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S. M. SHAH

WITH THE ASSISTANCE OF

P. T. BATEMAN P. L. BHATNAGAR HANSRAJ GUPTA
V. S. KRISHNAN S. MINAKSHISUNDARAM C. RACINE
K. G. RAMANATHAN M. S. RAMANUJAN B. R. SETH
J. A. SIDDIQI K. VENKATACHALIENGAR
M. VENKATARAMAN

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CLASSROOM NOTES

TURAN'S INEQUALITY FOR THE GENERAL LAGUERRE AND HERMITE FUNCTIONS

By S. K. LAKSHMANA RAO

0. It is well known ([2], [3], [4]) that the orthogonal polynomials $P_n^{(\lambda)}(x)$, $L_n^{(\alpha)}(x)$ and $H_n(x)$ satisfy the inequality of Turan, viz.

$$[f_n(x)]^2 - f_{n+1}(x) f_{n-1}(x) \geqslant 0$$

over suitable ranges of x, with appropriate restrictions on the parameters like λ and α. Recently K. Venkatachaliengar and the author have shown [4] that the inequality of Turan holds also for the solution $P_n^{(\lambda)}(x)$ of the differential equation

$$(1-x^2)\, y'' - (2\lambda+1)\, xy' + n(n+2\lambda)y = 0 \tag{0.1}$$

with $n =$ any real, positive number and $\lambda > 0$, by taking $P_n^{(\lambda)}(x)$ as the solution of (0.1) which is regular at $x = 1$ and corresponding to the conditions $(y)_{x=1} > 0$ and $(y')_{x=1} > 0$.

In the present paper we prove Turan's inequality for solutions $L_n^{(\alpha)}(x)$, $H_n(x)$ of the differential equations

$$xy'' + (\alpha + 1 - x)\, y' + ny = 0, \tag{0.2}$$

$$y'' - 2xy' + 2\, ny = 0, \tag{0.3}$$

when n is any *real*, positive index. We also observe incidentally that some of the relations to be noticed below are of characterizational nature when n is taken to be a positive integer.

1. **Turan's inequality for $L_n^{(\alpha)}(x)$.** We take the solution $L_n^{(\alpha)}(x)$ of the differential equation (0.2) in the form

$$L_n^{(\alpha)}(x) = \binom{n+\alpha}{n} \Phi(-n, \alpha+1; x), \tag{1.1}$$

where $\Phi(a, b; x)$ denotes the confluent hypergeometric function in Humbert's notation. This solution is regular at the origin for any real index $n > -1$ and real $\alpha > -1$ and reduces to the Laguerre polynomial $L_n^{(\alpha)}(x)$ when $n = 0, 1, 2, \ldots$. From the relations in confluent hypergeometric functions [1] we obtain

$$(n+1)\,L_{n+1}^{(\alpha)}(x) - (2n+\alpha+1-x)\,L_n^{(\alpha)}(x) + (n+\alpha)\,L_{n-1}^{(\alpha)}(x)$$
$$= 0,\ n > 0 \tag{1.2}$$

$$x\frac{d}{dx}L_n^{(\alpha)}(x) = n\,L_n^{(\alpha)}(x) - (n+\alpha)\,L_{n-1}^{(\alpha)}(x), \quad n > 0 \tag{1.3}$$

$$x\frac{d}{dx}L_n^{(\alpha)}(x) = (n+1)\,L_{n+1}^{(\alpha)}(x) - (n+\alpha+1-x)L_n^{(\alpha)}(x),\ n > -1 \tag{1.4}$$

$$\frac{d}{dx}\,[L_n^{(\alpha)}(x) - L_{n+1}^{(\alpha)}(x)] = L_n^{(\alpha)}(x), \quad n > -1. \tag{1.5}$$

Differentiating the Turan expression

$$\Delta_n(x) = [L_n^{(\alpha)}(x)]^2 - L_{n+1}^{(\alpha)}(x)\,L_{n-1}^{(\alpha)}(x)$$

and eliminating $\frac{d}{dx}L_{n-1}^{(\alpha)}(x)$ as also one of the terms $\frac{d}{dx}L_n^{(\alpha)}(x)$, we obtain

$$\frac{d}{dx}\Delta_n(x) - \Delta_n(x) = \begin{vmatrix} \frac{d}{dx}L_n^{(\alpha)}(x) & \frac{d}{dx}L_{n+1}^{(\alpha)}(x) \\ L_{n-1}^{(\alpha)}(x) - L_n^{(\alpha)}(x) & L_n^{(\alpha)}(x) - L_{n+1}^{(\alpha)}(x) \end{vmatrix}$$

Subtracting the second column from the first and using (1.5), (1.3) to rewrite the first row terms and then multiplying throughout by x, we have

$$x\frac{d}{dx}\Delta_n(x) - x\,\Delta_n(x)$$
$$= \begin{vmatrix} x\,L_n^{(\alpha)}(x) & (n+1)\,L_{n+1}^{(\alpha)}(x) - (n+\alpha+1)\,L_n^{(\alpha)}(x) \\ L_{n-1}^{(\alpha)}(x) - 2L_n^{(\alpha)}(x) + L_{n+1}^{(\alpha)}(x) & L_n^{(\alpha)}(x) - L_{n+1}^{(\alpha)}(x) \end{vmatrix}.$$

Using (1.2) to eliminate $L_{n+1}^{(\alpha)}(x)$ in the first row and then evaluating the determinant, we get after some simplification

$$x\frac{d}{dx}\Delta_n(x) - x\,\Delta_n(x)$$
$$= L_{n+1}^{(\alpha)}(x)\,L_n^{(\alpha)}(x) - (\alpha+1)\,(L_n^{(\alpha)}(x))^2 +$$
$$+ L_{n-1}^{(\alpha)}(x)\,\{L_{n-1}^{(\alpha)}(x) - (2n+\alpha-x)\,L_n^{(\alpha)}(x) + L_{n+1}^{(\alpha)}(x)\}$$

or finally, the relation

$$x \frac{d}{dx} \Delta_n(x) - (1 - \alpha + x)\Delta_n(x)$$

$$= L_n^{(\alpha)}(x)\{L_{n+1}^{(\alpha)}(x) - 2L_n^{(\alpha)}(x) + L_{n-1}^{(\alpha)}(x)\},\ n > 0,\ \alpha > -1. \quad (1.6)$$

We deduce from this that

$$\frac{d}{dx}(e^{-x} x^{\alpha-1} \Delta_n(x)) = e^{-x} x^{\alpha-2} L_n^{(\alpha)}(x)\{L_{n+1}^{(\alpha)}(x) - 2L_n^{(\alpha)}(x) + + L_{n-1}^{(\alpha)}(x)\},\ n > 0,\ \alpha > -1. \quad (1.7)$$

Now it follows that the relative extrema of the function $e^{-x} x^{\alpha-1} \Delta_n(x)$ occur at the zeros of $L_n^{(\alpha)}(x)$ and $L_{n+1}^{(\alpha)}(x) - 2L_n^{(\alpha)}(x) + L_{n-1}^{(\alpha)}(x)$. Since n and α are real, $L_n^{(\alpha)}(x)$ has only a finite number of real zeros [1]. Let $(\beta) : \beta_1, \beta_2, \beta_3, \ldots$ and $(\delta) : \delta_1, \delta_2, \delta_3 \ldots$ denote the zeros of $L_n^{(\alpha)}(x)$ and $L_{n+1}^{(\alpha)}(x) - 2L_n^{(\alpha)}(x) + L_{n-1}^{(\alpha)}(x)$ respectively, the zeros on the positive real axis only being considered. Clearly we have

$$\Delta_n(\beta) = - L_{n+1}^{(\alpha)}(\beta) L_{n-1}^{(\alpha)}(\beta)$$

$$= \frac{n+\alpha}{n+1}(L_{n-1}^{(\alpha)}(\beta))^2 \geqslant 0 \text{ for } n > 0,\ \alpha \geqslant 0,$$

and

$$\Delta_n(\delta) = (L_n^{(\alpha)}(\delta))^2 - (2L_n^{(\alpha)}(\delta) - L_{n-1}^{(\alpha)}(\delta)) L_{n-1}^{(\alpha)}(\delta)$$

$$= [L_n^{(\alpha)}(\delta) - L_{n-1}^{(\alpha)}(\delta)]^2 \geqslant 0.$$

Thus $\Delta_n(x)$ remains non-negative at all the relative extrema of the function $e^{-x} x^{\alpha-1} \Delta_n(x)$ on the positive x-axis and hence we have

$$\Delta_n(x) = (L_n^{(\alpha)}(x))^2 - L_{n+1}^{(\alpha)}(x) L_{n-1}^{(\alpha)}(x) \geqslant 0,\ x > 0,\ n > 0,\ \alpha \geqslant 0$$

which is Turan's inequality for the function $L_n^{(\alpha)}(x)$.

2. Turan's inequality for $H_n(x)$. We take the solution $H_n(x)$ of the equation (0.3) in the form

$$H_n(x) = 2^{n/2} e^{x^2/2} D_n(\sqrt{(2)}\, x),$$

where $D_n(x)$ is the parabolic cylinder function so that [1]

$$H_n(x) = 2^n \left\{ \frac{\Gamma(1/2)}{\Gamma(1/2 - n/2)} \Phi\left(-\frac{n}{2}, 1; x^2\right) + \right.$$
$$\left. + \frac{\Gamma(-1/2)}{\Gamma(-n/2)} x \Phi\left(\frac{1-n}{2}, \frac{3}{2}; x^2\right) \right\}. \tag{2.1}$$

This is regular at the origin and reduces to the polynomial $H_n(x)$ when n is a non-negative integer. From the relations on $D_n(x)$ [1] we notice that

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x), \tag{2.2}$$

$$\frac{d}{dx} H_n(x) = 2n H_{n-1}(x), \tag{2.3}$$

and therefore also the relation

$$H_n(x) = 2x H_{n-1}(x) - \frac{d}{dx} H_{n-1}(x). \tag{2.4}$$

We have then

$$\begin{aligned} \Delta_n(x) &= (H_n(x))^2 - H_{n+1}(x) H_{n-1}(x) \\ &= (H_n(x))^2 - [2xH_n(x) - 2nH_{n-1}(x)] H_{n-1}(x) \\ &= H_n(x) [H_n(x) - 2xH_{n-1}(x)] + 2n [H_{n-1}(x)]^2 \\ &= H_n(x) [- H'_{n-1}(x)] + H'_n(x) H_{n-1}(x) \end{aligned}$$

or

$$\Delta_n(x) = \begin{vmatrix} H_{n-1}(x) & H_n(x) \\ H'_{n-1}(x) & H'_n(x) \end{vmatrix} \tag{2.5}$$

expressing the Turan expression $\Delta\ (x)$ as the Wronskian of $H_{n-1}(x)$ and $H_n(x)$.

