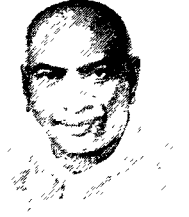




MADURAI KAMARAJ UNIVERSITY



(University with Potential for Excellence)

DISTANCE EDUCATION

B.Sc. SECOND YEAR

MATHEMATICS

PART III

PAPER - V

MECHANICS

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SYLLABUS
B.Sc., Second Year
Major Paper – V MECHANICS

UNIT-A
STATICS

UNIT I

Forces acting at a point – Resultant and components of forces – Parallelogram law of forces – Triangle law of forces – converse – Lami's theorem.

UNIT II

Resolution of force – Theorems resolved parts – Resultant of any number of coplanar forces – condition of equilibrium.

UNIT III

Force acting on a rigid body – Parallel forces – Resultant of two like and unlike parallel forces.

UNIT IV

Moment of a force – Varignon's theorem – Three forces acting on a rigid body and simple problems.

UNIT V

Friction – Laws of friction – coefficient of friction – Angle of friction – cone of friction – Problems.

UNIT B DYNAMICS

UNIT VI

Projectiles – path of projectiles a parabola – Range of projection – Range of a particle projected on an inclined plane etc.,

UNIT VII

Impact – impulses – Impact on a fixed plane – direct and oblique impacts.

UNIT VIII

Simple harmonic motion – composition of S.H.M. – related problems.

UNIT IX

Central orbits – components of velocity and acceleration along and perpendicular to the radius vector.

UNIT X

Differential equation of a central orbit equation.

Reference book

1. Statics – M.K. Venkatraman
2. Dynamics – M.K. Venkatraman

SCHEME

UNIT – A

STATICS

UNIT – 1

Forces Acting at a point

1. Resultant and Components
2. Parallelogram law of forces
3. Triangle law of forces
4. Converse of the triangle law of forces
5. Lami's Theorem.

UNIT – 2

1. Resolution of a force
2. Theorem of resolved parts.
3. Resultant of any number of coplanar forces acting at a point
4. Conditions of Equilibrium

UNIT – 3

1. Forces acting on a rigid body
2. Resultant of two like and unlike parallel forces

UNIT – 4

1. Moment of a force
2. Varignon's Theorem
3. Three forces acting on a rigid body

UNIT – 5

1. Friction
2. Coefficient of Friction
3. Laws of Friction
4. Angle of Friction
5. Cone of Friction

UNIT –B

DYNAMICS

UNIT – 6

1. Projectile
2. Path of a Projectile.
3. Range of Projectile
4. Range of a particle projected on an inclined plane.

UNIT – 7

1. Impact
2. Impact on a Fixed plane.
3. Impulses.
4. Direct impact of two smooth spheres.
5. Oblique impact of two smooth spheres.

UNIT – 8

1. Simple Harmonic Motion
2. Composition of two simple Harmonic Motions.

UNIT – 9

1. Central orbits
2. Components of velocity and Acceleration along and perpendicular to the radius vector

UNIT – 10

1. Differential Equation of central orbit
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MECHANICS
STATICS
UNIT – I
FORCES ACTING AT A POINT

1.1 Resultant and Components:

Definition: 1.1

If two or more forces $F_1, F_2, F_3 \dots$ etc act on a rigid body and if a single force R can be found whose effect on the body is the same as that of all the forces $F_1, F_2, F_3 \dots$ etc. and the forces F_1, F_2, F_3 etc. are called the components of the force R .

2. Simple cases of finding the Resultant:

If two forces P and Q act in the same direction simultaneously on a particle, the resultant is clearly equal to a force, $P+Q$ acting in the same direction on it. If however P and Q act in opposite directions, their resultant is clearly equal to $P-Q$ and acts in direction of the greater force.

When two forces acting at a point are in different directions (ie.) are inclined to each other their resultant can be found with help of a fundamental theorem in statics known as the law of the parallelogram of Forces.

1.2 Parallelogram law of Forces:

Theorem:

If two forces acting at a point be represented in magnitude and direction, by the sides of a parallelogram drawn from the point their resultant is represented both in magnitude and direction by the diagonal of the parallelogram drawn through that point.

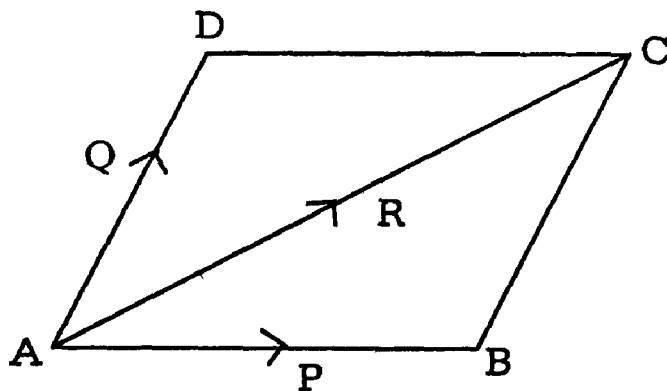


Fig.1

If the two forces P and Q acting at A are represented in magnitude and direction by the straight lines AB and AD and if the Parallelogram DAB be completed, then the diagonal AC will represent in magnitude and direction the resultant of P and Q.

In the language of vectors, the above law can be put as

$$\overline{AB} + \overline{AD} = \overline{AC}.$$

4. Analytical Expression for the Resultant of two forces acting at a point:-

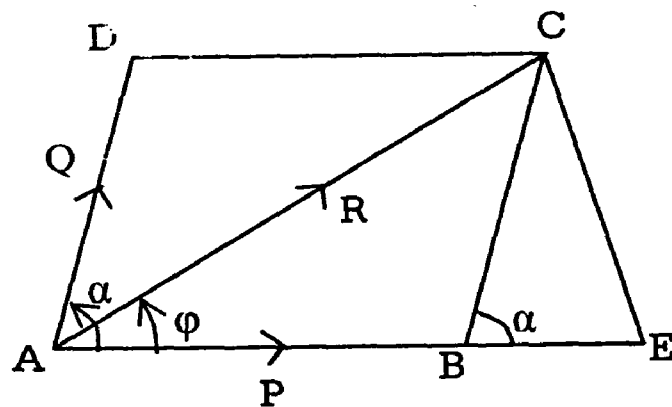


Fig.2

Let the two forces P and Q acting at A be represented by AB and AD and let the angle between them be α .

(ie) $\angle BAD = \alpha$

Complete the Parallelogram BAD.

Then the diagonal AC will represent the resultant. Let R be the magnitude of the result and let it make an angle ϕ with P.

(ie) $\angle CAB = \phi$

Draw $CE \perp$ to AB.

$BC = AD = Q$

From the right angled $\triangle CBE$

$$\sin \angle CBE = \frac{CE}{BC}$$

$$(ie) \sin \alpha = \frac{CE}{Q}$$

$$\therefore CE = Q \sin \alpha \quad \text{———— (i)}$$

$$\cos \alpha = \frac{BE}{BC} = \frac{BE}{Q}$$

$$\therefore BE = Q \cos \alpha \quad \text{———— (ii)}$$

$$\begin{aligned} \text{Now, } R^2 &= AC^2 = AE^2 + CE^2 = (AB + BE)^2 + CE^2 \\ &= (P + Q \cos \alpha)^2 + (Q \sin \alpha)^2 \\ &= P^2 + 2PQ \cos \alpha + Q^2 \end{aligned}$$

$$\therefore R = \sqrt{P^2 + 2PQ \cos \alpha + Q^2} \quad (1)$$

$$\text{Also } \tan \phi = \frac{CE}{AE} = \frac{Q \sin \alpha}{P + Q \cos \alpha} \quad (2)$$

(1) gives the magnitude and (2) the direction of the resultant in terms of P, Q and α .

Corollary 1:

If the forces P and Q are at right angles to each other, then $\alpha = 90^\circ$, $\cos \alpha = \cos 90^\circ = 0$ and

$$\sin \alpha = \sin 90^\circ = 1.$$

The above results become simpler & we have $R = \sqrt{P^2 + Q^2}$ and $\tan \phi = \frac{Q}{P}$

These results may be easily inferred.

Since the parallelogram becomes a rectangle.

Corollary 2:

If the forces are equal (ie) $Q = P$, then

$$R = \sqrt{P^2 + 2P^2 \cos \alpha + P^2} = \sqrt{2P^2(1 + \cos \alpha)}$$
$$= \sqrt{2P^2 \cdot 2 \cos^2 \frac{\alpha}{2}} = 2P \cos \frac{\alpha}{2}.$$

$$\text{and } \tan \phi = \frac{P \sin \alpha}{P + P \cos \alpha} = \frac{\sin \alpha}{1 + \cos \alpha}$$

$$= \frac{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{2 \cos^2 \frac{\alpha}{2}} = \tan \frac{\alpha}{2}$$

$$\text{(ie) } \phi = \frac{\alpha}{2}.$$

Thus the resultant of two equal forces P, P at an angle α is $2P \cos \frac{\alpha}{2}$ in a direction bisecting the angle between them.

This fact (that $\phi = \frac{\alpha}{2}$) is obvious otherwise, as the parallelogram becomes a rhombus.

Corollary: 3

Let the magnitudes P and Q of two forces acting at an angle α be given.

Then their resultant R is greatest when $\cos \alpha$ is greatest.

(ie) When $\cos \alpha = 1$ or $\alpha = 0^\circ$

In this case, the forces act along the same line in the same direction and $R = P + Q$.

The least value of R occurs when $\cos \alpha$ is least.

(ie) When $\cos \alpha = -1$ or $\alpha = 180^\circ$

In this case, the forces act along the same line but in opposite directions and $R = P - Q$.

Example: -1

The resultant of two forces P, Q acting at a certain angle X and that of P, R acting at the same angle is also X. The resultant of Q, R again acting at the same angle is Y. Prove that

$$P = (X^2 + QR)^{\frac{1}{2}} = \frac{QR(Q+R)}{Q^2 + R^2 - Y^2}$$

Prove also that, if $P + Q + R = 0$, $Y = X$.

Solution:-

Let P and Q act at an angle α . From the given data.

We have the following results

$$X^2 = P^2 + Q^2 + 2PQ \cos \alpha \tag{1}$$

$$X^2 = P^2 + R^2 + 2PR \cos \alpha \tag{2}$$

$$\text{and } Y^2 = Q^2 + R^2 + 2QR \cos \alpha \tag{3}$$

$$(1) - (2) \text{ gives } 0 = Q^2 - R^2 + 2P \cos \alpha (Q - R)$$

$$\text{(ie) } 0 = (Q - R) (Q + R + 2P \cos \alpha)$$

But $Q \neq R$ and So $Q - R$ is $\neq 0$.

$$\therefore Q + R + 2P \cos \alpha = 0$$

$$\text{or } \cos \alpha = (-) \frac{Q + R}{2P} \tag{4}$$

Substituting (4) in (1)

We have

$$X^2 = P^2 + Q^2 + 2PQ - \left(\frac{Q + R}{2P} \right) = P^2 + Q^2 - Q^2 - QR$$

$$\text{(or)} \quad P^2 = X^2 + QR, \text{ (ie)} \quad P = (X^2 + QR)^{\frac{1}{2}}$$

Substituting (4) in (3)

$$\text{We have } Y^2 = Q^2 + R^2 + 2QR - \left(\frac{Q+R}{2P} \right)$$

$$= Q^2 + R^2 - \frac{QR(Q+R)}{P}$$

$$\therefore \frac{QR(Q+R)}{P} = Q^2 + R^2 - Y^2$$

$$P = \frac{QR(Q+R)}{Q^2 + R^2 - Y^2}$$

If $P+Q+R = 0$. Then $Q + R = -P$

$$\therefore \text{From (4), } \cos \alpha = - \frac{Q+R}{2P} = \frac{P}{2P} = \frac{1}{2}$$

Putting $\cos \alpha = \frac{1}{2}$ in (2) and (3)

$$\text{We have } X^2 = P^2 + R^2 + PR. \tag{5}$$

$$\& \quad Y^2 = Q^2 + R^2 + QR \tag{6}$$

(5) - (6) gives

$$X^2 - Y^2 = P^2 - Q^2 + PR - QR$$

$$= (P - Q)(P + Q + R)$$

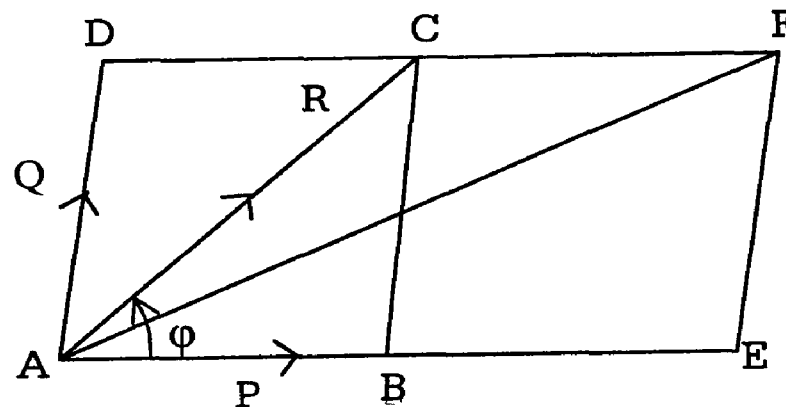
$$= (P - Q) \cdot 0$$

$$X^2 - Y^2 = 0$$

$$\therefore X = Y$$

Example: 2

If the resultant R of two forces P and Q inclined to one another at any given angle makes an angle ϕ with the direction of P , Show that the resultant of forces $(P + R)$ and Q acting at the same angle will make an angle $\frac{\phi}{2}$ with the direction of $P + R$.

Solution: First Method:

Let $\overline{AB} = P$ and $\overline{AD} = Q$

From $\parallel_{gm} ABCD$

$$\overline{AB} + \overline{AD} = \overline{AC} = R.$$

To mark the force $P + R$, Produce AB to E so that $BE = AC$.

In the $\parallel_{gm} DAEF$

\overline{AF} gives the new resultant.

In $\triangle CAF$, $CA = CF$ (each representing R in magnitude)

$$\therefore \angle CAF = \angle CFA$$

$$= \angle FAE \text{ (alternate angles)}$$

(ie) AF bisects $\angle CAB$

Second Method:

The resultant of $P + R$ and Q can be found in two stages. First the resultant of P along AB and Q along AD is a force R along AC . Secondly; we have to find the resultant of the forces R along AC with an extra force R along AB . As there are equal, the final resultant bisects the angle BAC .

Example: (U.Q)

Two forces P and Q acting at a point have a resultant R . If Q is doubled, R is doubled. Again if Q be reversed in direction, then also R is doubled. Show that $P^2: Q^2: R^2 = 2:3:2$.

Solution:

Let P and Q act at an angle. α

Their resultant R is given by

$$R^2 = P^2 + Q^2 + 2PQ \cos \alpha \quad (1)$$

When Q is doubled (So that the angle between the $2Q$ and P is again α), R is doubled.

$$\text{Hence } (2R)^2 = P^2 + (2Q)^2 + 2P \cdot 2Q \cos \alpha$$

$$\text{or } 4R^2 = P^2 + 4Q^2 + 4PQ \cos \alpha \quad (2)$$

When Q is reversed. [Angle between the original forces now changes to $(180^\circ - \alpha)$], R is doubled.

$$\text{Hence } (2R)^2 = P^2 + Q^2 + 2PQ \cos (180^\circ - \alpha)$$

$$\text{(or) } 4R^2 = P^2 + Q^2 - 2PQ \cos \alpha \quad (3)$$

Adding (1) & (3)

$$\text{We get } 5R^2 = 2P^2 + 2Q^2 \quad (4)$$

Multiplying (3) by 2 and adding with (2)

We get

$$12R^2 = 3P^2 + 6Q^2$$

$$\text{or } 4R^2 = P^2 + 2Q^2 \quad (5)$$

From (4) & (5)

We obtain on subtraction $R^2 = P^2$.

Substitution for R in (5) gives

$$4P^2 = P^2 + 2Q^2$$

$$3P^2 = 2Q^2$$

$$\text{(ie) } Q^2 = \frac{3}{2} P^2$$

$$\text{Thus } P^2 : Q^2 : R^2 = P^2 : \frac{3}{2} P^2 : P^2$$

$$P^2 : Q^2 : R^2 = 2 : 3 : 2.$$

Example:

Two forces of magnitudes $P \cos A$, $P \cos B$ act along the sides CA, CB of a triangle ABC. Prove that their resultant is $P \sin C$ and that it makes an angle $90^\circ - B$ with CA.

Solution:

The angle between the forces $P \cos A$, $P \cos B$ along CA and CB in C.

Hence their resultant R is given by

$$\begin{aligned} R^2 &= (P \cos A)^2 + (P \cos B)^2 + 2(P \cos A)(P \cos B) \cos C \\ &= P^2 [\cos^2 A + \cos^2 B + 2 \cos A \cos B \cos C] \end{aligned}$$

$$\text{But } \cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C.$$

$$\text{Hence } R^2 = P^2 [1 - \cos^2 C] = P^2 \sin^2 C$$

$$\text{or } R = P \sin C.$$

If ϕ is the angle made by the resultant with CA, then

$$\tan \phi = \frac{P \cos B \sin C}{P \cos A + P \cos B \cos C}$$

$$\tan \phi = \frac{\cos B \sin C}{\cos A + \cos B \cos C}$$

But $\cos A = \cos (180^\circ - B - C) = -\cos (B + C)$

$$\cos A = -[\cos B \cos C - \sin B \sin C]$$

or $\cos A + \cos B \cos C = \sin B \sin C$

$$\text{Hence } \tan \phi = \frac{\cos B \sin C}{\sin B \sin C} = \cot B.$$

$$= \tan (90 - B)$$

$$\therefore \phi = 90^\circ - B.$$

Example: (U.Q)

Two forces $P + Q$ and $P - Q$ make an angle 2α with one another and their resultant makes an angle θ with the bisector of the angle between them. Show that $P \tan \theta = Q \tan \alpha$.

Solution:

Since the resultant makes an angle θ with the bisector of the angle between the forces, we find it makes $\alpha - \theta$ with the first force

$$\begin{aligned} \text{Hence } \tan (\alpha - \theta) &= \frac{(P - Q) \sin 2\alpha}{(P + Q) + (P - Q) \cos 2\alpha} \\ &= \frac{(P - Q) \sin 2\alpha}{P(1 + \cos 2\alpha) + Q(1 - \cos 2\alpha)} \\ &= \frac{(P - Q) 2 \sin \alpha \cos \alpha}{P(2 \cos^2 \alpha) + Q(2 \sin^2 \alpha)} \end{aligned}$$

$$\text{(ie) } \frac{\tan \alpha - \tan \theta}{1 + \tan \alpha \tan \theta} = \frac{(P - Q) \tan \alpha}{P + Q \tan^2 \alpha}$$

Cross multiplying and simplifying.

We get

$$Q \tan \alpha (1 + \tan^2 \alpha) = P \tan \theta (1 + \tan^2 \alpha)$$

Since $1 + \tan^2 \alpha$ is non-zero, we get on division by $1 + \tan^2 \alpha$.

$$Q \tan \alpha = P \tan \theta$$

Example:

The greatest resultant that two forces can have is of magnitude P and the least is of magnitude Q . Show that, when they act at an angle α , their resultant is of magnitude $\sqrt{P^2 \cos^2 \frac{1}{2} \alpha + Q^2 \sin^2 \frac{1}{2} \alpha}$.

Solution:

If F_1 and F_2 are two forces, then their greatest resultant is obtained when they are in the same direction through a point.

$$\text{Hence } P = F_1 + F_2. \quad (1)$$

Their resultant is least, if they act in opposite directions through a point.

$$\text{(ie) } Q = F_1 - F_2 \quad (2)$$

Suppose now F_1 and F_2 act through a point at an angle α .

Then their resultant R is given by

$$R^2 = F_1^2 + F_2^2 + 2F_1 \cdot F_2 \cos \alpha \quad (3)$$

Consider

$$P^2 \cos^2 \frac{1}{2} \alpha + Q^2 \sin^2 \frac{1}{2} \alpha$$

$$= (F_1 + F_2)^2 \cos^2 \frac{1}{2} \alpha + (F_1 - F_2)^2 \sin^2 \frac{1}{2} \alpha \text{ by (1) \& (2)}$$

$$= F_1^2 \left(\cos^2 \frac{1}{2} \alpha + \sin^2 \frac{1}{2} \alpha \right) + F_2^2 \left(\cos^2 \frac{1}{2} \alpha + \sin^2 \frac{1}{2} \alpha \right) \\ + 2F_1 F_2 \left(\cos^2 \frac{1}{2} \alpha - \sin^2 \frac{1}{2} \alpha \right)$$

$$= F_1^2 + F_2^2 + 2F_1 \cdot F_2 \cos \alpha$$

$$= R^2 \text{ by (3)}$$

$$\text{Hence } R = \sqrt{P^2 \cos^2 \frac{1}{2} \alpha + Q^2 \sin^2 \frac{1}{2} \alpha}.$$

Exercises

- Two forces of given magnitudes P and Q act at a point at an angle α . What will be i) maximum value of the resultant?
- The resultant of two forces P and Q is at right angles to P . Show that the angle between the forces is $\cos^{-1} \left(-\frac{P}{Q} \right)$.
- The resultant of two forces P and Q is of magnitude P . Show that, if P be doubled, the new resultant is at right angles to Q and its magnitude will be $\sqrt{4P^2 - Q^2}$.
- Two equal forces act on a particle, find the angle between them when the square of their resultant is equal to three times their product.
- If the resultant of forces $3P$, $5P$ is equal to $7P$ find.
 - the angle between the forces.
 - the angle which the resultant makes with the first force.
- Two equal forces are inclined at an angle 2θ . Their resultant is 3 times as great as when they are inclined at an angle 2ϕ . Show that $\cos \theta = 3 \cos \phi$.
- The resultant of two forces P and Q acting at an angle θ is equal to $(2m+1) \sqrt{P^2 + Q^2}$; When they act at an angle $90^\circ - \theta$, the resultant is $(2m-1) \sqrt{P^2 + Q^2}$. Prove that $\tan \theta = \frac{m-1}{m+1}$.

Triangle law of Forces: 1.3

A simple deduction from the Parallelogram of forces is the following theorem, known as the Triangle of forces.

If three forces acting at a point can be represented in magnitude and direction by the sides of a triangle taken in order, they will be in equilibrium.

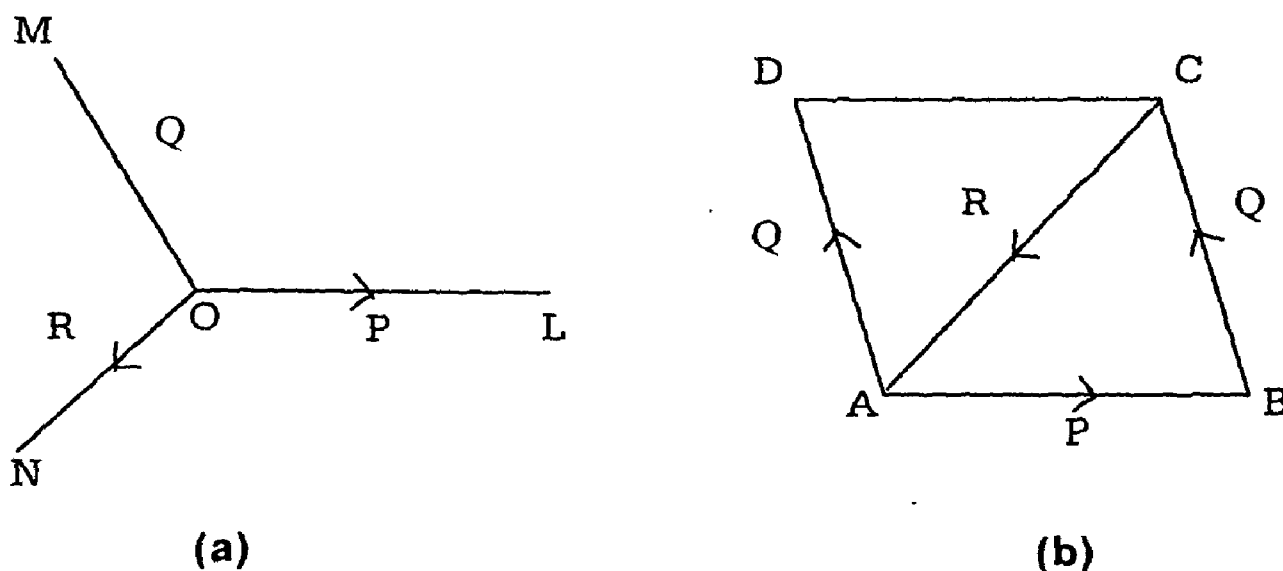


Fig. 1

Let the forces P, Q, R act at a point O and be represented in magnitude and direction by the sides AB, BC, CA of the triangle ABC. We have to prove that they will be in equilibrium.

Complete the Parallelogram BADC. As AD is equal and Parallel to BC, AD also represents Q in magnitude and direction.

$$\begin{aligned} P + Q &= \overline{AB} + \overline{AD} \\ &= \overline{AC} \text{ (by llgm law).} \end{aligned}$$

This shows that the resultant of the forces P and Q at O is represented in magnitude and direction by AC.

The Third force R acts at O and it is represented in magnitude and direction by CA.

$$\begin{aligned} \text{Hence } P + Q + R &= \overline{AC} \text{ at O} + \overline{CA} \text{ at O} \\ &= \overline{O} \text{ (as the two vectors at O are equal and opposite)} \end{aligned}$$

∴ The forces are in equilibrium.

Note:

In the above theorem, the forces P, Q, R are represented by the sides of the triangle A, B, C only in magnitude and direction but not in position. The forces act at a point and do not act along the sides of the triangle.

Corollary:

From the proof of the above theorem, it is clear that the resultant of the forces represented in magnitude and direction by the two sides AB and BC of the triangle ABC, is represented in magnitude and direction by AC.

This principle is stated as follows:

If two forces acting at a point are represented in magnitude and direction by two sides of a triangle taken in the same order, the resultant will be represented in magnitude and direction by the third side taken in the reverse order.

In the notation of vector, the above means that $\overline{AB} + \overline{BC} = \overline{AC}$

Perpendicular Triangle of Forces:-

If three forces acting at a point are such that their magnitudes are proportional to the sides of a triangle and their directions are perpendicular to the corresponding sides, all inwards or all outwards, then also the forces will be in equilibrium.

Let the forces P, Q, R meet at O.

ABC is a triangle such that magnitudes of P, Q, R are proportional to the sides BC, CA, and AB respectively of ΔABC and their directions are perpendicular to the corresponding sides all outwards.

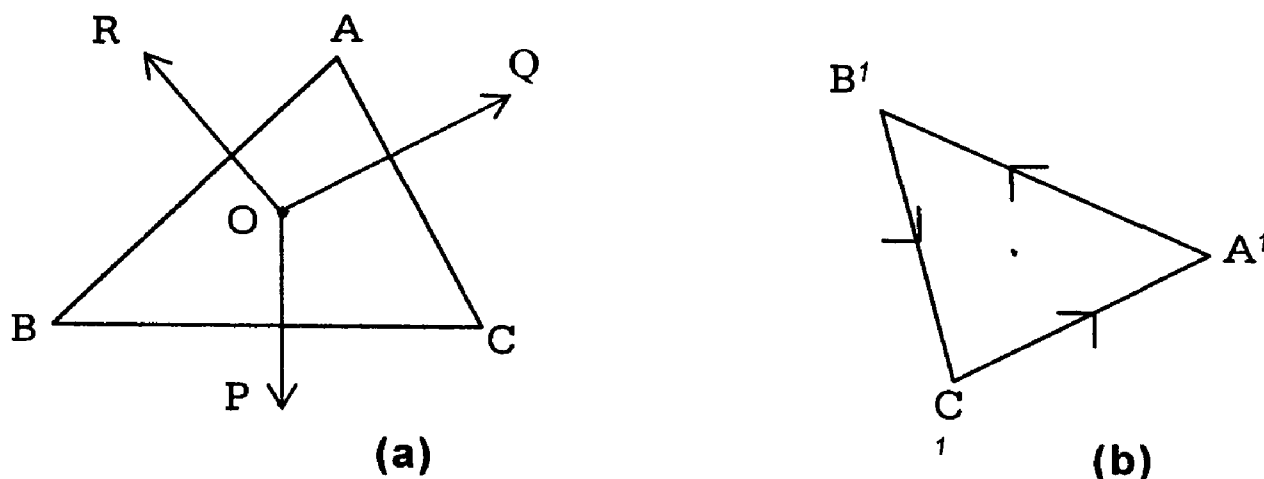


Fig. (2)

We have to prove that they will be in equilibrium.

If we rotate the $\triangle ABC$ through 90° in its own plane, we will get a new triangle A^1, B^1, C^1 whose sides are parallel to the given forces and represent the forces both in magnitude and direction. Hence by the triangle of forces p, Q, R are in equilibrium.

Note:

The above result will also be true, if the directions of the forces, instead of being perpendicular to the corresponding sides, make equal angles in the same sense with them. The proof is exactly similar.

1.4 Converse of the Triangle of Forces:

If three forces acting at a point are in equilibrium, then any triangle drawn so as to have its sides parallel to the directions of the forces shall represent them in magnitude also.

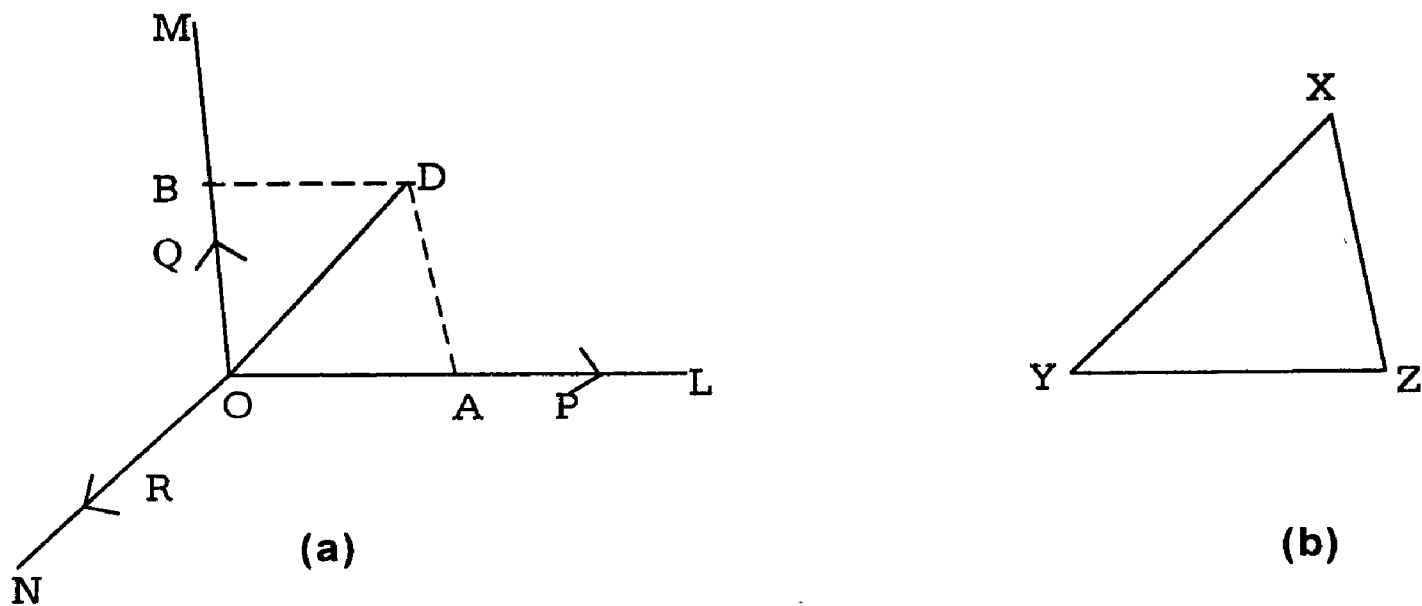


Fig. (3)

Let the three forces P, Q, R acting at O along the directions OL, OM and ON keep it in equilibrium. XYZ is a triangle such that the sides YZ, ZX and XY are parallel to the directions of P, Q, R respectively.

We have to prove that the sides of $\triangle XYZ$ are proportional to the magnitudes of P, Q and R given that $P + Q + R = 0$ (statically).

Along OL , cut off OA to represent the magnitude of P on some scale.

(ie) Let $\overline{OA} = P$.

On the same scale, make $\overline{OB} = Q$.

To get the resultant of P and Q.

Complete the llgm AOB.

$$\text{The } P + Q = \overline{OA} + \overline{OB} = \overline{OD}$$

$$\text{But } P + Q + R = \overline{O} \text{ (given)}$$

$$\text{(ie) } \overline{OD} + R = \overline{O} \text{ or } R = \overline{DO}$$

This shows that the third force R is represented in magnitude on the same scale by DO and that DON is a straight line.

Hence the three forces P, Q and R are Parallel and proportional to the sides of the triangle OAD.

Now any triangle like XYZ whose sides are parallel to the directions of P, Q and R will be similar to Δ OAD and hence

$$\frac{YZ}{OA} = \frac{ZX}{AD} = \frac{XY}{DO}$$

$$\text{But } \frac{P}{OA} = \frac{Q}{OB} = \frac{R}{DO}$$

$$\therefore \frac{YZ}{P} = \frac{ZX}{Q} = \frac{XY}{R}$$

(ie) The sides of Δ XYZ will be proportional to P, Q, R.

Polygon of Forces:-

If any number of forces acting at a point can be represented in magnitude and direction by the sides of a polygon taken in order, the forces will be in equilibrium.

Let the forces P_1, P_2, \dots, P_n acting at O be represented in magnitude and direction by the sides $B_1B_2, B_2B_3, \dots, B_nB_1$ of the polygon B_1, B_2, \dots, B_n .

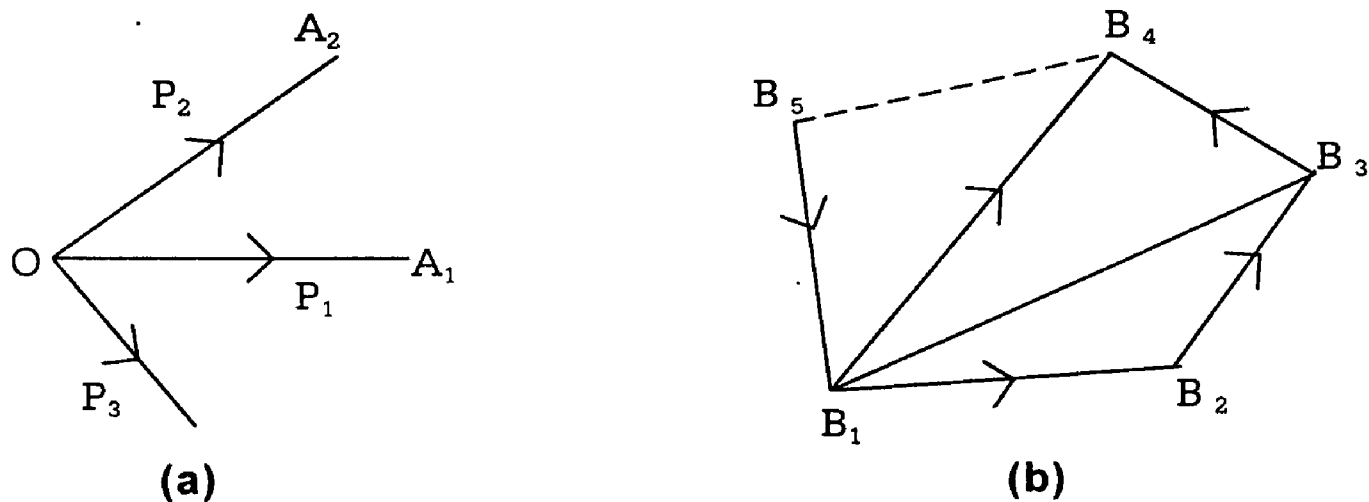


Fig. (4)

We have to prove that the forces will be equilibrium.

Compounding the forces by vector law, step by step.

$$\text{We have } P_1 + P_2 = \overline{B_1 B_2} + \overline{B_2 B_3} = \overline{B_1 B_3}$$

$$P_1 + P_2 + P_3 = \overline{B_1 B_3} + \overline{B_3 B_4} = \overline{B_1 B_4}$$

$$\text{and } P_1 + P_2 + P_3 + \dots + P_{n-1} = \overline{B_1 B_{n-1}} + \overline{B_{n-1} B_n} = \overline{B_1 B_n}$$

It is to be noted that in each of the equations above, the resultant on the right side, of the forces named on the left side, acts at the point O.

The last force P_n is represented by $\overline{B_n B_1}$.

$$\begin{aligned} \therefore P_1 + P_2 + \dots + P_{n-1} + P_n &= \overline{B_1 B_n} \text{ at } O + \overline{B_n B_1} \text{ at } O \\ &= \overline{O}. \end{aligned}$$

\therefore The forces are in equilibrium.

Note 1:

The above theorem is true even when the forces acting at O are not in the same plane.

Note: 2

The converse of the polygon of Forces is not true. The converse of the triangle of forces is true because when ever the directions of three forces acting at a point and keeping it in equilibrium are known, all triangles drawn with their

sides parallel to these directions, will be similar and hence represent the forces in magnitude also. But in the case of more than three forces acting at a point and keeping it in equilibrium, we cannot say that the sides of any polygon drawn with its sides parallel to the directions of the forces shall represent them in magnitude also.

If we draw two such polygons, they will be merely equiangular and not necessarily similar. All that we can say is that a polygon can be drawn with the sides parallel and proportional to the forces.

1.5 Lami's Theorem:

If three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two.

Refer to the result and figure : 1.4 (Page 20)

We have provided that the sides of the triangle OAD represent the forces P, Q, R in magnitude and direction.

Applying the sine rule to ΔOAD .

We have

$$\frac{OA}{\sin \angle ODA} = \frac{AD}{\sin \angle DOA} = \frac{DO}{\sin \angle OAD} \quad (1)$$

$$\text{But } \angle ODA = \text{alt. } \angle BOD = 180^\circ - \angle MON$$

$$\therefore \sin \angle ODA = \sin (180^\circ - \angle MON) = \sin \angle MON \quad (2)$$

$$\text{Also } \angle DOA = 180^\circ - \angle NOL$$

$$\therefore \sin \angle DOA = \sin (180^\circ - \angle NOL) = \sin \angle NOL \quad (3)$$

$$\& \angle OAD = 180^\circ - \angle BOD = 180^\circ - \angle LOM$$

$$\therefore \sin \angle OAD = \sin (180^\circ - \angle LOM) = \sin \angle LOM \quad (4)$$

Substituting (2), (3), (4) in (1)

We have

$$\frac{OA}{\sin \angle MON} = \frac{AD}{\sin \angle NOL} = \frac{DO}{\sin \angle LOM}$$

$$(ie) \frac{P}{\sin \angle MON} = \frac{Q}{\sin \angle NOL} = \frac{R}{\sin \angle LOM}$$

$$(or) \frac{P}{\sin(Q,R)} = \frac{Q}{\sin(R,P)} = \frac{R}{\sin(P,Q)}$$

Example : 1

Two forces act on a particle. If the sum and difference of the forces are at right angles to each other. Show that the forces are of equal magnitude.

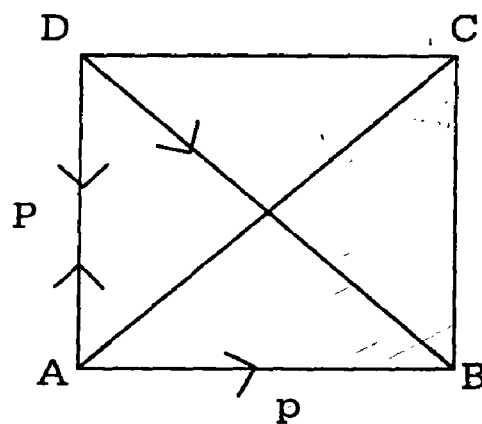
Solution:

Let the forces P and Q acting at A be represented in magnitude and direction by the lines AB and AD.

Complete the Parallelogram BAD.

Then $P + Q = \overline{AB} + \overline{AD} + \overline{AC}$ (||gm law)

$\therefore \overline{AC}$ is the sum of two forces.



$$P - Q = \overline{AB} - \overline{AD}$$

$$= \overline{AB} + \overline{DA}$$

$$= \overline{DA} + \overline{AB}$$

$$= \overline{DB} \text{ (by } \Delta^{le} \text{ law)}$$

$\therefore \overline{DB}$ is the difference of two forces.

It is given that \overline{AC} and \overline{DB} are at right angle.

(ie) In Parallelogram ABCD, the diagonals AC and BD cut at right angles.

\therefore ABCD must be a rhombus.

$\therefore AB = AD$.

(ie) $P = Q$ in magnitude.

Example : 2

A and B are two fixed points on a horizontal line at a distance C apart. Two fine light strings AC and BC of lengths b and a respectively support a mass at C.

Show that the tensions of the strings are in the ratio $b(a^2 + c^2 - b^2) : a(b^2 + c^2 - a^2)$.

Solution:

Let T_1 and T_2 be the tensions along the strings CA and CB and W the weight of the mass at C, acting vertically downwards along CE.

Produce EC to meet AB at D.

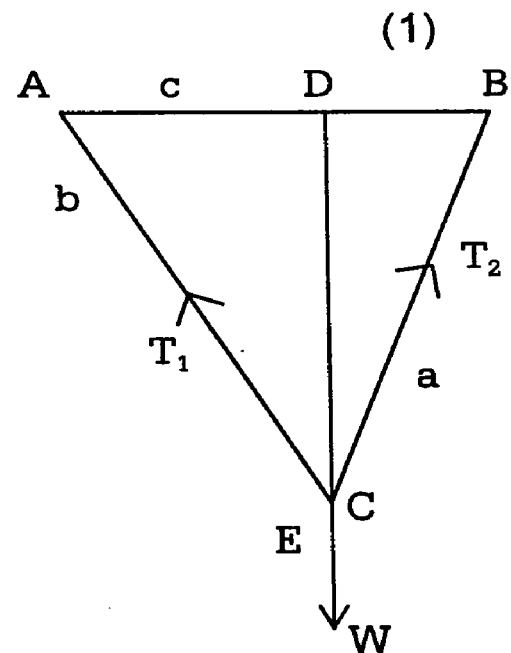
Since C is at rest under the action of three forces,

We have by Lami's theorem,

$$\frac{T_1}{\sin \angle ECB} = \frac{T_2}{\sin \angle ECA} \quad (1)$$

$$\begin{aligned} \text{Now } \sin \angle ECB &= \sin (180^\circ - \angle DCB) \\ &= \sin \angle DCB \\ &= \sin (90^\circ - \angle ABC) \\ &= \cos \angle ABC \end{aligned}$$

$$\sin \angle ECA = \sin (180^\circ - \angle ACD)$$



$$\begin{aligned}
&= \sin \angle ACD \\
&= \sin (90^\circ - \angle BAC) \\
&= \cos \angle BAC
\end{aligned}$$

Hence (1) becomes

$$\frac{T}{\cos \angle ABC} = \frac{T}{\cos \angle BAC}$$

$$\therefore \frac{T_1}{T_2} = \frac{\cos \angle ABC}{\cos \angle BAC} = \frac{\cos B}{\cos A} \quad (2)$$

In $\triangle ABC$,

We know that

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

and

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Hence (2) becomes

$$\frac{T_1}{T_2} = \left(\frac{c^2 + a^2 - b^2}{2ca} \right) \times \left(\frac{2bc}{b^2 + c^2 - a^2} \right)$$

$$\frac{T_1}{T_2} = \frac{b(c^2 + a^2 - b^2)}{a(b^2 + c^2 - a^2)}$$

\therefore The tensions of the strings are in the ratio $b(a^2 + c^2 - b^2) : a(b^2 + c^2 - a^2)$.

Example : 3

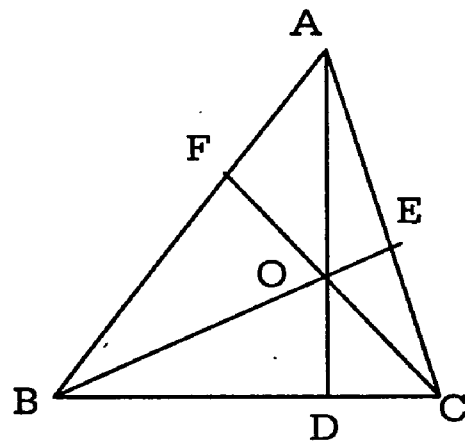
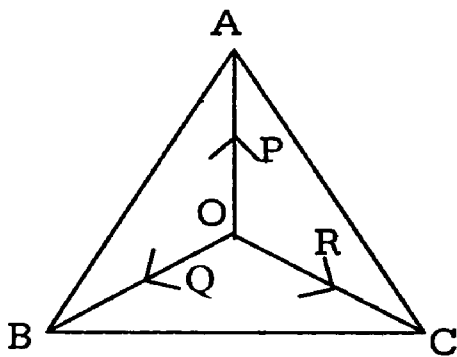
ABC is a given triangle. Forces P, Q, R acting along the lines OA, OB, OC are in equilibrium. Prove that

i) $P:Q:R = a^2(b^2 + c^2 - a^2) : b^2(c^2 + a^2 - b^2) : c^2(a^2 + b^2 - c^2)$ if O is the circum-centre of the triangle.

ii) $P : Q : R = \cos \frac{A}{2} : \cos \frac{B}{2} : \cos \frac{C}{2}$ if O is the in-centre of the triangle.

iii) $P : Q : R = a : b : c$ if O is the ortho-centre of the triangle.

iv) $P : Q : R = OA : OB : OC$ if O is the centroid of the triangle.



By Lami's theorem,

We have

$$\frac{P}{\sin \angle BOC} = \frac{Q}{\sin \angle COA} = \frac{R}{\sin \angle AOB} \quad (1)$$

i) What O is the circum-centre of the ΔABC ,

$$\angle BOC = 2 \angle BAC = 2A;$$

$$\angle COA = 2B$$

And $\angle AOB = 2C$.

$$\therefore (1) \text{ gives } \frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$$

$$(ie) \frac{P}{2 \sin A \cos A} = \frac{Q}{2 \sin B \cos B} = \frac{R}{2 \sin C \cos C} \quad (2)$$

But $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ and $\sin A = \frac{2\Delta}{bc}$ where Δ is the area.

$$\begin{aligned} \therefore 2 \sin A \cos A &= 2 \cdot \frac{2\Delta}{b} \cdot \frac{(b^2 + c^2 - a^2)}{2bc} \\ &= \frac{2\Delta(b^2 + c^2 - a^2)}{b^2c^2} \end{aligned}$$

$$Similarly \ 2 \sin B \cos B = \frac{2\Delta(c^2 + a^2 - b^2)}{c^2a^2}$$

$$\text{and } 2 \sin C \cos C = \frac{2\Delta(a^2 + b^2 - c^2)}{a^2b^2}$$

So (2) becomes

$$\frac{P \cdot b^2 \cdot c^2}{2\Delta(b^2 + c^2 - a^2)} = \frac{Q \cdot c^2 \cdot a^2}{2\Delta(c^2 + a^2 - b^2)} = \frac{R \cdot a^2 \cdot b^2}{2\Delta(a^2 + b^2 - c^2)}$$

Multiplying throughout by $\frac{2\Delta}{a^2b^2c^2}$

We get

$$\frac{P}{a^2(b^2 + c^2 - a^2)} = \frac{Q}{b^2(c^2 + a^2 - b^2)} = \frac{R}{c^2(a^2 + b^2 - c^2)}$$

ii) When O is the in centre of the triangle, OB and OC are the bisectors of $\angle B$ and $\angle C$.

$$\therefore \angle BOC = 180^\circ - \frac{B}{2} - \frac{C}{2} = 180^\circ - \left(\frac{B}{2} + \frac{C}{2} \right)$$

$$\angle BOC = 180^\circ - \left(90^\circ - \frac{A}{2} \right) = 90^\circ + \frac{A}{2}$$

$$\text{Similarly } \angle COA = 90^\circ + \frac{C}{2}$$

So (1) becomes.

$$\frac{P}{\sin\left(90^\circ + \frac{A}{2}\right)} = \frac{Q}{\sin\left(90^\circ + \frac{B}{2}\right)} = \frac{R}{\sin\left(90^\circ + \frac{C}{2}\right)}$$

$$(ie) \frac{P}{\cos \frac{A}{2}} = \frac{Q}{\cos \frac{B}{2}} = \frac{R}{\cos \frac{C}{2}}$$

iii) Let O be the orthocentre of the triangle In the fig.

In the fig.

AD, BE, CF are the altitudes.

Quadrilateral AFOE is Cyclic.

$$(\because \angle AFO + \angle AEO = 90^\circ + 90^\circ = 180^\circ)$$

$$\therefore \angle FOE + A = 180^\circ$$

$$(or) \therefore \angle FOE = 180^\circ - A$$

$$\text{Similarly } \angle COA = 180^\circ - B$$

$$\text{And } \angle AOB = 180^\circ - C$$

Hence (1) becomes

$$\frac{P}{\sin(180^\circ - A)} = \frac{Q}{\sin(180^\circ - B)} = \frac{R}{\sin(180^\circ - C)}$$

$$(ie) \frac{P}{\sin A} = \frac{Q}{\sin B} = \frac{R}{\sin C}$$

$$(ie) \frac{P}{a} = \frac{Q}{b} = \frac{R}{c} \left(\text{Since } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \right)$$

iv) When O is the centroid of the triangle.

we know that

$$\Delta BOC = \Delta COA = \Delta AOB \text{ and each} = \frac{1}{3} \Delta ABC.$$

$$\Delta BOC = \frac{1}{2} OB \cdot OC \sin \angle BOC = \frac{1}{3} \Delta ABC$$

$$\therefore \sin \angle BOC = \frac{2 \Delta ABC}{3 OB \cdot OC}$$

$$\text{Similarly } \sin \angle COA = \frac{2 \Delta ABC}{3 OC \cdot OA}$$

$$\text{and } \sin \angle AOB = \frac{2 \Delta ABC}{3 OA \cdot OB}$$

Hence (1) becomes

$$\frac{P \cdot 3 OB \cdot OC}{2 \Delta ABC} = \frac{Q \cdot 3 OC \cdot OA}{2 \Delta ABC} = \frac{R \cdot 3 OA \cdot OB}{2 \Delta ABC}$$

$$\text{(ie) } P \cdot OB \cdot OC = Q \cdot OC \cdot OA = R \cdot OA \cdot OB$$

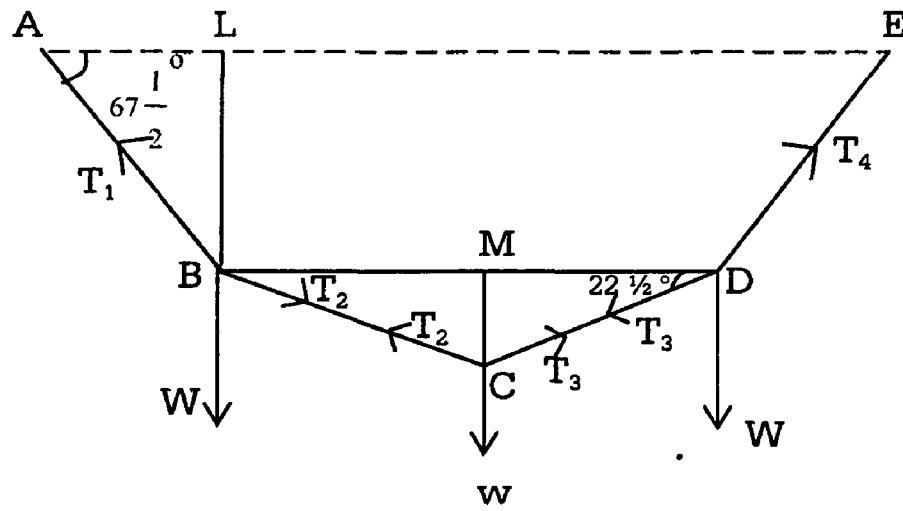
Dividing by OA. OB. OC throughout,

$$\frac{P}{OA} = \frac{Q}{OB} = \frac{R}{OC}$$

Example : 4

Weights W, w, W are attached at points B, C, D respectively of a light string AE where B, C, D divide the string into 4 equal lengths. If the string hangs in the form of 4 consecutive sides of a regular octagon with the ends A and E attached to point on the same level, show that $W = (\sqrt{2} + 1)w$.

Solution:



ABCDE is a part of regular octagon. We know that each interior angle of a regular polygon of n sides

$$= \left(\frac{2n-4}{n} \right) \times 90^\circ$$

Putting $n = 8$, each interior angle of ABCDE

$$= \left(\frac{2 \times 8 - 4}{8} \right) \times 90^\circ$$

$$= \frac{12}{8} \times 90^\circ = 135^\circ$$

Let the tensions in the portions AB, BC, CD, DE be T_1 , T_2 , T_3 , T_4 respectively. The String BC pulls B towards C and pulls C towards B, the tension being the same throughout its length. This fact is used to denote the forces acting at B, C, and D.

In $\triangle BCD$, $\angle BCD = 135^\circ$

$$\therefore \angle CBD = \angle CDB = \frac{45^\circ}{2} = 22 \frac{1}{2}^\circ$$

$\therefore \angle ABD = \angle ABC - \angle CBD$.

$$= 135^\circ - 22 \frac{1}{2}^\circ = 112 \frac{1}{2}^\circ$$

we know that every regular polygon is cyclic

∴ A, B, C, D, E lie on the same circle.

$$\angle EAB = 180^\circ - \angle BDE$$

$$\therefore \angle EAB = 180^\circ - \{ \angle CDE - \angle BDC \}$$

$$= 180^\circ - \left\{ 135^\circ - 22\frac{1}{2}^\circ \right\}$$

$$= 45^\circ + 22\frac{1}{2}^\circ = 67\frac{1}{2}^\circ$$

Equating the two values of T_2 from (1) and (2)

We have

$$\frac{W}{\sin 135^\circ} \sin 22\frac{1}{2}^\circ = \frac{w}{\sin 135^\circ} \cos 22\frac{1}{2}^\circ$$

$$(ie) \frac{w}{W} = \tan 22\frac{1}{2}^\circ = \sqrt{2} - 1$$

$$\therefore W = \frac{w}{\sqrt{2} - 1} = \frac{w(\sqrt{2} + 1)}{(\sqrt{2} - 1)(\sqrt{2} + 1)}$$

$$= \frac{w(\sqrt{2} + 1)}{1}$$

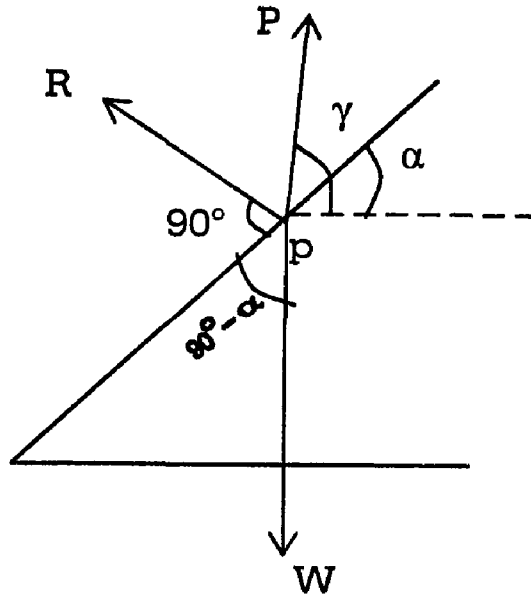
$$\therefore W = w(\sqrt{2} + 1)$$

Example: 5

A weight is supported on a smooth plane of inclination α by a string inclined to the horizon at an angle γ . If the slope of the plane be increased to β and the slope of the string unaltered, the tension of the string is doubled. Prove that $\cot \alpha - 2 \cos \beta = \tan \lambda$.

Solution:

P is the Position of the weight. The forces acting at P are i) its weight W downwards ii) the normal reaction $R \perp$ to the inclined plane and iii) the tension T along the string at



$$\therefore \angle EAB + \angle ABD = 67\frac{1}{2}^\circ + 112\frac{1}{2}^\circ$$

$$= 180^\circ$$

$$\therefore AE \parallel BD.$$

\therefore BD also is horizontal

Let the vertical line through B meet AE at L and the vertical line through C meet BD at M.

Applying Lami's theorem for the three forces at B.

$$\text{We get } \frac{W}{\sin \angle ABC} = \frac{T_2}{\sin (180^\circ - \angle ABL)}$$

$$(ie) \frac{W}{\sin 135^\circ} = \frac{T_2}{\sin \angle ABL} = \frac{T_2}{\sin 22\frac{1}{2}^\circ}$$

$$(\because \text{In rt } \triangle ABL, \angle ABL = 90^\circ - 67\frac{1}{2}^\circ)$$

$$\therefore \frac{T_2}{\sin 22\frac{1}{2}^\circ} = \frac{W}{\sin 135^\circ}$$

$$(ie) T_2 = \frac{W}{\sin 135^\circ} \sin 22\frac{1}{2}^\circ \quad (1)$$

Similarly applying Lami's Theorem for the three forces at C,

$$\text{We have } \frac{w}{\sin \angle BCD} = \frac{T_2}{\sin (180^\circ - \angle MCD)}$$

$$(ie) \frac{w}{\sin \angle 135^\circ} = \frac{T_2}{\sin \angle MCD} = \frac{T_2}{\sin \left(90^\circ - 22\frac{1}{2}^\circ \right)} = \frac{T_2}{\cos 22\frac{1}{2}^\circ}$$

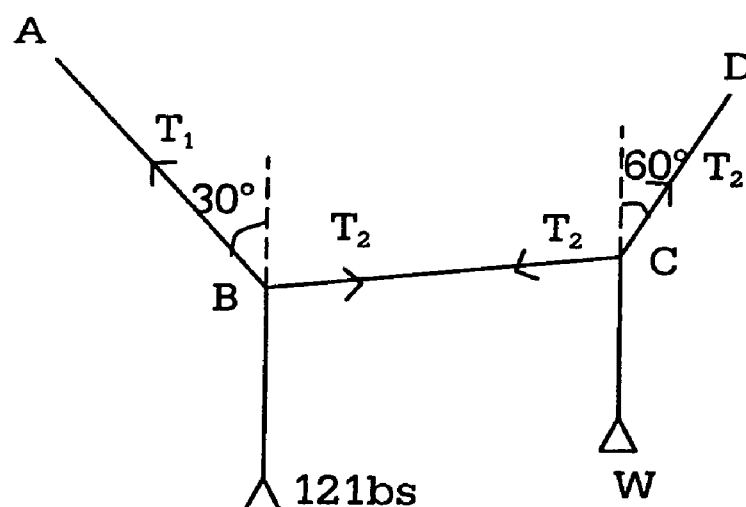
$$\therefore T_2 = \frac{w}{\sin 135^\circ} \cdot \cos 22\frac{1}{2}^\circ \quad (2)$$

Example:

A string ABCD hangs from fixed points, A, D carrying a weight of 12 lb. At B and A weight W at C. AB is inclined at 60° to the horizontal BC and CD is inclined at 30° to the horizontal. Find W.

Solution:

Let T_1 , T_2 , T_3 be the tensions in the Parts AB, BC, CD of the String ABCD hanging from the fixed points A and D.



Consider the equilibrium of the point B.

By Lami's theorem

$$\frac{T_1}{\sin 90^\circ} = \frac{T_2}{\sin (180^\circ - 30^\circ)}$$
$$= \frac{12}{\sin(90^\circ + 30^\circ)}$$

$$\text{or } T_2 = 12 \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{12}{\sqrt{3}}$$

Consider next the equilibrium of C.

By Lami's theorem

$$\text{We get } \frac{T_2}{\sin (180^\circ - 60^\circ)} = \frac{W}{\sin (90^\circ + 60^\circ)} = \frac{T_3}{\sin 90^\circ}$$

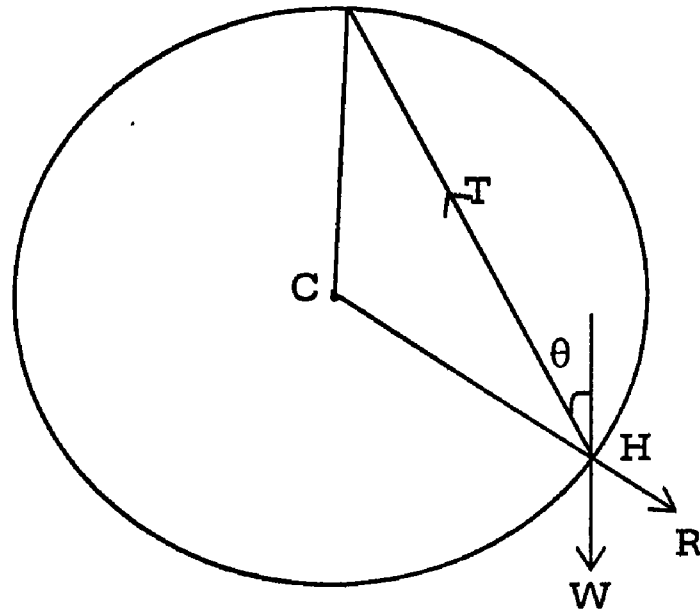
$$\text{or } W = T_2 \frac{\cos 60^\circ}{\sin 60^\circ} = \frac{12}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = 4 \text{ lbs.}$$

Example :

A bead of weight w can slide on a smooth circular wire in a vertical plane the bead is attached by a light thread to the highest point of the wire and in equilibrium the thread is taut and makes an angle θ with the vertical. Find the tension of the thread and reaction of the wire on the bead.

Solution:

The bead is kept in equilibrium at a point A of the wire by three forces, \overline{W} (the weight of the bead) the reaction \overline{R} at A and the tension \overline{T} in the string AB, where B is the highest point of the wire.



Also the wire being smooth the reaction \vec{R} at A passes through C, the centre of the circular wire.

Since AB is inclined at an angle θ with the vertical, $\angle CBA = \theta$.

But $BC = CA$, So that $\angle CBA = \theta$

By Lami's theorem

$$\frac{W}{\sin(180^\circ - \theta)} = \frac{R}{\sin(180^\circ - \theta)} = \frac{T}{\sin 2\theta}$$

$$\text{or } \frac{W}{\sin \theta} = \frac{R}{\sin \theta} = \frac{T}{2 \sin \theta \cos \theta}$$

This gives $T = 2W \cos \theta$ & $R = W$

Example : 6

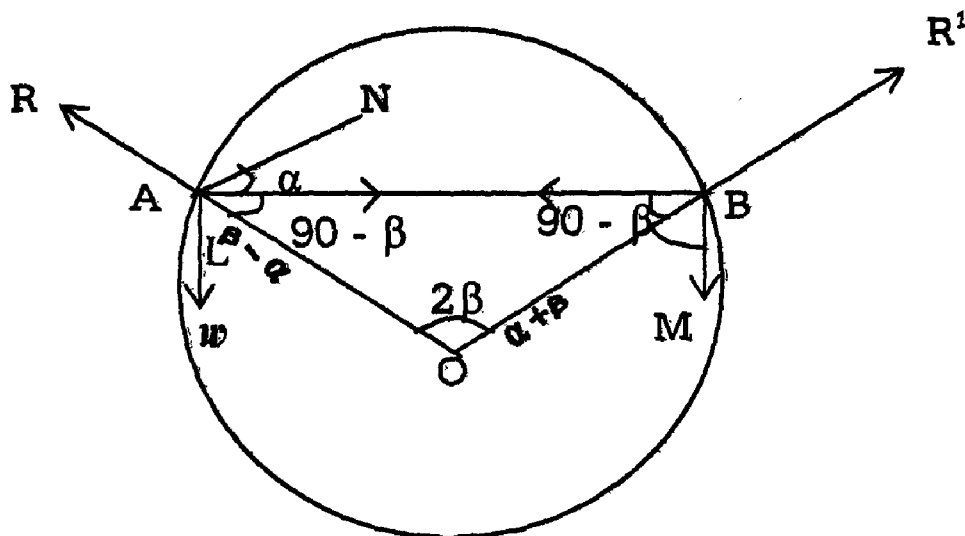
Two beads of weights w and w' can slide on a smooth circular wire in a vertical plane. They are connected by a light string which subtends an angle 2β at the centre of the circle when the beads are in equilibrium on the upper half of the wire. Prove that the inclination of the string to the horizontal is given by

$$\tan \alpha = \frac{W \sim W'}{w + w'} \tan \beta$$

Solution:

Let A and B be the beads of weights W and W' connected by a light string and sliding on a circular wire.

In equilibrium position $\angle AOB = 2\beta$



O being the centre of the circle.

$$\therefore \angle OAB = \angle OBA = 90^\circ - \beta.$$

Let AB make an angle α with the horizontal .

AL and BM are the vertical lines through A and B

$$\angle OAL = 90^\circ - \angle OAN = 90^\circ - (90^\circ - (90^\circ - \beta + \alpha)) = \beta - \alpha.$$

Since AL \parallel BM,

$$\angle ABM + \angle BAL = 180^\circ$$

$\therefore \angle ABM = 180^\circ - \angle BAL = 180^\circ - (90^\circ - \alpha) = 90^\circ + \alpha$ an angle γ to the horizontal. By Lami's theorem three forces at P,

$$\frac{T}{\sin(180^\circ - \alpha)} = \frac{W}{\sin(90^\circ - (\gamma - \alpha))}$$

$$(ie) \frac{T}{\sin \alpha} = \frac{W}{\cos(\gamma - \alpha)}$$

$$\therefore T = \frac{W \sin \alpha}{\cos(\gamma - \alpha)} \quad (1)$$

In the second case, the inclination of the plane is β .

There is no change in γ

If T_1 is the tension in the string.

$$\text{We have } T_1 = \frac{W \sin \beta}{\cos (\gamma - \beta)} \quad (2)$$

But $T_1 = 2T$ (given)

$$\therefore \frac{W \sin \alpha}{\cos (\gamma - \alpha)} = \frac{2W \sin \alpha}{\cos (\gamma - \alpha)}$$

$$\therefore \sin \beta \cos (\gamma - \alpha) = 2 \sin \alpha \cos (\gamma - \beta).$$

$$\text{(ie) } \sin \beta (\cos \gamma \cos \alpha + \sin \gamma \sin \alpha)$$

$$= 2 \sin \alpha (\cos \gamma \cos \beta + \sin \gamma \sin \beta)$$

$$\sin \beta \cos \gamma \cos \alpha = 2 \sin \alpha \cos \gamma \cos \beta + \sin \alpha \sin \beta \sin \gamma$$

$$= \sin \alpha (2 \cos \gamma \cos \beta + \sin \beta \sin \gamma)$$

$$\therefore \frac{\cos \alpha}{\sin \alpha} = \frac{2 \cos \gamma \cos \beta + \sin \beta \sin \gamma}{\sin \beta \cos \gamma}$$

$$\text{(ie) } \cot \alpha = 2 \cot \beta + \tan \gamma \quad (\text{or})$$

$$\cot \alpha - 2 \cot \beta = \tan \gamma.$$

$$\therefore \angle OBM = \angle ABM - \angle ABO$$

$$\Rightarrow 90^\circ + \alpha - (90^\circ - \beta) = \alpha + \beta.$$

The forces acting on the bead w and A are

- i) Weight w acting vertically downwards along AL
 - ii) normal reaction R due to contact with the wire along the radius OA outwards.
- and iii) tension T is the string along AB .

Similarly the force acting on the bead w^1 at B are

- i) Weight w^1 acting vertically downwards along BM .
- ii) normal reaction R^1 along the radius OB outwards and
- iii) tension T is the string along BA .

Applying Lami's theorem for the three forces at A

$$\frac{w}{\sin[180^\circ - (90 - \beta)]} = \frac{T}{\sin(180^\circ - \beta - \alpha)}$$

$$(ie) \quad \frac{w}{\cos \beta} = \frac{T}{\sin(\beta - \alpha)} \quad (1)$$

Similarly applying Lami's theorem for the three forces at B ,

$$\frac{w^1}{\sin[180^\circ - (90 - \beta)]} = \frac{T}{\sin(180^\circ - \beta - \alpha)}$$

$$(ie) \quad \frac{w^1}{\cos \beta} = \frac{T}{\sin(\beta - \alpha)} \quad (2)$$

Dividing (1) by (2)

We have

$$\frac{w}{w^1} = \frac{\sin(\beta + \alpha)}{\sin(\beta - \alpha)}$$

$$\frac{w - w^1}{w + w^1} = \frac{\sin(\beta + \alpha) - \sin(\beta - \alpha)}{\sin(\beta + \alpha) + \sin(\beta - \alpha)}$$

$$= \frac{2 \cos \beta \sin \alpha}{2 \sin \beta \cos \alpha} = \frac{\tan \alpha}{\tan \beta}$$

$$\therefore \tan \alpha = \left(\frac{w - w'}{w + w'} \right) \tan \beta$$

Hence the result.

Exercise.

1. Two forces act at a point and are such that if the direction of the resultant is turned through a right angle. Prove that the two forces must be equal in magnitude.
2. Three forces x, y, z , acting at the vertices A, B, C respectively of a triangle, each \perp to the opposite side keep it in equilibrium. Prove that $\frac{X}{a} = \frac{Y}{b} = \frac{Z}{c}$.
3. If three forces represented in magnitude and direction by the bisectors of the angles of a Δ^{1e} all acting from the vertices be in equilibrium, show that the triangle must be equilateral.
4. A string $ABCD$ hangs from fixed points, A, D carrying a weight of 12kgs at B and a weight W at C . AB is inclined at 60° to the horizontal, BC is horizontal and CD is inclined at 30° to the horizontal. Find W .

UNIT – II

RESOLUTION OF A FORCE

2.1 Resolution of a Force

Two forces given in magnitude and direction have only one resultant for we can construct only one parallelogram when two adjacent sides are given. Conversely, a single force can be resolved into two components in an infinite number of ways. Since any number of parallelograms can be constructed on a given line AC as diagonal. (See fig.1. page.5)

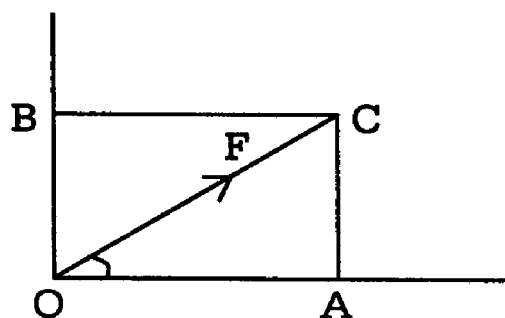
If BADC is any one of these, the force AC is equivalent to the two component forces AB and AD.

The most important case of resolution of a force occurs, when a given force is to be resolved in two directions at right angles, one of these directions being given. In this case, the magnitudes of the component forces are easily got as follows:

Let OC represent the given force F and OX be a line inclined at an angle θ to OC.

Let OY be perpendicular to OX.

Draw CA \perp to OX and complete the parallelogram OACB.



(a)

Fig. 1

Then the force OC is equivalent to the two component forces OA and OB.

$$\text{Also } OA = OC \cdot \cos \theta = F \cos \theta$$

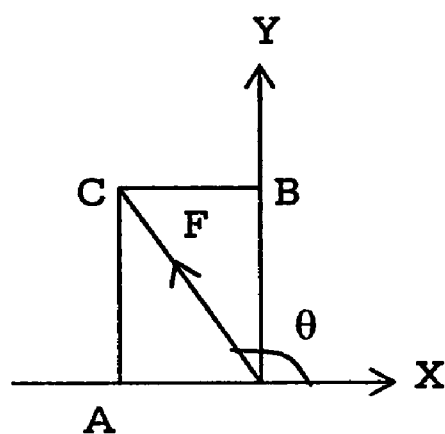
$$\text{And } OB = AC = OC \cdot \sin \theta = F \sin \theta$$

When a given force is resolved into two components two mutually perpendicular directions, the components are referred to as the resolved parts in the corresponding directions.

In fig. (a) OA is the resolved part of F along OX while OB is the resolved part of F along OY.

In fig. (b), θ is obtuse and OA is in a direction opposite to OX.

In this case, the resolved part of F along OX is negative.



(b)

Its value as before is $F \cos \theta$ Which is negative, as θ is obtuse.

Hence we have the following important proposition:

A force F is equivalent to a force $F \cos \theta$ along a line making an angle θ with its own direction together with a force $F \sin \theta$ Perpendicular the direction of the first component.

Corollary: 1

When $\theta = 0$, $\cos \theta = 1$.

The resolved part = F

(ie) The resolved part of a force in its own direction is the force itself.

Corollary: 2

When $\theta = 90^\circ$, $\cos 90^\circ = 0$

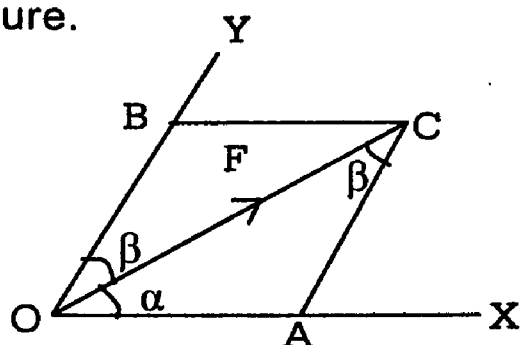
The resolved part = 0.

(ie) A force has no resolved part in a direction perpendicular to itself.

2.1.1 Components of a force along two given direction:

Let OC represent a given force F and OX, OY be two lines making angles α and β with OC.

Draw CA parallel to OY and CB parallel to OX, making the parallelogram OACB as shown in the figure.



Then OA and OB are the components of the force OC along OX & OY respectively.

From ΔOAC ,

$$\frac{OA}{\sin \angle OCA} = \frac{AC}{\sin \angle AOC} = \frac{OC}{\sin \angle OAC}$$

$$(ie) \quad \frac{OA}{\sin \beta} = \frac{AC}{\sin \alpha} = \frac{OC}{\sin \{180 - \alpha + \beta\}}$$

$$\frac{OA}{\sin \beta} = \frac{AC}{\sin \alpha} = \frac{F}{\sin(\alpha + \beta)}$$

$$\therefore OA = \frac{F \sin \beta}{\sin(\alpha + \beta)}$$

$$\text{and } OB = AC = \frac{F \sin \alpha}{\sin(\alpha + \beta)}$$

It should be noted that the component of force in a given direction is different from the resolved part of the force in that direction. To find the component of a force in any direction. We must be given the direction of the other component also. On the other hand to find the resolved part. We need only the given direction, since the other direction must be at right angles to it.

