}{\mathrm{Lt}} f(z) = 4$

Example 2 :

The function $f(z)=\frac{\bar{z}}{z}$ does not have a limit as $z\to 0$.

Solution :

$$f(z) = \frac{\bar{z}}{z} = \frac{x-iy}{x+iy}$$

Suppose $z\to 0$ along the path $y = mx$

Along this path $f(z) = \frac{x-imx}{x+imx} = \frac{1-im}{1+im}$ as $x \neq 0$.

Hence if $z\to 0$ along the path $y=mx$, f(z) tends to $\frac{1-im}{1+im}$ which is different for different values of m.

Hence f(z) does not have a limit as $z\to 0$.

Definition :

We say $\underset{z\to \infty}{\mathrm{Lt}} f(z) = l$ if given $\epsilon>0$ there exists a number $m>0$ such that $|z| > m$ $\Rightarrow |f(z)-l|<\epsilon$.

We say that $\underset{z\to\infty}{\text{Lt}} f(z) = \infty$ if for given n>0 there exists m>0 such that $|z|>m \Rightarrow |f(z)|>n$.

Theorems on Limit :

Let f and g be two functions whose limits at z_0 exist.

Let $\underset{z\to z_0}{\text{Lt}} f(z) = l$

and $\underset{z\to z_0}{\text{Lt}} g(z) = m$

Then

(i) $\underset{z\to z_0}{\text{Lt}} [f(z)+g(z)] = l+m$

(ii) $\underset{z\to z_0}{\text{Lt}} f(z)g(z) = lm$

(iii) $\underset{z\to z_0}{\text{Lt}} \dfrac{f(z)}{g(z)} = \dfrac{l}{m}$ provided $m \neq 0$.

(iv) If $\underset{z\to z_0}{\text{Lt}} f(z) = l$ then $\underset{z\to z_0}{\text{Lt}} \overline{f(z)} = \bar{l}$

Proof :

Let $\epsilon>0$ be given. Then there exists $\delta>0$ such that $0<|z-z_0|<\delta \Rightarrow |f(z)-l|<\epsilon$.

$$|\overline{f(z)}-\bar{l}| = |\overline{f(z)-l}| = |f(z)-l|$$

$$0<|z-z_0|<\delta \Rightarrow |\overline{f(z)}-\bar{l}| < \epsilon \text{ so that } \underset{z\to z_0}{\text{Lt}} \overline{f(z)} = \bar{l}.$$

(v) If $\underset{z\to z_0}{\text{Lt}} f(z) = l$ then $\underset{z\to z_0}{\text{Lt}} |f(z)| = |l|$

Proof :

$$||f(z)|-|l|| \leq |f(z)-l| \text{ and hence}$$

$$0<|z-z_0|<\delta \Rightarrow ||f(z)|-|l||<\epsilon$$

$$\therefore \quad \underset{z\to z_0}{\text{Lt}} |f(z)| = |l|$$

(vi) $\underset{z \to z_0}{\text{Lt}} f(z) = l$ iff $\underset{z \to z_0}{\text{Lt}} \text{Re } f(z) = \text{Re } l$ and $\underset{z \to z_0}{\text{Lt}} \text{Im } f(z) = \text{Im } (l)$.

Proof :

Let $\underset{z \to z_0}{\text{Lt}} f(z) = l$

since $\text{Re } f(z) = \frac{1}{2}[f(z) + f(\bar{z})$ we have

$$\underset{z \to z_0}{\text{Lt}} \text{Re } f(z) = \frac{1}{2}\left[\underset{z \to z_0}{\text{Lt}} f(z) + \underset{z \to z_0}{\text{Lt}} f(\bar{z})\right]$$

$$= \frac{1}{2}(l + \bar{l})$$

$$= \text{Re } l.$$

Similarly $\underset{z \to z_0}{\text{Lt}} \text{Im } f(z) = \text{Im } l.$

Conversely, let $\underset{z \to z_0}{\text{Lt}} \text{Re } f(z) = \text{Re } l$ and let $\underset{z \to z_0}{\text{Lt}} \text{Im } f(z) = \text{Im } l$. Since f(z)=Re f(z) + i Im f(z) it follows that $\underset{z \to z_0}{\text{Lt}} f(z) = \text{Re } l + i \text{ Im } l = l$.

Exercise :

1. Express each of the following functions in the form u(x, y)+iv(x, y)

   (i) $w = z^3$, (ii) $w = 2\bar{z}^2 + 1$, (iii) $w = \frac{z}{1+z}$

2. Use the definition of limit to prove

   $\underset{z \to z_0}{\text{Lt}} az + b = az_0 + b.$

3. Prove that $f(z) = \frac{xy}{x^2 + y^2}$, $z \neq 0$ does not have a limit as $z \to 0$.

4. Evaluate the following limits.

   (i) $\underset{z \to 2i}{\text{Lt}} \left(2x + iy^2\right)^2$

   (ii) $\underset{z \to -2i}{\text{Lt}} \frac{(z+3)(z-4)}{z^2 + 5z + 9}$

CONTINUOUS FUNCTIONS

Definition :

Let f be a complex valued function defined on a region D of the complex plane. Let $z_0 \in D$. Then f is said to be continuous at z_0 if $\underset{z \to z_0}{\text{Lt}} f(z) = f(z_0)$.

Thus f is continuous at z_0 if given $\in > 0$ there exists a $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \in$.

f is said to be continuous in D if it is continuous at each piont of D.

Theorems :

(i) If f and g are continuous at z_0 then f+g, fg and $\bar{f}$ are continuous at z_0 and $\frac{f}{g}$ is continuous at z_0 if $g(z_0) \neq 0$.

(ii) If f is continuous at z_0, then $|f|$ is also continuous at z_0.

(iii) If f is continuous at z_0 iff Re f and Im f are continuous at z_0.

(iv) Any polynomial P(z) is continuous at each point of the complex plane and any rational function $\frac{P(z)}{Q(z)}$ is continuous at all points where $Q(z) \neq 0$.

DIFFERENTIABILITY

Definition :

Let f be a complex function defined in a region D and let $z \in D$. Then f is said to be **differentiable** at z if $\underset{h \to 0}{\text{Lt}} \frac{f(z+h) - f(z)}{h}$ exists and is finite. This limit is denoted by $f'(z)$ or $\frac{df}{dz}$ and is called the **derivative** of f(z) at z.

The function is said to be differentiable in D if it is differentiable at all points of D.

Example 1 :

The function $f(z) = z^2$ is differentiable at every point and $f'(z) = 2z$.

Proof :

$$\frac{f(z+h)-f(z)}{h} = \frac{(z+h)^2-z^2}{h}$$

$$= 2z+h$$

$$\therefore \quad \underset{h\to 0}{\mathrm{Lt}} \frac{f(z+h)-f(z)}{h} = \underset{h\to 0}{\mathrm{Lt}} (2z+h) = 2z$$

$$\therefore \quad f'(z) = 2z$$

Example 2 :

The function $f(z) = \bar{z}$ is nowhere differentiable.

Proof :

$$\frac{f(z+h)-f(z)}{h} = \frac{(\overline{z+h})-\bar{z}}{h}$$

$$= \frac{\bar{z}+\bar{h}-\bar{z}}{h} = \frac{\bar{h}}{h}$$

$\underset{h\to 0}{\mathrm{Lt}} \frac{\bar{h}}{h}$ does not exist.

$f(z) = \bar{z}$ is nowhere differentiable.

Example 3 :

If f(z) is differentiative at a point z then it is continuous at that point.

Proof :

$$\underset{h\to 0}{\mathrm{Lt}}\left[f(z+h)-f(z)\right] = \underset{h\to 0}{\mathrm{Lt}}\left[\frac{f(z+h)-f(z)}{h}\right] \underset{h\to 0}{\mathrm{Lt}}\ h$$

$$= f'(z) \times 0$$

$$= 0$$

$\therefore \underset{h\to 0}{\mathrm{Lt}} f(z+h) = f(z)$ so that f is continuous at z.

The converse of the above result is ~~not~~ true. For example, $f(z) = \bar{z}$ is continuous everywhere but it is nowhere differentiable.

Theorems :

Let f(z) and g(z) be differentiable at a point z. Then

(i) $(f+g)'(z) = f'(z) + g'(z)$

(ii) $(fg)'(z) = f(z)g'(z) + f'(z)g(z)$

(iii) $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{[g'(z)]^2}$ provided $g(z) \neq 0$.

(iv) Suppose g is differentiable at z and f is differentiable at g(z). Let $F(z) = f(g(z))$. Then $F'(z) = f'(g(z))\, g'(z)$.

(v) Let n be any positive integer. The function $f(z) = z^n$ is differentiable at every point and $f'(z) = n\, z^{n-1}$.

(vi) The polynomial $P(z) = a_0+a_1z+a_2z^2+.....+a_nz^n$ is differentiable at every point and $P'(z) = a_1+2a_2z+....+n\, a_n\, z^{n-1}$

(vii) If n is a negative integer $f(z) = z^n$ is differentiable at every point $z \neq 0$ and $f'(z) = n\, z^{n-1}$.

Exercise :

1. Find the derivative of the following functions

(i) z^2+3z+1 (ii) $\frac{z+1}{2z+3}$

2. Prove that $f(z) = \frac{z-1}{z+1}$ is differentiable at every point $z \neq -1$ and find $f'(z)$.

3. Prove that $f(z) = \text{Re } z$ is not differentiable at any point.

THE CAUCHY - RIEMANN EQUATIONS

Theorem :

Let $f(z) = u(x, y)+iv(x, y)$ be differentiable at a point $z_0 = x_0+iy_0$. Then u(x, y) and v(x, y) have first order partial derivatives $u_x(x_0,y_0)$, $u_y(x_0, y_0)$, $v_x(x_0,y_0)$ and $v_y(x_0, y_0)$ at (x_0, y_0) and these partial derivatives satisfy the Cauchy-Riemann equations (C.R.equations) given by

$$u_x(x_0, y_0) = v_y(x_0, y_0)$$

and $$u_y(x_0, y_0) = -v_x(x_0, y_0)$$

Also $$f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0)$$

$$= v_y(x_0, y_0) - i\, u_y(x_0, y_0)$$

Proof :

Since $f(z) = u(x, y)+iv(x, y)$ is differentiable at $z_0 = x_0+iy_0$, $\underset{h\to 0}{\text{Lt}} \dfrac{f(z_0+h)-f(z_0)}{h}$ exists and hence the limit is independent of the path in which h approaches zero.

Let $h = h_1+ih_2$

Now $\dfrac{f(z_0+h)-f(z_0)}{h}$

$$= \frac{u(x_0+h_1, y_0+h_2)+iv(x_0+h_1, y_0+h_2)-u(x_0, y_0)-iv(x_0, y_0)}{h_1+ih_2}$$

$$= \left[\frac{u(x_0+h_1, y_0+h_2)-u(x_0, y_0)}{h_1+ih_2}\right]+i\left[\frac{v(x_0+h_1, y_0+h_2)-v(x_0, y_0)}{h_1+ih_2}\right]$$

Suppose $h\to 0$ along the real axis so that $h = h_1$.

Then $$f'(z_0) = \underset{h_1\to 0}{\text{Lt}}\left[\frac{f(z_0+h_1)-f(z_0)}{h_1}\right]$$

$$= \underset{h_1\to 0}{\text{Lt}}\left[\frac{u(x_0+h_1, y_0)-u(x_0, y_0)}{h_1}\right]+i\,\underset{h_1\to 0}{\text{Lt}}\left[\frac{v(x_0+h_1, y_0)-v(x_0, y_0)}{h_1}\right]$$

$= u_x(x_0, y_0) + i\, v_x(x_0, y_0)$ --------(1)

Now, suppose $h\to 0$ along the imaginary axis so that $h = i\, h_2$.

$$\therefore f'(z_0) = \underset{ih_2\to 0}{\text{Lt}}\left[\frac{f(z_0+h_2)-f(z_0)}{ih_2}\right]$$

$$= \underset{h_2\to 0}{\text{Lt}}\left[\frac{u(x_0, y_0+h_2)-u(x_0, y_0)}{ih_2}\right]+i\,\underset{h_2\to 0}{\text{Lt}}\left[\frac{v(x_0, y_0+h_2)-v(x_0, y_0)}{ih_2}\right]$$

$$= \left[\frac{u_y(x_0,y_0)}{i}\right] + i\left[\frac{v_y(x_0,y_0)}{i}\right]$$

$$= \frac{1}{i}u_y(x_0, y_0) + v_y(x_0, y_0)$$

$$= -i\, u_y(x_0, y_0) + v_y(x_0, y_0) \qquad \text{-------(2)}$$

From (1) and (2) we get

$$f'(z_0) = u_x(x_0, y_0)+i\, v_x(x_0, y_0)$$

$$= v_y(x_0, y_0)-i\, u_y(x_0, y_0)$$

Equating real and imaginary parts we get

$$u_x(x_0, y_0) = v_y(x_0, y_0)$$

$$u_y(x_0, y_0) = -v_x(x_0, y_0)$$

Remark 1 :

Since $f'(z) = u_x+i\, v_x = v_y-i\, u_y$ we have

$$|f'(z)|^2 = u_x^{\,2}+v_x^{\,2} = u_y^{\,2}+v_y^{\,2}$$

Also $$|f'(z)|^2 = u_x^{\,2}+u_y^{\,2} = v_x^{\,2}+v_y^{\,2}$$

Further $$|f'(z)|^2 = u_x\, v_y - u_y\, v_x$$

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)}$$

Remark 2 :

The Cauchy-Riemann equations provide a necessary condition for differentiability at a point. Hence if the C.R. equations are not satisfied for a complex function at any point then we can conclude that the function is not differentiable. For example, consider the function.

$$f(z) = \bar{z} = x-iy$$

$$u(x, y) = x$$

and $$v(x, y) = -y$$

$\therefore$ $$u_x(x, y) = 1$$

and $$v_y(x, y) = -1$$

$\therefore$ $u_x \neq v_y$ so that C.R. equations are not satisfied at any point z.

Hence the function $f(z) = \bar{z}$ is nowhere differentiable.

Remarks 3 :

The C.R. equations are not sufficient for differentiability at a point.

Example :

Let $$f(z) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

$$u(x, y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$v(x, y) = 0$$

$$u_x(0, 0) = \underset{h\to 0}{\text{Lt}}\left[\frac{u(h,0)-u(0,0)}{h}\right]$$

$$= \underset{h\to 0}{\text{Lt}}\left[\frac{0-0}{h}\right] = 0$$

Similarly $u_y(0, 0) = 0$

Also $v_x(0,0) = 0$ and $v_y(0, 0) = 0$.

Hence the C.R. equations are satisfied at z=0.

Now, along the path $y = mx$

$$f(z) = \frac{xmx}{x^2+m^2x^2} = \frac{m}{1+m^2} \text{ if } x \neq 0$$

Hence if $z\to 0$ along the path $y = mx$, $f(z)\to\dfrac{m}{1+m^2}$ which is different for different values of m.

Hence f(z) does not have a limit as $z\to 0$ so that f(z) is not even continuous at z=0.

In the following theorem we prove that C.R. equations together with the continuity of partial derivatives give a sufficient condition for differentiability of complex functions.

Theorem :

Let $f(z) = u(x, y)+i\, v(x, y)$ be a function defined in a region D such that u, v and their first order partial derivatives are continuous in D. If the first order partial derivatives of u, v satisfy the Cauchy-Riemann equations at a point $(x, y)\in D$ then f is differentiable at $z = x+iy$.

Proof :

Since $u(x, y)$ and its first order partial derivatives are continuous at (x, y) we have by the mean value theorem for functions of two variables.

$$u(x+h_1, y+h_2)-u(x, y) = h_1\, u_x(x,y)+h_2\, u_y(x, y)+h_1\epsilon_1+h_2\epsilon_2 \qquad \text{-------(1)}$$

where ϵ_1 and $\epsilon_2 \to 0$ as h_1 and $h_2\to 0$

Similarly

$$v(x+h_1, y+h_2)-v(x,y) = h_1\, v_x(x, y)+h_2\, v_y(x, y)+h_1\epsilon_3+h_2\epsilon_4 \qquad \text{-------(2)}$$

where $\epsilon_3, \epsilon_4\to 0$ as h_1 and $h_2\to 0$

Let $h = h_1 + ih_2$

Then $\dfrac{f(z+h)-f(z)}{h} = \dfrac{1}{h}[u(x+h_1, y+h_2)-u(x, y)+iv(x+h_1, y+h_2)-v(x, y)]$

$$= \frac{1}{h}[\{h_1u_x(x,y)+h_2u_y(x,y)+h_1\epsilon_1+h_2\epsilon_2\} + i\{h_1v_x(x,y)+h_2v_y(x,y)+h_1\epsilon_3+h_2\epsilon_4\}]$$

using (1) and (2)

$$= \frac{1}{h}[h_1\{u_x(x,y)+iv_x(x,y)\}+h_2\{u_y(x,y)+iv_y(x,y)\} + h_1(\epsilon_1+i\epsilon_3)+h_2(\epsilon_2+i\epsilon_4)]$$

$$= \frac{1}{h}[(h_1+ih_2)u_x(x,y)-i(h_1+ih_2)u_y(x,y) + h_1(\epsilon_1+i\epsilon_3)+h_2(\epsilon_2+i\epsilon_4)]$$

(using C.R. equations)

$$= \frac{1}{h}[hu_x(x,y)-ihu_y(x,y)+h_1(\epsilon_1+i\epsilon_3) + h_2(\epsilon_2+i\epsilon_4)]$$

$$= u_x(x,y)-iu_y(x,y)+\frac{h_1}{h}(\epsilon_1+i\,\epsilon_3)+\frac{h_2}{h}(\epsilon_2+i\,\epsilon_4)$$

Since $\left|\dfrac{h_1}{h}\right|\le 1$, $\dfrac{h_1}{h}(\epsilon_1+i\,\epsilon_3)\to 0$ as $h\to 0$

Similarly, $\frac{h_2}{h}(\epsilon_3 + i\,\epsilon_4) \to 0$ as $h \to 0$

$$\therefore \underset{h\to 0}{\text{Lt}} \frac{f(z+h)-f(z)}{h} = u_x(x, y) - i\, u_y(x, y)$$

Hence f is differentiable.

Example 1 :

Let $f(z) = e^x (\cos y + i \sin y)$

$\therefore$ $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$

Then $u_x(x, y) = e^x \cos y = v_y(x, y)$

and $u_y(x, y) = -e^x \sin y = -v_x(x, y)$

Thus the first order partial derivaties of u and v satisfy the C.R. equations at every point.

Further u(x, y) and v(x,y) and their first order partial derivatives are continuous at every point. Hence f is differentiable at every point of the complex plane.

Example 2 :

Let	$f(z)$	=	$\lvert z\rvert^2$
$\therefore$	$f(z)$	=	$u(x,y)+iv(x, y) = x^2+y^2$
$\therefore$	$u(x, y)$	=	x^2+y^2 and $v(x, y) = 0$
So	$u_x(x, y)$	=	$2x$;
	$u_y(x, y)$	=	$2y$
	$v_x(x, y)$	=	$0 = v_y(x, y)$

Clearly the C.R. equations are satisfied at $z = 0$.

u and v and their first order partial derivatives are continuous and hence f is differentiable at $z = 0$.

Also, the C.R. equations are not satisfied at any point $z \neq 0$ and hence f is not differentiable at $z \neq 0$. Thus f is differentiable only at $z = 0$.

Complex forms of C.R. equations

Let $f(z) = u(x, y) + iv(x,y)$ be differentiable. Then the C.R. equations can be put in the complex form as $f_x = -if_y$

Proof :

Let $f(z) = u(x,y) + iv(x, y)$

Then $f_x = u_x + iv_x$

and $f_y = u_y + iv_y$

Hence $f_x = -i fy$

$\Leftrightarrow \quad u_x + iv_x = -i(u_y + iv_y)$

$\Leftrightarrow \quad u_x + iv_x = v_y - iu_y$

$\Leftrightarrow \quad u_x = v_y$

and $v_x = -u_y$

Thus the two C.R. equations are equivalent to the equation $f_x = -if_y$.

C.R. equations in polar coordinates :

Let $f(z) = u(r, \theta) + iv(r, \theta)$ be differentiable at $z = re^{i\theta} \neq 0$.

Then $$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

and $$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Further $$f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

Proof :

We know that $x = r \cos \theta$ and $y = r \sin \theta$.

Hence $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x}.\cos\theta + \frac{\partial u}{\partial y}\sin\theta \quad \text{--------(1)}$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}.\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta)$$

$$\therefore \quad \frac{1}{r}\frac{\partial v}{\partial \theta} = -\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta$$

$$= \frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta \quad \text{(using C.R. equation)}$$

$$= \frac{\partial u}{\partial r} \quad \text{(using (1))}$$

$$\therefore \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Similarly we can prove that $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

$$r\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = r\left[\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right) + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}\right)\right]$$

$$= r\left[\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) + i\left(\frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\right]$$

$$= r\cos\theta\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + r\sin\theta\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + iy\left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right)$$

$$= xf'(z) + iyf'(z)$$

$$= (x+iy)f'(z)$$

$$= zf'(z)$$

$$\therefore \quad f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

Theorem :

If f(z) is a differentiable function, the C.R. equations can be put in the form $\frac{\partial f}{\partial \bar{z}} = 0$.

Proof :

Let $\quad f(z) = u(x, y) + iv(x,y)$

Since $\quad x = \frac{z+\bar{z}}{2}$

and $y = \dfrac{z-\bar{z}}{2i}$

We have $f(z) = u\left(\dfrac{z+\bar{z}}{2}, \dfrac{z-\bar{z}}{2i}\right) + iv\left(\dfrac{z+\bar{z}}{2}, \dfrac{z-\bar{z}}{2i}\right)$

$$\therefore \quad \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial f}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(-\frac{1}{2i}\right)$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

So, $\dfrac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \dfrac{\partial f}{\partial x} = -i\dfrac{\partial f}{\partial y}$ which is the complex form of C.R. equations.

Thus the C.R. equation can be put in the form $\dfrac{\partial f}{\partial \bar{z}} = 0$.

Worked Examples :

Example 1 :

Verify Cauchy - Riemann equation for the function $f(z) = z^3$

Solution :

$$f(z) = z^3 = (x+iy)^3$$

$$= (x^3-3xy^2)+i(3x^2y-y^3)$$

$\therefore \quad u(x, y) = x^3-3xy^2$

and $v(x, y) = 3x^2y-y^3$

$\therefore \quad u_x = 3x^2-3y^2$

and $v_x = 6xy$

$u_y = -6xy$

and $v_y = 3x^2-3y^2$

$u_x = vy$

and $u_y = -vx$

Hence the Cauchy-Riemann equations are satisfied.

Example 2 :

Prove that the function $f(z) = e^x (\cos y - i \sin y)$ is nowhere differentiable.

Solution :

$$\begin{aligned} f(z) &= e^x (\cos y - i \sin y) \\ &= e^x \cos y - ie^x \sin y \end{aligned}$$

$\therefore$ $u(x, y) = e^x \cos y$

and $v(x, y) = -e^x \sin y$

$\therefore$ $u_x = e^x \cos y$

and $v_x = -e^x \sin y$

$u_y = -e^x \sin y$

and $v_y = -e^x \cos y$

C.R. equations are not satisfied at any point and hence f(z) is nowhere differentiable.

Example 3 :

Prove that $f(z) = z \text{ Im } z$ is differentiable only at $z = 0$ and find $f'(0)$.

Solution :

$$\begin{aligned} f(z) &= z \text{ Im } z \\ &= (x+iy)y \end{aligned}$$

$\therefore$ $u(x, y) = xy$

and $v(x, y) = y^2$

$\therefore$ $u_x = y$; $u_y = x$; $v_x = 0$ and $v_y = 2y$.

Clearly the C.R. equations are satisfied only at $z = 0$.

All the first order partial derivatives are continuous.

Hence f(z) is differentiable at $z = 0$.

Also $f'(0) = u_x(0, 0) + iv_x(0, 0) = 0$

Example 4 :

Prove that the function $f(z)=\begin{cases} \dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z=0 \end{cases}$ satisfies C.R. equations at the origin but $f'(0)$ does not exist.

Solution :

$$f(z) = \begin{cases} \dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z=0 \end{cases}$$

Here $u(x, y) = \dfrac{x^3-y^3}{x^2+y^2}$

and $v(x, y) = \dfrac{x^3+y^3}{x^2+y^2}$ if $(x, y) \neq (0, 0)$

$$u(0, 0) = v(0, 0) = 0$$

$$u_x(0, 0) = \underset{h\to 0}{\text{Lt}} \frac{u(h,0)-u(0,0)}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \frac{\frac{h^3}{h^2}-0}{h} = 1$$

Similarly $u_y(0, 0) = 1$

$$u_x(0, 0) = 1$$

and $u_y(0, 0) = 1$

$$\frac{f(z)-f(0)}{z-0} = \frac{x^3-y^3}{(x^2+y^2)(x+iy)}+i\frac{x^3+y^3}{(x^2+y^2)(x+iy)}$$

Along the path $y = mx$ we have

$$\frac{f(z)-f(0)}{z-0} = \frac{x^3-m^3x^3}{(x^2+m^2x^2)(x+imx)}+i\frac{x^3+m^3x^3}{(x^2+m^2x^2)(x+imx)}$$

$$= \frac{1-m^3}{(1+m^2)(1+im)}+i\frac{1+m^3}{(1+m^2)(1+im)}$$

Hence the value of the limit depends on the path along which $z\to 0$.

Thus $\underset{z\to 0}{\text{Lt}} \dfrac{f(z)-f(0)}{z-0}$ does not exist. Hence f is not differentiable at 0.

Example 5 :

Find constants a and b so that the function $f(z) = a(x^2-y^2)+i\ bxy+c$ is differentiable at every point.

Solution :

$$f(z) = a(x^2-y^2)+i\ bxy+c$$

Here $u(x, y) = a(x^2-y^2)+c$

and $v(x, y) = bxy$

$u_x = 2ax$; $u_y = -2ay$; $v_x = by$ and $v_y = bx$.

Clearly $u_x = v_y$ and $u_y = -v_x$ iff $2a = b$.

∴ C.R. equations are satisfied at all points iff $2a = b$.

∴ The function f(z) is differentiable for all values of a, b with $2a = b$.

Example 6 :

Show that $f(z) = \sqrt{r}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$ where $r>0$ and $0<\theta<2\pi$ is differentiable and find $f'(z)$.

Solution :

$$f(z) = \sqrt{r}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$$

$$u = \sqrt{r}\cos\left(\frac{\theta}{2}\right)$$

and $$v = \sqrt{r}\sin\left(\frac{\theta}{2}\right)$$

∴ $$\frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}}\cos\frac{\theta}{2}$$

and $$\frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}}\sin\frac{\theta}{2}$$

$$\frac{\partial u}{\partial \theta} = -\frac{\sqrt{r}}{2}\sin\frac{\theta}{2}$$

and $$\frac{\partial v}{\partial \theta} = \frac{\sqrt{r}}{2}\cos\frac{\theta}{2}$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\left(\frac{\sqrt{r}}{2}\cos\frac{\theta}{2}\right)$$

$$= \frac{1}{2\sqrt{r}}\cos\frac{\theta}{2}$$

$$= \frac{\partial u}{\partial r}$$

$$\therefore \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Similarly $\quad \dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$

$$= \frac{1}{2\sqrt{r}}\sin\frac{\theta}{2}$$

Hence the C.R. equations in polar form are satisfied.

Further all the first order partial derivatives are continuous.

Hence f′(z) exists.

$$f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$= \frac{r}{z}\left(\frac{1}{2\sqrt{r}}\cos\frac{\theta}{2} + \frac{i}{2\sqrt{r}}\sin\frac{\theta}{2}\right)$$

$$= \frac{r}{2\sqrt{r}z}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

$$= \frac{1}{2z}\left[\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\right]$$

$$= \frac{1}{2z}\sqrt{z} = \frac{1}{2\sqrt{z}}$$

$$f'(z) = \frac{1}{2\sqrt{z}}$$

Exercise :

1. Verify C.R. equations for the following functions.

   (i) $f(z) = e^z$

   (ii) $f(z) = iz+2$

   (iii) $f(z) = \sin z$

2. Prove that the following are nowhere differentiable.

   (i) $f(z) = |z|$

   (ii) $f(z) = xy+iy$

   (iii) $f(z) = 2x+ixy2$

   (iv) $f(z) = z-\bar{z}$

3. Prove that for the following functions the C.R. equations are satisfied at z=0 but the function is not differentiable at z=0.