By differentiating $\Delta_n(x)$ we get

$$\frac{d}{dx} \Delta_n(x)$$

$$= \begin{vmatrix} H_{n-1}(x) & H_n(x) \\ H''_{n-1}(x) & H''_n(x) \end{vmatrix}$$

$$= \begin{vmatrix} H_{n-1}(x) & H_n(x) \\ 2xH'_{n-1}(x) - 2(n-1)H_{n-1}(x) & 2xH'_n(x) - 2nH_n(x) \end{vmatrix}$$

$$= 2x\Delta_n(x) - 2H_{n-1}(x)H_n(x),$$

or equivalently the relation

$$\frac{d}{dx}(e^{-x^2}\Delta_n(x)) = -2e^{-x^2}H_{n-1}(x)H_n(x). \tag{2.6}$$

Hence $e^{-x^2}\Delta_n(x)$ has its extrema values at the zeros of $H_{n-1}(x)$ and $H_n(x)$. Let $(\alpha): \alpha_1, \alpha_2, \alpha_3, \ldots$, and (β): $\beta_1, \beta_2, \beta_3, \ldots$ be the real zeros of $H_n(x)$ and $H_{n-1}(x)$ respectively. We find that

$$\Delta_n(\alpha) = -H_{n+1}(\alpha)H_{n-1}(\alpha) = 2n(H_{n-1}(\alpha))^2 \geqslant 0, \quad \text{for } n \geqslant 0$$

and

$$\Delta_n(\beta) = (H_n(\beta))^2 \geqslant 0.$$

Hence $e^{-x^2}\Delta_n(x)$ is non-negative at all its relative extrema on the x-axis when $n \geqslant 0$, and thus we have

$$\Delta_n(x) \geqslant 0 \text{ for all real } x \text{ and for } n \geqslant 0 \tag{2.7}$$

which is Turan's inequality for the general solution $H_n(x)$.

3. Case of n = a non-negative integer. Characteristic relations. The relations deduced in Sections 1, 2 are certainly valid when $n = 0, 1, 2, \ldots$ We find that relation (1.6) or (1.7) is characteristic for the Laguerre polynomial while the relations (2.5) or (2.6) are characteristic for the Hermite polynomial. Since the proofs are quite similar in all these cases we content ourselves by giving the proof in the last case only.

THEOREM. *If $f_0(x)$, $f_1(x)$, $f_2(x), \ldots$ is a set of polynomials such that*

(i) *$f_n(x)$ has degree n,*

(ii) $$\frac{d}{dx}\left(e^{-x^2}[(f_n(x))^2 - f_{n+1}(x)f_{n-1}(x)]\right) = -2e^{-x^2}f_{n-1}(x)f_n(x)$$

and if

(iii) *$f_0(x) = 1$ and $f_1(x) = 2x$, then*

$$f_n(x) \equiv H_n(x) \text{ for } n = 0, 1, 2, \ldots$$

To prove this, we take $f_r(x) = H_r(x)$, $0 \leqslant r \leqslant n$ and set $f_{n+1}(x) = H_{n+1}(x) + g(x)$ in (ii). Clearly $g(x)$ is to be a polynomial of degree $\not> n+1$. We obtain from (ii)

$$\frac{d}{dx}\left[e^{-x^2}\left(-g(x)\,H_{n-1}(x)\right)\right] = 0,$$

so that

$$g(x) = \frac{c\,e^{x^2}}{H_{n-1}(x)} = \text{a polynomial of degree} \leqslant n+1.$$

The integration constant c can only be zero, whence $g(x) = 0$ or $f_{n+1}(x) = H_{n+1}(x)$ and hence the theorem.

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SOME PROPERTIES OF QUADRATIC RESIDUES

By M. V. SUBBA RAO

1. In this note we obtain two results on quadratic residues which include as special cases Hansraj Gupta's generalization [1] of the two results of J. B. Kelly [2]. Let p be an odd prime, and R_i, L_i, $i = 1, 2, \ldots, n-1$, denote respectively the set of integers of the form $p^i q$, $(p, q) = 1$, for which q is a quadratic residue or non-residue modulo p^{n-i}. Let t be an arbitrary integer and S any set of integers, and $t \oplus S$ denote the set of integers obtained by adding t to each of the elements of S. Let r_i, l_i be arbitrary members of R_i, L_i respectively. Let $j = p^{n-i-1}[p/4]$, $k = p^{n-i-1}[(p-3)/4]$, where as usual $[x]$ stands for the greatest integer not exceeding x; $m = p^{n-i-1}$ or 0 according as $p = 1$ or $-1 \pmod 4$; $m_1 = p^{n-i-1} - m$. Then we will show

THEOREM 1. *Each of the sets $l_i \oplus R_i$; $r_i \oplus L_i$ exactly gives j members of each of the sets L_i, R_i together with m members which are multiples of p^{i+1}. These m numbers are made up of $\phi(p^{n-\alpha})$ numbers of the form $p^\alpha q$, $(p, q) = 1$, $\alpha = 1, 2, \ldots, n$.*

THEOREM 2. (*a*) *The set $r_i \oplus R_i$ exactly gives K members of R and $k + p^{n-i-1}$ numbers of L_i together with m_1, numbers which are multiples of p^{i+1}.*

(*b*) *The set $l_i \oplus L_i$ exactly gives k members of L_i and $k + p^{n-i-1}$ members of R_i and m_1 numbers which are multiples of p^{i+1}.*

In either case these m_1 numbers are made up of $\phi(p^{n-\alpha})$ numbers of the form $(p^\alpha q)$, $(p, q) = 1$, $\alpha = i+1, i+2, \ldots, n$.

The special cases of these theorems for $i = 0$ include, and in fact go beyond Gupta's results [1].

2. These results can be proved directly on the same lines as in [1]. A direct proof is therefore omitted. What is of interest here is the fact that these can be immediately deduced and are indeed implied in the class algebra worked out in [3]. Let us illustrate this by proving Theorem 1.

From the system of equations (A) in [3] we have, if $p \equiv 1 \pmod 4$,

$$R_i \oplus L_i = (1/4)\,\phi\,(p^{n-i})\,K_i = (1/4)\,\phi\,(p^{n-i})\,(R_i + L_i).$$

If x is a member of R_i, and if $x \oplus L_i$ contributes β terms to the term containing R_i on the right side (β is evidently the same for any choice of x of R_i by considerations of symmetry), noting that R_i, L_l each contain $(1/2)\phi(p^{n-i})$ terms, on equating the total number of terms on both sides, we have

$$\beta\,(1/2)\,\phi(p^{n-i}) = (1/4)\,\phi\,(p^{n-i}).\ (1/2)\,\phi\,(p^{n-i}),$$

$$\beta = (1/4)\,\phi\,(p^{n-1}).$$

A similar method applies for finding the contribution to L_i. The case when $p \equiv -1 \pmod 4$ can be dealt with similarly by using equations (B) of [3].

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Sri Venkateswara University
Tirupathi

ON THE CHARACTERISTIC EQUATION OF A RECTILINEAR CONGRUENCE*

By P. JHA†

1. The coefficients of the two forms of Kummer and two functions p and q introduced in connection with a rectilinear congruence [1] satisfy three partial differential equations. The elimination of p and q from these three equations leads to a single differential equation [2] connecting the coefficients of the two forms. Like the Gauss characteristic equation, this is not integrable. The object of the present paper is to integrate this equation, called the characteristic equation, in some special cases and to obtain the equation of the middle surface of the congruence. Some peculiar congruences have been constructed, e.g. a congruence whose limit distance is constant and another for which the product of the distances of a focus from the limits is constant. In some particular cases, it has been possible to construct the congruence. The notations are those of Weatherburn [3].

2. Taking $\mathbf{r}$ to be the vector position of any point on the director surface and $\mathbf{d}$ the unit vector along the ray at the point, we may write, $\mathbf{r}_1 = \alpha\mathbf{d}_1 + \beta\mathbf{d}_2 + p\mathbf{d}$ and $\mathbf{r}_2 = \alpha'\mathbf{d}_1 + \beta'\mathbf{d}_2 + q\mathbf{d}$, where the suffixes 1 and 2 denote partial differentiation with respect to u and v respectively.

For simplicity in calculation, we take $f = \mathbf{d}_1.\mathbf{d}_2 = 0$. Then $\alpha = a/e$, $\beta = b'/g$, $p = \mathbf{d}.\mathbf{r}_1$, $\alpha' = b/e$, $\beta' = c/g$, $q = \mathbf{d}.\mathbf{r}_2$.

† This paper is a part of my Ph.D. thesis submitted to Patna University in September 1956.

* We recall that, according to [3], the fundamental forms for a rectilinear congruence are defined as

$$e\,du^2 + 2f\,du\,dv + g\,dv^2\ (= d\mathbf{d}.\,d\mathbf{d})$$

and

$$a\,du^2 + (b + b')\,du\,dv + c\,dv^2\ (= d\mathbf{r}.\,d\mathbf{d}),$$

where $e = \mathbf{d}_1^2, f = \mathbf{d}_1.\,\mathbf{d}_2, g = \mathbf{d}_2^2, a = \mathbf{r}_1.\,\mathbf{d}_1, b = \mathbf{r}_2.\,\mathbf{d}_1, b' = \mathbf{r}_1.\,\mathbf{d}_2, c = \mathbf{r}_2.\,\mathbf{d}_2$.

As $\mathbf{r}_1 du + \mathbf{r}_2 dv$ $(= d\mathbf{r})$ is a perfect differential, we get

$$\frac{\partial}{\partial v}(\alpha \mathbf{d}_1 + \beta \mathbf{d}_2 + p\mathbf{d}) = \frac{\partial}{\partial u}(\alpha' \mathbf{d}_1 + \beta' \mathbf{d}_2 + q\mathbf{d}).$$

Taking dot product of both sides of this relation with $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}$ and simplifying, we get

$$q = \frac{a_2}{e} + \frac{be_1}{2e^2} - \frac{e_2}{2e^2 g}(ag + ce) - \frac{b_1}{e} - \frac{b' g_1}{2eg},$$

$$p = \frac{c_1}{g} + \frac{g_2 b'}{2g^2} - \frac{g_1}{2eg^2}(ag + ce) - \frac{b'_2}{g} - \frac{be_2}{2eg},$$

$$p_2 - q_1 = b' - b.$$

Hence, by eliminating p and q from these equations, we get [2, p. 497],

$$b' - b = \frac{\partial}{\partial v}\left[\frac{c_1}{g} + \frac{g_2 b'}{2g^2} - \frac{g_1}{2eg^2}(ag + ce) - \frac{b'_2}{g} - \frac{be_2}{2eg}\right] -$$

$$- \frac{\partial}{\partial u}\left[\frac{a_2}{e} + \frac{be_1}{2e^2} - \frac{e_2}{2e^2 g}(ag + ce) - \frac{b_1}{e} - \frac{b' g_1}{2eg}\right]. \qquad \text{(A)}$$

We call this equation (A) as the characteristic equation. When the parametric curves on the unit sphere are not orthogonal, there is a similar but somewhat complicated relation. This relation or as it is called the characteristic equation, and the given value of $b + b'$, give b and b' separately whence p and q can be determined and therefore $\mathbf{r} = \int(\mathbf{r}_1\, du + \mathbf{r}_2\, dv)$, the corresponding point on the director surface, can be obtained by quadrature. Given any two binary quadratic forms, one of which represents the square of the line element of a unit sphere, the corresponding rectilinear congruence can be constructed, because corresponding to any point on the unit sphere giving the direction of the ray we can, by solving the characteristic equation and getting the values of b and b' separately, get the corresponding point $\mathbf{r}$ on the director surface. Hence we get the result :

Given any two differential quadratic forms one of which is the square of the line element of a unit sphere, the corresponding rectilinear *congruence can be constructed by solving a certain partial differential equation of the second order (called the characteristic equation).*

3. We shall now, without loss of generality, simplify the equation by choosing (i) the parallels of latitude and longitude on the unit sphere as parametric curves and (ii) the middle surface of the congruence (which is always real) as the director surface.