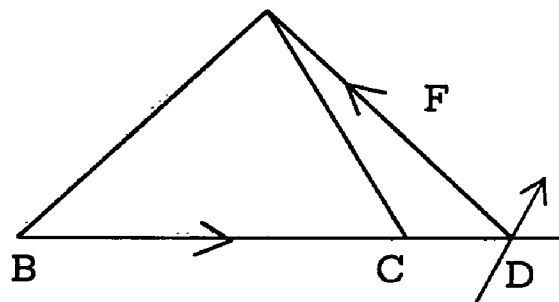
In other words the component of a force in a direction is a variable quantity, while the resolved part of the force in a direction is a fixed quantity its value being $F \cos \theta$.

Example: 1

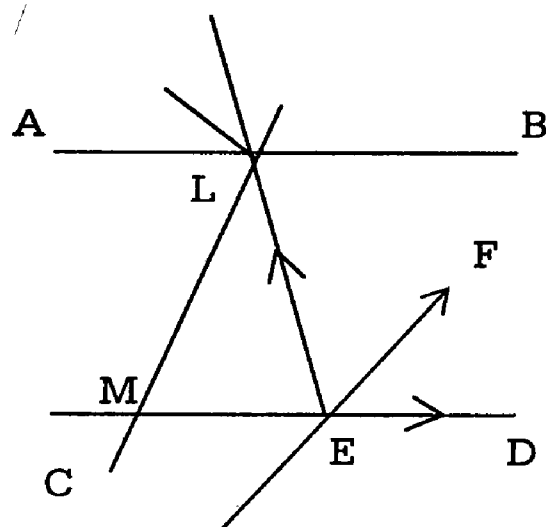
Show that a given force may be resolved into the components, acting in three given lines which are not all parallel or all concurrent.

Solution:

Let the three lines form a $\triangle ABC$ and let the given force F meet the side BC in D .



Then F can be resolved into two components acting BC and DA respectively. The component along DA can be resolved into two components along AB and AC respectively.



Suppose two of the lines AB and CD are parallel and LM is the third line. Let the given force F meet CD at E . F can be resolved into two components along CD and EL . The component along EL can be resolved into two components acting along BA and ML respectively.

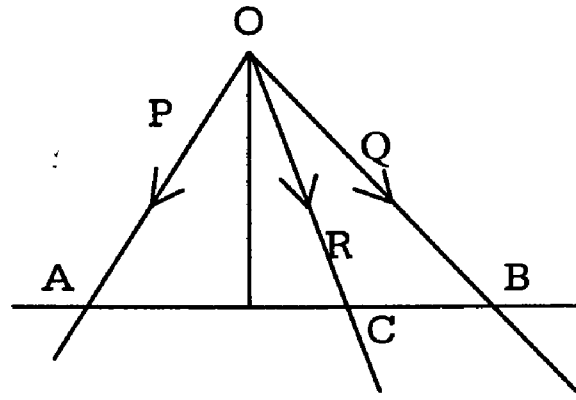
Example:

A line cuts the lines of action of three concurrent forces $\vec{P}, \vec{Q}, \vec{R}$ in A, B, C respectively. If \vec{R} be the resultant of \vec{P} & \vec{Q} Show that $\frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}$.

Solution:

Let OD be drawn \perp to the line ACB (the line cutting the lines of action of the forces) meeting it at D.

The resolved parts of $\vec{P}, \vec{Q}, \vec{R}$ along OD are $P \cos \angle AOD$, $Q \cos \angle BOD$ and $R \cos \angle COD$.



Since \vec{R} is the resultant of \vec{P} and \vec{Q}

We get

$$R \cos \angle COD = P \cos \angle AOD + Q \cos \angle BOD$$

$$\text{or } R \cdot \frac{OD}{OC} = P \cdot \frac{OD}{OA} + Q \cdot \frac{OD}{OB}$$

$$\text{(ie) } \frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}$$

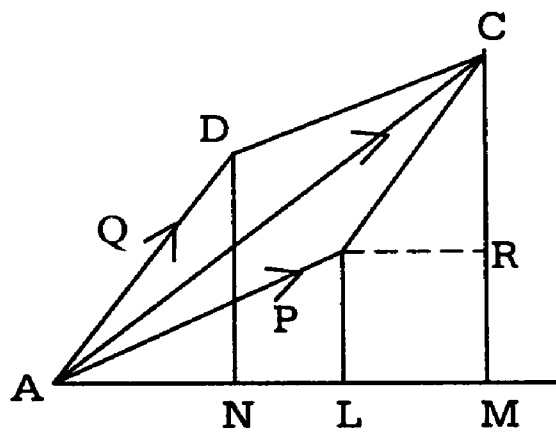
Exercises

1. If a force P be resolved into two forces making angles of 45° and 15° with its direction show that the latter force is $\frac{\sqrt{6}}{3} P$.
2. Find the components of force P along two directions making angles of 45° and 60° with P on opposite sides.
3. Two forces P and Q have a resultant R and the resolved part of R in the direction of P is of magnitude Q.

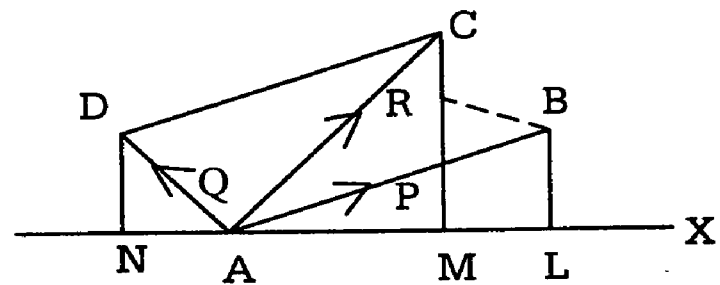
Show that the angle between the forces is $2 \sin^{-1} \sqrt{\frac{P}{2Q}}$.

2.2 THEOREM ON RESOLVED PARTS

The algebraic sum of the resolved parts of two forces in any direction is equal to the resolved part of the resultant in the same direction



(a)



(b)

Let AB and AD represent completely the forces P and Q and AX be the direction in which the forces are to be resolved. Complete the parallelogram ABCD so that the resultant R is represented by AC.

Draw BL, DN and CM perpendiculars to OX and BK \perp to CM.

Then AL, AN and AM are resolved parts of the forces P, Q, R along AX.

In fig.(b) AD makes an obtuse angle with AX and so the resolved part of Q is $-AN$.

We have to show that $AL \pm AN = AM$.

The triangles DAN and CBK are congruent and hence $AN = BK$

$$\therefore AL \pm AN = AL \pm BK$$

$$= AL \pm LM = AM.$$

Obviously the above theorem can be extended to the resultant of any number of forces acting at a point.

Suppose P_1, P_2, P_3 are three forces acting at O.

Let R_1 be the resultant of P_1 and P_2 and R_2 be the resultant of R_1 and P_3 .

Applying the theorem to the two sets of three forces.

$$P_1, P_2, R_1 \text{ and } R_1, P_3, R_2$$

We have

$$\text{resolved part of } R_1 \text{ along } OX = \text{resolved Part of } P_1 + \text{Resolved part of } P_2 \quad (1)$$

And

$$\text{resolved part of } R_2 \text{ along } OX = \text{Resolved Part of } R_1 + \text{Resolved part of } P_3 \quad (2)$$

Combining (1) and (2)

We have

$$\text{resolved part of } R_2 = \text{resolved part of } P_1 + \text{resolved part of } P_2 + \text{resolved part of } P_3 \text{ and so on.}$$

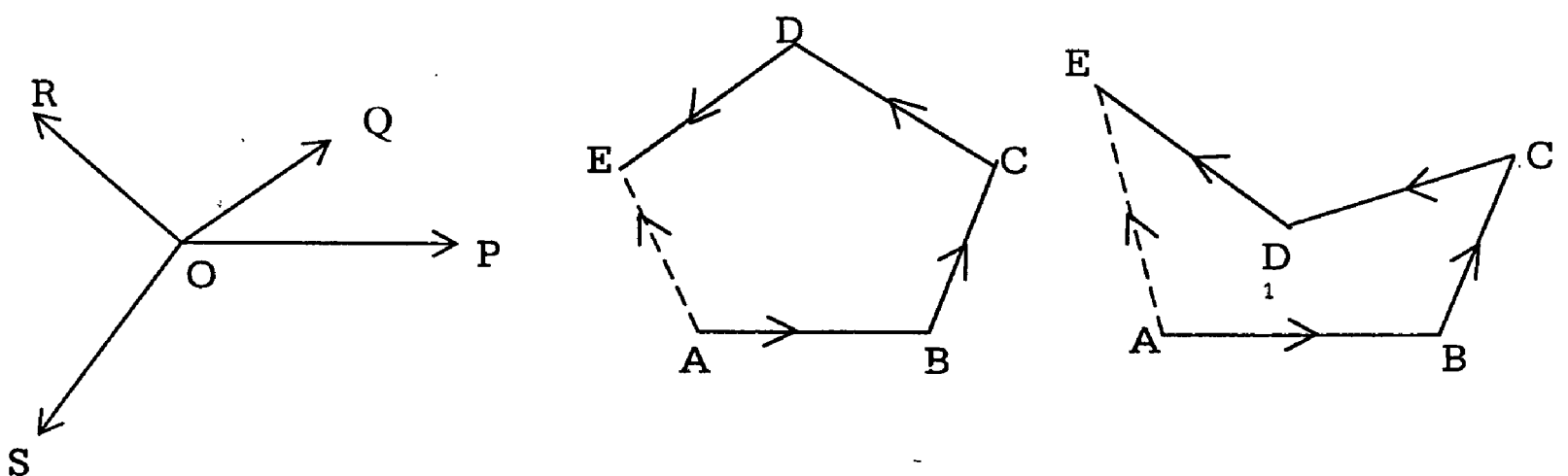
Hence in a generalised form

We have the theorem

The algebraic sum of the resolved parts of a number of forces in any direction is equal to the resolved part of the resultant in the same direction.

In the application of this theorem, it is to be noted that all the forces are resolved in the same direction and each resolved part has to be taken with its proper sign.

Resultant of any number of forces acting at a point: Graphical method.



Let P, Q, R, S be the forces acting at O.

Take a point A and draw lines AB, BC, CD and DE to represent successively the forces P, Q, R and S in magnitude and direction.

Compounding the forces by vector law, step by step we have

$$P + Q = \overline{AB} + \overline{AC} = \overline{AC}$$

$$P + Q + R = \overline{AC} + \overline{CD} = \overline{AD}$$

$$\text{and } P + Q + R + S = \overline{AD} + \overline{DE} = \overline{AE}$$

Hence the required resultant is represented in magnitude and direction by the line AE. The same construction will apply for any number of forces. The figure ABCDE is said to be the force-polygon.

The force-polygon can be constructed by drawing the vectors corresponding to the forces in any order. In fig (c) the order of the third and the fourth forces have been interchanged but AE is the same in each case.

Example: 1

ABC and A' B' C' are two triangles. Show that the resultant of the forces $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$ acting on a particle is $3 \overline{GG'}$ where G and G' are the centroids of the two triangles.

Solution:

Since G is the centroid of ΔABC , by definition of $\overline{GA} + \overline{GB} + \overline{GC} = 0$
 ||| ly from the definition of G'.

$$G'A' + \overline{G'B'} + \overline{G'C'} = 0.$$

$$\text{But } \overline{AA'} = \overline{AG'} + \overline{G'A'}$$

$$\overline{BB'} = \overline{BG'} + \overline{G'B'}$$

$$\text{and } \overline{CC'} = \overline{CG'} + \overline{G'C'}$$

$$\text{Adding } \overline{AA'} + \overline{BB'} + \overline{CC'} = (\overline{AG'} + \overline{BG'} + \overline{CG'}) + (\overline{G'A'} + \overline{G'B'} + \overline{G'C'})$$

$$= \overline{AG'} + \overline{BG'} + \overline{CG'} \text{ by (2)}$$

$$\begin{aligned}
&= (\overline{AG} + \overline{GG'}) + (\overline{BG} + \overline{GG'}) + (\overline{CG} + \overline{GG'}) \\
&= -(\overline{GA} + \overline{GB} + \overline{GC}) + 3 \overline{GG'} \\
&= 3 \overline{GG'} \text{ by (1)}
\end{aligned}$$

Example: 2

Find a point inside a quadrilateral such that if it is acted on by forces represented by the lines joining it to the vertices of the quadrilateral. It will be in equilibrium.

Solution:

Let P be a point within the quadrilateral ABCD such that the forces represented by \overline{PA} , \overline{PB} , \overline{PC} , \overline{PD} are in equilibrium.

Let E and F be the mid points of the diagonals AC and BD of the quadrilateral.

$$\begin{aligned}
\text{Then } \overline{PA} + \overline{PB} + \overline{PC} + \overline{PD} \\
&= (\overline{PA} + \overline{PC}) + (\overline{PB} + \overline{PD}) \\
&= 2 \cdot \overline{PE} + 2 \cdot \overline{PF} = 2(\overline{PE} + \overline{PF}) \\
&= 2 \cdot 2 \overline{PM}.
\end{aligned}$$

Where M is the mid point of EF.

Since the forces are in equilibrium.

$$\overline{PA} + \overline{PB} + \overline{PC} + \overline{PD} = 0$$

$$\text{or } 4 \overline{PM} = 0$$

So that

$$\overline{PM} = 0$$

(ie) P coincides with M.

∴ the required point is the mid-point of the line joining the midpoints of the diagonals.

Example: 3

ABCD is a quadrilateral and forces acting at a point are represented in direction and magnitude by \overline{BA} , \overline{BC} , \overline{CD} and \overline{DA} . Find their resultant.

Solution:

$$\text{We have } \overline{BC} + \overline{CD} + \overline{DA} = \overline{BA}$$

$$\begin{aligned} \therefore \overline{BA} + (\overline{BC} + \overline{CD} + \overline{DA}) &= \overline{BA} + \overline{BA} \\ &= 2 \overline{BA} \end{aligned}$$

Hence the resultant is $2\overline{BA}$, both in magnitude and direction.

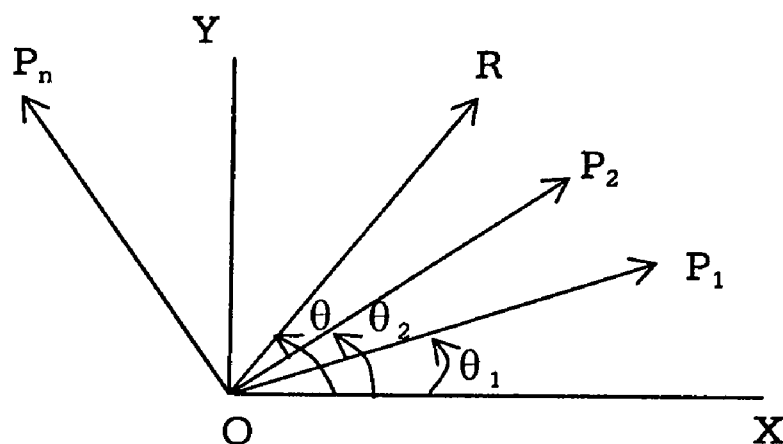
Exercises

1. ABCDE is a pentagon. Forces acting on a particle are represented in magnitude and direction by \overline{AB} , \overline{BC} , \overline{CD} , $2\overline{DE}$, \overline{AD} and \overline{AE} . Find their resultant.
2. a) ABCDE is a square and forces acting at a point are represented in magnitude and direction by \overline{AB} , \overline{BC} , \overline{CD} and \overline{DA} . Find their resultant.
b) Forces acting at a point are represented in magnitude and direction by \overline{AB} , $2\overline{BC}$, $2\overline{CD}$, \overline{DA} and \overline{DB} where ABCD is a square. Show that the forces are in equilibrium.

2.3 RESULTANT OF ANY NUMBER OF COPLANAR FORCES

Resultant of any number of coplanar forces acting at a point : Analytical Method.

Let forces $P_1, P_2, P_3, \dots, P_n$ act at O. Through O draw two lines OX and OY at right angles to each other in the plane of the forces.



Let the lines of action of $P_1, P_2 \dots P_n$ make angles $\alpha_1, \alpha_2 \dots \alpha_n$ with OX.

Let R be the resultant inclined at an angle θ to OX. Then

$R \cos \theta$ = resolved part of the resultant along OX.

= algebraic sum of the resolved parts of $P_1, P_2 \dots P_n$ along OX. (2.3)

$$\begin{aligned} &= P_1 \cos \theta_1 + P_2 \cos \theta_2 + \dots + P_n \cos \theta_n. \\ &= X \text{ (Say)} \end{aligned} \tag{1}$$

$R \sin \theta$ = resolved part of the resultant along OY.

= algebraic sum of the resolved parts of $P_1, P_2 \dots P_n$ along OY.

$$= Y \text{ (Say)} \tag{2}$$

Squaring (1) and (2) and adding

We have

$$R^2 = X^2 + Y^2$$

$$\text{(ie) } R = \sqrt{X^2 + Y^2} \tag{3}$$

Dividing (2) by (1)

$$\tan \theta = \frac{Y}{X}$$

$$\text{(ie) } \theta = \tan^{-1} \left(\frac{Y}{X} \right) \tag{4}$$

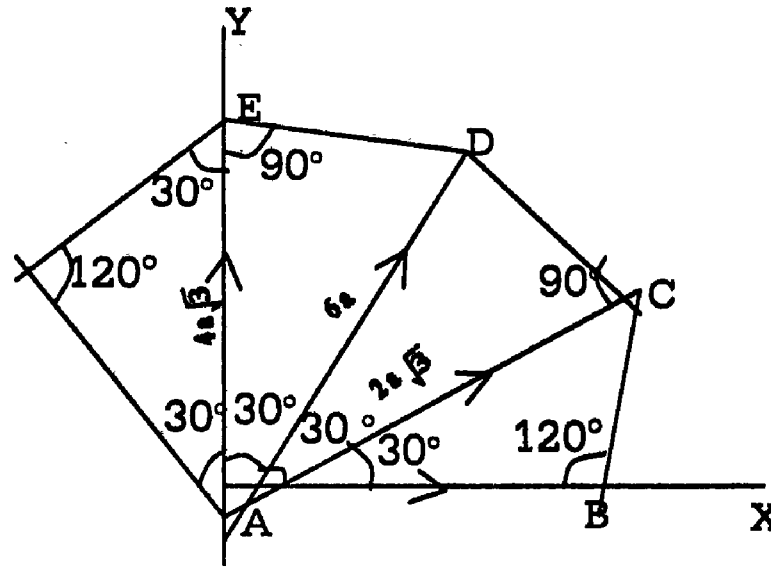
Equations (3) and (4) give respectively.

The magnitude and direction of the resultant.

Example: 1(U.Q)

ABCDEF is a regular hexagon and at A, act forces represented by \overline{AB} , $2\overline{AC}$, $3\overline{AD}$, $4\overline{AE}$ and $5\overline{AF}$. Show that the magnitude of the resultant is $AB \cdot \sqrt{351}$ and that it makes an angle $\tan^{-1} \left(\frac{7}{\sqrt{3}} \right)$ with AB.

Solution:



Let a be the side of the hexagon.

Each interior angle of a regular hexagon = 120°

$$\therefore \angle CAB = \angle ACB = 30^\circ$$

$$= \angle FAE = \angle FEA$$

From the the isosceles $\triangle ABC$

$$AC = 2AB \cdot \cos 30^\circ$$

$$AC = 2a \frac{\sqrt{3}}{2}$$

$$= a\sqrt{3} = AE$$

$$\angle AED = 90^\circ$$

$$\therefore AD^2 = AE^2 + ED^2$$

$$= 3a^2 + a^2 = 4a^2$$

$$AD = 2a.$$

Since the vertices of a regular hexagon lie on a circle.

$$\angle DAB + \angle DCB = 180^\circ$$

$$\therefore \angle DAB = 180^\circ - 120^\circ$$

$$= 60^\circ$$

$$\therefore \angle DAC = 30^\circ \text{ and } \angle EAD = 30^\circ$$

The magnitudes of the forces acting at A are a , $2a\sqrt{3}$, $6a$, $4a\sqrt{3}$ and $5a$ as shown in the figure.

Take AB and AE as axes of X and Y and let R be the resultant inclined at an angle θ to Resolving the forces along AB and AE.

We have

$$R \cos \theta = a + 2a\sqrt{3} \cos 30^\circ + 6a \cos 60^\circ + 5a \cos 120^\circ$$

$$= a + 2a\sqrt{3} \cdot \frac{\sqrt{3}}{2} + 6a \cdot \frac{1}{2} - 5a \cdot \frac{1}{2}$$

$$= \frac{9a}{2} \quad (1)$$

$$\text{and } R \sin \theta = 2a\sqrt{3} \cos 60^\circ + 6a \cos 30^\circ + 4a\sqrt{3} + 5a \cos 30^\circ$$

$$= 5a\sqrt{3} + \frac{11a\sqrt{3}}{2} = \frac{21a\sqrt{3}}{2} \quad (2)$$

Squaring (1) and (2)

Add adding

$$R^2 = \frac{9a}{2} + \left(\frac{21a\sqrt{3}}{2} \right)^2$$

$$= \frac{81a^2}{4} + \frac{441}{4} \times 3a^2$$

$$= \frac{1404}{4} a^2 = 351 a^2$$

$$R = a\sqrt{351}$$

Dividing (2) by (1)

$$\begin{aligned} \tan \theta &= \frac{21a\sqrt{3}}{2} \times \frac{2}{9a} \\ &= \frac{21\sqrt{3}}{9} = \frac{7\sqrt{3}}{3} = \frac{7}{\sqrt{3}} \end{aligned}$$

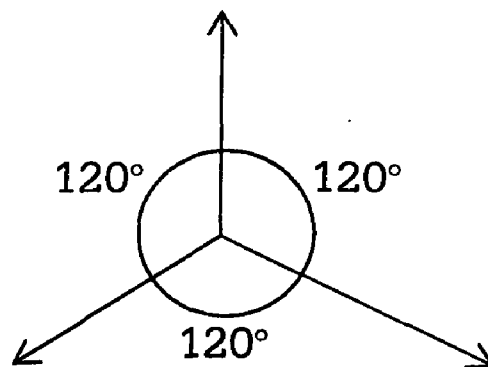
Hence the resultant is a force of magnitude $AB\sqrt{351}$, in a direction making an angle $\tan^{-1}\left(\frac{7}{\sqrt{3}}\right)$ with AB.

Example: 2

Find the magnitude and direction of the resultant of three coplanar forces P , $2P$, $3P$ acting at a point and inclined mutually at an angle of 120° .

Solution:

Resolving the force along and perpendicular to the direction of the first force \vec{P} ,



We get

$$X = P \cos 0^\circ + 2P \cos 120^\circ + 3P \cos 240^\circ$$

$$= P - 2P \cos 60^\circ - 3P \cos 60^\circ$$

$$= P - 2P \left(\frac{1}{2}\right) - 3P \left(\frac{1}{2}\right) = -\frac{3}{2} P.$$

$$Y = P \sin 0^\circ + 2P \sin 120^\circ + 3P \sin 240^\circ$$

$$= 2P \sin 60^\circ - 3P \sin 60^\circ = -P \sin 60^\circ$$

$$= -P \frac{\sqrt{3}}{2}$$

The resultant of the force has magnitude.

$$R = \sqrt{X^2 + Y^2}$$

$$= \sqrt{\frac{9}{4}P^2 + \frac{3}{4}P^2} = \sqrt{3}P$$

If θ is the angle made by this resultant with the first forces \overline{P} .

$$\text{Then } \tan \theta = \frac{Y}{X} = -\frac{\frac{\sqrt{3}}{2}P}{-\frac{3}{2}P}$$

$$= \frac{1}{\sqrt{3}} = \tan 30^\circ = \tan (180^\circ + 30^\circ)$$

Since X and Y are -ve, the

resultant lies in 3rd quadrant.

$$\therefore \theta = 210^\circ.$$

Example: 3 (U.Q)

Forces of magnitudes $P - Q$, P , $P + Q$ act at a point in directions parallel to the sides of an equilateral triangle taken in order. Show that the resultant is of magnitude $Q\sqrt{3}$ acting perpendicular to the direction of the second force.

Solution:

Let R be the resultant of force making an angle θ with BC. Resolving along BC.

$$R \cos \theta = P \cos 0^\circ + (P+Q) \cos 120^\circ - (P - Q) \cos 60^\circ$$

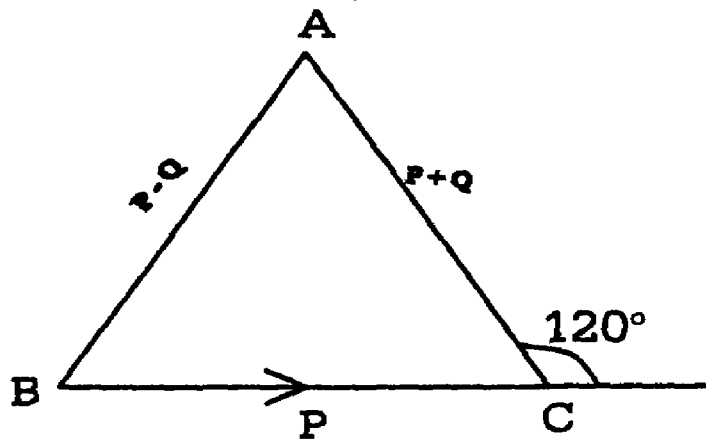
$$= P + (P+Q) \cos (180^\circ - 60^\circ) - (P - Q) \cos 60^\circ$$

$$= P + (P+Q) (-\cos 60^\circ) - (P - Q) \cos 60^\circ.$$

$$= P - 2P \cos 60^\circ$$

$$= P - P = 0$$

(1)



Resolving \perp^r to BC

$$\begin{aligned}
 R \sin \theta &= P \sin 0^\circ + (P + Q) \sin 120^\circ - (P - Q) \sin 60^\circ \\
 &= (P+Q) \sin (180^\circ - 60^\circ) - (P - Q) \sin 60^\circ \\
 &= (P+Q) \sin 60^\circ - (P - Q) \sin 60^\circ. \\
 &= 2Q \sin 60^\circ = 2Q \cdot \frac{\sqrt{3}}{2} = \sqrt{3} \cdot Q.
 \end{aligned}$$

Squaring (i) & (ii) and adding

$$R^2 (\cos^2 \theta + \sin^2 \theta) = 0 + 3Q^2$$

$$R^2 = 3Q^2$$

$$\therefore R = \sqrt{3} Q.$$

Also, Dividing $\frac{R \sin \theta}{R \cos \theta} = \frac{\sqrt{3} Q}{0}$

$$\tan \theta = \infty$$

$$\theta = 90^\circ$$

Resultant = $\sqrt{3} \cdot Q$ and acts \perp^r to BC (ie) \perp^r to the direction of P.

Exercises.

1. Forces P , $3P$, $\sqrt{3}P$ and $\sqrt{3}P$ act along the straight lines OA , OB , OC , OD lying in a plane. If $\angle AOB = 60^\circ$, $\angle BOC = 90^\circ$ and $\angle COD = 120^\circ$, find the magnitude and direction of their resultant.
2. Forces of 2 , $\sqrt{3}$, $5\sqrt{3}$, 2 kgs. Wt. Respectively act one of the angular points of a regular hexagon towards the five others in order. Find the direction and magnitude of the resultant.
3. Show that the resultant of forces equal to 7 , 1 , 1 and 3 kgs.wt. respectively acting at an angular point of a regular pentagon towards the other angular points, taken in order is $\sqrt{71}$ kgs.wt.
4. Three forces P , Q , R in one plane act on a particle the angles between R and Q , P and R and P and Q being α , β and γ respectively show that their resultant is equal to $\sqrt{P^2 + Q^2 + R^2 + 2QR \cos \alpha + 2RP \cos \beta + 2PQ \cos \gamma}$.
5. Three forces acting at a point are parallel to the sides of a triangle ABC , taken in order and proportional to the cosine of the opposite angles: Show that their resultant is proportional to $(1 - 8 \cos A \cos B \cos C)^{\frac{1}{2}}$.

2.4 CONDITIONS OF EQUILIBRIUM

Conditions of equilibrium of any number of forces acting upon a particle:

Forces acting at a point are in equilibrium when their resultant is zero. We shall now give the conditions which must be satisfied by a number of forces acting at a point of a rigid body or on a particle, in order that the body, or the particle may be at rest.

Geometrical or Graphical Conditions:

We have already studied the Triangle of Forces and the polygon of Forces.

If forces acting at a point are represented in magnitude and direction by lines forming the successive sides of a polygon, then for equilibrium, the polygon must be closed. When there are only three forces acting on a particle, the conditions of equilibrium are often most easily found by applying Lami's Theorem.

Analytical conditions:

If we resolve the forces in any two directions at right angles and the sums of the components in these directions be X and Y , the resultant R is given by

$$R^2 = X^2 + Y^2 \text{ [Refer equation (3) is 2.3]}$$

If the forces are in equilibrium, $R = 0$.

$$\text{Then } X^2 + Y^2 = 0$$

Now the sum of the squares of two real quantities cannot be zero unless each quantity is separately zero.

$$\therefore X = 0 \text{ and } Y = 0.$$

Hence, if any number of forces acting at a point are in equilibrium, the algebraic sums of the resolved parts of the forces in any two perpendicular directions must be zero separately.

Conversely,

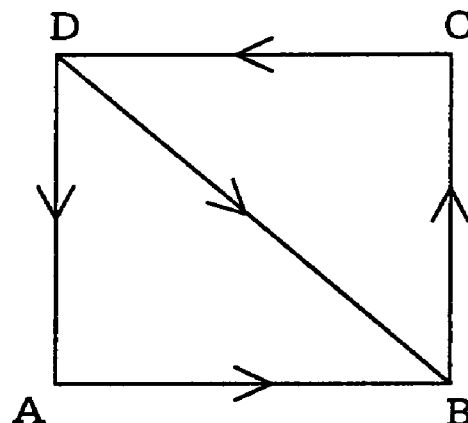
If the algebraic sum of the resolved parts of the forces acting at a point in any two perpendicular directions are zero separately the forces will be in equilibrium.

This is because when $X = 0$ and $Y = 0$ we must have $R = 0$.

Example : 1

Forces acting at a point are represented in magnitude and direction by \overline{AB} , \overline{BC} , \overline{CD} , \overline{DA} and \overline{DB} where $ABCD$ is a square. Show that the forces are in equilibrium.

Solution:



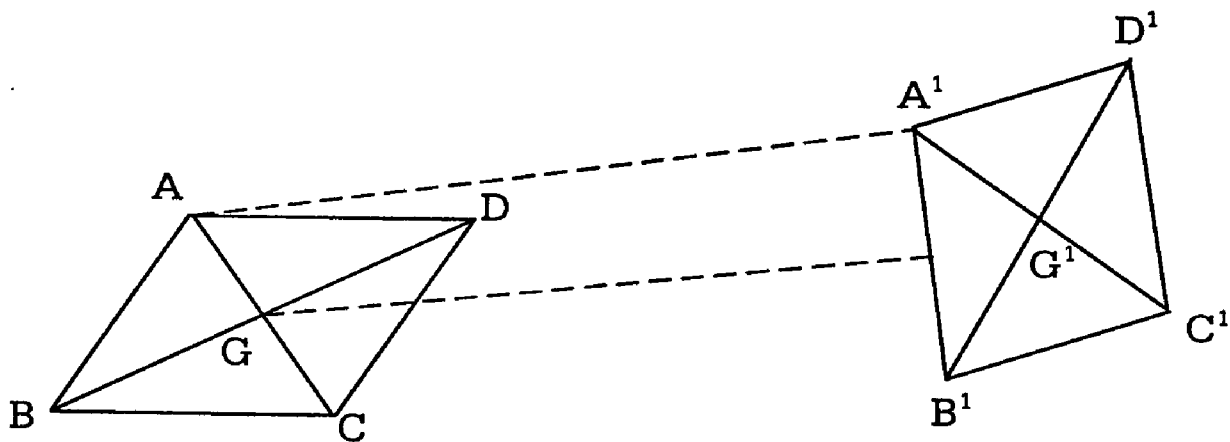
$$\begin{aligned} & \overline{AB} + 2\overline{BC} + 2\overline{CD} + \overline{DA} + \overline{DB} \\ &= (\overline{AB} + \overline{BC} + \overline{CD} + \overline{DA}) + (\overline{BC} + \overline{CD} + \overline{DB}) \\ &= \overline{0} \text{ [Since the forces in the first set brackets are in equilibrium by} \\ & \text{the polygon of forces (square ABCD) and the forces in the} \\ & \text{second set are in equilibrium by the triangle of forces (triangle} \\ & \text{BCD)].} \end{aligned}$$

Hence the given set of forces are in equilibrium.

Example : 2

ABCD and A¹ B¹ C¹ D¹ are Parallelograms prove that forces $\overline{AA^1}$, $\overline{B^1B}$, $\overline{CC^1}$ and $\overline{D^1D}$ acting at a point will keep it at rest.

Solution:-



Let G and G¹ be the points of intersection of the diagonals.

By the polygon of forces, from the quadrilateral AGG¹A¹.

$$\overline{AA^1} = \overline{AG} + \overline{GG^1} + \overline{G^1A^1}$$

Similarly

$$\overline{B^1B} = \overline{B^1G^1} + \overline{G^1G} + \overline{GB}$$

$$\overline{C^1C} = \overline{CG} + \overline{GG^1} + \overline{G^1C^1}$$

$$\overline{D^1D} = \overline{D^1G^1} + \overline{G^1G} + \overline{GD} \text{ (AG = AG}^1\text{)}$$

Adding up,

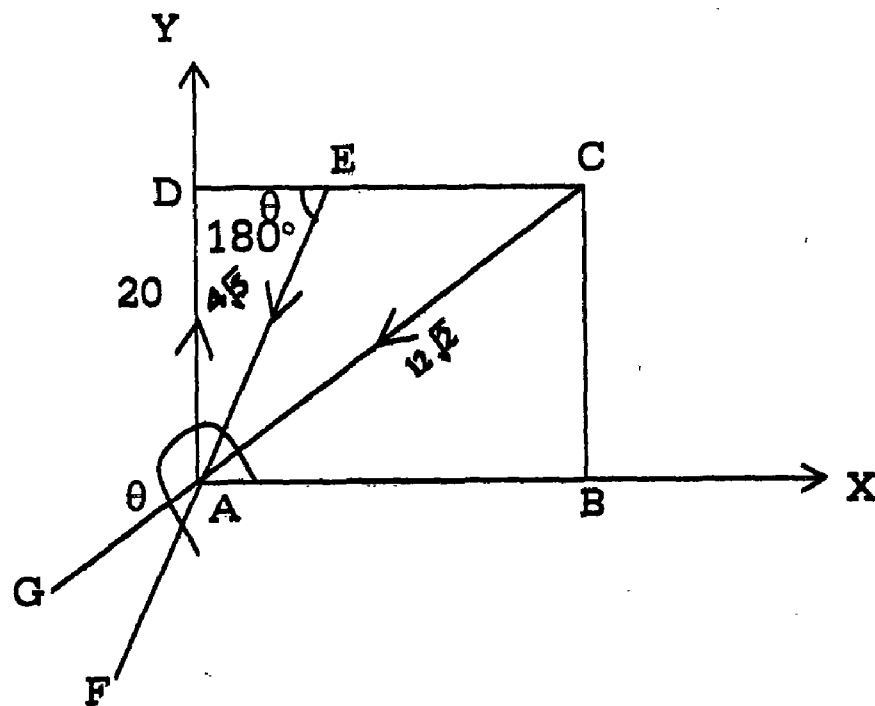
$$\overline{AA^1} + \overline{B^1B} + \overline{CC^1} + \overline{D^1D} = \overline{0}$$

[∴ the concerned vectors in the right side are equal and opposite].

Example: 3 (U.G)

E is the middle point of the side CD of a square ABCD. Forces 16, 20, $4\sqrt{5}$, $12\sqrt{2}$ kg. wt. Act along AB, AD, EA, CA in the directions indicated by the order of the letters. Show that they are in equilibrium.

Solution:



Take AB and AD as axes of X and Y. Produce EA to F and let $\angle BAF = \theta$.

Produce CA to G.

$$\begin{aligned} \angle BAG &= \angle BAC + \angle CAG \\ &= 45^\circ + 180^\circ = 225^\circ \end{aligned}$$

Let R be the resultant of the forces inclined at an angle θ to AB.

Resolving the forces along AB and AD.

We have

$$R \cos \theta = 16 + 12\sqrt{2} \cos 225^\circ + 4\sqrt{5} \cos \theta$$

$$= 16 + 12\sqrt{2} \cos (180^\circ + 45^\circ) + 4\sqrt{5} \cos \theta$$

$$= 16 + 12\sqrt{2} \times -\cos 45^\circ + 4\sqrt{5} \cos \theta$$

$$= 16 - 12\sqrt{2} \times \sqrt{\frac{1}{2}} + 4\sqrt{5} \cos \theta$$

$$= 4 + 4\sqrt{5} \cos \theta$$

(1)

$$\angle BAE = \angle BAF - \angle EAF$$

$$= \theta - 180^\circ$$

$$\& \angle DEA = \text{alt. } \angle BAE = \theta - 180^\circ$$

In rt. θ d. $\triangle AED$, $AE^2 = AD^2 + DE^2$

$$= a^2 + \frac{a^2}{4}, \text{ [a being the side of the square].}$$

$$= \frac{5a^2}{4}$$

$$AE = \frac{a\sqrt{5}}{2}$$

$$\cos (\theta - 180^\circ) = \frac{DE}{AE} = \frac{\left(\frac{a}{2}\right)}{\left(\frac{a\sqrt{5}}{2}\right)} = \frac{1}{\sqrt{5}}$$

$$\cos (\theta - 180^\circ) = \frac{1}{\sqrt{5}} = -\cos \theta$$

$$\cos \theta = -\frac{1}{\sqrt{5}}$$

From (1)

$$R \cos \theta = 4 + 4\sqrt{5} \times -\frac{1}{\sqrt{5}} = 0 \quad (2)$$

$$\begin{aligned} R \sin \theta &= 12\sqrt{2} \sin 225^\circ + 4\sqrt{5} \sin \theta + 20 \\ &= -12\sqrt{2} \sin (180^\circ + 45^\circ) + 4\sqrt{5} \sin \theta + 20 \\ &= -12\sqrt{2} \cdot \sin 45^\circ + 4\sqrt{5} \sin \theta + 20 \\ &= -12\sqrt{2} \cdot \frac{1}{\sqrt{2}} + 4\sqrt{5} \sin \theta + 20 \\ &= 8 + 4\sqrt{5} \sin \theta \end{aligned} \quad (3)$$

From rt \angle d $\triangle AED$.

$$\sin (\theta - 180^\circ) = \frac{AD}{AE} = \frac{a}{\left(\frac{a\sqrt{5}}{2}\right)} = \frac{2}{\sqrt{5}}$$

$$-\sin (180 - \theta) = \frac{2}{\sqrt{5}}$$

$$\text{(or) } \sin \theta = -\frac{2}{\sqrt{5}}$$

$$\begin{aligned} \text{From (3) } R \sin \theta &= 8 + 4\sqrt{5} \times -\frac{2}{\sqrt{5}} \\ &= 8 - 8 = 0 \end{aligned} \quad (4)$$

Squaring (2) and (4) and adding.

$$R^2 = 0 + 0 = 0 \quad (\text{ie) } R = 0$$

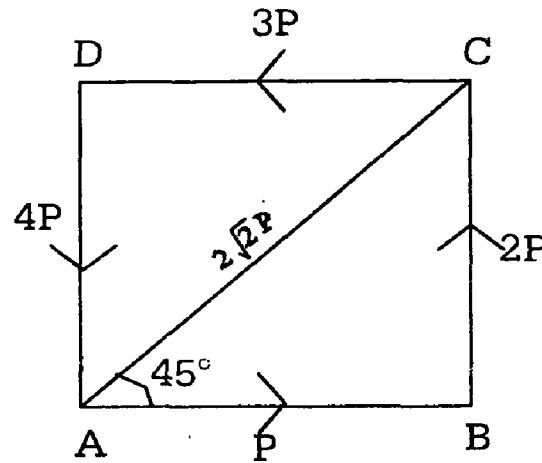
\therefore The forces are in equilibrium.

Example : 4 (U . Q)

Forces $P, 2P, 3P, 4P$ and $2\sqrt{2}P$ act at a point in direction of AB, BC, CD, DA and AC where $ABCD$ in a square : Show that they are in equilibrium.

Solution :

Let R be the resultant making an angle θ with AB .



Resolving along AB

$$\begin{aligned} R \cos \theta &= P - 3P + 2\sqrt{2}P \cos 45^\circ \\ &= -2P + 2\sqrt{2}P \cdot \frac{1}{\sqrt{2}} = 0 \end{aligned}$$

Resolving \perp to AB

$$\begin{aligned} R \sin \theta &= 2P - 4P + 2\sqrt{2}P \sin 45^\circ \\ &= -2P + 2\sqrt{2}P \cdot \frac{1}{\sqrt{2}} = 0 \end{aligned}$$

Squaring and adding $R^2 = 0$

$$\therefore R = 0$$

Hence the forces are in equilibrium.

Exercises

1. Fifteen Coplanar forces act at a point and are represented in magnitude and direction by the lines drawn from each of the vertices of a pentagon to the midpoints of those sides on which the vertex does not lie. Show that there are in equilibrium.
2. Three equal forces acting at a point are in equilibrium. Show that they are equally inclined to one another.
3. Three forces act perpendicularly to the sides of a triangle at their midpoints and are proportional to the sides. Prove that they are in equilibrium.

UNIT -III

FORCE ACTING ON A RIGID BODY

3.1 Introduction:

In the previous unit we have considered the method of finding the resultant of two forces which meet at a point. We shall now consider how to find the resultant of two parallel forces. Such forces do not meet in a point and so we cannot find their resultant by direct application of the law of parallelogram of forces.

Two parallel forces are said to be like when they act in the same direction. They are said to be unlike when they act in opposite parallel directions.

3.2 Resultant of Two like and Unlike Parallel Forces

To find the resultant of two like parallel forces acting on a rigid body:

Let like parallel forces P and Q act at the points A and B of the rigid body respectively and let them be represented by the lines AD and BL . At A and B , introduce two equal

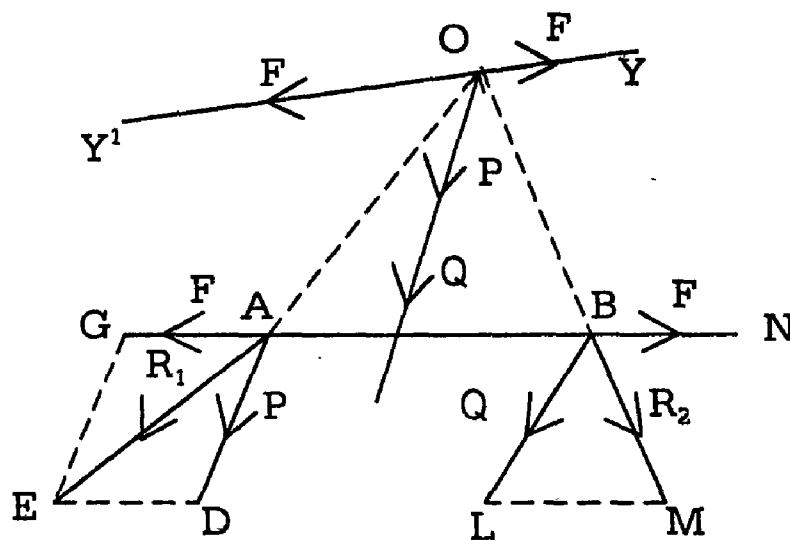


Fig. 1

and opposite forces F of arbitrary magnitude along the line AB and let them be represented by AG and BN . These two new forces will balance each other and hence will not affect the resultant of the system.

The two forces F and P acting at the point A can be compounded into a single force R_1 , represented by the diagonal AE of the parallelogram $ADEG$. Similarly the two forces F and Q acting at the point B will have a resultant R_2 represented by the diagonal BM of the parallelogram $BLMN$.

Produce EA and MB and let them meet at O. The two resultants R_1 and R_2 can be considered to act at O. At O draw $Y'OY \parallel$ to AB and $OX \parallel$ to the directions of P and Q. Resolve R_1 and R_2 at O into their original components.

R_1 at O is equal to a force F along OY' and a force P along OX. R_2 at O is equal to a force F along OY and a force Q along OX. The two F's at O cancel each other, being equal and opposite. We are now left with two forces P and Q acting along OX. Hence their resultant is a force (P+Q) acting along OX.

(ie) acting in a direction parallel to the original directions of P and Q.

Thus the magnitude of the resultant of two like parallel forces is their sum. The direction of the resultant is parallel to the components and in the same sense.

To find the position of the resultant:

Let OX meet AB at C.

Triangles OAC and AED are similar:

$$\therefore \frac{OC}{AD} = \frac{AC}{ED}, \text{ (ie) } \frac{OC}{P} = \frac{AC}{F}$$

$$\text{(or) } F \cdot OC = P \cdot AC \quad (1)$$

Triangles OCB and BLM are similar.

$$\therefore \frac{OC}{BL} = \frac{CB}{LM} \text{ (ie) } \frac{OC}{Q} = \frac{CB}{F}$$

$$\text{(or) } F \cdot OC = Q \cdot CB \quad (2)$$

From (1) and (2)

We have $P \cdot AC = Q \cdot CB$

$$\text{(i.e) } \frac{AC}{CB} = \frac{Q}{P}$$

(i.e) The point C divides AB internally in the inverse ratio of the forces.

3.2.1 To find the resultant of two unlike parallel forces acting on a rigid body: (U.Q)

Let P and Q be two unequal and unlike parallel forces acting at the points A and B of the rigid body.

Let $P > Q$ let them be represented by AD and BL . At A and B introduce two equal and opposite forces F of arbitrary magnitude along the line AB and let them be represented by AG and BN

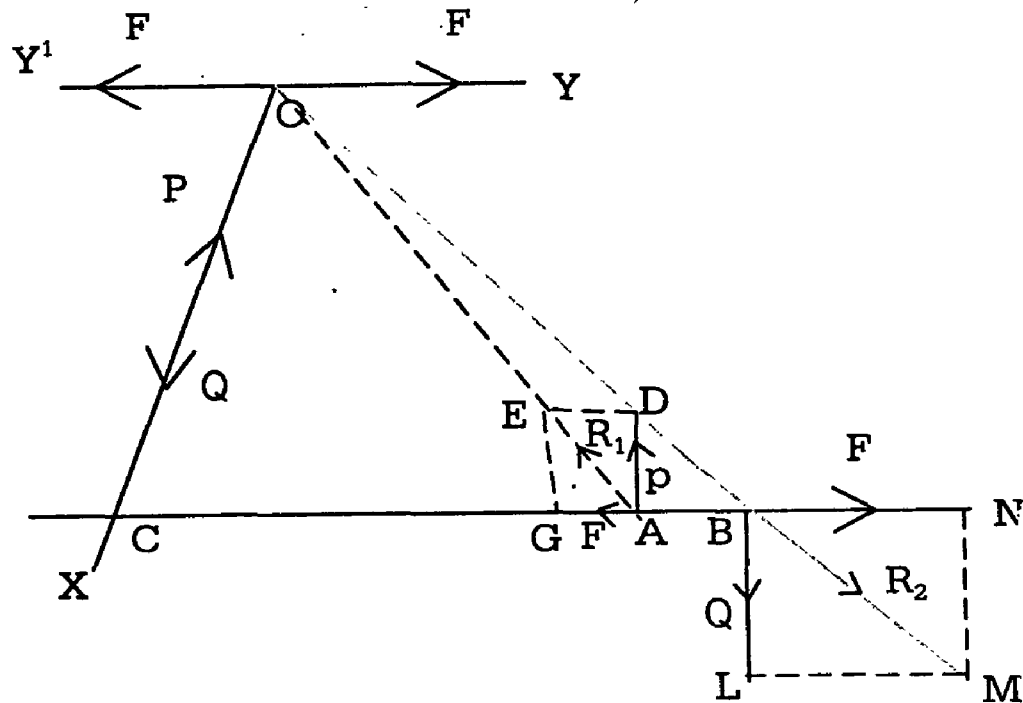


Fig. 2

These two new forces will balance each other and hence will not affect the resultant of the system.

The two forces F and P acting at A can be compounded into a single force R_1 , represented by the diagonal AE of the parallelogram $AGED$. Similarly the two forces F and Q acting at B have a resultant R_2 by the diagonal BM of the parallelogram $BLMN$.

Produce AE and MB and let them meet at O . The two resultants R_1 and R_2 can be considered to act at O . At O , draw $Y'OY \parallel$ to AB and OX parallel to the directions of P and Q . Reresolve R_1 and R_2 at O into their original components. R_1 at O is equal to a force F along OY' and a force P along OX . R_2 at O is equal to a force F along OY and a force Q along OX . The two F 's at O cancel each other being equal and opposite. We are now left with a force P along OX and a force Q along OX clearly the resultant is a force $P-Q$ (as $P > Q$) acting along OX .

(ie) acting in a direction parallel to that of P .

Thus the magnitude of the resultant of two unlike parallel forces is their difference. The direction of the resultant is parallel to and in the sense of the greater component.

To find the position of the resultant:

Let OX meet AB at C.

Triangles OCA and EGA are similar

$$\therefore \frac{OC}{EG} = \frac{CA}{GA}$$

$$(ie) \frac{OC}{P} = \frac{CA}{F}$$

$$(or) F \cdot OC = P \cdot CA \quad (1)$$

Triangles OCB and BLM are similar.

$$\therefore \frac{OC}{BL} = \frac{CB}{LM}$$

$$(ie) \frac{OC}{Q} = \frac{CB}{F}$$

$$(or) F \cdot OC = Q \cdot CB \quad (2)$$

From (1) and (2)

We have $P \cdot CA = Q \cdot CB$

$$(ie) \frac{CA}{CB} = \frac{Q}{P}$$

(i.e) The point C divides AB externally in the inverse ratio of the forces.

Note: As $P > Q$

CB must be $> CA$.

Hence the resultant passes nearer the greater force.

Failure of the above construction:

The construction for finding the resultant of two unlike parallel forces P and Q will fail.

If $P = Q$. (ie) if the forces are equal in magnitude. In that case, In. fig.2 triangles AGE and BNM will be congruent ;

$\angle GAE = \angle NBM$ and the lines AE and MB will be parallel.

There will be no such point as O.

Hence we conclude that the effect of two equal and unlike parallel forces that the effect of two equal and unlike parallel forces cannot be replaced by a single force. Such a pair of forces have no single resultant and they constitute what is called a couple, which will be considered later on.

3.2.2 Resultant of a number of parallel forces acting on a rigid body:-

If a number of parallel forces P, Q, R, \dots act on a rigid body, their resultant can be found by successive applications of law of the Parallelogram of Forces.

First, we find the resultant R_1 , of P and Q then we find the resultant R_2 of R_1 and R this process is continued, until the final resultant is obtained. If the parallel forces are all like, the magnitude of the final resultant is the sum of the forces. If the parallel forces are not all like, the magnitude of the resultant is the algebraic sum of the forces each taken with its proper sign.

3.2.3 Condition of equilibrium of three coplanar parallel forces:-

Let P, Q, R be the three forces parallel in one place and be in equilibrium.

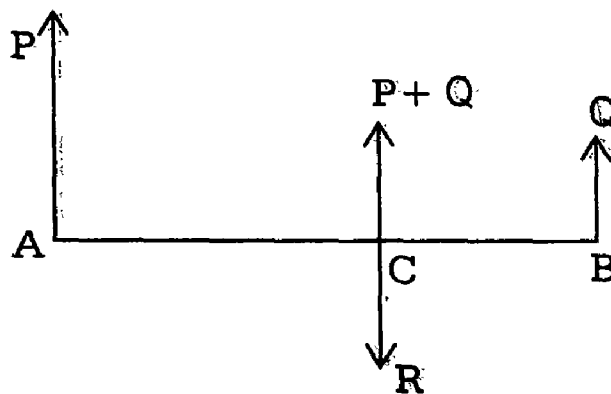


Fig. 3

Draw a line to meet the lines of action of these forces at A, B, and C respectively.

If all the three forces are in the same sense equilibrium will be like and the third R unlike.

The resultant of P and Q is (P+Q), Parallel to P or Q and hence, for equilibrium, R must be equal and opposite of (P+Q).

∴ R = P+Q and the line of action of P+Q must pass through C.

$$\therefore P \cdot AC = Q \cdot CB$$

$$\text{i.e) } \frac{P}{CB} = \frac{Q}{AC}$$

$$\text{and each} = \frac{P+Q}{CB+AC} = \frac{P+Q}{AB} = \frac{R}{AB}$$

$$\text{(ie) } \frac{P}{CB} = \frac{Q}{AC} = \frac{R}{AB}$$

Thus, if three parallel forces are in equilibrium, each is proportional to the distance between the other two.

Centre of two parallel forces:

Let P and Q be two parallel forces acting at two points A and B. Then, their resultant R passes through a point C. Which divides AB internally or externally in the ratio Q:P

$$\text{(i.e) } \frac{AC}{CB} = \frac{Q}{P} \tag{1}$$

The position of C given by (1) depends only upon the positions of A and B and then magnitudes of the forces P and Q.

It does not depend on the actual direction of P and Q. In other words, whatever be the common direction of parallelism of the forces P and Q their resultant will always pass through a certain fixed point. This fixed point is called the centre of the two parallel forces is a fixed point through which their resultant always passes whatever be the direction of parallelism.

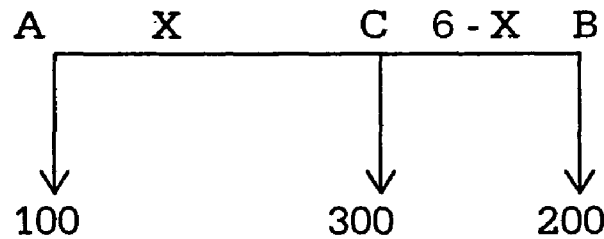
More generally, the resultant of a system of parallel forces of given magnitudes acting at given points of a body, will always pass through a fixed

point, for all directions of parallelism. This point is called the centre of parallel forces.

Example:

Two men, one stronger, than the other have to remove a block of stone weighting 300 kgs. With a light pole whose length is 6 metre. The weaker man cannot carry more than 100 kgs. Where must the stone be fastened to the pole. So as just to allow him his full share of weight?

Solution:



Let A be the weaker man bearing 100kgs.his full share of the weight of the stone and B the stronger man bearing 200 kgs. Let C be the point on AB where the stone is fastened to the pole, such that AC = X. Then the weight of the stone acting at C is the resultant of the parallel forces 100 and 200 at A and B respectively.

$$\therefore 100 AC = 200BC$$

$$(ie) 100 X = 200 (6-X)$$

$$100 X = 1200-200X$$

$$\therefore 300 X = 1200$$

$$(or) X = 4$$

Hence the stone must be fastened to the pole at the point distant 4 metres from the weaker man.

Example :2 (U.Q)

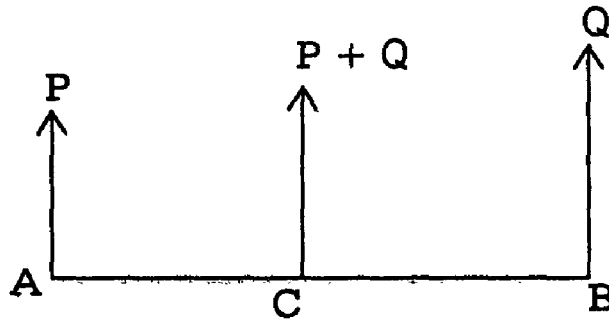
Two like parallel forces P and Q act on a rigid body at A and B respectively.

- a) If Q be changed to $\frac{P^2}{Q}$, show that the line of action of the resultant is the same as it would be if the forces were simply interchanged.

- b) If P and Q be interchanged in position, show that the point of application of the resultant will be displaced along AB through a distance d, where

$$d = \frac{P-Q}{P+Q} AB$$

Solution:



Let C be the centre of two parallel forces with P at A and Q at B.

$$\text{Then } P \cdot AC = Q \cdot CB \quad (1)$$

IF Q is changed to $\frac{P^2}{Q}$, (P remaining the same)

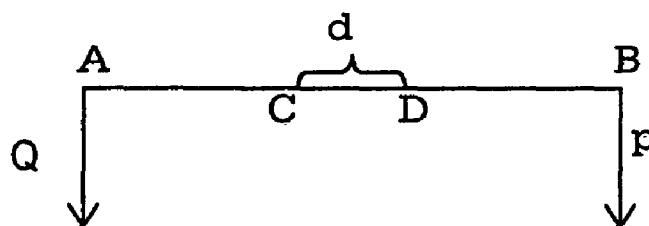
Let D be the new centre of parallel forces.

$$\text{Then } P \cdot AD = \frac{P^2}{Q} \cdot DB \quad (2)$$

$$PQ \cdot AD = P^2 \cdot DB$$

$$\text{(or) } Q \cdot AD = P \cdot DB \quad (3)$$

Relation (3) shows that D is the centre of two like parallel forces with Q and A and P at B



When the forces P and Q are interchanged in position, D is the new centre of parallel forces.

$$CD = d$$

$$\text{From (3) } Q \cdot (AC + CD) = P \cdot (CB - CD)$$

$$\text{(ie) } Q \cdot AC + Q \cdot d = P \cdot CB - P \cdot d$$

$$\text{(or) } (Q+P) \cdot d = P \cdot CB - Q \cdot AC$$

$$= P \cdot (AB - AC) - Q \cdot (AB - CB)$$

$$= P \cdot AB - P \cdot AC - Q \cdot AB + Q \cdot CB$$

$$(Q+P) \cdot d = (P-Q) \cdot AB \quad [\because P \cdot AC = Q \cdot CB \text{ from (1)}]$$

$$\text{(or) } d = \frac{P-Q}{P+Q} \cdot AB$$

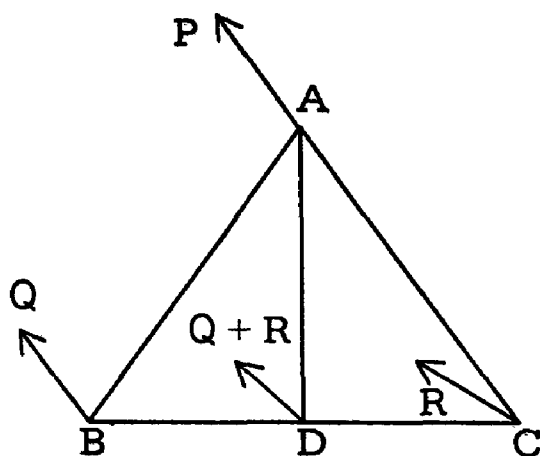
Example: 3

Three like parallel forces, acting at the vertices of a triangle, have magnitudes proportional to the opposite sides show that their resultant passes through the incentre of the triangle.

Solution:

Let like parallel forces P, Q, R act at A, B, C . It is given that

$$\frac{P}{a} = \frac{Q}{b} = \frac{R}{c} \quad (1)$$



Let the resultant of Q and R meet BC at D .

We know that the magnitude of the resultant is $Q + R$

$$\text{(ie) } Q \cdot BD = R \cdot DC$$

$$\text{Also } \frac{BD}{DC} = \frac{\text{force at C}}{\text{force at B}} = \frac{R}{Q}$$

$$= \frac{c}{b} \text{ from (1)}$$

$$= \frac{AB}{AC}$$

AD is the internal bisector of $\angle A$.

We have now to find the resultant of the two like parallel forces, $Q+R$ at D and P at A.

Let this resultant meet AD at I.

$$\text{Then } \frac{AI}{ID} = \frac{\text{force at D}}{\text{force at A}} = \frac{Q+R}{P}$$

$$= \frac{b+c}{a} \text{ from (1)}$$

(2)

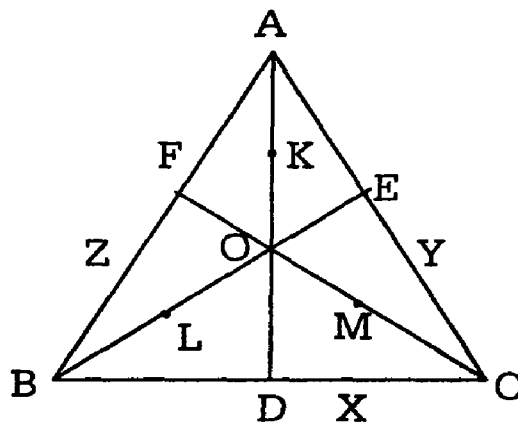
From result (2) it is clear that I is the incentre of the Δ .

Example: 4

O is the orthocentre of a ΔABC . Six equal like parallel forces pass, three of them through the mid points of the sides and the other three through the middle points of OA, OB, OC show that the resultant passes through the nine point centre of the triangle.

Solution:

Three forces each of magnitude P pass through the mid points X, Y, Z of the sides. Three other forces of magnitude P each pass through K, L, M the mid points of OA, OB and OC.



The nine points circle passes through the points D, X and K and $\angle KDX = 90^\circ$. Hence KX is a diameter of the circle. The nine point is a diameter of the circle. The nine point centre N is thus the middle point of KX. (ie) the resultant of like parallel forces \vec{P} at k and \vec{P} at X is $2\vec{P}$ acting at N. ||| N being the midpoint of LY and MZ, the resultant of forces \vec{P} each at L and Y as well as at M and Z will be \vec{QP} and $2\vec{P}$ at N. (ie) the resultant of the six forces is $2\vec{P} + 2\vec{P} + 2\vec{P} = 6\vec{P}$ at N the nine point centre.

Exercise:

- 1) If the magnitudes of two unlike parallel forces P, Q ($P > Q$) be increased by the same amount, show that the line of action of the resultant will move further off from P.
- 2) Three equal like parallel forces act at the middle points of the sides of a triangle, show that their resultant passes through the point of intersection.
- 3) Four equal like parallel forces act at the corners of a square, show that their resultant passes through the centre of square.

UNIT – IV

MOMENT OF A FORCE

Topic: 4.1

Moment of a Force:

When forces act on a particle, the only motion that can occur is a motion of translation. But a force acting on a rigid body may produce either a motion of translation or rotation or translation and rotation combined. When there is a motion of translation alone. The force is measured by the product of the mass of the particle and the acceleration produced on it by the force. In the case of rotation, the idea of the turning effect or moment of a force is introduced.

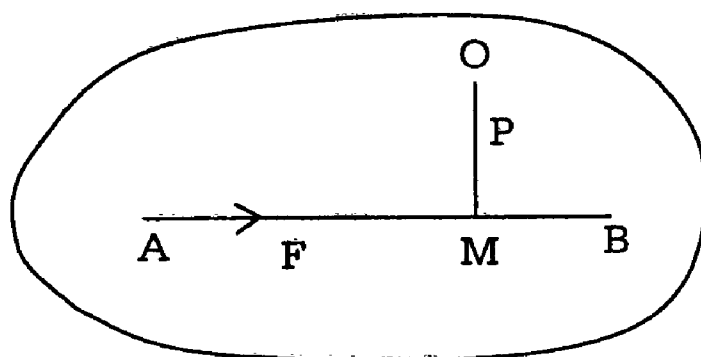


Fig. 1

Consider a sheet of cardboard pivoted freely at a fixed point O. If a force F acts along a straight line AB, it is clear that there will be no rotation if AB passes through O. If AB does not pass through O. The force will tend to rotate the sheet about O. This tendency to rotate the body will increase as the magnitude of the force increases and also as the perpendicular distance from O on the line of action of F. The tendency to rotate varies as F when ON is constant. It also varies ON when F is constant.

Hence it varies as $F \times ON$

(ie) the product of F and ON. When both these quantities vary.

This product is called the moment of F about O. Thus the moment of a force about a point is defined to be the product of the force and the perpendicular distance of the point from the line of action of the force.

The product F . ON will become Zero only if either F is Zero or ON is Zero.

When $ON = 0$ the point O is on the line of action of F .

Hence if the moment of a force about a point is zero, either.

- i) the force itself is zero, or
- ii) the line of action of the force passes through that point.

4.1.1 Physical significance of the moment of a force:-

From the definition of the moment of a force about a point, it is clear that it is a fitting measure for the turning effect of the force about that point. Thus the physical meaning for the moment of a force about a point is that it measures the tendency to rotate the body about the point.

4.1.2 Geometrical Representation of a moment:-

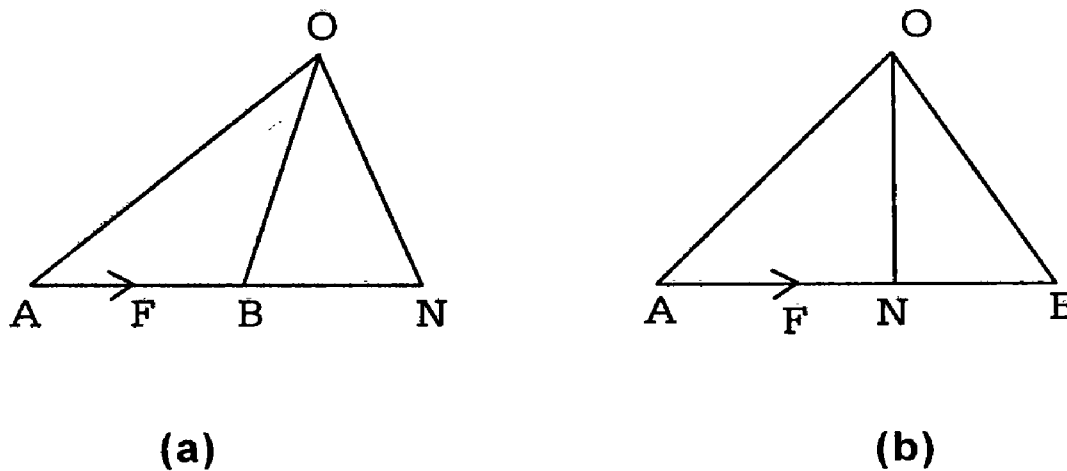


Fig. 2

Let a force F acting on a body be represented in magnitude direction and line of action by the line AB .

Let O be any given point and ON the perpendicular from O on AB or AB produced.

The moment of the force F about O .

$$= F \times ON = AB \times ON = 2 \Delta AOB.$$

Hence if a force is represented completely by a straight line, its moment about any point is given by twice the area of the triangle which the straight line subtends at the point.

4.1.3 Sign of the moment:

In fig.2 (a) when the force F acts along AB , it will tend to rotate the lamina in the anticlockwise direction.

(ie) in a direction opposite to that in which the hands of clock move. In such case, the moment is said to be positive. If the force tends to turn the body in a clock wise direction, its moment is said to be negative.

Thus the moment, of a force about a point has both magnitude and direction and is therefore a vector quantity.

4.1.4 Unit of moment:

The moment of a unit force about a point at a unit perpendicular distance from the line of action of force is defined as the unit for the measurement of moments. If the unit of force be a poundal and unit of distance be one foot the unit of moment is a poundal foot. If the unit of force be a dyne and unit of distance be one centimeter, the unit of moment is a dyne -cm.

4.2 VARIGON'S THEOREM

Varigon's theorem of moments: (U.Q)

The algebraic sum of the moments of two forces about any point in their plane is equal to the moment of their resultant about that point.

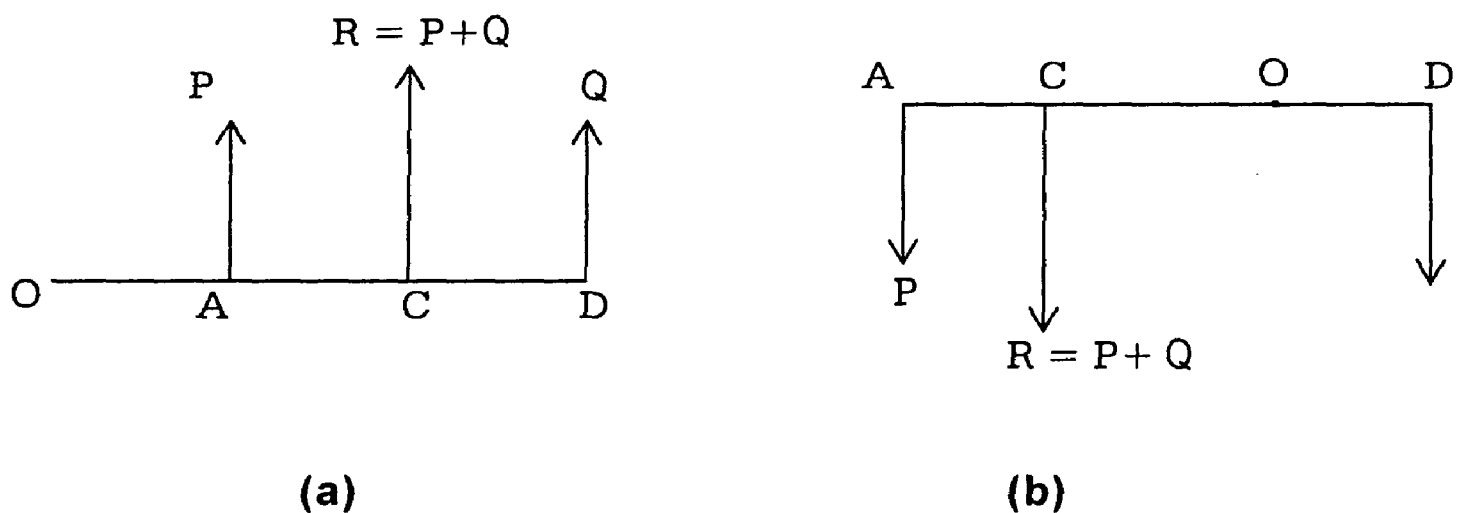


Fig. 3

To prove this theorem.

We consider two cases.

Case i) Let the forces be parallel.

Let P and Q be two parallel forces and O any point in their plane. Draw AOB perpendicular to the forces to meet their lines of action in A and B .

The resultant of P and Q is a force

$R = (P+Q)$ acting at C. Such that

$$P \cdot AC = Q \cdot CB.$$

The algebraic sum of the moments of P and Q about O

$$\begin{aligned} &= P \cdot OA + Q \cdot OB \\ &= P \cdot (OC-AC) + Q \cdot (OC +CB) \\ &= (P+Q) \cdot OC - P \cdot AC +Q \cdot CB \\ &= (P+Q) \cdot OC [\because P \cdot AC = Q \cdot CB] \\ &= R \cdot OC \\ &= \text{moment of R about O.} \end{aligned}$$

In fig. (a) and (b) , O is within AB and the algebraic sum of the moments of P and Q about O.

$$\begin{aligned} &= P \cdot OA - Q \cdot OB \\ &= P \cdot (OC+CA) - Q \cdot (CB -CO) \\ &= (P+Q) \cdot OC +P \cdot CA - Q \cdot CB \\ &= (P+Q) \cdot OC [\because P \cdot AC = Q \cdot CB] \\ &= R \cdot OC \\ &= \text{moment of R about O.} \end{aligned}$$

When the parallel forces P and Q are unlike and unequal, the theorem can be proved exactly in the same way

Case ii)

Let the two forces P and Q act at A as shown in fig as (a) and (b)

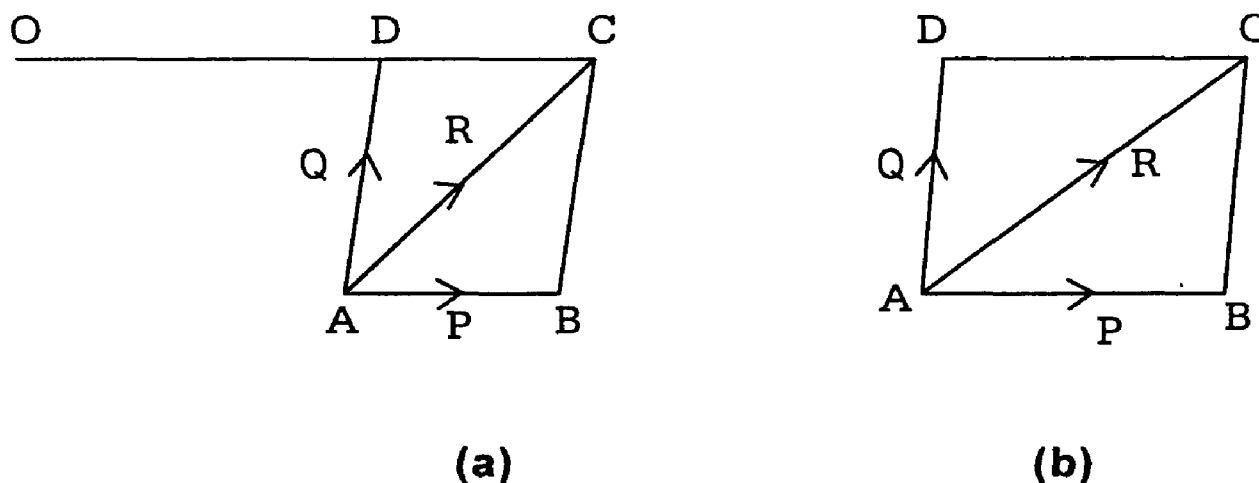


Fig. 4

And let O be any point in their plane. Through O, draw a line parallel to the direction of P meeting the line of action of Q at D.

Choose the scale of representation such that length AD represents Q in magnitude. On the same scale, let length AB represent P. Complete the parallelogram BAD so that the diagonal AC represents the resultant R of P and Q.

In either figure, the moments of P, Q, R about O are represented by $2 \Delta AOB$, $2 \Delta AOD$ and $2 \Delta AOC$ respectively.

If fig (a), O lies outside the $\angle BAD$ and the moments of P and Q are both positive.

$$\begin{aligned}
 &\text{The algebraic sum of the moments of P and Q} \\
 &= 2 \Delta AOB + 2 \Delta AOD \\
 &= 2 \Delta ACB + 2 \Delta AOD \\
 &\quad (\because \Delta AOB = \Delta ACB) \\
 &= 2 \Delta ADC + 2 \Delta AOD \\
 &\quad (\because \text{diagonal AC bisects the } \parallel_{gm}) \\
 &= 2(\Delta ADC + \Delta AOD) \\
 &= 2 \Delta AOC \\
 &= \text{moment of R about O.}
 \end{aligned}$$

In fig. (b) O lies inside the angle BAD.