   (i) $f(z) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

   (ii) $f(z) = \begin{cases} \dfrac{x^3y(y-ix)}{x^6+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

   (iii) $f(z) = \begin{cases} \dfrac{x^2y^5(x+iy)}{x^4+y^{10}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

4. Prove that the following functions are differentiable at every point.

   (i) $f(z) = iz+2$

   (ii) $f(z) = x^2-y^2-2xy+i(x^2-y^2+2xy)$

   (iii) $f(z) = (x^3-3xy^2)+i(3x^2y-y^3)$

   (iv) $f(z) = 2x-3y+i(3x+2y)$

5. Find constants, a, b and c so that the following functions are differentiable at every point.

   (i) $f(z) = x+ay-i(bx+cy)$ Ans. ($a=b$; $c=-1$)

   (ii) $f(z) = ax^2-by^2+icxy$ Ans. ($a= c/2 =b$)

   (iii) $f(z) = \cos x(\cosh y+a \sinh y)+i \sin x(\cosh y+b \sinh y)$ Ans. ($a=b=-1$)

ANALYTIC FUNCTIONS

Definition :

A function f defined in a region D of the complex plane is said to be **analytic at a point** $a \in D$ if f is differentiable at every point of some neighbourhood of a. Thus f is analytic at a if there exists $\epsilon>0$ such that f is differentiable at every point of the disc $S(a, \epsilon) = \{z / |z-a| < \epsilon\}$.

If f is analytic at every point of a region D then f is said to be analytic in D. A function which is analytic at every point of the complex plane is called an **entire function** or **integral function.**

Example :

Any polynomial is an entire function.

Remark 1 :

If f is analytic at a point a then f is differentiable at a.

But the converse is not true.

For example, $f(z) = |z|^2$ is differentiable only at z=0. Hence f is differentiable at z=0 but not analytic at z=0.

Remark 2 :

If f(z) is analytic at a then there exists $\epsilon>0$ such that f(z) is differentiable at each point of $S(a, \epsilon)$. Let $z \in S(a, \epsilon)$. Then we can find $\delta>0$ such that $S(z, \delta) \subseteq S(a, \epsilon)$. Hence f is differentiable at every point of $S(z, \delta)$ so that f is analytic at z.

Theorem :

An analytic function in a region D with its derivative zero at every point of the domain is a constant.

Proof :

Let $f(z) = u(x, y)+iv(x, y)$ be analytic in D and $f'(z) = 0$ for all $z \in D$.

Since $f'(z) = u_x+iv_x = v_y-iu_y$ we have $u_x = u_y = v_x = v_y = 0$

$\therefore$ u(x, y) and v(x, y) are constant functions and hence f(z) is constant.

Remark :

The above theorem is not true if the domain of f(z) is not a region.

For example led $D = \{z/|z|<1\} \cup \{z/|z|>2\}$

D is not a connected subset of C so that D is not a region.

Let $f: D \to C$ be defined by

$$f(z) = \begin{cases} 1 \text{ if } |z|<1 \\ 2 \text{ if } |z|>2 \end{cases}$$

Clearly $f'(z) = 0$ for all points $z \in D$ and f is not a constant function in D.

Worked Examples :

Example 1 :

An analytic function in a region with constant modulus is constant.

Solution :

Let $f(z) = u(x, y)+iv(x,y)$ be analytic in a domain D.

Since $|f(z)|$ is constant, we have $u^2+v^2=C$ where C is a constant.

Differentiating partially with respect to x

we get $2uu_x+2vv_x = 0$

i.e., $uu_x+vv_x = 0$ --------(1)

Similarly, differentiating partially with respect to y

we get $uu_y+vv_y = 0$ --------(2)

Using C.R. equations in (1) and (2) we get

$uu_x-vv_y = 0$ --------(3)

$uu_y+vv_x = 0$ --------(4)

Eliminating u_y from (3) and (4) we get $(u^2+v^2)u_x = 0$

Since $u^2+v^2 = C$ we get $u_x = 0$

Similarly we can prove that $v_x = 0$ so that

$$f'(z) = u_x+iv_x = 0$$

Hence f is constant.

Example 2 :

Any analytic function f(z) = u+iv with arg f(z) constant is itself a constant function.

Solution :

$$\arg f(z) = \tan^{-1}\left(\frac{v}{u}\right) = C \text{ where C is a constant}$$

$\therefore \quad \frac{v}{u} = K$ where K is a constant.

$\therefore \quad v = Ku$

Hence $\quad v_x = Ku_x$

and $\quad v_y = Ku_y$

Eliminating K from the above equations

we get $\quad u_x v_y = v_x u_y$

$\therefore \quad u_x v_y - u_y v_x = 0$

$\therefore \quad u_x^2 + u_y^2 = 0$ (using C.R. equations)

$\therefore$ $u_x = 0$ and $u_y = 0$ and hence u is constant. Similarly we can prove that v is constant.

$\therefore$ f = u+iv is constant.

Example 3 :

Prove that the function f(z) and $\overline{f(\bar{z})}$ are simultaneously analytic.

Solution :

Suppose f(z) = u(x, y)+iv(x,y) is analytic in a region D.

Then the first order partial derivatives of u and v are continuous and satisfy the C.R. equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{-------(1)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \text{-------(2)}$$

$$f(\bar{z}) = u(x, -y) - iv(x, -y)$$

$$= u_1(x, y) + iv_1(x, y)$$

where $\quad u_1(x, y) = u(x, -y)$

and $\quad v_1(x, y) = -v(x, -y)$

Hence $$\frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v_1}{\partial y} \qquad \text{(using (1))}$$

and $$\frac{\partial u_1}{\partial y} = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = -\frac{\partial v_1}{\partial x}$$

∴ The first order partial derivatives of u_1 and v_1 are continuous and satisfy the Cauchy-Riemann equations in D.

Hence $\overline{f(\bar{z})}$ is analytic in D.

Similarly if $\overline{f(\bar{z})}$ is analytic in D then f(z) is also analytic in D.

Example 4 :

If $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$ prove that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

Solution :

Let $\quad z = x + iy$

∴ $\quad x = \frac{1}{2}(z + \bar{z})$

and $\quad y = \frac{1}{2i}(z - \bar{z})$

Hence $$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z}$$

$$= \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\therefore \quad \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{2}\left[\left(\frac{\partial^2}{\partial x^2} + i\frac{\partial^2}{\partial x \partial y}\right)\frac{1}{2} + \left(\frac{\partial^2}{\partial y \partial x} + i\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial}{2i}\right)\right]$$

$$= \frac{1}{4}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + i\frac{\partial^2}{\partial x \partial y} + \frac{1}{i}\frac{\partial^2}{\partial y \partial x}\right]$$

$$= \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x \partial y}\left(i + \frac{1}{i}\right)\right]$$

$$= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

$$\therefore \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

Exercise :

1. Prove that an analytic function whose real part is constant is itself a constant.

2. If $f = u+iv$ is analytic in a region D and uv is constant in D then prove that f reduces to a constant.

3. If $f = u+iv$ is analytic in a region D and $v = u^2$ in D then prove that f reduces to a constant.

4. Determine the constants a and b in order that the function $f(z)=(x^2+ay^2-2xy) + i(bx^2-y^2+2xy)$ should be analytic. Find $f'(z)$.

HARMONIC FUNCTIONS

Definition :

Let u(x, y) be a function of two real variables x and y defined in a region D. u(x, y) is said to be a **harmonic function** if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and this equation is called.

Laplace's equation :

Theorem :

The real and imaginary parts of an analytic function are harmonic function.

Proof :

Let $f(z) = u(x, y)+iv(x, y)$ be an analytic function.

Then u and v have continuous prtial derivatives of first order which satisfy the C.R. equations given by

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Also $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

and $$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

Now $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial^2 v}{\partial x \partial y}-\frac{\partial^2 v}{\partial y \partial x}$$

$$= 0$$

So u is a harmonic function.

Similarly, we can prove that v is a harmonic function.

Note : Laplace's equation provides a necessary condition for a function to be the real or imaginary part of an analytic function.

For example if $u(x, y) = x^2+y$ we have

$$\frac{\partial^2 u}{\partial x^2} = 2; \quad \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 2$$

∴ u(x, y) is not harmonic function and hence it cannot be the real part of any analytic function.

Definition :

Let f = u+iv be an analytic function in a region D. Then v is said to be a **conjugate harmonic** function of u.

Theorem :

Let f = u+iv be an analytic function in a region D. Then v is a harmonic conjugate of u if and only if u is a harmonic conjugate of –v.

Proof :

Let v be a harmonic conjugate of u.

Then f = u+iv is analytic.

∴ if = iu–v is also analytic.

Hence u is a harmonic conjugate of –v.

The converse is similar.

Theorem :

Any two harmonic conjugates of a given harmonic function u in a region D differ by a real constant.

Proof :

Let u be a harmonic function.

Let v and v* be two harmonic conjugates of u.

u+iv and u+iv* are analytic in D.

By the Cauchy-Riemann equations we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v^*}{\partial y}$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\frac{\partial v^*}{\partial x}$$

∴ $$\frac{\partial v}{\partial y} = \frac{\partial v^*}{\partial y}$$

and $$\frac{\partial v}{\partial x} = \frac{\partial v^*}{\partial x}$$

Hence $$\frac{\partial}{\partial y}(v - v^*) = 0$$

and $$\frac{\partial}{\partial x}(v - v^*) = 0$$

∴ $v = v^* + C$ where C is a real constant.

Note : The Cauchy-Riemann equations can be used to obtain a harmonic conjugate of a given harmonic function.

For example, let $u(x, y) = x^2-y^2$

Then $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 2-2 = 0$ so that u is harmonic in the whole complex plane C.

Let v(x, y) be a harmonic conjugate of u.

Then $$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x \qquad \text{-------(1)}$$

and $$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -2y \qquad \text{-------(2)}$$

On integration of (1) with respect to y we get $v = 2xy+\phi(x)$ where $\phi(x)$ is a function of x alone.

From (2) $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ gives $2y+\phi'(x) = 2y$

$\therefore$ $\phi'(x) = 0$ so that $\phi(x) = C$ (a constant)

$\therefore$ $v = 2xy+C$

Thus the harmonic conjugate of $u(x, y) = x^2-y^2$ is given by $v(x, y) = 2xy+C$ and the corresponding entire function is given by

$$\begin{aligned} f(z) &= (x^2-y^2)+i(2xy+C) \\ &= z^2+iC \end{aligned}$$

MILNE - THOMPSON METHOD

Let u(x, y) be a given harmonic function. Let $f(z) = u(x, y)+iv(x,y)$ be an analytic function.

Then $$\begin{aligned} f'(z) &= u_x(x,y)+iv_x(x,y) \\ &= u_x(x, y)-iu_y(x,y) \end{aligned}$$

Let $$\phi_1(x, y) = u_x(x, y)$$

and $$\phi_2(x, y) = -u_y(x, y)$$

$$x = \frac{z+\bar{z}}{2}$$

and $$y = \frac{z-\bar{z}}{2i}$$

So $f'(z) = \phi_1\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) - i\phi_2\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$

Putting $z = \bar{z}$ we obtain $f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$

Hence $f(z) = \int[\phi_1(z,0) - i\phi_2(z,0)]dz + C$

Note : It can be proved in a similar way that the analytic function f(z) with a given harmonic function v(x, y) as imaginary part is given by

$$f(z) = \int[\psi_1(z,0) + i\psi_2(z,0)]dz + C \text{ where}$$

$$\psi_1(x, y) = v_y \text{ and } \psi_2(x, y) = v_x.$$

Worked Examples :

Example 1 :

Prove that $u = 2x - x^3 + 3xy^2$ is harmonic and find its harmonic conjugate. Also find the corresponding analytic function.

Solution :

$$u = 2x - x^3 + 3xy^2$$

$$\therefore \quad u_x = 2 - 3x^2 + 3y^2;$$

$$u_{xx} = -6x$$

$$u_y = 6xy$$

$$u_{yy} = 6x$$

$$\therefore \quad u_{xx} + u_{yy} = 0. \text{ Hence u is harmonic.}$$

Let v be a harmonic conjugate of u.

$\therefore$ $f(z) = u + iv$ is the analytic function where v is to be found out.

By Cauchy-Riemann equations we have

$$v_y = u_x = 2 - 3x^2 + 3y^2$$

$\therefore$ Integrating with respect to y we get

$$v = 2y - 3x^2y + y^3 + \lambda(x) \qquad \text{--------(1)}$$

where $\lambda(x)$ is an arbitrary function of x.

$$\therefore \quad v_x = -6xy + \lambda(x)$$

$$v_x = -u_y \text{ gives } -6xy+\lambda(x) = -6xy$$

Hence $\lambda(x) = 0$ so that $\lambda(x) = C$ where C is a constant

$$v = 2y-3x^2y+y^3+C \qquad \text{(from (1))}$$

$$f(z) = (2x-x^3+3xy^2)+i(2y-3x^2y+y^3)+iC$$

$$= 2(x+iy)-[(x^3-3xy^2)+i(3x^2y-y^3]+iC$$

$$= 2z-z^3+iC$$

$\therefore$ $f(z) = 2z-z^3+iC$ is the required analytic function.

Example 2 :

Show that $u(x, y) = \sin x \cosh y + 2\cos x \sinh y + x^2-y^2+4xy$ is harmonic. Find an analytic function f(z) with the given u for its real part.

Solution :

$$u_x = \cos x \cosh y - 2\sin x \sinh y +2x + 4y$$

$$u_{xx} = -\sin x \cosh y - 2\cos x \sinh y + 2$$

$$u_y = \sin x \sinh y + 2\cos x \cosh y - 2y + 4x$$

$$u_{yy} = \sin x \cosh y + 2\cos x \sinh y - 2$$

$\therefore$ $u_{xx}+u_{yy} = 0$

Hence u is harmonic.

Let $\phi_1(x, y) = u_x$

and $\phi_2(x, y) = u_y$

$\therefore$ $\phi_1(z, 0) = \cos z \cosh 0 - 2\sin z \sinh 0 + 2z$

$$= \cos z + 2z$$

Similarly $\phi_2(z, 0) = 2\cos z + 4z$

$\therefore$ $f(z) = \int[\phi_1(z,0)-i\phi_2(z,0)]dz$

$$= \int[(\cos z+2z)-i(2\cos z+4z)]dz$$

$$= -\sin z + z^2-2i \sin z - 2i z^2 + C$$

Example 3 :

given $v(x, y) = x^4-6x^2y^2+y^4$ find $f(z) = u(x, y)+iv(x, y)$ using Milne Thomson method such that f(z) is analytic.

Solution :

$$v(x, y) = x^4-6x^2y^2+y^4$$

$$v_x = 4x^3-12xy^2$$

$$v_{xx} = 12x^2-12y^2$$

$$v_y = -12x^2y+4y^3$$

$$v_{yy} = -12x^2+12y^2$$

$$\therefore \quad v_{xx}+v_{yy} = 0$$

Hence v is harmonic.

Let $\psi_1(x, y) = v_y$

and $\psi_2(x, y) = v_x$

$$\therefore \quad \psi_1(x, y) = -12x^2y+4y^3$$

and $\psi_2(x, y) = 4x^3-12xy^2$

$$\therefore \quad \psi_1(z, 0) = 0$$

and $\psi_2(z, 0) = 4z^3$

$$\therefore \quad f(z) = \int[\psi_1(z,0)+i\psi_2(z,0)]dz$$

$$= i\int 4z^3 dz = i\,z^4+C$$

$$f(z) = i\,z^4+C$$

Example 4 :

Find the constant a so that $u(x, y) = ax^2-y^2+xy$ is harmonic. Find an analytic function f(z) for which u is the real part. Also find its harmonic conjugate.

Solution :

$$u = ax^2-y^2+xy$$

Given that u is harmonic. Hence it satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial x} = 2ax+y$$

and
$$\frac{\partial^2 u}{\partial x^2} = 2a$$

$$\frac{\partial u}{\partial y} = -2y+x$$

and $$\frac{\partial^2 u}{\partial y^2} = -2$$

$\therefore$ $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow 2a-2 = 0$$

Hence $$a = 1$$

$\therefore$ $$u = x^2-y^2+xy$$

Hence $$u_x = 2x+y$$

and $$u_y = -2y+x$$

Let $$\phi_1(x, y) = u_x = 2x+y$$

and $$\phi_2(x, y) = u_y = -2y+x$$

$\therefore$ $$\phi_1(z, 0) = 2z$$

and $$\phi_2(z, 0) = z$$

$\therefore$ $$f(z) = \int[\phi_1(z,0)-i\phi_2(z,0)]dz$$

$$= \int(2z-iz)dz$$

$$= z^2-\frac{iz^2}{2}+C$$

$\therefore$ $$f(z) = z^2-\frac{iz^2}{2}+C$$

$$= (x+iy)^2-i\frac{(x+iy)^2}{2}+C$$

$$= \left(x^2-y^2+2ixy\right)-\frac{i}{2}\left(x^2-y^2+2ixy\right)+C$$

$$= \left(x^2-y^2+xy\right)+i\left(2xy+\frac{y^2-x^2}{2}\right)+C$$

$\therefore$ $v(x, y) = 2xy+\frac{y^2-x^2}{2}$ is the harmonic conjugate of u(x,y)

Example 5 :

Prove that the real and imaginary parts of an analytic function when expressed in polar form satisfy the equation $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$

Solution :

We know that Cauchy-Riemann equation in polar form is given by

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{-------(1)}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \qquad \text{-------(2)}$$

We eliminate v from (1) and (2)

Differentiating (1) partially with respect to r and (2) partially with respect to θ we have

$$\frac{\partial^2 v}{\partial r \partial \theta} = r\frac{\partial^2 u}{\partial r^2}+\frac{\partial u}{\partial r} \qquad \text{-------(3)}$$

$$\frac{\partial^2 v}{\partial \theta \partial r} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} \qquad \text{-------(4)}$$

Since $\quad \frac{\partial^2 v}{\partial r \partial \theta} = \frac{\partial^2 v}{\partial \theta \partial r}$

we have $\quad r\frac{\partial^2 u}{\partial r^2}+\frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}$

$$\therefore \quad \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Similarly, $\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$

Example 6 :

Show that if u and v are conjugate harmonic functions the product uv is a harmonic function.

Solution :

Since u and v are conjugate harmonic functions we have

$$u_{xx}+u_{yy} = 0 \quad \text{--------(1)}$$

$$v_{xx}+v_{yy} = 0 \quad \text{--------(2)}$$

$$u_x = v_y \quad \text{--------(3)}$$

$$u_y = -v_x \quad \text{--------(4)}$$

Let $\phi = uv$

$$\phi_x = uv_x+vu_x$$

$$\phi_{xx} = uv_{xx}+2u_xv_x+vu_{xx}$$

Similarly $\phi_{yy} = uv_{yy}-2v_xu_x+vu_{yy}$ (using (3) and (4))

$$\phi_{xx}+\phi_{yy} = u(v_{xx}+v_{yy})+v(u_{xx}+u_{yy})$$

$$= 0 \text{ (using (1) and (2))}$$

$\therefore$ $\phi = uv$ is a harmonic function.

Example 7 :

If f(z) is analytic prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f(z)|^2$

Solution :

Let $f(z) = u+iv$

$$|f(z)|^2 = u^2+y^2 = \phi(\text{say})$$

and $f'(z) = u_x+iv_x$

$$\frac{\partial\phi}{\partial x} = 2uu_x+2vv_x$$

$$\frac{\partial^2\phi}{\partial x^2} = 2\left[u_x^2+uu_{xx}+v_x^2+vv_{xx}\right] \quad \text{-------(1)}$$

Similarly $\frac{\partial^2\phi}{\partial y^2} = 2\left[u_y^2+uu_{yy}+v_y^2+vv_{yy}\right]$ -------(2)

Since u and v are harmonic $\left.\begin{matrix} u_{xx}+u_{yy}=0 \text{ and} \\ v_{xx}+v_{yy}=0 \end{matrix}\right\}$ -------(3)

Adding (1) and (2) using (3) we get

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = 4[u_x^2+v_x^2]$$

$$= 4|u_x+iv_x|^2$$

$$= 4|f'(z)|^2$$

Exercise :

1. Prove that the following functions are harmonic. Also find a harmonic conjugate.

   (i) $u = \sinh x \sin y$ (ii) $u = e^x \cos y$

2. Find the function $f(z) = u+iv$ such that $f(z)$ is analytic given that

   (i) $u = x$; (ii) $u = x^3-3xy^2$ (iii) $u = \cos x \cosh y$ (iv) $v = 3x^2y-y^3$

3. Prove that the functions $u(x, y)$ and $u(x^2-y^2, 2xy)$ are simultaneously harmonic.

4. Prove that $u(x,y) = x^2-y^2$ and $v(x, y) = -\frac{y}{x^2+y^2}$ are both harmonic but $u+iv$ is not analytic.

5. Find the analytic function $f(z) = u+iv$ if $u+v = \frac{x}{x^2+y^2}$ given $f(1) = 1$.

BILINEAR TRANSFORMATIONS

A function f:C→C can be thought of as a transformation from one complex plane to another complex plane. Hence the nature of a complex function can be described by the manner in which it maps regions and curves from one complex plane to another.

Elementary Transformations :

1. **Translation :** Consider the transformation $w = z+b$. If $z = x+iy$, $w = u+iv$ and $b = b_1+ib_2$ then the image of the point (x,y) in the z-plane is the point $(x+b_1, y+b_2)$ in the w-plane.

Under this transformation the image of any region is simply a **translation** of that region.

Hence the two regions have the same shape, size and orientation. In particular the image of a straight line is a straight line and the image of a circle with centre a and radius r is a circlewith centre a+b and radius r.

We note that ∞ is the only fixed point of this transformation when $b\neq 0$.

2. **Rotation :** Consider the transformation $w=az$ where $|a| = 1$.

Let $z = re^{i\theta}$ and $a= e^{i\alpha}$ so that $|a|=1$.

$\therefore w = az = e^{i\alpha}(re^{i\theta}) = re^{i(\theta+\alpha)}$

$\therefore$ A point with polar coordinates (r, θ) in the z-plane is mapped to the point $(r, \theta+\alpha)$ in the w-plane. Hence this transformation represents a **rotation** through an angle $\alpha=\arg a$ about the origin.

Under this transformation also straight lines are mapped into straight lines and circles are mapped into circles.

We note that 0 and ∞ are the two fixed points of this transformation.

3. **Magnification or Contraction :** Consider the transformation $w=bz$ where b is real and $b>0$.

Then a point with polar coordinates (r, θ) in the z-plane is mapped into the point (br, θ) in the w-plane. Hence this transformation represents a **magnification** or **contraction** by the factor according as $b>1$ or $b<1$.

Under this transformation also straight lines are mapped into straight lines and circles are mapped into circles.

We note that 0 and ∞ are the fixed points of this transformation.

In general the transformation w=bz where b is a non-zero complex number represents a rotation through an angle arg b followed by a magnification or a contraction by the factor |b|. Such a transformation is called a **homethatic transformation.**

4. **Inversion :** Consider the transformation $w = \frac{1}{z}$.

Put
$$z = re^{i\phi}$$
$$w = \frac{1}{r}e^{-i\theta}$$

This transformation can be expressed as a product of two transformations $T_1(z)=\frac{1}{r}e^{i\theta}$ and $T_2(z) = re^{-i\phi} = \bar{z}$.

For,
$$\begin{aligned}(T_1 \cdot T_2)(z) &= T_1(T_2(z)) \\ &= T_1(re^{-i\phi}) \\ &= \frac{1}{r}e^{-i\theta} = \frac{1}{z}\end{aligned}$$

The transformation $T_1(z) = \frac{1}{r}e^{i\theta}$ represents the **inversion** with respect to the unit circle $|z|=1$ and $T_2(z)=\bar{z}$ represents **reflection** about the real axis.

Hence the transformation $w = \frac{1}{z}$ is the inversion w.r.t. the unit circle followed by the reflection about the real axis.

Here points outside the unit circle are mapped into points inside the unit circle and vice versa. Points on the circle are reflected abuot the real axis.

In terms of cartesian coordinates the above transformation can be expressed in the form.

$$w = u+iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$\therefore \qquad u = \dfrac{x}{x^2+y^2}$

and $\qquad v = -\dfrac{y}{x^2+y^2}$

Similarly from $z = \dfrac{1}{w}$ we get $x = \dfrac{u}{u^2+v^2}$ and $y = \dfrac{-v}{u^2+v^2}$ --------(1)

Consider the equation $a\,(x^2+y^2)+bx+cy+d = 0$ --------(2)

where a, b, c, d are real.

This equation represents a circle or a straight line according as $a \neq 0$ or $a = 0$.

Using (1) in (2) we get

$d(u^2+v^2)+bu-cv+a = 0$ --------(3)

Suppose $a \neq 0$; $d \neq 0$

In this case both (2) and (3) represent circles not passing through the origin. Hence circles not passing through the origin are mapped into circles not passing through the origin.

Similarly a circle passing through the origin is mapped into a straight line not passing through the origin.

A straight line not passing through the origin is mapped into a circle passing through the origin.

A straight line passing thruogh the origin is again mapped into a line passing through the origin.

Thus we see that under the transformation $w = \dfrac{1}{z}$ the image of a circle need not be a circle and the image of a straight line need not be a straight line.

We note that the fixed points of the transformation $w = \dfrac{1}{z}$ are 1 and -1.

Worked Examples :

Example 1 :

Under the transformation $w = iz+i$ show that the half plane $x>0$ maps onto the half plane $v>1$.

Solution :

Let $z = x + iy$

and $w = u + iv$

$w = iz+i$

$\Rightarrow w = i(x+iy)+i$

$= -y+i(x+1)$

$\therefore u+iv = -y+i(x+1)$

$\therefore u = -y$

and $v = x+1$

$\therefore x > 0 \Leftrightarrow v > 1$

$\therefore$ The half plane $x>0$ is mapped into the half plane $v>1$.

Example 2 :

Show that by means of the inversion $w = \frac{1}{z}$ the circle given by $|z-3|=5$ is mapped into the circle $\left|w+\frac{3}{16}\right| = \frac{5}{16}$.

Solution :

The circle $|z-3| = 5$ is mapped into $\left|\frac{1}{w}-3\right| = 5$.

Now $\left|\frac{1}{w}-3\right| = 5 \Rightarrow \left|\frac{1}{u+iv}-3\right| = 5$

$\Rightarrow |(1-3u)-3iv| = 5|u+iv|$

$\Rightarrow (1-3u)^2+9v^2 = 25(u^2+v^2)$

$\Rightarrow 9u^2-6u+1+9v^2 = 25u^2+25v^2$

$\Rightarrow 16(u^2+v^2)+6u-1 = 0$

$\Rightarrow u^2+v^2+\frac{6}{16}u-\frac{1}{16} = 0$

This is a circle with centre $\left(-\frac{3}{16},0\right)$ and radius $\sqrt{\left(\frac{3}{16}\right)^2+\frac{1}{16}} = \frac{5}{16}$

Hence the image circle in the w-plane is given by the equation $\left|w+\frac{3}{16}\right| = \frac{5}{16}$.

Exercise :

1. Find the image of the strip $0<x<1$ under the transformation $w=iz$.
2. Find the image of the region $y>1$ under the transformation $w=(1-i)z$.
3. Show that by means of the inversion $w=\frac{1}{z}$ the circle given by $|z-2|=7$ is mapped into the circle $\left|w+\frac{2}{45}\right|=\frac{7}{45}$.
4. Find the image of the semi infinite strip $x>0$; $0<y<2$ under the transformation $w=iz+1$.

Bilinear Transforamtions :

A transformation of the form $w = T(z) = \frac{az+b}{cz+d}$ ------(1)

where a, b, c, d are complex constants and $ad-bc\neq 0$ is called a **bilinear transformation** or **Mobius transformation.**

We define $T(\infty) = \frac{a}{c}$ and $T\left(-\frac{d}{c}\right) = \infty$. Hence T becomes a 1–1 onto map of the extended complex plane onto itself.

The inverse of (1) is given by $z = T^{-1}(w) = \frac{-dw+b}{cw-a}$ which is also a bilinear transformation.

Theorem :

Any bilinear transformation can be expressed as a product of translation, rotation, magnification or contraction and inversion.

Proof :

Let $w = T(z) = \frac{az+b}{cz+d}$ where $ad-bc \infty 0$ -------(1)

be the given bilinear transformation.

Case (i) :

$c = 0$.

Hence $d \neq 0$ $(\because ad-bc \neq 0)$

$\therefore \quad (1) \Rightarrow w = \dfrac{az+b}{d}$

$$= \left(\frac{a}{d}\right)z + \frac{b}{d}$$

Let $\quad T_1(z) = \left(\dfrac{a}{d}\right)z$

and $\quad T_2(z) = z + \dfrac{b}{d}$

T_1 and T_2 are elementary transformation and

$$(T_2 \bullet T_1)(z) = T_2\left[\left(\frac{a}{d}\right)z\right] = \frac{a}{d}z + \frac{b}{d}$$

$$= T(z).$$

Case (ii) :

$$c \neq 0$$

$$\therefore \quad w = \frac{az+b}{cz+d} = \frac{a\left[z+\left(\frac{d}{c}\right)\right]+b-\left(\frac{ad}{c}\right)}{c\left[z+\left(\frac{d}{c}\right)\right]}$$

$$= \frac{a}{c} + \frac{b-\left(\frac{ad}{c}\right)}{cz+d}$$

Let $\quad T_1(z) = cz+d$

$$T_2(z) = \frac{1}{z}$$

$$T_3(z) = \left(b - \frac{ad}{c}\right)z$$

$$T_4(z) = z + \left(\frac{a}{c}\right)$$

Then $\quad T(z) = (T_4 \bullet T_3 \bullet T_2 \bullet T_1)(z)$

For $\quad (T_4 \bullet T_3 \bullet T_2 \bullet T_1)(z) = (T_4 \bullet T_3 \bullet T_2)(cz+d)$

$$= (T_4 \bullet T_3)\left(\frac{1}{cz+d}\right)$$

$$= T_4\left(\left(b - \frac{ad}{c}\right)\left(\frac{1}{cz+d}\right)\right)$$

$$= T_4\left(\frac{bc-ad}{c(cz+d)}\right)$$

$$= \frac{bc-ad}{c(cz+d)} + \frac{a}{c}$$

$$= \frac{bc-ad+acz+ad}{c(cz+d)}$$

$$= \frac{c(b+az)}{c(cz+d)}$$

$$= \frac{az+b}{cz+d}$$

$$= T(z)$$

Hence the theorem.