Let $p(u, v)$ be a point on the unit sphere, vector position $\mathbf{d}$, and $P(u, v)$ the corresponding point on the director surface (which is to be determined). The direction of the unit vector $\mathbf{d}$ along a ray of the congruence may be expressed as

$$\mathbf{d} = (\cos u \sin v, \cos u \cos v, \sin u).$$

$$\therefore \mathbf{d}_1 = (-\sin u \sin v, -\sin u \cos v, \cos u),$$

$$\mathbf{d}_2 = (\cos u \cos v, -\cos u \sin v, 0).$$

Hence

$$e = \mathbf{d}_1{}^2 = 1, \quad f = \mathbf{d}_1 . \mathbf{d}_2 = 0, \quad g = \mathbf{d}_2^2 = \cos^2 u$$

and

$$h = \cos u, \quad g_1 = -2 \sin u \cos u, \quad g_2 = 0 = e_1 = e_2.$$

As already mentioned, for the construction of the congruence it will be enough to find, corresponding to any point p on the unit sphere, the position of P on the director surface, which for the sake of definiteness is taken to be the middle surface of the congruence; so that $ag + ce = 0$, i.e. $a \cos^2 u + c = 0$, and the characteristic equation takes the form

$$b' - b = -2a_{12} + 2a_2 \tan u - \sec^2 u \, b'_{22} + b_{11} - b'_1 \tan u - b' \sec^2 u. \quad \text{(B)}$$

It is to be noted that with the chosen system of reference, if the coefficients of Kummer's second form be constants, a and c are separately zero as $a \cos^2 u + c = 0$. If $b + b' = 2m$ (constant) so that $b' = m + x$, $b = m - x$, the characteristic equation becomes

$$x_{22} \sec^2 u + x_{11} + x_1 \tan u + (\sec^2 u + 2)\, x + m \sec^2 u = 0.$$

If Kummer's second form is identically zero, i.e. $m = 0$, the characteristic equation takes the form

$$\sec^2 u\, b_{22} + b_{11} + b_1 \tan u + (\sec^2 u + 2)\, b = 0.$$

It is evident that such a congruence has the peculiar property of having the two sheets of the limit surface coincident and the focal surface imaginary.

4. The equation (B) cannot be simplified further without imposing restrictions. In order to study this we shall assume that the principal surfaces correspond to a family of latitudes and longitudes on the unit sphere, so that $b + b' = 0$. Now $ag + ce = 0$ may be written as

$$a/e = -c/g = \mu,$$

where 2μ (in this case) represents the distance between the limits.

It can be easily seen that $(b - b')/h = 2\sqrt{(L_1F_1 \,.\, F_1L_2)}$, where L_1, L_2 are the two limits and F_1, F_2 are the two foci on any ray.

If $\sqrt{(L_1F_1 . F_1L_2)} = k$, we have

$$2k = (b - b')/h = 2b/h \quad \text{or} \quad b = k \cos u.$$

Hence

$$p = 2\mu \tan u + k_2 \sec u - \mu_1 \quad \text{and} \quad q = \mu_2 - k_1 \cos u.$$

Now the characteristic equation changes into

$$2k \cos u = \frac{\partial}{\partial u}(\mu_2 - k_1 \cos u) - \frac{\partial}{\partial v}(2\mu \tan u + k_2 \sec u - \mu_1)$$

or

$$k_{22} \sec^2 u + k_{11} - k_1 \tan u + 2k = 2 \sec u \frac{\partial}{\partial v}(\mu_1 - \mu \tan u). \quad \text{(C)}$$

We have

$$\mathbf{r}_1 = \mu \mathbf{d}_1 - k \sec u\, \mathbf{d}_2 + (2\mu \tan u + k_2 \sec u - \mu_1)\mathbf{d},$$

$$\mathbf{r}_2 = k \cos u\, \mathbf{d}_1 - \mu \mathbf{d}_2 + (\mu_2 - k_1 \cos u)\mathbf{d}.$$

Therefore, the equation of the middle surface of the rectilinear congruence is given by

$$\mathbf{r} = \int [\{\mu\, \mathbf{d}_1 - k \sec u\, \mathbf{d}_2 + (2\mu \tan u + k_2 \sec u - \mu_1)\mathbf{d}\}\, du + $$
$$+ \{k \cos u\, \mathbf{d}_1 - \mu\, \mathbf{d}_2 + (\mu_2 - k_1 \cos u)\, \mathbf{d}\}\, dv].$$

If now $\mu(=a)$ be supposed to be a function of u alone (or constant) the value of k is determined by (C) which takes the form

$$k_{22} \sec^2 u + k_{11} - k_1 \tan u + 2k = 0. \tag{C'}$$

(It is to be noted that the characteristic equation remains the same even if $\mu =$ constant as for μ being a function of u alone.)

The equation (C′) in k does not appear to be easily solvable. It may be noted that k cannot be a function of v alone. If k be a function of u alone, (C′) changes into

$$k_{11} - k_1 \tan u + 2k = 0.$$

Clearly, $k = \sin u$ is a solution of this equation. The complete solution is given by

$$k = A \sin u + B \sin u \log \tan (\pi/4 + u/2) - B,$$

where A and B are arbitrary constants.

As $a = \mu$, $c = -u \cos^2 u$, $b = -b' = k \cos u$, $h = \cos u$, the foci are given by the equation

$$\rho^2 = \mu^2 - k^2.$$

Therefore, the foci are real or imaginary according as

$$\mu^2 - k^2 \gtrless 0,$$

i.e.

$$\mu^2 - [A \sin u + B \sin u \log \tan (\pi/4 + u/2) - B]^2 \gtrless 0.$$

Therefore, the roots of the equation

$$\mu^2 - [A \sin u + B \sin u \log \tan (\pi/4 + u/2) - B]^2 = 0$$

divide the unit sphere into zones such that on rays of the congruence correspondong to some zones, the foci are real and on rays of the congruence corresponding to the remaining zones, the foci are imaginary and on rays corresponding to the curves dividing the sphere into zones, the foci are coincident.

If k be a function of u alone and μ be supposed to be constant, the middle surface is given by

$$\mathbf{r} = \int [(\mu \mathbf{d}_1 - k \sec u\, \mathbf{d}_2 + 2\mu \tan u\, \mathbf{d})\, du + \\ + (k \cos u\, \mathbf{d}_1 - \mu\, \mathbf{d}_2 - k_1 \cos u\, \mathbf{d})\, dv]$$

or

$$x = A \cos u \cos v + B \cos u \cos v \log \tan (\pi/4 + u/2) - \mu \cos u \sin v,$$

$$y = - A \cos u \sin v - B \cos u \sin v \log \tan (\pi/4 + u/2) - \mu \cos u \cos v,$$

$$z = - Bv - \mu [\sin u - 2 \log \tan (\pi/4 + u/2)].$$

The congruence so obtained ($\mu =$ constant) has the peculiar property of having the limit distance always constant whether the foci are real, coincident or imaginary.

If $\mu = 0$, the congruence becomes isotropic. In this case, Kummer's second form is non-existent but Sannia's exists and we have

$$\mathbf{r}_1 = \sec u\, (- k\mathbf{d}_2 + k_2\mathbf{d})$$

$$\mathbf{r}_2 = \cos u\, (k\mathbf{d}_1 - k_1\mathbf{d}),$$

and the equation of the middle surfaces of all isotropic congruences is given by

$$\mathbf{r} = \int [(k_2\, \mathbf{d} - k\mathbf{d}_2) \sec u\, du - (k_1\mathbf{d} - k\mathbf{d}_1) \cos u\, dv],$$

where the value of k is determined by (C′).

As the limits coincide ($\mu = 0$), the foci if real must also coincide with the limits ($k = 0$) and then the congruence becomes normal. Kummer's second form is non-existent and the congruence becomes normal isotropic. We have $\mathbf{r}_1 = 0$, $\mathbf{r}_2 = 0$. Therefore $\mathbf{r} =$ constant, so that the middle surface is a point. Thus we have the well-known result "the only normal isotropic congruence is a system of rays through a point."

For the equation (C′) to be satisfied for $k \neq 0$, it follows that the foci are imaginary. Assuming k to be a function of u alone, we get

$$k = A \sin u + B \sin u \log \tan (\pi/4 + u/2) - B,$$

and

$$x = A \cos u \cos v + B \cos u \cos v \log \tan (\pi/4 + u/2)$$

$$y = -A \cos u \sin v - B \cos u \sin v \log \tan (\pi/4 + u/2)$$

$$z = -Bv.$$

In this case, the equation of the middle surface is given by

$$y/x = \tan z/B,$$

which is a right helicoid generated by lines parallel to the plane $z = 0$ which intersects the z-axis and a family of right circular helices.

The congruence so obtained will have coincident limits but imaginary foci.

5. If the product of the distances of the limits from a focus be a constant, i.e. if k be a constant, (C) changes into

$$\mu_{12} \cos u - \mu_2 \sin u = k \cos^2 u.$$

Integrating it with respect to u, we get

$$\mu_2 \cos u = \tfrac{1}{2} k (u + \sin u \cos u) + A_2(v),$$

where $A_2(v)$ is a function of v or an absolute constant. Integrating it with respect to v, we get

$$\mu \cos u = \tfrac{1}{2} k(uv + v \sin u \cos u) + A(v) + B(u),$$

i.e.

$$\mu = \tfrac{1}{2} k(uv \sec u + v \sin u) + \sec u \, [A(v) + B(u)],$$

where $B(u)$ is a function of u alone or an absolute constant. Hence the vector position of P on the middle surface corresponding to the point p on the unit sphere is given by

$$\mathbf{r} = \int (\mathbf{r}_1 \, du + \mathbf{r}_2 \, dv), \tag{D}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are now known functions of u and v.

As the value of μ involves two arbitrary functions, we get the result:

There are doubly infinitude of rectilinear congruences for which the product of the distances of a focus from the two limits is constant.

Another result follows immediately that the middle surfaces of all such normal congruences (for which the principal surfaces correspond to a family of latitudes and longitudes on the unit sphere) are represented by the equation (D), where $\mu = \sec u\,[A(v)+B(u)]$.

If with $k =$ constant, μ is also constant, then (C) gives $k = 0$ and hence the congruence is normal. In this case

$$\mathbf{r}_1 = \mu\mathbf{d}_1 + 2\mu \tan u\, \mathbf{d} \text{ and } \mathbf{r}_2 = -\mu\mathbf{d}_2.$$

Therefore

$$x = -\mu \cos u \sin v,$$

$$y = -\mu \cos u \cos v,$$

$$z = -\mu\, [\sin u - 2 \log \tan (\pi/4 + u/2)].$$

Thus we get the Cartesian co-ordinates of $P(u, v)$ on the director surface corresponding to the point $p(u, v)$ on the unit sphere. In this case, we can easily construct the congruence. In order to locate the point P we take the diameter of the unit sphere (whose centre is O) through p and measure a distance of μ units along pO from O and then measure a distance $2\mu \log \tan\ (\pi/4 + u/2)$ parallel to the z-axis. If we draw a line through P parallel to Op, we get a ray of the normal congruence for which the distance between the limits and the product of the distances of a focus from the limits are constants. It may be noted that this middle surface is a surface of revolution.