The moment of P about O is positive while that of Q is negative.

$$\begin{aligned}
 &\text{The algebraic sum of the moments of P and Q} \\
 &= 2 \Delta AOB - 2 \Delta AOD \\
 &= 2 \Delta ACB - 2 \Delta AOD \\
 &= 2 \Delta ADC - 2 \Delta AOD \\
 &= 2(\Delta ADC - \Delta AOD) \\
 &= 2 \Delta AOC \\
 &= \text{moment of R about O.}
 \end{aligned}$$

4.2.1 Generalised theorem of moments (principle of moments)

If any number of coplanar forces acting on a rigid body have a resultant, the algebraic sum of their moments about any point in their plane is equal to the moment of their resultant about the same point.

Let P_1, P_2, P_3, \dots etc., be any number of coplanar forces and O any point in their plane.

Let R_1 be the resultant of P_1 and P_2 , R_2 that of R_1 and P_3 , R_3 that of R_2 and P_4 and so on until the final resultant R is obtained.

Apply varignon's theorem to the forces P_1 , P_2 , and R_1

We have

$$\begin{aligned} \text{Moment of } P_1 \text{ about } O + \text{moment of } P_2 \text{ about } O \\ = \text{moment of } R_1 \text{ about } O \end{aligned} \quad (1)$$

Similarly, applying the theorem to the forces R_1 , P_3 , and R_2 ,

We have

$$\begin{aligned} \text{Moment of } R_1 \text{ about } O + \text{moment of } P_3 \text{ about } O \\ = \text{moment of } R_2 \text{ about } O \end{aligned} \quad (2)$$

Combining (1) and (2)

We have

$$\begin{aligned} \text{Moment of } P_1 \text{ about } O + \text{moment of } P_2 \text{ about } O + \text{moment of } P_3 \text{ about } O \\ = \text{moment of } R_2 \text{ about } O \end{aligned}$$

Proceeding thus, till all the forces are exhausted, We prove the above theorem.

Let p_1, p_2, p_3, \dots be the perpendicular distances of O from the lines of action of the forces P_1, P_2, P_3, \dots respectively and P . The perpendicular distance of O from the line of action of the resultant R .

$$P_1 p_1 + P_2 p_2 + \dots = pR$$

$$\text{(ie) } \sum Pp_1 = pR \quad (1)$$

From this theorem, we derive the following important corollaries:

Corollary: 1

If the basic point O about which moments is taken happens to lie on the line of action of the resultant R , then $p = 0$.

$$\therefore \text{From (1) } \sum P_1 p_1 = 0$$

Hence the algebraic sum of the moments of any number of coplanar forces about any point on the line of action of their resultant is zero.

Corollary 2:

Suppose $\sum P_1 p_1 = 0$

Then from (1) $pR = 0$

\therefore Either $p = 0$ and $R = 0$

If $p=0$, it means that the basic point O about which moment is taken, lies on the line of action of the resultant.

If $R=0$, it means there is no resultant for system (ie) the forces are equilibrium.

Thus if the algebraic sum of the moments of any number of forces about any point in their plane is zero, then either their resultant passes through the point about which moments are taken or the resultant is zero. In the latter case, the forces will be in equilibrium.

Corollary: 3

Suppose $R = 0$

(ie) the forces are in equilibrium.

Then from (1) $\sum P_1 p_1 = p \times 0 = 0$

Hence if a system of coplanar forces is in equilibrium the algebraic sum of their moments about any point in their plane is zero.

This theorem enables us to find points on the line of action of resultant of a system of forces. For, we have only to find a point which the algebraic sum of the moments of the forces is zero and then the resultant must pass through that point.

Example: 1

Two men carry a load of 224kg.wt which hangs from a light pole of length 8m. each end of which rests on a shoulder of one of the men. The point from which the load is hung is 2m. nearer to one man than the other what is the pressure on each shoulder?

Solution:

AB is the light pole of length 8m. C is the point from which the load of 224 kgs is hung

Let $AC = x$

Then $BC = 8 - x$

It is given that

$$(8-x) - x = 2$$

$$(ie) 8 - 2x = 2$$

$$(or) 2x = 6$$

$$\therefore x = 3$$

(ie) $AC = 3$ and $BC = 5$.

Let the pressures at A and B be R_1 and R_2 kg. wt respectively.

Since the pole is in equilibrium, the algebraic sum of the moments of the three forces R_1 , R_2 and 224 kg. Wt about any point must be equal to zero.

Taking moments about B,

$$224 \cdot CB - R_1 \cdot AB = 0$$

(as the moment of R_2 about B is 0)

$$(ie) 224 \times 5 - R_1 \cdot 8 = 0$$

$$\therefore R_1 = \frac{224 \times 5}{8}$$

$$R_1 = 140.$$

Taking moments about A,

$$R_2 \cdot AB - 224 \cdot AC = 0$$

$$(ie) 8R_2 - 224 \times 3 = 0$$

$$\therefore R_2 = \frac{224 \times 3}{8}$$

$$\therefore R_2 = 84$$

Note: 1

For equilibrium, the weight of 224 kgs must be equal and opposite to the resultant of R_1 and R_2 .

$$\therefore R_1 + R_2 = 224.$$

Hence from this relation, we may find R_2 after finding R_1 .

Note: 2

In practice, instead of equating the algebraic sum of the moments of the forces about any point to zero, it will be convenient to equate the sum of the moments in one direction to the sum of the moments in the other direction. Hence in the above, taking moments about B,

We have $R_1 \cdot AB = 224 \cdot BC$.

Example: 2

A uniform plank of length $2a$ and weight w is supported horizontally on two vertical props at a distance b apart. The greatest weights that can be placed at the two ends in succession without upsetting the plank are w_1 and w_2 respectively. Show that

$$\frac{w_1}{w + w_1} + \frac{w_2}{w + w_2} = \frac{b}{a}$$

Solution:

Let AB be the plank placed upon two vertical props at C and D. $CD = b$.

The weight w of the plank acts at G, the midpoints of AB,

$$AG = GB = a$$

When the weight W_1 is Placed at A, the contact with D is just broken and the upward reaction at D then is zero.

There is upward reaction R_1 at C.

Now, taking moments about C.

We have $W_1 \cdot AC = W \cdot CG$

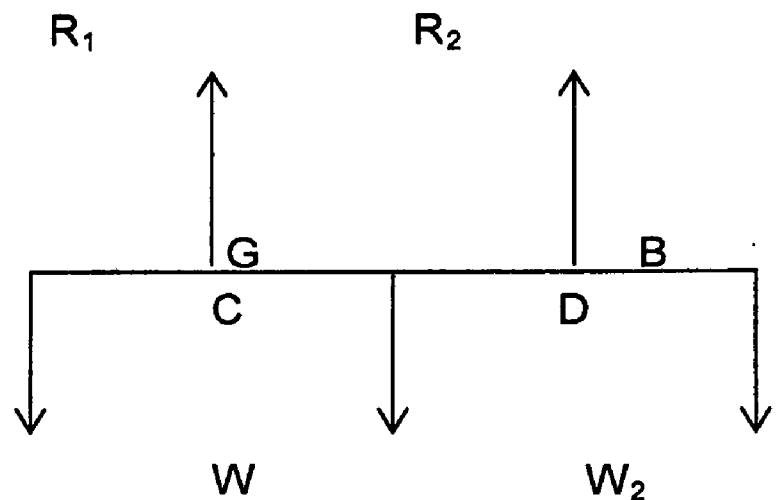
$$(ie) \quad W_1 (AG - CG) = W \cdot CG$$

$$W_1 AG = (W + W_1) CG$$

$$(ie) \quad W_1 a = (W + W_1) CG$$

$$CG = \frac{w_1 a}{w + w_1}$$

(1)



When the weight w_2 is attached at B,

There is loose contact at C.

The reaction at C becomes zero.

There is upward reaction R_2 about D.

Now taking moments about D,

We get $W \cdot GD = w_2 \cdot BD$

$$(ie) \quad W \cdot GD = w_2 (GB - GD)$$

$$GD (W + w_2) = w_2 \cdot GB = w_2 \cdot a$$

$$GD = \frac{w_2 a}{W + w_2} \quad (2)$$

But $CG + GD = CD = b$

$$\therefore \frac{w_1 a}{W + w_1} + \frac{w_2 a}{W + w_2} = b$$

$$\frac{w_1}{W + w_1} + \frac{w_2}{W + w_2} = \frac{b}{a}$$

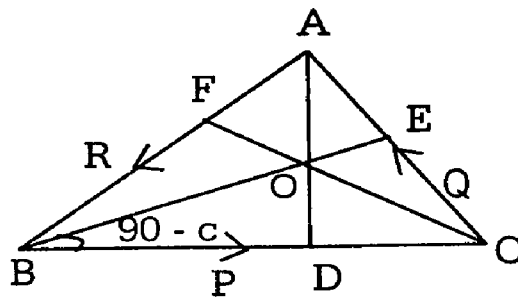
Example: 3

The resultant of three forces P, Q, R acting along the sides BC, and CA, AB of a triangle ABC passes through the orthocentre. Show that the triangle must be obtuse angled. If $\angle A = 120^\circ$, and $B=C$ show that $Q+R = P\sqrt{3}$.

Solution:

Let AD, BE and CF be the altitudes of the triangle intersecting at O. the orthocentre.

As the resultant passes through O, moment of the resultant about O = 0.



∴ Sum of the moments about P, Q, R about O is also = 0

Hence, taking moments about O.

We have $P \cdot OD + Q \cdot OE + R \cdot OF = 0$

(1)

In rt. $\triangle BOD$,

$$\angle OBD = \angle EBC = 90^\circ - C$$

$$\therefore \tan(90^\circ - C) = \frac{OD}{BD}$$

$$\therefore \cot C = \frac{OD}{BD}$$

$$\text{or } OD = BD \cot C$$

(2)

From rt. $\triangle ABD$,

$$\cos B = \frac{BD}{AB}$$

$$\therefore BD = AB \cdot \cos B = c \cdot \cos B$$

From (2) $OD = c \cos B \cdot \cot C$

$$= c \cos B \frac{\cos C}{\sin C}$$

$$= \frac{c}{\sin C} \cdot \cos B \cos C$$

$$= 2R^1 \cos B \cos C \left(\because \frac{c}{\sin C} = 2R^1 \text{ } R^1 \text{ being the circumradius of the } \triangle \right)$$

Similarly $OE = 2R^1 \cos C \cos A$.

And $OF = 2R^1 \cos A \cos B$.

Hence (1) becomes

$$P \cdot 2R^1 \cos B \cos C + Q \cdot 2R^1 \cos C \cos A + R \cdot 2R^1 \cos A \cos B = 0$$

Dividing by $2R^1 \cos A \cos B \cos C$

$$\text{We get } \frac{P}{\cos A} + \frac{Q}{\cos B} + \frac{R}{\cos C} = 0 \quad (3)$$

Now P, Q, R being magnitudes of the forces, are all positive

Hence in order that relation (3) may hold good, at least one of the terms must be negative.

Hence one of the cosines must be negative.

(ie) the triangle must be obtuse angled .

(Two of the cosine cannot be negative, as we cannot have two obtuse angles in the same triangle).

If $A = 120^\circ$ and the other angles equal

Then $B = C = 30^\circ$

Hence (3) becomes

$$\frac{P}{\cos 120^\circ} + \frac{Q}{\cos 30^\circ} + \frac{R}{\cos 30^\circ} = 0$$

$$\text{(ie) } \frac{P}{\left(-\frac{1}{2}\right)} + \frac{Q+R}{\left(\frac{\sqrt{3}}{2}\right)} = 0$$

$$\text{(ie) } P\sqrt{3} = Q+R$$

Example: - 4

Forces P, Q, R act along the sides BC, AC, BA respectively of an equilateral triangle. If their resultant is a force parallel to BC through the centroid of the triangle prove that $Q = R = \frac{1}{2} P$.

Solution:

$\triangle ABC$ being equilateral the medians AA^1 , BB^1 and CC^1 are also the altitudes meeting at G, the centroid.

Let DE be \parallel to BC through G.

It is given that DGE is the line of action of the resultant.

As the resultant passes through G, its moment about G = 0

\therefore sum of the moments of P, Q, R about G is also = 0.

$$(ie) P \cdot GA^1 - Q \cdot GB^1 - R \cdot GC^1 = 0.$$

$$(ie) P - Q - R = 0 \quad (GA^1 = GB^1 = GC^1) \quad (1)$$

Since the resultant passes through E also, sum of the moments of P, Q, R about E is = 0.

Draw $EL \perp$ to BC and $EM \perp$ to AB.

$$\therefore P \cdot EL - R \cdot EM = 0 \quad (2)$$

From the similar \triangle s ELC and AA^1C ,

$$\frac{EL}{AA^1} = \frac{EC}{AC} = \frac{1}{3}$$

$$\therefore DGE \parallel BC \text{ and } \frac{AD}{DB} = \frac{AE}{EC} = \frac{AG}{GA^1} = \frac{2}{1}$$

$$\therefore EC = \frac{1}{3} AA^1 \quad (3)$$

From the similar \triangle s AME and AC^1C .

$$\frac{EM}{CC'} = \frac{AE}{AC} = \frac{2}{3}$$

$$\therefore EM = \frac{2}{3} CC' \quad (4)$$

\therefore substituting (3) and (4) in (2)

We have

$$P \cdot \frac{1}{3} AA' - R \cdot \frac{2}{3} CC' = 0$$

$$(ie) P = 2R (\because AA' = CC')$$

$$(or) R = \frac{P}{2} \quad (5)$$

$$\text{Putting } R = \frac{P}{2} \text{ in (1)}$$

$$\text{We have } P - Q - \frac{P}{2} = 0$$

$$(ie) \frac{P}{2} - Q = 0$$

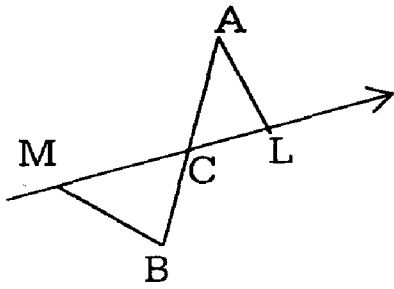
$$\frac{P}{2} = Q \quad (6)$$

$$\text{From (5) and (6) } R = \frac{P}{2} = Q$$

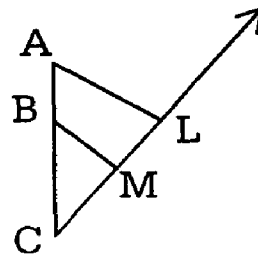
Example: 5

If ℓ and m are the moments of a given force p about two fixed points A and B respectively. Show that the line of action of divides AB in the ratio $\ell : m$ deduce that a force is completely known when the moments of the force about any three points not lying in the same straight line, are known.

Solution



(a)



(b)

Let ℓ and m be the moments of a given force P about A and B and let them be in

- i) Opposite directions, as shown in fig (a)
- ii) The same direction, as shown in fig (b)

Draw AL and BM perpendicular to the line of action of p .

Let the line of action of P meet AB or AB produced at C

$$\ell = P \cdot AL \text{ and } m = P \cdot BM$$

$$\frac{\ell}{m} = \frac{AL}{BM}$$

$$= \frac{AC}{CB} \text{ [}\because \Delta \text{ s } ACL \text{ and } BCM \text{ are similar]}$$

Hence the line of action of P divides AB in the ratio $\ell : m$ internally in fig (a) and externally in fig (b).

Let ℓ , m , n be the moments of a force P about three given non-collinear points A , B , C respectively. The line of action of P divides AB internally or externally in the ratio $\ell : m$. Similarly it divides BC internally or externally in the ratio $m : n$. Hence we can fix the points D and E in which the line of action of P intersects AB and BC respectively.

The line of action of P is therefore the straight line DE .

Let p be the perpendicular distance of A from DE .

$$\text{Then } P \cdot p = \ell.$$

$$\therefore P = \frac{\ell}{p}$$

As ℓ and p are known, we can calculate the magnitude of the force p .

Thus the force is known completely.

Example: 6

Find the locus of all points in a plane such that two forces through a point given in magnitude and position on the plane shall have equal moments about them in the same sense.

Solution:

Let \vec{OA} , \vec{OB} be two forces of given magnitude and position. Join AB and let ℓ be a line through O , parallel to AB .

Let P be any point on ℓ .

Then the moments about P of \vec{OA} and \vec{OB} are of the same sense and their magnitudes are equal to $2 \Delta POA$ and $2 \Delta POB$.

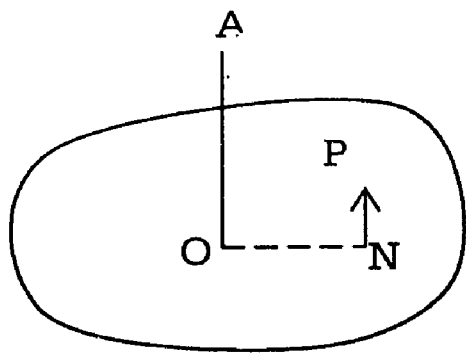
But $\Delta POA = \Delta POB$, as the triangles stand on the same base OP between the same parallels.

Hence the moments about p are equal in magnitude and sign.

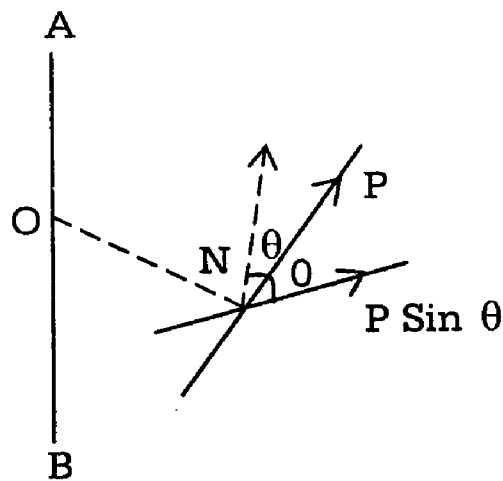
The locus of such points is hence the line ℓ .

4.2.2 Moment of a force about an axis:

We have considered only coplanar forces and their moments about a point in their plane. Let us now consider a rigid body which is capable of turning about some axis fixed in the body for instance a door capable of turning about the line of hinges. Now, any force whose line of action is not parallel to or does not pass through this axis, will tend to turn the body about it. To measure the tendency of rotation in such cases we introduce the idea of the moment of a force about an axis.



(a)



(b)

In fig. (a) a force P acts on a body in a direction perpendicular to a line AB in the body, but not intersecting it.

(ie) P acts in a plane \perp^r to AB . The moment of the force P about the line AB is defined to be $P \cdot ON$. Where ON is \perp^r distance between the line of action of P , and the line AB .

In fig. (b) the force P acts in any direction (not necessarily) \perp^r to AB .

Let ON is the shortest distance between AB and the line of action of P .

The force P can be considered to act at ON along its line of action. It can be resolved into two components:

- i) $P \cos \theta$ parallel to AB and
- ii) $P \sin \theta$ Perpendicular to it.

The component $P \cos \theta$ being \parallel^o AB has zero moment about AB .

The moment of $P \sin \theta$ about AB is $P \sin \theta \cdot ON$ and this is the moment of P about AB .

Note: 1

The moment of P about AB is zero if either (i) P is \parallel^o AB or else (ii) the line of action of P intersects AB .

Note: 2

As in the case of Varignon's theorem in two dimensions, we can show that, if a system of forces acting on a body have a resultant, the algebraic sum of their moments about any line in the body is equal to the moment of their resultant.

Note: 3

If a system of forces acting on a body keeps it in equilibrium, the algebraic sum of their moments about any line in the body is zero.

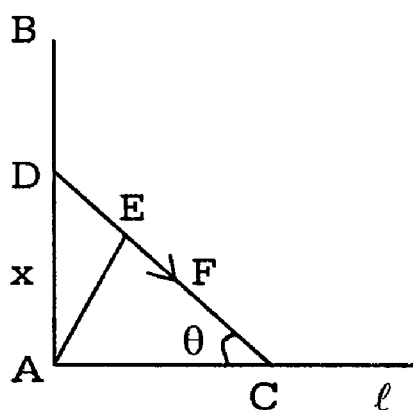
Example: 1

At what height from the base of a pillar must the end of a rope of a given length be fixed so that a man standing on the ground and pulling at its other end with a given force may have the greatest tendency to make the pillar overturn?

Solution:

Let AB be the pillar. A being the base and C the man of the ground.

Let the rope be fixed at D and $AD = x$. $CD = \ell =$ length of the rope.



Let F be the given force exerted by the man along DC . When the pillar overturns. The contact at A is broken. The man will have the best chance of overturning the pillar, if the moment of F about A is greatest.

Draw $AE \perp$ to DC and

Let $\angle DCA = \theta$

Moment of F about A

$$= F \cdot AE$$

$$= F \cdot AE$$

$$= F \cdot AC \sin \theta$$

$$= F \cdot DC \cos \theta \sin \theta \quad (\Delta ADC)$$

$$= F \ell \cos \theta \sin \theta$$

$$= \frac{F \ell}{2} 2 \cos \theta \sin \theta = \frac{F \ell}{2} \sin 2\theta$$

F and ℓ being given the above moment is greatest when $\sin 2\theta$ is greatest.

(ie) When $2\theta = 90^\circ$

$$\text{Or } \theta = 45^\circ$$

Then $X = AD = DC \sin \theta$

$$= \ell \sin 45^\circ = \frac{\ell}{\sqrt{2}}$$

Thus the rope is to be fixed at a height of $\frac{\ell}{\sqrt{2}}$ from the base.

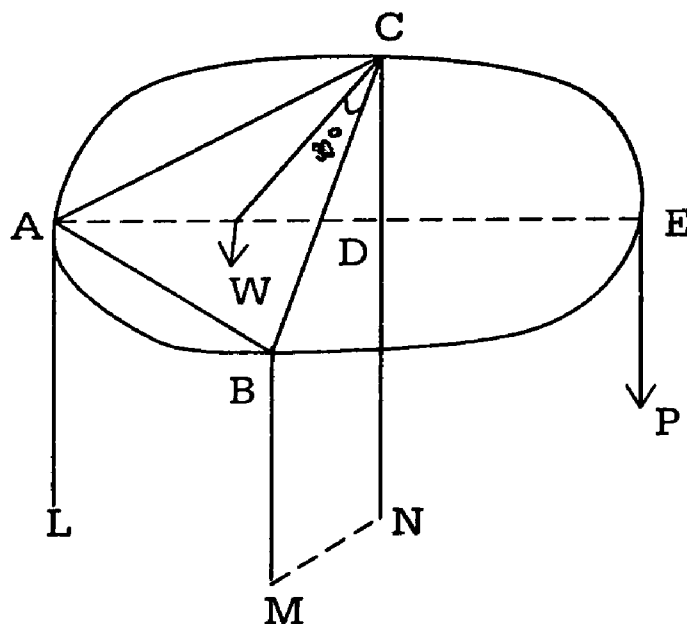
Example: 2

A round table of weight w stands on three legs of which the upper ends are attached to its rim so as to form an equilateral triangle show that a body whose weight does not exceed w may be placed anywhere on the table without the risk of toppling it over.

Solution:

Let ABC be the round table in which A, B, C are the upper ends of the legs attached to the rim such that ABC is an equilateral triangle and let L, M, N be the points of contact of legs with the floor.

Let G be the centre of the table through which its weight w acts. There is a possibility of overturning if any weight p is placed on the portion of the table outside the triangle ABC , say, in the portion, BEC .



If there is overturning the table will turn about the line MN and the leg AL will rest very lightly on the floor.

In that case, the end L just loses contact with the floor and the weights p and W will have equal moments about MN.

(ie) about BC. The weight will have the greatest turning effect when placed farthest away from BC.

(ie) When placed at E, the midpoint of the arc BEC.

Let D be the midpoint of BC

As ΔABC is equilateral, AGDE is \perp to BC.

Taking moments about BC.

We have

$$W.GD = P.DE \quad (1)$$

From right angled ΔGDC

$$\sin 30^\circ = \frac{GD}{GC}$$

$$\therefore GD = GC \sin 30^\circ$$

$$\frac{GC}{2} = \frac{GE}{2} \quad (G \text{ being the circumcentre of } \Delta ABC)$$

\therefore D is the midpoint of GE and $GD = DE$

\therefore From (1) $P = W$.

Hence w is the greatest weight that can be placed anywhere on the table without toppling it over. It can be noted that if any weight is placed within the ΔABC , its moment about BC or CA or AB will be in the same direction as that of w and hence there is no change of the table being overturned whatever the weight may be.

Example: 3

A uniform circular plate is supported horizontally at three points A, B, C of its circumference. Show that the pressures on the supports are in the ratio $\sin 2A : \sin 2B : \sin 2C$.

Solution:

Let $BC = a$, $CA = b$ and $AB = c$ the weight of the plate acts at O.

The centre of the circle and which is also the circumcentre of the triangle.

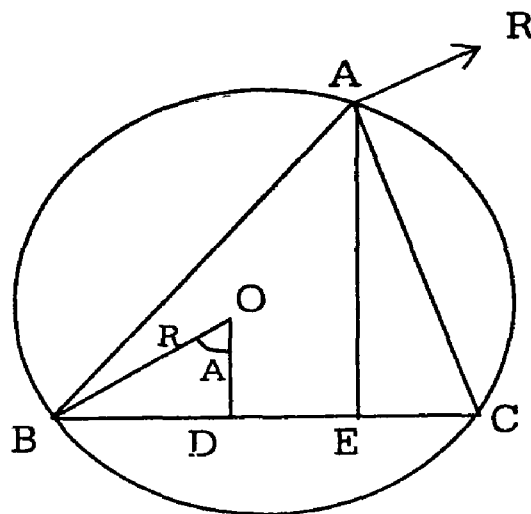
Let OD be perpendicular to BC

We know that $\angle BOD = A$

From rt. $\triangle BOD$

$$OD = OB \cdot \cos \angle BOD = R \cdot \cos A$$

R being the circum - radius of the triangle



Let AE be perpendicular to BC

$$AE = AC \cdot \sin \angle ACE = b \sin C$$

Let R_1 be the reaction at A.

Taking moments about BC, (to avoid the reactions at B and C)

We have at R_1 . $AE = W \cdot OD$

$$\therefore R_1 = \frac{W \cdot R \cos A}{b \sin C}$$

$$= \frac{WR \cos A}{2R \sin B \sin C} \quad (\because b = 2R \sin B)$$

$$\frac{W \cos A}{2 \sin B \sin C} = \frac{2W \sin A \cos A}{4 \sin A \sin B \sin C}$$

$$= \frac{W \sin 2A}{4 \sin A \sin B \sin C}$$

Similarly the reactions, R_2 and R_3 at the other two supports are

$$\frac{W \sin 2B}{4 \sin A \sin B \sin C} \text{ and } = \frac{W \sin 2C}{4 \sin A \sin B \sin C}$$

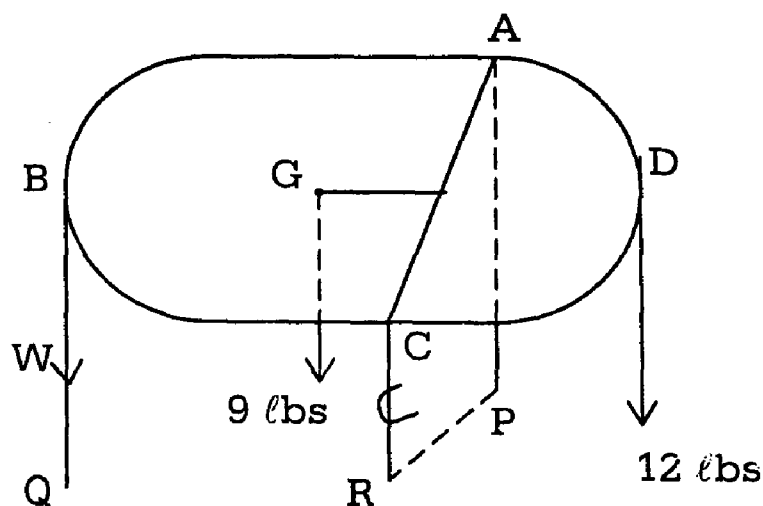
$$\therefore R_1 : R_2 : R_3 = \sin 2A : \sin 2B : \sin 2C.$$

Example: 4

A table consists of a uniform circular board supported by three vertical legs fixed at equal distances around the circumference. The weight of the board is 9 lbs. A weight of 12 lbs. Placed on the edge of the table, midway between two legs is just sufficient to cause the table to over turn. Find the weight of each leg.

Solution:-

The weight of the board is 9 lbs, and it acts through the centre G of the board vertically downwards.



The weight 12 lbs is placed at D which is mid-way between the legs at A and C and this causes the table to overturn about the line PR joining the feet of the legs at A and C.

Let w be the weight of each leg.

Taking moments about the line PR (or AC)

We get $9 GE + W \cdot EB = 12 ED$.

If $GD = r$

The radius of circular board

$$EB = \frac{3}{2} r,$$

$$GE = ED = \frac{1}{2} r,$$

$$\text{Hence } 9 \cdot \frac{1}{2} r + w \cdot \frac{3}{2} r = 12 \cdot \frac{1}{2} r$$

$$W = 1 \text{ lb.}$$

4.3 THREE FORCE ACTING ON A RIGID BODY

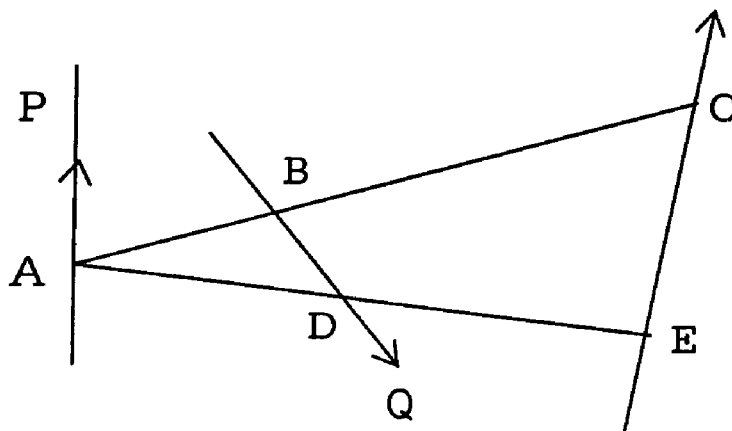
Rigid body subjected to any three forces:

There is a large class of problems in which a body is in equilibrium under the action of three forces. We shall first prove that if three forces acting on a rigid body are in equilibrium, they must be coplanar.

Let P, Q, R be three forces in equilibrium. Take any point A on the line of action of P and any point B on the line of action of Q such that AB is not parallel to R . Then the three forces being in equilibrium, the sum of their moments about the line AB is zero. But P and Q intersect AB and therefore their moments about AB are each zero.

$\therefore R$ is either parallel to AB or R intersects AB .

But we have chosen the points A and B such that R is not parallel to AB .



∴ R must intersect AB at a point, say C.

Similarly, if D is some other point on Q such that AD is not \parallel to R.

We can prove that R must intersect AD also at a point, say E.

Since the lines BC and DE intersect at A, BD and CE must lie in one plane and A is on this plane.

(ie) A is a point on the plane formed by Q and R.

But A is any point on the line of action of P.

∴ Every point on P is a point on the plane formed by Q and R.

(i.e) P, Q, R are in one plane.

4.3.1 Three coplanar Forces:

Theorem

If three coplanar forces acting on a rigid body keep it in equilibrium, they must either be concurrent or be all parallel.

Let P, Q, R be three coplanar forces acting on a body and keep it in equilibrium.

Then R must be equal and opposite to the resultant of P and Q.

Now, P and Q being coplanar must either be parallel or intersect.

Case: 1

If P and Q are parallel (like or unlike), their resultant is also a parallel force. As R balances the above resultant, it must act in the same line but in opposite direction. So R also is in the same direction as that of P and Q.

(i.e) P, Q, R are all parallel to one another.

Case 2:

Let P and Q meet at a point O. Then, by parallelogram law, their resultant is a force through O. As this is balanced by the third force R, the line of action of R must also pass through O.

(i.e) the three forces are concurrent.

Note:

In the above discussion P and Q can never form a couple. Since we know that a couple and a force can never be in equilibrium.

4.3.2 Condition of Equilibrium:

When the number of forces acting on rigid body in equilibrium is three and when the forces are not parallel, we can use the methods which apply to forces acting on a particle. Thus we can use Lami's theorem, or the triangle of forces or we can resolve the forces in two directions at right angles to each other.

When the 3 forces in equilibrium are parallel, we use the condition that each is proportional to the distance between the other two.

In all cases, it is important to draw a figure with the three forces clearly shown, either all parallel or meeting in a point.

Procedure to be followed in solving any statical

In solving any statical problem the student should proceed in the following manner:

1. First draw the figure according to the conditions given.
2. Mark all the forces acting on the body or bodies, bearing in mind the following fundamental points:
 - i) The weight of a body acts vertically downwards through its centre of gravity.
 - ii) When a body is leaning against a smooth surface, the reaction of the body is normal to the surface.
 - iii) When a rod is resting on a smooth peg, the reaction on the rod is perpendicular to the rod.
 - iv) The tension in a light string is the same throughout its length and this tension is unaffected by the string passing over smooth pegs or pulleys. If the pulley is rough, the tension is different on the two sides of the pulley.
 - v) The resultant of two equal forces bisects the angle between them.

3. In addition to the above considerations, we can use the fact that when there are only three non-parallel forces, they must meet in a point. Thus if three forces are in equilibrium and two of them meet at a point O, the third also must pass through O. This consideration will enable us to draw an accurate figure showing the position of the body.

4.3.3 Two Trigonometrical Theorems:

The following two important trigonometrical theorems will be found to be highly useful in the solution of many statical problems.

If D is any point on the base BC of triangle ABC such that $\frac{BD}{DC} = \frac{m}{n}$ and $\angle ADC = \theta$

$$\angle BAD = \alpha \text{ and } \angle DAC = \beta.$$

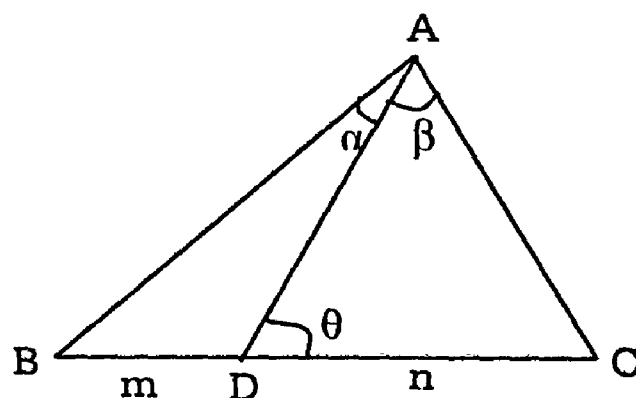
$$\text{Then } (m+n) \cot \theta = m \cot \alpha - \cot \beta \quad (1)$$

$$(m+n) \cot \theta = n \cot B - m \cot C \quad (2)$$

Proof:

$$\begin{aligned} (1) \quad \frac{m}{n} \frac{BD}{DC} &= \frac{BD}{DA} \cdot \frac{DA}{DC} \\ &= \frac{\sin \angle BAD}{\sin \angle ABD} \cdot \frac{\sin \angle ACD}{\sin \angle DAC} \\ &= \frac{\sin \alpha}{\sin (\theta - \alpha)} \times \frac{\sin (\theta + \beta)}{\sin \beta} \end{aligned}$$

$$[\because \angle ACD = 180^\circ - (\theta - \beta)]$$



$$= \frac{\sin \alpha (\sin \theta \cos \beta + \cos \theta \sin \beta)}{\sin \beta (\sin \theta \cos \alpha + \cos \theta \sin \alpha)}$$

$$= \frac{\cot \beta + \cot \theta}{\cot \alpha - \cot \theta}$$

[dividing the numerator and denominator by $\sin \alpha \sin \beta \sin \theta$]

$$\therefore n (\cot \beta + \cot \theta) = m (\cot \alpha - \cot \theta)$$

$$(m+n) \cot \theta = m \cot \alpha - n \cot \beta$$

(2) Again

$$\frac{m}{n} = \frac{\sin \angle BAD}{\sin \angle ABD} \cdot \frac{\sin \angle ACD}{\sin \angle DAC}$$

$$= \frac{\sin(\theta - B)}{\sin B \sin(C + \theta)} \quad [\because \angle DAC = 180^\circ - \theta - C]$$

$$= \frac{\sin C (\sin \theta \cos B - \cos \theta \sin B)}{\sin B (\sin C \cos \theta + \cos C \sin \theta)}$$

$$= \frac{\cot B - \cot \theta}{\cot \theta + \cot C}$$

(Dividing the numerator and the denominator by $\sin B \sin C \sin \theta$)

$$(ie) m (\cot \theta + \cot C) = n (\cot B - \cot \theta)$$

$$(or) (m+n) \cot \theta = n \cot B - m \cot C.$$

Example: 1 (U.Q)

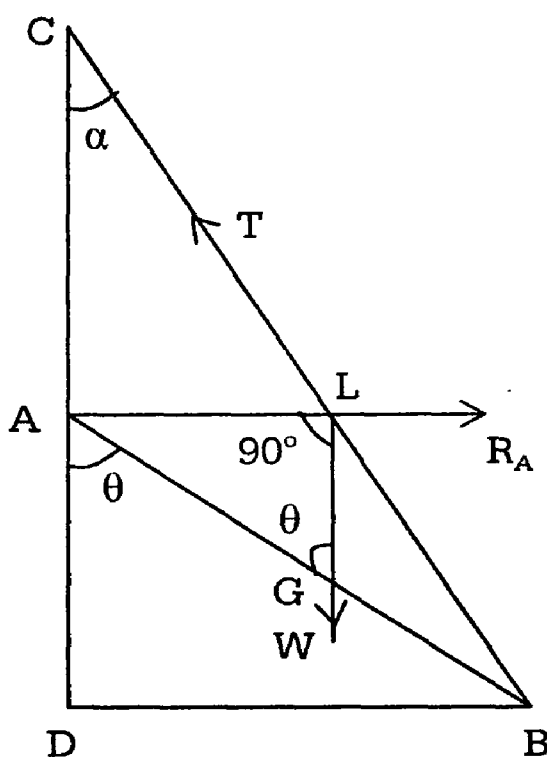
A uniform rod of length a hangs against a smooth vertical wall being supported by means of a string, of length ℓ tied to one end of the rod, the other end of the string attached to a point in the wall. Show that the rod can rest

$$\text{inclined to the wall at an angle } \theta \text{ given by } \cos^2 \theta = \frac{\ell^2 - a^2}{3a^2}$$

What are the limits of the ratio of $a:\ell$ in order that equilibrium may be possible?

Solution:

AB is the rod of length a with G its centre of gravity and BC is the string of length ℓ .



The forces acting on the rod are:

- i) its weight w acting vertically downwards through G.
- ii) the reaction R_A at A which is normal to the wall and therefore horizontal
- iii) the tension T of the string along BC

These three forces in equilibrium not being all parallel, must meet in a point L, as shown in the figure.

Let the string make an angle α with the vertical.

$$\therefore \angle ACB = \alpha = \angle GLB$$

$$\text{Also } \angle LGB = 180^\circ - \theta \text{ and } \angle ALG = 90^\circ$$

Using the first trigonometrical theorem

$$\Delta ALB \text{ and noting that } AG:GB = 1:1$$

We have

$$\begin{aligned} (1+1) \cot (180^\circ - \theta) &= 1 \cdot \cot 90^\circ - 1 \cdot \cot \alpha \\ -2 \cot \theta &= -\cot \alpha \\ \text{or } 2 \cot \theta &= \cot \alpha \end{aligned}$$

(1)

Draw $BD \perp$ to CA

From rt. $\triangle CDB$, $BD = BC \sin \alpha = \ell \sin \alpha$ and

From rt. $\triangle ABD$, $BD = AB \sin \theta = a \sin \theta$.

$$\therefore \ell \sin \alpha = a \sin \theta \quad (2)$$

Eliminate α between (1) and (2)

We know that $\operatorname{cosec}^2 \alpha = 1 + \operatorname{Cot}^2 \alpha$ (3)

From (2)

$$\sin \alpha = \frac{a \sin \theta}{\ell}$$

$$\therefore \operatorname{cosec} \alpha = \frac{\ell}{a \sin \theta} \quad (4)$$

Substituting (4) and (1) in (3)

We have

$$\frac{\ell^2}{a^2 \sin^2 \theta} = 1 + 4 \operatorname{Cot}^2 \theta = \frac{\ell^2}{a^2 \sin^2 \theta} = 1 + 4 \frac{\operatorname{Cos}^2 \theta}{\sin^2 \theta}$$

$$(ie) \frac{\ell^2}{a^2} = \sin^2 \theta + 4 \operatorname{Cos}^2 \theta$$

$$\frac{\ell^2}{a^2} = \sin^2 \theta + \operatorname{Cos}^2 \theta + 3 \operatorname{Cos}^2 \theta$$

$$\therefore 3 \operatorname{Cos}^2 \theta = \frac{\ell^2}{a^2} - 1$$

$$= \frac{\ell^2 - a^2}{a^2}$$

$$\therefore \cos^2 \theta = \frac{\ell^2 - a^2}{3a^2} \quad (5)$$

For the above equilibrium position to be possible, $\cos^2 \theta$ must be positive and less than 1.

$$\therefore \ell^2 - a^2 > 0.$$

$$(ie) \ell^2 > a^2 \text{ or } a^2 > \ell^2$$

$$\text{Also } \frac{\ell^2 - a^2}{3a^2} < 1$$

$$(ie) \ell^2 - a^2 < 3a^2 \text{ or } \ell^2 < 4a^2$$

$$(ie) a^2 > \frac{\ell^2}{4}$$

$$\therefore a^2 \text{ lies between } \frac{\ell^2}{4} \text{ and } \ell^2$$

$$(ie) \frac{a^2}{\ell^2} \text{ lies between } \frac{1}{4} \text{ and } 1$$

$$(or) \frac{a}{\ell} \text{ must lie between } \frac{1}{2} \text{ and } 1.$$

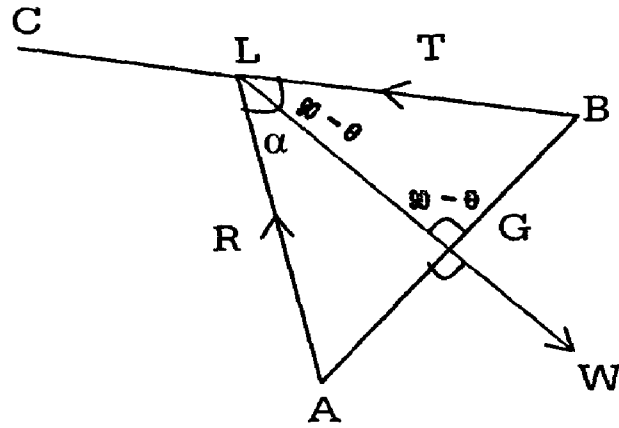
Example: 2

A beam of weight w hinged at one end is supported at the other end by a string so that the beam and the string are in a vertical plane and make the same angle θ with the horizon. Show that reaction at the hinge is

$$\frac{w}{4} \sqrt{8 + \operatorname{cosec}^2 \theta}$$

Solution:

Let AB be the beam of weight w and G its centre of gravity. BC is the string.



The forces acting on the beam are:

- i) its weight w acting vertically downwards at G .
- ii) the tension T along BC and.
- iii) The reaction R at the hinge at A .

Let the forces (i) and (ii) meet at L .

For equilibrium, the third force R also must pass through L .

(ie) the reaction at the hinge is a force along AL .

BC and AB make the same angle θ with the horizon.

\therefore They make the same angle $90^\circ - \theta$ with the vertical LG .

$$(ie) \angle BLG = 90^\circ - \theta = \angle LGB$$

$$\text{Let } \angle ALG = \alpha$$

Using the first trigonometrical theorem $\triangle ALB$

We have (as $AG : GB = 1:1$)

$$(1+1) \cot (90 - \theta) = 1 \cdot \cot \alpha - 1 \cdot \cot (90^\circ - \theta)$$

$$(ie) 2 \tan \theta = \cot \alpha - \tan \theta$$

$$(or) 3 \tan \theta = \cot \alpha$$

(1)

Applying Lami's theorem for the three forces at L ,

We have $\frac{R}{\sin(90^\circ - \theta)} = \frac{W}{\sin(90^\circ - \theta + \alpha)}$

$$(ie) \frac{R}{\cos \theta} = \frac{W}{\sin(90^\circ - \theta - \alpha)} = \frac{W}{\cos(\theta - \alpha)}$$

$$\therefore R = \frac{W \cos \theta}{\cos(\theta - \alpha)} = \frac{W \cos \theta}{\cos \theta \cos \alpha \sin \theta \sin \alpha}$$

$$= \frac{W \cos \theta}{\sin \alpha (\cos \theta \cot \alpha + \sin \theta)}$$

$$= \frac{W \cos \theta}{\sin \alpha (\cos \theta \cdot 3 \tan \theta + \sin \theta)} \text{ Using (1)}$$

$$= \frac{W \cos \theta \operatorname{cosec} \alpha}{3 \sin \theta + \sin \theta}$$

$$= \frac{W}{4} \cot \theta \operatorname{cosec} \alpha$$

$$= \frac{W}{4} \cot \theta \sqrt{1 + \cot^2 \alpha}$$

$$= \frac{W \cot \theta}{4} \sqrt{1 + 9 \tan^2 \alpha}$$

$$= \frac{W}{4} \sqrt{\cot^2 \theta + 9} = \frac{W}{4} \sqrt{\cot^2 \theta + 1 + 8}$$

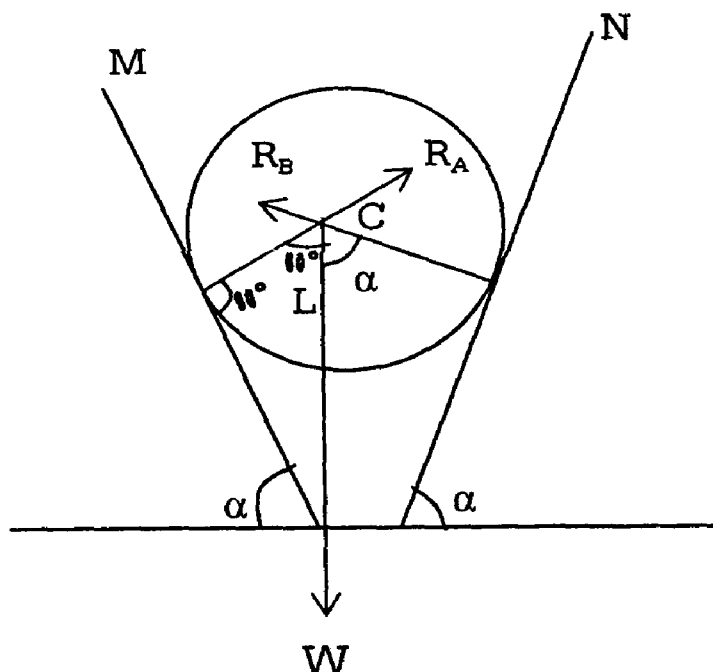
$$\therefore R = \frac{W}{4} \sqrt{\operatorname{cosec}^2 \theta + 8}$$

Example: 3

A heavy uniform sphere rests touching two smooth inclined planes one of which is inclined at 60° to the horizontal. If the pressure on this plane is one-half of the weight of the sphere, prove that the inclination of the other plane to the horizontal is 30°

Solution:

Let the sphere centre C rest on the inclined planes AM and BN. MA makes 60° with the horizontal and let NB make an angle α with the horizon.



The forces acting on the sphere are

- i) Reaction R_A at A \perp to the inclined plane AM and to the sphere and hence passing through C.
- ii) Reaction R_B at B which is normal to the inclined plane BN and to the sphere and hence passing through C.
- iii) W , the weight of the sphere acting vertically downwards at C along CL.

Clearly the above three forces meet at C.

$$\text{Also } \angle ACL = 60^\circ$$

$$\text{And } \angle BCL = \alpha$$

Applying Lami's theorem,

$$\frac{R_A}{\sin \alpha} = \frac{W}{\sin(60^\circ + \alpha)}$$

$$R_A = \frac{W \sin \alpha}{\sin(60^\circ + \alpha)}$$

$$\text{But } R_A = \frac{W}{2} \tag{2}$$

From (1) and (2)

We have

$$\frac{W \sin \alpha}{\sin(60^\circ + \alpha)} = \frac{W}{2}$$

$$(ie) \quad 2 \sin \alpha = \sin(60^\circ + \alpha)$$

$$= \sin 60^\circ \cos \alpha + \cos 60^\circ \sin \alpha$$

$$(ie) \quad 2 \sin \alpha = \frac{\sqrt{3}}{2} \cos \alpha + \frac{1}{2} \sin \alpha$$

$$(ie) \quad 4 \sin \alpha = \sqrt{3} \cos \alpha + \sin \alpha$$

$$(ie) \quad 3 \sin \alpha = \sqrt{3} \cos \alpha$$

$$(ie) \quad \frac{\sin \alpha}{\cos \alpha} = \frac{\sqrt{3}}{3}$$

$$(ie) \quad \tan \alpha = \frac{1}{\sqrt{3}}$$

$$(or) \quad \alpha = 30^\circ$$

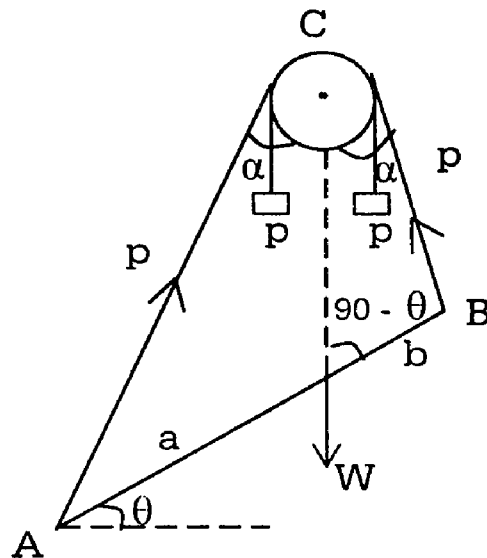
Example: 4

Equal weights P and P are attached to two strings ACP and BCP passing over a smooth peg C . AB is a heavy beam of weight W , whose centre of gravity is 'a' ft from A and 'b' ft from B . Show that AB is inclined to the horizon at an angle.

$$\tan^{-1} \left[\frac{a-b}{a+b} - \tan \left(\sin^{-1} \frac{W}{2p} \right) \right]$$

Solution:

Since the string ACP and BCP support the equal weights P at their ends the tensions along AC and BC must be each equal to P .



The forces acting on the rod are

- i) tension P along AC
- ii) Tension P along BC and
- iii) Its weight W acting vertically downwards through G , its centre of gravity.

For equilibrium, the above three forces must meet at a point C as in the figure.

Now the resultant of the two equal tensions P along AC and CB must balance the weight W .

Hence CG must bisect the angle ACB .

Let $\angle ACG = \alpha = \angle BCG$.

$$\text{Then } W = 2P \cos \alpha \tag{1}$$

Let AB make an angle θ with the horizontal.

Then $\angle CGB = 90^\circ - \theta$.

Using the first trigonometrical theorem.

To ΔACB .

We have $(a+b) \cot (90^\circ - \theta) = a \cot \alpha - b \cot \alpha$

$$(ie) (a+b) \tan \theta = (a-b) \cot \alpha$$

$$\therefore \tan \theta = \frac{a-b}{a+b} \cot \alpha.$$

$$= \frac{a-b}{a+b} \tan \left(\frac{\pi}{2} - \alpha \right)$$

$$= \frac{a-b}{a+b} \tan \left(\frac{\pi}{2} - \frac{\cos^{-1} W}{2p} \right)$$

Using (1)

$$= \frac{a-b}{a+b} \tan \left(\sin^{-1} \frac{W}{2p} \right)$$

$$\left(\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \right)$$

$$\therefore \theta = \tan^{-1} \left[\frac{a-b}{a+b} \tan \left(\sin^{-1} \frac{W}{2p} \right) \right]$$

Example: 5

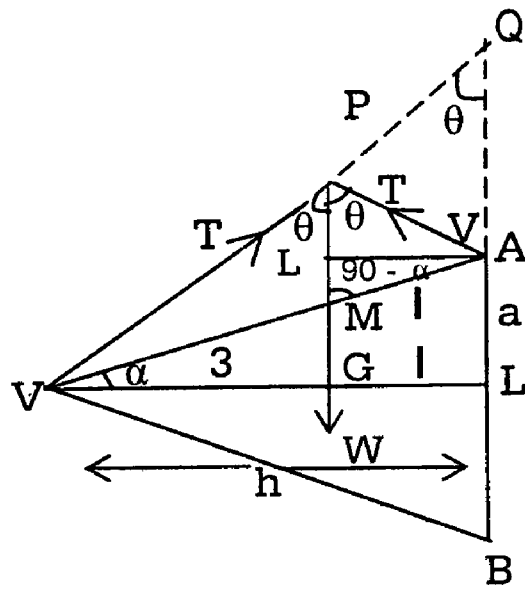
The attitude of a right cone is h and the radius of its base is a . A string is fastened to the vertex and to a point on the circumference of the circular base. The string is then put over a smooth peg. The cone rests with its axis horizontal. Show that the length of the string is $\sqrt{h^2 + 4a^2}$.

Solution:

VAB is the cone resting with the axis VL horizontal.

G is its centre of gravity.

We know that $\frac{VG}{GL} = \frac{1}{3}$.



Since the string passes over a smooth peg P, the tensions along VP and PA must be equal. Let them be T each.

Since the two tensions pass through P, for equilibrium, the third force, (ie) W, the weight of the cone must pass through P. Hence PG is vertical.

Since W balances the resultant of the equal tensions T, the line PG must bisect $\angle VPA$.

$$\text{Let } \angle VPG = \theta = \angle APG$$

Let α be the semi-vertical angle of the cone and let PG meet VA at M.

$$\angle PMA = 90^\circ - \alpha.$$

$$\frac{VM}{MA} = \frac{VG}{GL} = \frac{3}{1}$$

Using the first trigonometrical theorem.

ΔVPA

We have

$$(3+1) \cot(90^\circ - \alpha) = 3 \cot \theta + 1 \cot \theta$$

$$\text{(ie) } 4 \tan \alpha = 2 \cot \theta$$

$$\text{(or) } 2 \tan \alpha = \cot \theta$$

(1)

From ΔPVG ,

$$\sin \theta = \frac{VG}{VP}$$

$$\therefore VP = \frac{VG}{\sin \theta} \quad (2)$$

From $\triangle PNA$

$$\sin \theta = \frac{AN}{PA}$$

$$\therefore PA = \frac{AN}{\sin \theta} = \frac{GL}{\sin \theta} \quad (3)$$

Adding (2) & (3)

$$\begin{aligned} VP+PA &= \frac{VG}{\sin \theta} + \frac{GL}{\sin \theta} \\ &= \frac{VG+GL}{\sin \theta} = \frac{h}{\sin \theta} \end{aligned}$$

$$\therefore \text{Length of the string} = \frac{h}{\sin \theta}$$

$$= h \operatorname{cosec} \theta$$

$$= h\sqrt{1+\operatorname{Cot}^2\theta}$$

$$\left[\because \operatorname{Cosec}^2 \theta = 1 + \operatorname{Cot}^2 \theta \right]$$

$$= h\sqrt{1+4\tan^2 \alpha} \quad \text{using (1)}$$

$$= h\sqrt{1+\frac{4a^2}{h^2}}$$

$$\left[\because \tan \alpha = \frac{AL}{VL} = \frac{a}{h} \right]$$

$$= \sqrt{h^2 + 4a^2}$$

Aliter:

Produce VP to meet BA produced at O. $\angle PAQ = \theta$ and $\angle PQA = \theta$ ($\because PG \parallel AB$).

$$\therefore \angle PAQ = \angle PQA$$

and $PQ = PA$

Hence length of the string

$$= VP + PA = VP + PQ = VQ.$$

From (1) $\cot \theta = 2 \tan \alpha$.

$$\begin{aligned} &= \frac{2a}{h} \\ \therefore \tan \theta &= \frac{h}{2a} \end{aligned} \quad (4)$$

From rt. $\triangle VQL$

$$\tan \theta = \frac{VL}{QL} = \frac{h}{QL} \quad (5)$$

From (4) and (5) $QL = 2a$

Now, $VQ^2 = VL^2 + QL^2$

$$VQ^2 = h^2 + 4a^2$$

$$VQ = \sqrt{h^2 + 4a^2}$$

Example: 6

A string of length $2l$ has one end attached to the extremity of a smooth heavy uniform rod of length $2a$ and the other carries a weightless ring which slides on the rod. The rod is suspended by means of the string from a smooth peg, prove that, if θ be the angle which the rod makes with the vertical, then

$$l \cos \theta = a \sin^3 \theta.$$

Solution:

Let AB be the rod. G its centre of gravity, C the peg and D the ring.

The forces acting on the ring are

- i) tension T along DC and
- ii) the normal reaction R at D due to contact with rod, which is \perp to AB downwards.

Hence for equilibrium of the ring, $T = R$.

Also, DC is \perp to AB .

Since the string passes over a smooth peg. The tensions also DC and AC must be equal. Let them be T each.

Consider the equilibrium of the rod and the ring together.

The forces acting on the are:-

- i) tension T along AC
- ii) tension T along DC and
- iii) Weight W of the rod acting vertically downwards through G .

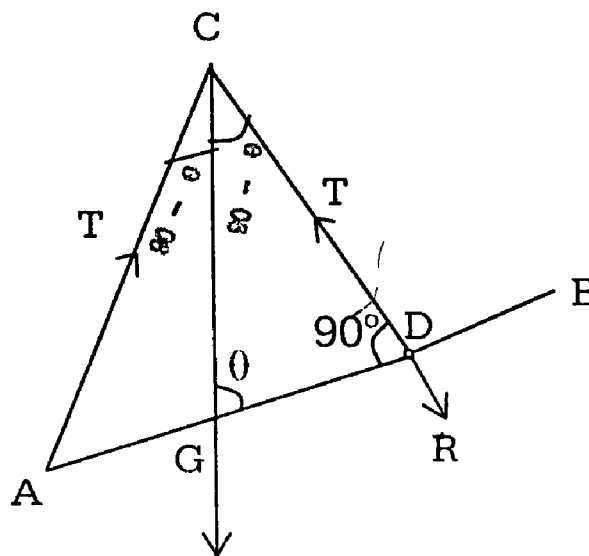
The mutual normal reactions cancel each other. As the forces (i) and (ii) pass through C for equilibrium, we also must pass through C .

\therefore CG is Vertical

Since W balances the resultant of the equal tensions T , the line CG must bisect $\angle ACD$.

$$\therefore \angle ACG = \angle GCD = 90^\circ - \theta$$

Using the first trigonometrical equation.



$\triangle ACD$

We have

$$(AG + GD) \cot \theta = AG \cdot \cot (90^\circ - \theta) - GD \cdot \cot (90^\circ - \theta)$$

$$(ie) (a + GD) \cot \theta = \tan \theta - GD \tan \theta$$

$$GD (\cot \theta + \tan \theta) = a (\tan \theta - \cot \theta)$$

$$\begin{aligned} \therefore GD &= \frac{a(\tan \theta - \cot \theta)}{(\cot \theta + \tan \theta)} \\ &= a \frac{\left(\frac{\sin \theta}{\cos \theta} - \frac{\cos \theta}{\sin \theta} \right)}{\left(\frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} \right)} \\ &= \frac{a(\sin^2 \theta - \cos^2 \theta)}{\cos \theta \sin \theta} \frac{\sin \theta \cos \theta}{(\cos^2 \theta + \sin^2 \theta)} \\ &= a(\sin^2 \theta - \cos^2 \theta) \end{aligned} \quad (1)$$

From $\triangle ACG$,

$$\frac{AC}{\sin(180^\circ - \theta)} = \frac{AG}{\sin(90^\circ - \theta)}$$

$$(ie) \frac{AC}{\sin \theta} = \frac{a}{\cos \theta} \text{ or } AC = \frac{a \sin \theta}{\cos \theta} \quad (2)$$

From $\triangle GCD$

$$\frac{CD}{\sin \theta} = \frac{GD}{\sin(90^\circ - \theta)}$$

$$CD = GD \frac{\sin \theta}{\cos \theta} = a(\sin^2 \theta - \cos^2 \theta) \frac{\sin \theta}{\cos \theta} \quad (3)$$

[Using (1)]

Adding (2) and (3)

$$AC + CD = \frac{a \sin \theta}{\cos \theta} + \frac{a(\sin^2 \theta - \cos^2 \theta)}{\cos \theta} \sin \theta$$

$$(ie) \quad 2l = \frac{a \sin \theta}{\cos \theta} [1 + \sin^2 \theta - \cos^2 \theta]$$

$$2l = \frac{2a \sin^3 \theta}{\cos \theta}$$

$$(or) \quad l \cos \theta = a \sin^3 \theta$$

Problems on parallel forces:

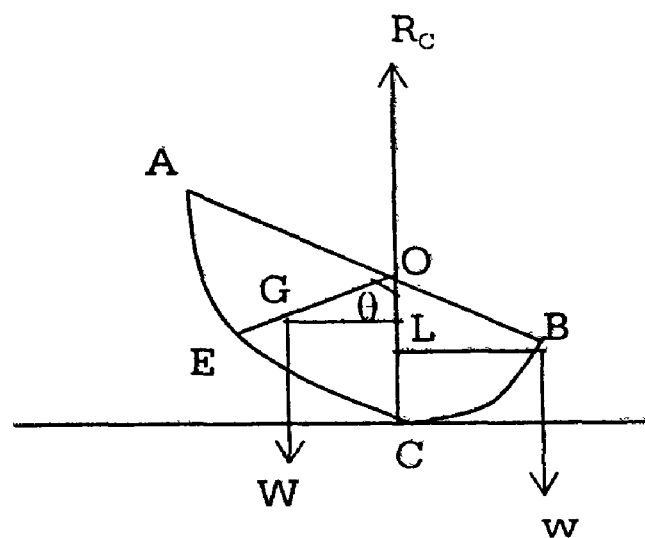
Example: 1

A uniform solid hemisphere of weight w rests with its curved surface on a smooth horizontal plane. A weight w is suspended from point on the rim of the hemisphere. If the plane base of the rim is inclined to the horizontal at an angle

$$\tan \theta = \frac{8w}{3w}$$

Solution:

The forces acting on the hemisphere are



- i) the reaction R_c at C \perp^r to the hemisphere. Hence it is along CO, O being the geometric centre of the hemisphere.

- ii) The weight w of the hemisphere acting vertically downwards at G , the centre of gravity of the body, which is on the radius $OE \perp^I$ to AB such that

$$OG = \frac{3r}{8}, \text{ } r \text{ being the radius.}$$

- iii) the weight w acting vertically downwards at 3 . These three forces are clearly parallel.

For equilibrium, we know that each of the three forces is proportional to the distance between the other two.

$$\therefore \frac{W}{BD} = \frac{\omega}{GL} \quad (1)$$

From the figure, $BD = OB \cdot \cos \theta = r \cos \theta$ and

$$\begin{aligned} GL &= OG \cdot \sin \theta \\ &= \frac{3r}{8} \sin \theta \end{aligned}$$

\therefore (1) becomes

$$\frac{w}{r \cos \theta} = \frac{\omega}{\left(\frac{3r}{8}\right) \sin \theta}$$

$$\text{(or)} \quad \frac{\sin \theta}{\cos \theta} = \frac{8\omega}{3w}$$

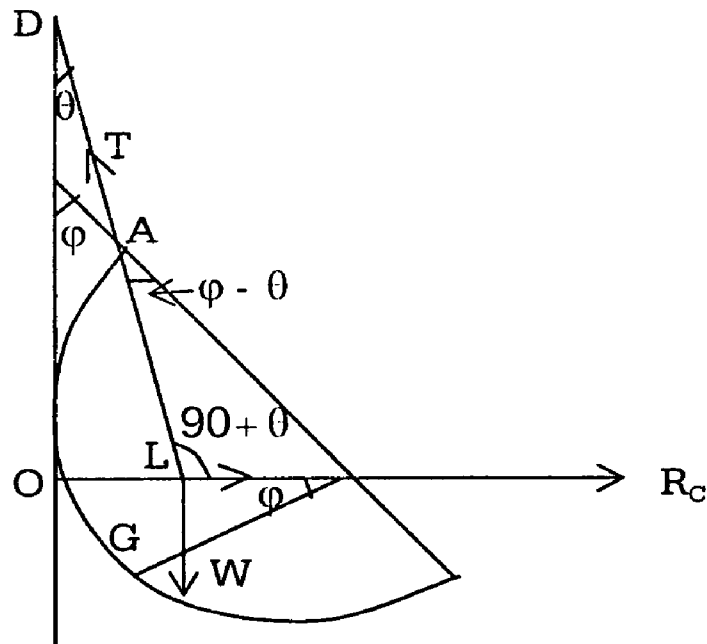
$$\therefore \tan \theta = \frac{8\omega}{3w}$$

Example: 2 (U.G)

A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ , ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that $\tan \phi = \frac{3}{8} + \tan \theta$.

Solution:

The figure represents a section of the hemisphere and the wall by a plane \perp^r to the wall and the base of the hemisphere. AB is the plane base and O is the centre of the hemisphere C is the point of contact with the wall.



We know that, G the centre of gravity of the hemisphere is on the radius \perp^r to AB such that $OG = \frac{3r}{8}$.

The forces acting on the hemisphere are:

- i) the reaction R_c at C perpendicular to the wall and the hemisphere and hence passing through O.
- ii) the tension T along the string AD and
- iii) the weight w of the hemisphere acting at G vertically downwards.

Let the forces (i) and (ii) meet at L.

For equilibrium, the force (iii) also must pass through L.

\therefore LG is vertical.

$$\angle LAO = \phi - \theta$$

$$\angle ALO = 90^\circ + \theta$$

$$\angle LOG = \phi$$

$$\text{From rt } \angle d, \Delta OLG, OL = OG. \cos \phi = \frac{3r}{8} \cos \phi \quad (1)$$

$$\text{From } \Delta OAL, \frac{OL}{\sin \angle OAL} = \frac{OA}{\sin \angle OLA}$$

$$(ie) \quad \frac{OL}{\sin(\phi - \theta)} = \frac{r}{\sin(90^\circ + \theta)}$$

$$\begin{aligned} \therefore OL &= \frac{r \sin(\phi - \theta)}{\cos \theta} \\ &= \frac{r(\sin \phi \cos \theta - \cos \phi \sin \theta)}{\cos \theta} \\ &= r(\sin \phi - \cos \phi \tan \theta) \quad (2) \end{aligned}$$

Equating (1) and (2)

$$\frac{3r}{8} \cos \phi = r(\sin \phi - \cos \phi \tan \theta)$$

Dividing by (1) $\cos \phi$

We have

$$\frac{3}{8} = \tan \phi - \tan \theta$$

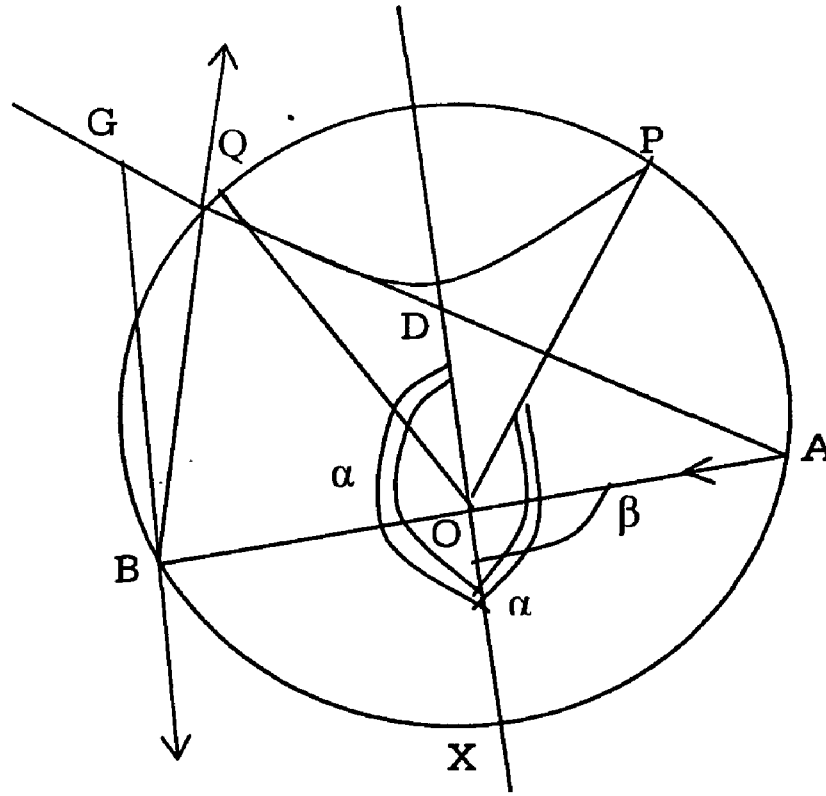
$$(or) \tan \phi = \frac{3}{8} + \tan \theta$$

Example: 3 (U.Q)

A bowl is formed from a hollow sphere, of radius 'a' and is so placed that the radius of the sphere, drawn to each point in the rim makes an angle α with the vertical whilst the radius drawn to appoint A of the bowl makes an angle β with the vertical if a smooth uniform rod remains at rest with one end at and a point of its length in contact with the rim, show that the length of the rod is $4a \sin \beta \cdot \sec \frac{\alpha - \beta}{2}$

Solution:-

PQ is the rim of the bowl and $\angle POX = \alpha = \angle QOX$



$\angle AOX = \beta$, $\angle AOB$ is the diameter.

G is the centre of gravity of the rod.

The forces acting on the rod are:

- i) the reaction at A normal to the spherical surface and which hence passes through O the centre.
- ii) The reaction at Q \perp^r to the rod AG.

Since $\angle AQB = 90^\circ$

(AB being a diameter this reaction passes through B.)

- iii) The weight of the rod acts vertically downwards through G.

The forces (i) and (ii) pass through B.

For equilibrium, the force (iii) also must pass through B and

Hence GB is vertical

$$\angle BOX = 180^\circ - \beta$$

$$\begin{aligned} \therefore \angle BOQ &= \angle QOX - \angle BOX = \alpha - (180^\circ - \beta) \\ &= \alpha + \beta - 180^\circ \end{aligned}$$

$$\therefore \angle BAQ = \frac{1}{2} \angle BOQ = \frac{1}{2} (\alpha + \beta - 180^\circ) = \frac{\alpha + \beta}{2} - 90^\circ$$

From rt. \angle d \triangle BAQ.

$$QA = AB \cdot \cos \angle BAQ.$$

$$= 2a \cos \left(\frac{\alpha + \beta}{2} - 90^\circ \right)$$

$$= 2a \cos \left(90^\circ - \frac{\alpha + \beta}{2} \right) = 2a \sin \frac{\alpha + \beta}{2} \quad (1)$$

$$QB = AB \sin \angle BAQ$$

$$= 2a \sin \left(\frac{\alpha + \beta}{2} - 90^\circ \right)$$

$$= 2a \sin \left[- \left(90^\circ - \frac{\alpha + \beta}{2} \right) \right]$$

$$= -2a \cos \frac{\alpha + \beta}{2} \quad (2)$$

Let the vertical line OX cut the rod AG at D

$$\angle DOA = 180^\circ - \angle AOX = 180^\circ - \beta$$

$$\angle OAD = \angle BAQ = \frac{\alpha + \beta}{2} - 90^\circ$$

$$\begin{aligned}
\therefore \angle ODA &= 180^\circ - [\angle DOA + \angle OAD] \\
&= 180^\circ - (180^\circ - \beta + \frac{\alpha + \beta}{2} - 90^\circ) \\
&= \beta - \frac{\alpha + \beta}{2} + 90^\circ \\
&= 90^\circ + \frac{\beta - \alpha}{2}
\end{aligned}$$

Since GB is \parallel to OX ..

$$\angle BGQ = \angle ODA = 90^\circ + \frac{\beta - \alpha}{2}$$

From rt \triangle BGQ

$$\tan \angle BGQ = \frac{BQ}{GQ}$$

$$(ie) \quad \tan \left(90^\circ + \frac{\beta - \alpha}{2} \right) = \frac{BQ}{GQ}$$

$$(ie) \quad \tan \left(90^\circ - \frac{\beta - \alpha}{2} \right) = \frac{BQ}{GQ}$$

$$(or) \quad \cot \frac{\alpha - \beta}{2} = \frac{BQ}{GQ}$$

$$\therefore GQ = \frac{BQ}{\cot \frac{\alpha - \beta}{2}}$$

$$= \frac{2a \cos \frac{\alpha + \beta}{2}}{\cot \frac{\alpha - \beta}{2}} \quad - (3) \text{ using } (2)$$

$$AG = AQ + QG$$

$$= 2a \sin \frac{\alpha + \beta}{2} - \frac{2a \cos \frac{\alpha + \beta}{2}}{\cot \frac{\alpha - \beta}{2}} \quad [\text{from (1) } \times \text{ (3)}]$$

$$= 2a \sin \frac{\alpha + \beta}{2} - \frac{2a \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}}{\cos \frac{\alpha - \beta}{2}}$$

$$= -\frac{2a}{\cos \frac{\alpha - \beta}{2}} \left[\sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} - \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} \right]$$

$$= -\frac{2a}{\cos \frac{\alpha - \beta}{2}} \sin \left(\frac{\alpha + \beta}{2} - \frac{\alpha - \beta}{2} \right)$$

$$= 2a \sec \frac{\alpha - \beta}{2} \sin \beta.$$

$$\therefore \text{Length of the rod} = 2 AG = 4 a \sec \frac{\alpha - \beta}{2} \sin \beta.$$

UNIT – V

FRICTION

Introduction:

When two bodies are in contact, with their surfaces of contact smooth one body is free to move on the other along the plane of contact this means that the surfaces cannot provide any force preventing motion along the plane of contact. (ie) the reaction at the point of contact is just normal to the plane of contact. However, if the surfaces are rough, a force is called into play along the plane of contact preventing motion along the plane. Such a tangential force is called the frictional force. This force is just the component of the reaction at the point of contact along the plane of contact (the other component, \perp to the plane being called normal reaction).