Corollary :

Under a bilinear transformation circles and lines are transformed into circles and lines.

Worked Examples :

Example 1 :

Show that the transformation $w = \frac{5-4z}{4z-2}$ maps the unit circle $|z|=1$ into a circle of radius unity and centre $-\frac{1}{2}$.

Solution :

$$w = \frac{5-4z}{4z-2}$$

$$\therefore \quad 4wz-2w = 5-4z$$

$$\therefore \quad (4w+4)z = 5+2w$$

$$\therefore \quad z = \frac{5+2w}{4w+4}$$

Now $\quad |z| = 1 \Rightarrow z\bar{z} = 1.$

$$\Rightarrow \left(\frac{5+2w}{4w+4}\right)\left(\frac{5+2\bar{w}}{4\bar{w}+4}\right) = 1$$

$$\Rightarrow 25+4w\bar{w}+10w+10\bar{w} = 16w\bar{w}+16+16(w+\bar{w})$$

$$\Rightarrow 12w\bar{w}+6w+6\bar{w}-9 = 0$$

$$\Rightarrow w\bar{w}+\frac{1}{2}w+\frac{1}{2}\bar{w}-\frac{3}{4} = 0$$

This represents the equation of the circle with centre $-\frac{1}{2}$ and radius $\sqrt{\frac{1}{4}+\frac{3}{4}} = 1.$

Hence the result.

Example 2 :

Show that the transformation $w = \frac{2z+3}{z-4}$ maps the circle $z\bar{z}-2(z+\bar{z})=0$ into a straight line given by $2(w+\bar{w})+3 = 0.$

Solution :

$$w = \frac{2z+3}{z-4}$$

$$\therefore \quad w(z-4) = 2z+3$$

$$\therefore \quad z(w-2) = 3+4w$$

$$\therefore \quad z = \frac{3+4w}{w-2}$$

The image of the circle $z\bar{z}-2(z+\bar{z})=0$ is

$$\left(\frac{3+4w}{w-2}\right)\left(\frac{3+4\bar{w}}{\bar{w}-2}\right)-2\left[\left(\frac{3+4w}{w-2}\right)+\left(\frac{3+4\bar{w}}{\bar{w}-2}\right)\right]= 0$$

$$(3+4w)(3+4\overline{w})-2[(3+4w)(\overline{w}-2)+(3+4\overline{w})(w-2)]=0$$

$$9+16w\overline{w}+12w+12\overline{w}-2[3\overline{w}+4w\overline{w}-6-8w+3w+4w\overline{w}-6-8\overline{w}]=0$$

$$9+22w+22\overline{w}+24=0$$

$$22w+22\overline{w}+33=0$$

$$2w+2\overline{w}+3=0$$

$2(w+\overline{w})+3=0$ which is obviously a straight line.

Exercise :

1. Express $w=\dfrac{z-1}{z+1}$ as a product of elementary transformation.
2. Prove that the transformation $w=\dfrac{i-iz}{1+z}$ maps the unit circle $|z|=1$ into the real axis of the w-plane.
3. Show that the transformation $w=\dfrac{iz+2}{4z+i}$ maps the real axis in the z plane to a circle in the w-plane. Find the centre and radius of the circle.
4. Prove that if a point on a circle is mapped into ∞ under a bilinear transformation then this circle is transformed into a straight line.

CROSS RATIO

Definition :

Let z_1, z_2, z_3, z_4 be four distinct points in the extended complex plane. The **cross ratio** of these four points denoted by (z_1, z_2, z_3, z_4) is defined by $(z_1, z_2, z_3, z_4)=$ $\dfrac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$ if none of z_1, z_2, z_3, z_4 is ∞.

$$=\frac{z_1-z_3}{z_1-z_4} \text{ if } z_2 \text{ is } \infty$$

$$=\frac{z_2-z_4}{z_1-z_4} \text{ if } z_3 \text{ is } \infty$$

$$=\frac{z_1-z_3}{z_2-z_3} \text{ if } z_4 \text{ is } \infty$$

$$=\frac{z_2-z_4}{z_2-z_3} \text{ if } z_1 \text{ is } \infty$$

Theorem :

Any bilinear transformation preserves cross ratio.

Proof :

Let $w = \frac{az+b}{cz+d}$, $ad-bc \neq 0$ be the given bilinear transformation. Let z_1, z_2, z_3, z_4 be four distinct points. Let their images under this transformation be w_1, w_2, w_3, w_4 respectively.

We assume that all the z_i and w_i are different from ∞.

We claim that $(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4)$

We have
$$w_i = \frac{az_i+b}{cz_i+d} \quad (i = 1, 2, 3, 4)$$

$$w_1-w_3 = \frac{az_1+b}{cz_1+d} - \frac{az_3+b}{cz_3+d}$$

$$= \frac{(ad-bc)(z_1-z_3)}{(cz_1+d)(cz_3+d)}$$

$$= K_1(z_1-z_3) \quad \text{(say)}$$

Similarly
$$w_2-w_4 = K_2(z_2-z_4)$$

$$(w_1-w_3)(w_2-w_4) = K_1K_2(z_1-z_3)(z_2-z_4)$$

$$= K(z_1-z_3)(z_2-z_4)$$

Similarly we can prove that

$$(w_1-w_4)(w_2-w_3) = K(z_1-z_4)(z_2-z_3)$$

$$\frac{(w_1-w_3)(w_2-w_4)}{(w_1-w_4)(w_2-w_3)} = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$$

The proof is similar if one of the z_i or w_i is ∞.

Note 1 : Four distinct points z_1, z_2, z_3, z_4 are collinear or concyclic iff (z_1, z_3, z_3, z_4) is real.

Note 2 : Any bilinear transformation preserves cross ratio. Hence it follows that circles and straight lines are mapped into circles and straight lines.

Worked Examples :

Example 1 :

Find the bilinear transformation which maps the points $z = -1, 1, \infty$ respectively on $w = -i, -1, i$.

Solution :

Let the image of any point z under the required bilinear transformation be w.

Since bilinear transformation preserves cross ratio we have $(z, -1; 1, \infty) = (w, -i; -1, i)$

$$\frac{z-1}{-1-1} = \frac{(w+1)(-i-i)}{(w-i)(-i+1)}$$

$$\therefore \quad (z-1)(w-iw-i-1) = +4iw + 4i$$

$$\therefore \quad w[z-1-i(z-1)-4i] = 4i+(i+1)(z-1)$$

$$\therefore \quad w = \frac{(i+1)z+3i-1}{(1-i)z-3i-1}$$

Example 2 :

Find the bilinear transformation which maps the points $z_1=0$, $z_2=-i$, $z_3=-1$ into the points $w_1=i$, $w_2=1$ and $w_3=0$.

Solution :

Let the image of any point z under the required bilinear transformation be w.

Since bilinear transformation preserves cross ratio we have

$$(z, 0; -i, -1) = (w, i; 1, 0)$$

$$\therefore \quad \frac{(z+i)(0+1)}{(z+1)(0+i)} = \frac{(w-1)(i-0)}{(w-0)(i-1)}$$

$$\therefore \quad (z+i)w(i-1) = (-w-1)(z+1)$$

$$\therefore \quad w(iz-i) = z+1$$

$$\therefore \quad w = +(-i)\frac{z+1}{z-1}$$

$$w = -i\left(\frac{z+1}{z-1}\right)$$

which is the required bilinear transformation.

Exercise :

1. Find the bilinear transformation which maps z_1, z_2, z_3 to w_1, w_2, w_3 respectively where

   (i) $z_1 = 2; z_2 = i; z_3 = -2$

   $w_1 = 1; w_2 = i; w_3 = -1$

   (ii) $z_1 = \infty, z_2 = i, z_3 = 0$

   $w_1 = 0, w_2 = i, w_3 = \infty.$

2. Find a bilinear transformation which maps the vertices 1+i, –i, 2–i of a triangle of the z-plane into the points 0, 1, i of the w-plane.

FIXED POINTS OF A BILINEAR TRANSFORMATION

Definition :

Consider a bilinear transformation given by $w = \frac{az+b}{cz+d}$ where $ad-bc \neq 0$.

The **fixed points** or **invariant points** of the bilinear transformation are given by the roots of the equation $z = \frac{az+b}{cz+d}$

i.e., $\quad cz^2+(d-a)z-b = 0$

Case (i) :

$c \neq 0$

In this case the fixed points are given by $z = \frac{(a-d)\pm\sqrt{(d-a)^2+4bc}}{2c}$

When $(d-a)^2+4bc \neq 0$, the given bilinear transformation has **two finite fixed points** and when $(d-a)^2+4bc = 0$ it has **only one finite fixed point.**

Case (ii) :

$c = 0$

In this case the bilinear transformation becomes

$$w = \frac{a}{d}z+\frac{b}{d}$$

Clearly ∞ is one fixed point.

Other fixed point is determined by the equation

$$z = \frac{a}{d}z + \frac{b}{d}$$

i.e., $(d-a)z-b = 0$

If $d-a \neq 0$ we get a finite fixed point $\frac{b}{d-a}$.

If $d-a = 0$ then ∞ is the only fixed point.

Thus we have

(i) $c \neq 0$; $(d-a)^2+4bc \neq 0 \Rightarrow$ 2 finite fixed points.

(ii) $c \neq 0$; $(d-a)^2+4bc = 0 \Rightarrow$ one finite fixed point

(iii) $c=0$; $a \neq d \Rightarrow \infty$ and one finite fixed point.

(iv) $c=0$; $a=d \Rightarrow \infty$ is the only fixed point.

Theorem :

Any bilinear transformation having two finite fixed points α and β can be written in the form $\frac{w-\alpha}{w-\beta} = K\left(\frac{z-\alpha}{z-\beta}\right)$

Proof :

Let T be the given bilinear transformation having α and β as fixed points. Let the image of any point r under T be δ.

Then the bilinear transformation T is given by

$$(w, \delta; \alpha, \beta) = (z, \gamma; \alpha, \beta)$$

$$\therefore \quad \frac{(w-\alpha)(\delta-\beta)}{(w-\beta)(\delta-\alpha)} = \frac{(z-\alpha)(\gamma-\beta)}{(z-\beta)(\gamma-\alpha)}$$

i.e., $$\frac{w-\alpha}{w-\beta} = K\left(\frac{z-\alpha}{z-\beta}\right) \text{ where } K = \frac{(\gamma-\beta)(\delta-\alpha)}{(\gamma-\alpha)(\delta-\beta)} \qquad \text{-----(1)}$$

Definition :

Let T be a bilinear transformation with two finite fixed points α, β. If K given by (1) is real T is called **hyperbolic** and if $|K|=1$, Tis called **elliptic**.

Theorem :

Any bilinear transformation having ∞ and $\alpha \neq \infty$ as fixed points can be written in the form $w-\alpha = K(z-\alpha)$.

Proof :

Let T be the given bilinear transformation having ∞ and α as fixed points. Let the image of any point γ under T be δ.

Then the bilinear transformation is given by

$$(w, \delta; \alpha, \infty) = (z, \gamma; \alpha, \infty)$$

$$\therefore \quad \frac{w-\alpha}{\delta-\alpha} = \frac{z-\alpha}{\gamma-\alpha}$$

$$\therefore \quad w-\alpha = K(z-\alpha) \text{ where } K = \frac{\delta-\alpha}{\gamma-\alpha}$$

Definition :

A bilinear transformation with only one finite fixed point is called **parabolic.**

Theorem :

Any bilinear transformation having ∞ as the only fixed point is a translation.

Proof :

Let $w = \dfrac{az+b}{cz+d}$ be the bilinear transformation having ∞ as the only fixed point. Then $c = 0$ and $a = d$.

$\therefore$ The bilinear transformation reduces to the form

$$w = \frac{az+b}{a}$$

$$\therefore \quad w = z+\frac{b}{a} \text{ which is a translation.}$$

Theorem :

Let C be a circle or a straight line and z_1, z_2 be inverse points or reflection points with respect to C. Let w_1, w_2 and C_1 be the images of z_1, z_2 and C under a bilinear transformation. Then w_1 and w_2 are inverse points or reflection points with respect to C_1 i.e., a bilinear transformation preserves inverse points.

Proof :

Let the equation of C be $pz\bar{z}+\alpha\bar{z}+\bar{\alpha}z+\beta = 0$ --------(1)

Since z_1 and z_2 are inverse points w.r.t. C we have

$$pz_1\overline{z_2}+\alpha\overline{z_2}+\bar{\alpha}z_1+\beta = 0 \quad \text{--------(2)}$$

Let the given bilinear transformation be

$$w = \frac{az+b}{cz+d} \text{ where } ad-bc \neq 0$$

$$\therefore \quad z = \frac{dw-b}{-cw+a}$$

$\therefore$ Under the given bilinear transformation (1) is transformed into

$$p\left(\frac{dw-b}{-cw+a}\right)\left(\frac{\overline{dw}-\bar{b}}{-\overline{cw}+\bar{a}}\right)+\alpha\left(\frac{\overline{dw}-\bar{\beta}}{-\overline{cw}+\bar{\alpha}}\right)+\bar{\alpha}\left(\frac{dw-b}{-cw+a}\right)+\beta = 0 \quad \text{--------(3)}$$

Also (2) is transformed into

$$p\left(\frac{dw_1-b}{-cw_1+a}\right)\left(\frac{\overline{dw_2}-\bar{b}}{-\overline{cw_2}+\bar{a}}\right)+\alpha\left(\frac{\overline{dw_2}-\bar{b}}{-\overline{cw_2}+\bar{a}}\right)+\bar{\alpha}\left(\frac{dw_1-b}{-cw_1+a}\right)+\beta=0 \quad \text{--------(4)}$$

Clearly (4) is the condition for w_1 and w_2 to be inverse points with respect to (3). Hence the theorem.

Note : We shall regard the centre of the circle and ∞ as inverse points with respect to the circle.

Exercise :

1. Prove that a bilinear transformation having origin as th fixed point can be written in the form $w = \frac{z}{cz+d}$

2. Prove that a bilinear transformation having 0 and ∞ as fixed points is of the form $w = az$.

3. Find the fixed points and the normal form of the following bilinear transformations. Also determine whether they are elliptic, hyperbolic or parabolic

   (i) $w = z+3$ (ii) $w = \frac{z-1}{z+1}$ (iii) $w = \frac{3iz+1}{z+i}$ (iv) $w = \frac{6z-9}{z}$

Special Bilinear Transformations

We shall determine the general form of the transformations which map

(i) the real axis onto itself

(ii) the unit circle onto itself

(iii) the real axis onto the unit circle.

Theorem 1 :

A bilinear transformation $w = \frac{az+b}{cz+d}$ where $ad-bc \neq 0$ maps the real axis into itself iff a, b, c, d are real.

Further this transformation maps the upper half plane $\text{Im } z \geq 0$ into the upper half plane $\text{Im } w \geq 0$ iff $ad-bc>0$.

Proof :

Suppose a, b, c, d, are real.

Then z is real $\Rightarrow$ w is also real.

$\therefore$ The real axis is mapped into itself.

Conversely suppose w is a bilinear transformation that maps the real axis into itself.

$\therefore$ There exist real numbers x_1, x_2, x_3 such that $T(x_1)=1$, $T(x_2)=0$ and $T(x_3)=\infty$.

$\therefore$ The bilinear transformation is given by

$$(z, x_1; x_2, x_3) = (w, 1; 0, \infty)$$

$$\therefore \quad \frac{(z-x_2)(x_1-x_3)}{(z-x_3)(x_1-x_2)} = w$$

$$\therefore \quad w = \frac{az+b}{cz+d} \text{ where } a = x_1-x_3;\ b = -x_2(x_1-x_3)$$

$$c = x_1-x_2$$

and $$d = -x_3(x_1-x_2)$$

Since x_1, x_2, x_3 are real a, b, c, d are also real.

$$2i\text{Im } w = w - \overline{w} = \frac{az+b}{cz+d} - \frac{a\overline{z}+b}{c\overline{z}+d}$$

$$= \frac{(ad-bc)(z-\bar{z})}{|cz+d|^2} = 2i\left(\frac{ad-bc}{|cz+d|^2}\right)\text{Im}\, z$$

$$\therefore \qquad \text{Im}\, w = \frac{ad-bc}{|cz+d|^2}\text{Im}\, z$$

$\therefore$ The upper half plane Im $z \geq 0$ is mapped onto the upper half plane Im $w \geq 0$ $\Leftrightarrow$ ad=bc>0.

Theorem 2 :

Any bilinear transformation which maps the unit circle $|z| = 1$ onto the unit circle $|w|=1$ can be written in the form $w = e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)$ where λ is real.

Further this transformation maps the circular disc $|z|\leq 1$ onto the circular disc $|z|\leq 1$ iff $|\alpha|<1$.

Proof :

Let $w = \frac{az+b}{cz+d}$ where $ad-bc\neq 0$ be any bilinear transformation which maps $|z|=1$ onto $|w|=1$.

0 and ∞ are inverse points with respect to the circle $|w| = 1$.

Hence their pre-images $-\left(\frac{b}{a}\right)$ and $-\left(\frac{d}{c}\right)$ are inverse points with respect to $|z|=1$.

$$\therefore \qquad \left(-\frac{b}{a}\right)\left(-\frac{\bar{d}}{c}\right) = 1$$

$$\therefore \text{ If } \qquad \alpha = \left(-\frac{b}{a}\right) \text{ then } \frac{1}{\bar{\alpha}} = -\frac{d}{c}$$

$$\therefore \qquad w = \frac{az+b}{cz+d} = \frac{a}{c}\left(\frac{z-\alpha}{z-\left(\frac{1}{\bar{\alpha}}\right)}\right)$$

$$= \frac{a\bar{\alpha}}{c}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)$$

Let $|z| = 1$. Hence $|w| = 1$.

$$\therefore \quad 1 = |w| = \left|\frac{a\bar{\alpha}}{c}\right|\left|\frac{z-\alpha}{\bar{\alpha}z-1}\right|$$

$$= \left|\frac{a\bar{\alpha}}{c}\right|\left|\frac{\bar{z}-\bar{\alpha}}{\bar{\alpha}z-z\bar{z}}\right| \quad \text{(since } z\bar{z}=1\text{)}$$

$$= \left|\frac{a\bar{\alpha}}{c}\right|\left|\frac{\bar{z}-\bar{\alpha}}{\bar{\alpha}-\bar{z}}\right| \quad \text{(since } |z| = 1\text{)}$$

$$= \left|\frac{a\bar{\alpha}}{c}\right|$$

Thus $\left|\frac{a\bar{\alpha}}{c}\right| = 1$

$$\therefore \quad \frac{a\bar{\alpha}}{c} = e^{i\lambda} \text{ for some real number } \lambda.$$

$$\therefore \quad w = e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right) \text{ where } \lambda \text{ is real.}$$

$$w\bar{w}-1 = e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)e^{-i\lambda}\left(\frac{\bar{z}-\bar{\alpha}}{\alpha\bar{z}-1}\right)-1$$

$$= \frac{(z-\alpha)(\bar{z}-\bar{\alpha})}{(\bar{\alpha}z-1)(\alpha\bar{z}-1)}-1$$

$$= \frac{(z-\alpha)(\bar{z}-\bar{\alpha})-(\bar{\alpha}z-1)(\alpha\bar{z}-1)}{|\alpha\bar{z}-1|^2}$$

$$= \frac{z\bar{z}-\alpha\bar{z}-\bar{\alpha}z+\alpha\bar{\alpha}-\alpha\bar{\alpha}z\bar{z}+\alpha\bar{z}+\bar{\alpha}z-1}{|\alpha\bar{z}-1|^2}$$

$$= \frac{z\bar{z}(1-\alpha\bar{\alpha})-(1-\alpha\bar{\alpha})}{|\alpha\bar{z}-1|^2}$$

$$= \frac{(1-\alpha\bar{\alpha})(z\bar{z}-1)}{|\alpha\bar{z}-1|^2}$$

ஃ The transformation maps $|z| \leq 1$ onto $|w| \leq 1$

$$\Leftrightarrow 1-\alpha\bar{\alpha} > 0$$

$$\Leftrightarrow \alpha\bar{\alpha} < 1$$

$$\Leftrightarrow |\alpha| < 1.$$

Theorem 3 :

Any bilinear transformation which maps the real axis onto unit circle $|w|=1$ can be written in the form $w = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right)$ where λ is real.

Further this transformation maps the upper half plane Im $z \geq 0$ onto the unit circular disc $|w| \leq 1$ iff Im $\alpha > 0$.

Proof :

Let $w = \frac{az+b}{cz+d}$ where $ad-bc \neq 0$ be any bilinear transformation which maps the real axis onto the unit circle $|w| = 1$.

0 and ∞ are inverse points with respect to the unit circle $|w| = 1$.

Hence their pre images $-(b/a)$ and $-(d/c)$ are reflection points with respect to the real axis.

ஃ If $\alpha = -(b/a)$ then $\bar{\alpha} = -(d/c)$

Now,

$$w = \frac{az+b}{cz+d}$$

$$= \left(\frac{a}{c}\right)\left[\frac{z+\left(\frac{b}{a}\right)}{z+\left(\frac{d}{c}\right)}\right]$$

$$= \frac{a}{c}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right)$$

Suppose z is real. Hence $|w| = 1$.

$$\left|\frac{a}{c}\right|\left|\frac{z-\alpha}{z-\bar{\alpha}}\right| = 1$$

Since z is real, $z = \bar{z}$ and hence

$$|z-\alpha| = |\overline{z-\alpha}| = |\bar{z}-\bar{\alpha}| = |z-\bar{\alpha}|$$

$\therefore$ $$\frac{|a|}{|c|} = 1. \text{ Hence } \frac{a}{c} = e^{i\lambda} \text{ where } \lambda \text{ is real.}$$

$\therefore$ $$w = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right) \text{ where } \lambda \text{ is real is the required transformation.}$$

$$w\bar{w}-1 = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right)e^{-i\lambda}\left(\frac{\bar{z}-\bar{\alpha}}{\bar{z}-\alpha}\right)-1$$

$$= \left(\frac{z-\alpha}{z-\bar{\alpha}}\right)\left(\frac{\bar{z}-\bar{\alpha}}{\bar{z}-\alpha}\right)-1$$

$$= \frac{-4\,\mathrm{Im}\,z\,\mathrm{Im}\,\alpha}{|z-\alpha|^2}$$

$\therefore$ The bilinear transformation maps the upper half plane Im $z \geq 0$ onto the disc $|w| \leq 1$ iff Im $\alpha > 0$.

Worked Examples :

Example 1 :

Find the general bilinear transformation which maps the unit circle $|z| = 1$ onto $|w| = 1$ and the points $z = 1$ to $w = 1$ and $z = -1$ to $w = -1$.

Solution :

We know any bilinear transformation which maps $|z|=1$ onto $|w|=1$ is of the form $w = e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)$ where λ is real.

Since 1 and -1 are again mapped to 1, -1, respectively we have

$$1 = e^{i\lambda}\left(\frac{1-\alpha}{\bar{\alpha}-1}\right) \quad \text{--------(1)}$$

$$-1 = e^{i\lambda}\left(\frac{-1-\alpha}{-\bar{\alpha}-1}\right) = e^{i\lambda}\left(\frac{1+\alpha}{1+\bar{\alpha}}\right) \quad \text{--------(2)}$$

Dividing (1) by (2) we get

$$-1 = \left(\frac{1-\alpha}{\bar{\alpha}-1}\right)\left(\frac{1+\bar{\alpha}}{1+\alpha}\right)$$

$$\therefore \quad -\bar{\alpha}-\alpha\bar{\alpha}+1+\alpha = 1+\bar{\alpha}-\alpha-\alpha\bar{\alpha}$$

$$\therefore \quad -2\bar{\alpha}+2\alpha = 0$$

$$\therefore \quad \alpha = \bar{\alpha} \qquad \text{--------(3)}$$

Using (3) in (1) we get $1 = -e^{i\lambda}$

$$\therefore \quad e^{i\lambda} = -1$$

∴ The required transformation is $w = \dfrac{\alpha - z}{\alpha z - 1}$

Example 2 :

Prove that the transformation, given by $\bar{a}wz - bw - \bar{b}z + a = 0$ maps the unit circle $|z|=1$ onto the unit circle $|w|=1$ if $|b| \neq |a|$.

Solution :

$$\bar{a}wz - bw - \bar{b}z + a = 0$$

$$\therefore \quad w = \frac{\bar{b}z-a}{\bar{a}z-b}$$

Now,

$$w\bar{w}-1 = \left(\frac{\bar{b}z-a}{\bar{a}z-b}\right)\left(\frac{b\bar{z}-\bar{a}}{a\bar{z}-b}\right)$$

$$= \frac{(z\bar{z}-1)\left(|b|^2-|a|^2\right)}{|\bar{a}z-b|^2}$$

If $|b| \neq |a|$ then $w\bar{w}-1 = 0 \Leftrightarrow z\bar{z}-1 = 0$.

∴ The unit circle $|z| = 1$ is mapped onto the unit circle $|w| = 1$ if $|b| \neq |a|$.

Example 3 :

Show that the bilinear transformation which maps the unit circle $|z|=1$ onto the unit circle $|z|=1$ onto the unit circle $|w|=1$ can be put in the form $w=e^{i\lambda}\left(\dfrac{az+b}{\bar{b}z+a}\right)$ where λ is real.

Further this transformation maps the circular disc $|z|\le 1$ onto the circular disc $|w|\le 1$ iff $|a|>|b|$.

Also if the point $z = 1$ is the only invariant point show that the transformation may be written as $\dfrac{1}{w-1} = \dfrac{1}{z-1}+\dfrac{1}{K}$ where $K = 1+\dfrac{\bar{a}}{b}$.

Solution :

We know that any bilinear transformation which maps $|z|=1$ onto $|w|=1$ can be written in the form $w = e^{i\mu}\left(\dfrac{z-\alpha}{\bar{\alpha}z-1}\right)$ where μ is real and this maps $|z|\le 1$ onto $|w|\le 1$ iff $|\alpha|<1$.

Choose $a = 1$ and $b = -\alpha$

$$\therefore \quad w = e^{i\mu}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)$$

$$= e^{i\mu}\left(\frac{az+b}{-\bar{b}z-\bar{a}}\right) = -e^{i\mu}\left(\frac{az+b}{\bar{b}z+\bar{a}}\right)$$

$$= e^{i\lambda}\left(\frac{az+b}{\bar{b}z+\bar{a}}\right)$$

where $\quad e^{i\lambda} = -e^{i\mu}$ and λ is real.

Further $\quad |\alpha|<1 \Leftrightarrow |-b|<a$

$$\Leftrightarrow |b|<|a|$$

$\therefore$ The transformation (1) maps $|z|\le 1$ onto $|w|\le 1$ iff $|a|>|b|$.

Suppose $z=1$ is the only fixed point of (1).