I am obliged to Dr. V. R. Chariar for his kind guidance.

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Science College
Patna

A FEW IDENTITIES ON HIGHER DIFFERENCES

By S. C. CHAKRABARTI

1. The following notations will be used throughout this paper.

(i) $(x)^{(n)} = x(x-1)\ldots(x-n+1)$; $(x)^{(0)} = 1$.

(ii) $(a^m)_n = (a^m - 1)(a^{m-1} - 1)\ldots(a^{m-n+1} - 1)$; $(a^m)_0 = 1$.

(iii) $(a^m)_{2,n} = (a^m - 1)(a^{m-2} - 1)(a^{m-4} - 1)\ldots n$ factors; $(a^m)_{2,0} = 1$.

(iv) ${}^nS_p =$ sum of the products of n factors, $1, a, a^2, \ldots a^{n-1}$ taken p at a time; ${}^nS_0 = 1$, ${}^nS_p = 0$ if $p < 0$ or $> n$.

2. INTRODUCTION. In this paper is given a generalization of the identity*

$$(x+r-1)^{(r)} = \sum_{p=0}^{r} (\delta + r - 1)^{(p)} (x-\delta)^{(r-p)} \binom{r}{p} \tag{1}$$

which the author obtained in 1922, while evaluating a factorable continuant. A few other identities including a determinant resolvable into factors, are also given here.

3. THEOREM.

$$(a^{x+r-1})_r = \sum_{p=0}^{r} (a^{\delta+r-1})_{r-p} (a^{x-\delta})_p \, a^{p\delta} \, {}^pS_p \, {}^rS_p. \tag{2}$$

This may be proved by a basic formula—Higher Differences. [When $a \to 1$, (2) reduces to (1).]

4. THEOREMS.

$$\sum_{p=0}^{r} (-)^p (a^{2r-1-2p})_{2,r-p} (a^{2r-2p} - 1) \, {}^{2r}S_{2p} = (-)^{r-1} (a^{2r})_2, \tag{3}$$

$$\sum_{p=0}^{r} (-)^p (a^{2r-1-2p})_{2,r-p} (a^{2r-2p} - 1) \, {}^{2r+1}S_{1+2p} = (-)^{r-1} (a^{2r+1})_2 \tag{4}$$

* Chakrabarti, S.C.: On the evaluation of some factorable continuants, *Bull. Calcutta Math. Soc.* 13 (1922-3), 71-84.

It has recently been found that the identity (1) is the classical Vander Monde's identity. (See, Jordon, *Calculus of Finite Differences*, p. 48).

PROOF. Left side of (3)

$$=a^{2r}\sum_{p=0}^{r}(-)^p\,(a^{2r-1-2p})_{2,r-p}\,a^{-2p}\,{}^{2r}S_{2p}-\sum_{p=0}^{r}(-)^p\,(a^{2r-1-2p})_{2,r-p}\,{}^{2r}S_{2p}.$$

$= a^{2r}(-)^r(-a^{2r-1}+1+1/a)-(-)^r$, by earlier theorems. †

Hence follows the result. Similarly (4) may be treated.

5. If

$$|\alpha_{11}\,\alpha_{22}\,\alpha_{33}| = \begin{vmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{vmatrix},$$

then similarly formed

$$|\alpha_{11}\,\alpha_{22}\cdots\alpha_{nn}| = \prod_{k=1}^{n}\left\{\sum_{p=0}^{k-1}(-)^p\ \alpha_{k-p,k}\ {}^{k-1}S_p\right\} \tag{5}$$

if the elements below the principal diagonal of any column (kth say), are expressed in terms of the elements of that column on and above the principal diagonal by the formula

$$\alpha_{rk}=a^{k-1}\sum_{p=1}^{k}(-)^{p-1}\,{}^{r-1}S_{k-p}\,{}^{r-k+p-2}S_{p-1}\,\alpha_{k-p+1,k}/{}^{k-p+1}S_{k-p+1},\ r>k.$$

PROOF. On the determinant, perform successively $n-1$ operations, viz.

$$\sum_{p=0}^{n-1}(-)^p\,\mathrm{row}_{n-p}\,{}^{n-1}S_p,\ \sum_{p=0}^{n-2}(-)^p\,\mathrm{row}_{n-1-p}\,{}^{n-2}S_p,\ldots\ \sum_{p=0}^{1}(-)^p\,\mathrm{row}_{2-p}\,{}^{1}S_p.$$

First operation will make all the elements, except the last, of the nth row, vanish. Because in the element on the nth row and kth column ($k = 1, 2,\ldots k-1$), the coefficient of $\alpha_{k-c,k}$ ($c = 0, 1, 2\ldots k-1$), is

† Chakrabarti. S.C: Some identities and recurrents, *Jour. Indian Math. Soc.* (2) 11 (3 & 4) (1947), 89-94.

$$\left[(-)^c \frac{a^{k-1}}{{}^{k-c}S_{k-c}} \sum_{p=0}^{n-k-1} (-)^p \, {}^{n-1-p}S_{k-c-1} \, {}^{n-k+c-1-p}S_c \, {}^{n-1}S_p \right] +$$

$$+ (-)^{n-k+c} \, {}^{n-1}S_{n-k+c}$$

$$= \left[(-)^c \frac{a^{\frac{1}{2}(c+1)c} \, (a^{n-1})_{n-1}}{(a^{k-c-1})_{k-c-1} \, (a^c)_c \, (a^{n-k-1})_{n-k-1}} \times \right.$$

$$\left. \times \sum_{p=0}^{n-k-1} (-)^p \frac{1}{a^{n-k+c-p} - 1} \, {}^{n-k-1}S_p \right] + (-)^{n-k+c} \, {}^{n-1}S_{n-k+c}$$

by substituting the values of S's,

$= (-)^{n-k+c-1} \, {}^{n-1}S_{n-k+c} + (-)^{n-k+c} \, {}^{n-1}S_{n-k+c}$, by H. D. formula,

$= 0.$

The last element of the nth row will be

$$\sum_{p=0}^{n-1} (-)^p \, \alpha_{n-p,n} \, {}^{n-1}S_p,$$

which is an element on the principal diagonal. Similarly the second operation will make all the elements, except the last two, of the $(n-1)$th row vanish and the last but one element, which is on the principal diagonal, is

$$\sum_{p=0}^{n-2} (-)^p \, \alpha_{n-p,n} \, {}^{n-2}S_p.$$

After all the operations are performed, all the elements below the principal diagonal vanish, and the elements of that diagonal are the factors on the right side of (3). Hence we get the result.

Jadavpur University
Calcutta

ON THE POSSIBILITY OF STEADY BELTRAMI FLOW IN A VISCOUS LIQUID

By RAM BALLABH

We know that Beltrami flows are represented analytically by the equation

$$\text{Curl } \mathbf{q} = \lambda \, \mathbf{q}, \tag{1}$$

where $\mathbf{q}$ is the velocity vector and λ a scalar point function which may be defined as the torsion of neighbouring vector lines.

In the case of steady Beltrami flow of a viscous homogeneous incompressible fluid the equations of motion take the simple form

$$\nabla \chi' = \nu \nabla^2 \, \mathbf{q}, \tag{2}$$

where ν is the kinematic coefficient of viscosity and χ' has the usual meaning.

We have

$$\text{Curl Curl } \mathbf{q} = \text{grad div } \mathbf{q} - \nabla^2 \mathbf{q} = -\nabla^2 \mathbf{q},$$

since the vector $\mathbf{q}$ is solenoidal.

Therefore, from (1), we have

$$\text{Curl } \lambda \mathbf{q} = -\nabla^2 \mathbf{q},$$

from which we get

$$\nabla \lambda \times \mathbf{q} = -(\nabla^2 + \lambda^2)\mathbf{q}.$$

Equation (2) can therefore be written as

$$\nabla \chi' = \nu (\mathbf{q} \times \nabla \lambda - \lambda^2 \mathbf{q}). \tag{2a}$$

Also, eliminating χ' from (2) and using (1) we get

$$\nabla^2(\lambda \mathbf{q}) = 0. \tag{2b}$$

We shall consider the following cases separately :

(i) Unbounded fluid with constant $\mathbf{q}$ at infinity.

(ii) Bounded fluid with fixed boundaries.

(iii) Other cases, not included in (i) or (ii).

For cases (i) and (ii) we shall assume that the functions concerned satisfy the conditions of Green's theorem.

CASE (i). From Green's theorem we have

$$\iiint \left\{\left(\frac{\partial}{\partial x}\lambda u\right)^2+\left(\frac{\partial}{\partial y}\lambda u\right)^2+\left(\frac{\partial}{\partial z}\lambda u\right)^2\right\}dx\,dy\,dz,$$

$$= -\iint \lambda u\,\frac{\partial(\lambda u)}{\partial n}\,d\sum - \iiint \lambda u\,\nabla^2(\lambda u)\,dx\,dy\,dz, \qquad (3)$$

where u is the x-component of fluid velocity, Σ a sphere of large radius R with its centre at the origin of coordinates and δn an element of the inwardly-directed normal to Σ. The volume integrals extend over the region of space enclosed within Σ.

From (2b), $\nabla^2(\lambda u) = 0$ and therefore the volume integral on the right side of (3) is zero.

Also, since $\mathbf{q}$ is constant at infinity, Curl $\mathbf{q}$ will be of a higher order than R^{-1} for points of Σ. The integrand in the surface integral will therefore be of a higher order than R^{-3}. Consequently when $R\rightarrow\infty$

$$\iint \lambda u\,\frac{\partial(\lambda u)}{\partial n}\,d\Sigma \rightarrow 0.$$

Therefore, we have λu = constant throughout the fluid.

Similarly, we have λv = constant throughout the fluid, and λw = constant throughout the fluid, where v and w are the y and z components of velocity.

From equation (1) we therefore have Curl $\mathbf{q}$ = constant throughout the fluid. But since Curl $\mathbf{q}$ vanishes at infinity it must be zero everywhere.

We thus conclude that steady Beltrami flow is not possible in this case.

CASE (ii). In the case of a bounded fluid with fixed boundaries, the double integral in (3) is taken over the given boundaries. But at each point of a fixed boundary, $u = 0, v = 0, w = 0$, We, therefore, conclude again that throughout the fluid λu, λv and λw must be constants.

Let $\lambda u = k_1, \lambda v = k_2, \lambda w = k_3$, where k_1, k_2, k_3 are constants.

From equation (1) we then have*

$$\begin{aligned} k_1\lambda_x + k_2\lambda_y + k_3\lambda_z &= 0, \\ -k_2\lambda_x + k_1\lambda_y - k_3\lambda^2 &= 0, \\ k_3\lambda_x - k_1\lambda_z - k_2\lambda^2 &= 0, \\ -k_3\lambda_y + k_2\lambda_z - k_1\lambda^2 &= 0, \end{aligned}$$

$\lambda \neq 0$.