5.1.1 Definition:-

If two bodies are in contact with one another, the property of the two bodies, by means of which a force is exerted between them at their point of contact to prevent one body from sliding on the other, is called friction the force exerted is called the force of friction.

5.1.2 Limiting Equilibrium:

Frictional force is a self-adjusting force it appears only when the external forces act and vanishes when they cease to act. The frictional force is exactly equal to the force that we exert on the body in pulling it and if we increase the pull, the frictional force also increases.

It must be noted that the frictional force between any two rough surfaces cannot go on increasing to balance the external forces. Ultimately a stage is soon reached when the frictional force available is no longer sufficient to overcome the effect of the external forces and the equilibrium is about to be disturbed. The body is then just on the point of motion.

The body is then said to be in limiting equilibrium.

5.1.3 Limiting Friction:

While the body is in limiting equilibrium the frictional force also reaches its maximum or limiting value that is, it is the frictional force at the time when the body is about to move the frictional force is called limiting friction.

5.2 Coefficient of Friction:

When the body is in limiting equilibrium the magnitude of the limiting friction bears a constant ratio to the normal reaction between the two bodies in contact. This ratio depends only on the materials of which the surface in contact are made and not on their shapes or areas.

This constant ratio is called the coefficient of friction. We denote it by μ .

Thus if \vec{F} denotes the frictional force and \vec{R} the normal reaction then when the equilibrium is limiting

$$\frac{F}{R} = \mu \text{ or } F = \mu R$$

(ie) the limiting friction is μR

5.3 Laws of Friction:

Friction is not a mathematical concept, it is a physical reality. The results of physical observation and experiment are formulated as the laws of Friction.

Law: 1

When two bodies are in contact, the direction of friction on one of them at the point of contact is opposite to the direction in which the point of contact would commence to move.

Law: 2

When there is equilibrium, the magnitude of friction is just sufficient to prevent the body from moving.

Law: 3

The magnitude of the limiting friction always bears a constant ratio to the normal reaction and this ratio depends only on the substances of which the bodies are composed.

Law: 4

The limiting friction is independent of the extent and shape of the surface in contact, so long as the normal reaction is unaltered.

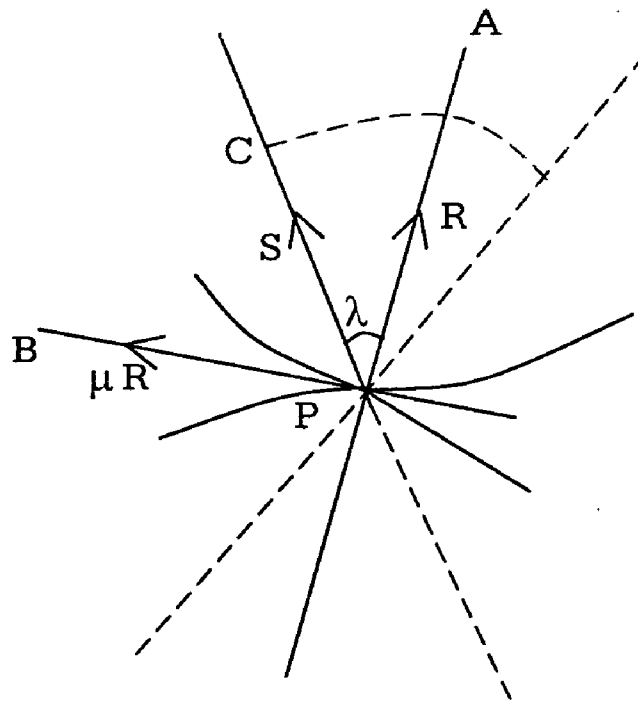
Law: 5 (Law of dynamical Friction)

When the motion ensues by one body sliding over the other the direction of friction is opposite to that of motion; the magnitude of the friction is independent of the velocity of the point of contact but the ratio of the friction to the normal

reaction is slightly less when the body moves than when it is in limiting equilibrium.

5.4 Angle of Friction:-

When two rough bodies in contact are in limiting equilibrium, limiting friction is called into play. The angle which the resultant of this limiting frictional force $\vec{\mu R}$ and the normal reaction \vec{R} makes with the normal to the surface in contact (PA in the diagram) is called 'angle of friction'. It is denoted by λ .



Note that R and μR are the resolved parts of \vec{S} along PA and PB respectively and so $R = S \cos \lambda$ and $\mu R = S \sin \lambda$, giving $\mu = \tan \lambda$.

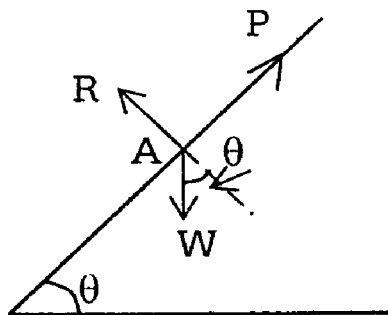
Thus the coefficient of friction is equal to the tangent of the angle of friction.

5.5 Cone of Friction:

For two given bodies, the limiting frictional force being $\vec{\mu R}$, the maximum inclination of the resultant reaction \vec{S} to the common normal at the point of contact of the surfaces is $\lambda = \tan^{-1} \mu$ with the normal PA as axis and the point of contact P as the vertex if we describe a cone with semi-vertical angle λ , the resultant reaction through p will always have a direction lying within the cone and in the limiting case on the surface of it. Such a cone is called the cone of friction.

5.2.4 Equilibrium of a particle on a rough inclined plane:

Let a particle of weight W be placed



at A on a rough inclined plane, whose inclination to the horizontal is θ .

The forces acting on it are:

- i) its weight w acting vertically downwards
- ii) the frictional force F acting along the inclined plane upwards. (If there had been no friction, the body would have a tendency to move downwards. Hence friction will act upwards.
- iii) The normal reaction R , perpendicular to the plane.

Resolving along and \perp^r to the plane,

$$\text{We get } F = W \sin \theta \quad (1)$$

$$\text{And } R = W \cos \theta \quad (2)$$

$$\therefore \frac{F}{R} = \tan \theta \quad (3)$$

We know that $\frac{F}{R}$ is always $< \mu$

Hence for equilibrium $\tan \theta < \mu$.

$$\text{(ie) } \tan \theta < \tan \lambda$$

λ being the angle of friction or $\theta < \mu$

Suppose θ , the inclination of the plane is gradually increased

$$\text{When } \theta = \lambda, \text{ then } \frac{F}{R} = \tan \lambda = \mu.$$

In this case, the equilibrium becomes limiting and the particle is just on the point of sliding down.

Hence we have the following theorem:

If a body be placed on a rough inclined plane and be on the point of sliding down the plane under the action of its weight and the reaction of the plane only, the angle of inclination of the plane to the horizon is equal to the angle of friction.

The inclination ($\lambda = \tan^{-1} \mu$) of the inclined plane when the body just begins to slip is called the angle of repose. Hence the above theorem is stated as:

The angle of repose of a rough inclined plane is equal to the angle of friction.

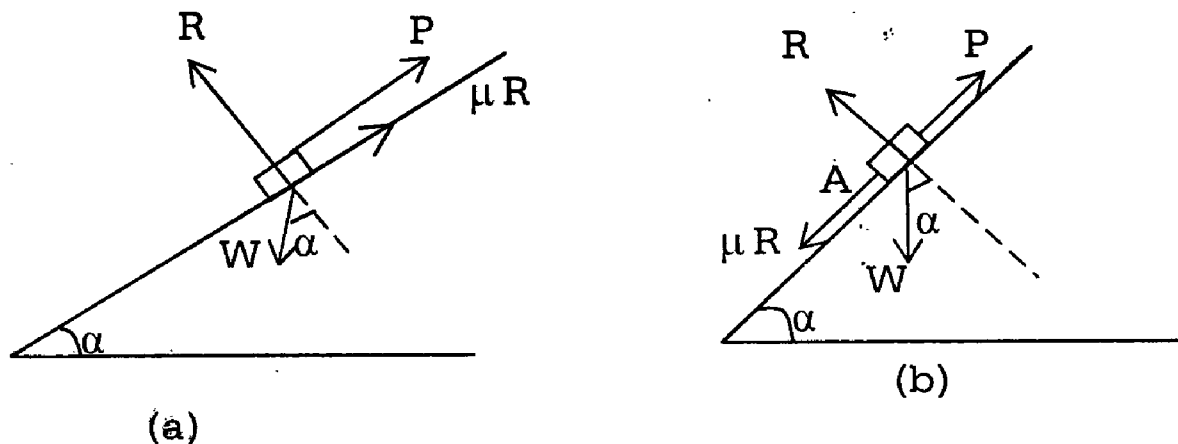
Note:

It should be noted that the angle of repose of a rough inclined plane is equal to the angle of friction, only when there are no external force acting on the body.

5.1.5 Equilibrium of a body on a rough inclined Plane under a force parallel to the plane:

Theorem:

A body is at rest on a rough plane inclined to the horizon at an angle greater than the angle of friction and is acted upon by a force, parallel to the plane and along the line of greatest slope, to find the limits between which the force must lie.



Let α be the inclination of the plane to the horizon, w the weight of the body and R the normal reaction.

Case .1 Refer to fig.(a).

Let the body be on the point of moving down the plan. Then limiting friction acts up the plane and $= \mu R$.

Let P be the force required to keep the body at rest.

Resolving along and perpendicular to the plane,

We have

$$P + \mu R = W \sin \alpha \quad (1)$$

$$\text{And } R = W \cos \alpha \quad (2)$$

Substituting for R from (2) in (1)

$$\text{We get } P = w \sin \alpha - \mu w \cos \alpha$$

If λ is the angle of friction.

We know that $\mu = \tan \lambda$.

$$\therefore P = W (\sin \alpha - \tan \lambda \cos \alpha).$$

$$= W \frac{(\sin \alpha \cos \lambda - \sin \lambda \cos \alpha)}{\cos \lambda}$$

$$= \frac{W \sin (\alpha - \lambda)}{\cos \lambda}$$

Let this value of P be P_1

$$\text{Then } P_1 = \frac{W \sin (\alpha - \lambda)}{\cos \lambda}$$

Since $\alpha > \lambda$

P_1 is positive

Case .2 As in fig (b)

Let the body be on the point of moving up the plane. Then limiting friction μR acts downwards.

Let P be the force required to keep the body at rest.

Resolving as before,

We get

$$P - \mu R = W \sin \alpha \quad (4)$$

$$\text{And } R = W \cos \alpha \quad (5)$$

Hence

$$P = \mu W \cos \alpha + W \sin \alpha$$

$$= W (\tan \lambda \cos \alpha + \sin \alpha)$$

$$= \frac{W(\sin \lambda \cos \alpha + \cos \lambda \sin \alpha)}{\cos \lambda}$$

$$= \frac{W(\sin \alpha + \lambda)}{\cos \lambda} = P_2 \text{ (Say)}$$

$$\therefore P_2 = \frac{W(\sin \alpha + \lambda)}{\cos \lambda} \quad (6)$$

Now if $P > P_2$, the body will move up the plane.

$\therefore P_2$ is the limiting value of P , which is necessary to keep the body in equilibrium, without moving upwards.

If $P < p$, the body will move down the plane.

$\therefore P_1$ is the limiting value of P , which is necessary to keep the body in equilibrium, without moving downwards.

Hence, if p lies between P_1 and P_2 the body will remain in equilibrium and in not is the point of motion is either direction.

Hence, for equilibrium, the force p must lie between the values $\frac{W(\sin \alpha - \lambda)}{\cos \lambda}$ and $\frac{W(\sin \alpha + \lambda)}{\cos \lambda}$.

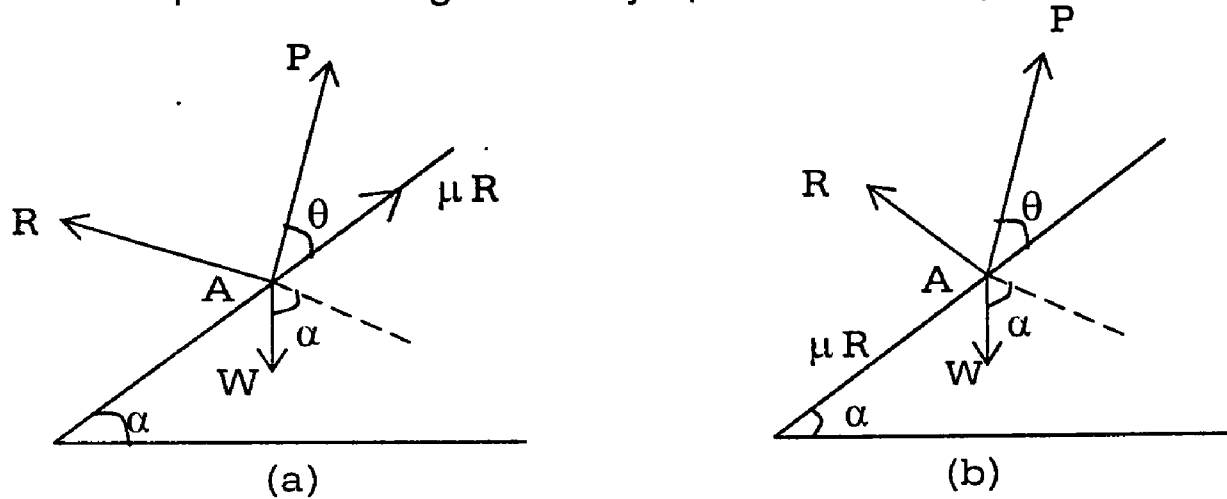
Note:

The value of P_2 may be obtained from that of P_1 by changing the sign of μ

5.6 Equilibrium of a body on a rough inclined plane under any force:

Theorem:

A body is at rest on a rough inclined plane of inclination α to the horizon being acted on by a force making an angle θ with the plane to find the limits between which the force must lie and also to find the magnitude and direction of the least force required to drag the body up the inclined plane.



Let W be the weight of the body, P the force acting at an angle θ with the plane and R the normal reaction.

Case 1:

In fig (a) the body is just on the point of moving down the plane. Then limiting friction μR acts upwards.

Resolving the force along and \perp to the plane,

$$\text{We get, } P \cos \theta + \mu R = W \sin \alpha \quad (1)$$

$$\text{and } P \sin \theta + R = W \cos \alpha \quad (2)$$

Substituting the value of R from (2) in (1)

we get

$$P \cos \theta + \mu (W \cos \alpha - P \sin \theta) = W \sin \alpha$$

$$\text{(ie) } P (\cos \theta - \mu \sin \theta) = (W \sin \alpha - \mu W \cos \alpha)$$

$$\therefore P = \frac{W(\sin \alpha - \mu \cos \alpha)}{\cos \theta - \mu \sin \theta}$$

If λ is the angle of friction,

We know that $\mu = \tan \lambda$

$$\begin{aligned} \therefore P &= \frac{W(\sin \alpha - \tan \lambda \cos \alpha)}{\cos \theta - \tan \lambda \sin \theta} \\ &= \frac{W(\sin \alpha \cos \lambda - \sin \lambda \cos \alpha)}{\cos \theta \cos \lambda - \sin \lambda \sin \theta} \\ P &= \frac{W \sin (\alpha - \lambda)}{\cos (\theta + \lambda)} \end{aligned}$$

Let this value of be P_1

$$\therefore P_1 = \frac{W \sin (\alpha - \lambda)}{\cos (\theta + \lambda)}$$

Case. 2

In fig. (b) the body is just on the point of moving up the plane. Then limiting friction μR acts downwards,
Resolving the forces as before.

$$P \cos \theta - \mu R = W \sin \alpha \quad (4)$$

$$P \sin \theta + R = W \cos \alpha \quad (5)$$

Substituting the value of R from (5) in (4)

We get

$$P \cos \theta - \mu (W \cos \alpha - P \sin \theta) = W \sin \alpha$$

$$(ie) \quad P (\cos \theta + \mu \sin \theta) = W (\sin \alpha + \mu \cos \alpha)$$

$$\begin{aligned} P &= \frac{W (\sin \alpha + \mu \cos \alpha)}{\cos \theta + \mu \sin \theta} \\ &= \frac{W (\sin \alpha + \tan \lambda \cos \alpha)}{\cos \theta + \tan \lambda \sin \theta} \\ &= \frac{W (\sin \alpha \cos \lambda + \sin \lambda \cos \alpha)}{\cos \theta \cos \lambda + \sin \lambda \sin \theta} \end{aligned}$$

$$\therefore P = \frac{W \sin(\alpha + \lambda)}{\cos(\theta - \lambda)}$$

Let this value of P be P_2 .

$$\therefore P_2 = \frac{W \sin(\alpha + \lambda)}{\cos(\theta - \lambda)} \quad (6)$$

P_1 and P_2 are the limiting values of the force P, necessary to keep the body in equilibrium.

Hence if lies between P_1 and P_2 , the body will remain in equilibrium.

Note:

- (1) the value of P_2 may be obtained from that of P_1 by changing the sign of μ .
- (2) By putting $\theta = 0$, the discussion of (3) gives the results of discussion of 2

Corollary:

We can find the direction and magnitude of the least force required to drag the body up the inclined plane.

From Case 2

$$P = \frac{W \sin(\alpha + \lambda)}{\cos(\theta - \lambda)}$$

Since α , W and λ are constants, P is least if $\cos(\theta - \lambda)$ is greatest.

(ie) if $\cos(\theta - \lambda)$ is greatest.

(ie) if $\cos(\theta - \lambda) = 1$.

This happens when $(\theta - \lambda) = 0$..

(ie) when $\theta = \lambda$.

In that case, value of

$$P = W \sin(\alpha + \lambda)$$

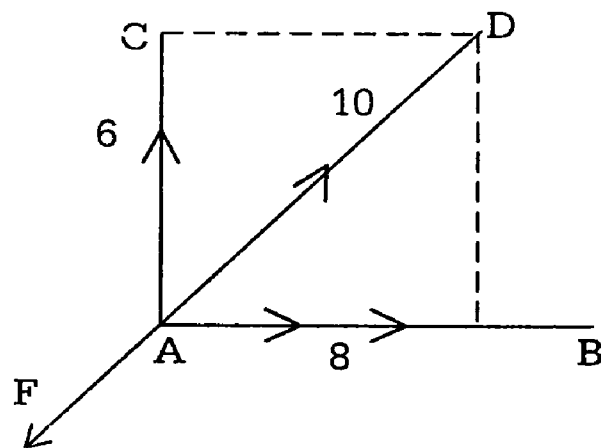
Hence the force required to move the body up the plane will be least when it is applied in a direction making with the inclined plane an angle equal to the angle of friction. This result is sometimes stated as:

“The best angle of friction up a rough inclined plane is the angle of friction”.

Example: 1 (U.Q)

A Particle of weight 30kgs resting on a rough horizontal plane is just on the point of motion when acted on by horizontal forces of 6kg wt and 8kg.wt at right angles to each other. Find the coefficient of friction between the particle and the plane and the direction in which the friction acts.

Solution:



Let AB = 8 AC = 6 represent the directions of the forces. A being the particle.

The resultant force = $\sqrt{8^2 + 6^2} = \sqrt{100} = 10$ kg. Weight and this acts along AD. Making an angle $\cos^{-1}\left(\frac{4}{5}\right)$ with the 8kg.force.

The particle tends to move in the direction AD of the resultant force and hence friction acts in the opposite direction DA.

Let F be the frictional force.

As motion just begins, magnitude of F is equal to that of the resultant force.

$$\therefore F = 10 \tag{1}$$

$$\text{If R is the normal reaction on the particle, } R = 30 \tag{2}$$

If μ is the coefficient of friction as the equilibrium is limiting, $F = \mu R$.

$$(ie) 10 = \mu \cdot 30 \text{ . (or) } \mu = \frac{10}{30} = \frac{1}{3}.$$

Example: 2 (U.Q)

A heavy particle rests in limiting equilibrium on a rough inclined plane. It is just on the point of moving up the plane along the line of greatest slope when a force equal to the weight is applied in that direction. Find the coefficient of friction.

Solution:

Let W be the weight of the particle resting at a point O on the line of greatest slope PQ is limiting equilibrium. Let α be the inclination of the plane to the horizon.

When a force equal to the weight W of the particle is applied to it along OQ , it is on the point of moving up the plane.

If \vec{R} is the normal reaction at O , the limiting friction $\mu\vec{R}$ acts down the plane along OP ,

Resolving the forces along and \perp to the plane.

We get,

$$\mu R + W \sin \alpha = W$$

$$R = W \cos \alpha$$

$$\text{Hence } \mu (W \cos \alpha) + W \sin \alpha = W$$

$$\text{giving } \mu = \frac{1 - \sin \alpha}{\cos \alpha}$$

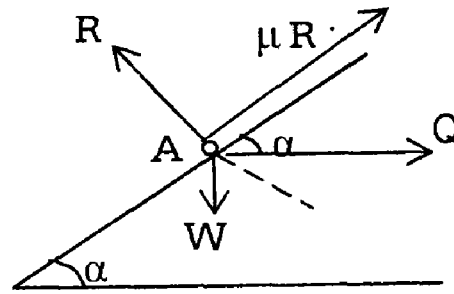
Example: 3

A weight can be supported on a rough inclined plane by a force P acting along the plane or by a force Q acting horizontally. Show that the weight is

$$\frac{PQ}{\sqrt{(Q^2 \sec^2 \lambda - P^2)}} \text{ where } \lambda \text{ is the angle of friction.}$$

Solution:

Let W be the weight and α the angle of inclination of the plane. R is the normal reaction. When the weight is just on the point of moving down, limiting friction. μR acts upwards. A horizontal force Q keeps the weight in equilibrium.



Resolving along and perpendicular to the plane.

$$\mu R + Q \cos \alpha = W \sin \alpha \quad (1)$$

$$R = W \cos \alpha + Q \sin \alpha$$

Putting the value of R from (2) in (1)

We have

$$\mu (W \cos \alpha + Q \sin \alpha) + Q \cos \alpha = W \sin \alpha$$

$$(ie) Q (\mu \sin \alpha + \cos \alpha) = W (\sin \alpha - \mu \cos \alpha) \quad (3)$$

The same weight W is supported by a force P acting along the plane.

From (2) by the theorem.

We have

$$P = \frac{W \sin (\alpha - \lambda)}{\cos \lambda} = \frac{W}{\cos \lambda} [\sin \alpha \cos \lambda - \cos \alpha \sin \lambda] \quad (4)$$

$$\text{From (3) } \cos \alpha (Q + \mu W) = \sin \alpha (W - \mu Q)$$

$$\therefore \frac{\cos \alpha}{W - \mu Q} = \frac{\sin \alpha}{Q + \mu W}$$

$$\begin{aligned}
\text{and each} &= \frac{\sqrt{\cos^2 \alpha + \sin^2 \alpha}}{\sqrt{(W - \mu Q)^2 + (Q + \mu W)^2}} \\
&= \frac{1}{\sqrt{W(1 + \mu^2) + Q^2(1 + \mu^2)}} = \frac{1}{\sqrt{1 + \mu^2} \sqrt{W^2 + Q^2}} \\
&= \frac{1}{\sqrt{1 + \tan^2 \lambda} \sqrt{W^2 + Q^2}} = \frac{1}{\sqrt{\sec^2 \lambda} \sqrt{W^2 + Q^2}} \\
\therefore \cos \alpha &= \frac{W - \mu Q}{\sec \lambda \sqrt{W^2 + Q^2}} \text{ and } \sin \alpha = \frac{Q + \mu W}{\sec \lambda \sqrt{W^2 + Q^2}}
\end{aligned}$$

Substituting these in (4)

We have

$$\begin{aligned}
P &= \frac{W}{\cos \lambda} \left[\frac{(Q + \mu W) \cos \lambda - (W - \mu Q) \sin \lambda}{\sec \lambda \sqrt{W^2 + Q^2}} \right] \\
&= \frac{W}{\sqrt{W^2 + Q^2}} [Q(\cos \lambda + \mu \sin \lambda) + W(\mu \cos \lambda - \sin \lambda)] \\
&= \frac{W}{\sqrt{W^2 + Q^2}} \left[Q \left(\cos \lambda + \frac{\sin \lambda}{\cos \lambda} \sin \lambda \right) + W \left(\frac{\sin \lambda}{\cos \lambda} \cos \lambda - \sin \lambda \right) \right] \\
&= \frac{W}{\sqrt{W^2 + Q^2}} \frac{Q(\cos^2 \lambda + \sin^2 \lambda)}{\cos \lambda} = \frac{WQ \sec \lambda}{\sqrt{W^2 + Q^2}}
\end{aligned}$$

$$P^2 = \frac{W^2 Q^2 \sec^2 \lambda}{W^2 + Q^2} = P^2 (W^2 + Q^2) = W^2 Q^2 \sec^2 \lambda^2$$

$$(\text{or}) W^2 (Q^2 \sec^2 \lambda - P^2) = P^2 Q^2$$

$$\therefore W^2 = \frac{P^2 Q^2}{Q^2 \sec^2 \lambda - P^2} \text{ or } W = \frac{PQ}{\sqrt{Q^2 \sec^2 \lambda - P^2}}$$

Example: 4

Show that the horizontal force, which can support a heavy mass on a smooth inclined plane of α to the horizon can just sustain it in limiting equilibrium on a rough plane inclined to the horizon at an angle of $(\alpha + \lambda)$ where λ is the angle of friction.

Solution:

Referring to equation (3) of the previous worked example:

We have

$$Q = \frac{W(\sin \alpha - \mu \cos \alpha)}{\mu \sin \alpha + \sec \alpha} \quad (1)$$

$$= \frac{W \left(\sin \alpha - \frac{\sin \lambda}{\cos \lambda} \cos \alpha \right)}{\frac{\sin \lambda}{\cos \lambda} \sin \alpha + \cos \alpha} \frac{W \sin(\alpha - \lambda)}{\cos(\alpha - \lambda)} \quad (2)$$

This gives the horizontal force Q which just keeps the weight W in limiting equilibrium on a rough plane of inclination α .

Let Q_1 be the horizontal force which keeps W in limiting equilibrium on a rough plane of inclination $\alpha + \lambda$.

Let Q_1 can be got from Q by changing α in to $\alpha + \lambda$.

$$Q_1 = \frac{W \sin(\alpha + \lambda - \lambda)}{\cos(\alpha + \lambda - \lambda)} = W \tan \alpha \quad (3)$$

Let Q_2 be the horizontal force which keeps w in equilibrium on a smooth plane of inclination α : Q_2 can be got from the value of Q gives in (1) by putting $\mu = 0$.

$$\therefore Q_2 = \frac{W \sin \alpha}{\cos \alpha} = W \tan \alpha \quad (4)$$

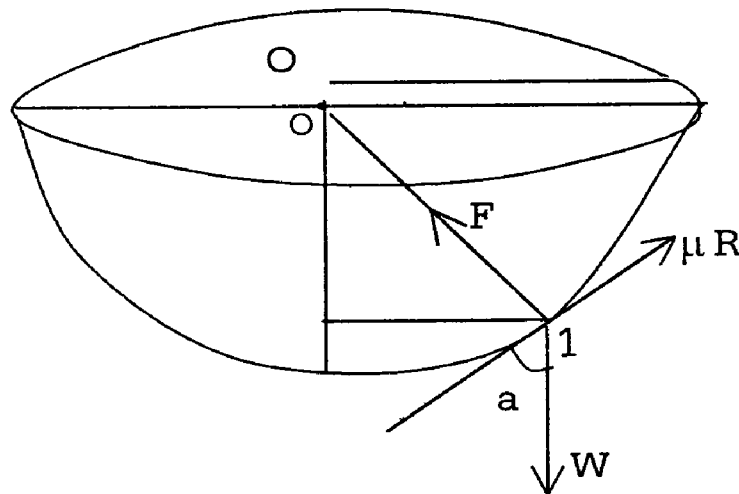
From (3) and (4)

We find that $Q_1 = Q_2$

Example: 5

An insect tries to crawl up the inside of fixed hemispherical bowl of radius R . How high can it travel? If the coefficient of friction between its feet and the bowl is $\frac{1}{3}$?

Let A be lowest point of the fixed hemispherical bowl of radius R . Let the insect of weight \vec{W} (Say), crawl upto I . When it is about to slip back. Then the maximum friction $\mu\vec{F}$ is called into play \vec{F} being the normal reaction through O , the centre of the bowl.



Since the three forces \vec{F} , $\mu\vec{F}$, \vec{W} meeting at I are in limiting equilibrium by Lami's theorem, we get

$$\frac{F}{\sin(180^\circ - \alpha)} = \frac{\mu F}{\sin(90^\circ + \alpha)} = \frac{W}{\sin 90^\circ}$$

Where α is the angle that $\mu\vec{F}$ makes with the vertical

$$\text{Hence } F = W \sin \alpha$$

$$\text{and } \mu F = W \cos \alpha$$

$$\text{So that } \mu = \cot \alpha$$

$$\text{Since } \mu = \frac{1}{3}$$

$$\text{Cot } \alpha = \frac{1}{3}$$

The height of the point I about A = AP = OA – OP = R – R Sin α

$$\text{as } \angle \text{OIP} = \alpha$$

$$= R (1 - \text{Sin } \alpha) = R \left(1 - \frac{3}{\sqrt{10}} \right) \text{ as Cot } \alpha = \frac{1}{3}$$

Exercise:-

1. A body of weight 4kgs. rests in limiting equilibrium on an inclined plane whose inclination is 30° . Find the coefficient of friction and the normal reaction.
2. A body of weight 4kgs. rests in limiting equilibrium on a rough plane whose slope is 30° , if the plane is raised to a slope of 60° , find the force along the plane required to support the body.
3. If the force, which acting parallel to a rough plane of inclination α to the horizon is just sufficient to draw a weight up be n times the force which will just let it be on the point of slipping down, show that

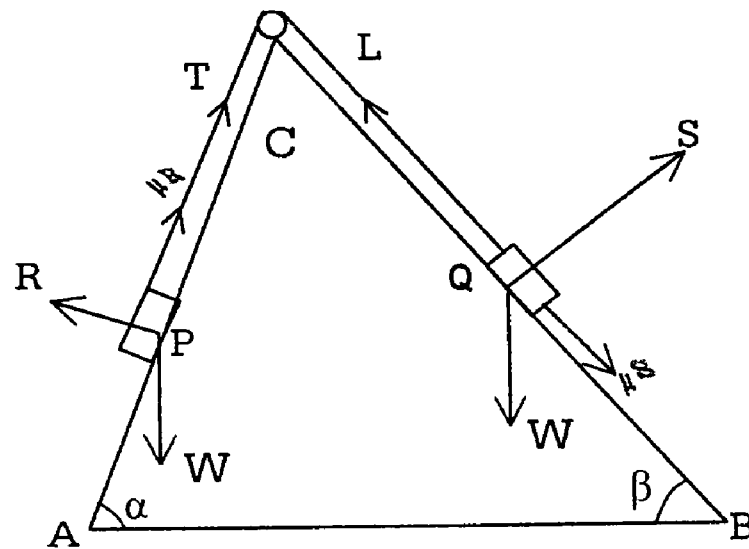
$$\tan \alpha = \mu \frac{n+1}{n-1}$$

Example:- 6

Two particles P and Q each of weight W on two equally rough inclined planes CA and CB of the same height, placed back to back are connected by a light string which passes over the smooth top edge C of the planes. Show that if the particles are on the point of slipping the difference of the inclinations of the planes is double the angle of friction.

Solution:-

Let α and β be the inclinations of the planes CA and CB; R,S be the normal reactions of the planes, T the tensions of the planes, T the tension of the string and μ the coefficient of friction (which is the same for both planes).



Let P be on the point of moving downwards. Then Q will be moving upwards.

Limiting friction μR will act on P upwards the inclined plane and limiting friction μS will act on Q downwards the inclined plane.

Considering the equilibrium of P and resolving along and \perp to the plane CA.

We have

$$\mu R + T = W \sin \alpha \quad (1)$$

$$R = W \cos \alpha \quad (2)$$

Substituting the value of R from (2) in (1)

$$T = W \sin \alpha - \mu W \cos \alpha \quad (3)$$

Resolving along and perpendicular to the plane CB.

$$\text{We have } T = W \sin \beta + \mu S \quad (4)$$

$$\text{And } S = W \cos \beta \quad (5)$$

Substituting (5) in (4)

$$\text{We get } T = W \sin \beta + \mu W \cos \beta \quad (6)$$

Equating the two values of T in (6) and (3)

$$W(\sin \beta + \mu \cos \beta) = W(\sin \alpha - \mu \cos \alpha)$$

$$\text{(ie) } \mu(\cos \alpha + \cos \beta) = \sin \alpha - \sin \beta$$

$$\begin{aligned} \therefore \mu &= \frac{\sin \alpha - \sin \beta}{\cos \alpha + \cos \beta} = \frac{2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}}{2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}} \\ &= \tan \frac{\alpha - \beta}{2} \end{aligned} \quad (7)$$

If λ is the angle of friction,

We know that

$$\mu = \tan \lambda \quad (8)$$

\therefore From (7) & (8)

$$\text{We get } \lambda = \frac{\alpha - \beta}{2}$$

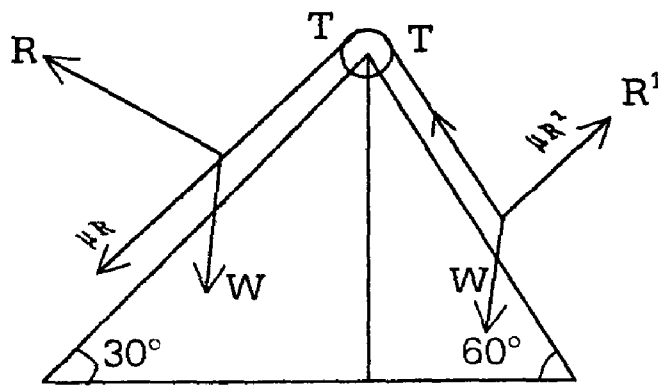
$$\text{(or) } 2\lambda = \alpha - \beta$$

Example: 7

Two equal weights are attached to two ends of a string which pass over a smooth pulley on the top of two equally rough planes having the same altitude and placed back to back the inclination of the planes to the horizontal being 30° and 60° respectively. If the system be in limiting equilibrium. Show that the coefficient of friction is $2 - \sqrt{3}$.

Solution:-

Since the inclination of the second plane. (namely, 60°) is greater, the weight w on this plane is on the point of moving down the plane and so the weight W on the other plane is on the point of moving up.



The pulley being smooth, the tension in the two parts of the string are the same, say \vec{T} .

The normal reaction of the planes are \vec{R} and \vec{R}' and $\mu\vec{R}$ acts down the plane (of inclination 30°) while $\mu\vec{R}'$ acts up the other plane.

Resolving the forces normal to the planes separately,

We get

$$R = W \cos 30^\circ = \frac{\sqrt{3}}{2} W$$

$$\text{And } R' = W \cos 60^\circ = \frac{1}{2} W$$

Resolving the forces on each plane along the plane,

We get

$$T - \mu R = W \sin 30^\circ$$

$$\text{(or) } T = \mu \cdot \frac{\sqrt{3}}{2} W + W \cdot \frac{1}{2}$$

$$\text{and } T + \mu R' = W \sin 60^\circ$$

$$\text{(Or) } T = W \cdot \frac{\sqrt{3}}{2} - \mu \cdot \frac{1}{2} W$$

$$\text{These give } \mu \cdot \frac{\sqrt{3}}{2} W + W \cdot \frac{1}{2} = W \cdot \frac{\sqrt{3}}{2} - \mu \cdot \frac{1}{2} W$$

$$\text{(or) } \mu(\sqrt{3} + 1) = \sqrt{3} - 1$$

Hence

$$\begin{aligned} \mu &= \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} \\ &= \frac{(\sqrt{3}-1)^2}{3-1} = \frac{3+1-2\sqrt{3}}{2} \\ &= \frac{4-2\sqrt{3}}{2} \\ &= \frac{2(2-\sqrt{3})}{2} \\ \mu &= 2 - \sqrt{3} \end{aligned}$$

Example: 8

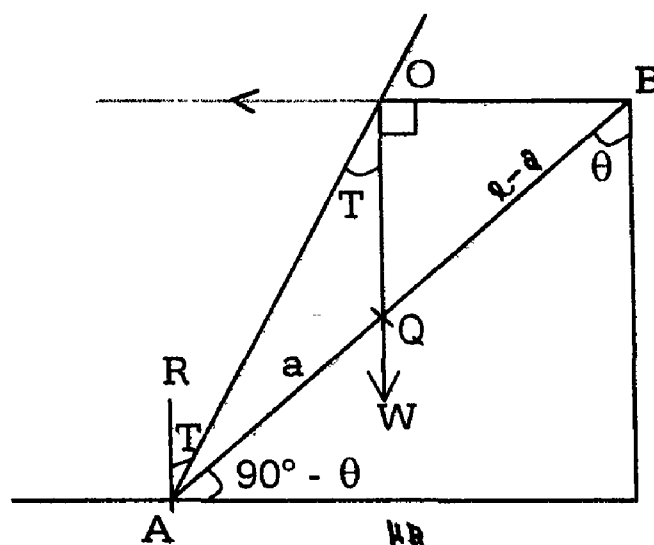
A straight rod rests with one end A on a rough horizontal plane and the other end B against a smooth vertical wall. If ℓ be the length of rod and a , the distance of its gravity from show that the centre of the inclination of the rod to the wall when on the point of slipping is $\tan^{-1} \left(\frac{\ell\mu}{a} \right)$ where μ is the coefficient of friction.

Solution:-

Since the wall is smooth, the reaction \vec{S} at B is perpendicular to the wall.

The rod being on the point of slipping the reaction at A on the ground makes an angle λ (angle of friction) with the vertical through A.

If G is C. G of the rod AB the line of action of the weight \vec{W} through G passes through O, the point where the reactions meet.



(ie) OG is Vertical

In $\triangle AOB$, $\angle AOG = \lambda$,

$$\angle GOB = 90^\circ$$

$$AG = a \text{ and } GB = \ell - a$$

Applying the trigonometrical result to $\triangle AOB$

We get

$$(AG + GB) \cot \angle OGB = AG \cot \angle AOG - GB \cot \angle GOB$$

$$(ie) \ell \cot \theta = a \cot \lambda$$

$$(or) \tan \theta = \frac{\ell}{a} \tan \lambda$$

$$= \frac{\ell \mu}{a}$$

$$\therefore \theta = \tan^{-1} \left(\frac{\mu}{a} \right)$$

Problems on Friction:-

Problems on the equilibrium of a rigid body or bodies involving friction are of varied types. They can be classified as follows.

- a) These in which the equilibrium of the rigid body can be broken only by sliding, the direction of motion being obvious.

For example,

Consider a ladder resting on rough ground against a rough wall in a vertical plane \perp^I to the wall. If no other forces act, it is clear that the lower end of the ladder slips away from the wall and the upper end slips downwards at the same time. One end cannot slip without the other. Other examples of this type are, a rod resting over two rough pegs, a rod resting within a rough hollow sphere etc.

- b) Those in which equilibrium is broken either by sliding or tilting. For example, a block of cylinder resting on a rough plane which is gradually tilted. If the vertical through the centre of gravity comes outside the base before sliding commences, the body will topple over before sliding.
- c) Those in which we have a non-rigid body such as two jointed rods AB, BC freely jointed at B and with the ends A and C resting on a rough horizontal plane. In this case, it is not necessary for slipping to occur at both A and C. It may happen that one of the rods may slip before the other does.

The method of dealing with such problems is illustrated in the following examples.

Example: 1

A uniform ladder is in equilibrium with one end resting on the ground and the other against a vertical wall: if the ground and wall be both rough, the coefficients of friction being μ and μ' respectively, and if the ladder be on the point of slipping at both ends, show that θ , the inclination of the ladder to horizon is given by

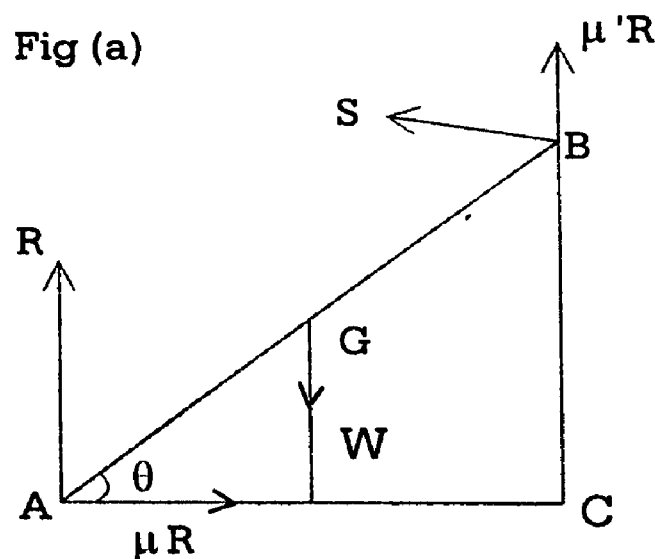
$$\tan \theta = \frac{1 - \mu\mu'}{2\mu}$$

Find also the reactions at the wall and ground.

Solution:-

First method: (Refer to 'a')

Let AB ($=2a$) be the ladder G its centre of gravity and W its weight. Let R and S be the normal and S be the normal reactions acting on the ladder at the ground and wall.



The lower end A will have a tendency to move away from the wall and hence friction there will be acting in the direction AC. The upper end B will have a tendency to move downwards and hence friction there will be acting upwards. Since both ends are slipping the frictions at A and B are limiting. They are μR and $\mu' S$ as marked in the figure.

$$\text{Resolving horizontally, } S = \mu R \quad (1)$$

$$\text{Resolving Vertically, } \mu' S + R = W \quad (2)$$

Taking moments about A,

$$S \cdot BC + \mu' S \cdot AC = W \cdot AE.$$

$$\text{(ie) } S \cdot 2a \sin \theta + \mu' S \cdot 2a \cos \theta = W \cdot a \cos \theta \quad (3)$$

Putting the value of S from (1) in (2)

We have

$$\mu' \mu R + R = W$$

$$\text{(ie) } R = \frac{W}{1 + \mu \mu'} \quad (4)$$

$$\text{Then from (1) } S = \frac{\mu W}{1 + \mu \mu'} \quad (5)$$

(4) and (5) give the reactions.

Putting the value of S from (5) in (3)

$$\text{We get } \frac{\mu W}{1 + \mu \mu'} \cdot 2 \sin \theta + \frac{\mu' \mu W}{1 + \mu \mu'} \cdot 2 \cos \theta = W \cos \theta$$

$$\text{(ie) } 2 \mu \sin \theta + 2 \mu \mu' \cos \theta = (1 + \mu \mu') \cos \theta$$

$$\text{(ie) } 2 \mu \sin \theta = (1 - \mu \mu') \cos \theta$$

$$\text{(or) } \frac{\sin \theta}{\cos \theta} = \frac{1 - \mu \mu'}{2 \mu} = \tan \theta \quad (6)$$

Second Method:- (Refer to fig.(b))

The two forces, friction and normal reaction at A can be compounded into a single force. Similarly the two forces, friction and normal reaction at B can be compounded into a single force. Let these two resultant reactions at L. For equilibrium, the third force the weight of the ladder must pass through L.

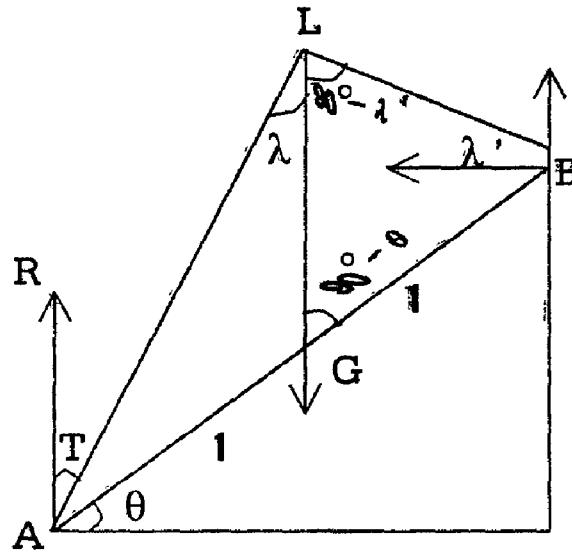


Fig (b)

Hence LG is vertical. As the equilibrium is limiting, these resultant reactions must make angles λ, λ' with the normal at A and B [where $\tan \lambda = \mu$ and $\tan \lambda' = \mu'$] In ΔALB ,

$$\angle LGB = 90^\circ - \theta$$

$$\angle ALG = \lambda, \angle BLG = 90^\circ - \lambda'$$

And $AG: GB = 1:1$

Using the first trigonometric equation

We have $(1+1) \cot (90^\circ - \theta) = 1 \cdot \cot \lambda - 1 \cdot \cot (90^\circ - \lambda)$

$$(ie) 2 \tan \theta = \cot \lambda - \tan \lambda'$$

$$= \frac{1}{\tan \lambda} - \tan \lambda'$$

$$= \frac{1}{\mu} - \mu' = \frac{1 - \mu\mu'}{\mu}$$

$$\therefore \tan \theta = \frac{1 - \mu\mu'}{2\mu}$$

Note:

- (1) By this method, we cannot find the reactions. They have to be found by resolving the forces.
- (2) This geometrical method is particularly useful when we require only the position of a body in limiting equilibrium as we obtain the result without introducing the reactions and having to solve the equations.

Example: 2 (U.Q)

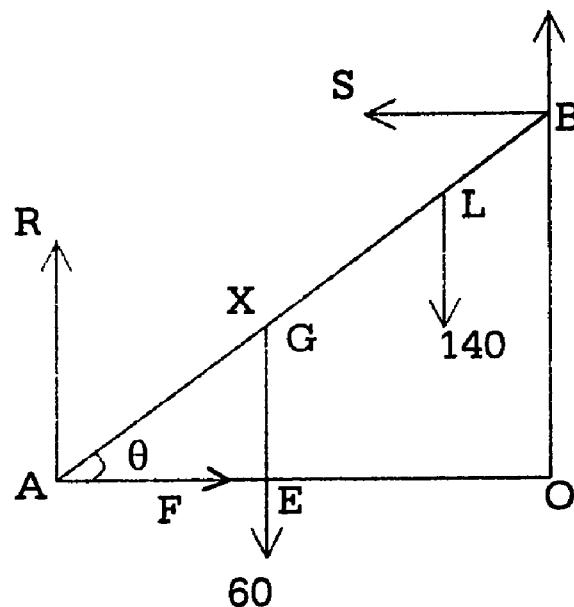
A ladder 20 meters long with its centre of gravity 8 meters up from the bottom, weights 60kgms. and rests at an angle of θ to the ground against a smooth vertical wall. The coefficient of friction between the ladder and the ground is $\frac{1}{2}$. Find the least value of θ which will enable a man weighing 140kgms to reach the top without the ladder slipping.

Solution:-

Let L be the position of the man on the ladder AB such that $AL = x$.

The forces acting on the ladder are:

- 1) its weight 60kgms acting vertically downwards at G where $AG = 8$.
- 2) Reaction R at A, \perp^r to the ground.
- 3) Friction F at A towards the wall as the end A will begin to slip in the direction OA.
- 4) Normal reaction S at B due to contact with the wall (As the wall is smooth, there is no friction at B.)
- 5) The weight of the man 140kgms at L acting vertically downwards



Resolving horizontally and vertically,

$$F = S \quad (1)$$

$$R = 140 + 60 = 200 \quad (2)$$

Taking moments about A.

$$60. \vec{AE} + 140 \vec{AF} = S. \vec{BC}$$

$$S. 20 \sin \theta = 60. 8 \cos \theta + 140. x \cos \theta \quad (3)$$

$$\therefore S = \frac{480 \cos \theta + 140 x \cos \theta}{20 \sin \theta}$$

$$= (24 + 7x) \cot \theta \quad (4)$$

$$\text{Now, } \frac{F}{R} = \frac{S}{200} = \frac{(24 + 7x) \cot \theta}{200}$$

For equilibrium,

$\frac{F}{R}$ Must be $\leq \mu$, the coefficient of friction

$$\text{(ie) } \frac{(24 + 7x) \cot \theta}{200} \geq \frac{1}{2} \text{ or } \cot \theta \leq \frac{100}{24 + 7x} \quad (5)$$

The condition (5) must be satisfied for values of x from 0 to 20, if the man is to reach the top safely.

It is enough if the value of θ is such that, there is no slipping when the man reaches the top.

\therefore Putting $x = 20$ in (5), the condition

$$\cot \theta \leq \frac{100}{24 + 140}$$

$$(ie) \cot \theta \leq \frac{100}{164}$$

$$\text{or } \tan \theta \leq \frac{100}{164}, (ie) \tan \theta \leq \frac{41}{25}$$

As θ increase, $\tan \theta$ also increases, Hence condition (6) is satisfied if

$$\theta \leq \tan^{-1}\left(\frac{41}{25}\right)$$

$$\therefore \text{Least value of } \theta = \tan^{-1}\left(\frac{41}{25}\right)$$

Example: 3 (U.Q)

A uniform ladder rests in limiting equilibrium with its lower end on a rough horizontal plane and its upper end against an equally rough vertical wall. If θ be the inclination of the ladder to the vertical, prove that

$$\tan \theta = \frac{2\mu}{-\mu^2}, \text{ where } \mu \text{ is the coefficient of friction.}$$

Solution:-

Let the ladder AB rest with its end A on a rough horizontal plane, its end B on the an equally rough vertical wall.

Since AB rests in limiting equilibrium, the reaction at A and B make angle λ (the angle of friction) with the perpendicular as A and B to horizontal plane and the vertical wall.

Let these forces of reaction meet at O. The line of action of the weight \vec{W} of the ladder acting through G, the mid-point of AB must pass, through O. There being only three forces is equilibrium.

(ie) OG is vertical.

Since AB is inclined to the vertical at θ ,

$$\angle OQB = \theta$$

$$\text{In } \Delta OAB, \angle AOG = \lambda \quad \angle GOB = 90^\circ - \lambda$$

Hence applying the trigonometrical result to the ΔOAB ,

We get,

$$(AG+GB) \cot \angle OGB = AG \cot \angle AOG - GB \cot \angle GOB$$

$$(or) \quad AB \cot \theta = \frac{1}{2} AB \cot \lambda - \frac{1}{2} AB \cot (90^\circ - \lambda)$$

$$(ie) \quad 2 \cot \theta = \cot \lambda - \tan \lambda$$

$$= \frac{1}{\mu} - \mu = \frac{1-\mu^2}{\mu}$$

$$\text{Hence } \tan \theta = \frac{2\mu}{1-\mu^2}$$

Example:4

A ladder AB rests with A resting on the ground and B against a vertical wall, the coefficients of friction on the ground and wall being μ and μ' respectively. The centre of gravity of the ladder is slipping at both ends. Show that the inclination to the ground is given by $\tan \theta = \frac{1-n\mu\mu'}{(n+1)\mu}$.

Solution:

The ladder AB of weight \vec{W} is in limiting equilibrium with its end A on the ground and the other end B on a vertical wall. The reactions \vec{R} and \vec{S} at A and B make angles λ and λ' . (the respective angles of friction) with the normal at A and B respectively.

Let \vec{R} and \vec{S} meet at O.

Then the line of action of the weight \vec{W} of the ladder through the C.G of the ladder, namely G must pass through O.

(ie) OG is vertical.

If θ is the angle that the ladder makes with the ground then in

$\triangle AOB$

$$\angle OGB = 90^\circ - \theta$$

$$\angle AOG = \lambda \quad \text{and} \quad \angle GOB = 90^\circ - \lambda$$

Hence applying the trigonometrical result to ΔAOB ,

We get,

$$(n+1) \cot \angle OGB = 1 \cot \angle AOG - n \cdot \cot \angle GOB.$$

as G divides AB in the ratio 1:n.

$$(ie) (n+1) \cot (90^\circ - \theta) = \cot \lambda - n \cot (90^\circ - \lambda')$$

$$(or) (n+1) \tan \theta = \cot \lambda - n \tan \lambda'$$

$$= \frac{1}{\mu} - n \mu' = \frac{1 - n\mu\mu'}{\mu}$$

$$\text{Hence } \tan \theta = \frac{1 - n\mu'}{(n+1)\mu}$$

Exercise:

1. A uniform ladder rests with its lower end on a rough horizontal ground and its upper end against a rough vertical wall, the ground and the wall being equally rough and the angle of friction being λ . Show that the greatest inclination of the ladder to the vertical is 2λ .
2. A ladder AB rests with A on a rough horizontal ground and B against an equally rough vertical wall. The centre of gravity of the ladder divides AB in the ratio a:b. If the ladder is on the point of slipping. Show that the inclination θ of the ladder to the ground is given by $\tan \theta = \frac{a - b\mu^2}{\mu(a + b)}$ where μ is the coefficient of friction.
3. A uniform ladder rests with its lower end on a rough ground and its upper end against a smooth vertical wall. When in limiting equilibrium, the ladder makes with the wall an angle 30° . Find the coefficient of friction μ . (U.Q).

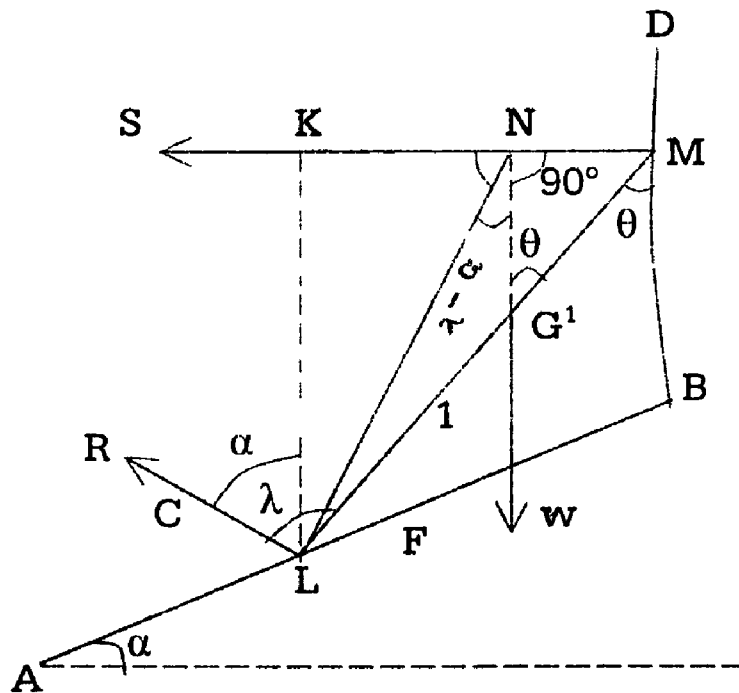
Example: 1

A uniform rod of weight W rests with its one end against a rough inclined plane AB of inclination α and the other end against a smooth vertical wall BD, B being at a higher level than A. If θ be the inclination of the rod to the vertical in

the position of limiting equilibrium. Prove that $\tan \theta = 2 \tan (\lambda - \alpha)$ where λ is the angle of friction.

Prove also that reaction of the wall is $W \tan (\lambda - \alpha)$ and the resultant reaction with ground is $W \sec (\lambda - \alpha)$.

Solution:-



LM is the rod, with G its midpoint the forces acting on it are

- i) its weight W acting vertically downwards at G .
- ii) the reaction R at $L \perp$ to the inclined plane AB
- iii) Limiting frictional force F at L along LB .
- iv) reactions S at M normal to the wall and hence horizontal.

As the equilibrium is limiting, the two forces F and R at L can be compound into one force, the resultant reaction P making an angle λ with LC , the normal at L .

Let this resultant reaction meet the line of action of S at N .

For equilibrium, NG must be vertical.

Draw $LK \parallel$ to NG meeting the forces S at K .

$\angle CLK =$ angle between CL and LK .

= angle between their \perp^{IS} , AB and the horizontal

$$= \alpha$$

$$\angle \text{KLN} = \lambda - \alpha$$

$$\therefore \angle \text{LNG} = \lambda - \alpha \text{ and LG: GM} = 1:1$$

From $\triangle \text{LNM}$, using the first trigonometrical equation

We have

$$(1+1) \text{ Cot } \theta = 1. \text{ Cot } (\lambda - \alpha) - 1. \text{ Cot } 90^\circ$$

$$\text{(ie) } 2 \text{ Cot } \theta = \text{Cot } (\lambda - \alpha).$$

$$\frac{2}{\tan \theta} = \frac{1}{\tan(\lambda - \alpha)}$$

$$\text{(or) } \tan \theta = 2 \tan (\lambda - \alpha)$$

By applying Lami's theorem for the three force P, S and W at N.

We get

$$\frac{P}{\sin 90^\circ} = \frac{S}{\sin(\lambda - \alpha)} = \frac{W}{\sin(90^\circ + \lambda - \alpha)}$$

$$\text{(ie) } \frac{P}{1} = \frac{S}{\sin(\lambda - \alpha)} = \frac{W}{\cos(\lambda - \alpha)}$$

$$\therefore P = \frac{W}{\cos(\lambda - \alpha)} = W \text{ Sec } (\lambda - \alpha)$$

$$\text{and } S = \frac{W \sin(\lambda - \alpha)}{\cos(\lambda - \alpha)} = W \tan(\lambda - \alpha)$$

Example: 2(U.Q)

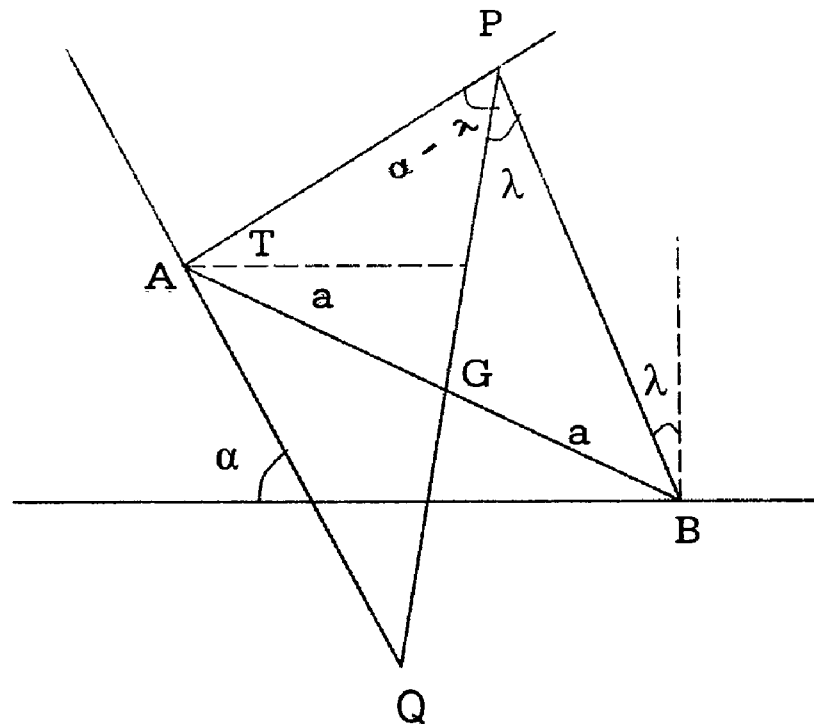
A uniform rod is in limiting equilibrium with one end resting on a rough horizontal plane and the other end on an equally rough plane inclined at an angle α to the horizon. If λ be the angle of friction and the rod be in a vertical plane, show that if θ be the inclination of the rod with the horizon then,

$$\tan \theta = \frac{\sin(\alpha - 2\lambda)}{2\sin \lambda \sin(\alpha - \lambda)}$$

Solution:

Let the rod AB of length 2a rest with its end A on the plane of inclination α with the horizon and the other end B on the horizontal plane.

Since the two planes, are equally rough, the equilibrium being limiting, the reactions at A and B make the same angle λ (the angle of friction) with the normal at A and B.



Let the reactions meet at P.

The line of action of the weight \vec{W} of the rod AB acting through G, the midpoint of AB, passes through P, there being only three forces in equilibrium.

$$\begin{aligned} \text{In } \triangle APQ, \angle APG & \\ &= 180^\circ - (\angle PAQ + \angle AQP) \\ &= 180^\circ - (90^\circ + \lambda + 90^\circ - \alpha) \end{aligned}$$

$$= \alpha - \lambda$$

$$\angle GPB = \lambda$$

$$\text{And } \angle PGA = 90^\circ - \theta$$

Hence applying the trigonometrical result to ΔAPB .

We get,

$$(AG + GB) \cot \angle PGA = BG \cot \angle GPB - AG \cot \angle APG$$

$$(\text{or}) 2a \cot (90^\circ - \theta) = a \cot \lambda - a \cot (\alpha - \lambda)$$

$$= a \left[\frac{\cos \lambda}{\sin \lambda} - \frac{\cos (\alpha - \lambda)}{\sin (\alpha - \lambda)} \right]$$

$$\therefore \tan \theta = \frac{1}{2} \left[\frac{\sin (\alpha - \lambda) \cos \lambda - \cos (\alpha - \lambda) \sin \lambda}{\sin \lambda \sin (\alpha - \lambda)} \right]$$

$$\therefore \tan \theta = \frac{\sin (\alpha - 2\lambda)}{2 \sin \lambda \sin (\alpha - \lambda)}$$

Example: 3 (U.Q)

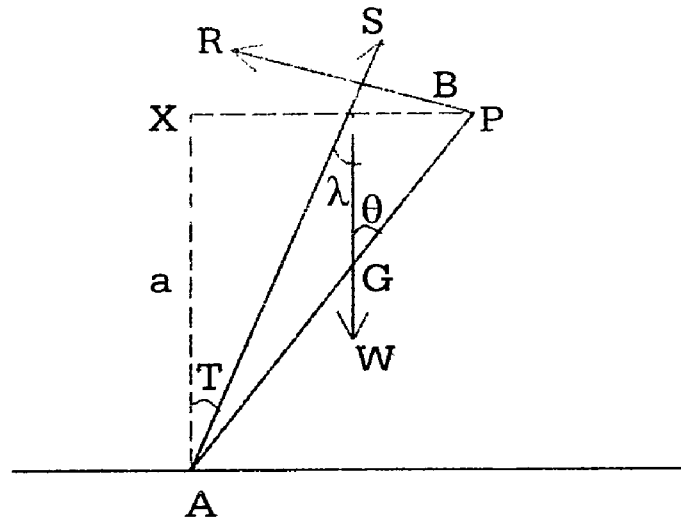
A uniform ladder of length ℓ rest on a rough horizontal ground with its a upper end projecting very slightly over a horizontal smooth rail at a height a above the ground. If the ladder is about to slip down and is the coefficient of friction, show that

$$\mu = \frac{a \sqrt{\ell^2 - a^2}}{\ell^2 + a^2}$$

Solution:-

The ladder AB rests with the end A on the ground and its upper end B is very slightly above the rail at P, initially.

When the ladder is about to slip down. B is at P and the reaction \vec{R} at B (or P) acts \perp^I to the ladder AB.



Let the weight \vec{W} of the ladder acting through the midpoint G meet \vec{R} at O . Then the reaction \vec{S} at A must pass through O there being limiting equilibrium.

The equilibrium being limiting and the ground being rough, \vec{S} makes with the vertical AX the angle λ (the angle of friction) Let AB make an angle θ with AX .

$$\text{Since } \angle GBO = 90^\circ; \angle GOB = 90^\circ - \angle OGB = 90^\circ - \theta$$

$$\text{Also } \angle AOG = \angle XAO = \lambda.$$

Applying the trigonometrical result to the ΔAOB .

We get

$$(AG + GB) \cot \angle OGB = AG \cot \angle AOG - GB \cot \angle GOB$$

$$l \cot \theta = \frac{1}{2} l \cot \lambda - \frac{1}{2} l \cot (90^\circ - \theta)$$

$$\text{(or)} \quad 2 \cot \theta = \cot \lambda - \tan \theta$$

$$\cot \lambda = 2 \cot \theta + \tan \theta = \frac{2 + \tan^2 \theta}{\tan \theta}$$

In ΔABX , $\cos \theta = \frac{a}{l}$ and So

$$\tan \theta = \sqrt{\frac{l^2 - a^2}{a}}$$

$$\text{Hence } \mu = \tan \lambda = \frac{\tan \theta}{2 + \tan^2 \theta}$$

$$= \frac{\frac{\sqrt{\ell^2 - a^2}}{a}}{2 + \frac{\ell^2 - a^2}{a^2}}$$

$$\mu = \frac{a \sqrt{\ell^2 - a^2}}{\ell^2 + a^2}$$

Exercise:

1. A rod is in limiting equilibrium resting horizontally with its ends on two inclined planes at right angles, one of which makes an angle α ($< 45^\circ$) with the horizontal. If μ the coefficient of friction is same for both ends, prove

$$\text{that } \mu = \frac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha}.$$

2. A uniform rod of length ℓ rests in a vertical plane against (and over) a smooth horizontal bar at a height h , the lower end of the rod being on level ground. Show that if the rod is on the point of slipping when its inclination to the horizontal is θ , then the coefficient of friction between the rod and

$$\text{ground is } \frac{\ell \sin 2\theta \sin \theta}{4h - \ell \sin 2\theta \cos \theta}$$

Example:- 1 (U.Q)

A uniform rod rests in limiting equilibrium within a rough hollow sphere. If the rod subtends an angle 2α at the centre of the sphere and if λ be the angle of friction, show that the inclination of the rod to the horizontal is

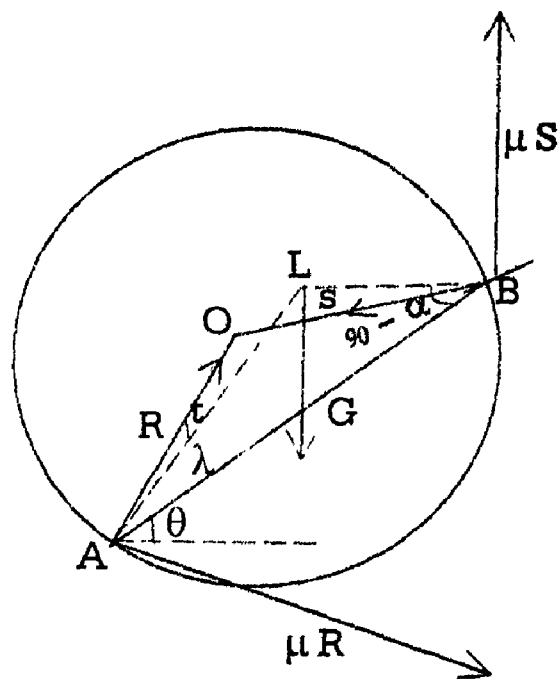
$$\tan^{-1} \left[\frac{\sin 2\lambda}{\cos 2\alpha + \cos 2\lambda} \right]$$

Solution:-

Let AB the rod, G its C.G and O is the centre of the sphere.

$$\text{Since } \angle AOB = 2\alpha$$

$$\angle OAB = \angle OBA = 90^\circ - \alpha.$$



Let R and S be the normal reaction at A and B acting along AO and BO respectively.

Since B is on the point of coming down and A is on the point of going up, limiting friction μS and μR act in the directions shown in the figure.

The forces $R, \mu R$ at A and $S, \mu S$ at B can be compounded into single forces. The resultant reactions acting along AL and BL respectively.

$$\text{Then } \angle OAL = \angle OBL = \lambda$$

For equilibrium, the 3rd force.

(ie) the weight of the rod must pass through L .

(ie) LG is vertical.

Let AB make an angle θ with the horizontal.

$$\text{In } \triangle ALB, \angle LAB = \angle OAB - \lambda = 90^\circ - \alpha - \lambda$$

$$\angle LBA = \angle OBA + \lambda = 90^\circ - \alpha + \lambda$$

$$\angle LGA = 90^\circ - \theta \text{ and } AG:GB = 1:1$$

\therefore Using the second trigonometric equation.

We have

$$(1+1) \cot (90^\circ - \theta) = 1 \cot (90^\circ - \alpha - \lambda) - 1 \cot (90^\circ - \alpha + \lambda)$$

$$= \cot (90^\circ - (\alpha + \lambda)) - \cot (90^\circ - (\alpha - \lambda))$$

$$(ie) 2 \tan \theta = \tan (\alpha + \lambda) - \tan (\alpha - \lambda)$$

$$= \frac{\sin(\alpha + \lambda)}{\cos(\alpha + \lambda)} - \frac{\sin(\alpha - \lambda)}{\cos(\alpha - \lambda)}$$

$$= \frac{\sin(\alpha + \lambda) \cos(\alpha - \lambda) - \cos(\alpha + \lambda) \sin(\alpha - \lambda)}{\cos(\alpha + \lambda) \cos(\alpha - \lambda)}$$

$$= \frac{\sin[(\alpha + \lambda) - (\alpha - \lambda)]}{\cos(\alpha + \lambda) \cos(\alpha - \lambda)}$$

$$= \frac{\sin 2\lambda}{\cos(\alpha + \lambda) \cos(\alpha - \lambda)}$$

$$\therefore \tan \theta = \frac{\sin 2\lambda}{2 \cos(\alpha + \lambda) \cos(\alpha - \lambda)}$$

$$= \frac{\sin 2\lambda}{\cos(\alpha + \lambda + \alpha - \lambda) + \cos((\alpha + \lambda) - (\alpha - \lambda))}$$

$$\therefore \tan \theta = \frac{\sin 2\lambda}{\cos 2\alpha + \cos 2\lambda}$$

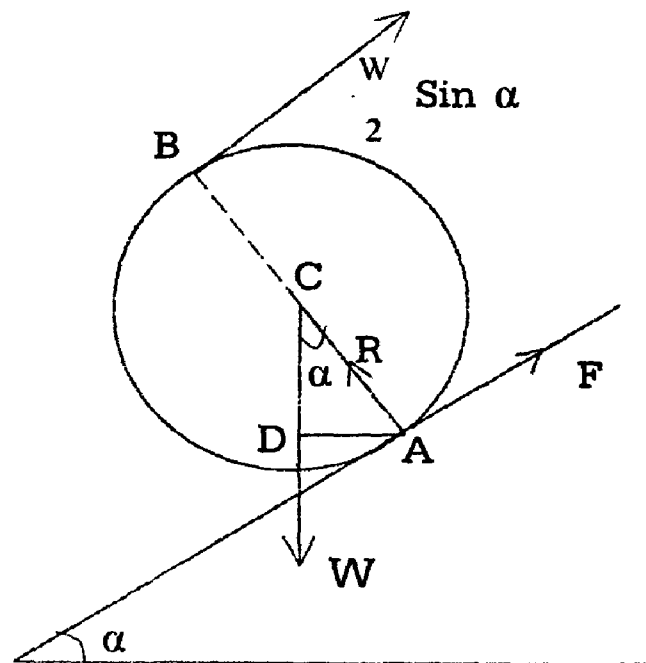
$$\theta = \tan^{-1} \frac{\sin 2\lambda}{\cos 2\alpha + \cos 2\lambda}$$

Example: 2 (U.Q)

A uniform sphere is held in equilibrium on a rough inclined plane of angle α by a force of magnitude $\frac{W}{2} \sin \alpha$ applied tangentially to its circumference, where w is the weight of the sphere. Prove that the force must act parallel to the plane and that the coefficient of friction must be not less than $\frac{1}{2} \tan \alpha$.

Solution:

Let C be the centre of the sphere, A the point of contact of the sphere with inclined plane and AB the diameter through A.



Let the radius be a .

The forces acting on the sphere are:

- 1) its weight W acting vertically downwards at C
- 2) the normal reaction R acting along AC .
- 3) Frictional force F acting upwards along the inclined plane as the sphere is on the point of moving down and
- 4) the force $\frac{W \sin \alpha}{2}$ applied tangentially.

Let X be the \perp^I distance of A from the line of action of $\frac{W}{2} \sin \alpha$

For equilibrium, taking moments about A

$$\frac{W}{2} \sin \alpha = W \cdot AD = W \cdot a \sin \alpha$$

$$(ie) \therefore x = 2a.$$

∴ The force $\frac{W \sin \alpha}{2}$ must be at a distance $2a$ from A. So it must act at B \parallel to the plane.

Resolving along and \perp to the plane ,

We have

$$F + \frac{1}{2} W \sin \alpha \cdot AD = W \cdot a \sin \alpha$$

$$(ie) F = \frac{W \sin \alpha}{2} \quad (1)$$

$$\text{and } R = W \cos \alpha \quad (2)$$

$$\therefore \frac{F}{R} = \frac{W}{2} \sin \alpha \div W \cos \alpha = \frac{\tan \alpha}{2}$$

For equilibrium $\frac{F}{R}$ must be $< \mu$

$$(ie) \frac{\tan \alpha}{2} < \mu$$

$$(or) \mu > \frac{\tan \alpha}{2}.$$

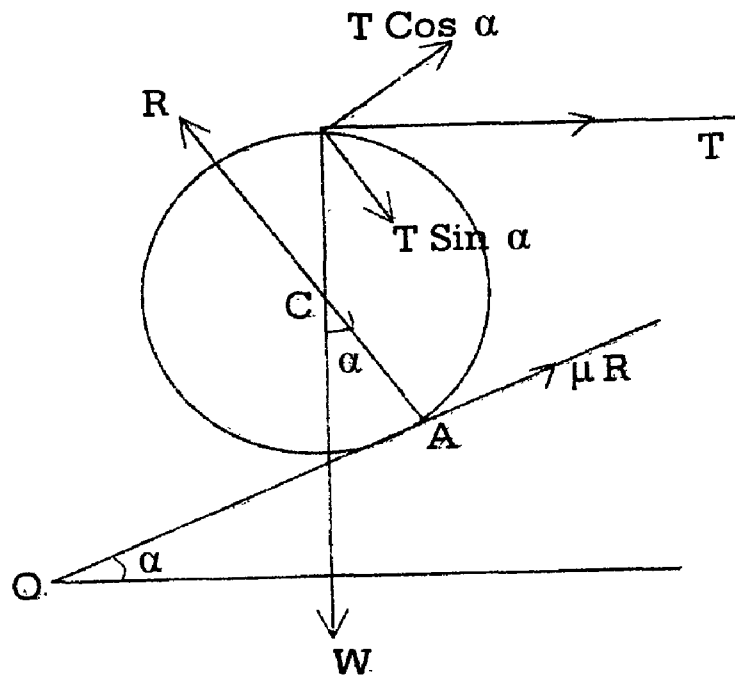
Example: 3

A sphere of weight W resting on a rough inclined plane at an angle α to the horizontal, is maintained in equilibrium by a horizontal string attached to the highest point of the sphere. Show that the angle of friction is greater than $\frac{1}{2} \alpha$ and that the tension of the string is $w \tan \frac{1}{2} \alpha$.