$\therefore$ $z = 1$ is the only root of the equation.

$$z = e^{i\lambda}\left(\frac{az+b}{\bar{b}z+\bar{a}}\right)$$

i.e., $\bar{b}z^2 + \left(\bar{a}-ae^{i\lambda}\right)z - be^{i\lambda} = \bar{b}(z-1)^2$

Equating the corresponding coefficient we get

$$\bar{a}-ae^{i\lambda} = -2\bar{b} \qquad \text{---------(2)}$$

$$be^{i\lambda} = \bar{b}$$

(2) can be written as $\bar{a}+\bar{b} = -\bar{b}+ae^{i\lambda}$ --------(4)

Using (3) we get $a-be^{i\lambda} = ae^{i\lambda}-\bar{b}$ --------(5)

$$w-1 = e^{i\lambda}\left(\frac{az+b}{\bar{b}z+\bar{a}}\right)-1$$

$$= \frac{e^{i\lambda}az+be^{i\lambda}-\bar{b}z-\bar{a}}{\bar{b}z+\bar{a}}$$

$$= \frac{\left(ae^{i\lambda}-\bar{b}\right)z+\left(be^{i\lambda}-\bar{a}\right)}{\bar{b}z+\bar{a}}$$

$$= \frac{\left(ae^{i\lambda}-\bar{b}\right)z-\left(\bar{a}-be^{i\lambda}\right)}{\bar{b}z+\bar{a}}$$

$$= \frac{(z-1)\left(ae^{i\lambda}-\bar{b}\right)}{(\bar{b}z+\bar{a})} \qquad \text{(using (5))}$$

$$= \frac{(z-1)\left(ae^{i\lambda}-\bar{b}\right)}{\bar{b}+\bar{a}+(z-1)\bar{b}}$$

$$= \frac{(z-1)(\bar{a}+\bar{b})}{(\bar{a}+\bar{b})+(z-1)\bar{b}} \qquad \text{(using (4))}$$

$$\therefore \quad \frac{1}{w-1} = \frac{(\bar{a}+\bar{b})+(z-1)\bar{b}}{(z-1)(\bar{a}+\bar{b})}$$

$$= \frac{1}{z-1}+\frac{\bar{b}}{\bar{a}+b}$$

$$= \frac{1}{z-1}+\frac{1}{1+\left(\frac{\bar{a}}{\bar{b}}\right)}$$

$$= \frac{1}{z-1}+\frac{1}{K} \text{ where } K = 1+\frac{\bar{a}}{\bar{b}}$$

Exercise :

1. Prove that any bilinear transformation which maps the imaginary axis onto the unit circle $|w|=1$ can be written in the form $w = e^{i\lambda}\left(\frac{z-\alpha}{z+\bar{\alpha}}\right)$. Further this transformation maps the upper half plane Re $z \geq 0$ onto the unit circular disc $|w| \leq 1$ iff Re $\alpha > 0$.

2. Show that the bilinear transformation $w = \frac{1+z}{1-z}$ maps the region $|z| \leq 1$ onto the half plane Re $w \geq 0$.

COMPLEX INTEGRATION

We define the integral of a complex valued function defined on [a, b] and the integral of a function $f: D \rightarrow C$ where D is a region in C, along a curve C lying in D. We prove Cauchy's fundamental theorem and study the various consequences of this theorem.

Definite Integral :

We start with the definition of definite integral for a continuous complex valued function of a real variable.

Definition :

Let $f(t) = u(t)+iv(t)$ be a continuous complex valued function defined on [a, b].

We define $\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$

Properties of the definite integral

1. $$\text{Re}\int_a^b f(t)dt = \int_a^b \text{Re}[f(t)]dt$$

2. $$\text{Im}\int_a^b f(t)dt = \int_a^b \text{Im}[f(t)]dt$$

3. $$\int_a^b [f(t)+g(t)]dt = \int_a^b f(t)dt + \int_a^b g(t)dt$$

4. $$\int_a^b Cf(t)dt = C\int_a^b f(t)dt$$ where C is any complex constant.

Lemma :

$$\left|\int_a^b f(t)dt\right| = \int_a^b |f(t)|dt$$

Proof :

Let $$\int_a^b f(t)dt = re^{i\theta}$$

$$\left|\int_a^b f(t)dt\right| = r = e^{-i\theta}\int_a^b f(t)dt$$

$$= \operatorname{Re}\left(e^{-i\theta}\int_a^b f(t)dt\right) \qquad \text{(since r is real)}$$

$$= \operatorname{Re}\int_a^b e^{-i\theta} f(t)dt \qquad \text{(using 4}^{th}\text{ property)}$$

$$= \int_a^b \operatorname{Re}\left(\left[e^{-i\theta} f(t)\right]dt\right) \qquad \text{(using (1))}$$

$$\leq \int_a^b \left|e^{-i\theta} f(t)\right|dt$$

$$= \int_a^b \left|e^{-i\theta}\right|\left|f(t)\right|dt$$

$$= \int_a^b |f(t)|dt$$

Thus $$\left|\int_a^b |f(t)|dt\right| \leq \int_a^b |f(t)|dt$$

Definition :

Let C be a piecewise differentiable curve given by the equation z=z(t) where $a \leq t \leq b$. Let f(z) be a continuous complex valued function defined in a region containing the curve C. We define $\int_C f(z)dz = \int_a^b f(z(t))\, z'(t)dt$

Example 1 :

Consider $\int_C f(z)dz$ where $f(z) = \frac{1}{z}$ and C is the circle $|z| = r$ described in the positive sense. The parametric equation of the circle $|z|=r$ is given by $z=r\, e^{it}$ where $0 \leq t \leq 2\pi$ and $z'(t) = ir\, e^{it}$.

$$\therefore \quad \int_C f(z)dz = \int_C \frac{dz}{z} = \int_0^{2\pi} \frac{ire^{it}}{re^{it}}dt$$

$$= i\int_0^{2\pi} dt = 2\pi i.$$

Note : In general $\int_C \frac{dz}{z-a} = 2\pi i$ where C is the circle with centre a radius r given by the equation $z=a+re^{it}$, $0 \le t \le 2\pi$.

Theorem :

$$\int_{-C} f(z)dz = -\int_C f(z)dz$$

Proof :

Suppose the equation of C is given by $z=z(t)$ where $a \le t \le b$. We know that the equation of $-C$ is given by

$$z(t) = z(b+a-t) \text{ where } a \le t \le b.$$

$$\int_{-C} f(z)dz = \int_a^b f(z(b+a-t))z'(b+a-t)(-dt)$$

Put $\quad b+a-t = u.$

Then $\quad -dt = du$

$t = a \Rightarrow u = b$ and $t = b \Rightarrow u = a$

$$\therefore \quad \int_{-C} f(z)dz = \int_a^b f(z(u))\, z'(u)du$$

$$= -\int_a^b f(z(u))\, z'(u)du$$

$$= -\int_C f(z)dz$$

Remark

1. Let α be a complex constant.

 Then $\quad \int_C \alpha f(z)dz = \alpha \int_C f(z)dz$

2. $$\int_C [f(z)+g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$$

Definition :

Let C_1 be a differentiable curve with origin z_1 and terminus z_2. Let C_2 be another differentiable curve with origin z_2 and terminus z_3. Then the curve C which consists of C_1 followed by C_2 is a **piecewise differentiable curve** with origin z_1 and terminus z_3. This curve is denoted by C_1+C_2.

Note :

If $C = C_1+C_2$ then $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$

In general if $C = C_1+C_2+......+C_n$ then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + ... + \int_{C_n} f(z)dz$$

Definition :

Let C be a piecewise differentiable curve given by the equation $z = z(t)$ where $a \leq t \leq b$. Then the length l of C is defined by

$$l = \int_a^b |z'(t)|dt$$

Example :

Consider the circle C with centre a and radius r. We know the parametric equation of C is given by $z = a+re^{it}$ where $0 \leq t \leq 2\pi$.

$$z'(t) = ire^{it}$$

$$t = \int_0^{2\pi} |z'(t)|dt$$

$$= \int_0^{2\pi} \left|ire^{it}\right|dt = \int_0^{2\pi} r\,dt$$

$$= 2\pi r$$

Theorem :

$\left|\int_C f(z)dz\right| \leq M\, l$ where $M = \max\{|f(z)|/z \in C\}$ and l is the length of C.

Proof :

Suppose C is given by the equation $z = z(t)$ where $a \leq t \leq b$.

By definition of M we have $|f(z(t))| \leq M$ for all t; $a \leq t \leq b$ -------(1)

$$\left|\int_C f(z)dz\right| = \left|\int_a^b f(z(t))z'(t)dt\right|$$

$$\leq \int_a^b |f(z(t))z'(t)|dt$$

$$= \int_a^b |f(z(t))||z'(t)|dt$$

$$\leq \int_a^b M|z'(t)|dt \quad \text{(using (1))}$$

$$= M\int_a^b |z'(t)|dt$$

$$= Ml$$

$$\therefore \quad \left|\int_C f(z)dz\right| \leq Ml.$$

Worked Examples :

Example 1 :

Evaluate $\int_C f(z)dz$ where $f(z) = y-x-i3x^2$ and C is the line segment from z=0 to z=1+i.

Solution:

The equation of the line segment C joining z=0 and z=1+i is given by y=x.

$\therefore$ The parametric equation of C can be taken as x=t and y=t where $0 \leq t \leq 1$.

Hence $z(t) = x(t)+iy(t)$

$= t+it$ so that $z'(t) = 1+i$.

$f(z(t)) = t-t-i3t^2 = -i3t^2$

$$\therefore \quad \int_C f(z)dz = \int_0^1 f(z(t))z'(t)dt$$

$$= \int_0^1 -i3t^2(1+i)dt$$

$$= -3i(1+i)\left(\frac{t^3}{3}\right)_0^1$$

$$= 1-i.$$

Example 2 :

Prove that $\int_C \frac{dz}{(z-a)^n} = \begin{cases} 0 & \text{if } n \neq 1 \\ 2\pi i & \text{if } n = 1 \end{cases}$ where C is a circle with centre a and radius r and $n \in Z$.

Solution :

The parametric equation of the circle C is given by $z-a = re^{it}$, $0 \le t \le 2\pi$.

$$z'(t) = ire^{it}$$

$$\int_C \frac{dz}{(z-a)^n} = \int_0^{2\pi} \frac{ire^{it}}{\left(re^{it}\right)^n} dt$$

$$= \frac{i}{r^{n-1}} \int_0^{2\pi} e^{i(1-n)t} dt$$

$$= \frac{i}{r^{n-1}} \left[\frac{e^{i(1-n)t}}{i(1-n)}\right]_0^{2\pi} \quad \text{when } n \neq 1$$

$$= \frac{1}{(1-n)r^{n-1}} \left[e^{i(1-n)2\pi} - 1\right]$$

$$= \frac{1}{(1-n)r^{n-1}} (1-1) = 0$$

If $n = 1$, $\quad \int_C \frac{dz}{z-a} = 2\pi i.$

Hence the result.

Example 3 :

Evaluate $\int_C \bar{z}dz$ from z=0 to z=4+2i along the curve C consisting of the line segment from z=0 to z=2i followed by the line segment from z=2i to z=4+2i.

Solution :

Let C_1 denote the line segment joining 0 to 2i and C_2 denote the line segment joining 2i to 4+2i. Then $C = C_1+C_2$.

The parametric equation of C_1 is given by x(t) = 0 and y(t) = t where $0 \le t \le 2$.

Hence z(t) = x(t)+iy(t) = it so that z'(t) = i.

Hence $\int_{C_1} \bar{z}dz = \int_0^2 (-it)i\, dt = \int_0^2 t\, dt = 2$

The parametric equation of C_2 is given by x(t) = t and y(t) = 2 where $0 \le t \le 4$.

Hence $\quad z(t) = t+2i$ and $z'(t) = 1$.

$$\therefore \int_{C_2} \bar{z}dz = \int_0^4 (t-2i)dt = \left[\frac{t^2}{2} - 2it\right]_0^4 = 8-8i$$

$$\int_C \bar{z}dz = \int_{C_1} \bar{z}dz + \int_{C_2} \bar{z}dz = 2+8-8i = 10-8i.$$

Example 4 :

Evaluate the integral $\int_C (x^2 - iy^2)dz$ where C is the parabola $y = 2x^2$ from (1, 2) to (2, 8).

Solution :

Let $f(z) = x^2 - iy^2$. The parametric equation of C is given by x=t and $y=2t^2$ where $1 \le t \le 2$.

$z(t) = x(t)+iy(t) = t+i2t^2$ and $z'(t) = 1+4it$

$$\int_C \left(x^2 - iy^2\right)dz = \int_1^2 \left(t^2 - 4it^4\right)(1+4it)dt$$

$$= \int_1^2 \left[\left(t^2 + 16t^5\right) + i\left(4t^3 - 4t^4\right)\right]dt$$

$$= \left[\left(\frac{t^3}{3} + \frac{16t^6}{6}\right) + i\left(t^4 - \frac{4t^5}{5}\right)\right]_1^2$$

$$= \frac{511}{3} - \frac{49}{5}i$$

Example 5 :

Evaluate $\int_C \frac{z+2}{z}dz$ where C is the semi circle $z=2e^{i\theta}$ where $0 \le \theta \le \pi$.

Solution :

$z'(\theta) = 2ie^{i\theta}$ so that $dz = 2ie^{i\theta}\, d\theta$

$$\int_C \frac{z+2}{z}dz = \int_0^\pi \left(\frac{2e^{i\theta}+2}{2e^{i\theta}}\right)\left(2ie^{i\theta}\right)d\theta$$

$$= 2i\int_0^\pi \left(1+e^{i\theta}\right)d\theta$$

$$= 2i\left[\theta + \frac{e^{i\theta}}{i}\right]_0^\pi$$

$$= 2i\left[\left(\pi - \frac{1}{i}\right) - \left(\frac{1}{i}\right)\right]$$

$$= 2i\left(\frac{\pi i - 2}{i}\right)$$

$$= -4+2\pi i.$$

Exercise :

1. Evalute $\int_C xdz$ where C is the circle $|z| = r$.

2. Find the vale of the integral $\int_{(0,0)}^{1+i} \left(x - y + ix^2\right)dz$ along the straight line from z=0 to z = 1+i.

3. Show that $\int_C xdz = \frac{i\pi}{2}$ and $\int_C ydz = -\frac{\pi}{2}$ where C is the semicircle $|z|=1$ and $0 \le \arg z \le \pi$ with initial point z=1.

4. Evaluate $\int_C \frac{z+2}{z}$ where C is the circle $z=2e^{i\theta}$ where $-\pi \le \theta \le \pi$.

5. Evaluate $\int_C z^2 dz$ along C where C is the segment joining the points (1, 1) and (2, 4).

CAUCHY'S THEOREM

Definition :

Let p(x, y) and q(x, y) be two real valued functions. Then the differential equation p(x, y)dx+q(x,y)dy = 0 is said to be exact if there exists a function u(x,y) such that $\frac{\partial u}{\partial x} = p$ and $\frac{\partial u}{\partial y} = q$.

We assume the following theorem without proof.

Theorem :

$\int_C pdx + qdy$ depends only on the end points of C if and only if the integrand is exact.

Note : The above theorem is true if p and q are compelx valued functions as well.

Theorem :

Let f(z) be a continuous complex valued function defined on a region D. Then $\int_C f(z)dz$ depends only on the end points of C if and onlyif there exists an analytic function F(z) such that F'(z) = f(z) in D.

Proof :

$$\int_C f(z)dz = \int_C f(z)(dx+idy) \quad \text{(since } z = x+iy)$$

$$= \int_C f(z)dx + if(z)dy$$

$\int_C f(z)dz$ depends only on the end points of C if and only if there exists a function F(z) defined on D such that $\frac{\partial F}{\partial x} = f(z)$ and $\frac{\partial F}{\partial y} = if(z)$.

$\therefore \frac{\partial F}{\partial x} = \frac{1}{i}\frac{\partial F}{\partial y}$ so that $\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$ which is the complex form of the Cauchy-Riemann equation for F(z).

Since f(z) is continuous the partial derivatives of F(z) are also continuous and hence F(z) is analytic in D and F'(z) = f(z).

Hence the theorem.

Corollary 1 :

Let f(z) be a continuous complex valued function defined on a region D then $\int_C f(z)dz = 0$ for every closed curve C lying in D iff there exists an analytic function F(z) such that F'(z) = f(z) in D.

Corollary 2 :

$\int_C (z-a)^n dz = 0$ for every closed curve C provided $n \geq 0$.

Proof :

Let
$$F(z) = \frac{(z-a)^{n+1}}{n+1}$$

$$F'(z) = (z-a)^n = f(z)$$

$\therefore$ By corollary (1), $\int_C f(z)dz = 0$

Hence $\int_C (z-a)^n dz = 0$ for all $n \geq 0$.

Lemma :

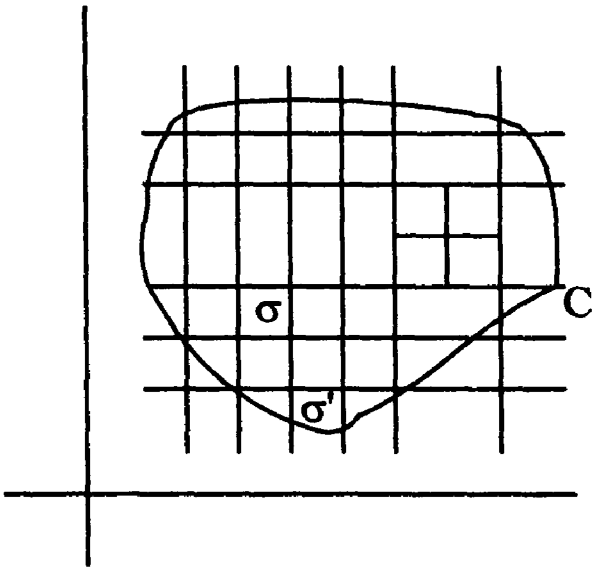
Let C be a simple closed curve. Let D denote the closed region consisting of all points interior to C together with the points on C. Let f be a function analytic in D. Then given $\in>0$ it is possible to cover D with a finite number of squares and partial squares whose boundaries are denoted by C_j such that there exists a point z_j lying inside or on each C_j satisfying

$$\left|\frac{f(z)-f(z_j)}{z-z_j}-f'(z_j)\right|<\in \quad (j=1,2,\ldots\ldots,n) \qquad \text{-------(1)}$$

for all points z distinct from each z_j and lying inside or on C_j.

Proof :

We subdivide the region D into squares and partial squares by drawing equally spaced lines parallel to the coordinate axes. A square is a closed region consisting of all points on and interior to it. If a particular square contains points which are not in D we remove those points and call what remains a partial square. Here σ is a square and σ' is a partial square. This gives a finite number of squares and partial squares which cover the region D.



Suppose the Lemma is false. Then in the covering constructed as above there exists a subregion with boundary C_j such that no point z_j exists satisfying (1)

Let σ_0 denote that subregion if it is a square. If it is a partial square let σ_0 denote the entire square of which it is a part.

We now subdivide σ_0 into four smaller squares by drawing line segments joining the mid points of the opposite sides. At least one of the four smaller squares say σ_1 is such that σ_1 contains points of D and no point z_j satisfying (1) exists.

Continuing this process we obtain a nested infinite sequence of squares σ_1, $\sigma_2, \ldots\ldots, \sigma_n, \ldots$ such that for each σ_n no z_j satisfying (1) exists.

Now there exists a point z_0 common to each σ_n such that for any $\delta>0$ the neighbourhood $|z-z_0|<\delta$ contains all the squares σ_n for all sufficiently large values of n.

Hence every neighbourhood of z_0 contains points of D distinct from z_0. Hence z_0 is a limit point of D. Since D is closed $z_0 \in D$.

Since f(z) is analytic at z_0 there exists $\delta>0$ such that

$$|z-z_0|<\delta \Rightarrow \left|\frac{f(z)-f(z_0)}{z-z_0}-f'(z_0)\right|<\epsilon \qquad \text{--------(2)}$$

Choose N such that th square σ_N is contained in the neighbourhood $|z-z_0|<\delta$. Then for every point z in σ_N(2) holds.

∴ z_0 serves as the point zj stated in the lemma. This is a contradiction since there is no z_j in σ_N satisfying (1).

This contradiction proves the lemma.

Theorem : Cauchy's Theorem

Let f be a function which is analytic at all points inside and on a simple closed curve C. Then $\int_C f(z)dz = 0$.

Proof :

Let D be the closed region consisting of all points interior to C together with the points on C.

Let $\epsilon>0$ be given.

Let C_j ($j = 1,2,\ldots\ldots,n$) denote the boundaries of the squares and partial squares covering D such that there exists a point z_j lying inside or on C_j satisfying

$$\left|\frac{f(z)-f(z_j)}{z-z_j}-f'(s_j)\right| < \epsilon \qquad \text{--------(1)}$$

for all z distinct from z_j and lying within or on C_j.

Let $$\delta_j(z) = \begin{cases} \dfrac{f(z)-f(z_j)}{z-z_j} - f'(z_j) & \text{if } z \neq z_j \\ 0 & \text{if } z = z_j \end{cases}$$

Clearly $\delta_j(z)$ is a continuous function and

$$f(z) = f(z_j) - z_j f'(z_j) + zf'(z_j) + (z-z_j)\delta_j(z)$$

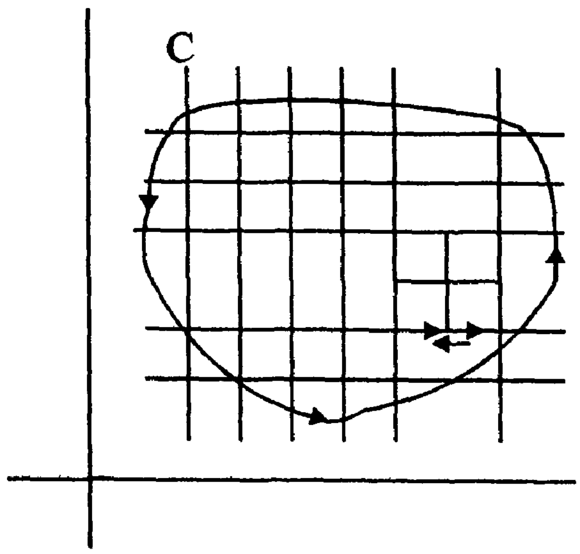
$$\int_{C_j} f(z)dz = \int_{C_j} f(z_j)dz - \int_{C_j} z_j f'(z_j)dz + \int_{C_j} zf'(z_j)dz + \int_{C_j} (z-z_j)\delta_j(z)dz$$

$$= f(z_j)\int_{C_j} dz - z_j f'(z_j)\int_{C_j} dz + f'(z_j)\int_{C_j} zdz + \int_{C_j} (z-z_j)\delta_j(z)dz$$

$$= \int_{C_j} (z-z_j)\delta_j(z)dz \qquad \left(\text{since } \int_{C_j} dz = 0 \text{ and } \int_{C_j} zdz = 0\right)$$

$$\therefore \sum_{j=1}^{n} \int_{C_j} f(z)dz = \sum_{j=1}^{n} \int_{C_j} (z-z_j)\delta_j(z)dz \qquad \text{--------(2)}$$

In the sum $\sum_{j=1}^{n} \int_{C_j} f(z)dz$ the integrals along the common boundary of every pair of adjacent subregions cancel each other. Since the integral is taken in one direction along that line segment in one subregion and in the opposite direction in the other.



Hence only the integrals along the arcs which are the parts of C remain.

$$\therefore \quad \sum_{j=1}^{n} \int_{C_j} f(z)dz = \int_C f(z)dz$$

$$\therefore \text{ From (2)} \quad \int_C f(z)dz = \sum_{j=1}^{n} \int_{C_j} (z-z_j)\delta_j(z)dz$$

$$\therefore \quad \left|\int_C f(z)dz\right| = \left|\sum_{j=1}^{n} \int_{C_j} (z-z_j)\delta_j(z)dz\right|$$

$$\leq \sum_{j=1}^{n} \int_{C_j} \left|(z-z_j)\delta_j(z)\right|dz$$

$$= \sum_{j=1}^{n} \int_{C_j} |z-z_j|\left|\delta_j(z)\right|dz$$

$$\therefore \quad \left|\int_C f(z)dz\right| \leq \sum_{j=1}^{n} \int_{C_j} |z-z_j|\left|\delta_j(z)\right|dz \qquad \text{-------(3)}$$

If C_j is a square and s_j is the length of its side then $|z-z_j|<\sqrt{2}\, s_j$ for all z on C_j. Also from (1) we have $\left|\delta_j(z)\right| < \epsilon$ and hence

$$\int_{C_j} |z-z_j|\left|\delta_j(z)\right|dz < \sqrt{2}(s_j)(\epsilon)(4s_j)$$

$$= 4\sqrt{2}\, A_j \, \epsilon \text{ where } A_j \text{ is the area of the square } C_j.$$

Similarly for a partial square with boundary C_j if l_j is the length of the arc of C which forms a part of C_j, we have

$$\int_{C_j} |z-z_j|\left|\delta_j(z)\right|dz < \sqrt{2}s_j \, \epsilon\left(4s_j + l_j\right)$$

$$< 4\sqrt{2}A_j \, \epsilon + \sqrt{2}Sl_j \qquad \text{--------(5)}$$

where S is the length of a side of some square containing the entire region D as well as all the squares originally used in covering D.

We observe that the sum of all the A_j's that occur in the right hand side of (4) and (5) do not exceed S^2 and the sum of all the l_j's is equal to L (the length of C).

Using (4) and (5) in (3) we obtain

$$\left|\int_C f(z)dz\right| < \left(4\sqrt{2}S^2 + \sqrt{2}SL\right)\epsilon$$

$$= K\epsilon \text{ where } K = 4\sqrt{2}S^2 + \sqrt{2}SL \text{ is a constant.}$$

Thus $$\left|\int_C f(z)dz\right| < K\epsilon$$

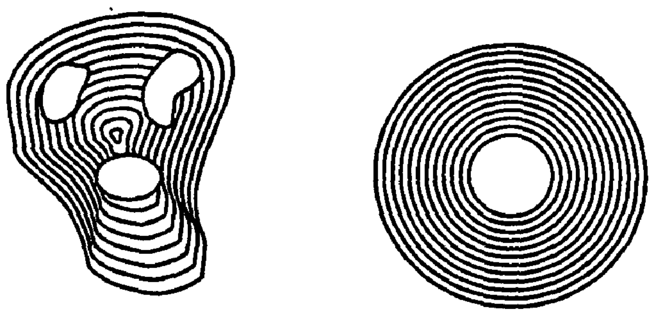
Since ϵ is arbitrary we have $\int_C f(z)dz = 0$.

Definition :

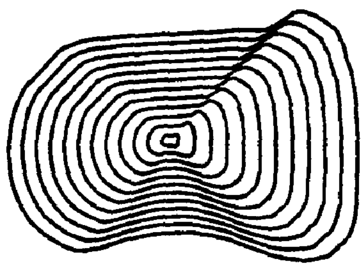
A region D is said to be **simply connected** if every simple closed curve lying in D encloses only points of D.

For example the interior of a simple closed curve is a simply connected region. The annular region enclosed by two concentric circle is not simply connected.

A region which is not a simply connected is said to be a **multiply connected region.**



Multiply connected region



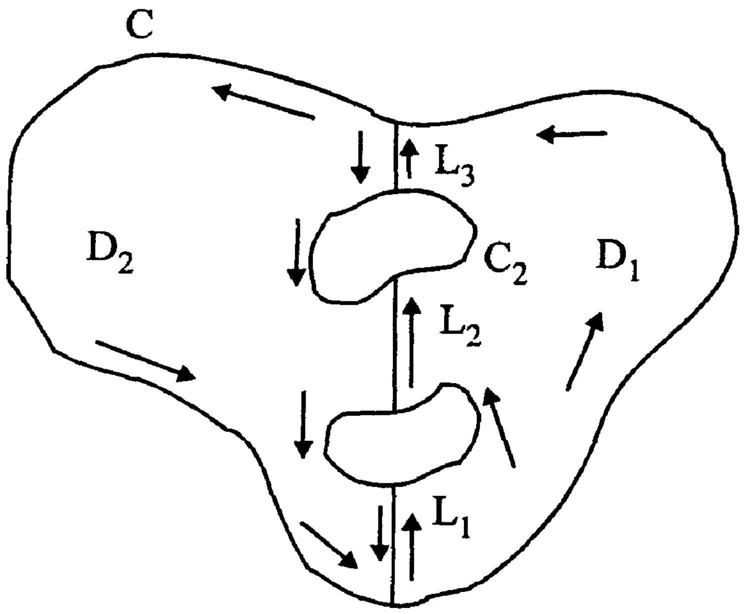
Simply connected region

Cauchy's Theorem for multiply connected regions

Let C be a simple closed curve. Let C_j(j =1,2,.....,n) be a finite number of simple closed curves lying in the interior of C such that the intriors of C_j's are disjoint. Let D be the closed region consisting of all points within and on C except the points interior to each C_j. Let B denote the entire oriented boundary of D consisting of C and all the C_j described in a direction such that the points of D are to the left of B. Let f be a function which is analytic in D. Then $\int_B f(z)dz = 0$.

Proof :

Let L_1 be a polygonal path joining a point of C to a point of C_1; L_2 a polygonal path joining a point of C_1 to a piont of C_2:........: L_i a polygonal path joining a point of C_{i-1} to a point of C_i and L_{n+1} a polygonal path joining a point of C_n to a point of C such that no two L_i's cross each other.



This divides the region D into two simply connected regions D_1 and D_2. Let B_1 and B_2 denote the boundaries of D_1 and D_2 respectively.

By Cauchy's theorem for simply connected region $\int_{B_1} f(z)dz = 0$ and $\int_{B_2} f(z)dz = 0$.

Also $\int_{B_1} f(z)dz + \int_{B_2} f(z)dz = \int_{B} f(z)dz$ since the integrals along L_j are taken twice in the opposite directions and cancel each other.

$$\therefore \quad \int_{B} f(z)dz = 0$$

We observe that $B = C - C_1 - C_2 - \ldots\ldots - C_n$ and hence the above theorem can also be written in the form

$$\int_{C} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots\ldots + \int_{C_n} f(z)dz$$

In particular if C is a simple closed curve and C_0 is another simple closed curve lying in the interior of C and f is analytic in the region D consisting of all points inside and on C excluding the points interior to C_0 then $\int_{C} f(z)dz = \int_{C_0} f(z)dz$.