Eliminating λ_x, λ_y, λ_z from the above equations, we get

$$\begin{vmatrix} k_1 & k_2 & k_3 & 0 \\ -k_2 & k_1 & 0 & -k_3\lambda^2 \\ k_3 & 0 & -k_1 & -k_2\lambda^2 \\ 0 & -k_3 & k_2 & -k_1\lambda^2 \end{vmatrix} = 0,$$

i.e. $(k_1^2 + k_2^2 + k_3^2)^2\lambda^2 = 0$, giving $\lambda = 0$.

Steady Beltrami flow is therefore not possible in this case either.

OTHER CASES. In other cases it is difficult to arrive at a conclusion, but the non-existence of steady Beltrami flows can be established in regions where the streamlines form a normal congruence of the surfaces $\chi' =$ constant.

From equation (2a) we have

$$\mathbf{q}\cdot\nabla\chi' = -\nu\lambda^2\mathbf{q}^2. \tag{4}$$

If, therefore, the streamlines of the flow coincide with the normals to the surfaces $\chi' =$ constant, $\mathbf{q}\cdot\nabla\chi' = 0$ and consequently $\lambda = 0$, denying the existence of Beltrami flow in such a case.

* The first equation is derived by taking the divergence of each side of equation (1). The other equations are obtained by putting the values of Curl $\mathbf{q}$ in equation (1) in terms of the derivatives of u, v, w replacing them by $k_1/\lambda, k_2/\lambda, k_3/\lambda$ respectively.

It is interesting to note that in cases where steady Beltrami flow is possible the streamlines are inclined to the surfaces $\chi' =$ constant at the same angle in fluids of different viscosities.

The angle θ between the normals to the surfaces $\chi' =$ constant and the streamlines is given by

$$\cos\theta = \frac{\nabla\chi' \cdot \mathbf{q}}{|\nabla\chi'|\,|\mathbf{q}|} = \frac{-\nu\lambda^2 \mathbf{q}}{|\nabla\chi'|} = -\frac{\lambda^2}{\sqrt{(\lambda_x^2 + \lambda_y^2 + \lambda_z^2 + \lambda^4)}},$$

using (2a) and the relation $\nabla\lambda . \mathbf{q} = 0$ derived from (1).

This clearly shows that θ is independent of the degree of viscosity of the fluid.

In conclusion I wish to express my gratitude to the referee for inviting my attention to cases (i) and (ii) which were not included in an earlier version of the paper.

Lucknow University

CLASSROOM NOTE

Bisectors of the angles between the lines $ax^2+2hxy+by^2=0$.

By N. L. Maria, *Govt. College, Ludhiana*

The following is an alternative method of finding the equation of the bisectors of the angles between the lines $ax^2+2hxy+by^2=0$. The method holds whether the axes are rectangular or oblique. Use is made of the property of a rhombus that its diagonals are at right angles to each other.

Let the axes be inclined at ω. Let OA and OB be the straight lines represented by the given equation. Let $P\ (\alpha,\beta)$ be any point on one of the bisectors. Through P draw lines parallel to OB and OA meeting OA and OB in M and N respectively. Then $OMPN$ is a rhombus and as such OP is at right angles to MN. The equation to the lines PM, PN is

$$a(x-\alpha)^2+2h(x-\alpha)(y-\beta)-b(y-\beta)^2=0.$$

Therefore the equation of MN is

$$ax^2+2hxy+by^2-[a(x-\alpha)^2+2h(x-\alpha)(y-\beta)+b(y-\beta)^2]=0,$$

or

$$2(a\alpha+h\beta)x+2(h\alpha+b\beta)y=a\alpha^2+2h\alpha\beta+b\beta^2.$$

Equation of OP is

$$\beta x-\alpha y=0.$$

Since OP is at right angles to MN, we have

$$\beta(a\alpha+h\beta)-\alpha(h\alpha+b\beta)-[\beta(h\alpha+b\beta)-\alpha(a\alpha+h\beta)]\cos\omega=0,$$

or

$$\alpha^2(h-a\cos\omega)-\beta^2(h-b\cos\omega)=(a-b)\alpha\beta.$$

Hence the locus of $P\ (\alpha,\beta)$, i.e. the equation of the two bisectors, is

$$x^2(h-a\cos\omega)-y^2(h-b\cos\omega)=(a-b)xy.$$

QUESTION AND SOLUTION

QN. 1836 (*C. Thebault*) *Let a tangent t to the incircle of a triangle ABC intersect AB, AC at C′ and B′ respectively. Let parallels to BB′ and CC′ through C′ and B′ respectively meet AC, AB, at B″ and C″. Show that B″ C″ is a tangent to the incircle of the triangle AB′C′ and is parallel to BC. Deduce that the centres of the circumcircles of the four triangles determined by the four vertices of a quadrangle circumscribing a circle are the four vertices of another quadrangle also circumscribing a circle.*

SOLUTION BY S. R. KHANWALKAR. Since $C'B''$ is parallel to BB', $\frac{AB''}{AB'} = \frac{AC'}{BA}$, similarly $\frac{AB'}{AC} = \frac{AC''}{AC'}$. Hence $\frac{AB''}{AC} = \frac{AC''}{AB}$ or $B''C''$ is parallel to BC.

The in-radius of $\triangle AB'C'$ is given by

$$2r = (AB' + AC' - B'C') \tan A/2$$

and the ex-radius of $\triangle AB''C''$ is given by

$$2r_1 = (AB'' + AC'' + B''C'') \tan A/2$$

Now

$$\begin{aligned} AB'' + AC'' + B''C'' &= AB'' + AC'' + B''B' + C''C' - B'C' \\ &= AC' + AB' - B'C'. \end{aligned}$$

Hence $r = r_1$ and $B''C''$ touches the in-circle of $\triangle AB'C'$.

The centres of the circles formed by taking three vertices of the quadrangle $BCB'C'$ lie at the points of intersection of the perpendicular bisectors of the adjacent sides of $BCB'C'$. Let E, F be the centres of circles BCC', BCB', G, H those of $CB'C'$, $BB'C'$. Now EF, FG, GH and HE are respectively at right angles to BC, CB', $B'C'$ and BC'. Since $B''C''$ is parallel to BC, sides of the figure $EFGH$ are at right angles to those of $C''B''B'C'$; and hence similar, and $C''B''B'C'$ is proved to be a circumscribing quadrilateral and hence $EFGH$ is also a circumscribing one.

BOOK REVIEW

Selections from modern abstract algebra. By R. V. Andree, Henry Holt and Company, New York, 1958. $6.50. pp. xii + 212.

THE author of this book sets himself, and definitely succeeds in, the task of giving an American undergraduate, as far as abstract algebra is concerned, (i) an elementary but far-reaching introduction to the basic concepts and a thorough drilling in the technique of their use and (ii) an effortless but revealing peep into the wide field of applications that are being made of the various parts of the subject. His methods of presentation include (i) provision, for every concept, of numerous examples ranging from the very trivial to the reasonably general; (ii) suggestions at every stage for allied or advanced reading, most of them being expository articles from the *American Mathematical Monthly* and the like; (iii) provision for the uninitiated reader of a number of opportunities and possibilities to 'discover' for himself, 'conjecture' for himself and 'prove or disprove' for himself; and, (iv) a vigorous style which is enjoyable.

Being only a judicious set of selections, the modest chapter headings are: 1. 'Number Theory and Proof', where the reader is given, besides the usual 'mysteries' of perfect numbers, game of Nim etc. a clear analysis of the nature and construction of a proof; 2. 'Equivalence and Congreuence', where an elaborate exposition is given as preparation for a maiden voyage into abstract thinking; 3. 'Boolean Algebra', which also includes a sound introduction to the design of switching electric circuits; 4. 'Groups', which takes one up to an acquaintance with the Jordan-Hölder theorem; 5. 'Matrices', which defines a matrix as an element of a properly defined matric algebra and, after a smooth development, ends with three examples illustrative of matrix applications, the first from industrial economics, the second, a simplified example of a Markov chain, from quantitative chemical analysis and the third, from matrix analysis of electrical networks; 6. 'Linear Systems', which helps even the least equipped student to understand the solving of systems of linear equations; 7. 'Determinants'; 8. 'Fields, Rings and Ideals', which slowly builds up the structural foundation of mathematics till the stage where the reader can appreciate the importance of residue class rings; and 9. 'More Matrix Theory', where topics such as Hamilton-Cayley theorem, the concept of eigenvector, infinite series of matrices are touched upon.

It is the reviewer's opinion that this book is well written and will be very useful to those interested in mathematics.

V. Krishnamurthy

NEWS AND NOTICES

DR. S. D. Chopra, Dr. Narayan Bahadūr Manandar and Prof. C. T. Rajagopal have been admitted as life-members of the Society.

The following persons have been admitted to the membership of the Society: H. S. Ahluwalia, Sri Nivas Bhatt, S. P. Bandyopadyay, B. K. Choudhry, V. K. Gangal, Miss Sulaxana Kumari, B. K. Lahiri, T. D. Minakshisundaram, B. Y. Oke, R. Raghavendran, Vikramaditya Singh, Miss Pramela Srivastava.

We regret to announce the death of Prof. N. M. Shah, a life member of the Society. He was also a secretary of the Society for some time.

The Council has accepted with thanks the gift of twenty books on higher mathematics, to the Library of the Society, by Professor H. G. S. Sharma, a life-member of the Society. We sincerely thank Professor Sharma for this gift.

The twenty-fourth Conference and the Golden Jubilee Celebrations of the Society will be held in Poona under the auspices of the University of Poona in the last week of December 1958. Members wishing to read papers are requested to send them in *full* together with two copies of abstracts of each paper to Professor S. M. Shah, Muslim University, Aligarh. The abstracts should be typewritten on special forms obtainable from the Secretary and should not exceed 200 words in length. Displayed formulas and complicated symbols likely to cause difficulties in printing should be avoided. These should be sent on or before 1st October 1958.

Dr. U. N. Singh of Muslim University, Aligarh has been appointed as Professor and Head of the Department of Mathematics, M. S. University of Baroda.

A summer school of mathematics has been organized by the professors of Delhi University from the 5th May to run for five weeks mostly as a refresher course comprising all aspects of mathematics and its applications.

We welcome the new quarterly *The Mathematics Seminar* edited by Professor Shantinarayan of Hansraj College, Delhi. The aim is to create interest in mathematics in schools and colleges and to help in the working of mathematical clubs and societies. The first issue is dated September 1957. We wish the new venture every success.

Professor W. K. Hayman, F.R.S., Professor, Imperial College of Science, London, delivered lectures on Meromorphic functions, on Symmetrization and on Schilicht functions, in Muslim University, Aligarh in March 1958.

The fortyfifth session of the Indian Science Congress was held in Madras in January 1958. Prof. M. S. Thacker was the General President and Prof. B. S. Madhava Rao was President of the Section on Mathematics.

The third Congress of Theoretical and Applied Mechanics was held in the Indian Institute of Science, Bangalore, in the last week of December 1957 under the Presidentship of Dr. S. R. Sen, Director, Indian Institute of Technology, Kharagpur.

Professor K. Chandrasekharan has been included in the team of 15 scientists from India headed by Dr. M. S. Thacker to visit Russia on the invitation of the Soviet Academy of Sciences.