Solution:-

Let the sphere rest on the inclined plane, touching it at A.

At the highest point on the sphere a string is attached and the sphere is pulled parallel to the horizontal plane



Let T be the tension in the string. Its components along and \perp to the inclined plane are $T \cos \alpha$ and $T \sin \alpha$.

Let us consider the case of limiting equilibrium:

Resolving the forces along and \perp to the inclined plane:

We get

$$T \cos \alpha + \mu R = W \sin \alpha$$

$$\text{and } T \sin \alpha + W \cos \alpha = R$$

$$\text{These give } T \cos \alpha + \mu (T \sin \alpha + W \cos \alpha) = W \sin \alpha$$

$$\text{(Or) } T = W \frac{\sin \alpha - \mu \cos \alpha}{\cos \alpha + \mu \sin \alpha} = W \frac{\sin (\alpha - \lambda)}{\cos (\alpha - \lambda)} \text{ as } \mu = \tan \lambda$$

Also taking moments about A.

We get

$$T (a + a \cos \alpha) = W \cdot a \sin \alpha$$

$$\text{(Or) } T = W \frac{\sin \alpha}{1 + \cos \alpha} = W \tan \frac{\alpha}{2}$$

Equating the two values of T

We get

$$W \frac{\sin(\alpha - \lambda)}{\cos(\alpha - \lambda)} = W \cdot \tan \frac{\alpha}{2}$$

$$\text{(Or) } \tan(\alpha - \lambda) = \tan \frac{\alpha}{2}$$

$$\text{(ie) } (\alpha - \lambda) = \frac{\alpha}{2} \text{ or } \lambda = \frac{1}{2} \alpha$$

(for limiting equilibrium)

For equilibrium,

$$\therefore \lambda > \frac{1}{2} \alpha$$

Example :4

A uniform rod of length 2ℓ rests within a hollow sphere of radius a in a vertical plane through the centre of the sphere. The sphere is rough, the angle of friction being λ . Prove that if $\ell < a \cos \lambda$ the greatest inclination to the horizontal at which the rod can rest is given by $(2\ell^2 - a^2) \sin \theta - a^2 \sin(\theta - 2\lambda) = 0$.

Solution:

The section of the hollow sphere by the vertical plane is a circle and the rod AB rests with its lower end A and upper end B on this circle.

Let the centre of the circle be C. Then

$$\angle ACB = 2\alpha \text{ Say.}$$

$$\text{So that } \angle CAB = \angle CBA = 90^\circ - \alpha$$

Let the inclination of the rod to the horizontal be θ when the rod is in limiting equilibrium. Evidently this value of θ is its greatest value.

The reactions at A and B make the same angle λ (the angle of friction) with the normal to the circle at these points.

Let the reactions meet at O. Then there being only three forces in equilibrium, the line of action of the weight \vec{W} of the rod acting at its midpoint G must pass through O.

(ie) OG is vertical.

$$\text{Hence } \angle OGB = 90^\circ - \theta$$

$$\text{In } \Delta OAB, \angle OAB = (90 - \alpha) - \lambda$$

$$\text{as } \angle CAB = 90^\circ - \alpha \text{ and } \angle OBA = 90^\circ - \alpha + \lambda$$

Hence applying the trigonometrical result to the ΔOAB ,

We get

$$(AG + GB) \cot (90^\circ - \theta) = GB \cot (90^\circ - \alpha - \lambda) - AG \cot (90^\circ - \alpha - \lambda)$$

$$(\text{or}) 2\ell \tan \theta = \ell [\tan (\alpha + \lambda) - \tan (\alpha - \lambda)].$$

$$(\text{ie}) \tan \theta = \frac{\sin 2\lambda}{\cos 2\alpha + \cos 2\lambda} \text{ on simplification}$$

$$\text{But } \sin \alpha = \frac{\ell}{a},$$

$$\text{So that } \cos 2\alpha = 1 - 2 \sin^2 \alpha = 1 - \frac{2\ell^2}{a^2}$$

$$\text{Hence } \frac{\sin \theta}{\cos \theta} = \frac{\sin 2\lambda}{1 - 2\ell^2/a^2 + \cos 2\lambda}$$

$$(\text{or}) \sin \theta (a^2 - 2\ell^2 + a^2 \cos 2\lambda) = a^2 \cos \theta \sin 2\lambda$$

$$\text{giving } (2\ell^2 - a^2) \sin \theta - a^2 (\sin \theta \cos 2\lambda - \cos \theta \sin 2\lambda) = 0$$

$$(\text{or}) (2\ell^2 - a^2) \sin \theta - a^2 \sin (\theta - 2\lambda) = 0$$

Note that in this position

$$\lambda < 90^\circ - \alpha \text{ which gives } \cos \lambda < \frac{1}{a} \text{ or } < a \cos \lambda$$

$$\text{(or) } \rho < a \cos \lambda.$$

Example: 5

A heavy circular disc whose plane is vertical is kept at rest on a rough inclined plane by a string parallel to the plane and touching the circle. Show that the disc will slip on the plane if the coefficient of friction be less than $\frac{1}{2} \tan a$ where a is the inclination of the plane.

Solution:-

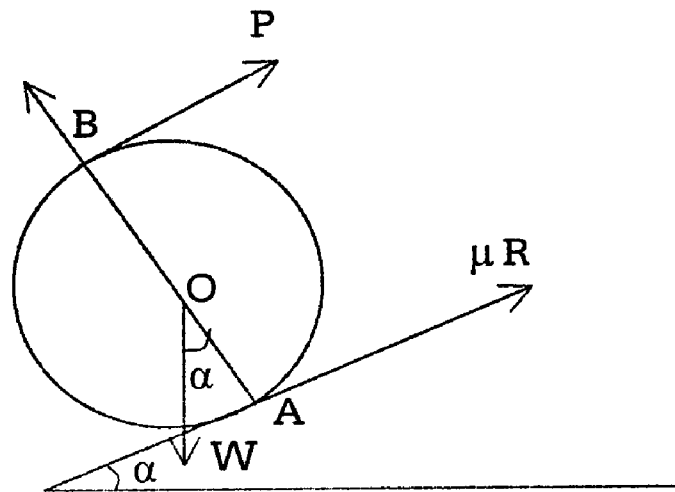
Let the circular disc centre O , be kept at rest by pulling the string in a direction parallel to the plane, as shown in the diagram.

Resolving the force that act on the disc along and \perp to the plane.

We get

$$R = W \cos a \text{ and } W \sin a = P + \mu R$$

Taking moments about A ,



We get

$$P \cdot AB = W \cdot OA \sin a$$

$$P = \frac{1}{2} W \cdot \sin a$$

The disc will slip (slide) if the component $W \sin a$ of the weight of the disc acting down the plane; exceeds the frictional and applied forces acting up the plane.

Hence the condition for the disc to slip is

$$W \sin a > \mu R + P.$$

$$\text{Or } W \sin a > \mu \cdot W \cos a + \frac{1}{2} W \sin a$$

giving $\tan a > 2\mu$

$$(\text{or}) \mu < \frac{1}{2} \tan a$$

Exercise:

1. How high can a particle rest inside a hollow sphere of radius a if the coefficient of friction be $\frac{1}{\sqrt{3}}$?
2. A uniform rod rests in a vertical plane within a fixed hemispherical bowl radius is equal to the length of the rod. If μ be the coefficient of friction between the rod and the bowl. Show that, in limiting equilibrium, the inclination of the rod to the horizontal is $\tan^{-1} \left(\frac{4\mu}{3 - \mu^2} \right)$.
3. A heavy uniform rod rests with its ends on the interior of a rough vertical circle and subtends a right angle at the centre. Show that in the position of limiting equilibrium, the rod is inclined to the horizontal at an angle equal to double the angle of friction.

Example: 1(U.Q)

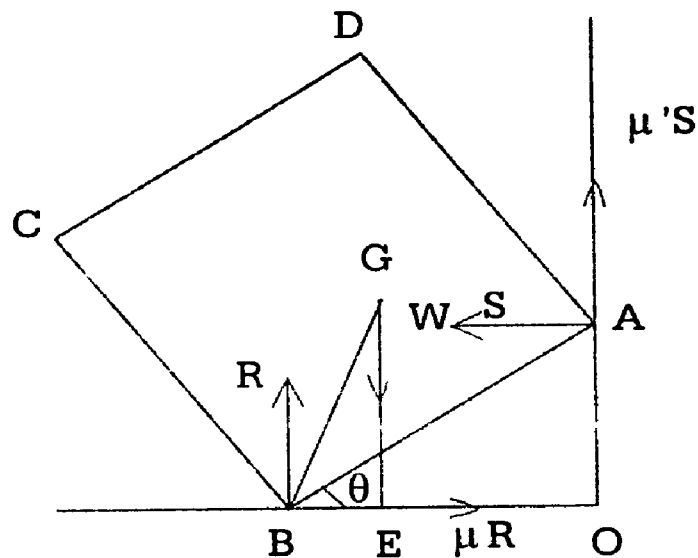
A square lamina whose plane is vertical rests with the ends of a side against a rough vertical wall and a rough horizontal ground. If the coefficients of friction for the ground and the wall be μ and μ' respectively prove that, when lamina is on the point of motion, the inclination of the side in question to the

horizontal is $\tan^{-1} \left(\frac{1 - \mu\mu'}{1 + 2\mu + \mu\mu'} \right)$

Solution:

Let ABCD be the square lamina and the side AB (=a) be in a contact with the wall at A and the ground at B. The forces acting on the lamina are:

- i) normal reaction R at B \perp to the ground
- ii) limiting friction μR at B as shown in the figure.
- iii) Normal reaction S at A \perp to the wall.



- iv) Limiting friction $\mu' s$ along the wall upwards and
- v) Its weight w acting vertically downwards at G its centre of gravity.

Let AB make an angle θ to the horizontal

Resolving horizontally and vertically,

$$S = \mu R \tag{1}$$

$$R + \mu' S = W \tag{2}$$

Taking the moments about B

$$S \cdot OA + \mu' S \cdot OB = W \cdot BE$$

$$(ie) S \cdot a \sin \theta + \mu' S \cdot a \cos \theta = W \cdot BG \cos (45^\circ + \theta)$$

$$= \frac{Wa}{\sqrt{2}} (\cos 45^\circ \cos \theta - \sin 45^\circ \sin \theta)$$

$$= \frac{Wa}{2} (\cos \theta - \sin \theta)$$

$$(ie) S (\sin \theta + \mu' \cos \theta) = \frac{W}{2} (\cos \theta - \sin \theta) \quad (3)$$

From (2) $R = W - \mu' S$ and putting this in (1)

We have $S = \mu (W - \mu' S)$

$$(ie) S = \frac{\mu W}{1 + \mu\mu'} \quad (4)$$

Substituting (4) in (3)

$$\frac{\mu W}{1 + \mu\mu'} (\sin \theta + \mu' \cos \theta) = \frac{W}{2} (\cos \theta - \sin \theta)$$

$$(ie) 2\mu (\sin \theta + \mu' \cos \theta) = (1 + \mu\mu') (\cos \theta - \sin \theta)$$

$$\sin \theta (2\mu + 1 + \mu\mu') = \cos \theta (1 + \mu\mu' - 2\mu\mu')$$

$$\therefore \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{1 - \mu\mu'}{1 + 2\mu + \mu\mu'}$$

$$\therefore \theta = \tan^{-1} \left(\frac{1 - \mu\mu'}{1 + 2\mu + \mu\mu'} \right)$$

Example: 2 (U.Q)

A solid homogeneous hemisphere rests on a rough horizontal plane and against a smooth vertical wall, show that if the coefficient of friction be greater than $\frac{3}{8}$, the hemisphere can rest in any position and, if it be less, the least angle

that the base of the hemisphere can make with the vertical is $\cos^{-1} \left(\frac{8\mu}{3} \right)$.

If the wall be rough (coefficient of friction μ') show that the angle is

$$\cos^{-1} \frac{8\mu(1 + \mu\mu')}{3(1 + \mu\mu')}$$

Solution:

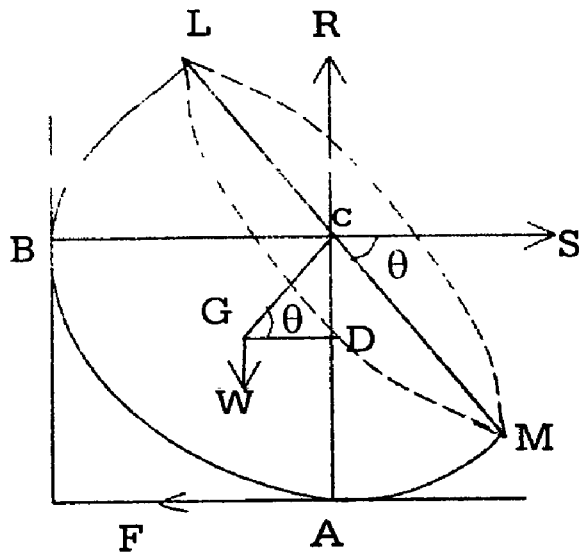


Fig. (a)

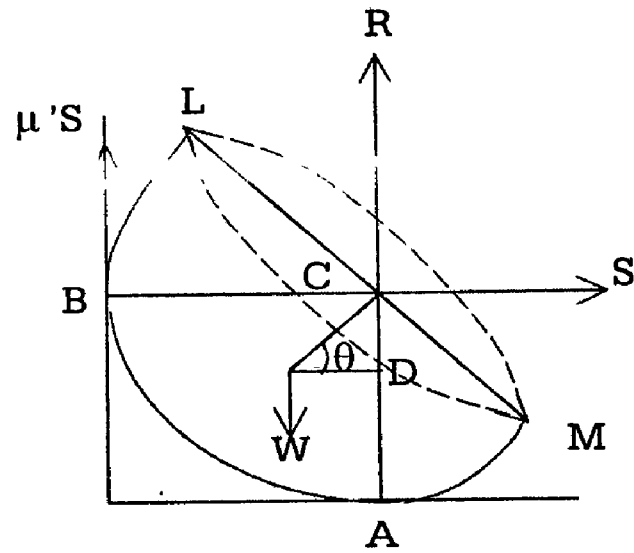


Fig. (b)

Figure represents a section of the hemisphere by the plane \perp^r to the wall. A is the point of contact with rough horizontal plane and B with the smooth wall LM is the base and C is the centre of hemisphere.

The centre of gravity is at G on the radius \perp^r to LM. Such the

$$CG = \frac{3r}{8}, \text{ r being the radius of the hemisphere}$$

The forces acting are,

- i) reaction R \perp^r to the ground at A
- ii) friction F at A.
- iii) reaction S \perp^r to the smooth wall at B and
- iv) Weight W of the hemisphere acting vertically downwards at G.

Let $\angle ACM = \theta$

Resolving horizontally and vertically,

$$S = F \tag{1}$$

$$R = W \tag{2}$$

Taking moments about A,

$$\begin{aligned} S \cdot CA &= W \cdot GD \\ &= W \cdot CG \cos \theta \end{aligned}$$

$$S \cdot r = W \cdot \frac{3r}{8} \cos \theta$$

$$\text{(or) } S = \frac{3W}{8} \cos \theta \quad (3)$$

From (2) and (3)

$$F = \frac{3W \cos \theta}{8} \text{ and } R = W \text{ from (2)}$$

$$\therefore \frac{F}{R} = \frac{3W \cos \theta}{8W} = \frac{3 \cos \theta}{8}$$

For equilibrium,

$\frac{F}{R}$ must be $\leq \mu$, the coefficient of friction.

$$\text{(ie) } \frac{3 \cos \theta}{8} \leq \mu,$$

$$\text{(or) } \mu \geq \frac{3}{8} \cos \theta \quad (4)$$

If μ is $\frac{3}{8}$, then as $\cos \theta$ is < 1 , μ will be automatically $\geq \frac{3}{8} \cos \theta$.

Hence condition (4) is satisfied and the hemisphere can rest in any position.

$$\text{From (4) } \cos \theta \leq \frac{8\mu}{3}.$$

$$\text{If } \mu < \frac{3}{8}, \frac{8\mu}{3} < 1$$

Let α be the angle such that $\text{Cos } \alpha = \frac{8\mu}{3}$

Then $\text{Cos } \theta \leq \text{Cos } \alpha$ for equilibrium.

$$\text{(ie) } \theta \geq \alpha$$

$$\text{(or) } \theta \geq \text{Cos}^{-1} \frac{8\mu}{3}$$

Hence the least value of θ is $\text{Cos}^{-1} \left(\frac{8\mu}{3} \right)$

When the wall is rough, limiting frictions μR and $\mu' S$ act at A and B refer to figure.

Resolving horizontally and vertically.

$$S = \mu R \tag{5}$$

$$R + \mu' S = W \tag{6}$$

Taking moments about A,

$$S \cdot CA + \mu' S \cdot BC = W \cdot GD$$

$$\text{(or) } Sr + \mu' S \cdot r = W \cdot \frac{3r}{8} \text{Cos } \theta$$

$$S (1 + \mu') = \frac{3W}{8} \text{Cos } \theta \tag{7}$$

Putting the value of from (6) in (5)

We have

$$S = \mu (W - \mu' S)$$

$$\text{(ie) } S = \frac{\mu W}{1 + \mu \mu'} \tag{8}$$

Substituting (8) in (7) θ

$$\text{We have } \frac{\mu W}{1 + \mu\mu'} (1 + \mu') = \frac{3W}{8} \cos \theta$$

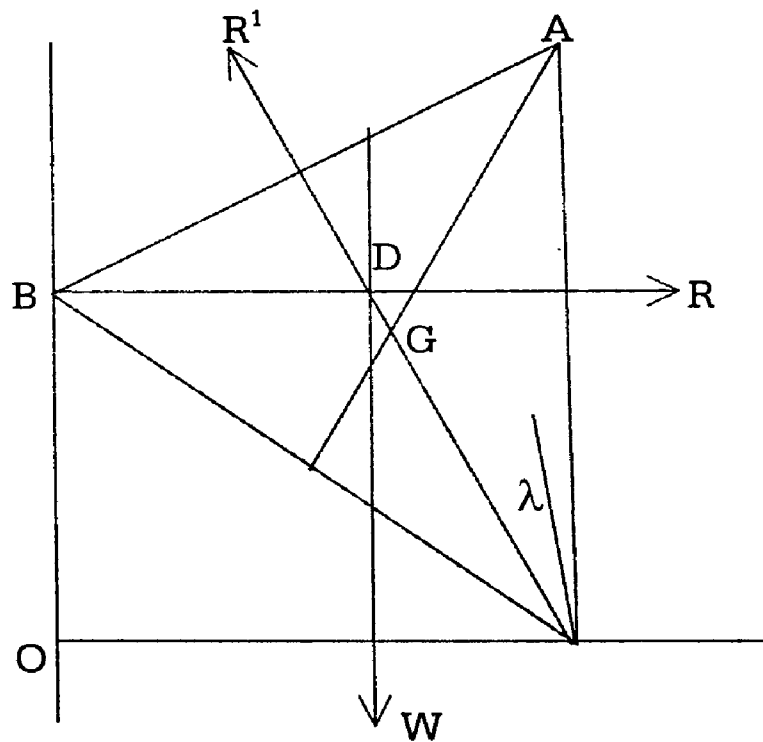
$$\text{(ie) } \cos \theta = \frac{8\mu(1 + \mu')}{3(1 + \mu\mu')} \text{ which gives the least angle } \theta$$

Example: 3 (U.Q)

A thin equilateral triangular plate of uniform density rests in a vertical plane with one end of its base on a rough horizontal floor and the other end against a smooth vertical wall. Show that the least angle its base can make with the horizontal is given by $\cot \theta = 2\mu + \frac{1}{\sqrt{3}}$, where μ is the coefficient of friction.

horizontal is given by $\cot \theta = 2\mu + \frac{1}{\sqrt{3}}$, where μ is the coefficient of friction.

Solution:



Let ABC be the plate of weight \vec{W} with its one vertex B on the smooth vertical wall and another end C on the horizontal floor.

The reaction \vec{R} at B is \perp^r to OB . Let the weight \vec{W} of the plate acting through G meet \vec{R} at D . Then the reaction \vec{R}' at C must pass through D , there being equilibrium.

Since the equilibrium is limiting, \vec{R}' makes the angle λ (the angle of friction with the vertical at C).

Let $\angle BCO = \theta$.

$$\text{From } \Delta BCD, \frac{BD}{\sin \angle BCD} = \frac{BC}{\sin \angle BDC}$$

$$\text{(Or)} \frac{BD}{\sin(90^\circ - \theta - \lambda)} = \frac{BC}{\sin(90^\circ + \lambda)} \text{ as } \angle CDG = \lambda$$

$$\text{(ie)} BD = BC \frac{\cos(\theta + \lambda)}{\cos \lambda}$$

$$\text{But } BD = BG \cos \angle GBD = \frac{1}{\sqrt{3}} BC \cos(\theta - 30^\circ)$$

$$\text{as } \angle GBD = \angle CBD - \angle CBG$$

$$\text{Hence } \frac{1}{\sqrt{3}} BC \cos(\theta - 30^\circ) = BC \cdot \frac{\cos(\theta + \lambda)}{\cos \lambda}$$

$$\text{(Or)} \frac{1}{\sqrt{3}} \left[\cos \theta \cdot \frac{\sqrt{3}}{2} + \sin \theta \cdot \frac{1}{2} \right] = \frac{\cos \theta \cos \lambda - \sin \theta \sin \lambda}{\cos \lambda}$$

$$\text{Thus } \frac{1}{2} \cos \theta = \sin \theta \left[\tan \lambda + \frac{1}{2\sqrt{3}} \right]$$

$$\text{(or)} \cot \theta = 2\mu + \frac{1}{\sqrt{3}}$$

Example:4

A solid hemisphere rests in equilibrium on a rough ground and against an equally rough vertical, the coefficient of friction being μ . Show that if the equilibrium is limiting, the inclination of the base to the horizon is

$$\sin^{-1} \left[\frac{8\mu(1+\mu)}{3(1+\mu^2)} \right]$$

Solution:-

Let the hemisphere of radius r have contact with the ground and the wall at A and B.

The weight reactions \vec{R} and \vec{S} at A and B act through the centre O of the base $\vec{\mu R}$ and $\vec{\mu S}$ act as in the diagram.

$$\angle LOB = \theta; \text{ then } \angle GOC = \theta$$

Resolving the forces horizontally and vertically,

We get

$$S = \mu R \text{ and } R + \mu S = W$$

$$S = \mu (W - \mu S) \text{ giving } S = \frac{\mu W}{1 + \mu^2}$$

Also taking moments about A,

We get

$$S \cdot r + \mu S \cdot r = W \cdot GC = W (OG \sin \theta)$$

$$= W \cdot \frac{3}{8} r \sin \theta$$

$$\text{(Or) } \sin \theta = \frac{8}{3W} (1 + \mu) S$$

$$= \frac{8}{3W} (1 + \mu) \cdot \frac{\mu W}{1 + \mu^2} = \left[\frac{8\mu(1 + \mu)}{3(1 + \mu^2)} \right]$$

$$\text{Thus } \theta = \sin^{-1} \left[\frac{8\mu(1 + \mu)}{3(1 + \mu^2)} \right]$$

Exercise:

1. A square Lamina rests in a vertical plane with one end of one side against a rough wall and the other end against an equally rough horizontal ground. Show that the edge of the lamina in question is inclined to the horizontal at angle $\tan^{-1} \left(\frac{1-\mu}{1+\mu} \right)$ where μ is the coefficient of friction at each end.
2. A uniform thin hemispherical bowl rests with its curved surface on a rough horizontal plane (coefficient of friction μ) and leans against a smooth vertical wall, prove that when the bowl is on the point of slipping the inclination of the axis of the bowl to the vertical is $\sin^{-1} (2\mu)$.
3. A hemispherical shell rests with its curved surface on a rough plane. Whose angle of friction is λ . Show that the inclination of the plane base of the rim to the horizon cannot be greater than $\sin^{-1} (2 \sin \lambda)$.

CONE OF FRICTION:

Cone:

Example:1

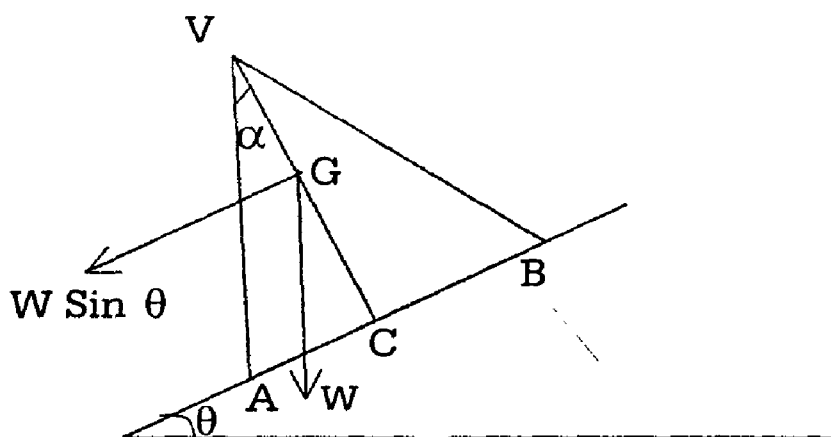
A uniform cone is placed on a rough inclined plane. As the inclination of the plane increases. Show that the cone will slide before it topples over if the coefficient of friction is less than $4 \tan \alpha$. Where α is the semi-vertical angle of the cone.

If $\frac{1}{\sqrt{3}}$ be the coefficient of friction, find the angle of the cone when it is on the point of both slipping and turning over.

Solution:

Let AVB represent the section of the cone by a vertical plane through its centre of gravity g.

$$\text{Then } GC = \frac{1}{4} VC.$$



When the inclination of the plane is θ . Let the cone be on the point of sliding.

The resultant normal reaction \vec{R} acts at a point on AB (the whole of AB being in contact with the plane) and so for sliding. We must have $W \sin \theta > \mu R$

But resolving \perp^r to the plane,

We get $R = W \cos \theta$.

Hence the cone will slide if $W \sin \theta > \mu \cdot W \cos \theta$

$$\text{(Or) } \tan \theta > \mu \quad (1)$$

In the case when toppling is about to take place (which can happen only about A), the normal reaction and the frictional force act through A.

Hence on taking moments about A,

We get the condition for the cone to topple as $W \sin \theta \cdot GC > W \cos \theta \cdot AC$

$$\text{(Or) } \tan \theta > \frac{AC}{GC} = 4 \frac{AC}{VC} \text{ as } GC = \frac{1}{4} VC$$

$$\text{(ie) } \tan \theta > 4 \tan \alpha \quad (2)$$

From (1) and (2)

We find the cone will slide before it topples over if $4 \tan \alpha < \mu$ or $\mu < 4 \tan \alpha$.

Sliding and toppling over will occur simultaneously if $\mu = 4 \tan \alpha$

$$\text{But } \mu = \frac{1}{\sqrt{3}}$$

$$\text{Hence } 4 \tan \alpha = \frac{1}{\sqrt{3}} \text{ (or) } \alpha = \tan^{-1} \frac{1}{4\sqrt{3}}$$

\therefore The angle of cone is

$$2\alpha = 2 \tan^{-1} \frac{1}{4\sqrt{3}}$$

Example: 2

A heavy solid right circular cone is placed with its base on a rough inclined plane, the inclination of which is gradually increased. Determine whether the initial motion of the cone will be one of sliding or tumbling.

Solution:

As the previous problem, we find that if the angle of inclination θ is steadily increased, the cone will slide initially if $\mu < \tan \theta$ (ie) sliding will occur before toppling over).

So long as $\mu < 4 \tan \alpha$, where α is the semi vertical angle of the cone there will be sliding only.

If θ is such that $\tan \theta = 4 \tan \alpha$, sliding and toppling over occur simultaneously.

Exercise:

1. A cone of vertical angle 2α , rests with its base on a rough plane inclined to the horizon. As the inclination of the plane is gradually increased, show that the cone will slide before it topples over, if the coefficient of friction be less than $4 \tan \alpha$.

Show that it will slide & topple over if the coefficient of friction is $< \frac{4r}{h}$, r being the base radius & h the height what happens if the coefficient.

2. A uniform solid cone with circular base is placed with its plane base on a rough inclined plane and the inclination to the horizon is gradually increased. Find the ratio of the radius of the base of the cone to its height if the cone is to topple before it slides. Coefficient of friction = μ
3. A uniform cone, resting with its circular base in contact with a rough inclined plane (angle α) is on the point of both slipping and toppling. If the angle of friction be λ , find α and the vertical angle of the cone in terms of λ .
4. A right cone is placed with its base on a rough inclined plane. If $\frac{1}{\sqrt{3}}$ be the coefficient of friction, find the angle of the cone when it is on the point of both slipping and turning over.

DYNAMICS

UNIT – VI

PROJECTILES

6.1 Projectiles:

Consider motion of a particle projected into the air in any direction and with any velocity. Such a particle is called a projectile.

The two forces that act on the projectile are its weight and the resistance of air.

For simplicity, we suppose the motion to take place with such a moderate distance from the surface of the earth that we can neglect the variations in the acceleration due to gravity. This means that g may be considered to be constant in magnitude throughout the motion of the projectile. Secondly we shall neglect the resistance of the air and consider the motion to take place in vacuum.

6.1.1. Definitions:

The following terms are used in connection with projectiles.

The angle of projection is the angle that the direction in which the particle is initially projected makes with the horizontal plane through the point of projection.

The velocity of projection is the velocity with which the particle is projected.

The trajectory is the path which the particle describes.

The Range on a plane through the point projection is the distance between the point of projection and the point where the trajectory meets that plane.

The time of flight is the interval of time that elapses from the instant of projection till the instant when the particle again meets the horizontal plane through the point of projection.

6.2 PATH OF PROJECTILE

To show that the path of a projectile is a parabola.

Let a particle be projected from O with a velocity u at an angle α to the horizon.

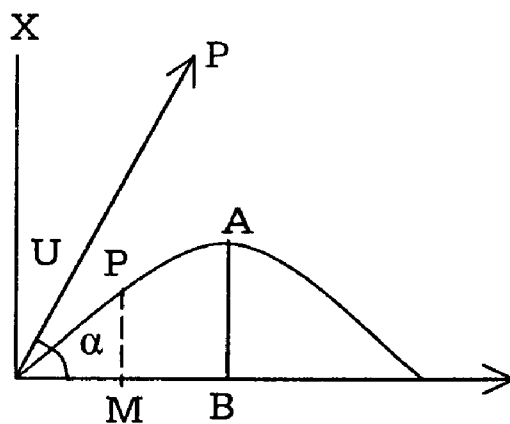


Fig (a)

Take O as the origin, the horizontal and the upward vertical through O as axes of X and Y respectively. The initial velocity u can be split into two components, which are $u \cos \alpha$ in the horizontal direction and $u \sin \alpha$ in the vertical direction. The horizontal component $u \sin \alpha$ subject to an acceleration g downwards.

Let $p(x, y)$ be the position of the particle at time t secs after projection. Then

$$x = \text{horizontal distance described in } t \text{ secs} = (u \cos \alpha).t \quad (1)$$

$$y = \text{vertical distance described in } t \text{ secs} = (u \sin \alpha) t - \frac{1}{2}gt^2 \quad (2)$$

(1) and (2) can be taken as the parametric equations of the trajectory. The equation to the path is got by eliminating t between them

From (1)

$$t = \frac{x}{u \cos \alpha}$$

And putting this in (2)

$$\text{We get } y = u \sin \alpha \cdot \frac{x}{u \cos \alpha} - \frac{1}{2}g \cdot \left(\frac{x}{u \cos \alpha} \right)^2$$

$$y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha} \quad (3)$$

Multiplying (3) by $2u^2 \cos^2 \alpha$.

$$2u^2 \cos^2 \alpha \lambda = 2u^2 \cos^2 \alpha x \frac{\sin \alpha}{\cos \alpha} - gx^2$$

$$(ie) x^2 - \frac{2u^2 \sin \alpha \cos \alpha}{g} x = - \frac{2u^2 \cos^2 \alpha}{g} y.$$

$$\begin{aligned} \left(x - \frac{u^2 \sin \alpha \cos \alpha}{g^2} \right)^2 &= \frac{u^4 \sin^2 \alpha \cos^2 \alpha}{g^2} - \frac{2u^2 \cos^2 \alpha}{g} y \\ &= \frac{2u^2 \cos^2 \alpha}{g} \left(y - \frac{u^2 \sin^2 \alpha}{2g} \right) \end{aligned}$$

Transfer the origin to the point

$$\left(\frac{u^2 \sin \alpha \cos \alpha}{g^2}, \frac{u^2 \sin^2 \alpha}{2g} \right)$$

The above equation then becomes

$$X^2 = - \frac{2u^2 \cos^2 \alpha}{g} Y \quad (4)$$

(4) is clearly the equation to a parabola latus rectum $\frac{2u^2 \cos^2 \alpha}{g}$ whose axis is vertical and downwards and whose vertex is the point

$$\left(\frac{u^2 \sin \alpha \cos \alpha}{g^2}, \frac{u^2 \sin^2 \alpha}{2g} \right)$$

Note:

The latus rectum of the above Parabola is

$$= \frac{2u^2 \cos^2 \alpha}{g} = \frac{2}{g} (u \cos \alpha)^2$$

$$= \frac{2}{g} \times (\text{Square of the horizontal velocity}).$$

So the latus rectum (ie) the size of the parabola) is independent of the initial vertical velocity and depends only on the horizontal velocity.

6.2 Characteristics of the motion of a Projectile:-

Refer to fig. (a)

Let a particle be projected from O with velocity u at an angle α to the horizontal OX. Let A be the highest point of the path & C the point through O. Using the two fundamental principles given in (i) we can drive the following results relating to the motion of a projectile.

1) Greatest height attained by a projectile.

At A the highest point, the particle will be moving only horizontally, having lost all its vertical velocity.

Let $AB = h =$ the greatest height reached. Considering vertical motion separately, initial upward vertical velocity = $u \sin \alpha$ and the acceleration in this direction is $-g$.

The final vertical velocity at A is = 0

$$\text{Hence } 0 = (u \sin \alpha)^2 - 2g.h.$$

$$\text{(ie) } h = \frac{u^2 \sin^2 \alpha}{2g}$$

(ie) The vertex of the parabola is the highest point of the path.

2) Time taken to reach the greatest height

Let T be the time from O to A. Then, in time T, the initial velocity $u \sin \alpha$ is reduced to zero, acted on by an acceleration $-g$.

$$\text{Hence } 0 = u \sin \alpha - gT.$$

$$T = \frac{u \sin \alpha}{g}$$

3) Time of flight (ie) the time taken to return to the same horizontal level as O.

When the particle arrives at O, the effective vertical distance it has described is zero.

Hence if t is the time of flight, considering vertical motion,

we have

$$0 = u \sin \alpha \cdot t - \frac{1}{2} g t^2$$

$$(ie) t = 0 \text{ or } t = \frac{2u \sin \alpha}{g}$$

$t = 0$ is the instant of projection when also the vertical distance travelled is zero.

$$\therefore \text{The time of flight} = \frac{2u \sin \alpha}{g}$$

We find that the time of flight is twice the time taken to reach the highest point, as we should expect from symmetry.

4. The range on the horizontal plane through the point of projection.

$$\text{The time of flight in } t = \frac{2u \sin \alpha}{g}$$

During this time, the horizontal velocity remains constant and is equal to $u \cos \alpha$.

Hence

OC = horizontal distance described in time t .

$$= u \cos \alpha \cdot t = u \cos \alpha \cdot \frac{2u \sin \alpha}{g}$$

$$= \frac{2u^2 \sin \alpha \cos \alpha}{g}$$

Hence

$$\text{The horizontal range } R = \frac{2u^2 \sin \alpha \cos \alpha}{g} = \frac{u^2 \sin 2\alpha}{2g}$$

Note:-

(1) The horizontal range can also be found thus:

The equation to the path is

$$Y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha} \quad (1)$$

The equation to the x – axis is $y = 0$. Putting $y = 0$ in (1)

$$\text{We have } x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha} = 0$$

$$\text{(ie) } x = 0 \text{ or } X = \frac{2u^2 \cos^2 \alpha \tan \alpha}{g} = \frac{2u^2 \sin \alpha \cos \alpha}{g}$$

$X = 0$, Corresponds to the point of projection and so the other value

$$\frac{2u^2 \sin \alpha \cos \alpha}{g} \text{ gives the horizontal range.}$$

$$2) \text{ Horizontal range} = \frac{2u^2 \sin \alpha \cos \alpha}{g}$$

$$= \frac{2(u \cos \alpha)(u \sin \alpha)}{g}$$

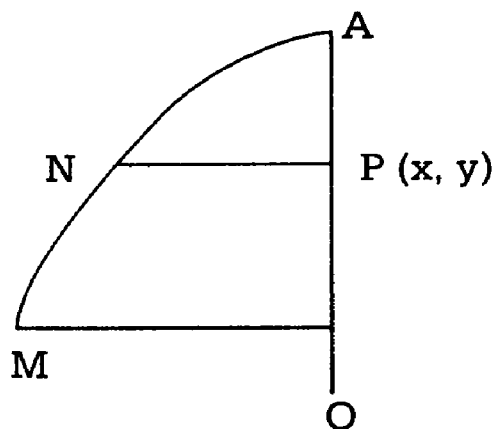
$$= \frac{2UV}{g} \text{ where } U \text{ and } V$$

are the initial horizontal and vertical velocities.

6.2.3 A particle is projected horizontally from a point at a certain height above the ground to show that the path described by it is a parabola.

Let a particle be projected horizontally with a velocity u from a point A at a height h above the ground level. Let it strike the ground at M. Take A as origin the horizontal through A as x – axis and the downwards vertical through A as Y

axis. Let P (X,Y) be the position of the particle at time t. As there is no horizontal acceleration, the horizontal velocity remains constant throughout the motion.



So $x =$ horizontal distance described in time $t = ut$ (1)

Due to gravity, the vertical acceleration during the motion is g downwards.

$Y =$ vertical distance, described in time $t = \frac{1}{2} gt^2$ (2)

Eliminate t between (1) & (2)

We have

$$Y = \frac{1}{2} g \cdot \frac{x^2}{y^2} \text{ (ie) } x^2 = \frac{2u^2}{g} y$$
 (3)

(3) shows that Y is a quadratic function of X .

So it represents a parabola with vertex at A and Axis AN .

Worked Examples

Example: 1

A body is projected with a velocity of 98 metres per sec in a direction making an angle $\tan^{-1}3$ with the horizon. Show that it rises to a vertical height of 441 metres and that its time of flight is about 19 secs. Find also the horizontal range through the point of projection. [$g = 9.8 \text{ m/sec}^2$]

Solution:

Here $u = 98_m$, $\alpha = \tan^{-1}3$. (ie) $\tan \alpha = 3$

$$\begin{aligned} \therefore \sin \alpha &= \frac{\sin \alpha}{\cos \alpha} \cdot \cos \alpha = \frac{\tan \alpha}{\sec \alpha} = \frac{\tan \alpha}{\sqrt{1 + \tan^2 \alpha}} \\ &= \frac{3}{\sqrt{1+9}} = \frac{3}{\sqrt{10}} \end{aligned}$$

$$\cos \alpha = \frac{\sin \alpha}{\tan \alpha} = \frac{1}{\sqrt{10}}$$

$$\begin{aligned} \text{greatest height reached} &= \frac{u^2 \sin^2 \alpha}{g} = \frac{98 \times 98 \times 9}{10 \times 2 \times 9.8} \\ &= 441 \text{ metres} \end{aligned}$$

$$\begin{aligned} \text{Time of flight} &= \frac{2u \sin \alpha}{g} = \frac{2 \times 98 \times 3}{\sqrt{10} \times 9.8} \\ &= 6 \sqrt{10} = 6 \times 3.162, \\ &= 18.972 \\ &= 19 \text{ secs, nearly} \end{aligned}$$

$$\begin{aligned} \text{Horizontal range} &= \frac{2u^2 \sin \alpha \cos \alpha}{g} \\ &= \frac{2 \times 98 \times 3}{9.8} \times \frac{3}{\sqrt{10}} \times \frac{1}{\sqrt{10}} \\ &= 588 \text{ metres.} \end{aligned}$$

Example: 2 (U.Q)

If the greatest height attained by the particle in a quarter of its range on the horizontal plane through the point of projection, find the angle of projection.

Solution:

Let u be the initial velocity and α the angle of projection.

Then the greatest height = $u^2 \sin^2 \alpha / 2g$.

And horizontal range = $2u^2 \sin \alpha \cos \alpha / 2g$.

It is given that $\frac{u^2 \sin^2 \alpha}{2g} = \frac{1}{4} \times \frac{2u^2 \sin \alpha \cos \alpha}{g}$

$$(ie) \frac{u^2 \sin^2 \alpha}{2g} = \frac{u^2 \sin \alpha \cos \alpha}{2g}$$

$$(ie) \sin \alpha = \cos \alpha \text{ (or) } \tan \alpha = 1$$

$$\therefore \alpha = 45^\circ.$$

Example: 3

A stone is thrown with a velocity of 39.2m/sec at 30° to the horizontal. Find at what times it will be at height of 14.7 ($g = 9.8 \text{ m/sec}^2$).

Solution:

$$\begin{aligned} \text{Initial vertical velocity} &= 39.2 \times \sin 30^\circ \\ &= 19.6 \text{ m/sec.} \end{aligned}$$

This is subject to an acceleration $-g$.

Let the particle be at a height 14.7m after time t sec.

Applying the formula " $s = ut + \frac{1}{2} at^2$ ".

$$\text{We have } 14.7 = 19.6t - \frac{1}{2}gt^2$$

$$= 19.6t - 4.9t^2$$

$$(ie) 3 = 4t - t^2 \text{ or } t^2 - 4t + 3 = 0$$

$$(ie) (t-3)(t-1) = 0; t=1 \text{ or } t=3.$$

Hence at the end of 1 sec. And again at the end of 3secs.

It will be at a height of 14.7m

Example: 4

A bomb was released from an aero plane when it was at a height of 1960m. above a point A on the ground and was moving horizontally with a speed of 100m per sec. Find the distance from A of the point where the bomb strikes the ground. ($g = 9.8\text{m/sec}^2$).

Solution:

Let us consider the motion of the bomb in the horizontal and the vertical directions separately. The dynamical details of each motion may be presented as follows:-

Horizontal Motion

initial velocity = 100m/Sec

Acceleration = 0

Vertical Motion

(upwards +ve)

Initial velocity = 0

Acceleration = -9.8m/Sec^2

Distance = -1960m (It is downwards)

Let t be the time taken by the bomb to strike the point on the ground.

Consider the vertical motion and applying the formula

$$s = ut + \frac{1}{2}at^2,$$

$$\text{We have } -1960 = 0 \times t - \frac{1}{2} \times 9.8t^2$$

$$\text{(ie) } 4.9t^2 = 1960 \Rightarrow t^2 = 400, \text{ (ie) } t = 20\text{secs.}$$

During this time, the horizontal velocity is constant.

$$\begin{aligned} \text{The horizontal distance described by the bomb in 20 secs} &= 20 \times 100 \\ &= 2000\text{m.} \end{aligned}$$

Example: 5

A particle is projected so as to graze the tops of two parallel walls, the first of height 'a' at a distance 'x' from the point of projection and the second of height 'b' at a distance 'y' from the point of projection. If the path of the particle lies in a plane \perp to both the walls, find the range on the horizontal plane and show that the angle of projection exceeds $\tan^{-1}3$.

Solution:

U being initial velocity and α the angle of projection, the equation to the path is

$$Y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha}$$

$$(ie) Y = xt - \frac{gx^2}{2u^2} (1+t^2) \text{ where } t = \tan \alpha \quad (1)$$

The tops of the two walls are (b,a) and (a,b) respectively and they lie on (1)

$$\therefore a = bt - \frac{gb^2}{2u^2} (1+t^2) \quad (2)$$

$$\& b = at - \frac{ga^2}{2u^2} (1+t^2) \quad (3)$$

$$\text{From (2), } a - bt = - \frac{gb^2}{2u^2} (1+t^2) \quad (4)$$

$$\text{From (3), } b - at = - \frac{ga^2}{2u^2} (1+t^2) \quad (5)$$

Divide (4) by (5)

$$\text{We get } \frac{a - bt}{b - at} = \frac{b^2}{a^2}$$

$$(ie) b^3 - ab^2t = a^3 - a^2bt$$

$$t(a^2b - ab^2) = a^3 - b^3$$

$$\therefore t = \frac{a^3 - b^3}{a^2b - ab^2} = \frac{(a-b)(a^2 + ab + b^2)}{ab(a-b)}$$

$$t = \frac{a^2 + ab + b^2}{ab}$$

$$\tan \alpha = \frac{a^2 + ab + b^2}{ab} = \frac{(a^2 - 2ab + b^2) + 3ab}{ab}$$

$$\therefore \tan \alpha = \frac{(a-b)^2}{ab} + 3$$

(6)

The first term in the right side of (6) is positive

$$\therefore \tan \alpha > 3 \text{ or } \alpha > \tan^{-1}3.$$

$$\text{From (4) } \frac{g(1+t^2)}{2u^2} = \frac{a-bt}{-b^2} = \frac{bt-a}{b^2}$$

$$= \frac{\frac{b(a^2+ab+b^2)-a}{ab}}{b^2} = \frac{a^2+ab+b^2-a^2}{ab^2}$$

$$= \frac{b(a+b)}{ab^2} = \frac{a+b}{ab}$$

$$\text{Horizontal range } R = \frac{u^2 \sin 2\alpha}{g} = \frac{2u^2 t}{g(1+t^2)}$$

$$= t \cdot \frac{ab}{a+b} \text{ Substituting from (7)}$$

$$= \frac{a^2+ab+b^2}{ab} \cdot \frac{ab}{a+b}$$

$$= \frac{a^2+ab+b^2}{a+b}$$

Example: 6 (U.Q)

A particle is thrown over a triangle from one end of a horizontal base and grazing the vertex falls on the other end of the base. If A, B are the base angles, and α the angle of projection, show that $\tan \alpha = \tan A + \tan B$.

Solution:-

Let u be the velocity and α the angle of projection and let t secs be the time from A to C.

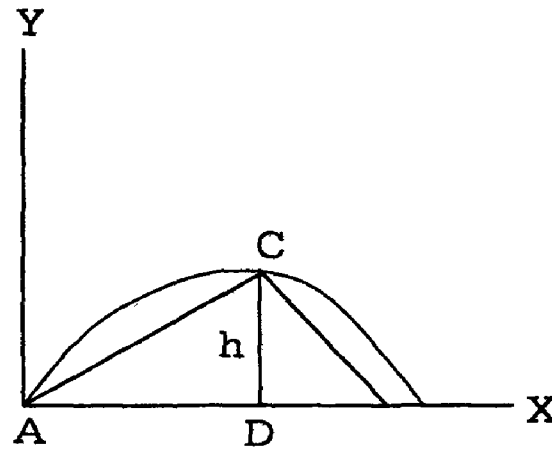
Draw $CD \perp AB$ and Let $CD = h$

Considering vertical motion

-h = vertical distance described in time t,

$$= u \sin \alpha . t - \frac{1}{2} g t^2$$

AD = horizontal distance described in time t = u Cos α . t



From Δ CAD,

$$\tan A = \frac{CD}{AD} = \frac{h}{AD} = \frac{u \sin \alpha . t - \frac{1}{2} g t^2}{u \cos \alpha . t}$$

$$\tan A = \tan \alpha - \frac{g t}{2 u \cos \alpha}$$

(1)

AB = horizontal range

$$= \frac{2 u^2 \sin \alpha \cos \alpha}{g}$$

\therefore DB = AB - AD

$$= \frac{2 u^2 \sin \alpha \cos \alpha}{g} - u \cos \alpha . t$$

From Δ CDB

$$\begin{aligned} \tan B &= \frac{CD}{DB} = \frac{h}{\left(\frac{2u^2 \sin \alpha \cos \alpha}{g} - u \cos \alpha \cdot t \right)} \\ &= \frac{u \sin \alpha \cdot t - \frac{1}{2}gt^2}{\left(\frac{2u^2 \sin \alpha \cos \alpha}{g} - u \sin \alpha \cdot t \right)} \\ &= \frac{gt \left(2u \sin \alpha - \frac{1}{2}gt \right)}{u \cos \alpha (2u \sin \alpha - gt)} \end{aligned}$$

$$\tan B = \frac{gt (2u \sin \alpha - gt)}{2u \cos \alpha (2u \sin \alpha - gt)} = \frac{gt}{2u \cos \alpha} \quad (2)$$

Adding (1) and (2)

$$\tan A + \tan B = \tan \alpha - \frac{gt}{2u \cos \alpha} + \frac{gt}{2u \cos \alpha}$$

$$\therefore \tan A + \tan B = \tan \alpha$$

Example: 7

Show that the greatest height which a particle with initial velocity V can reach on a vertical wall at a distance 'a' from the point of projection is $\frac{v^2}{2g} - \frac{ga^2}{2v^2}$.

Prove also that the greatest height above the point of projection attained by the particle in its flight is $\frac{v^6}{2g(v^4 + g^2 a^2)}$.

Solution:

In the usual notation, the equation to the path is

$$y = x \tan \alpha - \frac{gx^2}{2v^2 \cos^2 \alpha} \quad (1)$$

Putting $x = a$ in (1)

We get the value of y , which is the height reached on the vertical wall at a distance 'a' from the point of projection.

$$\begin{aligned} \therefore y &= a \tan \alpha - \frac{ga^2}{2v^2 \cos^2 \alpha} \\ &= at - \frac{ga^2}{2v^2} (1+t^2) \text{ where } t = \tan \alpha \end{aligned} \quad (2)$$

Now a and v are given and so y is a function of t .

$\therefore y$ is maximum when $\frac{dy}{dt} = 0$ and $\frac{d^2y}{dt^2}$ is negative

Differentiating (2) with respect to t ,

$$\frac{dy}{dt} = a - \frac{ga^2}{2v^2} \cdot 2t = a - \frac{ga^2 t}{v^2}$$

$$\frac{d^2y}{dt^2} = -\frac{ga^2}{v^2} = \text{negative}$$

Clearly

$$\text{So } Y \text{ is maximum when } a - \frac{ga^2 t}{v^2} = 0 \text{ or } t = \frac{v^2}{ga} \quad (3)$$

Putting this value of t in (2).

$$\begin{aligned} \text{Max. value of } Y &= a \cdot \frac{v^2}{ga} - \frac{ga^2}{2v^2} \left(1 + \frac{v^4}{g^2 a^2} \right) \\ &= \frac{v^2}{2g} - \frac{ga^2}{2v^2} - \frac{v^2}{2g} \\ &= \frac{v^2}{2g} - \frac{ga^2}{2v^2} \end{aligned}$$

This is the greatest height reached on the wall.

Greatest height attained during the flight.

$$= \frac{v^2 \sin^2 \alpha}{2g} = \frac{v^2}{2g} \cdot \frac{1}{\operatorname{Cosec}^2 \alpha} = \frac{v^2}{2g(1 + \cot^2 \alpha)}$$

$$= \frac{v^2}{2g \left(1 + \frac{ga^2}{v^2} \right)} \text{ putting the value of } \tan \alpha \text{ from (3)}$$

$$= \frac{v^6}{2g(v^4 + g^2 a^2)}$$

Example: 8

Show that the greatest height attained by a particle projected with a velocity u at an angle α to the horizontal is unaltered, if the velocity of projection is increased to ku and the angle of projection is decreased by λ , where

$$\operatorname{Cosec} \lambda = k (\cot \lambda - \cot \alpha),$$

Solution:-

The greatest height attained by the particle when the velocity of projection is u and the angle of projection is α is given by $\frac{u^2 \sin^2 \alpha}{2g}$.

It will be given by $\frac{(ku)^2 \sin^2 (\alpha - \lambda)}{2g}$ when the velocity of projection is increased to ku and the angle of projection is decreased by λ . These two are equal.

$$\frac{u^2 \sin^2 \alpha}{2g} = \frac{(ku)^2 \sin^2 (\alpha - \lambda)}{2g}$$

$$\text{(ie) if } \sin^2 \alpha = k^2 \sin^2 (\alpha - \lambda).$$

$$\text{(ie) if } \frac{\sin (\alpha - \lambda)}{\sin \alpha} = \frac{1}{k}$$

$$\frac{\sin \alpha \cos \lambda - \cos \alpha \sin \lambda}{\sin \alpha} = \frac{1}{k}$$

$$\frac{\sin \alpha \cos \lambda}{\sin \alpha} - \frac{\cos \alpha \sin \lambda}{\sin \alpha} = \frac{1}{k}$$

$$\text{If } \cos \lambda - \cot \alpha \sin \lambda = \frac{1}{k}$$

if $k (\cot \lambda - \cot \alpha) = \operatorname{cosec} \lambda$ which is required.

Exercise:

- 1) a) A projectile is thrown with a velocity of 20m/sec at an elevation 30° . Find the greatest height attained and horizontal range. B) A particle is projected with a velocity of 9.6 at an angle of 30° . Find
 - i) the time of flight.
 - ii) The greatest height of the particle.
2. If the time of flight of a shot in T seconds over a range of xm, show that the elevation is $\tan^{-1} \left(\frac{gT^2}{2x} \right)$ and determine the maximum height and the velocity of projection.
3. A body is projected at angle α to the horizontal, so as to clear two walls of equal height 'a' a distance 2a from each other. Show that the range is $2a \cot \frac{\alpha}{2}$.

6.3 Range of projection:

6.3.1 To determine when the horizontal range of a projectile is maximum, given the magnitude u of the velocity of projection.

If u is the initial velocity and α is the angle of projection, the range R on the horizontal plane through the point of projection is given by

$$R = \frac{2u^2 \sin \alpha \cos \alpha}{g} = \frac{u^2 \sin 2\alpha}{g} \quad (1)$$

Now g being a constant, for a given value of u, the value of R is greatest when $\sin 2\alpha$ is greatest

when $\sin 2\alpha = 1$

This happens when $2\alpha = 90^\circ$

(ie) $\alpha = 45^\circ$

Hence a maximum when the particle is projected at an angle of 45° to the horizontal (ie) the direction of projection for maximum horizontal range bisects the angle between the horizontal and the vertical. Also when $\alpha = 45^\circ$ from (1),

$$R = \frac{u^2}{g}.$$

(ie) The maximum horizontal range is $\frac{u^2}{g}$.

6.3.2 To show that for a given initial velocity of projection, there are in general two possible directions of projections so as to obtain a given horizontal range. (U.Q)

Let u be the velocity of projection of a particle, and α the necessary angle of projection so as to get a given horizontal range equal to k .

$$\text{Then } \frac{k}{1} = \frac{u^2 \sin 2\alpha}{g}$$

$$\therefore \sin 2\alpha = \frac{gk}{u^2} \quad (1)$$

Since u and k are given and g is a constant, the R.H.S of (1) is a known positive quantity. If $gk < u^2$.

We can determine an acute angle θ whose sine is exactly equal to $\frac{gk}{u^2}$.

Then (1) becomes $\sin 2\alpha = \sin \theta$ (2)

$$\therefore 2\alpha = \theta \text{ or } \alpha = \frac{\theta}{2}.$$

Since $\sin (180^\circ - \theta) = \sin \theta$, (2) can also be written as

$$\sin 2\alpha = \sin (180^\circ - \theta)$$

$$2\alpha = 180^\circ - \theta.$$

$$\alpha = \frac{\theta}{2} \quad (4)$$

From (3) and (4), we find that there are two values of α and so two directions of projection, each giving the same range k .

Let α_1 and α_2 be these two values of α .

$$\text{Then } \alpha_1 = \frac{\theta}{2} \text{ and } \alpha_2 = 90^\circ - \frac{\theta}{2},$$

$$\therefore \alpha_1 + \alpha_2 = 90^\circ$$

As $\theta < 90^\circ$, $\alpha_1 < 45^\circ$ and so $\alpha_2 > 45^\circ$

$$\text{Now } 45^\circ - \alpha_1 = 45^\circ - \frac{\theta}{2}.$$

$$\text{and } \alpha_2 - 45^\circ = 90^\circ - \frac{\theta}{2} - 45^\circ = 45^\circ - \frac{\theta}{2}$$

$$\text{(ie) } 45^\circ - \alpha_1 = \alpha_2 - 45^\circ \quad (5)$$

But 45° is the angle of projection to get maximum horizontal range will the same initial velocity. So (5) show that the two direction α_1 & α_2 are equally inclined to the direction of maximum range. This is shown in fig(a)

In fig (a) OT_1 and OT_2 are the directions α_1 & α_2 necessary to get a given range k . OT is the direction giving maximum horizontal range.

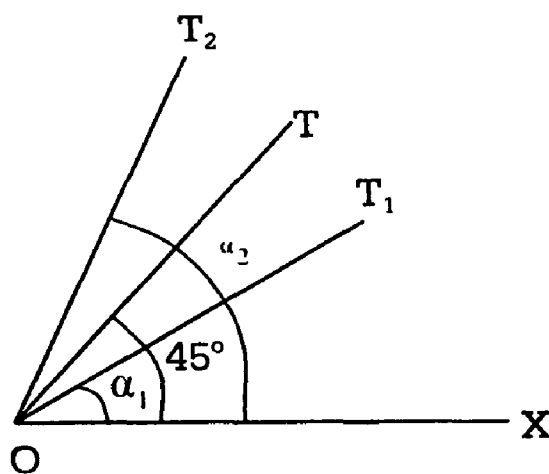


Fig. (a)

$$\angle T_1OT = \angle XOT - \angle XOT_1$$

$$= 45^\circ - \alpha_1$$

$$\begin{aligned} \angle T_2OT &= \angle XOT_2 - \angle XOT \\ &= \alpha_2 - 45^\circ \end{aligned}$$

$$\text{and } \angle T_1OT = \angle T_2OT$$

In other words, OT bisects the angle between OT_1 and OT_2 .

$$\text{If } u^2 = gk, \text{ from (1), } \sin 2\alpha = 1$$

Then $2\alpha = 90^\circ$ or $\alpha = 45^\circ$.

Only one value of α is possible and this corresponds to the case of maximum range.

If $u^2 < gk$, the R.H.S of (1) is greater than 1 and so we cannot get a real value for α .

(ie) There is no angle of projection to get a range greater than $\frac{u^2}{g}$, which is really the maximum range possible.

Note:

To get a given horizontal range k , we find that $u^2 > gk$. So the minimum value of $u = \sqrt{gk}$.

Example: 1

If h and h^1 be the greatest heights in the two paths of a projectile with a given velocity for a given range R , prove that $R = 4\sqrt{hh^1}$

Solution:

Let α and α^1 be the two angles of projection with a given velocity u to get a given range R .

Then we know that $\alpha + \alpha^1 = 90^\circ$

$$\text{(ie) } \alpha^1 = 90^\circ - \alpha \tag{1}$$

$$\text{Also } R = \frac{2u^2 \sin \alpha \cos \alpha}{g} \tag{2}$$

$$h = \frac{u^2 \sin^2 \alpha}{g} \quad (3)$$

$$h' = \frac{u^2 \sin^2 \alpha'}{2g} \quad (4)$$

$$\text{Hence } \sqrt{hh'} = \frac{u^3 \sin \alpha \sin \alpha'}{2g} = \frac{u^2 \sin \alpha \sin (90^\circ - \alpha)}{2g} \text{ Using (1)}$$

$$= \frac{u^2 \sin \alpha \cos \alpha}{2g} = \frac{R}{4} \text{ Using (2)}$$

$$\text{(ie) } R = 4 \sqrt{hh'}$$

Example: 2

A shell bursts on contact with the ground and pieces from it fly in all directions with all velocities upto 30metres per second. Show that a man 30m away is in danger for 5 secs nearly.

Solution:

Here the given velocity $u = 30\text{m}$, and the given range $R = 30\text{m}$

$$\text{Since } R = \frac{u^2 \sin 2\alpha}{g},$$

We have

$$\sin 2\alpha = \frac{gR}{u^2} = \frac{9.81 \times 30}{30 \times 30} = .327.$$

From the tables,

$$2\alpha = 19^\circ 5' \text{ or } 160^\circ 55'$$

$$\text{(ie) } \alpha = 9^\circ 32\frac{1}{2}' \text{ or } 80^\circ 27\frac{1}{2}'$$

These are the two angles of projection to get the given range.

Let t_1 and t_2 be the two times of flight for the two particular places which fly in the directions $9^\circ 32\frac{1}{2}'$ and $80^\circ 27\frac{1}{2}'$ respectively.

These two places will strike the man and so he is in danger for the interval $t_2 - t_1$ secs.

$$\text{Now, } t_1 = \frac{2u \sin 9^\circ 32\frac{1}{2}'}{g} = \frac{2 \times 30 \sin 9^\circ 32\frac{1}{2}'}{9.8} = 1.014$$

$$t_2 = \frac{2u \sin 80^\circ 27\frac{1}{2}'}{g} = \frac{2 \times 30 \sin 80^\circ 27\frac{1}{2}'}{9.81} = 6.032$$

$$\begin{aligned} \text{Period of danger} &= t_2 - t_1 \\ &= 6.032 - 1.014 = 5.018 \\ &= 5 \text{ secs. (nearly)} \end{aligned}$$

Example: 3

The range of a rifle bullet is 1000m. When α is the angle of projection. Show that if the bullet is fired with the same elevation from a car traveling 36km/h towards the target, the range will be increased by

$$\frac{1000 \sqrt{\tan \alpha}}{7} \text{ m } (g = 9.8 \text{ m/sec}^2)$$

Solution:

Let u m/sec be the velocity of projection.
The horizontal range.

$$R = \frac{2u^2 \sin \alpha \cos \alpha}{g} = 1000 \text{ (given)} \quad (1)$$

$$\text{Also } R = \frac{2}{g} (u \cos \alpha) \cdot (u \sin \alpha)$$

$$= \frac{2}{g} (\text{horizontal velocity}) \times (\text{initial vertical velocity})$$

When the bullet is fired from a moving car, the horizontal velocity is increased and the increases.

$$= 36 \text{ km/h} = \frac{36 \times 1000}{60 \times 60} = 10 \text{ m/sec.}$$

Now horizontal velocity = $u \cos \alpha + 10$

As there is no change in the vertical motion, new initial vertical velocity = $u \sin \alpha$.

Hence in the second case, horizontal range

$$R^1 = \frac{2}{g} (u \cos \alpha + 10) (u \sin \alpha) \text{ using the form given in (2)}$$

$$R^1 - R^0 = \frac{2}{g} (u \cos \alpha + 10) (u \sin \alpha) - \frac{2u^2 \sin \alpha \cos \alpha}{g}$$

$$= \frac{20u \sin \alpha}{g} \tag{3}$$

From (1)

$$U^2 = \frac{g \times 1000}{2 \sin \alpha \cos \alpha} = \frac{500g}{\sin \alpha \cos \alpha}$$

Putting this value of u in (3)

$$\begin{aligned} R^1 - R &= \frac{20 \sin \alpha}{g} \times \sqrt{\frac{500g}{\sin \alpha \cos \alpha}} \\ &= \frac{20 \times 10 \times \sqrt{\tan \alpha} \times 5}{\sqrt{g}} \\ &= \frac{200 \sqrt{\tan \alpha} \sqrt{5}}{\sqrt{98}} = \frac{200 \sqrt{\tan \alpha} \sqrt{5} \times \sqrt{10}}{\sqrt{98}} \\ &= \frac{200 \sqrt{\tan \alpha} \sqrt{5} \times \sqrt{5} \times \sqrt{2}}{7\sqrt{2}} \\ &= \frac{1000 \sqrt{\tan \alpha}}{7} \text{ m.} \end{aligned}$$

and this is the increase in the range.

Example:

A body is projected with the same velocity at two different angles cover the same horizontal range R . If T_1 and T_2 be the two times of flight prove that

$$R = \frac{1}{2} g T_1 T_2.$$

Solution:

We know that, with a given velocity of projection, w ,

We can obtain a given horizontal range R , with two angles of projection, α_1 or $90 - \alpha_1$, For the two paths, the times of flight are given by,

$$T_1 = \frac{2u \sin \alpha_1}{g}$$

$$\text{and } T_2 = \frac{2u \sin(90^\circ - \alpha_1)}{g} = \frac{2u \sin \alpha_1}{g}.$$

$$\therefore \frac{1}{2} g T_1 T_2 = \frac{1}{2} g \cdot \frac{2u \sin \alpha_1}{g} \cdot \frac{2u \sin \alpha_1}{g}$$

$$= \frac{u^2 \sin 2 \alpha_1}{g}$$

$$= R.$$

$$\therefore R = \frac{1}{2} T_1 T_2.$$

Exercise:-

- 1) If R be the horizontal range of a projectile and h its greatest height, prove that the maximum horizontal range with the same velocity of Projection is $\frac{16h^2 + R^2}{8h}$
- 2) Prove that in any trajectory over a horizontal plane, the horizontal range is a maximum when it is equal to four times the greatest height.
- 3) A shot projected with velocity V at an elevation of 45° reaches a point P on the horizontal plane through the point of projection. Show that in order to hit

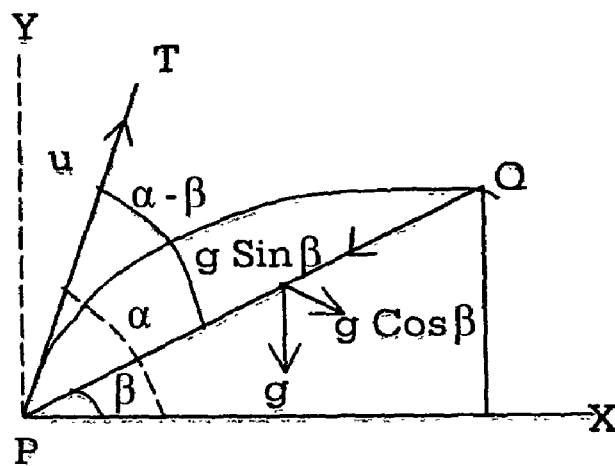
a mark is h feet above P , if the shot is projected at the same elevation, the velocity of projection must be increased to $\frac{v^2}{(v^2 - gh)^{1/2}}$

6.4 Range on a inclined plane:

6.4.1 From a point on a plane, which is inclined at an angle β with the horizon a particle is projected with a velocity u at an angle α with the horizontal in a plane passing through the normal to the inclined plane and the line of greatest slope. To find the range on the inclined plane ($U.Q$).

Solution:

Let P be the point of projection and the particle strike the inclined plane at Q . Then PQ is the range on the inclined plane.



Let $PQ = r$. Taking P as the origin and the horizontal and the vertical through P as the axes of X and Y respectively. The equation to the path is,

$$Y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha} \quad (1)$$

Draw $QN \perp$ to the horizontal plane through P . The coordinates of Q are $(r \cos \beta, r \sin \beta)$.

Substituting these in (1)

$$r \sin \beta = r \cos \beta \cdot \tan \alpha - \frac{gr^2 \cos^2 \beta}{2u^2 \cos^2 \alpha}$$

Multiplying by $2u^2 \cos^2 \alpha$ and canceling r throughout,

We have,

$$2u^2 \cos^2 \alpha \sin \beta = 2u^2 \cos \beta \sin \alpha \cos \alpha - g r \cos^2 \beta.$$

$$\therefore r = \frac{2u^2 \cos \beta \sin \alpha \cos \alpha - 2u^2 \cos^2 \alpha \sin \beta}{g \cos^2 \beta}$$

$$= \frac{2u^2 \cos \alpha (\sin \alpha \cos \beta - \cos \alpha \sin \beta)}{g \cos^2 \beta}$$

$$\text{(ie) } r = \frac{2u^2 \cos \alpha \sin(\alpha - \beta)}{g \cos^2 \beta}$$

Aliter;

We can study separately the motion of the particle along the inclined plane and the motion \perp to the plane. The initial velocity u can be resolved into two components i) $u \cos(\alpha - \beta)$ along PQ, the inclined plane. And ii) $u \sin(\alpha - \beta)$, \perp to the inclined plane.

The acceleration g can be resolved into two components i) $g \cos \beta$ \perp to the inclined plane in the downward direction and ii) $g \sin \beta$ along the inclined plane towards P. This resolution is shown in the figure.

Let T be the time which the particle takes to go from P to Q.

After time T , the particle is again on the inclined plane and so, during time T , the distance traveled \perp to the inclined plane is = 0.

$$\therefore 0 = u \sin(\alpha - \beta) T - \frac{1}{2} g \cos \beta T^2$$

$$\text{(ie) } T = \frac{2u \sin(\alpha - \beta)}{g \cos \beta}$$

This is the time of flight on the inclined plane. During this time, the horizontal velocity remains constant and $u \cos \alpha$. So, horizontal distance described in time $T = PN = u \cos \alpha T$.

But $PN = PQ \cos \beta$.

$\therefore PQ \cos \beta = u \cos \alpha T$.

$$\begin{aligned} \text{(ie) } PQ &= \frac{u \cos \alpha}{\cos \beta} \cdot T \\ &= \frac{u \cos \alpha}{\cos \beta} \cdot \frac{2u \sin(\alpha - \beta)}{g \cos \beta} \\ PQ &= \frac{2u^2 \sin(\alpha - \beta) \cos \alpha}{g \cos^2 \beta} \end{aligned}$$

6.4.2 To find the greatest distance of the projectile from the inclined plane and show that it is attained in half the total time of flight:

Let us consider the motion \perp to the inclined plane. As explained in range on an inclined plane. The initial velocity in this direction is $u \sin(\alpha - \beta)$ and this is subject to an acceleration $g \cos \beta$ in the same direction but acting downwards.

Let Y be the distance traveled by the particle in this direction in time T .

Then

$$Y = u \sin(\alpha - \beta) \cdot T - \frac{1}{2} g \cos \beta \cdot t^2 \quad (1)$$

Differentiating with respect to t .

$$\frac{dy}{dt} = u \sin(\alpha - \beta) - g \cos \beta \cdot t$$

$$\text{and } \frac{dy^2}{dt^2} = -g \cos \beta = \text{negative.}$$

So y is maximum. when $\frac{dy}{dt} = 0$.

(ie) when $u \sin(\alpha - \beta) - g \cos \beta \cdot t = 0$

$$(ie) t = \frac{u \sin (\alpha - \beta)}{g \cos \beta} \quad (3)$$

Substituting (3) in (1) maximum value of λ

$$\begin{aligned} &= u \sin (\alpha - \beta) \cdot \frac{u \sin (\alpha - \beta)}{g \cos \beta} - \frac{1}{2} g \cos \beta \frac{u^2 \sin^2 (\alpha - \beta)}{g^2 \cos^2 \beta} \\ &= \frac{u^2 \sin^2 (\alpha - \beta)}{g \cos \beta} - \frac{u^2 \sin^2 (\alpha - \beta)}{2g \cos \beta} \\ &= \frac{u^2 \sin^2 (\alpha - \beta)}{2g \cos \beta} \quad (4) \end{aligned}$$

(4) is the greatest distance of the projectile from the inclined plane. Also, from (3), time to this greatest distance

$$= \frac{u \sin (\alpha - \beta)}{g \cos \beta} \text{ and this is clearly half of the time of flight.}$$

Aliter:

When the particle is at the greatest distance from the inclined plane, it will have all its velocity only parallel to the inclined plane. Hence the component velocity \perp to the inclined plane is zero. So, if S is the greatest distance

$$\text{We have } 0 = [u \sin (\alpha - \beta)]^2 - 2g \cos \beta \cdot S$$

$$(ie) S = \frac{u^2 \sin^2 (\alpha - \beta)}{2g \cos \beta}$$

Also if t is the corresponding time,

$$0 = u \sin (\alpha - \beta) - g \cos \beta t$$

$$(Or) t = \frac{u \sin (\alpha - \beta)}{g \cos \beta}$$

6.4.3 To determine when the range on the inclined plane is maximum, given the magnitude u of the velocity of projection:-

From the Range of inclined plane

The range R on the inclined plane is given by

$$R = \frac{2u^2 \cos\alpha \sin(\alpha - \beta)}{g \cos^2\beta}$$
$$= \frac{u^2}{g \cos^2\beta} [\sin(2\alpha - \beta) - \sin\beta] \quad (1)$$

Now u and β are given.

The quantity out side the bracket, $\frac{u^2}{g \cos^2\beta}$ is constant.

So R is maximum, when the value of the expression inside the bracket is a maximum

(ie) when $\sin(2\alpha - \beta)$ is greatest.

(ie) When $2\alpha - \beta = \frac{\pi}{2}$

(ie) $\alpha = \frac{\pi}{4} + \frac{\beta}{2}$ for maximum range.

When α takes this value, $\alpha - \beta = (2\alpha - \beta) - \alpha$

$$= 90^\circ - \alpha \quad (2)$$

Referring to fig.

$$\alpha - \beta = \angle TPN - \angle QPN = \angle TPQ$$

$$\text{and } 90^\circ - \alpha = \angle YPT$$

$$\text{Hence from (2), } \angle TPQ = \angle YPT.$$

(ie) PT, the direction of projection for maximum range bisects the angle between the vertical and the inclined plane.