CAUCHY'S INTEGRAL FORMULA

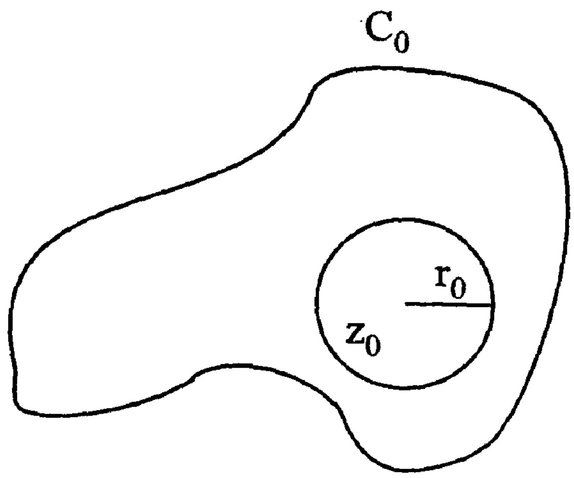
Theorem :

Let f(z) be a function which is analytic inside and on a simple closed curve C. Let z_0 be any point in the interior of C.

Then $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$

Proof :

Choose a circle C_0 with centre z_0 and radius r_0 such that C_0 lies in the interior of C.



Z_0 is the only point inside C at which the function $\frac{f(z)}{z-z_0}$ is not analytic and hence is analytic in the region D consisting of all points inside and on C except the points interior to C_0.

Hence
$$\int_C \frac{f(z)dz}{z-z_0} = \int_{C_0} \frac{f(z)dz}{z-z_0}$$

$$= \int_{C_0}\left(\frac{f(z)-f(z_0)+f(z_0)}{z-z_0}\right)dz$$

$$= \int_{C_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)dz + \int_{C_0}\frac{f(z_0)}{z-z_0}dz$$

$$= \int_{C_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)dz + f(z_0)\int_{C_0}\frac{dz}{z-z_0}$$

$$= \int_{C_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) dz + f(z_0) 2\pi i$$

Thus $$\int_C \frac{f(z)dz}{z - z_0} = \int_{C_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) dz + 2\pi i\, f(z_0)$$

We claim that $\int_{C_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) dz = 0.$

Since f(z) is analytic inside and on C it is continuous at z_0.

∴ Given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$$

If we choose $r_0 < \delta$, then $|z - z_0| < r_0 \Rightarrow |f(z) - f(z_0)| < \epsilon$.

Hence $$\left| \int_{C_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) dz \right| < \left(\frac{\epsilon}{r_0} \right) (2\pi r_0)$$

$$= 2\pi \epsilon$$

So $$\left| \int_{C_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) dz \right| < 2\pi \epsilon.$$

Since ϵ is arbitrary we have $\int_{C_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) dz = 0.$

∴ From (1) we get $\int_C \frac{f(z)}{z - z_0} dz = 2\pi i\, f(z_0)$

∴ $$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

Theorem :

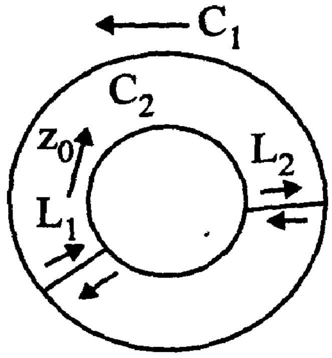
Let f(z) be analytic in a region D bounded by two concentric circles C_1 and C_2 and on the boundary. Let z_0 be any point in D. Then

$$f(z_0) = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)dz}{z - z_0} - \frac{1}{2\pi i} \int_{C_2} \frac{f(z)dz}{z - z_0}$$

Proof :

Let L_1 and L_2 be two disjoint line segments not passing through z_0 both joining a point of C_1 to a point of C_2. This divides the region D into two simply connected region D_1 and D_2. Let B_1 and B_2 denote the oriented boundary of D_1 and D_2 respectively. Then $B_1+B_2 = C_1-C_2$ ---------(1)

We assume without loss of generality that $z_0 \in D_1$.



By Cauchy's integral formula,

$$\frac{1}{2\pi i}\int_{B_1}\frac{f(z)dz}{z-z_0} = f(z_0) \qquad \text{---------(2)}$$

Also $\frac{f(z)}{z-z_0}$ is analytc in D_2 and hence by Cauchy's theorem

$$\frac{1}{2\pi i}\int_{B_1}\frac{f(z)}{z-z_0}dz = 0 \qquad \text{---------(3)}$$

Adding (2) and (3) and using (1) we get

$$f(z_0) = \frac{1}{2\pi i}\int_{C_1-C_2}\frac{f(z)}{z-z_0}dz$$

$$= \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}dz - \frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{z-z_0}dz$$

Example 1 :

Consider $\int_C \frac{dz}{z-3}$ C is the circle $|z-2| = 5$.

Let $f(z) = 1$.

The point z = 3 lies inside C.

Hence by Cauchy's integral formula

$$\int_C \frac{dz}{z-3} = 2\pi i\ f(3) = 2\pi i.$$

Example 2 :

Let C denote the unit circle |z|=1. Then $\int_C \frac{e^z dz}{z} = \int_C \frac{e^z}{z-0} dz = 2\pi i e^{o} = 2\pi i.$

THEOREM OF ARITHMETIC MEAN

Let f(z) be analytic inside and on the circle C with centre a and radius r. Then $f(a) = \frac{\int_0^l f(z) ds}{l}$ where s is the arc length and l is the circumference of the circle. i.e., The value of the function at the centre is equal to the mean of the value of the function on the circumference.

Proof :

By Cauchy's integral formula we have

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

The equation of the circle C is given by $z=a+re^{i\theta}$ where where $0 \le \theta \le 2\pi$.

$$\therefore \quad dz = ire^{i\theta} d\theta$$

$$\therefore \quad f(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f\left(a+re^{i\theta}\right)}{re^{i\theta}} \left(ire^{i\theta} d\theta\right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f\left(a+re^{i\theta}\right) d\theta$$

Also we have s = rθ and s varies from 0 to l.

$$\therefore \quad d\theta = \frac{ds}{r}$$

$$\therefore \quad f(a) = \frac{1}{2\pi r}\int_0^{l} f\left(a+re^{i\theta}\right)ds$$

$$= \frac{1}{l}\int_0^{l} f(z)ds$$

Hence the theorem.

MAXIMUM MODULUS THEOREM

Let f(z) be continuous in a closed and bounded region D and analytic and nonconstant in the interior of D. Then |f(z)| attains its maximum value on the boundary of D and never inthe interior of D.

Proof :

Since f is continuous in a closed and bounded region D, |f(z)| is bounded and attain its bound.

$\therefore$ There exists a positive real number M such that

$$|f(z)| \le M \text{ for all } z\in D \qquad \text{--------(1)}$$

and equality holds for at least one point z in D. Suppose that there exists an interior point $z_0 \in D$ such that

$$|f(z_0)| = M \qquad \text{--------(2)}$$

Choose a circle with centre z_0 and radius r such that the circular disc $|z-z_0|\le r$ is contained in D. Then we have

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0+re^{i\theta}\right)e^{i\theta}d\theta$$

$$\therefore \quad |f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi}\left|f\left(z_0+re^{i\theta}\right)\right|d\theta \qquad \text{--------(3)}$$

Also from (1) and (2) we have $|f(z_0+re^{i\theta})| \le |f(z_0)|$

$$\therefore \quad \int_0^{2\pi}\left|f\left(z_0+re^{i\theta}\right)\right|d\theta \le 2\pi|f(z_0)|$$

$$\therefore \quad f(z_0) \ge \int_0^{2\pi}\left|f\left(z_0+re^{i\theta}\right)\right|d\theta \qquad \text{--------(4)}$$

From (3) and (4) we get

$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi}\left|f\left(z_0 + re^{i\theta}\right)\right|d\theta$$

$$\therefore \quad 2\pi|f(z_0)| = \int_0^{2\pi}\left|f\left(z_0 + re^{i\theta}\right)\right|d\theta$$

$$\int_0^{2\pi}|f(z_0)|d\theta = \int_0^{2\pi}\left|f\left(z_0 + re^{i\theta}\right)\right|d\theta$$

$$\therefore \int_0^{2\pi}\left[|f(z_0)| - \left|f\left(z_0 + re^{i\theta}\right)\right|\right]d\theta = 0$$

Since the integrand in the above expression is continuous and non negative we have

$$|f(z_0)| - |f(z_0+re^{i\theta})| = 0$$

i.e., $|f(z_0)| = |f(z_0+re^{i\theta})|$ for all z in the circular disc $|z-z_0|<r$.

i.e., $|f(z_0)| = |f(z)|$ for all z in the circular disc.

$\therefore$ f(z) is constant in a neighbourhood of z_0.

Since f(z) is continuous it follows that f(z) is constant throughout D which is a contradiction.

$\therefore$ The maximum of $|f(z)|$ is not attained at any of the interior points of D.

Hence the theorem.

Worked Examples :

Example 1 :

Evaluate $\int_C \frac{z\,dz}{z^2-1}$ where C is the positiely oriented circle $|z| = 2$.

Solution :

$$\frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)}$$

$$= \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$$

$$\therefore \quad \int_C \frac{z}{z^2-1}dz = \frac{1}{2}\int_C \frac{z}{z-1}dz - \frac{1}{2}\int_C \frac{zdz}{z+1}$$

f(z) = z is analytic and 1, –1 lie inthe interior of C.

∴ By Cauchy's integral formula

$$\int_C \frac{zdz}{z-1} = 2\pi i(f(1)) = 2\pi i$$

$$\int_C \frac{zdz}{z+1} = 2\pi i(f(-1)) = -2\pi i$$

$$\therefore \quad \int_C \frac{zdz}{z^2-1} = \frac{1}{2}(2\pi i) - \frac{1}{2}(-2\pi i) = 2\pi i$$

Example 2 :

Evaluate $\int_C \frac{(\sin\pi z^2 + \cos\pi z^2)}{(z-1)(z-2)}dz$ where C is the circle $|z| = 3$.

Solution :

By partial fractions $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

Let $f(z) = \sin \pi z^2 + \cos \pi z^2$

Then f(z) is analytic inside and on C and the points 1 and 2 lie inside C.

Hence by Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-1}dz = 2\pi i(f(1))$$

$$= 2\pi i(\sin \pi + \cos \pi)$$

$$= -2\pi i$$

Similarly $\int_C \frac{f(z)}{z-2}dz = 2\pi i f(2)$

$$= 2\pi i\,(\cos 4\pi + i \sin 4\pi)$$

$$= 2\pi i$$

Hence $\int_C \frac{f(z)}{(z-1)(z-2)}dz = 2\pi i - (-2\pi i) = 4\pi i$

Example 3 :

Let C denote the boundary of the square whose sides lie along the lines x = ±2 and y = ±2 where C is described in the positive sense. Evaluate $\int_C \frac{\cos z}{z(z^2+8)}dz$

Solution :

Let $$f(z) = \frac{\cos z}{z^2+8}$$

The points where f(z) is not analytic are $\pm i2\sqrt{2}$ and these points lie outside C. Hence f(z) is analytic inside and on C.

∴ By Cauchy's integral formula.

$$\int_C \frac{\cos z}{z(z^2+8)}dz = \int_C \frac{f(z)}{z}dz = 2\pi i f(0)$$

$$= 2\pi i\left(\frac{1}{8}\right) = \frac{\pi i}{4}$$

Example 4 :

Evaluate $\int_C \frac{zdz}{(9-z^2)(z+i)}$ where C is the circle |z| = 2 taken in the positive sense.

Solution :

Let $$f(z) = \frac{z}{9-z^2}$$

Clearly f(z) is analytic within and on C.

∴ By Cauchy's integral formula

$$\int_C \frac{zdz}{(9-z^2)(z+i)} = \int_C \frac{f(z)}{z+i}dz$$

$$= 2\pi i f(-i)$$

$$= 2\pi i\left(-\frac{i}{10}\right) = \frac{\pi}{5}$$

Exercise :

1. Prove that $\int_C \frac{zdz}{z^2-1} = 2\pi i$ where C is the positively oriented circle $|z| = 2$.

2. Show that $\frac{1}{2\pi i}\int_C \frac{e^{zt}dz}{z^2+1} = \sin t$ if $t>0$ and C is the circle $|z| = 3$.

3. Evaluate $\int_C \frac{\sin 3z}{z+\frac{\pi}{2}}dz$ where C is the circle $|z| = 5$.

4. Evaluate $\int_C \frac{3z-1}{z^3-z}dz$ where C is (i) $|z| = \frac{1}{2}$, (ii) $|z| = 2$.

5. Prove $\int_C \frac{dz}{z^2+1} = 0$ where C is the positively oriented circle $|z| = 2$.

HIGHER DERIVATIVES

Here we shall prove that an analytic function has derivatives of all orders. It follows that the derivative of an analytic function is again an analytic function.

Consider a function f(z) which is analytic in a region D. Let $z \in D$. Let C be any circle with centre z such that the circle and its interior is contained in D. By Cauchy's integral formula we have $f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta$

We now prove that $f'(z)=\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^2}d\zeta$ and in general $f^{(n)}(z)=\frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta$

Theorem :

Let f be analytic inside and on a simple closed curve C. Let z be any point inside C. Then $f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^2}d\zeta$.

Proof :

By Cauchy's integral formula we have $f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta$.

$$\therefore \quad \frac{f(z+h)-f(z)}{h} = \frac{1}{h(2\pi i)}\int_C\left(\frac{f(\zeta)}{\zeta-z-h}-\frac{f(\zeta)}{\zeta-z}\right)d\zeta$$

$$= \frac{1}{h2\pi i}\int_C\left[\frac{hf(\zeta)}{(\zeta-z-h)(\zeta-z)}\right]d\zeta$$

$$= \frac{1}{2\pi i}\int_C\frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)} \qquad \text{-------(1)}$$

Now $$\int_C\frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)}-\int_C\frac{f(\zeta)d\zeta}{(\zeta-z)^2} = \int_C\left[\frac{f(\zeta)}{(\zeta-z-h)(\zeta-z)}-\frac{f(\zeta)}{(\zeta-z)^2}\right]d\zeta$$

$$= \int_C\frac{f(\zeta)}{(\zeta-z)}\left(\frac{1}{\zeta-z-h}-\frac{1}{\zeta-z}\right)d\zeta$$

$$= \int_C\frac{f(\zeta)}{(\zeta-z)}\left[\frac{h}{(\zeta-z-h)(\zeta-z)}\right]d\zeta$$

$$= h\int_C\frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)^2}$$

$$\therefore \frac{1}{2\pi i}\int_C\frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-h)}-\frac{1}{2\pi i}\int_C\frac{f(\zeta)d\zeta}{(\zeta-z)^2} = \frac{h}{2\pi i}\int_C\frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)^2}$$

$$\therefore \frac{f(z+h)-f(z)}{h}-\frac{1}{2\pi i}\int_C\frac{f(\zeta)d\zeta}{(\zeta-z)^2} = \frac{h}{2\pi i}\int_C\frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)^2} \quad \text{(using (1)) ----(2)}$$

Let M denote the maximum value of $|f(\zeta)|$ on C. Let L be the length of C and d be the shortest distance from z to any point on the curve C.

$\therefore$ For any point ζ on C we have

$|\zeta-z|\geq d$ and $|\zeta-z-h|\geq|\zeta-z|-|h|\geq d-|h|$

Hence $$\left|\frac{f(\zeta)}{(\zeta-z)^2(\zeta-z-h)}\right| \leq \frac{M}{d^2(d-|h|)}$$

From (2) we get

$$\left|\frac{f(z+h)-f(z)}{h}-\frac{1}{2\pi i}\int_C\frac{f(\zeta)d\zeta}{(\zeta-z)^2}\right| \leq \frac{|h|}{2\pi}\left(\frac{M}{d^2(d-|h|)}\right)$$

$$\therefore \quad \underset{h\to 0}{\text{Lt}}\left(\frac{f(z+h)-f(z)}{h}-\frac{1}{2\pi i}\int_C\frac{f(\zeta)d\zeta}{(\zeta-z)^2}\right) = 0$$

$$\therefore \quad \underset{h\to 0}{\text{Lt}}\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i}\int_C\frac{f(\zeta)d\zeta}{(\zeta-z)^2}$$

$$\therefore \quad f'(z) = \frac{1}{2\pi i}\int_C\frac{f(\zeta)d\zeta}{(\zeta-z)^2}$$

Remark :

By using induction on n we can prove that for any positive integer n we have

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta$$

Worked Examples

Example 1 :

$$\int_C\frac{e^z dz}{z^n} = \frac{2\pi i}{(n-1)!} \text{ where C is the circle } |z| = 1.$$

Solution :

Let $f(z) = e^z$. $f(z)$ is analytic and $f^{(n)}(z) = e^z$ for all n.

By the formula for higher derivatives

$$\int_C\frac{e^z}{z^n}dz = \int_C\frac{e^z}{(z-0)^n}dz = \frac{2\pi i}{(n-1)!}e^0 = \frac{2\pi i}{(n-1)!}$$

Example 2 :

$$\int_C\frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3}dz = \pi i \text{ where C is the circle } |z| = 1.$$

Solution :

Let $f(z) = \sin^2 z$

Then $f'(z) = 2\sin z\cos z = \sin 2z$

$f''(z) = 2\cos 2z$. Also $\frac{\pi}{6}$ lies inside C.

$$\therefore \int_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3}dz = \frac{2\pi i}{2!} f''\left(\frac{\pi}{6}\right)$$

$$= \pi i\left(2\cos\frac{\pi}{3}\right) = \pi i$$

CAUCHY'S INEQUALITY THEOREM

Let f(z) be analytic inside and on the circle C with centre z_0 and radius r. Let M denote the maximum of $|f(z)|$ on C. Then $\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{r^n}$

Proof :

We have $$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}$$

$$\therefore \left|f^{(n)}(z_0)\right| \leq \frac{n!}{2\pi}\left(\frac{M}{r^{n+1}}\right)(2\pi r) = \frac{n!M}{r^n}$$

Hence $$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{r^n}.$$

LIOUVILLE'S THEOREM

A bounded entire function in the complex plane is constant.

Proof :

Let f(z) be a bounded entire function.

Since f(z) is bounded there exists a real number M such that $|f(z)| \leq M$ for all z. Let z_0 be any complex number and $r \geq 0$ be any real number. By Cauchy's inequality we have $|f(z_0)| \leq \frac{M}{r}$.

Taking the limit as $r \to 0$ we get $f'(z_0) = 0$. Since z_0 is arbitrary $f'(z) = 0$ for all z in the complex plane.

∴ f(z) is a constant function.

FUNDAMENTAL THEOREM OF ALGEBRA

Every polynomial of degree ≥ 1 has atleast one zero (root) in C.

Proof :

Let f(z) be a polynomial of degree ≥ 1.

Suppose f(z) has no zero in C. Then $f(z) \neq 0$ for all z.

Further f(z) is an entire function in the complex plane.

∴ $\frac{1}{f(z)}$ is also an entire function.

Also as $z \to \infty$, $f(z) \to \infty$.

∴ $\frac{1}{f(z)} \to 0$ as $z \to \infty$.

∴ $\frac{1}{f(z)}$ is a bounded function.

Hence by Liouville's theorem $\frac{1}{f(z)}$ is a constant function.

∴ f(z) is a constant function and hence it is a polynomial of degree zero which is a contradiction.

Hence f(z) has atleast one root.

Hence the theorem.

MORERA'S THEOREM

If f(z) is continuous in a simply connected domain D and if $\int_C f(z)dz = 0$ for every simlpe closed curve C lying in D then f(z) is analytic in D.

(This theorem is the converse of Cauchy's theorem).

Proof :

There exists an analytic function F(z) such that F'(z) = f(z) in D.

Also we know the derivative of an analytic function is an analytic function.

∴ F'(z) = f(z) is analytic in D.

Hence f'(z) is analytic in D.

Worked Examples

Example 1 :

Evaluate $\int_C \frac{\sin z}{\left(z-\frac{\pi}{2}\right)^2} dz$ where C is the circle |z| = 2.

Solution :

Let f(z) = sin z. Hence f'(z) = cos z. Also π/2 lies inside |z|=2.

Hence $$\int_C \frac{\sin z\,dz}{\left(z-\frac{\pi}{2}\right)^2} = 2\pi i\, f'\left(\frac{\pi}{2}\right)$$

$$= 2\pi i\left(\cos\frac{\pi}{2}\right)$$

$$= 0$$

Example 2 :

Evaluate $\int_C \frac{z^3 dz}{(2z+i)^3}$ where C is the unit circle.

Solution :

$$\int_C \frac{z^3 dz}{(2z+i)^3} = \frac{1}{8}\int_C \frac{z^3\,dz}{\left(z+\frac{i}{2}\right)^3}$$

Let $f(z) = z^3$

Then $f'(z) = 3z^2$

$f''(z) = 6z$

Also $-i/2$ lies inside C.

Hence $$\int_C \frac{z^3\,dz}{(2z+i)^3} = \frac{1}{8}\frac{2\pi i}{2!}f''(-i)$$

$$= \frac{\pi i}{8}(-6i)$$

$$= \frac{3\pi}{4}$$

Example 3 :

Evaluate $\int_C \frac{(e^z + z\sinh z)}{(z-\pi i)^2}dz$ where C is the circle $|z| = 4$.

Solution :

Let $f(z) = e^z + z \sin h\, z$

∴ $f'(z) = e^z + z \cos h\, z + \sin h\, z$

Also πi lies inside C.

Hence $$\int_C \frac{f(z)}{(z-\pi i)^2}dz = 2\pi i\, f'(\pi i)$$

$$= 2\pi i[e^{\pi i} + \pi i \cosh \pi i + \sin h\, \pi i]$$

$$= 2\pi i(-1-\pi i)$$

$$= -2\pi i(1+\pi i)$$

Example 4 :

Show that when f is analytic within and on a simple closed curve C and z_0 is not on C then

$$\int_C \frac{f'(z)dz}{z-z_0} = \int_C \frac{f(z)dz}{(z-z_0)^2}$$

Solution :

Case (i) :

Suppose z_0 is in the exterior of C. Then both $\frac{f(z)}{(z-z_0)^2}$ and $\frac{f'(z)}{z-z_0}$ are analytic inside and on C.

∴ By Cauchy's theorem

$$\int_C \frac{f'(z)dz}{z-z_0} = \int_C \frac{f(z)dz}{(z-z_0)^2} = 0$$

Case (ii) :

z_0 lies in the interior of C.

Then by Cauchy's integral formula

$$\int_C \frac{f'(z)dz}{(z-z_0)} = 2\pi i\ f'(z_0)$$

Also by the formula for higher derivatives

$$\int_C \frac{f(z)}{(z-z_0)^2}dz = 2\pi i\ f'(z_0)$$

Hence $$\int_C \frac{f'(z)}{z-z_0}dz = \int_C \frac{f(z)}{(z-z_0)}dz$$

Example 5 :

Evaluate $\int_C \frac{\sin 2z dz}{\left(z-\frac{\pi i}{4}\right)^4}$ where C is $|z| = 1$.

Solution :

Let $f(z) = \sin 2z$.

$f(z)$ is analytic and $\frac{\pi i}{4}$ lies inside C.

$$\int_C \frac{\sin 2z}{(z-\pi i)^4}dz = \frac{2\pi i}{3!}f'''\left(\frac{\pi i}{4}\right)$$

$$f'(z) = 2\cos 2z$$

$$f''(z) = -4\sin 2z$$

$$f'''(z) = -8\cos 2z$$

Hence $$f'''\left(\frac{\pi i}{4}\right) = -8\cos\left(\frac{\pi i}{2}\right)$$

$$= -8\cosh\left(\frac{\pi}{2}\right)$$

$$\int_C \frac{\sin 2z}{(z-\pi i)^4}dz = -\frac{8\pi i}{3}\cosh\left(\frac{\pi}{2}\right)$$

Exercise :

1. Evaluate $\int_C \frac{e^{iz}}{z^3}dz$ where C is $|z| = 2$.

2. Evaluate $\int_C \frac{\tan z dz}{\left(z-\frac{\pi}{4}\right)^2}$ where C is $|z| = 1$

3. Prove that $\int_C \frac{dz}{\left(z^2+4^2\right)} = \frac{\pi}{6}$ where C is $|z-i| = 2$.

4. Prove that $\frac{1}{2\pi i}\int_C \frac{e^{zt}dz}{\left(z^2+1\right)^2} = \frac{t\sin t}{2}$ if $t>0$ and C is the circle $|z|=3$.

5. Evaluate $\int_C \frac{z^2dz}{(2z-1)^2}$ where C is $|z| = 1$.

6. If C is $|z| = 2$, prove that $\int_C \frac{e^z dz}{z-1} = 2\pi ie$

7. If C is a closed curve described in the positive sense and $\phi(z_0) = \int_C \frac{\left(z^4+z\right)dz}{(z-z_0)^3}$ show that $\phi(z_0) = 12\pi i z_0^2$ when z_0 is inside C and $\phi(z_0)=0$ when z_0 lies outside C.

SERIES EXPANSIONS

Here we consider the problem of representing a given function as a power series.

TAYLOR'S SERIES

Taylor's Theorem :

Let f(z) be analytic in a region D containing z_0. Then f(z) canbe represented as a power series in $z-z_0$ given by

$$f(z) = f(z_0)+\frac{f'(z_0)}{1!}(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+....+\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n+...$$

The expansion is valid in the largest open disc with centre z_0 contained in D.

Proof :

Let r>0 be such that the disc $|z-z_0|<r$ is contained in D.

Let $0<r_1<r$. Let C_1 be the circle $|z-z_0| = r_1$.

By Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)}{(\zeta-z)}d\zeta \quad \text{--------(1)}$$

Also by theorem on higher derivatives we have

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{(\zeta-z)^{n+1}} \quad \text{-------(2)}$$

$$\frac{1}{\zeta-z} = \frac{1}{(\zeta-z_0)-(z-z_0)}$$

$$= \frac{1}{(\zeta-z_0)\left[1-\frac{z-z_0}{\zeta-z_0}\right]}$$

$$= \frac{1}{\zeta-z_0}\left[1+\left(\frac{z-z_0}{\zeta-z_0}\right)+\left(\frac{z-z_0}{\zeta-z_0}\right)^2+...+\left(\frac{z-z_0}{\zeta-z_0}\right)^{n-1}+\frac{\left(\frac{z-z_0}{\zeta-z_0}\right)^n}{1-\left(\frac{z-z_0}{\zeta-z_0}\right)}\right]$$

$$\left(\text{using the identity } \frac{1}{1-\alpha} = 1+\alpha+\alpha^2+...+\alpha^{n-1}+\frac{\alpha^n}{1-\alpha}\right)$$

$$= \frac{1}{\zeta - z_0} + \frac{z - z_0}{(\zeta - z_0)^2} + \frac{(z - z_0)^2}{(\zeta - z_0)^3} + \dots + \frac{(z - z_0)^{n-1}}{(\zeta - z_0)^n} + \frac{(z - z_0)^n}{(\zeta - z_0)^n (\zeta - z)}$$

Multiplying thoughout by $\frac{f(\zeta)}{2\pi i}$, integrating over C_1 and using (1) and (2) we get

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots + \frac{f^{(n+1)}(z_0)}{(n-1)!}(z - z_0)^{n-1} + R_n \text{--(3)}$$

where $R_n = \frac{(z - z_0)^n}{2\pi i} \int_{C_1} \frac{f(\zeta)d\zeta}{(\zeta - z)(\zeta - z_0)^n}$

Here ζ lies on C_1 and z lies in the interior of C_1 so that $|\zeta - z_0| = r_1$ and $|z - z_0| < r_1$.

$$\begin{aligned} \therefore \quad |\zeta - z| &= |(\zeta - z_0) - (z - z_0)| \\ &\geq |\zeta - z_0| - |z - z_0| \\ &= r_1 - |z - z_0| \end{aligned}$$

$$\therefore \quad \frac{1}{|\zeta - z|} \leq \frac{1}{r_1 - |z - z_0|}$$

Let M denote the maximum value of $|f(z)|$ on C_1.

Then
$$\begin{aligned} |R_n| &\leq \frac{|z - z_0|^n}{2\pi} \frac{M(2\pi r_1)}{(r_1 - |z - z_0|)r_1^n} \\ &= \frac{M|z - z_0|}{(r_1 - |z - z_0|)} \left(\frac{|z - z_0|}{r_1} \right)^{n-1} \end{aligned}$$

Also $\left| \frac{z - z_0}{r_1} \right| < 1$. Hence $\underset{n \to \infty}{\text{Lt}} R_n = 0$.

$\therefore$ Taking Lt as $n \to \infty$ in (3) we get

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \dots$$

Note 1 : The above series is called the **Taylor series** of f(z) about the point z_0. Thus if f(z) is analytic at a point z_0 then f(z) can be represented as a Taylor's series about z_0 in non negative powers of $z-z_0$ in some neighbourhood of z_0.