THE INDIAN MATHEMATICAL SOCIETY

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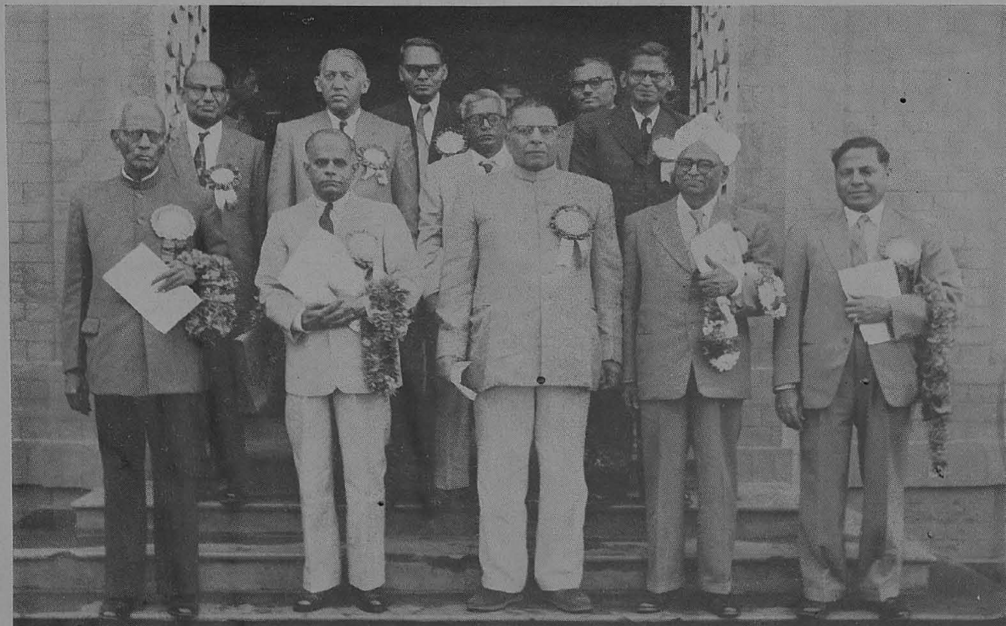
Errata :

Page	Line	For	Read
164	15	half-ray in Σ_0	half-ray along a fixed line L in Σ_0
164	last line	$\alpha_m m$	α_m
166	1	$r_m \cos^2 \beta_m$	$r_m^2 \cos^2 \beta_m$
166	6	$r_m^2 \sin^2 \beta_m^2$	$r_m^2 \sin^2 \beta_m$
166	9	$r_m^2 \sin^2 \beta_m^2$	$r_m^2 \sin^2 \beta_m$
167	19	$r_m^2 \sin^2 \alpha_m$	$r_m^2 \sin 2\, \alpha_m$
168	12	r_{n-2}	r_{n-2}^2
168	15	$r_m^2 e^{\alpha i m}$	$r_m^2 e^{2i\alpha_m}$

REPORT OF THE TWENTY-THIRD CONFERENCE OF THE INDIAN MATHEMATICAL SOCIETY

THE INDIAN MATHEMATICAL SOCIETY

TWENTY-THIRD CONFERENCE, CUTTACK, 1957



The President with Dr. P. Parija, Vice-Chancellor and Dr. H. K. Mahtab, Chief Minister and others.

THE INDIAN MATHEMATICAL SOCIETY

TWENTY-THIRD CONFERENCE, CUTTACK, 1957

LEFT TO RIGHT

Sitting : V. S. Krishnan, V. S. Huzurbazar, M. V. Subba Rao, M. Venkataraman, S. Mahapatra, S. Munzur Hussain, S. Minakshisundaram, S. Mahadevan (Secretary), B. C. Das (Vice-Chairman), V. Ganapathy Iyer (President), Ram Behari, R. Mohanty (Local Secretary), G. C. Rath, J. N. Panda (Asst. Local Secretary), D. K. Sen, H. M. Sengupta, Girdhari Lal Saini, T. V. Avadhani.

Standing 1st Row : Hari Krishnan Verma, P. C. Consul, V. Singh, S. K. Singh, P. Tiwari, P. L. Sharma, V. Lakshmi Kanth, U. N. Singh, H. G. S. Sharma, B. K. Lahiri, M. Raghavacharyulu, K. Sitaram, S. Ramakrishnan, M. Parameswara Ayyar, A. C. Choudhury, R. R. Umarji, M. V. Jambunathan.

Standing 2nd Row : Mahendra Pratap, Bhagwan Das Agrawal, M. K. Singal, P. C. Jain, H. S. Ahluwalia, R. K. Jha, Hari Shankar, P. S. Subramanian, N. Yegyanarayanan, M. S. Cheema, T. S. Venkatraman, S. S. Dubey, G. Sankaranarayanan, S. Swetharanyam, P. D. Gupta, S. D. Chopra, Shanti Narayan, S. Kumar Hindi.

Standing 3rd Row : Sukdeb Prasad Misra, Sachindra Chandra Das, Siva Kumar Panda, Ghanashyam Samal, Purusottam Das B. Patel, S. K. Abhyankar, S. S. Cheema, D. L. Sharma, S. P. Bandopadhyay, M. R. Parameswaran, K. N. Sundaresan, P. V. Chalapathi Rao, Asis Mukherjee, V. K. Balachandran, B. K. S. R. Somayajulu, Sahib Ram Mandan.

Standing 4th Row : M. K. Wadikhaye, J. N. Kapur, B. Y. Oke, V. S. Varadarajan, R. Ranga Rao, K. K. Deshpande, B. B. Bagi, U. Mohapatra, Suryanarayana. M., P. C. Misra, S. P. Misra, M. S. Ramanujan, P. C. Rath, D. R. Kaprekar, P. V. L. N. Rao.

Standing 5th Row : M. C. Agarwala, M. M. Satapathy, K. C. Panda, S. R. Sinha, T. Pati.

PROGRAMME

Thursday, December 26, 1957

10-00 A. M.—National anthem. Welcome address by Padma Bhushan Dr. P. Parija, Vice-Chancellor, and Chairman of the Reception Committee. Inauguration by Dr. H. K. Mahtab, Chief Minister, Orissa. Report by Prof. S. Mahadevan, Secretary. Presidential address by Prof. V. Ganapathy Iyer

12-30 P. M.—Lunch

2-00 P. M.—Business Meeting of the Council of the Indian Mathematical Society

2-30 P. M.—Annual General Meeting of the Indian Mathematical Society

3-00 P. M.—Invited address by Prof. V. S. Krishnan
Subject : *Topological Algebra*

3-30 P. M.—Invited Address by Dr. M. V. Subba Rao
Subject : *Closure Theorems*

4-30 P. M.—At Home by the Reception Committee (by special invitation)

6-00 P. M.—Popular lecture by Prof. V. S. Huzurbazar
Subject : *Is Mathematics Consistent?*

8-30 P. M.—Dinner

Friday, December 27, 1957

9-00 A. M.—Reading of Papers

12-00 NOON—Invited address by Prof. R. Mohanty
Subject : *Absolute Reisz Summability of Fourier Series*

1-00 P. M.—Lunch

2-00 P. M.—Symposium on 'Functional Analysis'

Participants —Prof. V. Ganapathy Iyer
Dr. M. Venkataraman
Dr. U. N. Singh
Shri M. S. Ramanujan
Shri. M. R. Parameswaran

4-30 P. M.—Tea and Photo

6-00 P. M.—Popular lecture by Shri J. N. Kapur
Subject : *Ballistics of guns and rockets*

8-30 P. M.—Dinner by the Chief Minister (By special invitation)

Saturday, December 28, 1957

9-00 A. M.—Reading of papers

11-00 A. M.—Symposium on 'Research and Mathematical Development in India'

1-00 P. M.—Lunch

2-00 P. M.—Reading of papers

3-00 P. M.—Excursion to Bhubaneshwar

7-00 P. M.—Variety Entertainment

8-30 P. M.—Dinner

Sunday, December 29, 1957

7-00 A. M.—Excursion to Konarak and Puri

12-00 NOON—Lunch at Konarak

3-00 P. M.—Tea at Puri

6-00 P. M.—Mahaprosad at Puri

REPORT OF THE CONFERENCE

INAUGURATION

THE Twenty-third Conference of the Indian Mathematical Society was held at Cuttack on December 26-28, 1957, on the invitation of the Utkal University. Over one hundred delegates were present.

The Conference was held in the spacious hall of the Ravenshaw College, which was tastefully decorated for the occasion. Padma Bushan Dr. P. Parija, Vice-Chancellor of the University and Chairman of the Reception Committee, while welcoming the delegates, expressed the hope that our deliberations would be of great help in securing objective thinking, so that we might avoid pitfalls regarding the present five-year plan. The Conference was formally inaugurated by Dr. Hare Krishna Mahtab, Chief Minister of Orissa.

SECRETARY'S ANNUAL REPORT

Professor S. Mahadevan, Secretary of the Society, then presented the report of the Society for the year 1957-58. He conveyed the thanks of the Society to the Utkal University for their kind invitation to hold the Conference at Cuttack and for the excellent arrangements the University had made for the same.

He referred to the loss sustained by the Society by the death of Sri. C. Bhaskaraiya, Retired Accountant-General, a life member of the Society, and conveyed the Society's condolences to the bereaved family.

He briefly reviewed the work of the Society for the last fifty years and said "looking back, starting with a few members, with no facilities for research and no financial backing, relying on the sympathy and active support of the members, we have built this Society on a firm footing as an All-India body and have developed its periodicals—the *Journal* and the *Mathematics Student*". Proceeding he said, " Of the twenty foundation members who started the Society in 1907, I am glad to state that two are still with us.

One is Dr. R. P. Paranjpye, who did much in his days to strengthen the Society and build its library, and who is now Vice-Chancellor of the Poona University. The other is Professor D. D. Kapadia, who rendered valuable service to the Society at its inception, as its Secretary for twelve years, 1910-1922". He stated that the Golden Jubilee would be celeberated next year at Poona and that the Vice-Chancellor had kindly extended the invitation for that purpose.

Proceeding to the work of the Society, the Secretary mentioned that the *Journal* was first issued in 1909 and that it completed the first series in 1933, ending with the Silver Jubilee volume. In 1933, the *Mathematics Student* was also started, to help young research workers and college students and contained expository articles, class-room notes, problems and solutions, book reviews and announcements. The *Journal* commenced the New Series in 1934 with research papers only. He said that the success of the periodicals was entirely due to the unselfish and devoted work of the editors. He mentioned that the present Editor, Professor K. Chandrasekharan had spared no pains in pulling up the standard both in quality and get up. In particular, he thanked the Commercial Printing Press, Bombay, for their excellent printing and their unfailing help.

Regarding the Society's Library the Secretary said that since the start, it was located in Poona with Dr. R. P. Paranjpye as the first librarian and that it was transferred to the Ramanujan Institute of Mathematics, Madras in 1951. He added that the Library contained books on higher mathematics and a more or less complete run of all the periodicals and thanked Professor C. T. Rajagopal, Director of the Ramanujan Institute and Librarian for the able management.

The Secretary regretted that the Narasinga Rao Medal could not be awarded that year as most of the issues of the periodicals contained the proceedings of the International Colloquium on Zeta Functions and the report of the South Asian Conference on Mathematical Education. He gave an assurance that two medals would be awarded the following year.