From (1), the value of maximum range

$$= \frac{u^2}{g \cos^2 \beta} (1 - \sin \beta) = \frac{u^2}{g (1 + \sin \beta)}$$

6.4.5 To show that, for a given initial velocity of projection, there are in general, two possible directions of projection so as to obtain a given range on an inclined plane:

Let u be the velocity of projection of a particle and α the necessary angle of projection so as to get a given range k on an inclined plane of inclination β to the horizontal.

$$\text{Then } k = \frac{2u^2 \sin(\alpha - \beta) \cos \alpha}{g \cos^2 \beta} = \frac{u^2 (\sin(2\alpha - \beta)) - \sin \beta}{g \cos^2 \beta} \quad (1)$$

$$\text{From (1) } \sin(2\alpha - \beta) = \frac{gk \cos^2 \beta}{u^2} + \sin \beta \quad (2)$$

Since k , u , β are given the R.H.S of (2) is a known positive quantity. So we can determine an acute angle θ whose sine is exactly to $\frac{gk \cos^2 \beta}{u^2} + \sin \beta$.

$$\text{Then (2) becomes, } \sin(2\alpha - \beta) = \sin \theta \quad (3)$$

$$\text{(ie) } 2\alpha - \beta = \theta \text{ or } \alpha = \frac{\theta}{2} + \frac{\beta}{2} \quad (4)$$

$$\text{(ie) } \alpha = 90^\circ - \frac{\theta}{2} + \frac{\beta}{2} \quad (5)$$

From (4) and (5)

We find that there are two values of α and so two directions of projection, each giving the same range k .

Let α_1 and α_2 these two values of α .

$$\text{Then } \alpha_1 = \frac{\theta}{2} + \frac{\beta}{2} \text{ and } \alpha_2 = 90^\circ - \frac{\theta}{2} + \frac{\beta}{2}$$

$$\text{Now } (45^\circ + \frac{\beta}{2}) - \alpha_1 = 45^\circ + \frac{\beta}{2} - \frac{\theta}{2} - \frac{\beta}{2} = 45^\circ - \frac{\theta}{2}$$

$$\text{and } \alpha_2 - (45^\circ + \frac{\beta}{2}) = 90^\circ - \frac{\theta}{2} + \frac{\beta}{2} - 45^\circ - \frac{\beta}{2} = 45^\circ - \frac{\theta}{2}$$

$$\therefore (45^\circ + \frac{\beta}{2}) - \alpha_1 = \alpha_2 - (45^\circ + \frac{\beta}{2}) \quad (6)$$

But $45^\circ + \frac{\beta}{2}$ is the angle of projection for maximum range on the inclined plane.

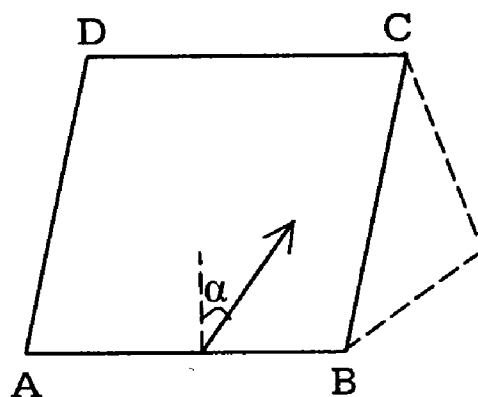
So (6) shows that the two directions α_1 and α_2 are equally inclined to the direction of maximum range.

Note:

The direction of projection is expressed as an elevation to the horizontal. We can also take the elevation relative to the inclined plane. In problems, it should be carefully found out which of these angles is given.

6.4.5 Motion on the surface of smooth inclined plane:

Let a particle be projected with velocity u on the surface of a smooth inclined plane ABCD of slope β in a direction inclined at an angle α to the line of greatest slope of the plane. The acceleration due to gravity can be resolved into two components, one, $g \sin \beta$ in the direction of the line of greatest slope and the other $g \cos \beta$ \perp to the inclined plane. But this component $g \cos \beta$ is opposed by an equal normal reaction of the inclined plane and so, the particle moves with an acceleration $g \sin \beta$ parallel to the line of greatest slope.

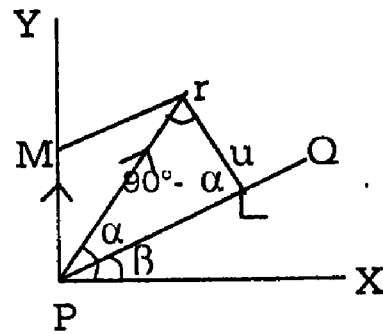


Example:1

If u and v be the oblique components of the initial velocity in the vertical direction and in the direction of the line of greatest slope, show that the range on

the inclined plane is $2 \frac{UV}{g}$.

Solution:



P is the point of projection and PX, PY are the horizontal and the vertical through P. PQ is the line of greatest slope of the inclined plane, with inclination β . PT is the initial direction of projection at an angle α to the horizontal. Let PT = the initial velocity u on some scale.

From T, draw $TL \parallel PY$ and $TM \parallel PQ$ as show is the figure.

The $PM(=TL)$ and $PL (=MT)$ are the oblique components of the initial velocity u in the vertical direction and is the direction of the line of greatest slope respectively.

From ΔPTL

We have

$$\frac{TL}{\sin/\underline{TPL}} = \frac{PL}{\sin/\underline{PTL}} = \frac{PT}{\sin/\underline{PLT}} \quad (1)$$

$$\underline{\angle TPL} = \underline{\angle TPX} = \underline{\angle LPX} = \alpha - \beta$$

$$\underline{\angle PTL} = \text{alternate } \underline{\angle MPT} = 90^\circ - \alpha$$

$$\begin{aligned} \underline{\angle PLT} &= 180^\circ - (\underline{\angle PTL} + \underline{\angle TPL}) \\ &= 180^\circ - 90^\circ - \alpha + \alpha - \beta = 90^\circ + \beta \end{aligned}$$

Also $PT = u$. $TL = U$ and $PL = V$.

Substituting in (1)

We have

$$\frac{U}{\sin(\alpha - \beta)} = \frac{V}{\sin(90^\circ - \alpha)} = \frac{u}{\sin(90^\circ + \beta)}$$

$$(ie) \quad \frac{U}{\sin(\alpha - \beta)} = \frac{V}{\cos\alpha} = \frac{u}{\cos\beta}$$

$$\therefore U = \frac{u \sin(\alpha - \beta)}{\cos\beta} \quad \& \quad V = \frac{u \cos\alpha}{\cos\beta}$$

$$2 \frac{UV}{g} = \frac{2u \sin(\alpha - \beta)}{g \cos\beta} \times \frac{u \cos\alpha}{\cos\beta} = \frac{u \cos\alpha}{\cos\beta}$$

$$= \frac{2u^2 \sin(\alpha - \beta) \cos\alpha}{g \cos^2\beta}$$

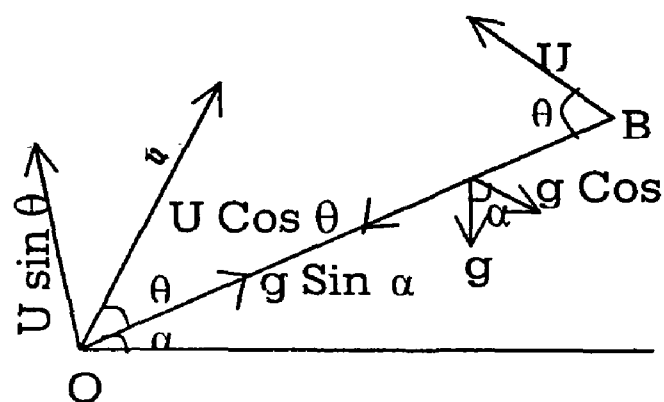
$$= 2 \frac{UV}{g}$$

Example: 2 (U.Q)

Show that, for a given velocity of projection the maximum range down an inclined plane of inclination α bears to the maximum range up the inclined plane the ratio $\frac{1 + \sin\alpha}{1 - \sin\alpha}$.

Solution:-

Let u be the given velocity of projection and θ the inclination of the direction of projection with the plane.



The velocity u can be resolved into two components $u \cos \theta$ along the upward inclined plane and $u \cos \theta \perp$ to the inclined plane and down wards.

Consider the motion perpendicular to the inclined plane. Let T be the time of flight. Distance traveled \perp to the inclined plane in Time T is = 0.

$$\therefore 0 = u \sin \theta \cdot T - \frac{1}{2} g \cos \alpha \cdot T^2$$

$$\text{(ie) } T = \frac{2u \sin \theta}{g \cos \alpha}$$

During this time, the distance traveled along the plane

$$= u \sin \theta \cdot T - \frac{1}{2} g \cos \alpha \cdot T^2$$

$$= u \cos \theta \cdot \frac{2u \sin \theta}{g \cos \alpha} - \frac{1}{2} g \sin \alpha \cdot \frac{4u^2}{g^2} \cdot \frac{\sin^2 \theta}{\cos^2 \alpha}$$

$$= \frac{2u^2 \sin \theta \cos \theta}{g \cos^2 \alpha} - \frac{2u^2 \sin \alpha \cdot \sin^2 \theta}{g \cos^2 \alpha}$$

$$= \frac{2u^2 \sin \theta}{g \cos^2 \alpha} (\cos \alpha \cos \theta - \sin \alpha \sin \theta)$$

$$= \frac{2u^2 \sin \theta}{g \cos^2 \alpha} \cos (\theta + \alpha)$$

$$= \frac{u^2}{g \cos^2 \alpha} 2 \cos (\theta + \alpha) \sin \theta$$

$$= \frac{u^2}{g \cos^2 \alpha} [\sin (2\theta + \alpha) - \sin \alpha]$$

This is the range R , up the inclined plane.

R_1 is maximum, when $\sin (2\theta + \alpha) = 1$

\therefore Maximum range up the plane

$$= \frac{u^2}{g \cos^2 \alpha} (1 - \sin \alpha) = \frac{u^2}{g (1 + \sin \alpha)}$$

When the particle is projected down the plane from B at the same angle to the plane, the time of flight has the same value $\frac{2u \sin \theta}{g \cos \alpha}$. But the component of the $u \cos \theta$ downwards and the component acceleration $g \sin \alpha$ also downwards.

Hence range down the plane

$$R_2 = \text{distance traveled along the plane in time } T$$

$$= u \cos \theta \cdot T + \frac{1}{2} g \sin \alpha \cdot T^2$$

$$= \frac{2u^2 \sin \theta}{g \cos^2 \alpha} (\cos \alpha \cos \theta + \sin \alpha \sin \theta)$$

$$= \frac{2u^2 \sin \theta}{g \cos^2 \alpha} \cos (\theta - \alpha)$$

$$= \frac{u^2}{g \cos^2 \alpha} [\sin (2\theta - \alpha) - \sin \alpha]$$

$$R_2 \text{ is maximum, when } \sin (2\theta - \alpha) = 1$$

So maximum range down the plane

$$= \frac{u^2}{g \cos^2 \alpha} (1 + \sin \alpha) = \frac{u^2}{g (1 - \sin \alpha)}$$

$$\therefore \frac{\text{Max. range down the plane}}{\text{Max. range up the plane}} = \frac{u^2}{g (1 - \sin \alpha)} \cdot \frac{g (1 + \sin \alpha)}{u^2}$$

$$= \frac{1 + \sin \alpha}{1 - \sin \alpha}$$

Note:

The range R, down the plane can be get from the range R, up the plane by changing α into $-\alpha$

Example: 3 (U.Q)

A particle is projected at an angle α with a velocity u and it strikes up an inclined plane of inclination β at right angles to the plane. Prove that

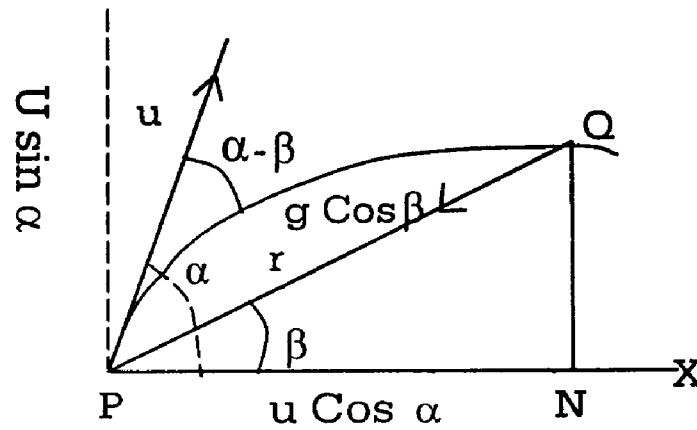
- i) $\text{Cot } \beta = 2 \tan (\alpha - \beta)$
- ii) $\text{Cot } \beta = \tan \alpha - 2 \tan \beta$.

If the plane is struck horizontally, show that $\tan \alpha = 2 \tan \beta$.

Solution:

The initial velocity and acceleration are split into components along the plane and \perp to the plane as explained we have shown that the time of

time of flight in $T = \frac{2u \sin (\alpha - \beta)}{g \cos \beta}$ (1)



Since the particle strikes the inclined plane normally its velocity parallel to the inclined normally, its velocity parallel to the inclined plane at the end of the time T is $= 0$

(ie) $0 = u \cos (\alpha - \beta) - g \sin \beta \quad T$

or $T = \frac{u \cos (\alpha - \beta)}{g \sin \beta}$ (2)

Equating (1) and (2)

We have

$$\frac{2u \sin (\alpha - \beta)}{g \cos \beta} = \frac{u \cos (\alpha - \beta)}{g \sin \beta}$$

(ie) $\text{Cot } \beta = 2 \tan (\alpha - \beta)$ (i)

$$(u) \cot \beta = \frac{2(\tan \alpha - \tan \beta)}{1 + \tan \alpha \tan \beta}$$

Cross multiplying

$$\cot \beta + \tan \alpha = 2 \tan \alpha - 2 \tan \beta \quad (\text{or})$$

$$\cot \beta = \tan \alpha - 2 \tan \beta \quad (ii)$$

If the plane is stuck horizontally, the vertical velocity of the projectile at the end of time T is = 0.

$$\text{Initial vertical velocity} = u \sin \alpha$$

and acceleration in this direction = g downwards

$$\text{Vertical velocity in time T} = u \sin \alpha - gT$$

$$\therefore u \sin \alpha - gT = 0 \text{ or } T = \frac{u \sin \alpha}{g} \quad (3)$$

Equating (1) and (3)

We have

$$\frac{2u \sin (\alpha - \beta)}{g \cos \beta} = \frac{u \sin \alpha}{g}$$

$$\text{or } 2 \sin (\alpha - \beta) = \sin \alpha \cos \beta$$

$$(ie) 2 (\sin \alpha \cos \beta - \cos \alpha \sin \beta) = \sin \alpha \cos \beta$$

$$(ie) \sin \alpha \cos \beta = 2 \cos \alpha \sin \beta \text{ or } \tan \alpha = 2 \tan \beta.$$

Exercise

1) A particle is projected from the top of a plane inclined at 60° to the horizontal. If the direction of projection is i) 30° above the horizontal and ii) 30° below the horizontal, Show that the range down the plane in the first case is doubles that in the second.

2) A particle projected with velocity u strikes at right angles a plane through the point of projection inclined at an angle β to the horizon. Show that the height of the point struck above the horizontal plane through the point of

$$\text{projection is } \frac{2u^2 \sin^2 \beta}{g(1 + 3 \sin^2 \beta)} \text{ and that the time of flight is } \frac{2u}{g \sqrt{1 + 3 \sin^2 \beta}}.$$

UNIT – VII

IMPACT

7.1 Introduction :

A solid body has a definite shape. When a force is applied at any point of it tending to change its shape, in general, all solids which we meet with in nature yield slightly and get more or less deformed near the point. Immediately, internal forces come into play tending to restore the body to its original form and as soon as the disturbing force is removed, the body regains its original shape. The internal force which acts, when a body tends to recover its original shape after a deformation or compression is called the force of restitution. Also the property which causes a solid body to recover its shape is called elasticity. If a body does not tend to recover its shape, it will cause no force of restitution and such a body is said to be inelastic.

Suppose a ball is dropped from any height h upon a hard floor. It strikes the floor with a velocity $u = \sqrt{2gh}$ and makes an impact. Soon it rebounds and moves vertically upwards with a velocity V . The height h_1 to which it rebounds is given by $h_1 = \frac{V^2}{2g}$. (ie) $V = \sqrt{2gh_1}$.

Generally we find that $h_1 < h$. So $v < u$. As soon as the ball strikes the floor, the impulsive action of the floor rapidly stops the downward velocity of the ball and at the same time causes a temporary compression near the point of contact. Due to the elastic property of the solid, the ball tends to regain its original form quickly. It presses the floor and receives an equal and opposite impulsive reaction from it and with a new upward velocity, it rebounds.

Now, bodies made of various materials are elastic in different degrees. If balls of different materials (like, iron, glass, lead, etc) are dropped from the same height upon floors of different constitution (like wooden floor, marble floor etc) it will be found that the heights to which they rebound after striking the floor will be different. In all these cases, the velocity of the ball on reaching the floor is the same, as it is dropped from the same height. But the velocity of the ball after impact is not the same in each case as the height to which it rebounds is different. Thus due to the elastic property of solid bodies, a change in velocity takes place when they strike each other.

If $v = u$, the velocity with which the ball leaves the floor is the same as that with which it strikes it. In this case, the ball is said to be elastic. If $v = 0$, the ball does not rebound at all. It is said to be inelastic. More generally, when a body completely regains its shape after a collision, it is said to be perfectly elastic. If it does not come to its original shape, it is said to be perfectly inelastic. These two cases of bodies are only ideal.

In this chapter, we shall study some simple cases of the impact of elastic bodies. We shall consider the cases of particles in collision with particles, or planes and of sphere in collision with planes or spheres. In all cases, we consider the impinging bodies to be smooth, so that the only mutual action they can have on each other will be along the common normal at the point where they touch.

7.1.1 Definitions :

Two bodies are said to impinge directly when the direction of motion of each before impact is along the common normal at the point where they touch.

They are said to be impinge obliquely, if the direction of motion of either body or both or both is not along the common normal at the point where they touch.

The common normal at the point of contact is called the line of impact. Thus in the case of two spheres, the line of impact is the line joining their centres.

7.1.2 Fundamental Laws of Impact :

The following three general principles hold good when two smooth moving bodies make an impact.

1. Newton's Experimental Law :

Newton studies the rebound of elastic bodies experimentally and the result of his experiments is embodied in the following law :-

When two bodies impinge directly, their relative velocity after, impact bears a constant ratio to their relative velocity before impact, resolved in the same direction and is of opposite sign.

The constant ratio depends on the material of which the bodies are made and is independent of their masses. It is generally denoted by e , and is called the coefficient (or modulus) of elasticity (or restitution or resilience).

This law can be put symbolically as follows :

If u_1 , u_2 are the components of the velocity of two impinging bodies along their common normal before impact and v_1 , v_2 their components velocities along the same line after impact, all components being measured in the same direction and e is the coefficient of restitution.

$$\text{Then } \frac{v_2 - v_1}{u_2 - u_1} = -e.$$

The quantity e , which is a positive number, is never greater than unity. It lies between 0 and 1. Its value differs widely for different bodies; for two glass balls it is about 0.9 ; for ivory 0.8; while for lead it is 0.2. For two balls, one of lead and the other of iron, its value is about 0.13. Thus, when one or both the bodies are altered, e becomes different but so long as both the bodies remain the same, e is constant. Bodies for which $e = 0$ are said to be inelastic while for perfectly elastic bodies, $e = 1$. Probably, there are no bodies in nature coming strictly under either of these headings. Newton's law is purely empirical and is true only approximately, like many experimental laws.

Motion of two smooth bodies \perp to the line of Impact:

When two smooth bodies impinge, the only force between them at the time of impact in the mutual reaction which acts along the common normal. There is no force acting along the common tangent and hence there is no change of velocity in that direction. Hence the velocity of either body resolved in a direction \perp to the line of impact is not affected by impact.

7.1.3 Principle of conservation of momentum :

We can apply the law of conservation of momentum in the case of two impinging bodies. The algebraic sum of the momenta of the impinging bodies after impact is equal to the algebraic sum of their momenta before impact all momenta, being measured along the common normal.

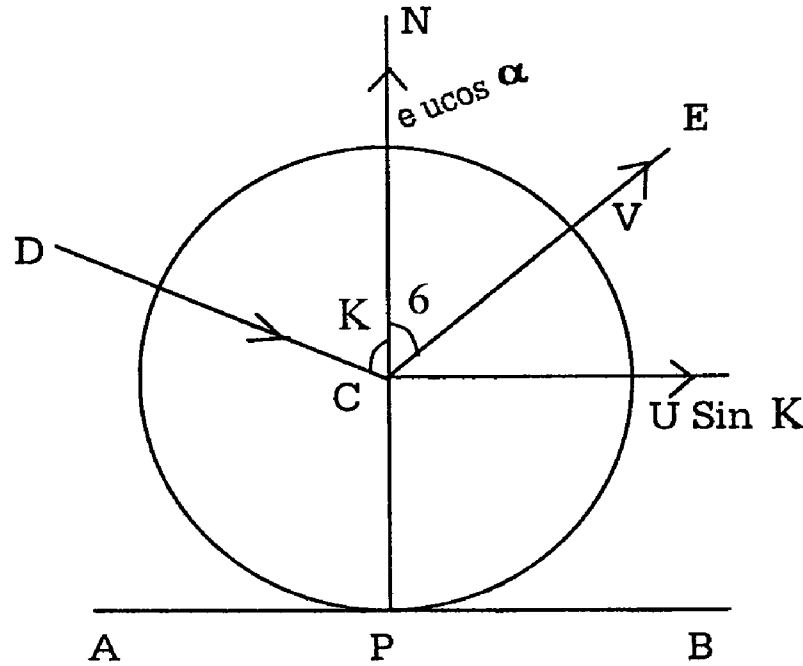
The above three principles are sufficient to study the changes in the motion of two impinging elastic bodies.

We shall now proceed to discuss particular cases.

Impact on a fixed plane

7.2 Impact of a smooth sphere on a fixed smooth plane :

A smooth sphere, or particle, whose mass is m and whose coefficient of restitution is e , impinges obliquely on a smooth fixed plane; to find its velocity and direction of motion after impact.



Let AB the plane and P the point at which the sphere strikes it. The common normal at P is the vertical line at P passing through the centre of the sphere. Let it be PC . This is the line of impact. Let the velocity of the sphere before impact be u at an angle α with CP and v its velocity after impact at an angle θ with CN as shown in the figure. Since the plane and the sphere are smooth, the only force acting impact is the impulsive reaction and this is along the common normal. There is no force parallel to the plane during impact. Hence the velocity of the sphere resolved in a direction parallel to the plane is unaltered by the impact.

$$\text{Hence } V \sin \theta = u \sin \alpha \quad (1)$$

By Newton's experimental law, the relative velocity of the sphere along the common normal after impact is $(-e)$ times its relative velocity along the common normal before impact. Hence

$$v \cos \theta - 0 = -e (-u \cos \alpha - 0)$$

$$\text{(ie) } v \cos \theta = eu \cos \alpha \quad (2)$$

Squaring (1) and (2) and adding

We have

$$V^2 = u^2 (\text{Sin}^2 \alpha + e^2 \text{cos}^2 \alpha)$$

$$\text{(ie) } v = u \sqrt{\text{Sin}^2 \alpha + e^2 \text{Cos}^2 \alpha} \quad (3)$$

Dividing (2) by (1)

$$\text{We have } \text{Cot } \theta = e \text{ Cot } \alpha \quad (4)$$

Hence (3) and (4)

give the velocity and direction of motion after impact

Corollary : 1

If $e = 1$, we find that from (3), $v=u$ and from (4). $\theta = \alpha$,

Hence if a perfectly elastic sphere impinges on a fixed smooth plane, its velocity is not altered by impact and the angle of reflection is equal to the angle of incidence.

Corollary: 2

If $e = 0$, then from (2)

$$V \cos \theta = 0 \text{ and}$$

$$\text{From (3), } V = u \sin \alpha$$

Hence $\text{Cos } \theta = 0$.

$$\text{(ie) } \theta = 90^\circ$$

Hence the inelastic sphere slides along the plane with velocity $u \sin \alpha$.

Corollary: 3

If the impact is direct, we have $\alpha = 0$. Then $\theta = 0$ and from (3),

$$V = eu$$

Hence if an elastic sphere strikes a plane normally with velocity u , it will rebound in the same direction with velocity eu .

Corollary: 4

The impulse of the pressure on the plane is equal and opposite to the impulse of the pressure on the sphere. The impulse I on the sphere is measured by the change in momentum of the sphere along the common normal.

$$I = mV \cos \theta - (mu \cos \alpha) = m (r \cos \theta + u \cos \alpha)$$

$$= m (eu \cos \alpha + u \cos \alpha) = mu \cos \alpha (1 + e)$$

Corollary: 5

Loss of kinetic energy due to the impact

$$= \frac{1}{2} mu^2 - \frac{1}{2} mv^2$$

$$= \frac{1}{2} mu^2 - \frac{1}{2} mu^2 (\sin^2 \alpha + e^2 \cos^2 \alpha)$$

$$= \frac{1}{2} mu^2 (1 - \sin^2 \alpha - e^2 \cos^2 \alpha)$$

$$= \frac{1}{2} mu^2 (\cos^2 \alpha - e^2 \cos^2 \alpha)$$

$$= \frac{1}{2} (1 - e^2) mu^2 \cos^2 \alpha.$$

If the sphere is perfectly elastic.

$e = 1$ and loss of kinetic energy is zero.

Example: 1 (U.Q)

A smooth circular table is surrounded by a smooth rim whose interior surface is vertical. Show that a ball projected along the table from a point A on the rim in a direction making an angle α with the radius through A will return to the point of projection after two impacts if

$$\tan \alpha = \frac{e^{(3/2)}}{\sqrt{1+e+e^2}}$$

Prove along that, when the ball returns to the point of projection, its velocity is to its original velocity as $e^{(3/2)} : 1$

Solution:-

Let the ball starting from A return to it after two reflections at B and C. At B, the point of the first impact, the common normal is the radius OB and at C, the point of the second impact, the common normal is OC.

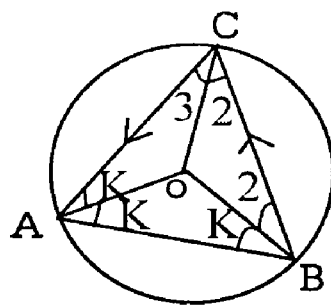
Let $\angle OBC = \beta$ and $\angle OCA = \gamma$.

Then $\angle OCB = \beta$ and $\angle OAC = \gamma$

Considering the impact at B, and applying equation (4). Of Impact of a smooth sphere.

We have $\text{Cot } \beta = e \text{ Cot } \alpha$

$$\text{(ie) } \tan \beta = \frac{1}{e} \tan \alpha \quad (1)$$



Similarly,

Considering the impact at C,

$$\text{Cot } \gamma = e \text{ Cot } \beta.$$

$$\therefore \tan \gamma = \frac{1}{e} \tan \beta = \frac{1}{e^2} \tan \alpha \quad (2)$$

Now, In $\triangle ABC$, $\angle A + \angle B + \angle C = 2\alpha + 2\beta + 2\gamma = 180^\circ$

$$\text{(ie) } \therefore \alpha + \beta + \gamma = 90^\circ \text{ or } \alpha = 90^\circ - (\beta + \gamma)$$

$$\therefore \tan \alpha = \tan (90^\circ - \overline{\beta + \gamma}) = \text{Cot } (\beta + \gamma)$$

$$= \frac{1}{\tan(\beta + \gamma)} = \frac{1 - \tan\beta \tan\gamma}{\tan\beta + \tan\gamma}$$

$$(ie) \tan \alpha (\tan \beta + \tan \gamma) = 1 - \tan \beta \tan \gamma .$$

$$(ie) \tan \alpha \left(\frac{1}{e} \tan \alpha + \frac{1}{e^2} \tan \alpha \right) = 1 - \frac{1}{e} \tan \alpha \frac{1}{e^2} \tan \alpha$$

$$(ie) \tan^2 \alpha \left(\frac{1}{e} + \frac{1}{e^2} \right) = 1 - \frac{\tan^2 \alpha}{e^3} \text{ Using (1) \& (2)}$$

$$(ie) \tan^2 \alpha \left(\frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} \right) = 1$$

$$(or) \tan^2 \alpha \left(\frac{1 + e + e^2}{e^3} \right) = 1$$

$$(ie) \tan^2 \alpha = \frac{e^3}{1 + e + e^2} \tag{3}$$

(or)

$$\tan \alpha = \frac{e^{(3/2)}}{\sqrt{1 + e + e^2}}$$

Let u be the velocity of projection from A, V be the velocity of the ball after the first impact at B. and W be the velocity after the second impact at C.

Applying equation (3)

$$V^2 = u^2 (\sin^2 \alpha + e^2 \cos^2 \alpha) \text{ and}$$

$$W^2 = v^2 (\sin^2 \beta + e^2 \cos^2 \beta)$$

$$W^2 = u^2 (\sin^2 \alpha + e^2 \cos^2 \alpha) (\sin^2 \beta + e^2 \cos^2 \beta)$$

$$= u^2 \cos^2 \alpha (\tan^2 \alpha + e^2) \cdot \cos^2 \beta (\tan^2 \beta + e^2)$$

$$\begin{aligned}
&= \frac{u^2(\tan^2 \alpha + e^2)(\tan^2 \beta + e^2)}{(1 + \tan^2 \alpha)(1 + \tan^2 \beta)} \\
&= \frac{u^2(\tan^2 \alpha + e^2)\left(\frac{1}{e^2} \tan^2 \alpha + e^2\right)}{(1 + \tan^2 \alpha)\left(1 + \frac{1}{e^2} \tan^2 \alpha\right)} \quad \text{Using (1)} \\
&= \frac{u^2(\tan^2 \alpha + e^2)(\tan^2 \alpha + e^4)}{(1 + \tan^2 \alpha)(e^2 + \tan^2 \alpha)} \\
&= \frac{u^2(\tan^2 \alpha + e^4)}{(1 + \tan^2 \alpha)} \\
&= \frac{u^2\left(\frac{e^4}{1 + e + e^2} + e^4\right)}{\left[1 + \frac{e^4}{1 + e + e^2}\right]} \quad \text{Substituting from(2)} \\
&= \frac{u^2(e^3 + e^4 + e^5 + e^6)}{1 + e + e^2 + e^3} = u^2 e^3
\end{aligned}$$

$$W = u.e^{(3/2)} \text{ or } W : u = e^{(3/2)} : 1$$

Example: 2 (U.Q)

A particle falls from a height h upon a fixed horizontal plane; let e be the coefficient of the restitution. Show that the whole distance described before the particle has finished rebounding is $h\left(\frac{1+e^2}{1-e^2}\right)$. Show also that the whole time

$$\text{taken is } \frac{1+e}{1-e} \sqrt{\frac{2h}{g}}.$$

Solution:

Let the velocity of the particle on first hitting the plane. Then $u^2=2gh$.

After the first impact, the particle rebounds with a velocity eu and ascends a certain height, retraces its path and makes a second impact with the plane. With velocity eu . After the second impact, it rebounds with a velocity e^2u and the process is repeated a number of times. The velocities after the third, fourth etc

impacts are e^3u , e^4u etc. The height ascended after the first impact with velocity

$$eu = \frac{(\text{velocity})^2}{2g} = \frac{e^2u^2}{2g}$$

The height ascended after the second impact with velocity $e^2u = e^4u^2/2g$ and so on.

∴ Total distance traveled before the particle stops rebounding

$$\begin{aligned} &= h + 2 \left(\frac{e^2u^2}{2g} + \frac{e^4u^2}{2g} + \frac{e^6u^2}{2g} + \dots \right) \\ &= h + \frac{2 \cdot e^2u^2}{2g} (1 + e^2 + e^4 + \dots \text{to } \infty) \\ &= h + \frac{e^2u^2}{g} \cdot \frac{1}{1-e^2} = h + \frac{e^2 \cdot 2gh}{h} \cdot \frac{1}{1-e^2} \\ &= h \left(1 + \frac{2e^2}{1-e^2} \right) = h \cdot \left(\frac{1+e^2}{1-e^2} \right) \end{aligned}$$

Considering the motion before the first impact we have the initial velocity = 0, acceleration = g, final velocity = u and so if t is the time taken,

$$u = 0 + gt.$$

$$\therefore t = \frac{u}{g} = \frac{\text{velocity}}{g}$$

Time interval between the first and second impact is

$$\begin{aligned} &= 2 \times \text{time taken for gravity to reduce the velocity } eu \text{ to } 0. \\ &= 2 \cdot \text{velocity}/g = 2eu/g. \end{aligned}$$

Similarly time interval between the second and third impacts = $2e^2u/g$ and so on.

So total time taken

$$= \frac{u}{g} + 2 \left(\frac{eu}{g} + \frac{e^2u}{g} + \frac{e^3u}{g} + \dots \text{to infinity} \right)$$

$$= \frac{u}{g} + \frac{2eu}{g} (1 + e + e^2 + \dots \text{to } \alpha) = \frac{u}{g} + \frac{2eu}{g} \cdot \frac{1}{1-e} = \frac{u}{g} \left[1 + \frac{2e}{1-e} \right]$$

$$= \frac{u}{g} \left(\frac{1+e}{1-e} \right) = \sqrt{\frac{2gh}{g}} \left(\frac{1+e}{1-e} \right)$$

$$\text{Total time} = \left(\frac{1+e}{1-e} \right) \sqrt{\frac{2h}{g}}$$

Example: 3

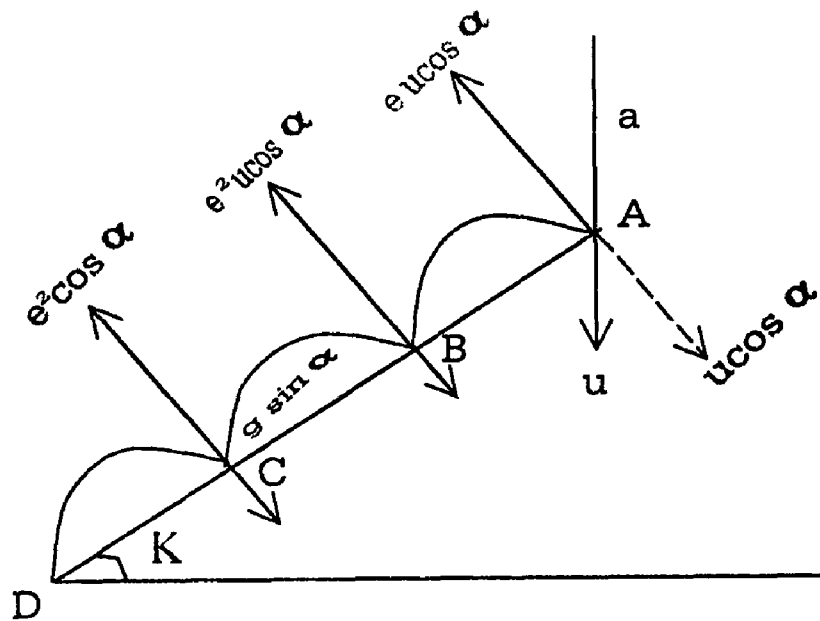
A particle of elasticity 'e' is dropped from a vertical height 'a' upon the highest point of a plane which is of length b and is inclined at an angle α to the horizon and descends to the bottom in three jumps. Show that

$$b = 4ae(1+e)(1+e^2)(1+e+e^2) \sin \alpha$$

Solution:

The downwards vertical velocity at A before string = $\sqrt{2ga}$ and let this be = u.

This can be resolved into



Two components as $u \cos \alpha \perp$ to the inclined plane and $u \sin \alpha$ parallel to the plane. At the impact at A, there is no force parallel to the plane and there is only the impulsive reaction normal to the plane. So the component $u \sin \alpha$ is not affected by impact while the component $u \cos \alpha$ is reversed as $e u \cos \alpha$. Hence the particle describes a parabola and strikes the plane at B with a velocity $e u \cos \alpha \perp$ to it after impact this is reversed as $e^2 u \cos \alpha$. The particle strikes the

inclined plane at C with a velocity $e^2u \cos \alpha$ normal to it, which is reversed as $e^3u \cos \alpha$.

Let t_1, t_2, t_3 be the times taken to describe the paths AB, BC and CD respectively.

Consider the motion, \perp to the inclined plane. Distance traveled is that direction in the time $t_1 = 0$. Applying the formula " $S = ut + \frac{1}{2}ft^2$ "

We have

$$0 = eu \cos \alpha \cdot t_1 - \frac{1}{2} g \cos \alpha \cdot t_1^2$$

$$t_1 = \frac{2eu \cos \alpha}{g \cos \alpha} = \frac{2eu}{g}$$

Similarly, $t_2 = \frac{2eu}{g}$ and $t_3 = \frac{2e^3u}{g}$.

Hence the total time taken from A to D

$$= t_1 + t_2 + t_3 = \frac{2e^3u}{g} (1+e+e^2)$$

In this period, the particle has described a distance b down the inclined plane, starting with an initial velocity $u \sin \alpha$ and acted on by an acceleration $g \sin \alpha$.

$$\therefore b = u \sin \alpha (t_1 + t_2 + t_3) + \frac{1}{2} g \sin \alpha (t_1 + t_2 + t_3)^2$$

$$(ie) b = u \sin \alpha \cdot \frac{2eu}{g} (1+e+e^2) + \frac{1}{2} g \sin \alpha \cdot \frac{4e^2u^2}{g} \times (1+e+e^2)^2$$

$$= \frac{2eu^2 \sin \alpha}{g} (1+e+e^2) + \frac{2e^2u^2 \sin \alpha}{g} (1+e+e^2)^2$$

$$= \frac{2eu^2 \sin \alpha}{g} (1+e+e^2) + \frac{2e^2u^2 \sin \alpha}{g} (1+e+e^2)^2$$

$$= \frac{2eu^2 \sin\alpha}{g} (1+e+e^2) + (1+e(1+e+e^2))$$

$$= \frac{2e \cdot 2ga \cdot \sin\alpha}{g} (1+e+e^2) (1+e)(1+e^2)$$

$$b = 4ae \sin \alpha (1+e) (1+e^2) (1+e+e^2)$$

Example: 4(U.Q)

A particle is projected from a point on an inclined plane and at the r^{th} impact, it strikes the plane \perp and at the n^{th} impact is at the point of projection. Show that $e^n - 2e^r + 1 = 0$.

Solution:

Let α be the inclination of the plane to the horizontal and u the velocity of projection at an angle θ to the inclined plane. This velocity can be resolved into two components, $u \cos\theta$ along the upward inclined plane and $u \sin\theta \perp$ to the inclined plane. The acceleration g can be resolved into two components, $g \sin\alpha$ along the downward inclined plane and $g \cos\alpha \perp$ to the inclined plane and downwards.

Consider motion \perp to the plane.

Let t_1 be the time upto the first impact Distance traveled \perp to the plane in time t_1 is 0. (ie) $0 = u \sin \theta \cdot t_1 - \frac{1}{2} g \cos \alpha \cdot t_1^2$

$$\therefore t_1 = \frac{2u \sin \theta}{g \cos \alpha}$$

The particle strikes the plane the first time with a velocity $u \sin \theta \perp$ to it and after this impact this component is reversed as $eu \sin \theta$. Hence time interval between the first and second impacts.

$$= 2eu \sin \theta / g \cos \alpha .$$

The particle strikes the plane a second time with a velocity $eu \sin \theta \perp$ to it and after the second impact, this component is reversed as $e^2u \sin \theta$.

Hence time interval between the second and third impacts = $2e^2u \sin \theta / g \cos \alpha$ and so on.

Time till the r^{th} impact

$$\begin{aligned}
 &= \frac{2u \sin \theta}{g \cos \alpha} + \frac{2eu \sin \theta}{g \cos \alpha} + \frac{2e^2u \sin \theta}{g \cos \alpha} + \dots \text{to } r \text{ terms.} \\
 &= \frac{2u \sin \theta}{g \cos \alpha} (1 + e + e^2 + \dots \text{to } r \text{ terms}) \\
 &= \frac{2u \sin \theta}{g \cos \alpha} \cdot \left(\frac{1 - e^r}{1 - e} \right) \tag{1}
 \end{aligned}$$

At the end of this time, the particle strikes the plane \perp^r . So the velocity parallel to the plane at that instant = 0

$$\text{Hence } u \cos \theta - g \sin \alpha \cdot \frac{2u \sin \theta}{g \cos \alpha} \cdot \frac{1 - e^r}{1 - e} = 0$$

$$\text{(ie) } \cos \theta \cos \alpha (1 - e) = 2 \sin \alpha \sin \theta (1 - e^r) \tag{2}$$

Putting $r = n$ is (1)

$$\text{Time till the } n^{\text{th}} \text{ impact} = \frac{2u \sin \theta}{g \cos \alpha} \cdot \frac{1 - e^n}{1 - e}$$

Now, it is at the point of projection.

Hence the distance traveled parallel to the plane upto this time = 0.

$$\text{(ie) } u \cos \theta \cdot \frac{2u \sin \theta}{g \cos \alpha} \cdot \frac{1 - e^n}{1 - e} - \frac{1}{2} g \sin \alpha \left(\frac{2u \sin \theta}{g \cos \alpha} \cdot \frac{1 - e^n}{1 - e} \right)^2 = 0.$$

$$\text{(ie) } \cos \theta \cos \alpha (1 - e) = \sin \alpha \sin \theta (1 - e^n) \tag{3}$$

Dividing (2) by (3)

We have

$$= \frac{2(1-e^r)}{1-e^n}$$

$$(ie) 2 - 2e^r = 1 - e^n$$

$$or e^n - 2e^r + 1 = 0$$

Example :5

A ball is thrown from a point on a smooth horizontal ground with a speed V at an angle α to the horizon. If e be the coefficient of restitution, Show that the total time for which the ball rebounds on the ground is $\frac{2V \sin \alpha}{g(1-e)}$ and the horizontal

distance traveled by it is $\frac{V^2 \sin 2\alpha}{g(1-e)}$

Solution:

The initial horizontal and vertical components of the velocity are $V \cos \alpha$ and $V \sin \alpha$. The particle describes a parabola and strikes the horizontal plane normally with velocity $u \sin \alpha$.

Due to impact, the horizontal velocity is not affected while the vertical component is reversed as $e^v \sin \alpha$.

Similarly the vertical components of the velocity after the second, third etc., impacts are $e^2V \sin \alpha$, $e^3 V \sin \alpha$ etc.

Let t_1, t_2, t_3 etc be the time for the successive trajectories. $t_1 = \frac{2V \sin \alpha}{g}$

$t_2 = \frac{2eV \sin \alpha}{g}$, $t_3 = \frac{2e^2V \sin \alpha}{g}$ and so on. So total time that elapses before the party stops rebounding.

$$= \frac{2V \sin \alpha}{g} + \frac{2eV \sin \alpha}{g} + \frac{2e^2 \sin \alpha}{g} + \dots$$

$$= \frac{2v \sin \theta}{g} (1 + e + e^2 + \dots \text{to } \alpha)$$

$$= \frac{2V \sin \theta}{g} \cdot \frac{1}{1-e} = \frac{2V \sin \alpha}{g(1-e)}$$

Through out this time, the horizontal component $V \cos \alpha$ is not affected. So horizontal distance described during this time.

$$= V \cos \alpha \cdot \frac{2V \sin \alpha}{g(1-e)}$$

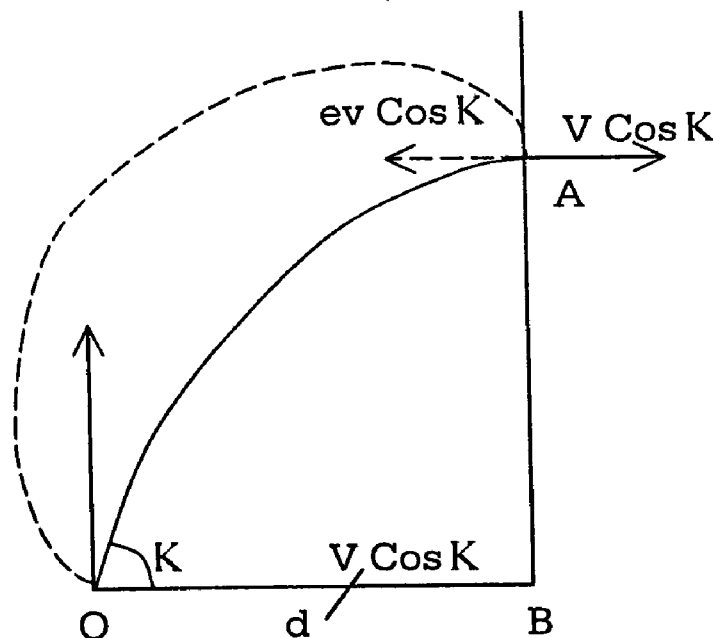
$$= \frac{V^2 \sin 2\alpha}{g(1-e)}$$

Example: 6

An elastic sphere is projected from a given point O with given velocity V at an inclination α to the horizontal and after hitting a smooth vertical wall at a distance d from O returns to O. Prove that $d = \frac{V^2 \sin 2\alpha}{g} \cdot \frac{e}{1+e}$ where e is the coefficient of restitution.

Solution:

Let the particle strike the wall at A. From O to A, the particle describes a parabola under gravity with constant horizontal velocity $V \cos \alpha$. Let t_1 be the time for this $V \cos \alpha \cdot t_1 = d$ (1)



At the impact at A there is no force parallel to the wall. The component $V \cos \alpha$ being \perp to the wall is reversed as $ev \cos \alpha$. The particle will describe another parabola with constant horizontal velocity $ev \cos \alpha$ and return to O. Let t_2 be the time for this return journey. Then $ev \cos \alpha \cdot t_2 = d$ (2)

But the vertical motion is not affected by impact and throughout the interval $t_1 + t_2$ it is subject to retardation by g only.

As the particle returns to O , vertical distance described in time $t_1 + t_2 = 0$

$$\therefore 0 = V \sin \alpha (t_1 + t_2) - \frac{1}{2} g (t_1 + t_2)^2 \quad \text{or}$$

$$\text{or } t_1 + t_2 = \frac{2V \sin \alpha}{g} \quad (3)$$

Substituting for t_1, t_2 from (1) and (2) is (3).

We have

$$\frac{d}{V \cos \alpha} + \frac{d}{eV \cos \alpha} = \frac{2V \sin \alpha}{g}$$

$$(ie) \frac{d(e+1)}{eV \cos \alpha} = \frac{2v \sin \alpha}{g}$$

$$\text{or } d = \frac{2ev^2 \sin \alpha \cos \alpha}{g(1+e)}$$

$$\therefore d = \frac{V^2 \sin 2\alpha}{g} \cdot \frac{e}{1+e}$$

Exercise

- 1) a) An elastic ball of mass m falls from a height h on a fixed horizontal plane and rebounds show that the loss of K.E. by the impact is $mgh(1 - e^2)$.
 b) A particle falls from a height h in time t upon a fixed horizontal plane. Prove that it rebounds and reaches a maximum height e^2h in time etc.
- 2) A Particle dropped through a vertical distance of 20 metres, Strikes the highest point of a plane which is of length 12m and is inclined at 30° to the horizon. If the particle descends to the bottom in three jumps. Show that $e(1+e)(1+e^2)(1+e+e^2) = 0.3$.

IMPULSES

7.3 Impulsive Forces

7.3.1 Impulse:

The term impulse of force is defined as follows:

- i) The impulse of a constant F during a time interval T is defined to be the product FT .

Let f be the constant acceleration produced on a particle of mass m on which F acts and u, v be respectively the velocity at the beginning and end of the period T . Then

$$V - U = f \cdot T \text{ and } F = mf$$

Hence the impulse $I = FT = mfT = m(v - u)$

= Change of momentum produced.

- 3) The impulse of a variable force F during a time interval T is defined to be the time integral of the force for the interval.

(ie) Impulse $I = \int_0^T F dt$.

This is got as follows

During a short interval of time Δt , the force F can be taken to be constant and hence elementary impulse in this interval = $F \cdot \Delta t$.

Hence the impulse during the whole time T for which the force F acts in the sum of such impulses and

$$= \lim_{\Delta t \rightarrow 0} \sum_{t=0}^T F \Delta t = \int_0^T F \cdot dt$$

Since F is variable. $F = m \cdot \frac{dv}{dt}$.

So impulse = $\int_0^T m \cdot \frac{dv}{dt} dt = [mv]_{t=0}^T = mv - mu$

Where u and v are the velocities at the beginning and end of the interval and hence this is also equal to the change of momentum produced. Thus whether a force is a variable or constant, its impulse = Change of momentum produced.

7.32 Impulsive Force:

The change of momentum produced by a variable force P acting on a body of mass m from time $t = t_1$ to $t = t_2$ is $\int_{t_1}^{t_2} P \cdot dt$.

Suppose P is very large but the time interval $t_2 - t_1$ during which it acts is very small. It is quite possible that the above definite integral tends to a finite limit. Such a force is called an impulsive force. This an impulsive force is one of large magnitude which acts for a very short period of time and yet produces a finite change of momentum.

Theoretically an impulsive force should be infinitely great and the time during which it acts must be very small. This of course is never realised in practice, but approximate examples are (1) the force produced by a hammer blow (2) the impact of a bullet on a target in such cases the measurement of the magnitude of the actual force is impracticable but the change in momentum produced may be easily measured. Thus an impulsive force is measured by its impulse.

(ie) The change of momentum it produces.

Since an impulsive force acts only for short time on a particle, during this time the distance traveled by a particle having a finite velocity is negligible. Also suppose a body is acted upon by a impulsive and finite forces simultaneously. Since the time of action of the impulsive forces is very short, during this time the effect of the ordinary finite forces can be neglected.

7.34 Impact of two bodies:

If two bodies A and B impinge on each other then we know, by Newton's third law that the action of A on B is equal and opposite to that of B on A , during the period in which they are in contact, and further these forces of action and reaction act along the common normal to the surfaces which are in contact. Hence the impulse of the force exerted by A on B is equal and opposite to that of the forces exerted by B on A . It follows that the change in momentum of A is equal and opposite to the change in momentum of B , the moments being measured along the common normal.

Hence taken together, the total change of moments of A and B is zero. In other words the sum of the momenta of the bodies, measured along the common normal is not altered by impact. This is called the principle of conservation of Linear Momentum which is used in dealing with problems in which impacts or impulsive forces occur.

Loss of Kinetic Energy in Impact:

Let a mass m , moving with velocity v , strike a mass M , which is free to move in the direction of m 's motion. After impact, let the two move together as a single body with velocity V . As there is no loss of momentum due to the impact.

$$(m + M) V = mV \quad (1)$$

The kinetic energy before impact = $\frac{1}{2} mV^2$

The K.E after impact = $\frac{1}{2} (m + M)V^2$

$$= \frac{1}{2} (m+M) \cdot \frac{m^2V^2}{(m+M)^2} = \frac{1}{2} m \left(\frac{m}{m+M} \right) V^2$$

Since $\frac{m}{m+M}$ is < 1 , the K.E after impact is clearly less than $\frac{1}{2} mV^2$ which is the K.E before impact. Hence there is a loss of kinetic energy due to impact and hence the principle of conservation of energy must never be used in problems where impulsive forces occur.

7.3.5 Motion of a Shot and gun:

When a gun is fired, power is immediately converted into a gas at a very high pressure and this gas in trying to expand, forces the shot forwards. An equal and opposite reaction is exerted on the gun. The forward momentum generated in the shot at the instant when it leaves the barrel is equal to the backward momentum generated in the gun. This, if m and M be the masses of the shot and the gun, V being the muzzle velocity with which the shot emerges from the gun, the gun will recoil with a velocity given by $MV = mv$.

When the barrel of the gun is elevated, we cannot say that the momenta of the shot and the gun are equal and opposite. In this case, the horizontal momentum of the gun will be equal and opposite to the horizontal momentum of the shot. The vertical momentum imparted to the gun will be at once destroyed by the impulsive pressure of the plane on which it stands.

7.3.4 Impact of water on a surface:

In the case of a jet of water impinging against a fixed surface or a continuous fall of rain on the ground we are dealing with a series of successive impacts or impulsive forces. We can calculate the amount of momentum destroyed per second and this will give us the average force on the surface.

Worked Examples

Example: 1

A 100gm. Cricket ball moving horizontally at 24 m./sec was hit straight back with a speed of 15 m/sec if the contact lasted $\frac{1}{20}$ second find the average force exerted by the bat.

Let F dynes be the average force exerted by bat on the ball and I be its impulse.

$$I = \text{Change of momentum produced.}$$

$$= 100 [24 - (-15)] = 3900 \text{ units}$$

we know that $I = ft$

$$\therefore F = \frac{I}{t} = \frac{3900}{\frac{1}{20}} = 78,000 \text{ dynes}$$

Example: 2

A jet of water leaves a nozzle of 3cm. diameter at a speed of 2m/sec. and impinges normally on a plane elastic wall so that the velocity of the water is destroyed on reaching the wall. Calculate in gm. Weight the thrust on the wall.

$$\text{Area of cross section on the nozzle} = \pi \left(\frac{3}{2}\right)^2 = 7.07 \text{ cm}^2$$

As water issues forth with a velocity of 200cm/sec a column of length 200cm is discharged every second

\therefore Volume of water discharged per second

$$= 7.07 \times 200 = 1414 \text{ cm}^3$$

Since density of water = 1 gm/cm^3 , mass of water discharged per.sec. = 1414gms, and its velocity in 200cm/sec which is reduced to zero after striking the wall.

\therefore The momentum destroyed parsec.

$$= 1414 \times 200 = 2,82,8000 \text{ units (absolute)}$$

\therefore Thrust on the wall = 2828000

$$= 2828000/981 \text{ or } 288.2 \text{ gm/ut}$$

Example 3:

8 centimeters of rain fall in certain district is 24 hours. Assuming that the drops fall freely from a height of 109meter find the pressure in the ground per square kilometer of the district.

The velocity of rain in striking the ground

$$= \sqrt{2g \times 109} = \sqrt{2 \times 981 \times 109}$$

$$= \sqrt{2 \times 9 \times 109 \times 109}$$

$$= 327 \sqrt{2} \text{ cm./sec.}$$

Volume of vain that falls on a sq.cm in 24 hours

$$= 1^2 \times 8 = 8\text{cc.}$$

\therefore Mass of rain that falls on a sq.cm in one sec.

$$= 8 \times 1 \times \frac{1}{24 \times 60 \times 60} = \frac{1}{10,800} \text{ gms}$$

\therefore Momentum destroyes parsec. Due to reaction of the ground on the rain drops

$$= \frac{1}{10,800} \times 327 \sqrt{2} \text{ units.}$$

The impulsive pressure on the ground is equal to the number of units of momentum destroyed parsec.

$$\therefore \text{pressure on the ground per sq.cm} = \frac{327\sqrt{2}}{10800} \text{ dynes}$$

$$\text{Pressure per sq.km} = \frac{327\sqrt{2}}{10800} \times 10^{10} \text{ dynes}$$

$$= \frac{327\sqrt{2} \times 10^{10}}{10800 \times 981}$$

$$\text{Pressure per sq.km} = \frac{\sqrt{2} \times 10^8}{324} \text{ gms.wt.}$$

Example: 4

A shot of mass m penetrates a thickness of a fixed plate of mass M . If M were free to move and the resistance supposed to be uniform. Show that the thickness penetrated is $\frac{Mt}{M+m}$.

Solution:

Let u denote the initial velocity of the shot and F the force of resistance of the plate to the shot which is equal and opposite to that on the plate by the shot.

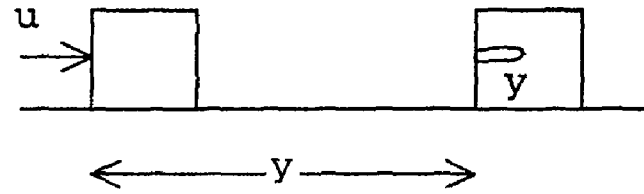
Let f be the retardation due to the resistance. In the first case, when the plate is fixed, the velocity u of the shot is reduced to 0 by retardation f in a distance t .

$$\therefore 0 = u^2 - 2ft \text{ or } f = \frac{u^2}{2t}$$

$$\therefore F = mf = \frac{mu^2}{2t} \tag{1}$$

In the second case when the plate is free to move, as the shot penetrates the plate, the velocity of the shot diminishes due to the resisting force F and the velocity of the plate increases from zero due to the equal and opposition reaction acting on it. The penetration will last as long as the velocity of the shot is greater than the velocity of the plate and it will stop when both the plate and the shot acquire a common velocity v (say).

Let X be the distance moved by the plate upto this instant and Y the thickness penetrated by the shot into the plate. Then $X + Y$ is the distance traveled by the shot in space.



By the principle of conservation of momentum for the shot and the plate considered as a compound body,

We have

$$(M + m) V = mu \quad (2)$$

The acceleration due to F on the plate of mass

$$M = \frac{F}{M} = \frac{mu^2}{2tM} \quad \text{Using (1)}$$

Due to this acceleration, the velocity of the plate increases from 0 to V in a distance X .

$$\therefore V^2 = \frac{2mu^2}{2tM} X = \frac{mu^2}{tM} X \quad (\text{or})$$

$$X = \frac{V^2 t M}{mu^2} \quad (3)$$

Considering the motion of the shot in space, its velocity decreases from u to v in a distance $X + Y$ due to the retardation f .

$$\therefore v^2 = u^2 - 2f \cdot (x + y)$$

$$\text{or } x + y = \frac{u^2 - v^2}{2f} = \frac{u^2 - v^2}{2\left(\frac{u^2}{2t}\right)} = \frac{t(u^2 - v^2)}{u^2} \quad (4)$$

(4) - (3) gives

$$Y = \frac{t(u^2 - v^2)}{u^2} - \frac{v^2 t M}{mu^2}$$

$$\begin{aligned}
&= t - \frac{v^2 t}{u^2} - \frac{v^2 t M}{m u^2} \\
&= t - \frac{v^2 t}{u^2} - \frac{v^2 t M}{m u^2} \\
&= t - \frac{v^2 t}{u^2} (m + M) \\
&= t - \frac{t}{m u^2} (m + M) \left(\frac{m u}{m + M} \right)^2
\end{aligned}$$

substituting for v from (2)

$$y = t - \frac{t m}{m + M} = \frac{M t}{m + M} \text{ and this is the thickness penetrated.}$$

Example: 5

A shot of mass 100kg is fired with a velocity of 250m. Per sec from a gun of mass 400 quintals. Find the velocity of recoil of the gun. If the gun is resting on an incline of 3 in 5 and the shot is fired horizontally, find the velocity of recoil of the gun.

Solution:

Let V cm per sec, be the velocity of recoil of the gun. In the first case since the momentum of the gun is equal and opposite to that of the shot,

We have

$$\begin{aligned}
400 \times 100 \times 100 \times v &= 100 \times 1000 \times 250 \times 100 \\
\text{(ie) } 4 \times 10^7 &= 25 \times 10^8 \text{ or } V = \frac{250}{4} = 6.25 \text{ cm/sec.}
\end{aligned}$$

In the second case the gun is resting on an inclined plane whose inclination to the horizon is α . It is given that

$$\sin \alpha = \frac{3}{5}$$

$$\therefore \cos \alpha = \frac{4}{5}$$

Momentum of the shot parallel to the plane

$$= 100 \times 1000 \times 250 \times 100 \cos \alpha$$

$$= 25 \times 10^8 \times \frac{4}{5} \times 20 \times 10^8$$

$$= 2 \times 10^9$$

Momentum of the gun parallel to the plane

$$= 400 \times 100 \times 1000 \times V$$

$$= 4 \times 10^7 v = 2 \times 10^9$$

$$\therefore V = \frac{2 \times 10^9}{4 \times 10^7} = \frac{10^2}{2} = 50 \text{ cm/sec.}$$

Example: 6

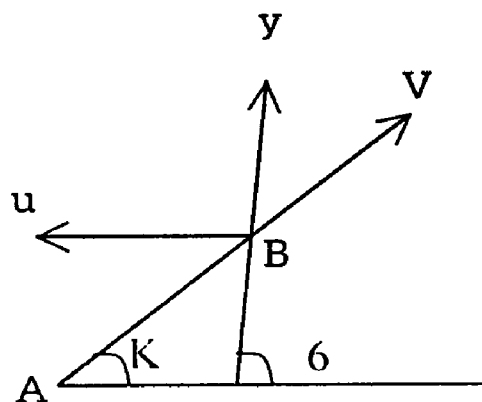
A gun of mass M fires a shell of mass m , the elevation of the gun being α . If the gun can recoil freely in the horizontal direction show that the angle θ which the path of the shell initially makes with the horizontal is given by the equation

$\tan \theta = \left(1 + \frac{m}{M}\right) \tan \alpha$. Further, assuming that the whole energy of the explosion is

transferred to the shell and the gun, show that the muzzle energy of the shell is less than what it would be if the gun were fixed, in the ratio $M : M + m \cos^2 \theta$.

Solution:

Let AB represent the barrel of the gun. As the shot the barrel, the gun is moving backward. Let U be this backward velocity of gun. Let v be the velocity of the shot relative to the gun. The velocity will be in the direction of the barrel AB . Let v be the actual velocity of the shot at an angle θ to the horizontal as shown in the figure. We know that v is the resultant of V and U .



Resolving V horizontally and vertically.

$$\text{We have } V \cos \theta = V \cos \alpha - u \quad (1)$$

$$\text{and } V \sin \theta = V \sin \alpha \quad (2)$$

$$\text{Dividing (2) by (1) , } \tan \theta = \frac{V \sin \alpha}{V \cos \alpha - u} \quad (3)$$

Also the horizontal forward momentum of the shot is the horizontal backward momentum of the gun

$$\text{i.e. } mu \cos \theta = Mu \quad (4)$$

$$\text{i.e. } m(v \cos \alpha - u) = Mu \text{ using (1)}$$

$$mv \cos \theta = (m+M) u \text{ (or) } v = \frac{(m+M)u}{m \cos \alpha} \quad (5)$$

Putting this value of v from (5) in (3), we have

$$\begin{aligned} \tan \theta &= \frac{\frac{(m+M)u}{m \cos \alpha} \sin \alpha}{\frac{(m+M)u}{m \cos \alpha} \cos \alpha - u} = \frac{(m+M) \sin \alpha}{(m+M) \cos \alpha - m \cos \alpha} \\ &= \frac{(m+M) \sin \alpha}{m \cos \alpha} = \left(\frac{m+M}{m} \right) \tan \alpha = \left(\frac{m}{M} + 1 \right) \tan \alpha \end{aligned}$$

Energy of explosion = energy of shot + energy of gun

$$= \frac{1}{2} mv^2 + \frac{1}{2} mu^2$$

If the gun had been fixed all the energy to explosion would have been transferred to the shot.

∴ Muzzle energy of the shot

when the gun is fixed $\approx \frac{1}{2} mv^2$

$$\text{Required ratio} = \frac{\frac{1}{2}mv^2}{\frac{1}{2}mv^2 + \frac{1}{2}Mu^2} = \frac{mv^2}{mv^2 + Mu^2}$$

$$= \frac{m \left(\frac{M^2 u^2}{m^2 \cos^2 \theta} \right)}{m \frac{M^2 u^2}{m^2 \cos^2 \theta} + Mu^2} \quad \text{Substituting for } V \text{ from (4)}$$

$$= \frac{mM^2U^2}{mM^2u^2 + M^2m^2u^2 \cos^2 \theta} = \frac{M}{M + m \cos^2 \theta}$$

Example: 7

A mass m after falling freely through a distance 'a' begins to raise a mass M greater than itself and connected with it by means of an inextensible string passing over a fixed pulley. Show that M will have returned to its original position

on the end of time $\frac{2m}{M-m} \sqrt{\frac{2a}{g}}$.

Find also what fraction of the kinetic energy of m is destroyed at the instant when M is jerked into motion.

Solution:

Velocity of m when it has fallen through a distance 'a' $= \sqrt{2ga} = u$ (say).

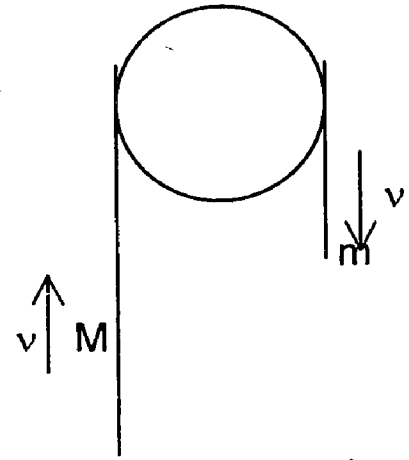
Now the string becomes tight and there is a jerk in the string. This jerk is in the nature of an impulsive force, being a very great force acting for a short time. As a result of this impulsive action, the system acquires a common velocity V and then finite motion begins. M rise upwards with this velocity.

By the principle of conservation of momentum.

$$(M+m)v = mu = m\sqrt{2ga} \quad (1)$$

Let f be the common acceleration of the system.

$$\text{We know that } f = \frac{(M-m)}{M+m} g$$



This acceleration will be acting on M downwards. M has a velocity upwards.

So it will rise to a certain height from its first position for a time t given by $0 = V - ft$.

$$\text{i.e. } t = \frac{V}{f} = \frac{V(M+m)}{(M-m)g} = \frac{m\sqrt{2ag}}{(M-m)g} \quad \text{using (1)}$$

Subsequently M will trace its path and reach its previous position after a further interval of time.

\therefore Total time taken by M to reach its original position

$$= 2t = \frac{2m\sqrt{2ag}}{(M-m)g} = \frac{2m}{M-m} \sqrt{\frac{2a}{g}}$$

$$\text{K.E. of the system before the jerk} = \frac{1}{2} mu^2 = mag$$

$$\text{K.E. of the system after the jerk} = \frac{1}{2} (M+m)v^2$$

$$= \frac{1}{2} (m+m) \cdot \frac{m^2 u^2}{(M+m)^2} = \frac{m^2 ag}{M+m}$$

$$\text{Hence Loss of K.E} = mag - \frac{m^2 ag}{M+m} = \frac{mMag}{M+m}$$

\therefore Fraction of the K. E. destroyed

$$= \frac{mMag}{m+M} \div mag = \frac{M}{m+M}$$

Exercise:

1) a) A shell explodes and breaks into two fragments of masses m_1 , m_2 moving with initial velocities U_1 and U_2 respectively in opposite directions. Show that there is a gain in kinetic energy of magnitude (U.Q) $\frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (u_1 + u_2)^2$.

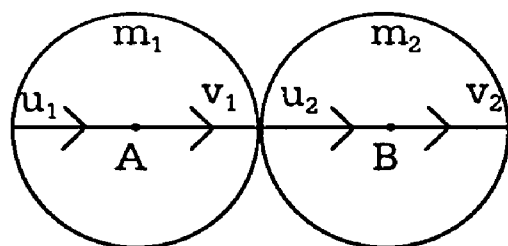
2) b) A shell lying in a straight smooth horizontal tube suddenly explodes and breaks into portions of masses m_1 and m_2 if d is the distance apart of the masses after a time t , show that the work done by the explosion is $\frac{1}{2} \cdot \frac{m_1 m_2}{m_1 + m_2} \frac{d^2}{t^2}$.

3) A shot of mass m is discharged from a gun of mass nm with a relative velocity V . Find the velocities of the shot and the gun and show that the total kinetic energy generated is $\frac{1}{2} \frac{nm}{n+1} v^2$.

7.4 Direct Impact of two smooth spheres:

A smooth sphere of mass m_1 impinges directly with velocity v_1 on another smooth sphere of mass m_2 moving in the same direction with velocity u_2 ; if the coefficient of restitution is e , to find their velocities after the impact.

AB is the line of impact, i.e. the common normal. Due to the impact, there is no tangential force and hence, for either sphere the velocity along the tangential direction is not altered by impact. But before impact, the spheres had been moving only along the line AB (as this is a case of direct impact). Hence for either sphere, tangential velocity after impact = its tangential velocity before impact = 0. So after impact, the spheres will move only in the direction AB. Let their velocities be V_1 and V_2 .



By Newton's experimental law, the relative velocity of m_2 with respect to m_1 after impact is $(-e)$ times the corresponding relative velocity before impact.

$$V_2 - V_1 = -e (u_2 - u_1) \quad (1)$$

By the principle of conservation of momentum, the total momentum along the common normal after impact is equal to the total momentum in the same direction before impact.

$$\therefore m_1 u_1 + m_2 v_2 = m_1 u_1 + m_2 u_2 \quad (2)$$

(2) - (1) \times m_2 gives

$$V_2 (m_1 + m_2) = m_1 v_1 + m_2 v_2 + e m_2 (u_2 - u_1)$$

$$= m_2 u_2 (1 + e) + (m_1 - e m_2) u_1$$

$$\therefore V_1 = \frac{m_2 u_2 (1 + e) + (m_1 - e m_2) u_1}{m_1 + m_2} \quad (3)$$

(1) \times m_1 + (2) gives

$$V_2 (m_1 + m_2) = -e m_1 (u_2 - u_1) + m_1 u_1 + m_2 u_2$$

$$= m_1 u_1 (1 + e) + (m_2 - e m_1) u_2$$

$$\therefore V_2 = \frac{m_1 u_1 (1 + e) + (m_2 - e m_1) u_2}{m_1 + m_2} \quad (4)$$

Equation (3) and (4) give the velocities of the spheres after impact.

Note:-

If one sphere say m_1 , is moving originally in a direction opposite to that of m_2 the sign of u_2 will be negative. Also it is most important that the directions of v_1 and v_2 must be specified clearly. Usually we take the positive direction as from left to right and then assume that both v_1 and v_2 are in the direction. If either of them is actually in the opposite direction the value obtained for it will turn to the negative.

In writing equation (1) corresponding to Newton's law the velocities must be subtracted in the same order on the both sides. In all problems it is better to draw a diagram showing clearly the positive direction and the directions of the velocities of the bodies.

Corollary. 1

If the two spheres are perfectly elastic and of equal mass, then $e = 1$ and $m_1 = m_2$. then from equations(3) and (4), we have

$$V_1 = \frac{m_1 \cdot u_2 \cdot 2 + 0}{2m_1} = u_2 \quad \text{and} \quad V_2 = \frac{m_1 \cdot u_1 \cdot 2 + 0}{2m_1} = u_1$$

i.e. if two equal perfectly elastic spheres impinge directly, they interchange their velocities.

Corollary. 2

The impulse of the blow on the sphere

A of mass $m_1 =$ change of momentum of A
 $= m_1 (u_1 - u_2)$.

$$\begin{aligned} &= m_1 \left[\frac{m_2 u_2 (1 + e) + (m_1 e m_2) u_1}{m_1 + m_2} - u_1 \right] \\ &= m_1 \left[\frac{m_2 u_2 (1 + e) + m_1 u_1 - e m_2 u_1 - m_1 u_1 - m_2 u_1}{m_1 + m_2} \right] \\ &= \frac{m_1 [m_2 u_2 (1 + e) - m_2 u_1 (1 + e)]}{m_1 + m_2} \end{aligned}$$

The impulsive blow on m_2 will be equal and opposite to the impulsive blow on m_1 .

7.4.1 Loss of kinetic energy due to direct impact of two smooth spheres:

Two spheres of given masses with given velocities impinge directly; to show that there is a loss of kinetic energy and to find the amount.

Let m_1, m_2 be the masses of the spheres, u_1 and u_2, v_1 and v_2 be their velocities before and after impact and e the coefficient of restitution.

By Newton's law,

$$v_2 - v_1 = -e(u_2 - u_1) \tag{1}$$

By the principle of conservation of momentum,

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2 \quad (2)$$

$$\text{Total kinetic energy before impact} = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$$

$$\text{And total kinetic energy after impact} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

Change in K.E = initial K.E - Final K.E.

$$= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - \frac{1}{2} m_1 u_1^2 - \frac{1}{2} m_2 u_2^2$$

$$= \frac{1}{2} m_1 (u_1 - v_1) (u_1 + v_1) - \frac{1}{2} m_2 (u_2 - v_2) (u_2 + v_2)$$

$$= \frac{1}{2} m_1 (u_1 - v_1) (u_1 + v_1) + \frac{1}{2} m_2 (v_2 - u_2) (u_2 + v_2)$$

$$[\because m_2 (u_2 - v_2) = m_1 (u_1 - v_1) \text{ from (2) }]$$

$$= \frac{1}{2} m_1 (u_1 - v_1) [u_1 + v_1 - (u_2 + v_2)]$$

$$= \frac{1}{2} m_1 (u_1 - v_1) [u_1 - v_2 - (u_2 - v_1)]$$

$$= \frac{1}{2} m_1 (u_1 - v_1) [u_1 - u_2 + e (u_2 - u_1)] \text{ using (1)}$$

$$= \frac{1}{2} m_1 (u_1 - v_1) (u_1 - u_2) (1 - e) \quad (3)$$

Now from (2), $m_1 (u_1 - v_1) = m_2 (v_2 - u_2)$

$$\therefore \frac{u_1 - v_1}{m_2} = \frac{v_2 - u_2}{m_1} \text{ and each} = \frac{u_1 - v_1 + v_2 - u_2}{m_1 + m_2}$$

$$\begin{aligned} \text{ie. Each} &= \frac{(u_1 - u_2) + (v_2 - v_1)}{m_1 + m_2} \\ &= \frac{(u_1 - u_2) - e(u_2 - u_1)}{m_1 + m_2} \text{ using (1)} \\ &= \frac{(u_1 - u_2)(1 + e)}{m_1 + m_2} \end{aligned}$$

$$\therefore u_1 - v_1 = \frac{m_2(u_1 - u_2)(1 + e)}{m_1 + m_2} \text{ and substituting this in (3)}$$

$$\begin{aligned} \text{Change in K.E.} &= \frac{1}{2} \frac{m_1 m_2 (u_1 - u_2)(1 + e)(u_1 - u_2)(1 - e)}{m_1 + m_2} \\ &= \frac{1}{2} \frac{m_1 m_2 (u_1 - u_2)^2 (1 - e^2)}{m_1 + m_2} \end{aligned} \tag{4}$$

As $e < 1$, the expression (4) is always positive and so the initial K.E. of the system is greater than the final K.E. So there is actually a loss of total K.E. by a collision. Only in the case, when $e = 1$ (ie). Only when the bodies are perfectly elastic, the expression (4) becomes zero and hence the total K.E. is unchanged by impact.

Example: 1

A ball of mass 8gm. Moving with a velocity of 10cm per sec. impinges directly on another of mass 24gms, moving at 2 cm per sec in the same direction. If $e = \frac{1}{2}$ find the velocities after impact. Also calculate the loss in kinetic energy.