Note 2 : TheTaylor series expansion of f(z) about the point zero is called the **Maclaurin's series**. Thus the Maclaurin's series of f(z) is given by

$$f(z) = f(0)+\frac{z}{1!}f'(0)+\frac{z^2}{2!}f''(0)+....+\frac{z^n}{n!}f^n(0)+...$$

Example 1 :

The Taylor's series for $f(z) = \frac{1}{z}$ abotu z=1 is given by

$$\frac{1}{z} = f(1)+\frac{f'(1)}{1!}(z-1)+\frac{f''(1)}{2!}(z-1)^2+\frac{f'''(1)}{3!}(2-1)^3+....$$

$$f(z) = \frac{1}{z} \Rightarrow f(1) = 1$$

$$f'(z) = -\frac{1}{z^2} \Rightarrow f'(1) = -1$$

$$f''(z) = \frac{2}{z^3} \Rightarrow f''(1) = 2$$

$$f'''(z = -\frac{6}{z^4} \Rightarrow f'''(1) = -6$$

Hence the Taylor's series expansion for $\frac{1}{z}$ about 1 is

$$\frac{1}{z} = 1-(z-1)+(z-1)^2-(z-1)^3+.....$$

This expansion is valid in the disc $|z-1|<1$. Similarly the Taylor's series for $f(z)=\frac{1}{z}$ about z=i is given by $\frac{1}{z} = \frac{1}{i}-\frac{z-i}{i^2}+\frac{(z-i)^2}{i^3}-\frac{(z-i)^3}{i^4}+....$ and the expansion is valid int he disc $|z-i|<1$.

Example 2 :

Let $f(z) = e^z$.

Then $f^{(n)}(z) = e^z$ for all n and hence $f^{(n)}(0) = 1$.

Hence the Maclaurin's series for e^z is given by $e^z = 1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+....+\frac{z^n}{n!}+....$ and the expansion is valid in the entire complex plane.

Maclaurin's series expansion of some of the standard functions are :

1. $$e^{-z} = 1-\frac{z}{1!}+\frac{z^2}{2!}-.....+(-1)^n\frac{z^n}{n!}+.....(|z|<\infty)$$

2. $$\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}-.....+(-1)^{n-1}\frac{z^{2n-1}}{(2n-1)!}+....(|z|<\infty)$$

3. $$\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-....+(-1)^{n-1}\frac{z^{2n-2}}{(2n-2)!}+....(|z|<\infty)$$

4. $$\sinh z = \frac{z}{1!}+\frac{z^3}{3!}+\frac{z^5}{5!}+....+\frac{z^{2n-1}}{(2n-1)!}+....(|z|<\infty)$$

5. $$\cosh z = 1+\frac{z^2}{2!}+\frac{z^4}{4!}+....+\frac{z^{2n}}{(2n)!}+....(|z|<\infty)$$

6. $$\frac{1}{1+z} = 1-z+z^2-z^3+....+(-1)^n z^n+...(|z|<1)$$

7. $$\frac{1}{1-z} = 1+z+z^2+z^3+....+z^n+...(|z|<1)$$

8. $$\log(1+z) = z-\frac{z^2}{2}+\frac{z^3}{3}-.....+(-1)^{n-1}\frac{z^n}{n}+......(|z|<1)$$

9. $$\log(1-z) = -z-\frac{z^2}{2}-\frac{z^3}{3}-.....-\frac{z^n}{n}-......(|z|<1)$$

Worked Examples :

Example 1 :

Expand cos z into a Taylor's series about the point $z = \frac{\pi}{2}$ and determine the region of convergence.

Solution :

Let f(z) = cos z

The Taylor's series for f(z) about $z = \frac{\pi}{2}$ is

$$f(z) \quad = \quad f\left(\frac{\pi}{2}\right) + \frac{\left(z-\frac{\pi}{2}\right)}{1!} f'\left(\frac{\pi}{2}\right) + \frac{\left(z-\frac{\pi}{2}\right)^2}{2!} f''\left(\frac{\pi}{2}\right) + \ldots + \frac{\left(z-\frac{\pi}{2}\right)^3}{3!} f'''\left(\frac{\pi}{2}\right) + \ldots.$$

$$f(z) \quad = \quad \cos z \Rightarrow f\left(\frac{\pi}{2}\right) = 0$$

$$f'(z) \quad = \quad -\sin z \Rightarrow f'\left(\frac{\pi}{2}\right) = -1$$

$$f''(z) \quad = \quad -(\cos z) \Rightarrow f''\left(\frac{\pi}{2}\right) = 0$$

$$f'''(z) \quad = \quad \sin z \Rightarrow f'''\left(\frac{\pi}{2}\right) = 1$$

The Taylor's series for cos z about $z = \frac{\pi}{2}$ is

$$\cos z \quad = \quad -\frac{\left(z-\frac{\pi}{2}\right)}{1!} + \frac{\left(z-\frac{\pi}{2}\right)^3}{3!} - \frac{\left(z-\frac{\pi}{2}\right)^5}{5!} + \ldots.$$

The expansion is valid throughuot the complex plane.

Example 2 :

Expand $f(z) = \frac{z-1}{z+1}$ as a Taylor's series

(i) about the point z = 0

(ii) about the point z = 1. Determine the region of convergence in each case.

Solution :

(i)
$$f(z) \quad = \quad \frac{z-1}{z+1}$$

$$= \quad (z-1)\,(1+z)^{-1}$$

$$= (z-1)(1-z+z^2-z^3+.....) \text{ if } |z|<1$$

$$= (z-z^2+z^3-.....)-(1-z+z^2-z^3-.....)$$

$$= -1+2z-2z^2+2z^3-.....$$

The region of convergence is $|z| < 1$.

(ii)
$$f(z) = \frac{z-1}{z+1}$$

$$= \frac{z-1}{2+z-1}$$

$$= \frac{z-1}{2\left(1+\frac{z-1}{2}\right)}$$

$$= \frac{z-1}{2}\left(1+\frac{z-1}{2}\right)^{-1}$$

$$= \frac{z-1}{2}\left[1-\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2-\left(\frac{z-1}{2}\right)^3+...\right] \text{ if } \left|\frac{z-1}{2}\right|<1$$

$$= \frac{z-1}{2}-\left(\frac{z-1}{2}\right)^2+\left(\frac{z-1}{2}\right)^3-....$$

The region of convergence is given by $\left|\frac{z-1}{2}\right|<1$ which is same as the circular disc $|z-1|<2$.

Example 3 :

Show that

(i) $\frac{1}{z^2} = 1+\sum_{n=1}^{\infty}(n+1)(z+1)^n$ when $|z+1| < 1$.

(ii) $\frac{1}{z^2} = \frac{1}{4}+\frac{1}{4}\sum_{n=1}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n$ when $|z-2|<2$.

Solution :

(i)
$$\frac{1}{z^2} = \frac{1}{[1-(z+1)]^2}$$

$$= [1-(z+1)]^{-2}$$

$$= 1+2(z+1)+3(z+1)^2+4(z+1)3+\ldots \text{ if } |z+1|<1.$$

$$= 1+\sum_{n=1}^{\infty}(n+1)(z+1)^n \quad \text{when } |z+1|<1.$$

(ii)
$$\frac{1}{z^2} = \frac{1}{(z-2+2)^2}$$

$$= \frac{1}{\left[2\left(1+\frac{z-2}{2}\right)\right]^2}$$

$$= \frac{1}{4}\left(1+\frac{z-2}{2}\right)^{-2}$$

$$= \frac{1}{4}\left[1-2\left(\frac{z-2}{2}\right)+3\left(\frac{z-2}{2}\right)^2-\ldots\right] \text{ if } \left|\frac{z-2}{2}\right|<1$$

$$= \frac{1}{4}-\frac{1}{4}\times 2\left(\frac{z-2}{2}\right)+\frac{1}{4}\times 3\left(\frac{z-2}{2}\right)^2\ldots\ldots$$

$$= \frac{1}{4}+\frac{1}{4}\sum_{n=1}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n$$

Here the region of convergence is $\left|\frac{z-2}{2}\right|<1$ which is the same as the circular disc $|z-2|<2$.

Example 4 :

Expand ze^{2z} in a Taylor's series about $z = -1$ and determine the region of convergence.

Solution :

$$\text{Let } f(z) = ze^{2z}$$

$$= ze^{2(z+1)}\, e^{-2}$$

$$= \frac{1}{e^2}\left[(z+1)e^{2(z+1)} - e^{2(z+1)}\right]$$

$$= \frac{1}{e^2}\left[(z+1)\left\{1+\frac{2(z+1)}{1!}+\frac{4(z+1)^2}{2!}+\ldots\right\}-\left\{1+\frac{2(z+1)}{1!}+\frac{4(z+1)^2}{2!}+\ldots\right\}\right]$$

$$= \frac{1}{e^2}\left[\left\{(z+1)+\frac{2(z+1)^2}{1!}+\frac{2^2(z+1)^3}{2!}+.....\right\}-\left\{1+\frac{2(z+1)}{1!}+\frac{2^2(z+1)^2}{2!}+...\right\}\right]$$

$$= \frac{1}{e^2}\left[-1+\left(1-\frac{2}{1!}\right)(z+1)+\left(\frac{2}{1!}-\frac{2^2}{2!}\right)(z+1)^2+\left(\frac{2^2}{2!}-\frac{2^3}{3!}\right)(z+1)^3+....\right]$$

The expansion is invalid thruoghout the complex plane.

Exercise :

1. Show that $\frac{1}{z^2} = 1-2(z-1)+3(z-1)^2-4(z-1)^3+...$ for all z in $|z-1|<1$.

2. Expand $\frac{z}{z-3}$ as a Taylor's series about $z = 1$.

3. Find the Taylor's series for ze^z about $z = 1$.

4. Show that $\sin z^2 = z^2-\frac{z^6}{3!}+\frac{z^{10}}{5!}-\frac{z^{14}}{7!}+....$ for $|z|<\infty$.

LAURENT'S SERIES

A series of the form $\sum_{n=1}^{\infty}\frac{b_n}{z^n}$ -------(1)

can be considered as an ordinary power series in the variable $\frac{1}{z}$. Hence if the radius of convergence of the power series $\sum_{n=1}^{\infty}b_n z^n$ is r and $r<\infty$ then the series $\sum_{n=1}^{\infty}\frac{b_n}{z^n}$ converges in the region $|z|>r$. The convergence is uniform in every region $|z|\geq\rho>r$ and the series represent an analytic function in $|z|>r$.

If the series (1) is combined with the usual power series we get a more general series of the form $\sum_{-\infty}^{\infty}a_n z^n$ --------(2)

This series is said to converge at a point if the part of the series consisting of the negative powers of z and the part of the series consisting of non-negative powers of z are separately convergent. We know that the series consisting of non-negative powers of z converges in a disc $|z|<r_2$ and the series consisting of negative powers of z converges in a region $|z|>r_1$.

$\therefore$ If $r_1 < r_2$ the series represented by (2) converges in the region $r_1<|z|<r_2$ and in this annulus region it represents as analytic function.

We shall prove that the converse situation is also true.

i.e., any function which is analytic in a region containing the annulus $r_1<|z-z_0|<r_2$ can be represented in a series of the form $\sum_{-\infty}^{\infty} a_n(z-z_0)^n$

LAURENT'S THEOREM

Let C_1 and C_2 denote respectively the concentric circles $|z-z_0|=r_1$ and $|z-z_0|=r_2$ with $r_1<r_2$. Let f(z) be analytic in a region containing the circular annulus $r_1<|z-z_0|<r_2$. Then f(z) can be represented as a convergent series of positive and negative powers of $z-z_0$ given by

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

where
$$b_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)d\zeta}{(\zeta-z_0)^{-n+1}} \text{ and}$$

$$a_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n+1}}$$

Proof :

Let z be any point in the circular annulus $r_1<|z-z_0|<r_2$. Then we have

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)d\zeta}{\zeta-z} - \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)d\zeta}{\zeta-z}$$

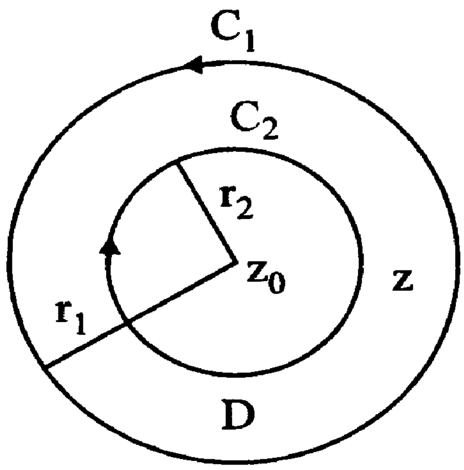
$$\therefore \quad f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)d\zeta}{\zeta-z} + \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)d\zeta}{z-\zeta} \quad \text{------(1)}$$

As in the proof of Taylor's theorem we have

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta-z}d\zeta = a_0+a_1(z-z_0)+a_2(z-z_0)^2+\ldots+a_{n-1}(z-z_0)^{n-1}+R_n(z) \quad \text{--(2)}$$

$$\text{where } a_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta \text{ and}$$

$$R_n(z) = \frac{(z-z_0)^n}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^n(\zeta-z)}$$



$$\frac{1}{z-\zeta} = \frac{1}{z-z_0+z_0-\zeta}$$

$$= \frac{1}{(z-z_0)-(\zeta-z_0)}$$

$$= \frac{1}{(z-z_0)\left[1-\frac{\zeta-z_0}{z-z_0}\right]}$$

$$= \frac{1}{z-z_0}\left[1+\left(\frac{\zeta-z_0}{z-z_0}\right)+\left(\frac{\zeta-z_0}{z-z_0}\right)^2+\ldots.+\left(\frac{\zeta-z_0}{z-z_0}\right)^{n-1}+\frac{\left(\frac{\zeta-z_0}{z-z_0}\right)^n}{1-\left(\frac{\zeta-z_0}{z-z_0}\right)}\right]$$

Multiplying by $\frac{f(\zeta)}{2\pi i}$ and integrating over C_1 we get

$$\int_{C_1}\frac{f(\zeta)d\zeta}{z-\zeta} = \frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\ldots.+\frac{b_{n-1}}{(z-z_0)^{n-1}}+s_n(z) \quad \text{----(3)}$$

$$b_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^{-n+1}};$$

$$s_n = \frac{1}{2\pi i(z-z_0)^n}\int_{C_1}\frac{f(\zeta)(\zeta-z_0)^n d\zeta}{z-\zeta}$$

From (1), (2) and (3) we get

$$f(z) = a_0+a_1(z-z_0)+\ldots.+a_{n-1}(z-z_0)^{n-1}$$

$$+\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\ldots.+\frac{b_{n-1}}{(z-z_0)^{n-1}}+R_n(z)+s_n(z) \quad \text{-------(4)}$$

The required result follows if we can prove that $R_n \to 0$ and $S_n \to 0$ as $n \to \infty$.

If $\zeta \in C_1$ then $|\zeta - z_0| = r_1$ and

$$|z-\zeta| = |(z-z_0)-(\zeta-z_0)| \geq |z-z_0|-r_1.$$

If $\zeta \in C_2$ then $|\zeta - z_0| = r_2$ and

$$|\zeta-z| = |(\zeta-z_0)-(z-z_0)| \geq r_2-|z-z_0|$$

Let M denote the maximum value of $|f(z)|$ in $C_1 \cup C_2$. Then

$$|R_n| \leq \frac{|z-z_0|^n}{2\pi} \frac{M(2\pi r_2)}{r_2^n(r_2-|z-z_0|)}$$

$$\leq \frac{M|z-z_0|}{(r_2-|z-z_0|)}\left(\frac{|z-z_0|}{r_2}\right)^{n-1}$$

Since $\frac{|z-z_0|}{r_2} < 1$, $R_n \to 0$ as $n \to \infty$.

Also $$|S_n| \leq \frac{1}{(z-z_0)^n 2\pi} \frac{M\, r_1^n(2\pi r_1)}{(|z-z_0|-r_1)}$$

$$\leq \frac{M r_1}{(|z-z_0|-r_1)}\left(\frac{r_1}{|z-z_0|}\right)^n$$

Since $\frac{r_1}{|z-z_0|} < 1$, $S_n \to 0$ as $n \to \infty$.

Hence by taking limit as $n \to \infty$ in (4) we get

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

Remark :

The formulae for the coefficients a_n and b_n in the Laurent's series expansion are given by

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)db}{(\zeta-z_0)^{n+1}} \quad \text{--------(1)}$$

and $$b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)db}{(\zeta-z_0)^{-n+1}} \quad \text{--------(2)}$$

Since the integrands in the integrals of (1) and (2) are analytic functions of ζ throughout the annular region, any simple closed curve C in the annulus can be used as the path of integration in place of C_1 and C_2.

Hence Laurent's series can be written as

$$f(z) = \sum_{-\infty}^{\infty} A_n(z-z_0)^n, \ (r_1 < |z-z_0| < r_2)$$

$$A_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n+1}}$$

Worked Examples :

Example 1 :

Expand $\frac{1}{z(z-1)}$ as Laurent's series

(i) about z = 0 in powers of z and

(ii) about z = 1 in powers of z–1. Also state the region of validity.

Solution :

(i) The only points where f(z) is not analytic are 0 and 1. Hence f(z) can be expanded as a Laurent's series in the annulus $0<|z|<1$.

$$f(z) = \frac{1}{z(z-1)}$$

$$= -\frac{1}{z}(1-z)^{-1}$$

$$= -\frac{1}{z}\left(1+z+z^2+....+z^n+.....\right) \quad \text{(since } |z|<1)$$

$$= -\left(\frac{1}{z}+1+z+z^2+....+z^n+....\right)$$

This is the Laurent's series expansion of f(z) in $0<|z|<1$.

(ii) f(z) is analytic in $0<|z-1|<1$ and hence can be expanded as a Laurent's series in powers of $z-1$ in this region.

$$\frac{1}{z(z-1)} = \frac{1}{z-1}\left[\frac{1}{1+(z-1)}\right]$$

$$= \frac{1}{z-1}\left[1+(z-1)\right]^{-1}$$

$$= \frac{1}{z-1}\left[1-(z-1)+(z-1)^2-(z-1)^3+.....\right]$$

(since $|z-1|<1$)

$$= \frac{1}{z-1}-1+(z-1)-(z-1)^2+.....$$

This gives the Laurent's series expansion in $0<|z-1|<1$.

Example 2 :

Find the Laurent's series for $\frac{z}{(z+1)(z+2)}$ about $z = -2$.

Solution :

$f(z) = \frac{z}{(z+1)(z+2)}$ is analytic in $0<|z+2|<1$. Hence f(z) canbe expanded as a Laurent's series in powers of $z+2$ in this region.

$$f(z) = \frac{z}{(z+1)(z+2)}$$

$$= \frac{(z+2)-2}{-[1-(z+2)](z+2)}$$

$$= \frac{2-(z+2)}{z+2}\left[1-(z+2)\right]^{-1}$$

$$= \frac{2-(z+2)}{z+2}\left[1+(z+2)+(z+2)^2+.....+(z+2)^n+....\right]$$

(since $|z+2|<1$)

$$= \left(\frac{2}{z+2}-1\right)\left[1+(z+2)+(z+2)^2+....+(z+2)^n+...\right]$$

$$= \frac{2}{z+2}+1+(z+2)+...+(z+2)^n+....$$

This is the required Laurent's series expansion of f(z) in $0<|z+2|<1$.

Example 3 :

Expand $\dfrac{-1}{(z-1)(z-2)}$ as a power series in z in the region

(i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$.

Solution :

Let
$$f(z) = \frac{-1}{(z-1)(z-2)}$$

By splitting into partial fraction, we have
$$f(z) = \frac{1}{z-1}-\frac{1}{z-2}$$

(i) The only points where f(z) is not analytic are 1 and 2. Hence f(z) is analytic in $|z|<1$ and hence can be represented as a Taylor's series in $|z|<1$.

$$\therefore \quad f(z) = \frac{1}{z-1}-\frac{1}{z-2}$$
$$= -\frac{1}{z-1}+\frac{1}{2-z}$$
$$= -(1-z)^{-1}+\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$
$$= -\left(1+z+z^2+\ldots+z^n+\ldots\right)+\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\ldots+\frac{z^n}{2^n}+\ldots\right)$$
$$= \sum_{n=0}^{\infty}\left[-z^n+\frac{1}{2}\left(\frac{z}{2}\right)^n\right]$$
$$= \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}}-1\right)z^n$$

(ii) f(z) is analytic in the annular region $1<|z|<2$ and hence can be expanded as a Laurent's series in this region.

$$f(z) = \frac{1}{z-1}-\frac{1}{z-2}$$
$$= \frac{1}{z\left(1-\frac{1}{z}\right)}+\frac{1}{2\left(1-\frac{z}{2}\right)}$$

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}+\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+.....\right]+\frac{1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+...\right]$$

$$\left(\text{since } \left|\frac{1}{z}\right|<1 \text{ and } \left|\frac{z}{2}\right|<1\right)$$

$$= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}+\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

This gives the Laurent's series expansion in $1<|z|<2$.

(iii) f(z) is analytic in the domain $|z|>2$ and in this domain we have $\left|\frac{2}{z}\right|<1$.

Hence $$f(z) = \frac{1}{Z}\left[\frac{1}{1-\left(\frac{1}{Z}\right)}\right]-\frac{1}{Z}\left[\frac{1}{1-\left(\frac{2}{Z}\right)}\right]$$

$$= \frac{1}{Z}\left[1-\left(\frac{1}{Z}\right)\right]^{-1}-\frac{1}{Z}\left[1-\left(\frac{2}{Z}\right)\right]^{-1}$$

$$= \frac{1}{z}\left[\left\{1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+....\right\}-\left\{1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+....\right\}\right]$$

$$= \sum_{n=0}^{\infty}\left(\frac{1-2^n}{z^{n+1}}\right)$$

Example 4 :

Expand $f(z) = \frac{e^{2z}}{(z-1)^3}$ about z=1 as a Laurent's series. Also indicate the region of convergence of the series.

Solution :

$$f(z) = \frac{e^{2(z-1)+2}}{(z-1)^3}$$

$$= \frac{e^2 e^{2(z-1)}}{(z-1)^3}$$

$$= \frac{e^2}{(z-1)^3}\left[1+\frac{2(z-1)}{1!}+\frac{2^2(z-1)^2}{2!}+\frac{2^3(z-1)^3}{3!}+....\right]$$

$$= e^2\left[\frac{1}{(z-1)^3}+\frac{2}{(z-1)^2}+\frac{2}{z-1}+\frac{4}{3}+\frac{2}{3}(z-1)+...\right]$$

This series converges for all values of z except z = 1.

Exercise :

1. Prove that $\frac{1+2z}{z^2+z^3} = \frac{1}{z^2}+\frac{1}{z}-1+z-z^2+z^3-.....$ where $0<|z|<1$.

2. Expand in Laurent's series $\frac{1}{z(z-1)^2}$ at the point z = 1.

3. Expand $\frac{1}{z^2(z-3)^2}$ as a Laurent's series at z=3 and state the region of validity.

4. Represent the function $\frac{z+1}{z-1}$ by its Larent's series in powers of z for the region $|z|>1$.

ZEROS OF AN ANALYTIC FUNCTION

Definition :

Let f(z) be a function which is analytic in a region D. Let $a \in D$. Then a is said to be a **zero of order** r (where r is a positive integer) for f(z) if $f(z) = (z-a)^r \phi(z)$ where $\phi(z)$ is analytic at a and $\phi(a) \neq 0$.

Example 1 :

Consider $f(z) = \sin z$

We know that
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$$
$$= z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots\right)$$
$$= z\phi(z)$$

Obviously $\phi(z)$ is analytic and $\phi(0)=1\neq 0$. $z=0$ is a zero of order 1 for sin z.

Example 2 :

Let $f(z) = (z-2i)^2(z+3)^3e^z$

2i is a zero of order 2 and –3 is a zero of order 3 for f(z).

Theorem :

Suppose f(z) is analytic in a region D and is not identically zero in D. Then the set of all zeros of f(z) is isolated.

Proof :

Let $a \in D$ be a zero for f(z). We shall prove that there exists a neighbourhood $|z-a|<\delta$ such that this neighbourhood does not contain any other zero for f(z).

Suppose a is a zero of order r for f(z).

Then $$f(z) = (z-a)^r \phi(z) \qquad \text{-------(1)}$$

$|\phi(a)| > 0$ where $\phi(z)$ is analytic at a and $\phi(a) \neq 0$.

Since ϕ is analytic at a ϕ is continuous at a.

∴ We can find a $\delta > 0$ such that

$$|z-a|<\delta \Rightarrow |\phi(z)-\phi(a)| < \frac{|\phi(a)|}{2}$$

We claim that the neighbourhood $|z-a|<\delta$ does not contain any other zero of f(z).

Suppose $b \neq a$ is another zero for f(z) in this neighbourhood.

$$\text{Then} \qquad |b-a| < \delta$$

$$\text{and} \qquad f(b) = 0$$

$$\therefore \qquad (b-a)^r \phi(b) = 0 \qquad \text{(from (1))}$$

Now since $b \neq a$, $(b-a)^r \neq 0$

$$\therefore \qquad \phi(b) = 0$$

$$\text{Further} \qquad |b-a|<\delta \Rightarrow |\phi(b)-\phi(a)| < \frac{|\phi(a)|}{2}$$

$$\Rightarrow |\phi(a)| < \frac{|\phi(a)|}{2} \text{ which is a contradiction.}$$

Thus the neighbourhood $|z-a|<\delta$ contains no other zero of f(z) and hence the set of all zeros of f(z) is isolated.

Corollary 1 :

Let f(z) be analytic in a region D. Suppose f(z) = 0 on a subset of D which has a limit point in D. Then f(z) is identically zero in D.

Corollary 2 :

Let f(z) and g(z) be two functions which are analytic in a region D. Suppose f(z) = g(z) on a subset of D which has a limit point in D. Then f(z) = g(z) in D.

Exercise :

1. Find all the zeros of cos z.
2. Prove that there is no analytic functions whose zeros are precisely the points $1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{n}, \ldots\ldots$

SINGULARITIES

Definition :

A point a is called a **singular point** or a **singularity** of a function f(z) if f(z) is not analytic at some point of every disc $|z-a|<r$.

Example 1 :

Consider the function $f(z) = \frac{1}{z}$

Then $f'(z) = -\frac{1}{z^2}$ for all $z \neq 0$

Thus f(z) is analytic except at $z = 0$

$\therefore$ $z = 0$ is a singular point of f(z).

Example 2 :

Consider the function $f(z) = \frac{1}{z(z-i)}$; 0 and i are singular points for f(z).

Definition :

A point a is called an **isolated singularity** for f(z) if (i) f(z) is not analytic at $z=a$ and (ii) there exists $r>0$ such that f(z) is analytic in $0<|z-a|<r$.

i.e., the neighbourhood $|z-a|<r$ contains no singularity of f(z) except a.

Example 1 :

$f(z) = \frac{z+1}{z^2(z^2+1)}$ has three isolated singularities $z = 0, i, -i$.

Example 2 :

Consider the principal branch of logarithm given by $\log re^{i\theta} = \log r + i\theta$ where $-\pi < \theta \leq \pi$.

All points on the negative real axis are singular points of the function. These singularities are not isolated.

Example 3 :

Consider the function $f(z) = \frac{1}{\sin z}$. The singular points are 0, $\pm\pi$, $\pm 2\pi$,..... and these are isolated singular points.

We now classify the isolated singularities of a function.

Let a be an isolated singularity for a function f(z). Let r>0 be such that f(z) is analytic in $0<|z-a|<r$. In this domain the function f(z) can be represented as a Laurent series given by

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} + \sum_{n=0}^{\infty} a_n (z-a)^n$$

where $$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}}$$

and $$b_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{(\zeta-a)^{-n+1}}$$

The series consisting of the negative powers of z–a in the above Laurent series expansion of f(z) is given by $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ and is called the **principal part or singular part** of f(z) at z=a.

The singular part of f(z) at z=a determines the character of the singularity.

There are three types of singularities. They are

(i) Removable singularities

(ii) Poles

(iii) Essential singularities.

Definition :

Let a be an isolated singularity for f(z). Then a is called a **removable singulrity** if the principal part of f(z) at z=a has no terms.

Note : If a is a removable singularity for f(z) then the Laurent's series expansion of f(z) about z = a is given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

$$= a_0 + a_1(z-a) + \ldots. + a_n(z-a)^n + \ldots..$$

Hence $\underset{z \to a}{\text{Lt}} f(z) = a_0$

Hence by defining $f(a) = a_0$ the function f(z) becomes analytic at a.

Example 1 :

Let $f(z) = \frac{\sin z}{z}$. Clearly 0 is an isolated singular point for f(z).

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots$$

Here the principal part of f(z) at z=0 has no terms.

Hence z=0 is a removable singularity.

Also $\underset{z \to 0}{\text{Lt}} \frac{\sin z}{z} = 1$. Hence the singularity can be removed by defining f(0)=1 so that the extended function becomes analytic at z=0.