Proceeding, the Secretary observed, " active research is being done in the Tata Institute, Bombay, which is now recognised by the Government of India as the national centre for advanced study and fundamental research in Mathematics. There is the Ramanujan Institute of Mathematics at Madras, financed by the Government of India and managed by the University of Madras, and the Institute is doing excellent work. We hope that the Government of India will give it increased aid to carry out the schemes of the Director for improving and widening the activities of the Institute. It is not enough if the Government gives financial aid ; it should open more institutes and at least two more in different places. If technological studies which are so vital to us are to take root in our country, then research in the fundamental subject of mathematics should be the chief item in the programme of the Government ".

The Secretary congratulated the Tata Institute of Fundamental Research, Bombay and the Sir Dorabji Tata Trust for publishing the *facsimile* edition in two volumes of the famous notebooks of the late Srinivasa Ramanujan.

Concluding the Secretary said, " we maintain our budget mostly from dues paid by our members and subscribers. We find it difficult to manage ; printing bills are mounting up, the cost of books and periodicals is also increasing. We request the Government of India to help us to meet the deficit. I wish to thank the various Universities which have been giving us annual grants and also the Tata Institute, Bombay, the National Institute of Sciences of India and the Government of India for their grants. The best help can come by many of you joining the Society in large numbers so that all of us may feel satisfied that we have contributed our bit towards the progress of mathematical research in India ".

Address

Professor V. Ganapathy Iyer of Annamalai University, President of the Conference, delivered the address, which is printed separately.

Vote of Thanks

Professor B. C. Das, Director of Public Instruction, Cuttack, and Vice-Chairman of the Reception Committee proposed a vote of thanks, bringing the proceedings of the inaugural session to a close.

Meeting of the Society

The Council of the Society met in the afternoon of December 26. At the meeting of the General Body which followed, a resolution of condolence on the death of Sri. C. Bhaskaraiya and Professor N. M. Shah was passed. The members evinced a lot of interest and a number of questions were asked. The replies of the Secretary covered the following points : (1) A list of books and periodicals will be printed in the *Mathematics Student*, as these are added to the library. (2) It is proposed to publish the author index of all papers which appeared in the *Journal*, Old and New Series and also a complete catalogue of books and periodicals in the Library and (3) Every effort will be made to expedite the 1957 issues of the *Journal* and the *Student*.

Proceedings of the Conference

The mathematical programme consisted of presentation of papers, invited addresses and two symposia. Two sessions were devoted to the reading of papers and abstracts of these papers appear elsewhere. Among the invited addresses were one by Professor V. S. Krishnan on 'Topological Algebra', one by Dr. M. V. Subba Rao on 'Closure theorems' and a third by Professor R. Mohanty on 'Absolute Riesz Summability of Fourier Series'. A symposium on 'Functional Analysis' was organised on December 27, in which Professor V. Ganapathy Iyer, Dr. M. Venkataraman, Professor U. N. Singh, Sri. M. R. Parameswaran and Sri. M. S. Ramanujan participated. On December 28, there was a symposium on 'Research and Mathematical Development in India', in which professors representing various universities participated. They gave in detail the facilities for research in their universities and suggestions for improving the present state of affairs. On December 26, Professor V. S. Huzurbazar, Head of the Department of Mathematics, University of Poona, gave a very interesting and humorous talk on "Is mathematics consistent ?". Though this was a difficult topic he explained in an extremely simple language the notion of consistency in mathematics drawing apt illustrations from the facts of everyday life. The next day Prof. J. N. Kapur, Head of the Department of Mathematics, Hindu College, Delhi, delivered a thought-provoking lecture on a matter of topical interest—Ballistics of guns and rockets. He explained the distinction between the interior and exterior ballistics, the various types of rockets and the principles underlying their construction. Though this was a technical subject the audience very much appreciated the lecture.

Social Programme

The delegates were entertained by the Reception Committee at an 'At Home' on the 26th December. On the 27th a grand dinner was given by the Chief Minister, besides the other receptions. There was an excursion to Bhubaneswar, the capital and the temple on the 28th. There was a variety entertainment consiting of classical dances and music on the same evening. The next day there was a full-day excursion to Konarak and Puri. The authorities of the local college at Puri entertained the delegates on a lavish scale.

Thanks of the Secretary

On the final day the Secretary thanked the authorities of the University, the participants in the symposia, those who gave invited addresses and popular lectures, the local secretaries and the volunteers for the excellent arrangements and unstinted service to the delegates.

PRESIDENTIAL ADDRESS

By V. GANAPATHY IYER

I DEEM it a great honour that I have been given the opportunity to preside over and address this Conference just when the Indian Mathematical Society is completing fifty years of its existence. Started, under the designation of Mathematical Club in 1907 by a few enthusiastic mathematicians, the Society is now a fully representative all-India body with a membership strength nearing 500 and with several distinguished foreign scholars on its roll. The Society has been holding conferences once in two years till 1951 and annually thereafter giving opportunities for mathematicians from different parts of the country to meet together. The Society is publishing two quarterlies. One is *The Journal of the Indian Mathematical Society*, devoted to the publication of research articles while the second, *The Mathematics Student* publishes articles of interest to College teachers and beginners in research. The Editor, Professor Chandrasekharan, informs me that the flow of good papers by Indian scholars is on the increase. All these stand on the credit side of the achievements of the Society.

But a contemplation of the overall picture of mathematical development in India leaves little room for complacency. After a rough survey of the situation, it appears to me that the proposition "In every important branch of mathematics in which research is carried on nowadays, there is at least one first-rate authority in India" is far from true and, as things stand today, not likely to become true even in another fifteen years. Concerted and well-planned efforts of all lovers of mathematics in India will be needed if we are to approach even the above moderate target. In mathematically advanced countries, there are scores of first-rate mathematicians, working in each important branch of mathematics and this is the ideal to be aimed at. To an audience of mathematicians, it is hardly necessary to point out that the true index of a country's position in the world of Science and Technology is the state of development of mathematics in that country.

I do not wish to devote this address to a detailed analysis of the causes and remedies for this state of affairs. But I would like to touch upon one or two points. It is hard for me to believe that there is dearth of mathematical talent in this country with centuries of inherited aptitute for abstract speculation. Broadly speaking there are two types of persons with mathematical talent. For one type, it is inevitable that he should devote himself to mathematical research —he is born for it and he will contend against all odds to achieve his end. But such persons are few and far between. The majority with an aptitude for mathematics require favourable environment and opportunities for their talent to flower. It is my view that during the last twenty-five years during which we are slowly recognising the need for mathematical research, many such persons have been sidetracked mainly due to economic forces and want of opportunities. It is out of this second type, that overall mathematical development can be expected in the country. So it behoves Universities and other bodies interested in the matter to devise suitable machinery to discover and exploit such talent for the benefit of the country.

Closely related to the discovery of mathematical talent are the recent trends in Collegiate education. The introduction of the Pre-University course followed by a three year degree course purporting to give a liberal and all round education has been hailed as remedy for the deteriorating standard of university education. But a study of the regulations and curricula for these leads me to just the opposite conclusion. For instance, in the old Intermediate Science group there are usually three subjects in which a student can specialize later and there are two years in which a teacher and the student himself can discover his aptitude. But now, the student undergoing the Pre-University course can choose only two fields of specialization later and there is only one year in which he has to decide the matter. Moreover in the name of all-round and liberal education, a good deal of matter is shoved into a pupil's brain during the Pre-University and three year degree course and the whole system seems to be based on the assumption that acquisition and assimilation of knowledge and acquisition of information are synonymous. Whatever be its merits and demerits, I feel sure that these reforms in University education will definitely bring down the standard of attainment in mathematics of the pupils coming out with a Master's degree under this new system and one has to remember that even now the standard is low enough when compared with the equipment necessary for any kind of research work. Probably a three year degree course after the present intermediate with a two years' Master's degree course, thereafter would have been a better conceived reform. I have placed before you what I have to say in the matter since, in my opinion, the reforms will retard the growth of mathematics in the Country.

I now turn to the mathematical part of the address for which I have chosen the topic *A survey of Analysis, classical and modern.*

During the last 30 years, mathematicians have been devoting more and more of their time to the investigation of inter-relation between abstract ideas and structures. These speculations though apparently unconnected with the physical world have been exploited and the result is the phenominal advance in other sciences and technology. This paradoxical situation reminds one of how our seers of old analysed the phenominal universe ultimately into thought and form, *Nama* and *Rupa*. The two broad divisions into which these abstract mathematical investigations fall are known as abstract algebra and topology. The former concerns itself with operational structures and the latter with the notions of limit and continuity so fundamental in classical analysis.

The foundation of classical analysis is the real number system. In this system, three abstract notions blend themselves harmoniously. Firstly the real numbers are closed with respect to two operations known as addition and multiplication. Addition is associative and commutative and has zero for its identity element with a unique inverse with respect to zero. Any system with such a binary operation is known as an abelian group. If zero (the identity element for addition) is omitted, the remaining numbers constitute an abelian group with respect to multiplication with one as the identity element and moreover multiplication distributes addition. Any abstract system endowed with two such operations having the properties mentioned above is known as a field. So algebraically, the real numbers form a field. Secondly, the real numbers are ordered. That is, there is a binary relation $\geqslant$ with the properties : $a \geqslant a$, if $a \geqslant b$ and $b \geqslant a$ then $a = b$ and if $a \geqslant b$ and $b \geqslant c$ then $a \geqslant c$. A system possessing a binary relation between some of its element pairs and having the above properties is known as a partially ordered system and if $a \geqslant b$ or $b \geqslant a$ holds for every pair of elements, the system is said to be simply ordered with respect to this relation. So the real numbers form a simply ordered system. Moreover this order is compatible with the field operations, that is, if $a \geqslant b$ then $a + c \geqslant b + c$ and $ax \geqslant bx$ provided $x \geqslant 0$. A field of this type is called an ordered field. A simply ordered structure is said to be complete with respect to that order if every bounded set has a greatest lower bound and a lowest upper bound. Now the real number system is a completely ordered field. Using the order to define open intervals, taking as open any set which is the union of open intervals, a topology leading to the fundamental notions of limit and continuity can be defined on the real number system. A family of subsets of a given set closed with respect to arbitrary unions and finite intersections is said to define a topology on the set and the sets of the family are said to constitute the open sets of the topology. The topology mentioned above for the real number system has several other properties. The field operations, addition and multiplication are continuous in the topology. There is an enumerable dense subset (the rational numbers) and the system is connected in the topology, that is, cannot be obtained as the union of two disjoint open sets. Finally it is locally compact, that is, every point has a neighbourhood whose closure has the Heine-Borel property that every open covering contains a finite covering. Thus the real number system is a connected, locally compact, completely connected topological field. The reason why the real number system occupies such a fundamental position in mathematical investigations is that it possesses several properties as indicated above which all blend together harmoniously.

I have briefly indicated above, how the real number system forms the model for several abstract systems like a field and topological space. Other important concepts in classical analysis have led to corresponding abstract systems. The limit of a sequence of real numbers, the limit and continuity at a point of a real valued function of a real variable, derivatives and integrals of such functions are some of the fundamental notions in analysis. The real number system has enabled the precise formulation of these concepts forming models for abstract generalizations.

The notion of a function from a set of real numbers taking real values is fundamental in Analysis. This leads to the general notion of functions or mappings defined on abstract sets taking values in another abstract set. And it is no exaggeration to say that classes of functions, transformations or mappings form the subject of study in all branches of modern mathematical research.