Solution:

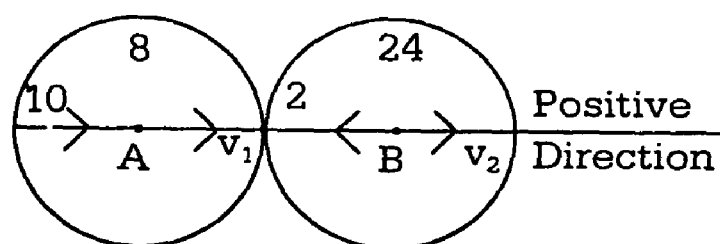


Fig (a)

Let V_1 and V_2 cm per sec. Be the velocities of the masses 8gm and 24gm respectively, after impact.

$$\text{By Newton's law, } V_2 - V_1 = -\frac{1}{2} (2 - 10) = 4 \quad (1)$$

By the principle of momentum ,

$$24 V_2 + 8 V_1 = 24 \times 2 + 8 \times 10 = 128$$

$$\text{i.e } 3 V_2 + V_1 = 16 \quad (2)$$

Solving (1) and (2), $V_1 = 1\text{cm/sec.}, V_2 = 5\text{cm/sec.}$

$$\begin{aligned} \text{The K.E. before impact} &= \frac{1}{2} \cdot 8 \cdot 10^2 + \frac{1}{2} \cdot 24 \cdot 2^2 \\ &= 448 \text{ dynes} \end{aligned}$$

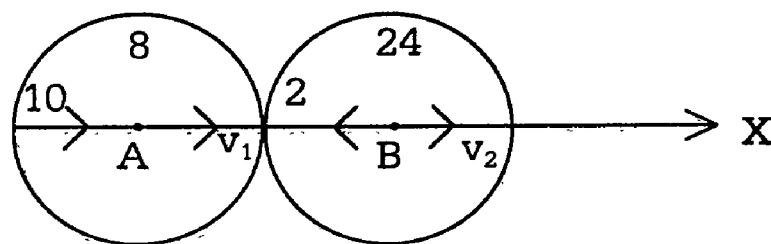
$$\text{The K.E. after impact} = \frac{1}{2} \cdot 8 \cdot 1^2 + \frac{1}{2} \cdot 24 \cdot 5^2 = 304 \text{ dynes}$$

\therefore Loss in K.E = 144 dynes

Example: 2

If the 24 gm, Mass in the preview question be moving in a direction opposite to that of the 8gm mass finds the velocities after impact.

Solution:



Let V_1 and V_2 cm/sec. Be the velocities of the 8gm and 24gms mass respectively after impact.

By Newton's law

$$V_2 - V_1 = -\frac{1}{2}(-2-10) = 6 \quad (1)$$

By conservation of momentum,

$$24 V_2 + 8 V_1 = 24 \times (-2) + 8 \times 10 = 32 \text{ ie. } 3v_2 + V_1 = 4 \quad (2)$$

Solving (1) and (2), $V_1 = \frac{1}{2}$ cm/sec, $V_2 = \frac{5}{2}$ cm/sec.

The negative sign of v_1 shows that the direction of motion of the 8gm.mass is reversed as we had taken the direction left to right as positive and assumed V_1 to be in this direction. Since V_2 is positive the 24 gm. Ball moves from left to right after impact, so that its direction of motion is also reversed.

Example:3 (U.Q)

A ball overtakes another ball of m times its mass, which is moving with $\frac{1}{n}$ th of its velocity in the same direction if the impact reduces the first ball to rest, prove that the coefficient of elasticity is $\frac{m+n}{m(n-1)}$.

Deduce that $m > \frac{n}{n-2}$.

Solution:

Taking A B in fig.(a).(Ex.1) as the positive direction. Let the mass of the first ball be k and u its velocity along AB before impact. Then, for the second ball, mass is mk and $\frac{u}{n}$ is the velocity before impact. After impact the first ball is reduced to rest and let V be the velocity of the second ball.

By Newton's law of impact, we have

$$V-0 = -e \left(\frac{u}{n} - u \right) \text{ (ie) } V = \frac{eu(n-1)}{n} \quad (1)$$

By principle of conservation of momentum along AB,

$$k \times 0 + mk.v = Ku + mk. \frac{1}{n} u.$$

$$\text{i.e. } mv = u + \frac{m}{v} u = \frac{u(m+n)}{n} \quad (2)$$

substituting the value of V from (1) and (2), we have

$$\frac{meu(n-1)}{n} = \frac{u(m+n)}{n} \text{ or } e = \frac{(m+n)}{m(n-1)}$$

Now e is positive and less than

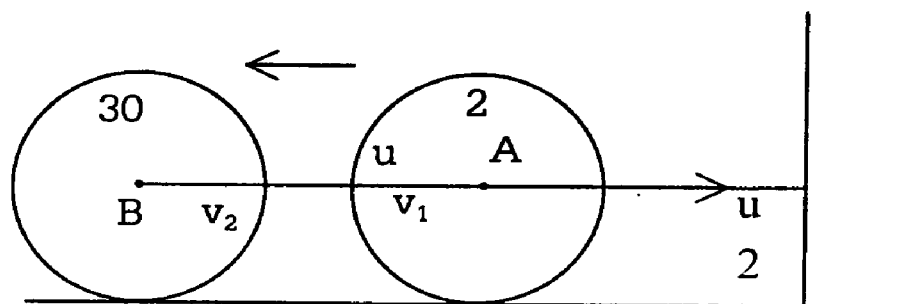
$$\therefore m(n-1) > m+n \text{ i.e. } mn - 2m > n$$

$$\therefore m(n-2) > n \text{ or } m > \frac{n}{n-2}$$

Example:4

Two equal spheres A and B of masses 2 gm. and 30 gm respectively lie on a smooth floor. So that their line of centers is perpendicular to a fixed vertical wall. A being nearer to the wall. A is projected towards B. Show that, if the coefficient of restitution between the two spheres and that between the first sphere and the wall is $\frac{3}{5}$ then. A will be reduced to rest after its second impact with B.

Solution:



Consider the impact between A and B. Taking AB as the positive direction.

Let the velocity of A before impact be u. B is at rest.

After the impact, Let the velocities of A & B be V_1 & V_2 respectively in the same direction.

$$\text{By Newton's rule, } v_2 - v_1 = -e(O-u) = \frac{3}{5}u \quad (1)$$

By conservation of momentum along AB,

$$30 v_2 + 2 v_1 = 30 \times 0 + 2 u \text{ (i.e) } 15 v_2 + v_1 = u \quad (2)$$

Solving (1) and (2), we get $v_1 = -\frac{u}{2}$ and $v_2 = \frac{u}{10}$.

Since v_1 is negative, the velocity A after the impact towards the wall and $= \frac{u}{2}$ while the velocity of B is $\frac{u}{10}$ away from the wall.

Now A strikes the wall with a velocity $\frac{u}{2}$. After this impact, its velocity will be reversed as a e. $\left(\frac{u}{2}\right) = \frac{3}{5} \cdot \frac{u}{2} = \frac{3u}{10}$ with the velocity, A moves in the direction AB, away from the wall and strike B a second time let the velocities of A and B be v_3 and v_4 after this impact in the direction AB.

For convenience the velocity distribution can be noted as follows:-

	A (2)	B(30)
Before impact	$\frac{3u}{10}$	$\frac{u}{10}$
After impact	v_3	v_4

By Newton's rule,

$$v_4 - v_3 = -e \left(\frac{u}{10} - \frac{3u}{10} \right) = \frac{3u}{25} \quad (3)$$

By conservation of momentum,

$$30 v_4 + 2v_3 = 30 \cdot \frac{u}{10} + 2 \cdot \frac{3u}{10} = \frac{18u}{5}$$

$$\text{i.e. } 15v_4 + v_3 = \frac{9u}{5} \quad (4)$$

multiplying (3) by 15 we have

$$15v_4 - 15v_3 = \frac{9u}{5} \quad (5)$$

subtracting (5) from (4), $16v_3 = 0$ or $v_3 = 0$.

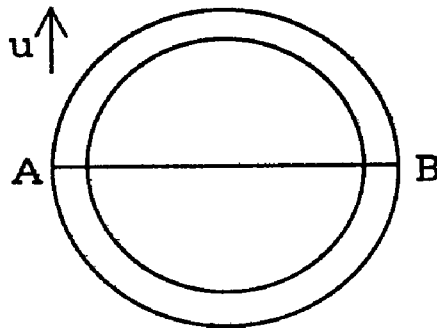
i.e A is reduced to rest after its second impact with R.

Example: 5

Two equal marble balls A, B lie in a horizontal circular groove at the opposite ends of a diameter; A is projected along the groove and after time t impinges on B; show that a second impact takes place after a further interval $\frac{2t}{e}$.

Solution:

Let the ball A move with velocity u . As there is no tangential force acting on A at any point of its path, its speed remains the same through-out. Hence it impinges on B with a velocity u .



Since the time from A to B is $= t$

we get

$$ut = \pi r u = \frac{\pi r}{t} \quad (1)$$

Let v and v' be the velocity of A and B respectively after impact.

Then by the principle of momentum.

$$mv + mv' = mu \quad (m \text{ being the mass of each ball})$$

$$(i.e) v + v' = u \quad (2)$$

Also, by Newton' $v - v' = -e(v - 0)$

$$(i.e) v - v' = -eu \quad (3)$$

Solving (2) and (3), we get $V = \frac{u}{2} (1-e)$; $v^1 = \frac{u}{2} (1+e)$

Clearly v^1 is greater than V . Hence B will move in advance of A. Let it strike again t_1 secs. After the first impact.

The velocity of B relative to A, after the first impact

$$= v^1 - v = eu \text{ from (3)}$$

Before striking again, B should cover a distance equal length to the circumference relative to A.

$$\therefore (v^1 - v) \cdot t_1 = 2\pi r \text{ (ie) } eu \cdot t_1 = 2\pi r$$

$$t_1 = \frac{2\pi r}{eu} = \frac{2\pi r}{e \left(\frac{\pi r}{t} \right)} \text{ using (1)}$$

$$= \frac{2t}{e}$$

\therefore The second impact occurs $\frac{2t}{e}$ secs. after the first.

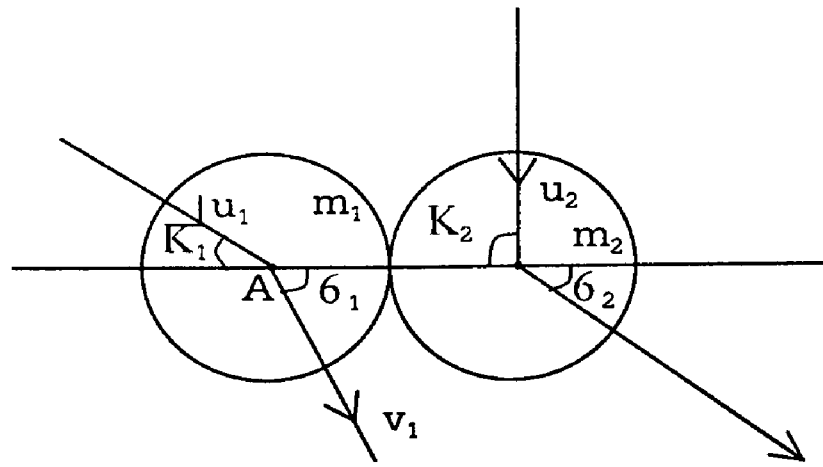
Exercise:

- 1) a) Two balls impinge directly and interchange their velocities after impact. Prove that they are perfectly elastic and are of equal masses.
- 2) A ball A impinges directly on a exactly equal and similar ball B lying on a smooth horizontal plane. If the coefficient of restitution is e , prove that after impact, the velocity of B will be to that of A as $1+e:1-e$.
- 3) A ball impinges directly on a second ball of twice its mass which is moving in the same direction as the first but with one – seventh of its velocity. Given that coefficient of restitution is $\frac{3}{4}$ Show that first ball will come to rest after the impact?

7.5 Oblique impact of two smooth spheres:

A smooth sphere of mass m_1 impinge obliquity with velocity and another smooth sphere of mass m_2 moving with velocity u_2 . If the direction of motion before impact make angles α_1 and α_2 respectively with the line joining the centre

of the spheres and if the co-efficient of restitution be e . To find the velocities and directions of motion after impact.



Let the velocities of the spheres after impact be v_1 and v_2 in directions inclined at angle θ_1 and θ_2 respectively to the line of centres. Since the spheres are smooth, there is no force perpendicular to the line of centers and therefore, for each sphere the velocities in the tangential direction are not affected by impact.

$$\therefore v_1 \sin \theta_1 = u_1 \sin \alpha_1 \quad (1)$$

$$\text{and } v_2 \sin \theta_2 = u_2 \sin \alpha_2 \quad (2)$$

By Newson's law concerning velocities along the

common normal AB,

$$\begin{aligned} V_2 \cos \theta_2 - V_1 \cos \theta_1 \\ = -e (u_2 \cos \alpha_2 - u_1 \cos \alpha_1) \end{aligned} \quad (3)$$

By the principle of conservation of momentum along AB,

$$m_2 \cdot v_2 \cos \theta_2 + m_1 \cdot v_1 \cos \theta_1 = m_2 \cdot u_2 \cos \alpha_2 + m_1 \cdot u_1 \cos \alpha_1 \quad (4)$$

(4) - (3) $\times m_2$ gives

$$\begin{aligned} V_1 \cos \theta_1 (m_1 + m_2) = m_2 u_2 \cos \alpha_2 + m_1 u_1 \cos \alpha_1 + e m_2 (u_2 \cos \alpha_2 - u_1 \cos \alpha_1) \\ \text{i.e. } v_1 \cos \theta_1 = \frac{u_1 \cos \alpha_1 (m_1 - e m_2) + m_2 u_2 \cos \alpha_2 (1 + e)}{m_1 + m_2} \end{aligned} \quad (5)$$

(4) + (3) $\times m_1$ gives

$$v_2 \cos \theta_2 = \frac{u_2 \cos \alpha_2 (m_2 - e m_1) + m_1 u_1 \cos \alpha_1 (1 + e)}{m_1 + m_2} \quad (6)$$

From (1) and (5) by squaring and adding, we obtain v_1^2 and by division, we have $\tan \theta_1$. Similarly from (2) and (6), we get v_2^2 and $\tan \theta_2$. Hence the motion after impact is completely determined.

Corollary 1:

If the two spheres are perfectly elastic and of equal mass, then $e = 1$ and $m_1 = m_2$.

Then, from equation (5) and (6), we have

$$v_1 \cos \theta_1 = \frac{0 + m_1 u_2 \cos \alpha_2}{2m_1} = u_2 \cos \alpha_2$$

$$\text{and } v_2 \cos \theta_2 = \frac{0 + m_1 u_1 \cos \alpha_1}{2m_1} = u_1 \cos \alpha_1$$

Hence if two equal perfectly elastic spheres impinge they interchange their velocities in the direction of the line of centres.

Corollary 2:

Usually, in most problems on oblique impact one of the spheres is at rest. Suppose m_2 is at rest i.e. $u_2 = 0$.

From equation (2), $v_2 \sin \theta_2 = 0$ i.e. $\theta = 0$.

Hence m_2 moves along AB after impact. This is seen independently, since the only force on m_2 during impact is along the line of centres.

Corollary 3:

The impulse of the blow on the sphere A of mass m_1

= Change of momentum of A along the common normal

$$= m_1 (v_1 \cos \theta_1 - u_1 \cos \alpha_1)$$

$$= m_1 \left[\frac{u_1 \cos \alpha_1 (m_1 - em_2) + m_2 u_2 \cos \alpha_1 (1 + e)}{m_1 + m_2} - u_1 \cos \alpha_1 \right]$$

$$= \frac{m_1 [m_1 u_1 \cos \alpha_1 - em_2 u_1 \cos \alpha_1 + m_2 u_2 \cos \alpha_2 + em_2 u_2 \cos \alpha_2 - m_1 u_1 \cos \alpha_1 - m_2 u_1 \cos \alpha_1]}{m_1 + m_2}$$

$$= \frac{m_1 [m_2 u_2 \cos \alpha_2 (1+e) - m_2 u_1 \cos \alpha_1 (1+e)]}{m_1 + m_2}$$

$$= \frac{m_1 m_2 (1+e)}{m_1 + m_2} (u_2 \cos \alpha_2 - u_1 \cos \alpha_1)$$

The impulsive blow on m_2 will be equal and opposite of the impulsive blow on m_1 .

7.5.1 Loss of kinetic energy due to oblique impact of two smooth spheres:

Two spheres of masses m_1 and m_2 , moving with velocities u_1 and u_2 at angles α_1 and α_2 with their line of centers come into collision. To find an expression for the loss of kinetic energy: -

The velocities perpendicular to the line of centres are not altered by impact. Hence the loss of kinetic energy in the case of oblique impact is therefore the same as in the case of direct impact if we replace in the expression (4) on page. No 256 the quantities u_1 and u_2 by $u_1 \cos \alpha_1$ and $u_2 \cos \alpha_2$ respectively.

Therefore the loss is $\frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (1-e^2) (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)^2$.

We shall now derive this independently. Let v_1 and v_2 be the velocities of the spheres after impact in directions inclined at angles θ_1 and θ_2 respectively to the line of centres. As explained in the tangential velocity of each sphere is not altered by impact.

$$\therefore V_1 \sin \theta_1 = u_1 \sin \alpha_1 \dots\dots(1) \text{ and } V_2 \sin \theta_2 = u_2 \sin \alpha_2 \quad (2)$$

By Newton's rule

$$v_2 \cos \theta_2 - v_1 \cos \theta_1 = -e (u_2 \cos \alpha_2 - u_1 \cos \alpha_1) \quad (3)$$

By conservation of momenta

$$m_2 v_2 \cos \theta_2 + m_1 v_1 \cos \theta_1 = m_2 u_2 \cos \alpha_2 + m_1 u_1 \cos \alpha_1$$

$$\text{i.e. } m_1 (u_1 \cos \alpha_1 - v_1 \cos \theta_1) = m_2 (v_2 \cos \theta_2 - u_2 \cos \alpha_2) \quad (4)$$

Change in K.E.

$$\begin{aligned}
&= \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 - \frac{1}{2} m_1 v_1^2 - \frac{1}{2} m_2 v_2^2 \\
&= \frac{1}{2} m_1 u_1^2 (\cos^2 \alpha_1 + \sin^2 \alpha_1) + \frac{1}{2} m_2 u_2^2 (\cos^2 \alpha_2 + \sin^2 \alpha_2) \\
&\quad - \frac{1}{2} m_1 v_1^2 (\cos^2 \theta_1 + \sin^2 \theta_1) - \frac{1}{2} m_2 v_2^2 (\cos^2 \theta_2 + \sin^2 \theta_2) \\
&= \frac{1}{2} m_1 u_1^2 \cos^2 \alpha_1 + \frac{1}{2} m_2 u_2^2 \cos^2 \alpha_2 - \frac{1}{2} m_1 v_1^2 \cos^2 \theta_1 \\
&\quad - \frac{1}{2} m_2 v_2^2 \cos^2 \theta_2 \text{ (using (1) and (2))} \\
&= \frac{1}{2} m_1 (u_1^2 \cos^2 \alpha_1 - v_1^2 \cos^2 \theta_1) + \frac{1}{2} m_2 (u_2^2 \cos^2 \alpha_2 - v_2^2 \cos^2 \theta_2) \\
&= \frac{1}{2} m_1 (u_1 \cos \alpha_1 + v_1 \cos \theta_1)(u_1 \cos \alpha_1 - v_1 \cos \theta_1) \\
&\quad + \frac{1}{2} (u_2 \cos \alpha_2 + v_2 \cos \theta_2)(u_2 \cos \alpha_2 - v_2 \cos \theta_2) \\
&= \frac{1}{2} m_1 (u_1 \cos \alpha_1 + v_1 \cos \theta_1) [u_1 \cos \alpha_1 - u_2 \cos \alpha_2] - \frac{1}{2} (u_2 \cos \alpha_2 + v_2 \cos \theta_2) \\
&\quad m_1 (u_1 \cos \alpha_1 - u_1 \cos \theta_1) \text{ using (3)} \\
&= \frac{1}{2} m_1 (u_1 \cos \alpha_1 - v_1 \cos \theta_1) (u_1 \cos \alpha_1 - v_1 \cos \theta_1) \\
&\quad - u_2 \cos \alpha_2 - v_2 \cos \theta_2 \\
&= \frac{1}{2} m_1 (u_1 \cos \alpha_1 - v_1 \cos \theta_1) [u_1 \cos \alpha_1 - u_2 \cos \alpha_2] \\
&\quad + e (u_2 \cos \alpha_2 - u_1 \cos \alpha_1) \text{] Using (3)} \\
&= \frac{1}{2} m_1 (u_1 \cos \alpha_1 - v_1 \cos \theta_1) (u_1 \cos \alpha_1 - u_2 \cos \alpha_2) (1-e) \quad \text{(5)}
\end{aligned}$$

Now from

$$\frac{u_1 \cos \alpha_1 - V_1 \cos \theta_1}{m_2} = \frac{V_2 \cos \theta_2 - u_2 \cos \alpha_2}{m_1}$$

$$\text{and each} = \frac{u_1 \cos \alpha_1 - V_1 \cos \theta_1 + V_2 \cos \theta_2 - u_2 \cos \alpha_2}{m_1 + m_2}$$

$$= \frac{(u_1 \cos \alpha_1 - u_2 \cos \alpha_2 + (V_2 \cos \theta_2 - V_1 \cos \theta_1))}{m_1 + m_2}$$

$$= \frac{u_1 \cos \alpha_1 - u_2 \cos \alpha_2 - e(u_2 \cos \alpha_2 - u_1 \cos \alpha_1)}{m_1 + m_2} \text{ Using (3)}$$

$$= \frac{(u_1 \cos \alpha_1 - u_2 \cos \alpha_2)(1 + e)}{m_1 + m_2}$$

$$\therefore u_1 \cos \alpha_1 - V_1 \cos \theta_1 = \frac{m_2(1 + e)}{m_1 + m_2} (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)$$

Substituting in (5)

$$\begin{aligned} \text{Change in K.E} &= \frac{1}{2} \frac{m_1 m_2 (1 + e)}{m_1 + m_2} (u_1 \cos \alpha_1 - u_2 \cos \alpha_2) \\ &\quad \times (u_1 \cos \alpha_1 - u_2 \cos \alpha_2) (1 - e) \\ &= \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (1 - e)^2 (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)^2 \end{aligned}$$

If the spheres are perfectly elastic, $e = 1$ and the loss of kinetic energy is zero.

7.5.2. Dissipation of energy due to impact:

We have found that in any impact, except where the coefficient of restitution of unity, some kinetic energy is lost. This missing kinetic energy is converted into other forms of energy and chiefly reappears in the shape the heat. Hence the principle of conservation of energy will not hold good in problem of impact.

Example: 1

A ball of mass 8gms. Moving with velocity 4cms. Per sec. Impinges on a ball of mass 4gms. Moving with velocity 2cm. Per.sec.if their velocities before impact be inclined at angle 30° and 60° to the line joining their centres at the moment of impact find their velocities after impact when

$$e = \frac{1}{2}$$

Refer to oblique impact. $m_1 = 8$; $u_1 = 4$; $\alpha_1 = 30^\circ$ $m_2 = 4$; $u_2 = 2$; $\alpha_2 = 60^\circ$
Let V_1 and V_2 be the velocities after impact in directions making θ_1 and θ_2 respectively with AB. The tangential velocity of each sphere is not affected by impact.

$$\therefore V_1 \sin \theta_1 = 4 \sin 30^\circ = 2 \quad (1)$$

$$\text{and } V_2 \sin \theta_2 = 2 \sin 60^\circ = \sqrt{3} \quad (2)$$

By Newton's law,

$$\begin{aligned} V_2 \cos \theta_2 - V_1 \cos \theta_1 &= -e (2 \cos 60^\circ - 4 \cos 30^\circ) \\ &= -\frac{1}{2} \left(2 \cdot \frac{1}{2} - 4 \cdot \frac{\sqrt{3}}{2} \right) \\ &= \frac{1}{2} (2\sqrt{3} - 1) \end{aligned} \quad (3)$$

By conservation of momenta along AB,

$$\begin{aligned} 4V_2 \cos \theta_2 + 8V_1 \cos \theta_1 &= 4 \cdot 2 \cos 60^\circ + 8 \cdot 4 \cos 30^\circ \\ &= 4 + 16\sqrt{3} \end{aligned} \quad (4)$$

$$\begin{aligned} \therefore 3V_1 \cos \theta_1 &= 1 + 4\sqrt{3} - \frac{1}{2}(2\sqrt{3} - 1) = \frac{3 + 6\sqrt{3}}{2} \\ \text{i.e. } V_1 \cos \theta_1 &= \frac{1 + 2\sqrt{3}}{2} \end{aligned} \quad (5)$$

$$\text{From (4), } V_2 \cos \theta_2 = 1 + 4\sqrt{3} - 1 - 2\sqrt{3} = 2\sqrt{3} \quad (6)$$

$$\text{From (1) and (5) } V_1^2 = 2^2 + \left(\frac{1+2\sqrt{3}}{2} \right)^2$$

$$= 4 + \frac{1+4\sqrt{3}+12}{4} = \frac{29+4\sqrt{3}}{4}$$

$$\therefore V_1 = \sqrt{\frac{29+4\sqrt{3}}{4}} \text{ cm. per sec.}$$

Dividing (1) by (5)

$$\tan \theta_1 = \frac{4}{1+2\sqrt{3}}$$

From (2) and (6)

$$V_2^2 = 3 + 12 = 15 \text{ and}$$

$$\therefore V_2 = \sqrt{15} \text{ cm/sec}$$

Dividing (2) by (6)

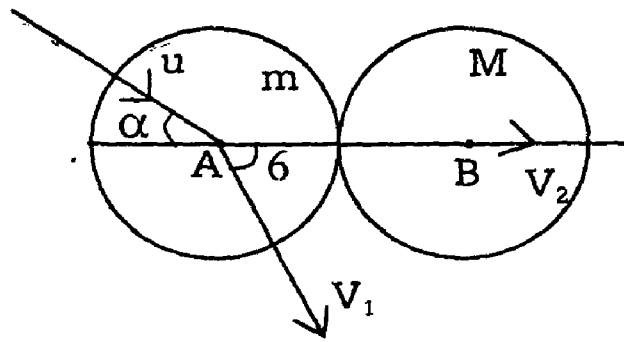
$$\tan \theta_2 = \frac{1}{2}.$$

Example : (U.Q)

A smooth sphere of mass m impinges obliquely on a smooth sphere of mass M which is at rest. Show that if $m = eM$ the directions of motion after impact are at right angles. (e is the coefficient of restitution).

Solution:

Considering the sphere M , its tangential velocity before impact is zero and hence after impact also, its tangential velocity is zero. (\because During impact, there is no force acting along the common tangent). Hence after impact, M will move along AB . Let its velocity be V_2 . Let the velocity of m be V at an angle θ to AB , after impact.



By Newton's rule,

$$V_2 - V_1 \cos \theta = -e (0 - u \cos \alpha)$$

$$(ie) \quad V_2 - V_1 \cos \theta = eu \cos \alpha \quad (1)$$

By conservation of momenta along AB,

$$M.V_2 + mV_1 \cos \theta = M.0 + m.u \cos \alpha \quad (2)$$

Multiplying (1) by M and subtracting from (2),

$$mV_1 \cos \theta + MV_1 \cos \theta = mu \cos \alpha - Meu \cos \alpha$$

$$(ie) \quad V_1 \cos \theta = \frac{u \cos \alpha (m - eM)}{m + M} = \frac{u \cos \alpha \cdot 0}{m + M} \quad (\because m = eM)$$

$$= 0.$$

$$\therefore \cos \theta = 0 \text{ or } \theta = 90^\circ$$

(ie) The direction of motion of m is perpendicular to AB.

Example: 2

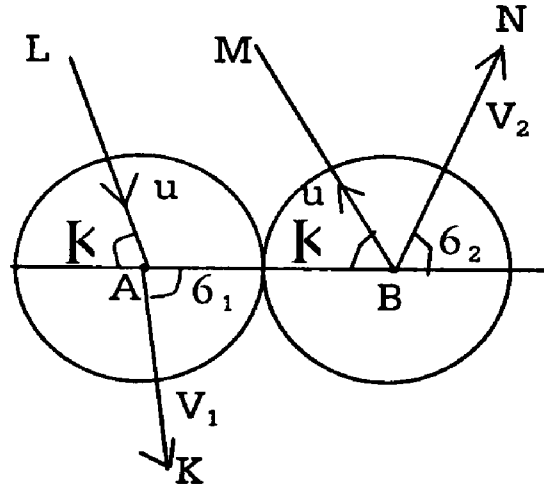
Two equal elastic balls moving in opposite parallel direction with equal speeds impinge on one another. If the inclination of their direction of motion to the line of centres be $\tan^{-1}(\sqrt{e})$ where e is the coefficient of restitution, show that their direction of motion will be turned through a right angle.

Solution:

Let m be the mass of either sphere. AB is the line of impact.

Before impact, the directions of motion are LA and BM making the same acute angle α with AB as shown in the figure. Let u be their velocity.

After impact, let the sphere A proceed in the direction AK with velocity V_1 at an angle θ_1 to AB and the sphere B proceed in the direction BN with velocity V_2 at an angle θ_2 to AB



The tangential velocity of either sphere is not affected by impact

$$\therefore V_1 \sin \theta_1 = u \sin \alpha \quad (1)$$

$$V_2 \sin \theta_2 = u \sin \alpha \quad (2)$$

By Newton's law, (resolving all velocities along AB)

$$V_2 \cos \theta_2 - V_1 \cos \theta_1 = -e (-u \cos \alpha - u \cos \alpha)$$

$$(ie) \quad V_2 \cos \theta_2 - V_1 \cos \theta_1 = 2eu \cos \alpha \quad (3)$$

By conservation of momenta along AB

$$m(V_2 \cos \theta_2) + m.V_1 \cos \theta_1 = m (-u \cos \alpha) + mu \cos \alpha.$$

$$(ie) \quad V_2 \cos \theta_2 + V_1 \cos \theta_1 = 0 \quad (4)$$

$$(4) - (3) \text{ gives } 2V_1 \cos \theta_1 = -2eu \cos \alpha$$

$$\therefore V_1 \cos \theta_1 = -eu \cos \alpha \quad (5)$$

$$\text{From (4) } V_2 \cos \theta_2 = -V_1 \cos \theta_1 = eu \cos \alpha \quad (6)$$

Dividing (1) by (5)

$$\tan \theta_1 = -\frac{1}{e} \tan \alpha = -\frac{1}{e} \sqrt{e} \quad (\because \alpha = \tan^{-1} \sqrt{e}) \text{ given}$$

$$= -\frac{1}{\sqrt{e}} = -\frac{1}{\tan \alpha} = -\cot \alpha$$

$$= \tan (90^\circ + \alpha)$$

$$\therefore \theta_1 = 90^\circ + \alpha.$$

$$\text{Dividing (2) by (6) } \tan \theta_2 = \frac{1}{e} \tan \alpha = \cot \alpha = \tan (90^\circ - \alpha)$$

$$\therefore \theta_2 = 90^\circ - \alpha.$$

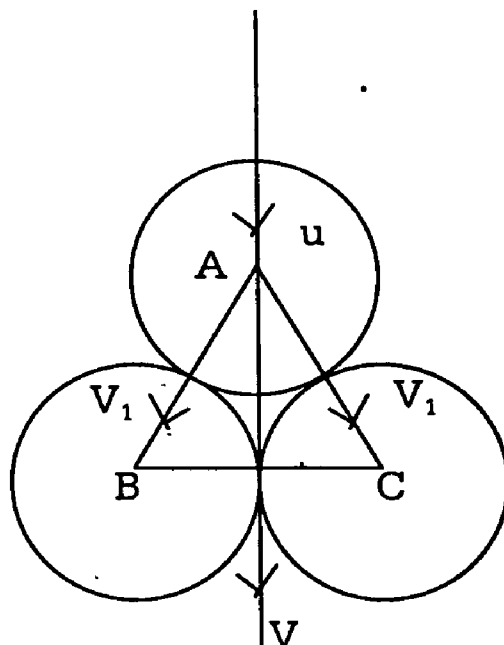
Hence their directions of motion are turned through a right angle.

Example: 3

Two equal billiard balls are in contact on a smooth table and a third equal ball, moving along their common tangent, strikes them simultaneously. Prove that $\frac{3}{5} (1 - e^2)$ of its kinetic energy is lost by the impact, e being the coefficient of restitution for each pair of balls.

Solution:-

Let A be the centre of the impinging ball, and B, C the centres of the other two balls at rest. Since the balls are equal, $\triangle ABC$ is equilateral. Let u be the velocity of A before impact. After impact, let V be the velocity of A and V_1 the common velocity of B and C. V will be in the same line as u . While B and C. Will move along AB and AC respectively. Let m be the mass of each ball. We apply the principle of conservation of momentum for all the 3 balls along the direction of motion of A.



$$mV + 2m.V_1 \cos 30^\circ = mu.$$

$$(ie) V + V_1\sqrt{3} = u \quad (1)$$

Applying Newton's Law along AB for the spheres A and B.

We have

$$V_1 - V \cos 30^\circ = -e(0 - u \cos 30^\circ)$$

$$(ie) V_1 - \frac{V\sqrt{3}}{2} = \frac{eu\sqrt{3}}{2}$$

$$or -V\sqrt{3} + 2V_1 = eu\sqrt{3} \quad (2)$$

Multiplying (1) by $\sqrt{3}$ and adding to (2)

We have

$$5V_1 = u\sqrt{3}(1+e) \text{ or } V_1 = \frac{u\sqrt{3}(1+e)}{5} \quad (3)$$

$$\text{From (1), } V = u - V_1\sqrt{3} = u - \frac{3u(1+e)}{5} = \frac{u(2-3e)}{5} \quad (4)$$

The loss of kinetic energy.

$$= \text{Initial K.E} - \text{final K.E.}$$

$$= \frac{1}{2}mu^2 - \left(\frac{1}{2}mV^2 + \frac{1}{2}mV_1^2 + \frac{1}{2}mV_1^2\right)$$

$$= \frac{1}{2}mu^2 - \frac{1}{2}mV^2 - mV_1^2$$

$$= \frac{1}{2}mu^2 - \frac{1}{2}mu^2 \cdot \frac{(2-3e)^2}{25} - \frac{m.3u^2(1+e)^2}{25}$$

$$= \frac{1}{2}mu^2 \left[1 - \frac{(4-12e+9e^2)}{25} - \frac{6}{25}(1+2e+e^2) \right]$$

$$\begin{aligned}
&= \frac{1}{2}mu^2 \left[\frac{25 - 4 + 12e - 9e^2 - 6 - 12e - 6e^2}{25} \right] \\
&= \frac{1}{2}mu^2 \cdot \frac{(15 - 5e^2)}{25} = \frac{1}{2}mu^2 \cdot \frac{3}{5}(1 - e^2) \\
&= \frac{3}{5}(1 - e^2) \times \text{the original K.E.}
\end{aligned}$$

Exercise:

- 1) A sphere of mass m moving on a horizontal plane with velocity impinges obliquely on a sphere of mass m' at rest on the same plane. If $e = 1$ and $m = m'$ prove that the directions of motion after impact are at right angles.
- 2) If two equal perfectly elastic sphere impinge obliquely, Prove that they interchange their velocities in the direction of the line of centres.
- 3) Two equal perfectly elastic balls impinge if their directions of motion before impact be at right angles. Show that their directions of motion after impact are also at right angles.

UNIT – VIII

SIMPLE HARMONIC MOTION

8.1 Introduction:

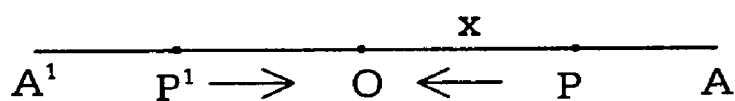
A very common and important type of motion occurring in nature is that which involves oscillations backwards and forwards about some fixed point. For instance, suppose one end of an elastic string is tied to a fixed point and a heavy particle is attached to the other end. If the particle is disturbed vertically from its position of equilibrium, it is found that it oscillates to and from about this position clearly the particle cannot be moving under constant acceleration. It is found that it has an acceleration, which is always directed towards the equilibrium position and varies in magnitude as the distance of the particle from that position. This kind of motion occurs frequently in nature and since it is of the type which produces all musical notes. It is called **Simple Harmonic Motion** (S . H . M). The oscillations of a simple pendulum and the transverse vibrations of a plucked violin string are examples of Simple Harmonic motion.

8.1.1 Simple Harmonic Motion in a straight line :

Definition :

When a particle moves in a straight line so that its acceleration is always directed towards a fixed point in the line and proportional to the distance from that point, its motion is called Simple Harmonic Motion.

Let O be a fixed point on the straight line A'OA on which a particle is having Simple Harmonic motion. Take O as the Origin and OA as the x-axis.



Let P be the position of the particle at time t. Such that $OP = x$. The magnitude of the acceleration at P = μx where μ is a positive constant. As this acceleration acts towards O, the acceleration at p is the positive direction of the x-axis is $-\mu x$.

Hence the equation of motion of p is

$$\frac{d^2x}{dt^2} = -\mu x \quad (1)$$

Here it must be noted that for a position of P to the right of O. The x – coordinate x is positive and so the acceleration $\frac{d^2x}{dt^2}$ is negative directed towards O. If P' is a position of the particle to the left of O, x is negative and so the acceleration $\frac{d^2x}{dt^2}$ is positive, again towards O.

Hence the same equation of motion (1) holds good for all positions of P on the line.

Equation (1) is the fundamental differential equation representing a S . H . M . we now proceed to solve it.

If v is the velocity of the particle at time t (1) can be written as

$$\frac{d}{dt} \left(\frac{dx}{dt} \right) = \frac{d}{dt} (v) = \frac{d}{dx} \cdot v \cdot \frac{dx}{dt}$$

$$\Rightarrow \frac{dv}{dt} \cdot \frac{dx}{dt} = v \cdot \frac{dv}{dx}$$

$$\therefore v \cdot \frac{dv}{dx} = -\mu X.$$

$$v dv = -\mu X dx$$

(2)

Integrating (2)

$$\text{We have } \frac{v^2}{2} = -\frac{\mu X^2}{2} + C \quad (3)$$

Where C is the constant of integration.

Initially let the particle start from rest at the point A where OA = a and let us measure time also from this instant

Hence when $x = a$, $v = 0$.

Putting these in (3)

$$0 = -\frac{\mu a^2}{2} + C \quad (\text{or}) \quad C = \frac{\mu a^2}{2}$$

$$v^2 = -\mu x^2 + 2c$$

$$v^2 = -\mu x^2 + \frac{2\mu a^2}{2}$$

$$v^2 = -\mu x^2 + \mu a^2 = \mu (a^2 - x^2). \quad (4)$$

$$\therefore v = \pm \sqrt{\mu (a^2 - x^2)}$$

Equation (4) gives the velocity v corresponding to any displacement x .

Now, as t increases, x decreases.

$$\text{So } \frac{dx}{dt} = v = -\sqrt{\mu (a^2 - x^2)} \quad (5)$$

$$(\text{or}) \quad \frac{-dx}{\sqrt{a^2 - x^2}} = \sqrt{\mu} dt.$$

integrating $\text{Cos}^{-1} x/a = \sqrt{\mu} t + A$.

initially when $t = 0$, $x = a$

$$\therefore \text{Cos}^{-1} 1 = 0 + A$$

$$(\text{ie}) \quad A = 0.$$

$$\therefore \text{Hence } \text{Cos}^{-1} \frac{x}{a} = \sqrt{\mu} t$$

$$\therefore \frac{x}{a} = \text{Cos} \sqrt{\mu} t$$

$$\therefore x = a \text{Cos } \sqrt{\mu} t \quad (6)$$

Equation (6) gives the displacement x in terms of time t .

When the particle comes to O , $x = 0$ and by (5) .

$$\text{Its velocity then} = - a \sqrt{\mu}$$

So the particle passes through O and immediately the acceleration alters its direction and tends to decrease the velocity.

From (5) $v = 0$ when $x = -a$

So the particle comes to rest at a point A' to the left of O such that $OA = OA'$

If then retraces its path passes through O, and again is instantaneously at rest at A. The whole motion of the particle is an oscillation from A to A' and back.

To get the time from A to A' .

Put $x = -a$ in (6)

$$\text{We have } \cos \sqrt{\mu} t = -1 = \cos \pi$$

$$\therefore t = \frac{\pi}{\sqrt{\mu}}$$

$$\text{The time from A to A' and back} = \frac{2\pi}{\sqrt{\mu}}$$

We have Equation (6) can be written as

$$X = a \cos \sqrt{\mu} t = a \cos (\sqrt{\mu} t + 2\pi)$$

$$= a \cos (\sqrt{\mu} t + 4\pi) \text{ etc}$$

$$= a \cos \left(\sqrt{\mu} \left(t + \frac{2\pi}{\sqrt{\mu}} \right) \right)$$

$$= a \cos \sqrt{\mu} \left(t + \frac{4\pi}{\sqrt{\mu}} \right) \text{ etc.}$$

This shows that the displacement of the particle at any particular time t , is repeated at times $t_2 + \frac{2\pi}{\sqrt{\mu}}$, $t_1 + \frac{4\pi}{\sqrt{\mu}}$ etc.

Differentiating (6)

$$\begin{aligned} \frac{dx}{dt} &= -a \sqrt{\mu} \sin \sqrt{\mu} t \\ &= -a \sqrt{\mu} \sin (\sqrt{\mu} t + 2\pi) \\ &= -a \sqrt{\mu} \sin (\sqrt{\mu} t + 4\pi) \text{ etc.} \\ &= -a \sqrt{\mu} \sin \sqrt{\mu} \left(t + \frac{2\pi}{\sqrt{\mu}} \right) \\ &= -a \sqrt{\mu} \sin \sqrt{\mu} \left(t + \frac{4\pi}{\sqrt{\mu}} \right) \text{ etc.} \end{aligned}$$

This shows that the values of $\frac{dx}{dt}$ are the same if t is increased by $\frac{2\pi}{\sqrt{\mu}}$ or by any multiple of $\frac{2\pi}{\sqrt{\mu}}$. Hence after a time $\frac{2\pi}{\sqrt{\mu}}$ the particle is again at the same particular point moving with the same velocity in the same direction as before, having covered the whole path of the motion just once. The particle is said to be the period $\frac{2\pi}{\sqrt{\mu}}$

8.1.2 Definitions:

The period or the periodic time of a simple harmonic motion, is the interval of time that elapses from any instant till a subsequent instant when the particle is again moving through the same position with the same velocity in the same direction. The frequency of the oscillation is the number of complete oscillations that the particle makes in one second. So frequency is the reciprocal of the period and is equal to $\sqrt{\mu} / 2\pi$.

The distance through which the particle moves away from the centre of motion on either side of it is called the amplitude of the oscillation.

Thus in the above case amplitude = OA = OA' = a.

We notice that the periodic time being = $\frac{2\pi}{\sqrt{\mu}}$ is independent of the

amplitude which is the distance from the centre at which the particle started. It depends only on the constant, which is the acceleration at unit distance from the centre.

Note:

1) Since $\frac{d^2x}{dt^2} = -\mu x$, maximum acceleration corresponds to the greatest value of x and so it is numerically = $\mu \cdot a = \mu \cdot$ (amplitude).

2) Since $v = \sqrt{\mu(a^2 - x^2)}$, the greatest value of v is got at x = 0 and it is = $a\sqrt{\mu} = \mu \cdot$ (amplitude)

General solution of the S . H . M equation :-

The S . H .M equation is $\frac{d^2x}{dt^2} = -\mu x$.

$$(ie) \frac{d^2x}{dt^2} + \mu x = 0 \tag{1}$$

(1) is a linear differential equation of the second order with constant coefficients. Its most general solution is of the form

$$x = A \text{ Cos } \sqrt{\mu} t + B \text{ Sin } \sqrt{\mu} t \tag{2}$$

where A and B are arbitrary constants.

Other forms of the solution equivalent to (2) are $x = C \text{ Cos } (\sqrt{\mu} t + \epsilon)$ (3)

and $x = D \text{ Sin } (\sqrt{\mu} t + \alpha)$ (4)

The constants A and B in (2), C and M in (3) and D and α in (4) are known if we know the values of x and $\frac{dx}{dt}$ corresponding to a given time t .

From (3) and (4) , the maximum value of x = C or D.

Hence if a is the amplitude of the motion, the forms (3) and (4) can be respectively put as.

$$X = a \cos (\sqrt{\mu} t + \varepsilon) \quad (5)$$

$$X = a \sin (\sqrt{\mu} t + \alpha) \quad (6)$$

When the solution of the S .H . M equation is expressed as $X = a \cos (\sqrt{\mu} t + \varepsilon)$ the quantity M is called the epoch. The phase of a S . H .M at any instant is the time that has elapsed since the particle was at its maximum distance in the positive direction.

From the equation (5) , x is maximum when $\cos (\sqrt{\mu} t + \varepsilon) = 1$.

If t_0 is the value of t

If t_0 is the value of t ,

$$\sqrt{\mu} t_0 = \varepsilon = 0$$

$$(ie) t_0 = - \frac{\varepsilon}{\sqrt{\mu}}$$

Hence phase at time t = t - t_0

$$= t + \frac{\varepsilon}{\sqrt{\mu}}$$

$$= \frac{\sqrt{\mu} t + \varepsilon}{\sqrt{\mu}}$$

Note

Two simple harmonic motions of the same period can be represented by

$$X_2 = a_1 \cos (\sqrt{\mu} t + \varepsilon_1)$$

$$X_2 = a_2 \cos(\sqrt{\mu} t + \epsilon_2)$$

$$\text{Then difference in phase} = \frac{\epsilon_1 - \epsilon_2}{\sqrt{\mu}}$$

If $\epsilon_1 = \epsilon_2$, the motions are in the same phase.

If $\epsilon_1 - \epsilon_2 = \pi$, they are in opposite phases.

8.1.3 Geometrical Representation of a Simple Harmonic Motion

Let a point Q describe with uniform angular velocity w , a circle of radius a and centre O of which A'O A is fixed diameter. Let P be the foot of the perpendicular from O on AA₁.

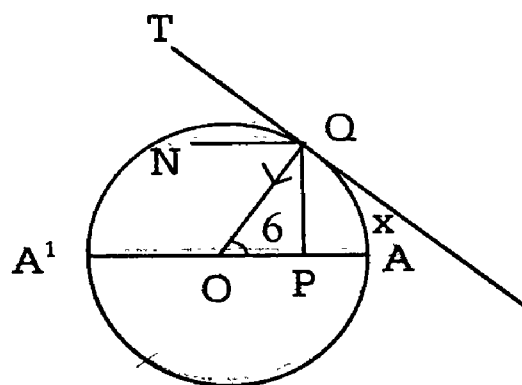
As O moves round the circle, P will move to and fro on the diameter.

We can show that the motion of P along AA' is simple harmonic, with O at centre.

Let Q move in the direction AOA' as shown in the figure. As Q moves uniformly in a circle, its only acceleration is w^2 , OQ along QO.

The velocity of Q in the circle is w . QO along the tangent QT.

The velocity and acceleration of P must be the same as the resolved parts along AA' of the velocity and acceleration of Q.



$$\text{Hence acceleration of P} = w^2 \cdot QO \cdot \cos \angle POQ$$

$$= w^2 \cdot QO \cdot \frac{PO}{QO}$$

$$= w^2 \cdot PO \text{ towards O.}$$

(ie) the acceleration of P is always directed towards O and proportional to its distance from O. Hence the motion of P is simple harmonic.

The various formulae of a S.H.M derived can be deduced by considering the motion of Q along the circle.

Taking O as the origin and OA as the positive direction of measuring displacement, let $OP = X$ and $\angle QOP = \theta$

velocity of Q = $a\omega$ along QT.

Hence velocity of P = resolved part of velocity of Q along AA' .

$$\begin{aligned}
 &= a\omega \cdot \cos \angle TQN = a\omega \sin \angle OQN \\
 &= a\omega \sin \theta = a\omega \cdot \frac{PQ}{a} \\
 &= \omega \sqrt{OQ^2 - OP^2} \\
 &= \omega \sqrt{a^2 - x^2} \quad (\text{in magnitude}) \quad (1)
 \end{aligned}$$

and this velocity of P is along AO towards O .

As Q moves round the circle from A to A' and back to A, P moves from A to A' through O and back A .

Hence the periodic time of the S . H . M described by P

= Time taken for Q to describe the circle

$$= \frac{2\pi}{\omega} \quad (2)$$

Also, if t is the time from A to Q

$$t = \frac{\theta}{\omega} = \frac{\cos^{-1}\left(\frac{x}{a}\right)}{\omega}$$

$$w = \cos^{-1} \left(\frac{x}{a} \right) \text{ or } x = a \cos \omega t \quad (3)$$

The acceleration of P towards O = $\omega^2 \cdot PO$ and putting $\omega^2 = \sqrt{\mu}$.

(ie) $\omega = \sqrt{\mu}$ in (1), (2), (3)

We get

(i) the velocity of P = $\sqrt{\mu} \cdot \sqrt{a^2 - x^2}$ in magnitude

(ii) the periodic time of P = $\frac{2\pi}{\sqrt{\mu}}$

(iii) the displacement $x = a \cos \sqrt{\mu} t$

These are the formulae derived.

8.1.4 Change now the equation :

A differential equation of the form $\frac{d^2x}{dt^2} = -\mu x$ where μ is a positive number, always represents a simple harmonic motion of period $\frac{2\pi}{\sqrt{\mu}}$ which is independent of the amplitude. The centre of the S.H.M is the origin from where the displacement x is measured.

Consider now the equation.

$$\frac{d^2x}{dt^2} = -\mu x + \alpha \quad (1)$$

This can be written as $\frac{d^2x}{dt^2} = -\mu \left(x - \frac{\alpha}{\mu} \right)$ (2)

Put $x - \frac{\alpha}{\mu} = x$ (3)

When $x = \frac{\alpha}{\mu}$, $x = 0$.

So this means that we are transferring the origin for measuring displacement, to the point distance $\frac{\alpha}{\mu}$ from the original origin.

Differentiating (3) twice

$$\frac{d^2x}{dt^2} = \frac{d^2x}{dt^2} \text{ and hence (2) becomes } \frac{d^2x}{dt^2} = -\mu x \quad (4)$$

(4) clearly represents a simple harmonic motion about the new origin.

Worked Examples

Example: 1

A particle is moving with S.H.M and while making an oscillation from one extreme position to the other, its distance from the centre of oscillation at 3 consecutive seconds are x_1, x_2, x_3 . Prove that the period of oscillation is

$$\frac{2\pi}{\cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right)}$$

Solution :

If a is the amplitude, μ : the constant of the S.H.M and X is the displacement at time t , we know that $x = a \cos \sqrt{\mu} t$ (1)

Let at three consecutive seconds t_1, t_1+1, t_1+2 the corresponding displacements be X_1, X_2, X_3 .

$$\text{Then } X_1 = a \cos \sqrt{\mu} t_1 \quad (2)$$

$$X_2 = a \cos \sqrt{\mu} (t_1 + 1) = a \cos (\sqrt{\mu} t_1 + \sqrt{\mu}) \quad (3)$$

$$\text{and } X_3 = a \cos \sqrt{\mu} (t_1 + 2) = a \cos (\sqrt{\mu} t_1 + 2\sqrt{\mu}) \quad (4)$$

$$\therefore X_1 + X_3 = a [\cos (\sqrt{\mu} t_1 + 2\sqrt{\mu}) + \cos \sqrt{\mu} t_1]$$

$$= a \cdot 2 \cos \frac{\sqrt{\mu} t_1 + 2\sqrt{\mu} + \sqrt{\mu} t_1}{2} \cos \frac{\sqrt{\mu} t_1 + 2\sqrt{\mu} - \sqrt{\mu} t_1}{2}$$

$$= 2a \cos(\sqrt{\mu}t_1 + \sqrt{\mu}) \cos \sqrt{\mu}$$

$$= 2x_2 \cos \sqrt{\mu}$$

$$\frac{X_1 + X_2}{2 \times 2} = \cos \sqrt{\mu} \quad \text{or} \quad \sqrt{\mu} = \cos^{-1} \left(\frac{X_1 + X_2}{2 \times 2} \right)$$

$$\therefore \text{Period} = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\cos^{-1} \left(\frac{X_1 + X_2}{2 \times 2} \right)}$$

Example : 2

If the displacement of a moving point at any time given by an equation of the form $X = a \cos \omega t + b \sin \omega t$, Show that the motion is a simple harmonic motion.

If $a = 3$, $b = 4$, $\omega = 2$ determine the period amplitude, maximum velocity and maximum acceleration of the motion.

Solution:

$$X = a \cos \omega t + b \sin \omega t \tag{1}$$

We have to show that the acceleration varies directly as the displacement .

Differentiating (1) w.r to t.

$$\frac{dx}{dt} = -a\omega \sin \omega t + b\omega \cos \omega t \tag{2}$$

$$\begin{aligned} \frac{d^2x}{dt^2} &= -a\omega^2 \cos \omega t - b\omega^2 \sin \omega t \\ &= -\omega^2 [a \cos \omega t + b \sin \omega t] = -\omega^2 x \end{aligned} \tag{3}$$

(3) shows that the motion is simple harmonic

The constant μ of the S.H.M = ω^2 .

$$\therefore \text{Period} = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi \text{ Secs}$$

Amplitude is greatest value of X .

When X is maximum $\frac{dx}{dt} = 0$

$$\therefore -a\omega \sin \omega t + b\omega \cos \omega t = 0$$

$$\text{(ie) } a \sin \omega t = b \cos \omega t$$

or $\tan \omega t = \frac{b}{a} = \frac{4}{3}$ using the given values. When $\tan \omega t = \frac{4}{3}$, $\sin \omega t = \frac{4}{5} \times \cos \omega t \frac{3}{5}$

Putting these values in (1) greatest value of x

$$= a \times \frac{3}{5} + b \times \frac{4}{5} = \frac{3a + 4b}{5} = \frac{3.3 + 4.4}{5} = 5$$

$$\therefore \text{amplitude} = 5 .$$

Using the formulae.

$$\text{Max . acceleration} = \mu . \text{amplitude} = 4 \times 5 = 20$$

$$\text{Max . velocity} = \sqrt{\mu} . \text{amplitude} = 2 \times 5 = 10$$

Example : 3

A horizontal shelf moves vertically with S.H.M whose complete period is one second; find the greatest amplitude in centimeters. It can have, so that an object resting on the shelf may always remain in contact.

Solutions :

Let m be the mass of an object lying on the shelf, O the centre of the S.H.M and P the position of m at time t .

Let OP = X.

The forces acting on the mass at P are

- i) its weight mg acting vertically downwards and
- ii) the normal reaction R due to the shelf acting upwards.

Resultant force on the mass = $mg - R$ and

so the acceleration on it = $\frac{mg - R}{m}$

and this acts towards O .

Since the particle is moving with S.H.M towards O. acceleration at

$P = \mu \cdot PO = \mu x$.

$$\therefore \mu x = \frac{mg - R}{m}$$

$$(ie) R = mg - m\mu x = m(g - \mu x) \tag{1}$$

$$\text{Period of the S.H.M} = \frac{2\pi}{\sqrt{\mu}} = 1 \text{ (given)}$$

$$\therefore \sqrt{\mu} = 2\pi \quad (\text{or}) \quad \mu = 4\pi^2$$

$$\therefore \text{From (1) , } R = m(g - 4\pi^2 x)$$

For the mass to remain always in contact with the shelf, reaction R must not be negative.

$$\therefore m(g - 4\pi^2 x) \geq 0$$

$$(ie) (g - 4\pi^2 x) \geq 0 \quad \text{or} \quad x \leq \frac{g}{4\pi^2}$$

$$\therefore \text{Greatest value of } X = \frac{g}{4\pi^2} = \frac{981}{4\pi^2} = 24.8 \text{ cms}$$

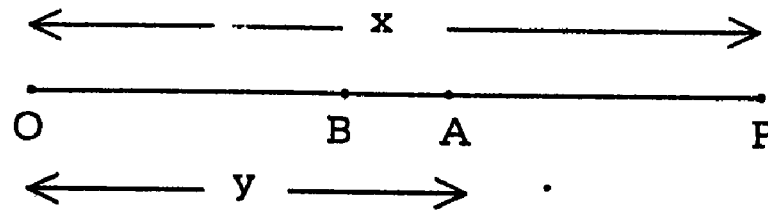
Example : 4

A particle P, of mass m , moves in a straight line OX under a force $m\mu$ (distance) directed towards a point A which moves in the straight line OX with

constant acceleration α . Show that the motion of P is simple harmonic of period $2\pi/\sqrt{\mu}$ about a moving centre which is always at a distance $a/\sqrt{\mu}$ behind A.

Solution :

Let at time t, the particle be at P where $OP = x$.



and A be such that $OA = y$.

The equation of motion of P is

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\mu \cdot PA \quad (\text{since the acceleration is towards A}) \\ &= -\mu(x - y) \end{aligned} \tag{1}$$

$$\text{The equation of motion of A is } \frac{d^2y}{dt^2} = \alpha \tag{2}$$

Subtracting (2) from (1)

We have

$$\frac{d^2x}{dt^2} - \frac{d^2y}{dt^2} = -\mu(x - y) - \alpha = -\mu \left(x - y + \frac{\alpha}{\mu} \right) \tag{3}$$

$$\text{Put } x - y + \frac{\alpha}{\mu} = Z \tag{4}$$

$$\text{Differentiating (4) } \frac{d^2x}{dt^2} - \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2}$$

Hence (3) becomes

$$\frac{d^2z}{dt^2} = -\mu z \tag{5}$$

∴ The displacement Z is simple harmonic and the period is $= \frac{2\pi}{\sqrt{\mu}}$.

The centre of the S.H.M represented by (5) is clearly the new origin from where Z is measured.

$$\text{Now } Z = x - y + \frac{\alpha}{\mu} = AP + \frac{\alpha}{\mu}.$$

If B is a point behind A such that

$$BA = \frac{\alpha}{\mu}.$$

We have $Z = AP + BA = BP$.

(ie) Z denotes the displacement of P measured from B.

∴ The motion of P is simple harmonic about B, a moving centre which is always at a distance $\frac{\alpha}{\mu}$ behind A.

Example : 5

A particle of mass m is oscillating in a straight line about a centre of force O, towards which when at a distance r, the force is $m \cdot n^2 r$ and 'a' is the amplitude of the oscillation. When at a distance $\frac{a\sqrt{3}}{2}$ from O, the particle receives a blow in the direction of motion which generates a velocity na. If this velocity be away from O, show that the new amplitude is $a\sqrt{3}$.

Solution :

Here the constant μ of the S.H.M = n^2 .

If V is the velocity at a distance x from o,

$$V^2 = n^2 (a^2 - x^2) \text{ since amplitude} = a.$$

$$\text{When } x = \frac{a\sqrt{3}}{2}, v^2 = n^2 \left(a^2 - \frac{3a^2}{4} \right) = \frac{n^2 a^2}{4}$$

$$\text{(ie) } v = \pm na/2$$

Since the additional velocity now given to the particle is away from O is the direction of its previous motion,

We must take the positive sign for v .

$$\begin{aligned} \text{Hence the new total velocity} &= \frac{na}{2} + na \\ &= \frac{3na}{2} \end{aligned}$$

The subsequent motion is again simple harmonic.

If V is the velocity at any distance x from O, is the subsequent motion,

$$v^2 = n^2 (A^2 - X^2) \tag{1}$$

where A is the new amplitude.

The new initial condition is that

$$\text{When } X = \frac{a\sqrt{3}}{2}, v = \frac{3na}{2}$$

Substituting in (1)

$$\frac{9n^2 a^2}{4} = n^2 \left(A^2 - \frac{3a^2}{4} \right)$$

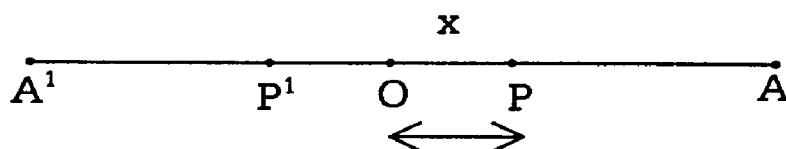
$$\text{(ie) } A^2 = \frac{9a^2}{4} + \frac{3a^2}{4} = 3a^2$$

or $A = a\sqrt{3}$ which is the new amplitude

Example : 6

Show that the energy of a system executing S.H.M is proportional to the square of the amplitude and of the frequency.

Solution:



The acceleration of the particle executing S.H.M at a distance x from

$$O = \mu x.$$

Hence the force acting on the particle

$$= m \mu x, \text{ m being its mass .}$$

If the particle is given a slight displacement dx from P, work done against the force

$$= m \mu x \cdot dx .$$

Hence total work done is displacing the particle to a distance

$$x = \int_0^x m \mu x \, dx = m \mu \frac{x^2}{2} \quad (1)$$

This work done is stored as the potential energy of the particle at P.

If V is the velocity at P.

We know that $V^2 = \mu (a^2 - x^2)$, a being the amplitude

$$\therefore \text{Kinetic energy at P} = \frac{1}{2} m v^2 = \frac{1}{2} m \mu (a^2 - x^2) \quad (2)$$

Adding (1) and (2)

The total energy at P

$$= \frac{m \mu x^2}{2} + \frac{m \mu}{2} (a^2 - x^2) = \frac{m \mu a^2}{2} \quad (3)$$

If n is the frequency.

$$\text{We know that } n = \frac{1}{\text{period}} = \frac{1}{\frac{2\pi}{\sqrt{\mu}}} = \frac{\sqrt{\mu}}{2\pi}$$

$$\therefore \sqrt{\mu} = 2\pi n \quad \text{or} \quad \sqrt{\mu} = 4\pi^2 n^2 .$$

Putting this values of in (3)

The total energy = $\frac{1}{2} m \cdot 4 \pi^2 n^2 a^2 = 2 \pi^2 m a^2 n^2$ and this is proportional to a^2 and n^2 .

Example : 7 (U . Q)

Show that in S.H.M of amplitude a and period T the velocity v at a distance x from the centre is given by the relation $v^2 T^2 = 4 \pi^2 (a^2 - x^2)$.

Find the new amplitude if the velocity were doubled when the particle is at a distance $a/2$ from the centre, the period remaining the same.

Solution :

We know that $v^2 = \mu (a^2 - x^2)$

And periodic is $T = \frac{2\pi}{\sqrt{\mu}}$.

$$\therefore \mu = \frac{4\pi^2}{T^2}$$

$$\therefore v^2 = \frac{4\pi^2}{T^2} (a^2 - x^2)$$

$$\text{or } v^2 T^2 = 4 \pi^2 (a^2 - x^2) .$$

The velocity v_1 of the particle at $x = \frac{a}{2}$ is given by

$$v_1^2 = \mu \left(a^2 - \frac{a^2}{4} \right) = 3\mu \frac{a^2}{4}$$

$$\therefore v_1 = \left(\sqrt{3\mu} \right) \frac{a}{2}$$

Now the initial condition will be that when $x = \frac{a}{2}$

$$V = 2v_1 = 2 \left(\sqrt{3\mu} \right) \frac{a}{2} = \left(\sqrt{3\mu} \right) a$$

From the equation of S.H.M

We get $\frac{d^2x}{dt^2} = v \frac{dv}{dx} = -\mu x$

$$Vdv = -\mu x dx$$

Or Integrating $V^2 = -\mu x^2 + A$.

Using the condition that at $x = \frac{a}{2}$, $v = 2v_1 = \sqrt{3\mu} a$

$$\therefore \sqrt{3\mu} a^2 = -\mu \frac{a^2}{4} + A$$

$$\therefore A = 13\mu \frac{a^2}{4}$$

$$\therefore V^2 = -\mu x^2 + 13\mu \frac{a^2}{4} = \mu \left(13 \frac{a^2}{4} - x^2 \right)$$

Amplitude is that value of x where velocity is zero.

$$13 \frac{a^2}{4} - x^2 = 0 \quad \text{or} \quad x = \frac{\sqrt{13}}{2} a$$

Example 8 (U . Q)

If v_1 and v_2 are the velocities of a particle moving in S.H.M at distance x_1 and x_2 from the centre. Show that the time of complete oscillation is

$$2\pi \left(\sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}} \right) \quad [\text{Nov. 93}]$$

Solution :

We know that $v^2 = \mu (a^2 - x^2)$

$$\therefore v_1^2 = \mu (a^2 - x_1^2); v_2^2 = \mu (a^2 - x_2^2)$$

$$\therefore v_2^2 - v_1^2 = \mu (x_1^2 - x_2^2)$$

$$\therefore \mu = \frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}$$

(1)

$$\text{Also periodic time} = \frac{2\pi}{\sqrt{\mu}} = 2\pi \sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}} \quad \text{by (1)}$$

Example :

The position of a particle moving in a straight line is given $x = a \cos nt + b \sin nt$

Prove that it executes S.H.M of period $\frac{2\pi}{n}$ and amplitude $\sqrt{a^2 + b^2}$.

Solution :

Differentiating the given relation.

We get

$$v = \frac{dx}{dt} = -an \sin nt + bn \cos nt$$

$$\therefore \frac{d^2x}{dt^2} = -an^2 \cos nt - bn^2 \sin nt = -n^2x$$

Hence the motion is S.H.M of period

$$\frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{n}$$

$$\therefore \mu = n^2$$

Amplitude is the distance between the centre and the position of instantaneous rest where velocity is zero.

Putting $v = 0$

We get

$$-an \sin nt + bn \cos nt = 0$$

$$\therefore \tan nt = \frac{b}{a} \text{ or } \sin nt = \frac{b}{\sqrt{a^2 + b^2}}$$

and $\text{Cos } nt = \frac{a}{\sqrt{a^2 + b^2}}$

Putting for Sinnt and Cosnt in

$$X = a \text{ cosnt} + b \text{ Sinnt}$$

We get Amplitude = $a \frac{a}{\sqrt{a^2 + b^2}} + b \frac{b}{\sqrt{a^2 + b^2}}$

$$\text{Amplitude} = \sqrt{a^2 + b^2}$$

Example : 9

A particle moves in a straight line. The velocity v at a distance x from a fixed point in the line is given by $v^2 = \alpha - \beta x^2$. Where K and 2 are positive constants. Show that the motion is simple harmonic. Find period and amplitude

Solution :

We have $v^2 = \alpha - \beta x^2$

Differentiating both sides w. r. to t .

We get

$$2v \frac{dv}{dt} = -\beta 2x \frac{dx}{dt}$$

(ie) $2v\ddot{x} = -2\beta xv$

or $\ddot{x} = -\beta x$

This is of the form $\ddot{x} = -n^2 x$ where $n^2 = \beta$. Hence the motion is simple harmonic of period $2\pi/n = 2\pi/\sqrt{\beta}$.

Further, the velocity vanishes when $\alpha - \beta x^2 = 0$

(ie) when $x = \pm \sqrt{\alpha/\beta}$

Hence the motion is confined to a distance $\sqrt{\alpha/\beta}$ on either side of the origin.

Hence the amplitude is $\sqrt{\alpha/\beta}$.

Example : 10

The velocity of a particle moving in a straight line is given by the equation

$V = k \sqrt{a^2 - x^2}$ where K and a are constants and x is the distance of the particular from a fixed point on the line. Prove that the motion is simple harmonic and find its amplitude.

Solution :-

$$V = K \sqrt{a^2 - x^2}$$

$$V^2 = K^2 (a^2 - x^2)$$

Differentiating both sides w. r. to x ,

We get

$$2v \cdot \frac{dv}{dx} = K^2 (-2x).$$

$$(or) \quad v \cdot \frac{dv}{dx} = -K^2 x \quad (ie) \quad \frac{d^2x}{dt^2} = -k^2 x$$

Above equation shows that acceleration varies as the distance of the particle from a fixed point and is directed towards the fixed point and is directed towards the fixed point.

\therefore The motion is simple harmonic of period $\frac{2\pi}{k}$.

Example : 11

A body moving in a straight line OAB with S.H.M has zero velocity at the points A and B whose distance from O are a and b respectively and has a velocity v when half way between them show that complete period is $\pi(b - a)/v$.

Solution :-

Since the velocities at A and B are zero and as such they are the positions of instantaneous rest and $AB = OB - OA = b - a$

$$\therefore \text{Amplitude} = \frac{1}{2} AB = \frac{b-a}{2} = A \text{ say.}$$

Now in S.H.M $v^2 = \mu (A^2 - x^2)$

At the centre $x = 0$ and $V = v$ given.

$$\therefore v^2 = \mu A^2 = \mu \left(\frac{b-a}{2} \right)^2 \text{ or } \sqrt{\mu} = \frac{2v}{b-a}$$

$$\therefore T = \frac{2\pi}{\sqrt{\mu}} = 2\pi \frac{(b-a)}{2v} = \pi \frac{(b-a)}{v}$$

Example :

The maximum velocity of a particle executing S.H.M is 2ft / sec and its period is $\frac{1}{5}$ sec. Find the amplitude.

Solution :-

If $\ddot{x} = -n^2 x$ be the equation of S.H.M and a be the amplitude, then the velocity v at a distance x is given by

$$V^2 = n^2 (a^2 - x^2)$$

The velocity is a maximum when $x = 0$ ((ie) at the origin) the maximum velocity is na .

Further , the period is $\frac{2\pi}{n}$

$$\text{Here } na = 2 \text{ and } \frac{2\pi}{n} = \frac{1}{5}$$

These equations give $a = (1/5\pi)$ ft.

Example :12

A particle moving with S.H.M of period T sec has a maximum velocity of 2.4 metres / sec. Find the amplitude and the velocity at a distance of 0.9 metres from the central position.

Solution :

$$\text{We have } \frac{2\pi}{n} = T$$

$$\text{and } na = 2.4 \quad \text{and}$$

$$\text{hence } a = 1.2 \text{ metres.}$$

$$\text{Also } v^2 = n^2 (a^2 - x^2)$$

$$\therefore \text{ when } x = 0.9$$

$$\text{we have } v^2 = 2^2 [(1.2)^2 - (0.9)^2]$$

$$\therefore v = \sqrt{2.52} \text{ metres / sec}$$

Example :13

A particle is performing S.H.M of period T about a centre O and it passes through a point P with velocity v in the direction OP . Show that the time elapses before it returns to P is $\frac{T}{\pi} \tan^{-1} (vt/2\pi \cdot op)$

Solution :

Let the equation of motion be

$$\ddot{x} = -n^2 x \text{ where } \frac{2\pi}{n} = T$$

The equation can be written as

$$(D^2 + n^2) x = 0 \text{ where } D = \frac{d}{dt}$$

The solution of this second order linear equation with constant coefficients is given by

$$X = A \cos nt + B \sin nt \quad (1)$$

Where A and B are arbitrary constants.

$$\text{Further } \ddot{x} = -An \sin nt + Bn \cos nt \quad (2)$$

If we measure time from the instant when the particle is at P. we have $x = OP$ and $\dot{x} = v$ when $t = 0$.

Equations (1) and (2) with $t = 0$ then give, $OP = A$ and $V = Bn$

$$\therefore x = -n \cdot OP \sin nt + v \cos nt$$

Time to reach the extreme position in the direction OP is obtained. When $\dot{x} = 0$

$$\therefore \text{ we have } \tan nt = \frac{V}{n \cdot OP}$$

$$t = \frac{1}{n} \tan^{-1} [v/n \cdot OP]$$

(ie)

$$= \frac{T}{2\pi} \tan^{-1} [VT/2\pi \cdot OP]$$

\therefore Time that elapses before the particle returns to P = $2t$

$$= \frac{T}{\pi} \tan^{-1} [VT/2\pi \cdot OP]$$

Example : 14

A mass m lie on a horizontal shelf making vertical simple harmonic oscillations of amplitude 1 foot and period 1 sec. Show that the mass leaves the shelf at a height of $8/\pi^2$ ft above the mean position that it then finally reaches a

height of $\left(\frac{\pi^2}{16} + \frac{4}{\pi^2}\right)$ ft above the mean position.

Solution: -

At a height x above the mean position O. the acceleration of the shelf (and hence that of the mass m on the shelf) is $n^2 x$ towards O. the forces acting on the mass are its own weight mg vertically downwards and the reaction R of the shelf vertically upwards.

∴ The equation of motion of the mass is given by $m(n^2 x) = mg - R$

$$(ie) R = mg - mn^2 x.$$

The particle leaves the shelf when $R = a$

$$(ie) \text{ when } x = \frac{g}{n^2}$$

$$\text{But } \frac{2\pi}{n} = \text{period} = 1 \text{ sec}$$

$$\therefore n = 2\pi$$

Hence the particles leaves the shelf at a height $x = \frac{g}{n^2} = \frac{3^2}{4\pi^2} = \frac{8}{\pi^2}$ feet above the mean position. Further the velocity v of the mass just when it leaves the shelf is given by

$$\begin{aligned} v^2 &= n^2 (a^2 - x^2) = 4\pi^2 \left(1 - \left(\frac{8}{\pi^2} \right)^2 \right) \\ &= \left(4\pi^2 - \frac{256}{\pi^2} \right) \end{aligned}$$

with v as initial velocity the mass travels upwards a further distance y under the retardation g before its velocity vanishes.

$$\therefore \text{ we have } 0 = v^2 - 2gy$$

$$y = \frac{v^2}{2g} = \frac{4\pi^2 - \frac{256}{\pi^2}}{64}$$

(ie)

$$= \left(\frac{\pi^2}{16} - \frac{4}{\pi^2} \right) \text{ ft}$$

∴ The total height to which the mass rises above the mean position

$$\begin{aligned} &= \frac{8}{\pi^2} + y = \frac{8}{\pi^2} + \frac{\pi^2}{16} - \frac{4}{\pi^2} \\ &= \frac{4}{\pi^2} + \frac{\pi^2}{16} \end{aligned}$$

Exercise:

1. A mass of 1 gm vibrates through a millimeter on each side of the midpoint of its path 256 times per sec; if the motion be simple harmonic, find the maximum velocity.
2. In a S.H.M if f be the acceleration and v the velocity at any time and T is the periodic time, prove that $f^2 T^2 + 4\pi^2 v^2$ is constant.
3. If a body of mass m executing S.H.M makes n complete oscillations per sec. Show that the difference of its K.E. When at the centre and when at a distance x from the centre is given by $2m\pi^2 n^2 x^2$
4. A particle moving in S.H.M along a straight line between the points A, A' . P is a point in AA' such that $AP : PA' = 1 : 3$. Show that the time from A to P is half the time taken from P to A' .
5. At the end of 3 consecutive seconds, the distances of a point moving with S.H.M from its mean position measured in the same direction are 1, 5, 5 units. Show that the period of one complete oscillation is $\frac{2\pi}{2}$ secs where $\cos \theta = \frac{3}{5}$.