Example 2 :

Let $$f(z) = \frac{z - \sin z}{z^3}$$

z = 0 is an isolated singularity.

$$\frac{z - \sin z}{z^3} = \frac{1}{z^3}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)\right]$$

$$= \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \ldots$$

∴ z = 0 is a removable singularity.

By defining $f(0) = \frac{1}{6}$ the function becomes analytic at z = 0.

Definition :

Let a be an isolated singularity of f(z). The point a is called a **pole** if the principal part of f(z) at z=a has a finite number of terms. If the principal part of f(z) at z=a is given by

$\frac{b_1}{z-a}+\frac{b_2}{(z-a)^2}+.....+\frac{b_r}{(z-a)^r}$ where $b_r \neq 0$, we say that a is a **pole of order r** for f(z).

Note : A pole of order 1 is called a **simple pole** and a pole of order2 is called a **double pole.**

Example 1 :

Consider $f(z) = \frac{e^z}{z}$.

$$\frac{e^z}{z} = \frac{1}{z}+1+\frac{z}{2!}+\frac{z^2}{3!}+....$$

Here the principal part of f(z) at z=0 has a single term $\frac{1}{z}$. Hence z=0 is a simple pole of f(z).

Example 2 :

Let $f(z) = \tan z = \frac{\sin z}{\cos z}$

The singularities of f(z) are $\frac{\pi}{2}+n\pi$ where $n \in z$. All the singularities are poles of order 1.

Example 3 :

Let $f(z) = \frac{\cos z}{z^2}$ has a double pole at z = 0.

For,
$$\frac{\cos z}{z^2} = \frac{1}{z^2}\left[1-\frac{z^2}{2!}+\frac{z^4}{4!}-.....\right]$$

$$= \frac{1}{z^2}-\frac{1}{2!}+\frac{z^2}{4!}-.....$$

Example 4 :

Let
$$f(z) = \frac{z^2-2z+3}{z-2}$$

$$f(z) = 2+(z-2)+\frac{3}{z-2}$$

Here f(z) has a simple pole at z = 2.

Definition :

Let a be an isolated singularity of f(z). The point a is called an **essential singularity** of f(z) at z=a if the principal part of f(z) at z=a has an infinite number of terms.

Example 1 :

Let $f(z) = e^{1/z}$. Obviously z=0 is an isolated singularity for f(z).

$e^{1/z} = 1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+....$ The principal part of f(z) has infinite number of terms.

Hence $e^{1/z}$ has an essential singularity at z=0.

Example 2 :

Let $f(z) = z^2 \sin\left(\frac{1}{z}\right)$. $f(z)$ has essential singularity at z=0.

Theorem :

Let f(z) be a function defined in a region D of the complex plane except possibly at a point $a \in D$ and let a be an isolated singulrity for f(z). Then a is a removable singularity for f(z) if and only if there exists a unique complex number a_0 such that by defining $f(a) = a_0$ the extended function becomes analytic at a.

Proof :

Suppose a is a removable singularity for f(z).

Then
$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n \quad 0<|z-a<r$$
$$= a_0+a_1(z-a)+a_2(z-a)^2+....$$

$\therefore$ By defining $f(z) = a_0$, f becomes analytic at a. Conversely, suppose there exists a unique complex number a_0 such that by defining $f(a) = a_0$. f becomes analytic in $|z-a|<r$.

Hence f can be represented as a Taylor's series, in power of z–a in this neighbourhood, given by $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$. This shows that the principal part of f(z) at z=a has no terms. Hence a is a removable singularity for f(z).

Riemann's Theorem :

Let f be a function which is bounded and analytic throughout a domain $0<|z-z_0|<\delta$. Then either f is analytic at z_0 or else z_0 is a removable singular point of f.

Proof :

Consider the Laurent's series for the function in the given domain about z_0. The coefficient b_n of $\frac{1}{(z-z_0)^n}$ is given by $b_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{-n+1}}$ where C is the circle $|z-z_0| = r$ where $r<\delta$.

Since f is bounded there exists a positive real number M such that $|f(z)|\leq M$ in $0<|z-z_0|<\delta$.

$$\therefore \quad |b_n| \leq \frac{1}{2\pi}\frac{M(2\pi r)}{r^{-n+1}}$$

$$= Mr^n.$$

Since it is true for every r such that $0<r<\delta$ taking limit $r\to 0$ we get b=0.

Hence the Laurent's series for f(z) has no principal part.

Theorem :

Let f(z) be a function having a as an isolated singular point. Then the following are equivalent.

(i) a is a pole of order r for f(z).

(ii) f(z) can be written in the form $f(z) = \frac{1}{(z-a)^r}\theta(z)$ where $\theta(z)$ has a removable singularity at z=a and $\underset{z\to a}{Lt}\ \theta(z)\neq 0$.

(iii) a is a zero of order r for $\frac{1}{f(z)}$.

Proof :

(i) $\Rightarrow$ (ii)

Let a be a pole of order r for f(z).

Then the Laurent's series expansion of f(z) about a is given by

$$f(z) = \sum_{n=1}^{r} \frac{b_n}{(z-a)^n} + \sum_{n=0}^{\infty} a_n (z-a)^n \text{ where } b_r \neq 0.$$

$$\therefore f(z) = \frac{1}{(z-a)^r}\left[b_r + b_{r-1}(z-a) + \ldots + b_1(z-a)^{r-1} + a_0(z-a)^r + \ldots\right]$$

$$= \frac{1}{(z-a)^r}\theta(z) \text{ where } \theta(z) = b_r + b_{r-1}(z-a) + \ldots.$$

Clearly $\underset{z \to a}{\text{Lt}}\ \theta(z) = b_r \neq 0$ and $\theta(z)$ has a removable singularity at z = a

(ii) $\Rightarrow$ (iii)

Let $$f(z) = \frac{1}{(z-a)^r}\theta(z)$$

and by suitably defining $\theta(a)$ we may assume that $\theta(z)$ is analytic at a and $\theta(a) \neq 0$.

$\therefore \frac{1}{f(z)} = (z-a)^r \frac{1}{\theta(z)}$ and $\frac{1}{\theta(z)}$ is analytic at a and $\frac{1}{\theta(a)} \neq 0$.

Hence a is a zero of order r for $\frac{1}{f(z)}$.

(iii) $\Rightarrow$ (i). Let a be a zero of order r for $\frac{1}{f(z)}$.

Then $\frac{1}{f(z)} = (z-a)^r g(z)$ where g(z) is analytic at a and $g(a) \neq 0$.

$\therefore f(z) = \frac{g_1(z)}{(z-a)^r}$ where $g_1(z)$ is analytic at a and $g_1(a) \neq 0$.

Let $g_1(z) = a_0 + a_1(z-a) + \ldots.. + a_n(z-a)^n + \ldots.$

so that $a_0 \neq 0$.

$$\therefore f(z) = \frac{a_0}{(z-a)^r} + \frac{a_1}{(z-a)^{r-1}} + \ldots + a_n + a_{n+1}(z-a) + \ldots. \text{ in } 0<|z-a|<r.$$

$\therefore$ The principal part of f(z) at z = a is $\frac{a_0}{(z-a)^r} + \frac{a_1}{(z-a)^{r-1}} + \ldots + \frac{a_{n-1}}{z-a}$ and $a_0 \neq 0$.

$\therefore$ a is a pole of order r for f(z).

Theorem :

An isolated singularity a of f(z) is a pole if and only if $\underset{z \to a}{\mathrm{Lt}}\, f(z) = \infty$.

Proof :

If a is a pole of order r for f(z) then $f(z) = \dfrac{g(z)}{(z-a)^r}$ with $g(a) \neq 0$.

$\therefore \underset{z \to a}{\mathrm{Lt}}\, f(z) = \infty$.

Conversely let a be an isolated singularity for f(z) and let $\underset{z \to a}{\mathrm{Lt}}\, f(z) = \infty$.

Let $\qquad \theta(z) = \dfrac{1}{f(z)}$

Then $\qquad \underset{z \to a}{\mathrm{Lt}}\, \theta(z) = 0$

Hence a is a removable singularity for θ(z) and by defining θ(z)=0, θ becomes analytic at a. Let a be a zero of order r for the function θ(z). Then a is a pole of order r for f(z).

Definition :

A function f(z) is said to be a **meromorphic function** if it is analytic except at a finite number of points and these finite set of points are poles.

Examples :

1. Let $f(z) = \dfrac{1}{z(z-1)^2}$

   f(z) is analytic except at z=0 and z=1.

   Also 0 and 1 are poles of order 1 and 2 respectively. Hence f(z) is a meromorphic function.

2. $\dfrac{e^z}{z} = \dfrac{1}{z} + 1 + \dfrac{z}{2!} + \dfrac{z^2}{3!} + \ldots$ is a meromorphic function.

3. $e^{1/z}$ is not a meromorphic function since z=0 is an essential singularity for $e^{1/z}$.

Theorem (Weierstrass Theorem on an essential singularity)

Let z_0 be an essential singularity for a function f(z). Let c be any complex number. Then given $\in$, $\delta>0$ there exists a point z_0 such that $z-z_0<\delta$ and $f(z)-c|<\in$.

(i.e.,) The function f(z) comes arbitrarily close to any complex number in every neighbourhood of an essential singularity.

Proof :

Suppose the theorem is false. Then there exists $\in$, $\delta>0$ such that for every point z satisfying $0<|z-z_0|<\delta$ we have $|f(z)-c|\geq\in$.

Consider the function $g(z) = \dfrac{1}{f(z)-c}$

$$\therefore |g(z)| = \frac{1}{|f(z)-c|} < \in$$

Hence g(z) is bounded and further g(z) is analytic in $0<|z-z_0|<\delta$.

Hence by Riemann's theorem $z=z_0$ is a removable singularity for g(z).

If $g(z_0)\neq 0$ then $\dfrac{1}{g(z)} = f(z)-C$ is analytic at z_0.

$\therefore$ By suitably defining $g(z_0)$, the function g(z) becomes analytic at z_0.

If $g(z_0) = 0$ then let z_0 be a zero of order r for g(z).

Then z_0 is a pole of order r for $\dfrac{1}{g(z)} = f(z)-c$. Thus f(z) is either analytic at z_0 or else z_0 is a pole of f(z) which is a contradiction to the hypothesis that z_0 is an essential singularity for f(z).

Hence the theorem.

Worked Examples

Example 1 :

Determine and classify the singular points of $f(z) = \dfrac{z}{e^z - i}$

Solution :

The singularities of f(z) are given by the values of z for which $e^z - 1 = 0$. Hence $z = 2n\pi i$, $n \in z$, are the singularities of f(z).

$$e^z - 1 = \left(1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots\right) - 1$$

$$= z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots$$

$$\therefore \quad \underset{z \to 0}{\text{Lt}} \frac{z}{e^z - 1} = 1$$

Hence 0 is a removable singularity for f(z).

Also $\underset{z \to zn\pi i}{\text{Lt}} \frac{z}{e^z - 1} = \infty$ if $n \neq 0$ and hence $2n\pi i$, $n \neq 0$ are simple poles of f(z).

Example 2 :

Determine and classify the singularities of $f(z) = \sin\left(\frac{1}{z}\right)$

Solution :

Clearly 0 is the only singularity of f(z).

Also $f(z) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \dots$

Thus the principal part of f(z) at z=0 has infinitely many terms and hence 0 is an essential singularity for f(z).

Example 3 :

Determine and classify the singular points of $\frac{1}{(2\sin z - 1)^2}$.

Solution :

The singularities of f(z) are given by the values of z for which $2 \sin z - 1 = 0$.

$\therefore$ The singularities of f(z) are given by

$z = \frac{\pi}{6} + 2n\pi$, $n \in z$ and they are double poles.

Exercise :

1. Find the singularities of the following functions and classify the singularities.

(i) $\frac{z^2}{1+z}$ (ii) $\sin\left(\frac{1}{1-z}\right)$ (iii) $\frac{z^2-2z+3}{z-2}$

2. Show that the singular points of each of the following functions are poles. Determine the order of each pole.

(i) $\frac{1}{z^2+1}$ (ii) $\frac{1}{z^2(z-3)^2}$ (iii) $(z-i)\sin\frac{1}{z+2i}$

3. Find the order of the pole z=0 for the following functions.

(i) $\frac{e^z}{z}$ (ii) $\frac{e^z}{z^2}$ (iii) $\frac{1-\sin z}{z^5}$

RESIDUES

Definition :

Let a be an isolated singularity for f(z). Then the **residue** of f(z) at a is defined to be the coefficietn of $\frac{1}{z-a}$ in the Laurent's series expansion of f(z) about a and is denoted by Res{f(z); a}.

Thus Res{f(z); a} $= \frac{1}{2\pi i}\int_C f(z)dz = b_1$ where C is a circle $|z-a| = r$ such that f is analytic in $0<|z-a|<r$.

Example :

Consider $f(z) = \frac{e^z}{z^2}$

$$\frac{e^z}{z^2} = \frac{1}{z^2}\left(1+\frac{z}{1!}+\frac{z^2}{2!}+.....\right)$$

$$= \frac{1}{z^2}+\frac{1}{z}+\frac{1}{2!}+\frac{z}{3!}+\frac{z^2}{4!}+....$$

∴ f(z) has a double pole at z = 0

∴ Res{f(z); 0} = coefficient of $\frac{1}{z} = 1$.

Lemma 1 :

If z = a is a simple pole for f(z) then $\text{Res}\{f(z); a\} = \underset{z\to a}{\text{Lt}}\,(z-a)f(z)$

Proof :

Since z=a is a simple pole for f(z) the Laurent's series expansion for f(z) about z=a is given by

$$
\begin{aligned}
f(z) &= \frac{b_1}{z-a} + a_0 + a_1(z-a) + \ldots. \\
(z-a)f(z) &= b_1 + a_0(z-a) + a_1(z-a)^2 + \ldots.. \\
\underset{z\to a}{\text{Lt}}\,(z-a)f(z) &= b_1 \\
&= \text{Res}\{f(z); a\}
\end{aligned}
$$

Lemma 2 :

If a is a simple pole for f(z) and $f(z) = \frac{g(z)}{z-a}$ where g(z) is analytic at a and $g(a) \neq 0$ then $\text{Res}\{f(z); a\} = g(a)$.

Proof :

By lemma 1,

$$
\begin{aligned}
\text{Res}\{f(z); a\} &= \underset{z\to a}{\text{Lt}}\,(z-a)f(z) \\
&= \underset{z\to a}{\text{Lt}}\,g(z) \\
&= g(a)
\end{aligned}
$$

Lemma 3 :

If a is a simple pole for f(z) and if f(z) is of the form $\frac{h(z)}{k(z)}$ where h(z) and k(z) are analytic at a and $h(a) \neq 0$ and $k(a) = 0$ then

$$\text{Res}\{f(z); a\} = \frac{h(z)}{k'(z)}$$

Proof :

$$\text{Res}\{f(z);\, a\} = \underset{z\to a}{\text{Lt}}\,(z-a)f(z)$$

$$= \underset{z\to a}{\text{Lt}}\,(z-a)\frac{h(z)}{k(z)}$$

$$= \underset{z\to a}{\text{Lt}}\,h(z)\,\underset{z\to a}{\text{Lt}}\,\frac{(z-a)}{k(z)}$$

$$= \underset{z\to a}{\text{Lt}}\,h(z)\,\underset{z\to a}{\text{Lt}}\left[\frac{z-a}{k(z)-k(a)}\right] \quad \text{(since } k(a)=0\text{)}$$

$$= h(a)\left[\frac{1}{k'(a)}\right]$$

$$= \frac{h(a)}{k'(a)}$$

Lemma 4 :

Let a be a **pole of order m>1** for f(z) and let $f(z) = \frac{g(z)}{(z-a)^m}$ where g(z) is analytic at a and $g(a)\neq 0$. Then $\text{Res}\{f(z);\, a\} = \frac{g^{(m-1)}(a)}{(m-1)!}$

Proof :

$$g^{(m-1)}(a) = \frac{(m-1)!}{2\pi i}\int_C \frac{g(z)dz}{(z-a)^m}$$

(by theorem on higher derivatives) where C is a circle $|z-a| = r$ such that f(z) is analytic in $0<|z-a|<r$.

$$\therefore \quad \frac{g^{(m-1)}(a)}{(m-1)!} = \frac{1}{2\pi i}\int_C f(z)dz$$

$$= \text{Res}\{f(z);\, a\}$$

Worked Examples :

Example 1 :

Calculate the residue of $\frac{z+1}{z^2-2z}$ at its poles.

Solution :

Let $$f(z) = \frac{z+1}{z^2-2z} = \frac{z+1}{z(z-2)}$$

z = 0 and z = 2 are simple poles for f(z).

$$\text{Res}\{f(z);\, 0\} = \underset{z\to 0}{\text{Lt}}\,(z-0)\left[\frac{z+1}{z(z-2)}\right]$$

$$= \underset{z\to 0}{\text{Lt}}\,\frac{z+1}{z-2} = \frac{-1}{2}$$

$$\text{Res}\{f(z);\, 2\} = \underset{z\to 2}{\text{Lt}}\,(z-2)\left[\frac{z+1}{z(z-2)}\right]$$

$$= \underset{z\to 2}{\text{Lt}}\,\frac{z+1}{z} = \frac{3}{2}$$

Aliter :

f(z) can be written as $f(z) = \frac{h(z)}{k(z)}$ where h(z) = z+1 and $k(z) = z^2-2z$ so that $k'(z) = 2z-2$.

$$\therefore \quad \text{Res}\{f(z);\, 0\} = \frac{h(0)}{k'(0)} \text{ (by lemma 3)}$$

$$= -\frac{1}{2}$$

$$\text{Res}\{f(z);\, 2\} = \frac{h(2)}{k'(2)} = \frac{3}{2}$$

Example 2 :

Find the residue of cot z at z = 0

Solution :

z = 0 is a simple pole for cot z.

Let $$f(z) = \frac{\cos z}{\sin z} = \frac{h(z)}{k(z)}$$

$$\therefore \quad \text{Res}\{f(z);\, 0\} = \frac{h(0)}{k'(0)} = \frac{\cos 0}{\cos 0} = 1$$

Example 3 :

Find the residue of $\frac{e^z}{z^2(z^2+9)}$ at its poles.

Solution :

Let $$f(z) = \frac{e^z}{z^2(z^2+9)}$$

Here z = 0 is a double pole and z = 3i and z = –3i are simple poles for f(z).

To find the Res{f(z); 0} let $g(z) = \frac{e^z}{z^2+9}$

Clearly g(z) is analytic at z=0 and g(0)≠0

Also $$g'(z) = e^z\left[\frac{(z^2+9)-2z}{(z^2+9)^2}\right]$$

$$\therefore \quad \text{Res}\{f(z);\, 0\} = \frac{g'(0)}{1!} \text{ (by lemma 4)}$$

$$= \frac{1}{9}$$

To find Res{f(z); 3i}, let $f(z) = \frac{f(z)}{k(z)}$ so that $h(z) = e^z$ and $k(z) = z^2(z^2+9)$

$$k'(z) = 4z^3+18z$$

$$\therefore \quad \text{Res}\{f(z);\, 3i\} = \frac{h(3i)}{k'(3i)}$$

$$= \frac{e^{3i}}{4(3i)^3+18(3i)}$$

$$= \frac{e^{3i}}{-108i+54i}$$

$$= -\frac{e^{3i}}{54i} = i\frac{e^{3i}}{54}$$

$$= \frac{i(\cos 3 + i\sin 3)}{54}$$

$$\text{Similarly Res}\{f(z);\, -3i\} = -\frac{(\sin 3 + i\cos 3)}{54}$$

Example 4 :

Find the residue of $\frac{ze^z}{(z-1)^3}$ at its pole

Solution :

Let $\quad f(z) = \frac{ze^z}{(z-1)^3}$

z = 1 is a pole of order 3 for f(z).

Let $g(z) = ze^z$ so that $g'(z) = e^z(z+1)$

$$g''(z) = e^z(z+2)$$

$$\therefore \quad \text{Res}\{f(z);\, 1\} = \frac{g''(1)}{2!} = \frac{3e}{2}$$

Exercise :

1. Find the order of each pole and find the residue at the poles for each of the following functions.

(i) $\frac{z}{z^2+1}$ (ii) $\frac{z+1}{z^2-2z}$ (iii) $\frac{e^{2z}}{(z-1)^2}$ (iv) $\frac{2z}{(z+4)(z-1)^2}$

2. Find the residue of $\frac{z^2-2z}{(z+1)^2(z^2+4)}$ at all its poles.

3. Find the residue of $\frac{1+e^z}{\sin z+z\cos z}$ at the pole $z = 0$.

4. Find the residue of $\frac{1}{(1+z^2)^n}$ at $z = i$.

5. Prove that

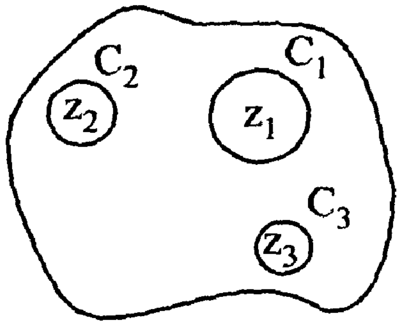
(i) $\text{Res}\left\{\tan z; \frac{\pi}{2}\right\} = -1$

(ii) $\text{Res}\left\{\frac{1-\cos z}{z^3}; 0\right\} = \frac{1}{2}$

6. Show that all the singular points of $\frac{1}{z(e^z-1)}$ are poles. Find the order of poles and find the radius at the poles.

CAUCHY'S RESIDUE THEOREM :

Let f(z) be a function which is analytic inside and on a simple closed curve C except for a finite number of singular points $z_1, z_2, z_3, \ldots, z_n$ inside C.



Then $\int_C f(z)dz = 2\pi i \sum_{j=1}^{n} \text{Res}\{f(z); z_j\}$.

Proof :

Let $C_1, C_2, \ldots, C_n$ be circles with centres $z_1, z_2, \ldots, z_n$ respectively such that all circles are interior to C and are disjoint with each other.

By Cauchy's theorem for multiply connected regions we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots + \int_{C_n} f(z)dz$$

$$= 2\pi i \text{Res}\{f(z); z_1\} + 2\pi i \text{Res}\{f(z); z_2\} + \ldots + 2\pi i \text{Res}\{f(z); z_n\}$$

(by definition of residue)

$$= 2\pi i \sum_{j=1}^{n} \text{Res}\{f(z); z_j\}$$

Example :

Evaluate $\int_C \frac{z^2 dz}{(z-2)(z+3)}$ where C is the circle $|z|=4$.

Let $$f(z) = \frac{z^2}{(z-2)(z+3)}$$

$z=2$ and $z=-3$ are simple poles for f(z) and both of them lie inside $|z|=4$.

$$\text{Res}\{f(z); 2\} = \underset{z\to 2}{\text{Lt}} (z-2)\left[\frac{z^2}{(z-2)(z+3)}\right] = \frac{4}{5}$$

$$\text{Res}\{f(z); -3\} = \underset{z\to -3}{\text{Lt}} (z+3)\left[\frac{z^2}{(z-2)(z+3)}\right] = -\frac{9}{5}$$

∴ By residue theorem $\int_C f(z)dz = 2\pi i\left[\frac{4}{5} + \left(-\frac{9}{5}\right)\right]$

$$\therefore \quad \int_C \frac{z^2 dz}{(z-2)(z+3)} = -2\pi i$$

ARGUMENT THEOREM :

Let f be a function which is analytic inside and on a simple closed curve C except for a finite number of poles inside C. Also let f(z) have no zeros on C. Then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz = N-P$ wher N is the number of zeros of f(z) inside C and P is the number of poles of f(z) inside C (A pole or zero of order m is counted m times).

Solution :

We observe that the singularities of the function $\frac{f'(z)}{f(z)}$ inside C are the poles and zeros of f(z) lying inside C.

Let z_0 be a zero of order n for f(z).

Let C_1 be a circle with centre z_0 such that it is the only zero of f(z) inside C_1.

Then $f(z) = (z-z_0)^n g(z)$ where g(z) is analytic and non zero inside C_1.

Hence $f'(z) = n(z-z_0)^{n-1} g(z)+(z-z_0)^n g'(z)$

$$\therefore \quad \frac{f'(z)}{f(z)} = \frac{n}{z-z_0}+\frac{g'(z)}{g(z)} \qquad \text{--------(1)}$$

Since g(z) is analytic and non zero inside C_1 $\frac{g'(z)}{g(z)}$ is also analytic and hence can be expanded as a Taylor's series about z_0.

$$\therefore \quad \text{Res}\left\{\frac{f'(z)}{f(z)};z_0\right\} = \text{coefficient of } \frac{1}{z-z_0} \text{ in (1)}$$

$$= n$$

Similarly if z_1 is a pole of order p for f(z), then $\text{Res}\left\{\frac{f'(z)}{f(z)};z_1\right\} = -p$.

Hence by Cauchy's residue theorem,

$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N-P$ where N is the number of zeros and P is the number of poles of f(z) within C.

Corollary :

If f(z) is analytic inside and on C and not zero on C, then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz=N$ where N is the number of zeros lying inside C.

Proof :

Since the number of poles is zero we have P=0.

Hence the result.

Rouche's Theorem :

If f(z) and g(z) are analytic inside and on a simple closed curve C and if $|g(z)|<|f(z)|$ on C then f(z)+g(z) and f(z) have the same number of zeros inside C.

Proof :

$$f(z)+g(z) = f(z)\left[1+\frac{g(z)}{f(z)}\right] = f(z)\phi(z) \text{ where } \phi(z) = 1+\frac{g(z)}{f(z)}.$$

Hence

$$[f(z)+g(z)]' = f'(z)+g'(z)$$

$$= f'(z)\phi(z)+f(z)\phi'(z)$$

$$\therefore \quad \frac{f'(z)+g'(z)}{f(z)+g(z)} = \frac{f'(z)\phi(z)+f(z)\phi'(z)}{f(z)\phi(z)}$$

$$= \frac{f'(z)}{f(z)}+\frac{\phi'(z)}{\phi(z)}$$

$$\therefore \quad \frac{1}{2\pi i}\int_C \frac{f'(z)+g'(z)}{f(z)+g(z)}dz = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz+\frac{1}{2\pi i}\int_C \frac{\phi'(z)}{\phi(z)}dz \qquad \text{-------(1)}$$

By hypothesis $|g(z)| < |f(z)|$ and hence

$$\left|\frac{g(z)}{f(z)}\right| < 1 \text{ on C}$$

$$\therefore \quad |\phi(z)-1| < 1 \text{ on C.}$$

Hence by maximum modulus theorem, $|\phi(z)-1|<1$ for every point z inside C.

$\therefore$ $\phi(z)\neq 0$ for every point inside C.

Hence

$$\int_C \frac{\phi'(z)}{\phi(z)}dz = \text{Number of zeros of } \phi(z) \text{ within C.}$$

$$= 0.$$

Hence from (1), we have

$$\frac{1}{2\pi i}\int_C \frac{f'(z)+g'(z)}{f(z)+g(z)}dz = \frac{1}{2\pi i}\int_C \frac{f'(z)dz}{f(z)}$$

$\therefore$ $N_1 = N_2$ where N_1 and N_2 denote respectively the number of zeros of f(z)+g(z) and f(z) inside C.

Note : We can deduce the Fundamental theorem of Algebra from Rouche's theorem.

Fundamental Theorem of Algebra :

A polynomial of degree n has n zeros.

Proof :

Let $a_0+a_1z+a_2z^2+...+a_nz^n$, where $a_n \neq 0$ be a polynomial of degree n.

Let $f(z) = a_n z^n$

and $g(z) = a_0+a_1+....+a_{n-1}z^{n-1}$

Clearly $\underset{z\to\infty}{\text{Lt}} \frac{g(z)}{f(z)} = 0$

Hence there exists a positive real number r such that $\left|\frac{g(z)}{f(z)}\right| < 1$ for all z with $|z|>r$.

Hence by Rouche's theorem f(z) and f(z)+g(z) have the same number of zeros inside the circle $|z| = r+1$. But 0 is a zero of multiplicity n for f(z). Hence the given polynomial f(z)+g(z) also has n zeros.

Worked Examples :

Example 1 :

Evaluate $\int_C \frac{2+3\sin\pi z}{z(z-1)^2}dz$ where C is a square having vertex at 3+3i, 3–3i, –3+3i, –3–3i.

Solution :

Let $f(z) = \frac{2+3\sin\pi z}{z(z-1)^2}$. Here z = 0 is a simple pole and z=1 is a double pole for f(z) and both of them lie within C.

$$\text{Res}\{f(z); 0\} = \underset{z\to 0}{\text{Lt}}\, z\left[\frac{2+3\sin\pi z}{z(z-1)^2}\right]$$

$$= 2$$

$$\text{Res}\{f(z); 1\} = \frac{g'(1)}{1!} \text{ where } g(z) = \frac{2+3\sin\pi z}{z}$$

$$\therefore \quad g'(z) = \frac{z3\pi\cos\pi z-(2+3\sin\pi z)}{z^2}$$

$$\therefore \quad g'(1) = -3\pi-2$$

$$\therefore \quad \text{Res}\{f(z);\, 1\} = -3\pi-2$$

$$\therefore \quad \int_C f(z)dz = 2\pi i[2-3\pi-2]$$

$$= -6\pi^2 i$$

Example 2 :

Evaluate $\int_C \tan z dz$ where C is $|z| = 2$.