8.2 Composition of Simple Harmonic Motion

8.2.1 Composition of two Simple Harmonic Motions of the same period and in the same straight line:

Since the period is dependent only on the constant μ . The two separate S.H.M are expressed by the same differential equations.

$$\frac{d^2x}{dt^2} = -\mu x$$

Let x_1 and x_2 be the displacements for the separate motions. Then we can take

$$X_1 = a_1 \cos(\sqrt{\mu}t + \epsilon_1) \text{ and } X_2 = a_2 \cos(\sqrt{\mu}t + \epsilon_2).$$

Let X be the resultant displacement

Then $x = x_1 + x_2$

$$= a_1 \cos (\sqrt{\mu t} + \epsilon_1) + a_2 \cos (\sqrt{\mu t} + \epsilon_2).$$

$$= \cos \sqrt{\mu t} (a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2)$$

$$- \sin \sqrt{\mu t} (a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2).$$

$$= \cos \sqrt{\mu t} \cdot A \cos \epsilon - \sin \sqrt{\mu t} \cdot A \sin \epsilon \quad (1)$$

$$\text{where } A \cos \epsilon = a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2 \quad (2)$$

$$\text{and } A \sin \epsilon = a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2 \quad (3)$$

we can find the new constants A and ϵ

Squaring (2) and (3) and adding

$$A^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\epsilon_1 - \epsilon_2) \quad (4)$$

Dividing (3) by (2)

$$\tan \epsilon = \frac{a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2}{a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2} \quad (5)$$

Now (1) becomes $x = A (\cos \sqrt{\mu t} \cos \epsilon - \sin \sqrt{\mu t} \sin \epsilon)$

$$= A \cos (\sqrt{\mu t} + \epsilon) \quad (6)$$

The resultant displacement given by (6) also represents a S.H.M of the same period as the individual motions. A new amplitude is the diagonal of the parallelogram whose sides are the original amplitudes a_1 and a_2 inclined to one another at an angle $\epsilon_1 - \epsilon_2$, the difference of the epochs.

8.2.2 Composition of two Simple Harmonic Motions of the same period in two perpendicular directions:

If a particle possesses two S.H.M in \perp directions and of the same period, we can prove that its path is an ellipse. Take the two \perp lines as the axes of X and Y. The displacements of the particle due to the separate motions can be taken as

$$X = a_1 \cos \sqrt{\mu t} \quad (1)$$

$$Y = a_2 \cos (\sqrt{\mu t} + \epsilon) \quad (2)$$

The path of the particle is obtained by eliminating t between (1) and (2)

From (2)

$$y = a_2 \cos \sqrt{\mu t} \cdot \cos \epsilon - a_2 \sin \sqrt{\mu t} \sin \epsilon$$

$$= a_2 \cos \epsilon \frac{x}{a_1} - a_2 \sin \epsilon \sqrt{1 - \frac{x^2}{a_1^2}}$$

$$(ie) \frac{y}{a_2} - \frac{x \cos \epsilon}{a_1} = - \sin \epsilon \sqrt{1 - \frac{x^2}{a_1^2}}$$

Squaring

$$\frac{y^2}{a_2^2} + \frac{x^2 \cos^2 \epsilon}{a_1^2} - \frac{2xy \cos \epsilon}{a_1 a_2} = \sin^2 \epsilon - \frac{x}{a_1} \sin^2 \epsilon$$

$$(ie) \frac{x^2}{a_1^2} - \frac{2xy}{a_1 a_2} \cos \epsilon + \frac{y^2}{a_2^2} = \sin^2 \epsilon \quad (3)$$

$$\text{This is of the form } ax^2 + 2hxy + by^2 = \lambda \quad (4)$$

$$\text{Where } a = \frac{1}{a_1^2}, \quad h = -\frac{\cos \epsilon}{a_1 a_2}, \quad b = \frac{1}{a_2^2}$$

Clearly (4) represents conic with centre at the origin.

$$\text{Also, } ab - h^2 = \frac{1}{a_1^2 a_2^2} - \frac{\cos^2 \epsilon}{a_1^2 a_2^2} = \frac{\sin^2 \epsilon}{a_1^2 a_2^2} = +ve$$

Hence (4) represents an ellipse.

If $\epsilon = 0$, equation (3) gives $\frac{x}{a_1} - \frac{y}{a_2} = 0$ which is a straight line.

If $\epsilon = \pi$, (3) gives $\frac{x}{a_1} + \frac{y}{a_2} = 0$ which is also a straight line.

If $\epsilon = \frac{\pi}{2}$, (3) gives $\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} = 1$ which is an ellipse whose principal axes are along the axes of x and y .

If $\epsilon = \frac{\pi}{2}$ and $a_1 = a_2$, the path is the circle $x^2 + y^2 = a_1^2$.

Example : 1

Show that the resultant of two S.H.M in the same direction and of equal periodic time, the amplitude of one being twice that of the other and its phase a quarter of a period in advance, is a S.H.M of amplitude $\sqrt{5}$ times that of the first and whose phase is in advance of the first by $\frac{\tan^{-1} 2}{2\pi}$ of a period.

Solution:

Referring to composition of two S.H.M of the same period and in the same straight line.

Let the separate displacements be

$$X_1 = a_1 \cos (\sqrt{\mu}t + \epsilon_1) \quad \text{and} \quad (1)$$

$$X_2 = a_2 \cos (\sqrt{\mu}t + \epsilon_2) \quad (2)$$

Here $a_2 = 2a_1$, and $\frac{\epsilon_2 - \epsilon_1}{\sqrt{\mu}} = \text{phase difference}$

$$= \frac{1}{4} \times \frac{2\pi}{\sqrt{\mu}}$$

$$\therefore \epsilon_2 - \epsilon_1 = \frac{\pi}{2} \quad \text{or} \quad \epsilon_2 = \frac{\pi}{2} + \epsilon_1$$

we know that the resultant displacement is

$$x = A \cos (\sqrt{\mu t} + \epsilon) \quad (3)$$

$$\text{where } A_2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos (\epsilon_1 - \epsilon_2)$$

$$= a_1^2 + 4a_1^2 + 4a_1^2 \cos(-90^\circ) = 5a_1^2$$

$$\therefore \text{Amplitude of the resultant motion} = A = a_1 \sqrt{5}$$

$$\text{Also } \tan \epsilon = \frac{a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2}{a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2}$$

$$= \frac{a_1 \sin \epsilon_1 + 2a_1 \sin (90^\circ + \epsilon_1)}{a_1 \cos \epsilon_1 + 2a_1 \cos (90^\circ + \epsilon_2)}$$

$$(ie) \frac{\sin \epsilon}{\cos \epsilon} = \frac{\sin \epsilon_1 + 2 \cos \epsilon_1}{\cos \epsilon_1 - 2 \sin \epsilon_1}$$

$$\sin \epsilon \cos \epsilon_1 - 2 \sin \epsilon \sin \epsilon_1 = \sin \epsilon_1 \cos \epsilon + 2 \cos \epsilon_1 \cos \epsilon$$

$$\text{or } \sin (\epsilon - \epsilon_1) = 2 \cos (\epsilon - \epsilon_1)$$

$$(ie) \tan (\epsilon - \epsilon_1) = 2$$

$$(or) \epsilon - \epsilon_1 = \tan^{-1} 2$$

$$\therefore \frac{\epsilon - \epsilon_1}{\sqrt{\mu}} = \frac{\tan^{-1} 2}{\sqrt{\mu}} = \frac{\tan^{-1} 2}{2\pi} \left(\frac{2\pi}{\sqrt{\mu}} \right)$$

$$= \frac{\tan^{-1} 2}{2\pi} \text{ of a period .}$$

This is the phase difference of the resultant Simple harmonic motion.

Exercise: -

- 1) Two simple harmonic motions in the same straight line of equal period and differing in phase by $\frac{\pi}{2}$ are impressed simultaneously on a particle. If the amplitudes are 4 and 6, find the amplitude and phase of the resulting motion.
- 2) A particle possesses two S.H.M of the same period with amplitudes a and b and phase difference $\frac{\pi}{2}$ in two \perp directions. Show that the particle traces an ellipse whose semi major and minor axes are a and b.

UNIT – IX

CENTRAL ORBITS

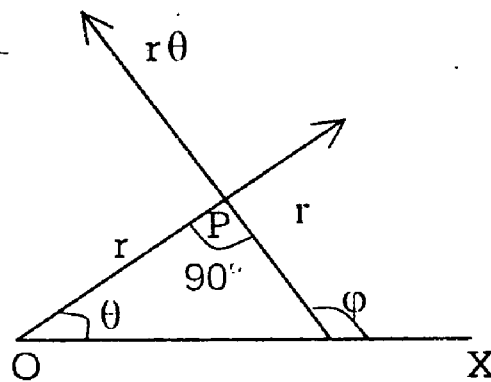
9.1 Introduction:

In the previous chapters, we have considered some particular case of motion of a particle in two dimensions. To fix the position of a particle in a plane, we require two coordinates and to study the motion of the particle. We require its component velocities and accelerations in two mutually \perp directions. We had previously used cartesian coordinates. In this chapter we shall use polar coordinates

9.2 Components of velocity and Acceleration Velocity and Acceleration in polar coordinates:

Let P be the position of moving particle at time t. Taking O as the pole and OX as the initial line, let the polar coordinates of P be (r, θ) . $\overline{OP} = r$ is the position vector of P. Hence the velocity of P = $\frac{d}{dt} (r)$ since r has modulus r and amplitude

θ , $\frac{d}{dt} (r)$ will have components \dot{r} along OP and $r\dot{\theta}$ to OP.



Hence the velocity vector v at P has components \dot{r} along OP in the direction in which r increases and $r\dot{\theta}$ \perp to OP.

In the direction in which θ increases. These are respectively called the radial and transverse components of v.

The acceleration vector at P is the derivative of the velocity vector V. The radial component of v is a vector which modulus r and amplitude $\dot{\theta}$. Hence the derivative of \dot{r} will have components

$$i) \frac{d}{dt}(\dot{r}) = \ddot{r}$$

along OP in the direction in which r increases and

$$ii) r \frac{d}{dt}(\dot{\theta}) = r \ddot{\theta}$$

\perp^r to OP in the direction in which θ increases. This is shown in fig. (a).

The transverse component of v is a vector with modulus $r \dot{\theta}$ and amplitude

$$\phi = \frac{\pi}{2} + \theta.$$

Hence the derivative of $r \dot{\theta}$ will have components.

$$i) \frac{d}{dt}(r \dot{\theta}) = \dot{r} \dot{\theta} + r \ddot{\theta}$$

along the line of $r \dot{\theta}$

(ie) in the direction \perp^r to OP and

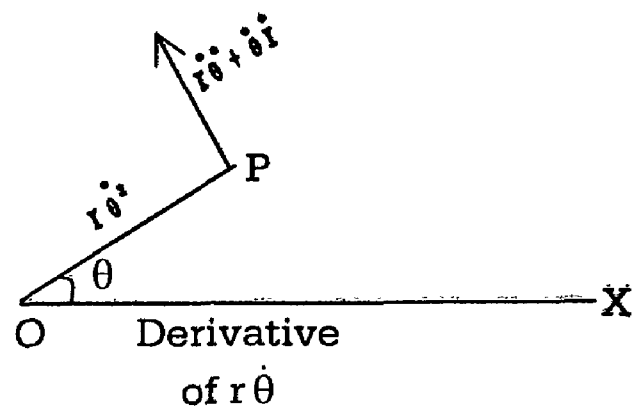
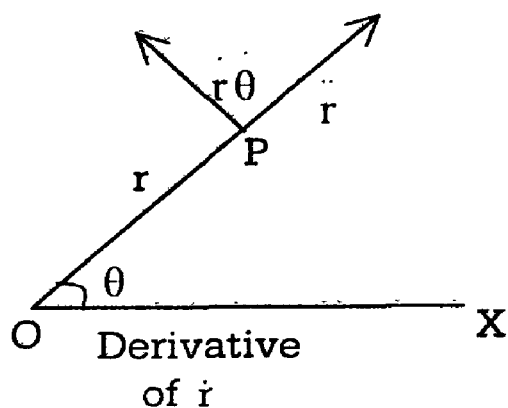
$$ii) r \dot{\theta} \frac{d}{dt} \left(\frac{\pi}{2} + \theta \right) = r \dot{\theta}^2$$

in the direction \perp^r to the line of $r \dot{\theta}$

(ie) in the direction PO.

(This component is towards O, as it is in the direction in which ϕ increases)

This is shown in fig. (b).



Hence the totals of the components of acceleration are $r - r\dot{\theta}^2$ in direction OP and $r\ddot{\theta} + 2r\dot{\theta}$ in the \perp^r direction.

$$\begin{aligned} \text{Now, } \frac{1}{r} \cdot \frac{d}{dt}(r^2 \dot{\theta}) &= \frac{1}{r}(r^2 \ddot{\theta} + 2r\dot{r}\dot{\theta}) \\ &= r\ddot{\theta} + 2r\dot{\theta} \end{aligned}$$

$$\therefore \text{Acceleration } \perp^r \text{ to OP is also } = \frac{1}{r} \frac{d}{dt}(r^2 \dot{\theta}).$$

The above results are collected in a table of reference.

	Magnitude	Direction	Sense
1. Radial component of velocity	r	Along the radius vector	In the direction in which r increases.
2. Transverse component Of velocity	$r\dot{\theta}$	Perpendicular to the radius vector.	In the direction in which θ increases.
3. Radial component of acceleration	$r - r\dot{\theta}^2$	Along the radius vector	In the direction in which r increases
4. Transverse component Of acceleration	$\frac{1}{r} \frac{d}{dt}(r^2 \dot{\theta})$	Perpendicular to the radius vector.	In the direction in which θ increases.

Theorem:-

9.2.1 The velocities of a particle along and \perp^r to the radius vector from a fixed point are a and b . Find the acceleration components along \perp^r to the radius vector and the equation of the path.

Solution:-

We are given that

$$r = a \quad \text{and} \quad r\dot{\theta} = b.$$

$$\therefore \frac{r}{r\dot{\theta}} = \frac{a}{b}$$

$$(ie) \frac{dr}{rd\theta} = \frac{a}{b}$$

$$(ie) \frac{dr}{r} = \frac{a}{b} d\theta$$

On integration, we get

$$\log r = \frac{a}{b} \theta + \log K$$

$$(ie) \log \left(\frac{r}{K} \right) = \frac{a}{b} \theta$$

(ie) $r = Ke^{a\theta/b}$ which is an equiangular spiral.

Further, Since $\dot{r} = a$

We have $r = 0$

$$\begin{aligned} \therefore \text{the radial acceleration} &= \ddot{r} - r\dot{\theta}^2 \\ &= 0 - \frac{1}{r} (r\dot{\theta})^2 \\ &= -\frac{b^2}{r} \end{aligned}$$

and the transverse acceleration.

$$= \frac{1}{r} \cdot \frac{d}{dt} (r^2 \dot{\theta})$$

$$\begin{aligned} \text{Transverse acceleration} &= \frac{1}{r} \cdot \frac{d}{dt} (r \cdot r\dot{\theta}) \\ &= \frac{1}{r} \frac{d}{dt} (r \cdot b) = \frac{1}{r} b \dot{r} \\ &= \frac{1}{r} ba = \frac{ab}{r} \end{aligned}$$

Corollary:

The magnitude of the resultant velocity of P

$$= \sqrt{r^2 + (r\dot{\theta})^2} = \sqrt{r^2 + r^2 \dot{\theta}^2}$$

and the magnitude of the resultant acceleration

$$= \sqrt{\left(\ddot{r} - r\dot{\theta}^2\right)^2 + \left\{\frac{1}{r} \cdot \frac{d}{dt} (r^2 \dot{\theta})\right\}^2}$$

9.22 Equations of Motion in Polar Coordinates:

If R and S are the components of the external force acting on a particle of mass m in the radial and transverse directions, we have the equation

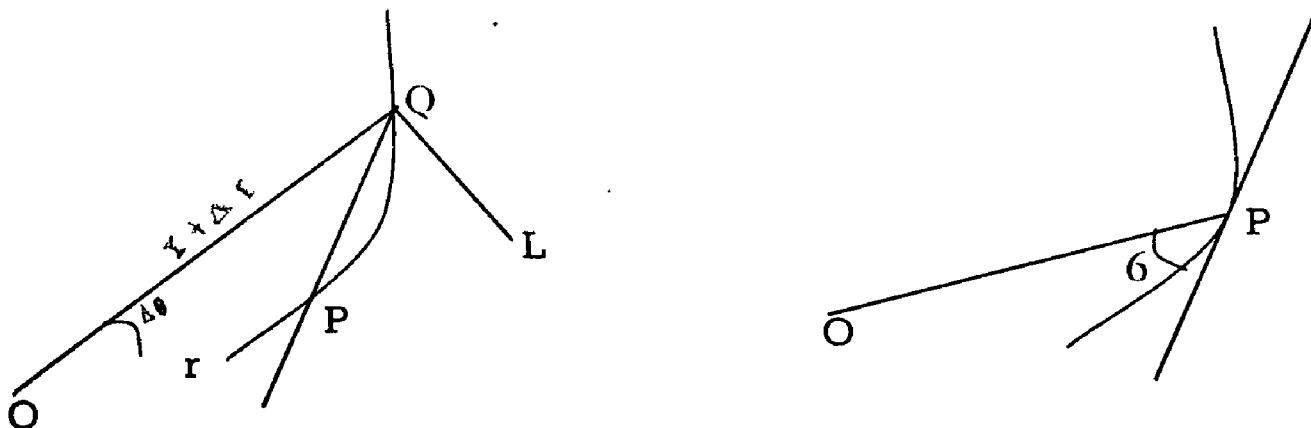
$$R = m (\ddot{r} - r\dot{\theta}^2) \tag{1}$$

and
$$S = m \cdot \frac{1}{r} \cdot \frac{d}{dt} (r^2 \dot{\theta})$$

If R and S are known functions of the coordinates r, θ and the time t, the differential equations (1) and (2) can be solved to find r and θ as functions of t and by eliminating t, the polar equation to the path is got.

Note on the equiangular spiral:

Some equations in this chapter will relate to the curve called the equiangular spiral. This curve has the important property that the tangent at any point P on it makes a constant angle with the radius vector OP.



Let OP (=r) and OQ (= r + Δr) be two consecutive radii vectors such that the included angle POQ = Δθ.

Draw $QL \perp r$ to OP .

Then $OL = (r + \Delta r) \cos \Delta \theta = r + \Delta r$ approximately.

Hence $PL = OL - OP = \Delta r$

and $LQ = (r + \Delta r) \sin \Delta \theta = (r + \Delta r) \Delta \theta$ nearly

$= r \Delta \theta$ to the first order of smallness .

Hence $\tan \angle QPL = \frac{QL}{PL} = r \frac{\Delta \theta}{\Delta r}$.

In the limit as Δr and $\Delta \theta$ both $\rightarrow 0$ the point Q tends to coincide with P . The chord QP becomes in the limiting position, the tangent at P . Let ϕ be the angle made by the tangent at P with OP .

Then $\phi = \lim_{Q \rightarrow P} \angle QPL$

Hence $\tan \phi = \lim_{\Delta r \rightarrow 0} \tan \angle QPL = \lim_{\Delta r \rightarrow 0} r \frac{\Delta \theta}{\Delta r} = r \frac{d\theta}{dr}$

$$(ie) \tan \phi = r \frac{d\theta}{dr}$$

This formula is an important one in dealing with curves in polar coordinates and it gives the angle between the radius vector and the tangent. Now for the equiangular spiral, at any point P on it the angle ϕ is constant.

Let $\phi = \alpha$. Then $\tan \phi = \tan \alpha$.

$$(ie) r \frac{d\theta}{dr} = \tan \alpha$$

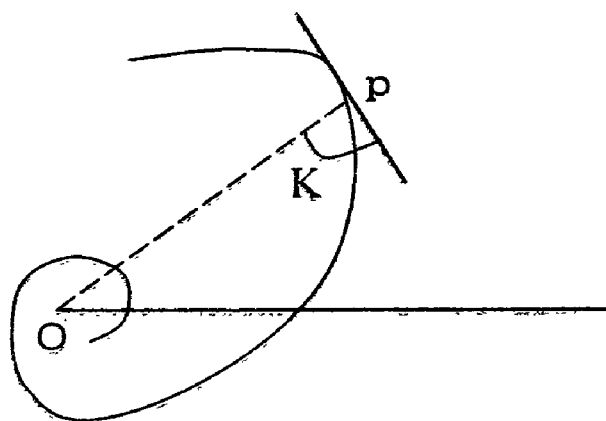
$$(or) \frac{dr}{r} = \cot \alpha \cdot d\theta$$

Integrating, $\log r = \theta \cot \alpha + \text{constant}$

$$(ie) r = ae^{\theta \cot \alpha}$$

This is the polar equation to the equiangular spiral.

This shape of the curve is as shown below.



Worked Examples

Example : 1

The velocities of a particle along and \perp to a radius vector from a fixed origin are λr^2 and $\mu\theta^2$ where μ and λ are constants. Show that the equation to the path of the particle is $\frac{\lambda}{\theta} + C = \frac{\mu}{2r^2}$ where C is a constant.

Show also that the accelerations along and perpendicular to the radius vector are $2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r}$ and $\mu \left(\lambda r\theta^2 + \frac{2\mu\theta^3}{r} \right)$

Solution:

$$\text{Radial velocity} = \frac{dr}{dt} = \lambda r^2 \quad (1)$$

$$\text{Transverse velocity} = r \frac{d\theta}{dt} = \mu\theta^2 \quad (2)$$

Dividing (2) by (1)

We have

$$r \frac{d\theta}{dr} = \frac{\mu\theta^2}{\lambda r^2} \quad (\text{ie}) \quad \lambda \frac{d\theta}{\theta^2} = \frac{\mu}{r^3} dr$$

$$\text{Integrating, } -\frac{\lambda}{\theta} = -\frac{\mu}{2r^2} + C$$

$$(ie) \frac{\mu}{2r^2} = \frac{\lambda}{r} + C$$

(3)

(3) is the equation to the path.

$$\text{Differentiating (1), } \frac{d^2r}{dt^2} = \lambda \quad 2r \frac{dr}{dt}$$

$$= 2\lambda^2 r^3 \text{ using (1)}$$

Radial acceleration

$$= r - r\theta^2 = \frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2$$

$$= 2\lambda^2 r^3 - r \left(\frac{\mu\theta^2}{r} \right)^2 = 2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r} \text{ using (2)}$$

Transverse acceleration

$$= \frac{1}{r} \cdot \frac{d}{dt} (r^2 \dot{\theta}) = \frac{1}{r} \cdot \frac{d}{dt} \left(r^2 \frac{\mu\theta^2}{r} \right)$$

$$= \frac{1}{r} \cdot \frac{d}{dt} (\mu r \theta^2) = \frac{\mu}{r} \left[r \cdot 2\theta \frac{d\theta}{dt} + \theta^2 \frac{dr}{dt} \right]$$

$$= \frac{\mu}{r} \left[2r \cdot 0 \cdot \frac{\mu\theta^2}{r} + \theta^2 \cdot \lambda r^2 \right] = \mu \left[\frac{2\mu\theta^3}{r} + \lambda r\theta^2 \right]$$

Example : 2

Show that the path of a point P which possesses two constant velocities u and v , the first of which is in a fixed direction and the second of which is perpendicular to the radius OP drawn from a fixed point O is a conic whose focus is O and whose eccentricity is $\frac{u}{v}$.

Solution:

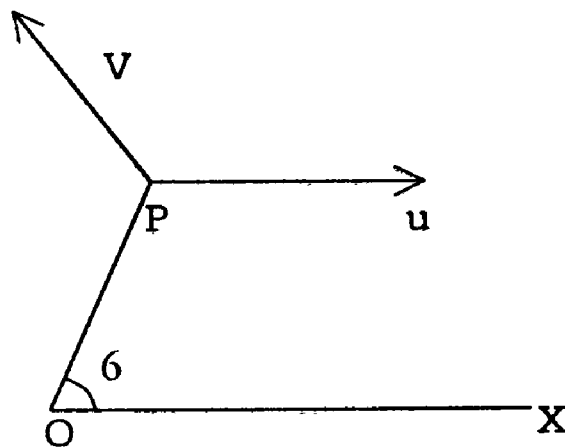
Take O as the pole and the line OX parallel to the given direction as the initial line P has two velocities u parallel to OX and v perpendicular to OP.

Resolving the velocities along and perpendicular to OP,

$$\text{We have } \frac{dr}{dt} = u \cos \theta \quad (1)$$

$$r \frac{d\theta}{dr} = v - u \sin \theta \quad (2)$$

To get the equation to the path, we have to eliminate t



Dividing (2) by (1)

We have

$$r \frac{d\theta}{dr} = \frac{v - u \sin \theta}{u \cos \theta}$$

$$\text{(ie) } \frac{u \cos \theta}{v - u \sin \theta} d\theta = \frac{dr}{r} \text{ or } \frac{d(u \sin \theta)}{v - u \sin \theta} = \frac{dr}{r}$$

Integrating,

$$- \log (v - u \sin \theta) + \log A = \log r \text{ where } \log A \text{ is the constant of integration}$$

$$\text{(ie) } \log r + \log (v - u \sin \theta) = \log A$$

$$\text{(ie) } r (v - u \sin \theta) = A \text{ or } \frac{A}{r} = v - u \sin \theta$$

$$\text{(ie) } \left(\frac{A}{v} \right) \frac{1}{r} = 1 - \frac{u}{v} \sin \theta = 1 + \frac{u}{v} \cos (90^\circ + \theta) \quad (1)$$

This is the form $\frac{\ell}{r} = 1 + e \cos(\theta + \alpha)$

(2)

Comparing (1) and (2)

We have $\ell = \frac{A}{V}$, $e = \frac{u}{v}$ and $\alpha = 90^\circ$

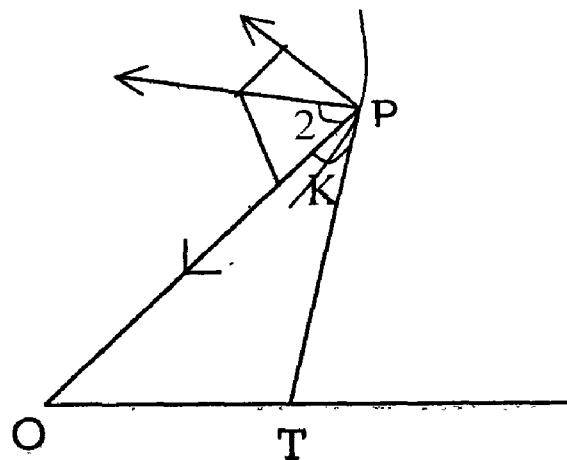
We know from Analytical Geometry that (2) is the polar equation to a conic whose focus is at the pole, semi-latus rectum is ℓ eccentricity is e and whose major axis makes an angle α , with the initial line. Hence (1) is a conic whose focus is at O , semi-latus rectum is $\frac{A}{V}$, eccentricity is $\frac{u}{v}$ and whose major axis is perpendicular to the initial line

Example : 3

A point P describes a curve with constant velocity and its angular velocity about a given fixed point O varies inversely as the distance from O ; Show that the curve is an equiangular spiral whose pole is O and that the acceleration of the point is along the normal at P and varies inversely as OP .

Solution :

Taking O as the pole, Let P be (r, θ) .



Resultant velocity of P

$$= \sqrt{r^2 + r^2 \theta^2} = \text{constant (given)}$$

If this velocity = k

We have

$$r^2 + r^2 \dot{\theta}^2 = k^2 \quad (1)$$

Also angular velocity about O

$$\dot{\theta} = \frac{\lambda}{r} \text{ (given)} \quad (2)$$

From (1) and (2)

$$r^2 + r^2 \frac{\lambda^2}{r^2} = k^2$$

$$\text{(ie) } r^2 = k^2 - \lambda^2 \quad \text{or} \quad r = \sqrt{k^2 - \lambda^2} \quad (3)$$

Eliminate t from (2) and (3)

$$\frac{dr}{d\theta} = \frac{\left(\frac{dr}{dt}\right)}{\left(\frac{d\theta}{dt}\right)} = \frac{r}{\dot{\theta}} = \frac{\sqrt{k^2 - \lambda^2}}{\lambda} r$$

$$\text{(ie) } \frac{dr}{r} = \frac{\sqrt{k^2 - \lambda^2}}{\lambda} d\theta$$

$$\text{Integrating } \log r = \frac{\sqrt{k^2 - \lambda^2}}{\lambda} \theta + B$$

$$\text{(or) } r = e^{\frac{\sqrt{k^2 - \lambda^2}}{\lambda} \theta + B} = e^B \cdot e^{\frac{\sqrt{k^2 - \lambda^2}}{\lambda} \theta}$$

$$\text{Putting } e^B = a \quad \text{and} \quad \frac{\sqrt{k^2 - \lambda^2}}{\lambda} = \cot \alpha$$

$$\text{The above becomes } r = a e^{\theta \cot \alpha} \quad (4)$$

Hence the path is an equiangular spiral, whose pole is O.

Differentiating (3), $r = 0$

$$\text{Radial acceleration} = r - r\theta^2 = -r \cdot \frac{\lambda^2}{r^2} = -\frac{\lambda^2}{r}$$

The negative sign shows that the radial acceleration at P is along PO.

Transverse acceleration

$$= \frac{1}{r} \cdot \frac{d}{dt} (r^2 \theta) = \frac{1}{r} \cdot \frac{d}{dt} \left(r^2 \frac{\lambda}{r} \right) = \frac{1}{r} \cdot \frac{d}{dt} (r\lambda)$$

$$= \frac{\lambda}{r} r = \frac{\lambda}{r} \sqrt{k^2 - r^2} \text{ from (3)}$$

Resultant acceleration of P

$$\begin{aligned} &= \sqrt{\left(\frac{-\lambda^2}{r} \right)^2 + \left[\frac{\lambda}{r} \sqrt{k^2 - \lambda^2} \right]^2} \\ &= \sqrt{\frac{\lambda^4}{r^2} + \frac{\lambda^2}{r^2} (k^2 - \lambda^2)} = \sqrt{\frac{\lambda^2 k^2}{r^2}} = \frac{\lambda k}{r} \end{aligned}$$

Thus the resultant acceleration varies inversely as r .

(ie) as OP . Let this acceleration be along PN making an angle β with PO .

$$\tan \beta = \frac{\text{Component } \perp^r \text{ to } PO}{\text{Component along } PO}$$

$$= \frac{\frac{\lambda}{r} \sqrt{k^2 - \lambda^2}}{\frac{\lambda^2}{r}} = \frac{\sqrt{k^2 - \lambda^2}}{\lambda}$$

$$\text{But } \text{Cot } \alpha = \frac{\sqrt{k^2 - \lambda^2}}{\lambda} \text{ from equation (4)}$$

Hence $\tan \beta = \text{Cot } \alpha = \tan (90^\circ - \alpha)$

(ie) $\beta = 90^\circ - \alpha$ or $\beta + \alpha = 90^\circ$

Hence $\angle NPT = 90^\circ$ where PT is the tangent at P.

Hence PN is normal at P.

Example : 4

A smooth straight thin tube revolves with uniform angular velocity ω in a vertical plane about one extremity which is fixed. If at zero time the tube be horizontal, and a particle inside it be at a distance 'a' from the fixed end, and be moving with velocity V along the tube Show that the distance at time t is

$$a \cos \omega t + \left(\frac{v}{\omega} - \frac{g}{2\omega^2} \right) \sin \omega t + \frac{g}{2\omega^2} \sin \omega t$$

Solution :

Let at time t, P be the position of the particle of mass m on the tube OB.

The forces acting at P are

- i) its weight mg vertically downwards and
- ii) normal reaction $R \perp$ to OB.

Let P be (r, θ)

$$\text{Angular velocity} = \dot{\theta} = \frac{d\theta}{dt} = \omega \text{ (given)}$$

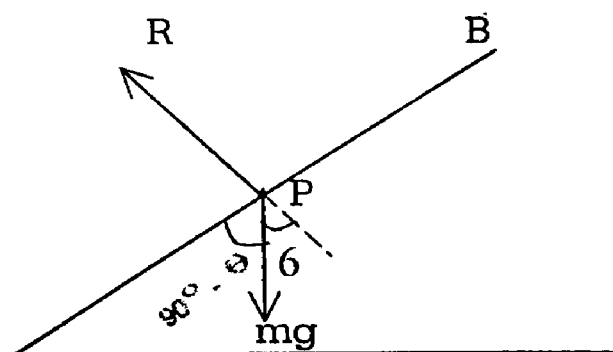
$$\text{Integrating } \theta = \omega t + A$$

$$\text{Initially when } t = 0, \theta = 0$$

$$\therefore A = 0$$

$$\therefore \theta = \omega t$$

(1)



Resolving along the radius vector OB,

$$m(\ddot{r} - r\dot{\theta}^2) = -mg \cos(90^\circ - \theta) = -mg \sin \theta$$

$$\therefore \ddot{r} - r\omega^2 = -g \sin \theta = -g \sin \omega t \quad (\text{using (1)})$$

$$\text{(ie) } (D^2 - \omega^2)r = -g \sin \omega t \quad (2)$$

$$\text{where } D = \frac{d}{dt}$$

The complementary function Y is found such that $(D^2 - \omega^2)Y = 0$

$$\text{The solution of this differential equation is } Y = Ae^{\omega t} + Be^{-\omega t} \quad (3)$$

Where A and B are constants.

The particular integral u of the equation (2) is given by

$$(D^2 - \omega^2)u = -g \sin \omega t$$

$$\therefore u = -\frac{g}{D^2 - \omega^2} \sin \omega t = -\frac{g}{-\omega^2 - \omega^2} \sin \omega t$$

$$u = \frac{g}{2\omega^2} \sin \omega t \quad (4)$$

Hence the general solution of (2) is

$$r = Y + u = Ae^{\omega t} + Be^{-\omega t} + \frac{g}{2\omega^2} \sin \omega t$$

The initial conditions are: when $t = 0$ (5)

$$r = a \quad \text{and} \quad \dot{r} = v$$

Hence (5) gives $A + B = a$ (6)

Differentiating (5)

$$\dot{r} = A\omega e^{\omega t} - B\omega e^{-\omega t} + \frac{g}{2\omega} \cos \omega t \quad (7)$$

Putting $t = 0$ and $\dot{r} = v$ in (7)

We have

$$A\omega - B\omega + \frac{g}{2\omega} = v \quad (\text{or}) \quad A - B = \frac{v}{\omega} - \frac{g}{2\omega^2} \quad (8)$$

Solving (6) and (8)

$$A = \frac{1}{2} \left(a + \frac{v}{\omega} - \frac{g}{2\omega^2} \right) \quad \text{and} \quad B = \frac{1}{2} \left(a - \frac{v}{\omega} + \frac{g}{2\omega^2} \right)$$

Substituting these values in (5)

$$r = \frac{1}{2} \left(a + \frac{v}{\omega} - \frac{g}{2\omega^2} \right) e^{\omega t} + \frac{1}{2} \left(a - \frac{v}{\omega} + \frac{g}{2\omega^2} \right) e^{-\omega t} + \frac{g}{2\omega^2} \sin \omega t$$

$$= \frac{a(e^{\omega t} + e^{-\omega t})}{2} + \left(\frac{v}{\omega} - \frac{g}{2\omega^2} \right) \left(\frac{e^{\omega t} - e^{-\omega t}}{2} \right) + \frac{g}{2\omega^2} \sin \omega t$$

$$r = a \cosh \omega t + \left(\frac{v}{\omega} - \frac{g}{2\omega^2} \right) \sinh \omega t + \frac{g}{2\omega^2} \sin \omega t$$

Example : 5

A particle moves with uniform speed v along a cardioid $r = a(1 + \cos \theta)$ show that its angular velocity about the pole is $\{ v \sec(\theta/2) / 2a \}$ and that the radial acceleration is constant.

Solution :

Since $r = a(1 + \cos \theta)$

We have $\dot{r} = -a \sin \theta \dot{\theta}$

The radial velocity $\dot{r} = -a \sin \theta \dot{\theta}$

The transverse velocity $= r\dot{\theta}$

$$= a(1 + \cos \theta) \dot{\theta}$$

We are given that

$$r^2 + r^2 \dot{\theta}^2 = v^2$$

$$(ie) \quad a^2 \sin^2 \theta \dot{\theta}^2 + a^2 (1 + \cos \theta)^2 \dot{\theta}^2 = v^2$$

$$(ie) \quad \dot{\theta}^2 a^2 [\sin^2 \theta + (1 + \cos \theta)^2] = v^2$$

$$or \quad \dot{\theta}^2 \cdot 2a^2 [1 + \cos \theta] = v^2$$

$$or \quad \dot{\theta}^2 \cdot 4a^2 \cos^2 (\theta/2) = v^2$$

$$or \quad \dot{\theta}^2 \cdot 2a \cos^2 (\theta/2) = v$$

$$(ie) \quad \dot{\theta} = \frac{v}{2a \cos (\theta/2)} = \frac{v \sec (\theta/2)}{2a}$$

\therefore The angular velocity about the pole is $[v \sec (\theta/2) / 2a]$

Substituting for $\dot{\theta}$ is (1)

We get

$$r = -a \sin \theta \cdot \frac{v \sec (\theta/2)}{2a}$$

$$= -a \cdot 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cdot \frac{v \sec (\theta/2)}{2a}$$

$$r = -v \sin (\theta/2)$$

$$r = -v \cos (\theta/2) \cdot 1/2 \theta$$

$$r = -v \cos (\theta/2) \cdot 1/2 \cdot \frac{v \sec (\theta/2)}{2a}$$

$$r = -\frac{v^2}{4a}$$

The radial acceleration

$$\begin{aligned}
 &= \dot{r} - r\dot{\theta}^2 \\
 &= -\frac{V^2}{4a} - a(1 + \cos \theta)\dot{\theta}^2 \\
 &= -\frac{V^2}{4a} - 2a \cos^2\left(\frac{\theta}{2}\right) \left\{ \frac{V \sec(\theta/2)}{2a} \right\}^2 \\
 &= -\frac{V^2}{4a} - \frac{V^2}{2a} = -\frac{3V^2}{4a}
 \end{aligned}$$

which is a constant.

Example : 6 (U . Q)

A particle of mass m moves along the curve $r = a(1 + \cos \theta)$ under the action of a force Q in the positive direction of the initial line. If the angular velocity of the particle about the pole be constant and is equal to ω , find the values of P and Q and show that the K.E of the particle at any point of the path is $a \left(\frac{2P + 3Q}{8} \right)$

Solution :

Given that $r = a(1 + \cos \theta)$ and $\dot{\theta} = \omega$

\therefore we have $\dot{r} = -a \sin \theta \cdot \dot{\theta} = -a \omega \sin \theta$

and $\ddot{r} = -a \omega \cos \theta \cdot \ddot{\theta} = -a \omega^2 \cos \theta$

The equations of motion of the particle are

$$m(\dot{r} - r\dot{\theta}^2) = -P + Q \cos \theta$$

$$\text{and } m \cdot \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) = -Q \sin \theta$$

$$\dot{\theta} = \omega, \text{ a constant and } \ddot{r} = -a\omega^2 \cos \theta$$

These equations give

$$\begin{aligned} -P + Q \cos \theta &= -m\omega^2 [a \cos \theta + r] \\ &= -m\omega^2 [a \cos \theta + a(1 + \cos \theta)] \\ &= -m\omega^2 a [2 \cos \theta + 1] \end{aligned} \quad (1)$$

$$\text{and } -Q \sin \theta = -2m\omega^2 a \sin \theta \quad (2)$$

Eliminating Q from (1) and (2)

We get

$$\begin{aligned} -P \sin \theta &= -m\omega^2 a [2 \sin \theta \cos \theta + \sin \theta] - 2m\omega^2 a \sin \theta \cos \theta \\ &= -m\omega^2 a \sin \theta [4 \cos \theta + 1] \end{aligned}$$

$$\begin{aligned} \text{(ie) } P &= -m\omega^2 a [1 + 4 \cos \theta] \\ &= m\omega^2 a \left[1 + 4 \cdot \frac{r-a}{a} \right] \text{ Since } r = a(1 + \cos \theta) \\ &= m\omega^2 a [4r - 3a] . \end{aligned}$$

Substituting the value of P in (1)

We get

$$\begin{aligned} Q \cos \theta &= P - m\omega^2 a (2 \cos \theta + 1) \\ &= m\omega^2 a (1 + 4 \cos \theta) - m\omega^2 a (2 \cos \theta + 1) \end{aligned}$$

$$Q \cos \theta = 2m\omega^2 a \cos \theta$$

$$\text{(ie) } Q = 2m\omega^2 a$$

Thus the values of P and Q are determined

Further,

$$\begin{aligned} \frac{2P + 3Q}{8} &= \frac{2m\omega^2 [4r - 3a] + 6m\omega^2 a}{8} \\ &= m\omega^2 r \end{aligned} \quad (3)$$

The velocity components of the particle along the radial and transverse direction are \dot{r} and $r\dot{\theta}$

$$\begin{aligned} \therefore \text{The K.E} &= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) \\ &= \frac{1}{2} m [(a\omega \sin \theta)^2 + a(1 + \cos \theta)^2 \omega^2] \\ &= \frac{1}{2} m \omega^2 a^2 [\sin^2 \theta + (1 + \cos \theta)^2] \\ &= \frac{1}{2} m \omega^2 a^2 [2 + 2 \cos \theta] \\ &= \frac{1}{2} m \omega^2 a^2 2 [1 + \cos \theta] \end{aligned}$$

$$\text{K.E} = m\omega^2 ar$$

Using (3) in this we see that

$$\text{K.E} = a \cdot \left(\frac{2P + 3Q}{8} \right)$$

Example : 7 (U.Q)

A point P describes an equiangular spiral with constant angular velocity about O. Show that the acceleration varies as OP and has a direction making with the tangent at P the same constant angle which OP makes.

Solution:-

Let $r = ae^{\theta \cot \alpha}$ be the equation of the equiangular spiral.

So that α is the constant angle which the radius vector OP makes with the tangent at P.

We have

$$= \dot{r} a e^{(\theta \cot \alpha)} \cdot \theta \cot \alpha = r \dot{\theta} \cot \alpha = r \omega \cot \alpha$$

where ω is the constant angular velocity about O.

Differentiating once again.

$$\text{We get } \ddot{r} = \dot{r} \omega \cot \alpha = r \omega^2 \cot^2 \alpha$$

\therefore The radial acceleration of the particle

$$\begin{aligned} &= \dot{r} - r \theta^2 \\ &= r \omega^2 \cot^2 \alpha - r \omega^2 \\ &= r \omega^2 (\cot^2 \alpha - 1) \end{aligned}$$

$$\text{The transverse acceleration} = \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta})$$

$$\begin{aligned} &= \frac{1}{r} \frac{d}{dt} (r^2 \omega) \\ &= \frac{1}{2} \omega 2r \dot{r} = 2\omega r = 2\omega (r \omega \cot \alpha) \end{aligned}$$

If ϕ be the angle which the resultant acceleration making with the radius vector OP

$$\begin{aligned} \text{Then } \tan \phi &= \frac{2\omega^2 r \cot \alpha}{r\omega^2 (\cot^2 \alpha - 1)} \frac{2 \cot \alpha}{\cot^2 \alpha - 1} \\ &= \tan 2\alpha \end{aligned}$$

$$\therefore \phi = 2\alpha$$

(ie) The direction of the acceleration makes with the tangent at P an angle $= (2\alpha - \alpha) = \alpha$.

(ie) The same constant angle α which OP makes with the tangent at P.

Exercise:

- 1) The velocities of a particle along and \perp to the radius vector from a fixed origin are a and b . Find the path and the accelerations along and perpendicular to the radius.
- 2) If the angular velocity of a particle about a point in its plane of motion be constant, prove that the transverse component of its acceleration is proportional to the radial component of its velocity.
- 3) A point P moves in a parabola in such a manner that the component velocity at right angles to the radius vector from the focus is constant. Vector from the focus is constant. Show that the acceleration of the point is constant in magnitude.

UNIT – X

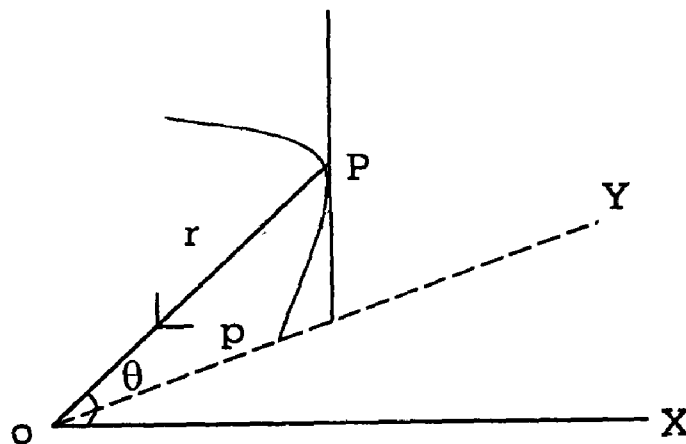
Differential Equation of a Central Orbit

10.1. Differential Equation of Central orbit:

A particle of mass m moves in a plane under the action of a force mF directed towards a fixed point in its plane. Show that its angular momentum is constant and that the differential equation of its path is $\frac{d^2u}{d\theta^2} + u = F/h^2u$, where $u = 1/r$, r be being the radius vector from the fixed point to the particle. θ is the vectorial angle and h , a constant.

Solution:

Let P be the position of the particle at any time t and let (r, θ) be the polar co-ordinates of P with respect to the fixed centre of force as pole. The acceleration of the particle along OP and perpendicular to OP are $\ddot{r} - r\dot{\theta}^2$ and $\frac{1}{r} \frac{d}{dt}(r^2\dot{\theta})$. The only force acting on P is mF towards C . Therefore the equations of motion are,



$$m(\ddot{r} - r\dot{\theta}^2) = -mF \quad \text{and} \quad (1)$$

$$m \cdot \frac{1}{r} \frac{d}{dt}(r^2\dot{\theta}) = 0 \quad (2)$$

$$\text{The Second equation gives } r^2\dot{\theta} = \text{a constant say } = h \quad (3)$$

Put $r = \frac{1}{u}$. Then we have

$$\begin{aligned}
r &= -\frac{1}{u^2} \frac{du}{dt} = -\frac{1}{u^2} \frac{du}{d\theta} \dot{\theta} \\
&= -\frac{1}{u^2} \frac{du}{dt} \frac{h}{r^2} \text{ [using (3)]} \\
&= -h \frac{du}{d\theta} .
\end{aligned}$$

$$\begin{aligned}
\text{Further, } \ddot{r} &= -h \left(\frac{du}{d\theta} \right) \\
&= -h \frac{d}{d\theta} \left(\frac{du}{d\theta} \right) \frac{du}{d\theta} \\
&= -h \frac{d^2u}{d\theta^2} \cdot \frac{h}{r^2} \\
&= -h^2 u^2 \frac{d^2u}{d\theta^2}
\end{aligned}$$

Substituting for \ddot{r} and $\dot{\theta}$ in equ (1), we have

$$\begin{aligned}
-h^2 u^2 \frac{d^2u}{d\theta^2} - r \left(\frac{h}{r^2} \right)^2 &= -F \\
\text{ie, } h^2 u^2 \frac{d^2u}{d\theta^2} + h^2 u^3 &= F
\end{aligned}$$

$$\text{ie, } \frac{d^2u}{d\theta^2} + u = \frac{F}{h^2 u^2} ,$$

Which gives the differential equation of the central orbit in polar co-ordinates. The velocity components of the particle along OP and \perp^r to OP are \dot{r} and $r\dot{\theta}$ respectively and hence the linear moments in those directions are $m\dot{r}$ and $mr\dot{\theta}$ respectively. Taking moments of these vectors about O , we see that $m\dot{r}$ has no moment about O and $mr\dot{\theta}$ has moment given by $r(mr\dot{\theta}) = mr^2\dot{\theta}$.

.. The moment of momentum (ie, the angular momentum) of the particle about O is $mr^2\dot{\theta}$. But $r^2\dot{\theta} = \text{a constant} = h$. Hence the angular momentum of the particle about O is a constant = mh .

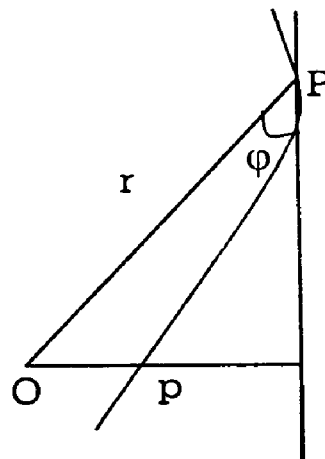
10.1.1 Perpendicular from the pole on the tangent Formulae in polar Coordinates:

Let ϕ be the angle made by the tangent at P with the radius vector OP.

We know that $\tan \phi = r \frac{d\theta}{dr}$ (1)

From O draw OL \perp to the tangent at P and let OL = p.

Then $\sin \phi = \frac{OL}{OP} = \frac{p}{r}$ or $p = r \sin \phi$ (2)



Let us eliminate ϕ between (1) and (2)

From (2), $\frac{1}{p^2} = \frac{1}{r^2 \sin^2 \phi} = \frac{1}{r^2} \operatorname{Cosec}^2 \phi$

$$= \frac{1}{r^2} (1 + \cot^2 \phi)$$

$$= \frac{1}{r^2} \left[1 + \frac{1}{r^2} \left(\frac{dr}{d\theta} \right)^2 \right] \quad \text{Substituting from (1)}$$

(ie) $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2} \left(\frac{dr}{d\theta} \right)^2$ (3)

Using $r = \frac{1}{u}$, $\frac{dr}{d\theta} = \frac{dr}{du} \cdot \frac{du}{d\theta} = -\frac{1}{u^2} \cdot \frac{du}{d\theta}$

Hence (3) becomes

$$\frac{1}{p^2} = u^2 + u^4 \cdot \frac{1}{r^4} \left(\frac{du}{d\theta} \right)^2 \quad \text{(ie)} \quad \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta} \right)^2 \quad (4)$$

10.2 Pedal Equation of the central orbit:

A particle P is describing a central orbit with a central acceleration F towards the pole O. If p be the length of the perpendicular from the pole on the tangent at the point P, Prove that $F = \frac{h^2 dp}{p^3 dr}$, h being the constant angular momentum of the particle about O.

Solution:

From the geometry of a plane curve, we have

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2} \left(\frac{dr}{d\theta} \right)^2$$

Taking $u = \frac{1}{r}$, we have $\frac{du}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta}$

Therefore, $\frac{1}{p^2} = u^2 + \left(\frac{dr}{d\theta} \right)^2$.

Differentiating both sides w.r.t.r, we have

$$\begin{aligned} -\frac{2}{p^3} \frac{dp}{dr} &= 2u \frac{du}{dr} + 2 \frac{du}{d\theta} \cdot \frac{d}{dr} \left(\frac{du}{d\theta} \right) \\ &= 2u \frac{du}{dr} + 2 \frac{du}{dr} \cdot \frac{d^2u}{d\theta^2} \cdot \frac{d\theta}{dr} \\ &= 2u \frac{du}{dr} + 2 \frac{du}{dr} \cdot \frac{d^2u}{d\theta^2} \\ &= 2 \frac{du}{dr} \left[\frac{d^2u}{d\theta^2} + u \right] \end{aligned}$$

$$= -\frac{2}{r^2} \left[\frac{d^2u}{d\theta^2} + u \right] \quad (\because u = 1/r)$$

But $\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2n^2}$, where h is the constant angular momentum of the particle about O . Therefore

$$-\frac{2}{p^3} \frac{dp}{dr} = -\frac{2}{r^2} \frac{F}{h^2u^2}$$

$$\text{ie, } \frac{h^2}{p^3} \frac{dp}{dr} = F.$$

Problem: 1

Show that the velocity V at any point $P(r, \theta)$ in a central orbit is given by $V = h/p$, where h is the constant angular momentum of the particle about O .

Solution:

At the point $P(r, \theta)$, the particle has a velocity V whose components along the radius vector and perpendicular to the radius vector are \dot{r} and $r\dot{\theta}$ respectively. Therefore,

$$V^2 = \dot{r}^2 + r^2 \dot{\theta}^2$$

$$\text{But } r^2 \dot{\theta} = h \quad \therefore \dot{\theta} = h/r^2 \text{ and therefore}$$

$$V^2 = \dot{r}^2 + r^2 (h/r^2)^2$$

$$= \dot{r}^2 + \frac{h^2}{r^2}$$

$$\text{But } \dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = \frac{dr}{d\theta} \cdot \frac{h}{r^2}$$

$$V^2 = \left\{ \frac{h}{r^2} \left(\frac{dr}{d\theta} \right) \right\}^2 + \frac{h^2}{r^2}$$

$$= h^2 \left\{ \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2 \right\} \quad \dots \dots (1)$$

$$= h^2 \cdot \frac{1}{p^2} \quad \text{ie} \quad V = h/p.$$

Note:

From (1), we have

$$V^2 = h^2 \left[\frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2 \right]$$

$$= h^2 \left[u^2 + \left(\frac{dr}{d\theta} \right)^2 \right]$$

$$u = 1/r.$$

10.2.2 Pedal equation of some of the well-known curves:

(1) Circle-pole at any point:

Let C be the centre 'a' the radius, O the pole where $OC = a$.

Let P be any point on the circle and OL be the \perp from O on the tangent at P.

$$OP = r \text{ and } OL = p.$$

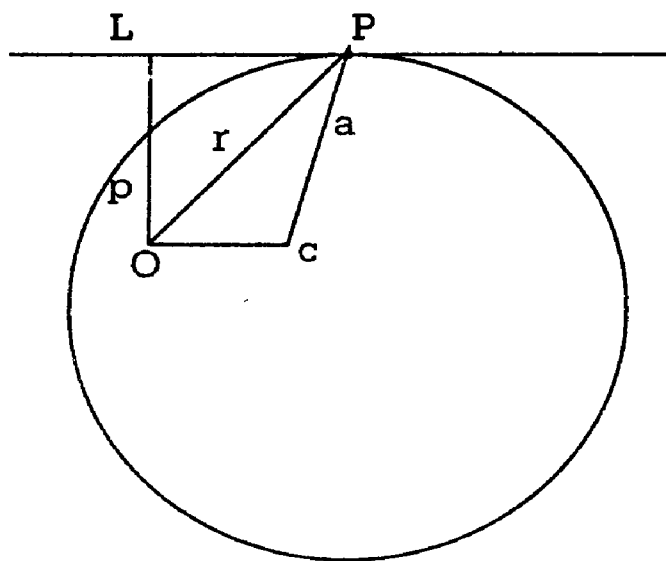
From $\triangle OPC$,

$$C^2 = r^2 + a^2 - 2ra \cos \angle OPC$$

$$= r^2 + a^2 - 2ra \cos \angle POL$$

$$= r^2 + a^2 - 2ra \cdot \frac{p}{r}$$

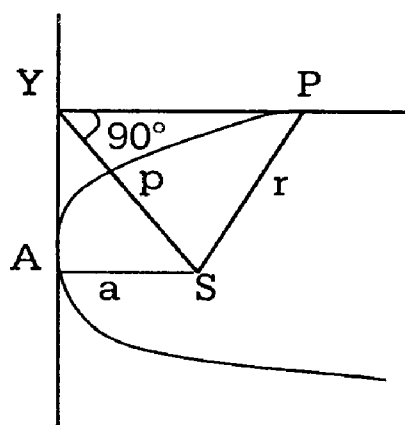
$$C^2 = r^2 + a^2 - 2ap$$



Hence the pedal equations of the circle for a general position of the pole is $c^2 = r^2 + a^2 - 2ap$. When $C = a$, the pole is on the circumference and the equation.

(2) Parabola – Pole at focus:-

To get the (p,r) equation to a parabola, we assume the geometrical property that if the tangent at P meets the tangent at the vertex A in Y and S is the focus, then SY is \perp to PY and the triangles SAY and SYP are similar.



$$\text{Hence } \frac{SA}{SY} = \frac{SY}{SP}$$

$$\text{(ie) } \frac{a}{p} = \frac{p}{r} \text{ or } p^2 = ar.$$

(3) Ellipse or Hyperbola Pole at focus:

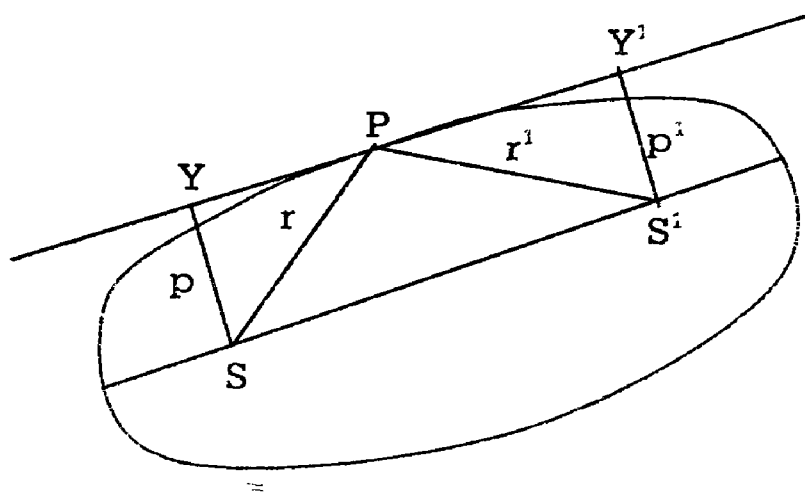
Let S and S' be the foci of the ellipse and SY and SY, S'Y' be the \perp^{rs} to the tangent at P. Taking S as the pole,

Let $SP = r$,

$$S'P = r', \quad SY = p, \quad S'Y' = p'$$

Let a and b be the semi-axes.

To find the (p,r) equation we assume the following geometrical properties of the ellipse.



i) $SP + S'P = 2a$

(ie) $r + r' = 2a$

ii) $SY \cdot S'Y' = b^2$

(ie) $PP' = b^2$

iii) The tangent at P is equally inclined to the focal distances so that SPY and $S'PY'$ are similar triangles.

So we have $\frac{p}{r} = \frac{p'}{r'}$

Now $\frac{h^2}{p^2} = \frac{pp'}{p^2}$ Using (ii)

$= \frac{p'}{p} = \frac{r'}{r}$ (Using iii)

$= \frac{2a-r}{r}$ Using (1)

$= \frac{2a}{r} - 1$

Hence $\frac{b^2}{p^2} = \frac{2a}{r} - 1$ is the (p,r) equation to the ellipse.

By a similar argument, the (p,r) equation of the branch of the hyperbola nearer to the focus is $\frac{b^2}{p^2} = \frac{2a}{r} + 1$.

(4) Equiangular Spiral:

In any curve $P = r \sin \phi$ in the usual rotation.

In the equilateral spiral, $\phi = \text{Constant} = \alpha$ (say)

Hence $P = r \sin \alpha = Kr$ is the (p,r) equation to the spiral.

10.2.3 Velocities in a central orbit :

In every central orbit the areal velocity is constant and the linear velocity varies inversely as the \perp^r from the centre upon the tangent to the path.

Let at time t the particle be at $P(r, \theta)$ and at time $t + \Delta t$,

Let it be at $Q(r + \Delta r, \theta + \Delta \theta)$

Sectorial area OPQ described by the radius vector OP

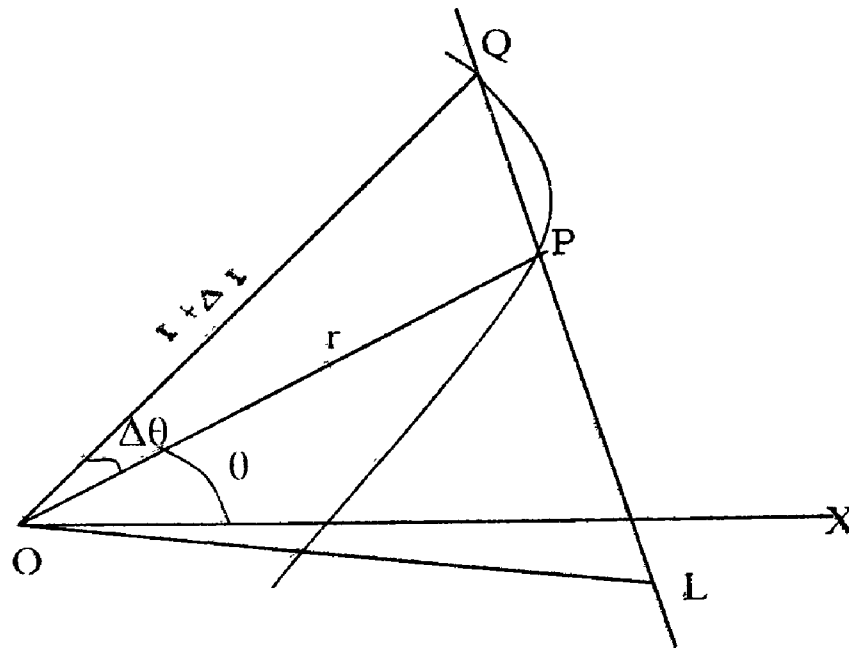
= Area of ΔOPQ nearly

$$= \frac{1}{2} OP \cdot OQ \sin \angle POQ$$

$$= \frac{1}{2} r (r + \Delta r) \sin \Delta \theta$$

$$= \frac{1}{2} r^2 \Delta \theta, \text{ to the first order of smallness.}$$

The rate of description of the area traced out by the radius vector joining the particle to a fixed point is called the areal.



Hence in the central orbit, areal velocity of P

$$= \lim_{\Delta t \rightarrow 0} \frac{1}{2} r^2 \frac{\Delta \theta}{\Delta t} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} h \quad (1)$$

Since $r^2 \dot{\theta} = \text{constant} = h$ from equation (3) of Differential equation of central orbit.

Hence $h = \text{twice the areal velocity}$ and as h is a constant, the areal velocity is constant. In other words, equal areas are described by the radius vector in equal times.

We can get another expression for the areal velocity.

Let Δs be the length of the arc PQ. Draw $OL \perp$ to PQ.

Sectorial area POQ = Δ POQ nearly

$$= \frac{1}{2} PQ \cdot OL$$

As $\Delta t \rightarrow 0$, Q tends to coincide with P along the curve and the chord QP becomes the tangent at P. Length PQ = Δs nearly and OL becomes the \perp from O on the tangent at P. Let $OL = p$.

Hence areal velocity

$$= \lim_{\Delta t \rightarrow 0} \frac{1}{2} \frac{\Delta s}{\Delta t} \cdot p = \frac{1}{2} p \frac{ds}{dt} = \frac{1}{2} p v \quad (2)$$

and $\frac{ds}{dt}$ is the rate of describing S and so in the linear velocity of P.

Hence combining (1) and (2).

$$\text{areal velocity} = \frac{1}{2}h = \frac{1}{2}PV.$$

$$\text{or } h = PV \quad (\text{ie }) \quad V = \frac{h}{p}$$

∴ Hence Linear velocity varies inversely as OP.

10.2.4 Two fold problems in central orbits :

It is clear that two types of problems arise in connection with central orbits. They are

- i) Given the orbit, to find the law of force to the pole.
- ii) Given the law of force, to find the path.

We shall first take up (1)

The differential equation to the central orbit in polar coordinates is

$$u + \frac{d^2 u}{d\theta^2} = \frac{P}{h^2 u^2}$$

$$\text{Hence } P = h^2 u^2 \left(u + \frac{d^2 u}{d\theta^2} \right)$$

Since the orbit is given, u is known as a function of θ hence by differentiation, P can be got from the above equation.

In a few cases, we may know the (p, r) equation to the path. To find P we can use the equation $P = \frac{h^2}{p^3} \cdot \frac{dp}{dr}$

Example: 1 (U.Q)

Find the law of force towards the pole under which the curve $r^n = a^n \cos n\theta$ can be described .

Solution

$$\text{Let } r^n = a^n \cos n\theta$$

$$\text{Since } r = \frac{1}{u}, \text{ the equation is } u^n a^n \cos n\theta = 1 \quad (1)$$

Taking logarithms.

$$n \log u + n \log a + \log \cos n\theta = 0 \quad (2)$$

Differentiating (2) with respect to θ .

$$n \cdot \frac{1}{u} \frac{du}{d\theta} - \frac{n \sin n\theta}{\cos n\theta} = 0$$

$$\text{(ie) } \frac{du}{d\theta} = u \tan n\theta \quad (3)$$

Differentiating (3) with respect to θ .

$$\frac{d^2 u}{d\theta^2} = u n \sec^2 n\theta + \tan n\theta \cdot \frac{du}{d\theta}$$

$$= nu \sec^2 n\theta + u \tan^2 n\theta \text{ using (3)}$$

$$u + \frac{d^2 u}{d\theta^2} = u + nu \sec^2 n\theta + u \tan^2 n\theta$$

$$= nu \sec^2 n\theta + u \sec^2 n\theta$$

$$= (n + 1) u \sec^2 n\theta$$

$$= (n + 1) u \cdot u^{2n} \cdot a^{2n}$$

using (1) to

Substitute for $\sec^2 n\theta$

$$= (n + 1) a^{2n} u^{2n+1}$$

$$P = h^2 u^2 \left(u + \frac{d^2 u}{d\theta^2} \right) = h^2 u^2 (n+1) a^{2n} \cdot u^{2n+1}$$

$$= (n+1) a^{2n} h^2 u^{2n+3} = (n+1) a^{2n} h^2 \cdot \frac{1}{r^{2n+3}} \quad (4)$$

(ie) $P \propto \frac{1}{r^{2n+3}}$ Which means that the central acceleration varies inversely as the $(2n+3)$ rd power of the distance.

Note :

From (4), P is positive only when $n+1 > 0$.

(ie) $n > -1$

For values of $n < -1$, P will be negative and in such cases, the central forces will be a repulsive one. The above case is a comprehensive one giving the law of force for describing the following well known curves corresponding to particular values of n .

i) when $n = 1$, the equation is $r = a \cos \theta$. The curve is a circle and

$$P \propto \frac{1}{r^5}.$$

ii) when $n = 2$, the equation is $r^2 = a^2 \cos 2\theta$. This is the Lemniscate of

Bernowli and $P \propto \frac{1}{r^7}$.

iii) When $n = \frac{1}{2}$, the equation is $r^{1/2} = a^{1/2} \cos \theta/2$.

$$(ie) \quad r = a \cos^2 \theta/2 = \frac{a}{2} (1 + \cos \theta)$$

This is a cardioid and $P \propto \frac{1}{r^4}$.

iv) When $n = -\frac{1}{2}$, the equation is $r^{-1/2} = a^{-1/2} \cos \frac{\theta}{2}$

$$(ie) \quad a^{1/2} = r^{1/2} \cos \frac{\theta}{2}.$$

$$\text{So } r = \frac{a}{\cos^2 \theta/2} = \frac{2a}{1+\cos\theta}$$

$$\text{(ie) } \frac{2a}{r} = 1 + \cos\theta$$

This is a parabola and $P \propto \frac{1}{r^2}$

v) When $n = -2$, the equation is

$$r^{-2} = a^{-2} \cos 2\theta$$

$$\text{(ie) } r^2 \cos 2\theta = a^2$$

This is a rectangular hyperbola. In this case the actual value of $P = -a^{-4} h^2 r$

The negative sign of P shows that the central force is a repulsive one.

Example :2

Find the law of force to an internal point under which a body will describe a circle.

Solution:

From pedal equation of some of the well – known curves the pedal equation of the circle for a general position of the pole is

$$C^2 = r^2 + a^2 - 2aP \quad (1)$$

Differentiating with respect to r ,

$$0 = 2r - 2a \frac{dp}{dr}$$

$$\text{(ie) } \frac{dp}{dr} = \frac{r}{a}$$

Now the central acceleration.

$$P = \frac{h^2}{p^3} \cdot \frac{dp}{dr} = \frac{h^2 r}{p^3 a} = \frac{8h^2 a^2 \cdot r}{(r^2 + a^2 - c^2)} \quad \text{Substituting for } \beta \text{ from (1)}$$

Example: 3(U.Q)

A Particle moves in an ellipse under a force which is always towards its focus. Find the law of force, the velocity at any point of the path and its periodic time.

Solution:

Referred to the focus as the pole, the polar equation to the ellipse is $\frac{\ell}{r} = 1 + e \cos\theta$ (1)

Where e is the eccentricity and ℓ is the semi-latus rectum.

$$\text{From (1), } u = \frac{1}{r} = \frac{1 + e \cos\theta}{\ell}$$

$$\text{Hence } \frac{du}{d\theta} = -\frac{e \sin\theta}{\ell} \text{ and } \frac{d^2u}{d\theta^2} = -\frac{e \cos\theta}{\ell}$$

$$u + \frac{d^2u}{d\theta^2} = \frac{1 + e \cos\theta}{\ell} - \frac{e \cos\theta}{\ell} = \frac{1}{\ell}$$

$$\text{We know that } \frac{P}{h^2 u^2} = u + \frac{d^2u}{d\theta^2} = \frac{1}{\ell}$$

$$\text{Hence } P = \frac{h^2 u^2}{\ell} = \frac{\mu}{r^2} \text{ where } \mu = \frac{h^2}{e}.$$

(ie) The force varies inversely as the square of the distance from the pole.

We have the result

$$\begin{aligned} \frac{1}{P^2} &= u^2 + \left(\frac{du}{d\theta}\right)^2 \left[\text{Refer } \perp \text{ from the pole on the tangent Formula} \right. \\ &\quad \left. \text{in polar coordinates} \right] \\ &= \frac{(1 + e \cos\theta)^2}{\ell^2} + \left(\frac{e \sin\theta}{\ell}\right)^2 \end{aligned}$$

$$= \frac{1 + 2e \cos\theta + e^2}{\rho^2}$$

Also $h = Pv$ where v is the linear velocity

$$\text{Hence } v^2 = \frac{h^2}{p^2} = \frac{h^2 (1 + 2e \cos\theta + e^2)}{\ell^2}$$

$$= \frac{\mu \ell}{\ell^2} \left[1 + e^2 + 2 \left(\frac{\ell}{r} - 1 \right) \right]$$

$$= \frac{\mu}{\ell} \left(e^2 + \frac{2\ell}{r} - 1 \right)$$

$$= \frac{\mu}{\ell} \left(\frac{2\ell}{r} - (1 - e^2) \right)$$

$$= \mu \left(\frac{2}{r} - \frac{(1 - e^2)}{\ell} \right)$$

(2)

Now if a and b are the semi-axes of the ellipse,

We know that

$$\ell = \frac{b^2}{a} = \frac{a^2(1 - e^2)}{a} = a(1 - e^2).$$

Hence Putting $\ell = a(1 - e^2)$ is (2),

$$v^2 = \mu \left[\frac{2}{r} - \frac{1}{a} \right] \text{ giving the velocity } v.$$

A real velocity in the orbit = $\frac{1}{2} h$ and this is constant.

The total area of the ellipse = πab .

$$\text{Periodic time } T = \frac{\pi ab}{(\frac{1}{2}h)} = \frac{2\pi ab}{h}$$

$$= \frac{2\pi ab}{\sqrt{\mu \ell}}, \text{ Since } \mu = \frac{h^2}{\ell}$$

$$= \frac{2\pi ab}{\sqrt{\mu} \cdot b} \cdot \sqrt{a} \text{ Since } \ell = \frac{b^2}{a}$$

$$T = \frac{2\pi}{\sqrt{\mu}} \cdot a^{3/2}$$

Example: 4

A Particle moves in a curve under a central attraction so that its velocity at any point is equal to that in a circle at the same distance and under the same attraction. Show that the path is an equiangular spiral and that the law of force is that of the inverse cube.

Solution:

Let the central acceleration be P . If V is the velocity in a circle at a distance r under the normal acceleration P , then

$$\frac{V^2}{r} = P \text{ (ie) } V^2 = Pr \tag{1}$$

Since V is also the velocity in the central orbit, $h = pV$ or $V = \frac{h}{p}$.

Putting this in (1), $\frac{h^2}{p^2} = Pr$ (2)

We know that

$$P = \frac{h^2}{p^3} \cdot \frac{dp}{dr} \tag{3}$$

Substituting (3) in (2)

$$\frac{h^2}{p^2} = \frac{h^2}{p^3} \cdot \frac{dp}{dr} \cdot r$$

(ie) $\frac{dP}{P} = \frac{dr}{r}$

Integrating,

$$\log P = \log r + \log A$$

$$(ie) P = Ar$$

(4) is clearly the (p,r) equation to an equiangular spiral.

$$\text{From (4), } \frac{dp}{dr} = A$$

Substituting this in (3),

$$P = \frac{h^2}{p^3} \cdot A = \frac{Ah^2}{A^3 r^3} \text{ using (4)}$$

$$= \frac{h^2}{A^2} \left(\frac{1}{r^3} \right)$$

$$(ie) P \propto \frac{1}{r^3}$$

Exercise:

1) Find the law of force towards the pole under which the following curves can be described

$$i) \quad r^2 = a^2 \cos 2\theta \quad ii) \quad r^{1/2} = a^{1/2} \cos \frac{\theta}{2}$$

$$iii) \quad r^n \cos n\theta = a^n \quad iv) \quad r^n = A \cos n + B \sin n\theta$$

[Hint: The equation can be taken as $r^n = \lambda \cos(n\theta + \alpha)$]

$$v) \quad a = r \sin n\theta$$

$$vi) \quad r = a \sin n\theta$$

$$vii) \quad \frac{a}{r} = e^{n\theta} \cdot s$$

2) Find the central acceleration under which the conic $\frac{\ell}{r} = 1 + e \cos \theta$ can be described.

3) The velocity at any point of a central orbit is one half of what it would be for a circular orbit at the same distance obtain the law of forces.