Solution :

Let $$f(z) = \tan z = \frac{\sin z}{\cos z} = \frac{h(z)}{k(z)}$$

cos z has zeros at $z = (2n+1)\frac{\pi}{2}$, $n \in z$.

$\therefore$ f(z) has simple poles at $z = -\frac{\pi}{2}$ and $z = \frac{\pi}{2}$ inside the circle $|z|=2$.

$$\text{Res}\left\{f(z); \frac{\pi}{2}\right\} = \frac{h\left(\frac{\pi}{2}\right)}{k'\left(\frac{\pi}{2}\right)} = \frac{\sin\left(\frac{\pi}{2}\right)}{-\sin\left(\frac{\pi}{2}\right)} = -1$$

$$\text{Res}\left\{f(z); -\frac{\pi}{2}\right\} = \frac{h\left(-\frac{\pi}{2}\right)}{k'\left(-\frac{\pi}{2}\right)} = \frac{\sin\left(-\frac{\pi}{2}\right)}{-\sin\left(-\frac{\pi}{2}\right)} = -1$$

$\therefore$ By residue theorem

$$\int_C \tan z dz = 2\pi i[(-1)+(-1)]$$

$$= -4\pi i$$

Example 3 :

Prove that $\int_C \frac{e^{2z}}{(z+1)^3}dz = \frac{4\pi i}{e^2}$ where C is $|z| = \frac{3}{2}$.

Solution :

Let $$f(z) = \frac{e^{2z}}{(z+1)^3}$$

f(z) has a pole of order 3 at $z = -1$.

$$\text{Res}\{f(z); -1\} = \frac{g''(-1)}{2!} \text{ where } g(z) = e^{2z}$$

$$g'(z) = 2e^{2z} \text{ and } g''(z) = 4e^{2z}$$

$$\therefore \quad \text{Res}\{f(z); -1\} = \frac{4e^{-2}}{2!} = \frac{2}{e^2}$$

∴ By residue theorem

$$\int_C f(z)dz = 2\pi i\frac{2}{e^2} = \frac{4\pi i}{e^2}$$

Example 4 :

Let $$f(z) = \frac{z^2+1}{(z^2+2z+2)^2}$$

Evaluate $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$ where C is the circle $|z| = 4$.

Solution :

i and –i are zeros of order 1 and –1+i and –1–i are poles of order 2 for f(z). Also these zeros and poles lie inside C.

Hence number of zeros of f(z) = N = 2 and number of poles of f(z) = P = 4. (Poles are counted according to their multiplicity)

∴ By Argument theorem $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P = 2 - 4 = -2$.

Exercise :

1. Evaluate $\int_C \frac{3z-4}{z(z-1)}dz$ where C is the circle $|z| = 2$.

2. Prove that $\int_C \coth z\, dz = 0$ where C is the circle $|z| = 1$.

3. Prove that $\int_C z e^{1/z} dz = \pi i$ where C is the circle $|z| = 5$.

4. Evaluate $\int_C \frac{dz}{z^3(z-1)}$ where C is the circle $|z| = 3$.

5. Evaluate $\int_C \frac{e^z dz}{z(z-1)^2}$ where C is the circle $|z| = 2$.

6. Evaluate $\int_C \frac{e^{-z}}{z^2} dz$ where C is the circle $|z| = 1$.

7. Evaluate $\int_C \frac{dz}{z^3(z+4)}$ where C is (a) $|z|=2$ (b) $|z+2|=3$.

8. Prove that $\int_C \frac{e^z dz}{\cosh z} = 8\pi i$ where C is the circle $|z| = 5$.

EVALUATION OF DEFINITE INTEGRALS

Type I :

$\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$ where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

To evaluate this type of integral we substitute $z=e^{i\theta}$. As θ varies from 0 to 2π, z describes the unit circle $|z| = 1$.

Also,
$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$$

and
$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$$

Substituting these values in the given integrand the integral is transformed into $\int_C \theta(z)dz$ where $\theta(z) = f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)$ and C is the positively oriented unit circle $|z|=1$. The integral $\int_C \theta(z)dz$ can be evaluated using the residue theorem.

Worked Examples :

Example 1 :

Evaluate $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$

Solution :

Let $I = \int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$

Put $z = e^{i\theta}$

Then $dz = iz\,d\theta$

and $\sin\theta = \frac{z-z^{-1}}{2i}$

The given integral is transformed to

$$I = \int_C \frac{dz}{iz\left[5+4\left(\frac{z-z^{-1}}{2i}\right)\right]}$$ where C is the unit circle $|z|=1$

$$I = \int_C \frac{dz}{2z^2+5iz-2}$$

Let $f(z) = \frac{1}{2z^2+5iz-2}$

$$= \frac{1}{2(z+2i)\left(z+\frac{i}{2}\right)}$$

$\therefore$ $-2i$ and $-\frac{i}{2}$ are simple poles of f(z) and the pole $-\frac{i}{2}$ lies inside C.

Also $$\text{Res}\left\{f(z); -\frac{i}{2}\right\} = \underset{z\to -\frac{i}{2}}{\text{Lt}} \frac{1}{2(z+2i)} = \frac{1}{3i}$$

Hence by Cauchy's residue theorem

$$I = 2\pi i\left(\frac{1}{3i}\right) = \frac{2\pi}{3}$$

Example 2 :

Prove that

$$\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}} \qquad (-1<a<1)$$

Solution :

Put $$z = e^{i\theta}$$

Then $$\sin\theta = \frac{z-z^{-1}}{2i}$$

and $$dz = iz\, d\theta$$

$$\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \int_C \frac{dz}{iz\left[1+a\left(\frac{z-z^{-1}}{2i}\right)\right]}$$ where C is the unit circle

$$= \int_C \frac{2dz}{z\left[2i+a\left(z-z^{-1}\right)\right]}$$

$$= \int_C \frac{2dz}{az^2+2iz-a}$$

Let $$f(z) = \frac{2}{az^2+2iz-a}$$

The poles of f(z) are given by

$$z = \frac{-2i \pm \sqrt{-4+4a^2}}{2a}$$

$$= \frac{-i \pm i\sqrt{1-a^2}}{a} \quad (\because\ -1<a<1)$$

Let $$z_1 = \frac{-i + i\sqrt{1-a^2}}{a}$$

and $$z_2 = \frac{-i - i\sqrt{1-a^2}}{a}$$

We note that $|z_2| = \dfrac{1+\sqrt{1-a^2}}{|a|} > 1$ (since $-1<a<1$).

Also, since $|z_1\ z_2| = 1$ it follows that $|z_1|<1$. Hence there are no singular points on C and $z = z_1$ is the only simple pole inside C.

$$\text{Res}\{f(z); z_1\} = \underset{z\to z_1}{\text{Lt}} (z-z_1)\left[\frac{2/a}{(z-z_1)(z-z_2)}\right]$$

$$= \frac{2/a}{z_1 - z_2}$$

$$= \frac{1}{i\sqrt{1-a^2}}$$

By residue theorem

$$\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = 2\pi i\left[\frac{1}{i\sqrt{1-a^2}}\right]$$

$$= \frac{2\pi}{\sqrt{1-a^2}}$$

Example 3 :

Prove that $$I = \int_0^{\pi} \frac{a\,d\theta}{a^2 + \sin^2\theta}$$

$$= \frac{\pi}{\sqrt{a^2+1}} \quad (a>0)$$

Solution :

$$I = \int_0^{\pi} \frac{a\,d\theta}{a^2+\left(\frac{1-\cos 2\theta}{2}\right)}$$

$$= \int_0^{\pi} \frac{2a\,d\theta}{2a^2+1-\cos 2\theta}$$

$$= \int_0^{2\pi} \frac{a\,d\phi}{2a^2+1-\cos\phi}$$

$$= \frac{1}{i}\int_C \frac{a\,dz}{z\left[2a^2+1-\left(\frac{z+z^{-1}}{2}\right)\right]} \quad \text{(putting } z = e^{i\phi}\text{)}$$

$$= \frac{2a}{i}\int_C \frac{dz}{\left[2\left(2a^2+1\right)z-z^2-1\right]}$$

$$= 2ai\int_C \frac{dz}{z^2-2\left(2a^2+1\right)z+1}$$

$$= 2ai\int_C f(z)dz \qquad \text{-------(1)}$$

where $f(z) = \frac{1}{z^2-2\left(2a^2+1\right)z+1}$ and C is the unit circle $|z|=1$.

Poles of f(z) are the roots of $z^2-2(2a^2+1)z+1 = 0$.

$\therefore \quad z = \left(2a^2+1\right) \pm 2a\sqrt{a^2+1}$

Let $z_1 = \left(2a^2+1\right)+2a\sqrt{a^2+1}$;

$z_2 = \left(2a^2+1\right)-2a\sqrt{a^2+1}$

Clearly $|z_1|>1$ and $|z_1 z_2| = 1$ so that $|z_2| < 1$.

Hence the only pole inside C is $z = z_2$.

$$\text{Res}\{f(z); z_2\} = \operatorname*{Lt}_{z\to z_2} (z-z_2)\frac{1}{(z-z_1)(z-z_2)}$$

$$= \frac{1}{z_2 - z_1}$$

$$= \frac{1}{(-4a)\sqrt{a^2+1}}$$

From (1), $$I = 2\pi i\left[\frac{2ai}{-4a\sqrt{a^2+1}}\right]$$

$$= \frac{\pi}{\sqrt{a^2+1}}$$

Exercise :

1. Show that $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \frac{\pi}{2}$.

2. Show that $\int_0^{\pi} \frac{d\theta}{a+\cos\theta} = \frac{\pi}{\sqrt{a^2-1}} (a > 1)$

3. Show that $\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}} \left(a^2 < 1\right)$

4. Show that $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = 0$

Type 2 :

$\int_{-\infty}^{\infty} f(x)dx$ where $f(x) = \frac{g(x)}{h(x)}$ and g(x), h(x) are polynomials in x and the degree of h(x) exceeds that of g(x) by atleast two.

To evaluate this type of integral we take $f(z) = \frac{g(z)}{h(z)}$.

The poles of f(z) are determined by the zeros of the equation h(z) = 0.

Case (i) No pole of f(z) lies on the real axis.

We choose the curve C consisting of the interval [–r, r] on the real axis and the semi circle |z|=r lying in the upper half of the plane.

Here r is chosen sufficiently large so that all the poles lying in the upper half of the plane lie in the interior of C. Then we have

$$\int_C f(z)dz = \int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz$$ where C_1 is the semi-circle.

Since deg h(x)–deg f(x)≥2 it follows that $\int_{C_1} f(z)dz \to 0$ and $r\to\infty$ hence $\int_C f(z)dz$

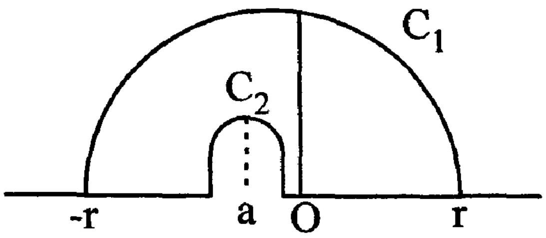
$= \int_{-\infty}^{\infty} f(x)dx.$

∴ $\int_{-\infty}^{\infty} f(x)dx$ can be evaluated by evaluating $\int_C f(z)dz$ which in turn can be evaluated by using Cauchy's residue theorem.

Case (ii) :

f(z) has poles lying on the real axis

Suppose a is a pole lying on the real axis. In this case we indent the real axis by a semi-circle C_2, of radius $\in$ with centre a lying in the upper half plane where $\in$ is chosn to be sufficiently small. Such an indenting must be done for every pole of f(z) lying on the real axis.



It can be proved that $\int_{C_2} f(z)dz = -\pi i \,\mathrm{Res}\{f(z);a\}$

By taking limit as $r\to\infty$ and $\in\to 0$ we obtain the rules of $\int_{-\infty}^{\infty} f(x)dx.$

Worked Examples :

Example 1 :

Evaluate $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}dx$

Proof :

Let $$f(z) = \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}$$

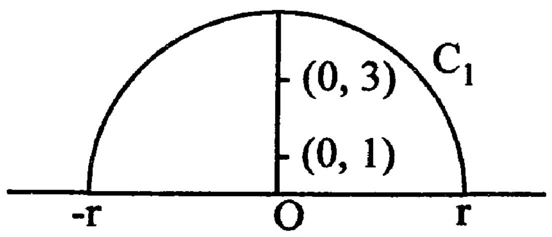
Poles of f(z) are the zeros of $z^4+10z^2+9 = 0$

$$z^4+10z^2+9 = (z^2+9)(z^2+1)$$

$\therefore$ $$z = \pm 3i; \pm i$$

$\therefore$ $z = 3i, -3i, i, -i$ are the simple poles of f(z).

Choose the contour C consisting of the interval $[-r, r]$ on the real axis and the semi-circle C_1, $|z|=r$ lying in the upper half of the plane.



$$\int_C f(z)dz = \int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz \quad \text{-------(1)}$$

The poles of f(z) lying within C are i and 3i and both of them are simple poles.

$\text{Res}\{f(z);\ i\} = \frac{h(i)}{k'(i)}$ where $h(z) = z^2-z+2$ and $k(z) = z^4+10z^2+9$ so that $k'(z) = 4z^3+20z$.

$\therefore$ $$\text{Res}\{f(z);\ i\} = \frac{-1-i+2}{-4i+20i} = \frac{1-i}{16i}$$

Similarly $$\text{Res}\{f(z);\ 3i\} = \frac{7+3i}{48i}$$

$\therefore$ $$\int_C f(z)dz = 2\pi i \times (\text{sum of the residues at the poles})$$

$$= 2\pi i\left(\frac{1-i}{16i} + \frac{7+3i}{48i}\right) = 2\pi i\left(\frac{10}{48i}\right)$$

$$= \frac{5\pi}{12}$$

From (1), $\int_{-r}^{r} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx + \int_{C_1} \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz = \frac{5\pi}{12}$

As $r \to \infty$ the integral over $C_1 \to 0$

$$\therefore \quad \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}$$

Example 2 :

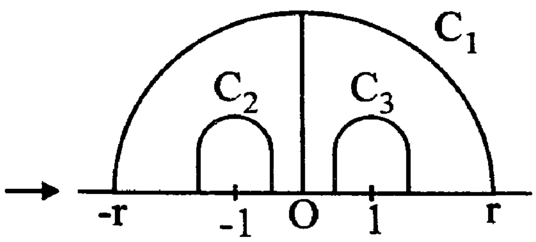
Prove that $\int_0^{\infty} \frac{x^4 dx}{x^6 - 1} = \frac{\pi\sqrt{3}}{6}$

Solution :

Let $$f(z) = \frac{z^4}{z^6 - 1}$$

The poles of f(z) are given by the sixth roots of unity namely, $e^{2n\pi i/6}$, n=0,1,2,3,4,5.

$\therefore$ f(z) has 2 simple poles on the real axis namely 1 and -1 and the two poles $e^{\pi i/3}$ and $e^{2\pi i/3}$ lie on the upper half of the plane.



Choose the contour C like this

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{-r}^{-1-\epsilon_1} f(x)dx + \int_{C_2} f(z)dz$$

$$+ \int_{-1+\epsilon_1}^{1-\epsilon_1} f(x)dx + \int_{C_3} f(z)dz + \int_{1+\epsilon_2}^{r} f(x)dx \quad \text{-----(1)}$$

$$\int_{C_2} f(z)dz = -\pi i \operatorname{Res}\{f(z);-1\}$$

$$= -\pi i\left[\frac{h(-1)}{k'(-1)}\right], \; h(z)=z^4 \text{ and } k(z)=z^6-1$$

$$= -\pi i\left[-\frac{1}{6}\right]$$

$$= \frac{\pi i}{6} \qquad \text{-------(2)}$$

Similarly $\int_{C_3} f(z)dz = -\pi i \operatorname{Res}\{f(z);1\}$

$$= -\pi i\left[\frac{h(1)}{k'(1)}\right]$$

$$= -\pi i\left(\frac{1}{6}\right)$$

$$= -\frac{\pi i}{6} \qquad \text{--------(3)}$$

Also $\int_{C} f(z)dz = 2\pi i\left[\operatorname{Res}\{f(z);e^{i\pi/3}\} + \operatorname{Res}\{f(z);e^{2\pi i/3}\}\right]$

$$= 2\pi i\left[\frac{h(e^{i\pi/3})}{k'(e^{i\pi/3})} + \frac{h(e^{i2\pi/3})}{k'(e^{i2\pi/3})}\right]$$

$$= 2\pi i\left[\frac{e^{i4\pi/3}}{6e^{i5\pi/3}} + \frac{e^{i8\pi/3}}{6e^{i10\pi/3}}\right]$$

$$= \frac{\pi i}{3}\left(e^{-i\pi/3} + e^{-i2\pi/3}\right)$$

$$= \frac{\pi i}{3}\left(e^{i\pi/3} - e^{i\pi/3}\right)$$

$$= \frac{\pi i}{3}\left(-2i\sin\frac{\pi}{3}\right)$$

$$= \frac{\pi\sqrt{3}}{3} \qquad \text{-------(4)}$$

Substituting (2), (3), (4) and (1) and taking limit as $\in_1$ and $\in_2 \to 0$ $r \to \infty$ we get

$$\int_{-\infty}^{\infty} \frac{x^4}{x^6-1} dx + \frac{\pi i}{6} - \frac{\pi i}{6} = \frac{\pi\sqrt{3}}{3}$$

$$\therefore \quad 2\int_{0}^{\infty} \frac{x^4}{x^6-1} dx = \frac{\pi\sqrt{3}}{3}$$

$$\therefore \quad \int_{0}^{\infty} \frac{x^4}{x^6-1} dx = \frac{\pi\sqrt{3}}{6}$$

Example 3 :

Evaluate $\quad I = \int_{0}^{\infty} \frac{dx}{\left(x^2+a^2\right)^2}$

Solution :

Since $\frac{1}{\left(x^2+a^2\right)^2}$ is an even function we have

$$\int_{-\infty}^{\infty} \frac{dx}{\left(x^2+a^2\right)^2} = 2\int_{0}^{\infty} \frac{dx}{x^2+a^2}$$

Let $\quad f(z) = \frac{1}{\left(z^2+a^2\right)^2}$

Poles of f(z) are the roots of $(z^2+a^2)^2 = 0$

Now, $(z^2+a^2)^2 = (z+ai)^2 (z-ai)^2$

$\therefore$ ai and $-ai$ are double poles of f(z).

Choose the contour C consisting of the interval $[-r, r]$ on the real axis and the semi circle C_1 with centre 0 and radius that lies in the upper half plane.

$$\therefore \int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz = \int_{C} f(z)dz \qquad \text{--------(1)}$$

The poles of f(z) lying within C is $z = ai$

$$\text{Res}\{f(z);ai\} = \frac{1}{1!}g'(ai) \quad \text{where } g(z) = \frac{1}{(z+ai)^2}$$

$$g'(z) = -2(z+ai)^{-3}$$

$$\therefore \quad g'(ai) = \frac{1}{4a^3i}$$

$$\therefore \quad \text{Res}\{f(z);ai\} = \frac{1}{4a^3i}$$

$$\therefore \quad \int_C f(z)dz = 2\pi i\left(\frac{1}{4a^3i}\right) = \frac{\pi}{2a^3}$$

$$\therefore \int_{-r}^{r} \frac{dx}{\left(x^2+a^2\right)^2} + \int_{C_1} f(z)dz = \frac{\pi}{2a^3}$$

When $r \to \infty$ the integral over $C_1 \to 0$

$$\therefore \quad \int_{-\infty}^{\infty} \frac{dx}{\left(x^2+a^2\right)^2} = \frac{\pi}{2a^3}$$

$$\therefore \quad \int_{0}^{\infty} \frac{dx}{\left(x^2+a^2\right)^2} = \frac{\pi}{4a^3}$$

Exercise :

Prove the following by using Cauchy's residue theorem.

(i) $\displaystyle\int_0^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2}$

(ii) $\displaystyle\int_0^{\infty} \frac{x^2dx}{\left(x^2+1\right)^2} = \frac{\pi}{4}$

(iii) $\displaystyle\int_0^{\infty} \frac{dx}{x^4+x^2+1} = \frac{\pi\sqrt{3}}{6}$

(iv) $\displaystyle\int_0^{\infty} \frac{2x^2-1}{x^4+5x^2+4}dx = \frac{\pi}{4}$

(v) $\int_{0}^{\infty}\frac{x^2dx}{\left(x^2+a^2\right)^3}=\frac{\pi}{16a^3}$

(vi) $\int_{-\infty}^{\infty}\frac{dx}{(x-1)\left(x^2+2\right)}=\frac{-\sqrt{2}\pi}{6}$

Types 3 :

$\int_{-\infty}^{\infty}\frac{g(x)}{h(x)}\cos ax\,dx$ or $\int_{-\infty}^{\infty}\frac{g(x)}{h(x)}\sin ax\,dx$ where g(x) and h(x) are real polynomials such that degree of h(x) exceeds that of g(x) by atleast one and a>0.

Case (i)

h(x) has no zeros on the real axis.

In this case take $f(z)=\frac{g(z)}{h(z)}e^{iaz}$

∴ f(z) has no poles on the real axies.

Choose the contour as in type 2 and proceeding as in type 2 we get the value of $\int_{-\infty}^{\infty}\frac{g(x)}{h(x)}e^{iax}dx$.

Taking the real and imaginary parts of $\int_{-\infty}^{\infty}\frac{g(x)}{h(x)}e^{iax}dx$ we obtain the value $\int_{-\infty}^{\infty}\frac{g(x)}{h(x)}\cos ax\,dx$ and $\int_{-\infty}^{\infty}\frac{g(x)}{h(x)}\sin ax\,dx$

Case (ii)

h(x) has zeros of order one on the real axis

Take $f(z)=\frac{g(z)}{h(z)}e^{iaz}$. We notice that f(z) has real poles. Suppose a is a real zero of h(x) on the real axis. In this case we indent the real axis at a as in type 2 case (ii) and evaluate the integral.

To prove that the integral over the upper semicircle tends to zero as $r \to \infty$, we use the following Lemma.

Jordon's Lemma :

Let f(z) be a function of the complex variable z satisfying the following conditions.

(i) f(z) is analytic in upper half plane except at a finite number of poles.

(ii) $f(z) \to 0$ uniformly as $|z| \to \infty$ with $0 \le \arg z \le p$

(iii) a is a positive integer.

Then $\underset{z \to \infty}{\text{Lt}} \int_C f(z) e^{iaz}\, dz = 0$ where C is a semi circle with centre at the origin and radius r.

Worked Examples :

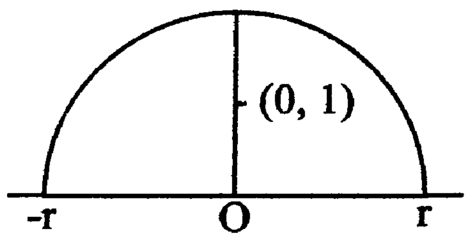
Example 1 :

Prove that $\int_0^{\infty} \frac{\cos x\, dx}{1+x^2} = \frac{\pi}{2e}$

Solution :

Let $f(z) = \frac{e^{iz}}{1+z^2}$

The poles of f(z) are given by i and –i. Choose the contour C as shown in the figure.



The pole of f(z) that lies within C is i. Hence by residue theorem

$$\int_C f(z) dz = 2\pi i \, \text{Res}\{f(z); i\}$$

$$= 2\pi i \frac{h(i)}{k'(i)} \text{ where } h(z) = e^{iz}$$

and $k(z) = 1+z^2$

$$\int_C f(z)dz = \frac{2\pi i e^{-1}}{2i} = \frac{\pi}{e}$$

$$\therefore \int_{-r}^{r} \frac{e^{iax}}{x^2+1}dx + \int_{C_1} \frac{e^{iaz}}{z^2+1}dz = \frac{\pi}{e}$$

When $r \to \infty$ the integral over C_1 tends to zero.

$$\therefore \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}dx = \frac{\pi}{e}$$

Equating real parts we get

$$\int_{-\infty}^{\infty} \frac{\cos x \, dx}{1+x^2} = \frac{\pi}{e}$$

$$\therefore 2\int_{0}^{\infty} \frac{\cos x}{1+x^2}dx = \frac{\pi}{e} \left(\text{since } \frac{\cos x}{1+x^2} \text{ is an even function}\right)$$

$$\therefore \int_{0}^{\infty} \frac{\cos x}{1+x^2}dx = \frac{\pi}{2e}$$

Example 2 :

Prove that $\int_{-\infty}^{\infty} \frac{\sin x \, dx}{x^2+4x+5} = -\frac{\pi \sin 2}{e}$

Solution :

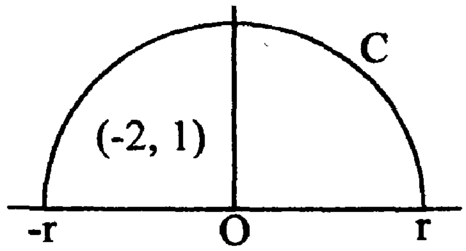
Let $$f(z) = \frac{e^{iz}}{z^2+4z+5}$$

The poles of f(z) are the roots of the equation

$$z^2+4z+5 = 0$$

They are given by $$z = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$$

Choose the contour C as this : $-2+i$ is the only pole of f(z) that lies within C and it is a simple pole.



Hence by Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i \operatorname{Res}\{f(z); -2+i\}$$

$$= 2\pi i \frac{h(-2+i)}{k'(-2+i)} \text{ where } h(z) = e^{iz}$$

and $$k(z) = z^2+4z+5$$

$$\therefore \int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz = \frac{\pi e^{-2i}}{e}$$

Since the integral over C_1 tends to zero as $r \to \infty$ we have

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi e^{-2i}}{e} = \frac{\pi}{e}(\cos 2 - i \sin 2)$$

Equating imaginary parts we get

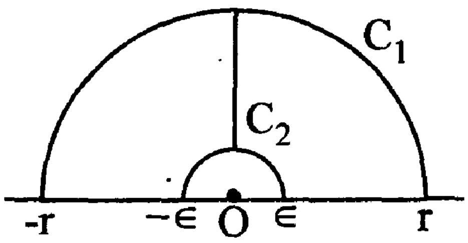
$$\int_{-\infty}^{\infty} \frac{\sin x\, dx}{x^2+4x+5} = \frac{-\pi \sin 2}{e}$$

Example 3 :

Prove that $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$

Solution :

Let $f(z) = \frac{e^{iz}}{z}$. The only singular point of f(z) is 0 which is a simple pole and it lies on the real axis. Choose the contour C as shown in the figure.



Then $\int_C f(z)dz = \int_{-r}^{-\epsilon} f(x)dx + \int_{C_2} f(z)dz + \int_{+\epsilon}^{r} f(x)dx + \int_{C_1} f(z)dz$ ------(1)

Since f(z) is analytic within C, $\int_C f(z)dz = 0$ ------(2)

Also $\int_{C_1} f(z)dz = -\pi i(\text{Res}\{f(z);0\})$

$= -\pi i e^0 = -\pi i$ ------(3)

Further the integral over C_1 tends to 0 as $r \to \infty$.

Hence using (2) and (3) in (1) and taking limit as $r \to \infty$ we get

$$0 = \int_{-\infty}^{0} f(x)dx - \pi i + \int_{0}^{\infty} f(x)dx$$

$$\therefore \int_{-\infty}^{\infty} f(x)dx = \pi i$$

Equating the imaginary parts we get

$$\int_{-\infty}^{\infty} \frac{\sin x dx}{x} = \pi$$

$$\therefore \int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} \left(\text{since } \frac{\sin x}{x} \text{ is an even function}\right)$$

Exercise :

1. Evaluate the following integration with the help of residues.

(i) $\int_{-\infty}^{\infty} \frac{\sin x\, dx}{x^2 - 2x + 5}$ $\left(\text{Ans.:} \frac{\pi \sin 1}{2e^2}\right)$

(ii) $\int_{0}^{\infty} \frac{\cos ax\, dx}{\left(x^2 + b^2\right)^2}$ $\left(\text{Ans.:} \frac{\pi}{4b^3}(1 + ab)e^{-ab}\right)(a > 0, b > 0)$

(iii) $\int_{0}^{\infty} \frac{x \sin x\, dx}{x^2 + a^2}$ $\left(\text{Ans.:} \frac{\pi e^{-a}}{2}\right)$

(iv) $\int_{-\infty}^{\infty} \frac{(a \cos x + x \sin x)}{x^2 + a^2} dx$ $\left(\text{Ans.:} 2\pi e^{-a}\right)$

(v) $\int_{-\infty}^{\infty} \frac{x \sin ax}{x^4 + 4} dx$ $\left(\text{Ans.:} \frac{\pi e^{-a} \sin a}{2}\right)$