Next let us take the notion of a sequence. A sequence of real numbers can be regarded as a function on positive integers to real numbers. A sequence (x_n) of real numbers is said to converge to the number l if every open interval round l contains all but a finite number of terms of the sequence. Now the closure of a set E of real numbers (that is, the union of E and its limit points) can be specified as the set of numbers x such that there is a sequence of points of E converging to x. In a topological space where the topology is defined by a distance function, the closure of a set can be specified as above but this is not true in general topological space. This leads to the notion of directed limits. A directed system X is a partially ordered set (order denoted by $\geqslant$) with the property that given a and b in X there is a $c \geqslant$ to both a and b. A directed net is a function f from X to a topological space Y. A point y_0 of Y is said to be the limit of the net f following X if for every open set U containing y_0 there is an $a \in X$ such that $f(x)$ belongs to U for all x in X with $x \geqslant a$. This generalized notion of the limit is adequate to specify the closure of a set as described above. The notion of filter used by the French school of mathematicians is equivalent to directed sets, set inclusion being the order relation.

We consider now the notion of the derivative. Interpreted as a rate measurer, it is the basis of applications of mathematics to mechanics. Interpreted as the slope of the tangent to a curve it is the basis of differential geometry. The derivative of a complex valued function of a complex variable has led to the rich theory of analytic functions. This in turn has led to functions of several complex variables, analytic functions with both the domain of definition and the range of values in general normed vector spaces and normed algebras. These are rich fields of mathematical speculations in which research work is being carried on by several noted mathematicians.

For continuous functions, the notion of a definite integral regarded as the increment of its primitive in an interval and the limit of finite sums coincide. The latter under the name of the "method of exhaustion", was used by Greek geometers to evaluate areas and volumes of simple curved figures. It is found that the definite integral as the limit of sum has significance even when the function has discontinuities. This led to the formulation of the theory of Riemann Integral. Though this notion is enough for all practical applications, it was found wanting in theoretical investigations. For instance, it is not necessary that the pointwise limit of a sequence of Riemann Integrable functions should be Riemann Integrable. The formulation of the theory of Lebesgue Integral which to a large extent remedies this defect should be regarded as an important landmark in the development of the notion of an integral which in turn has revolutionalized mathematical investigations in several other fields, for instance, the precise formulation of the mathematical theory of probability was made possible and several representation theorems in the theory of function spaces depend upon the general notion of measure and integral. After digressing a little to draw attention to a few other points in classical analysis I shall revert again to a brief survey of general integration and related ideas.

As we have remarked, the real number system is something of a perfection in several respects. But it is not algebraically closed if every polynomial equation with coefficients in the field has a root in the field. A field not algebraically closed can always be imbedded in an algebraically closed field. The complex number system which can be built up from the real number system following familiar methods is algebraically closed and contains a field ismorphic to the real number field. But in thus extending the number system, one has to sacrifice the notion of order as it is understood for real numbers. It is significant to note in this connection that the rational numbers form an ordered subfield of the real numbers but it is not completely ordered. The complex number field has its own peculiar properties. It is something like a culmination in the process of extending the number system step by step starting with Peano's axioms for positive integers and preserving at each stage of the extension, the properties of the previous extension as far as it is possible in the nature of things. For instance, it is not possible to imbed the complex number system in a larger system preserving commutativity of multiplication along with the remaining properties. Again, if a complex valued function of a complex variable defined in a domain has a derivate defined on the model of real valued functions of a real variable, such a function possesses derivatives of all orders—a result not true for real functions. Such complex valued functions are called analytic and the above property endows the theory of analytic functions with wonderful beauty and richness. As already stated, the notion of analytic functions based on the model of complex valued functions have been extended in several directions, the most recent being the theory of Pseudo-analytic functions formulated with a view to solving certain types of partial differential equations similar to Laplace's equation which lead to analytic functions.

Another achievement of classical analysis is the precise formulation of the intuitive geometric notions of a curve and its length and a surface and its area. With space filling curves on one side and different but apparently natural definitions of areas leading to different values for the areas of even simple surfaces, this problem bristles with difficulties. Thanks to the work of a large number of mathematicians interested in this problem, a good measure of success has been achieved in making precise the notions of length and area and solving apparently knotty problems related to them. The two monographs, *Length and Area* by T. Rado and *Surface Area* by L. Cesari give a good account of the results achieved so far in this direction.

As already mentioned, abstract algebra dealing with operational structures and topology dealing with limit and continuity form two broad divisions of mathematical investigations in recent times. But it is not to be imagined that these have been kept in watertight compartments. As a matter of fact, most fruitful work has been done in what are called topological algebraic structures. I shall briefly indicate how such structures are defined and bring the talk to a close.

Just as the same collection of bricks can be used to build different types of houses, it is possible to define several topologies on the same set of elements. Let X be any set and T_1 and T_2 two topologies on X. We say that T_1 is weaker than T_2 if the family of open sets defining T_1 is contained in that defining T_2. The notion of weaker (or its opposite stronger) defines a partial order among the set of topologies on X and the latter forms a complete lattice with respect to this order, that is, given any family of topologies on X, there is a topology just weaker than and another just stronger than all the topologies of the family. Given any collection M of subsets of X, there is a weakest topology containing M among its open sets and this is called the topology generated by M. If X and Y are topological spaces, a function f on X into Y is said to be continuous if the counter image of every open set in Y is open in X. If X is any set, Y a topological space, f a map on X into Y, there is a weakest topology on X for which f is continuous, namely, the topology generated in X by the counter images of the open sets in Y. Similarly there is one such topology if several maps are given. If X and Y are two sets, $X \times Y$ the cartesian products consisting of all pairs (x, y), x and y varying over X and Y respectively, the maps $(x, y) \to x$ and $(x, y) \to y$ are respectively called the projections of the cartesian product into X and Y respectively. If X and Y are topological spaces, their topological product is their cartesian product endowed with the weakest topology for which the two projections are both continuous. Similarly, the topological product of any family of topological spaces can be defined.

Now let us examine how the mixed structures are defined. We have already defined an abelian group. If the postulate of commutativity of the binary operation is dropped we get the notion of a general abstract group. Let G be a group with the binary operation written multiplicatively. Suppose G is endowed with a topology such that the map $(x, y) \to xy^{-1}$ (y^{-1} denoting the inverse of y) of the topological product $G \times G$ into G is continuous. Then G is called a topological group. We say that the group operation is compatible with the topology. A topological field is one in which both group operations are compatible with the topology.

A vector space or linear space over a field F is a set of elements X forming an abelian group with respect to an operation denoted by addition and is closed with respect to multiplication by the elements of F, that is for $a \in F$ and $x \in X$, ax is defined as a unique element of X and this multiplication satisfies the usual commutative and associative laws and distributes addition. Now let F be a topological field. Suppose the vector space X over F is endowed with a topology for which it is a topological group and, in addition, the map $(a, x) \to ax$ of $F \times X$ into X is continuous. Then X is a topological vector space. Finally if X is a topological vector space in which the third operation of multiplication among its elements is defined and this multiplication is also continuous, we call X a topological algebra or a topological ring.

The most widely studied topological vector spaces and rings are the function spaces. On any set X we can define real or complex valued functions. With the usual definitions of sum, product and multiplication by real or complex numbers, the class of all such functions forms a ring. Various sub-classes of this class forming vector spaces or algebras over the real or complex number field with suitable compatible topologies converting them into topological vector spaces or algebras have been the subject of intense study by a large number of mathematicians in recent years. Several classical results in analysis now find their place as properties of such function spaces in their general setting. To cite only one instance, the theorem that the limit of a uniformly convergent sequence of continuous functions is continuous now appears as the completeness of a suitable metric space.

Finally, I shall indicate how the notion of an integral has been generalized to be of use in general situations. A family Γ of subsets of a given set can conveniently be called a Borel field if it contains the empty set and is closed with respect to complementation and countable unions. A non-negative real function $m(E)$ defined for sets $E \in \Gamma$ with the property that it is zero when E is the empty set and $m(E) = \Sigma m(E_n)$ when E is the union of the sequence (E_n) of disjoint sets of Γ, is called a measure on Γ. A real valued function f defined on X is called measurable with respect to Γ if the subset of points $x \in X$ where $f(x) > a$ belongs to Γ for every real number a. Now for any given measure on Γ, it is possible to define an integral for every non-negative measurable function in a natural way following a procedure similar to that followed in defining Riemann integrals of functions in an interval, the value of the integral being a non-negative real number or $+\infty$. Every measurable function f can be written as $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ are non-negative measurable functions. If one at least of the integrals of f^+ and f^- be finite, we can define the integral of f as the difference of the integrals of these two functions. This definition includes the ordinary Lebesgue integrals when the measure is the ordinary Lebesgue measure and the Lebesgue Stieltjes integrals when the measure is defined by a non-negative increasing function on the set of real numbers. This is one approach to the general notion of an integral. Within the last decade, another method of approach has been formulated by M. H. Stone and the Bourbaki school. To explain this briefly, consider a vector space V over the real number field whose elements are real valued functions defined on a set X. A partial order can be introduced on V by defining $f \geqslant g$ when $f(x) \geqslant g(x)$ for every $x \in X$. Suppose V contains along with f and g the functions max (f, g) and min (f, g). Then V is called a vector lattice of functions defined on X. A linear functional on V is a real valued-function F defined on V such that $F(af+bg) = aF(f)+bF(g)$, where a and b are real numbers and f and g belong to V. F is called non-negative if $F(f) \geqslant 0$ if $f \geqslant$ the identically zero function. Now any non-negative linear functional on V is called an integral defined for functions of V and for $f \in V$, $F(f)$ is called the value of the integral of f. Usually the set X and the vector lattice V are chosen with special properties and by general limiting processes, the definition of an integral is extended to wider class of functions. In all cases of importance, the two methods of approach lead to equivalent definitions of integrals.

In the previous paragraphs, I have attempted, though in a rather sketchy manner, to present to you some of the fundamental notions in classical analysis and how they have formed the models for the later developments in topology and topological algebraic structures and in theories of measure and integration. In India, scholars working in abstract algebra and topology and their applications to other fields are very small indeed. Even in the subjects in which Indian scholars are working, the number is very small and not at all anywhere near the ideal as obtaining in other mathematically advanced countries. It behoves lovers of mathematics interested in its development in this country to take all steps to encourage research in as many of these modern topics as possible. Those in charge of guiding research can do this as individuals, taking into account the aptitudes of the pupils. The Indian Mathematical Society as an all-India body should take steps to find ways and means of discovering and encouraging mathematical talent and prevent their being side-tracked. I hope that constructive suggestions will be forthcoming at the symposium on " Research and Mathematical development in India ".

TOPOLOGICAL ALGEBRA*

By V. S. KRISHNAN

IN this address, we shall be considering examples of the way that the two fundamental branches, Algebra and Topology, have been intermingling and enriching each other. While Algebra considers structures defined by means of finitary operations on a basic set, Topology considers the notions of convergence in spaces and of continuity of functions from one space to another. While there are structures which have both types of considerations, it is the purpose of this address to show how methods of algebra help in certain questions of topology and vice-versa.

Algebraic methods in topology. Among the early objects of study under topology were the curves and surfaces in Euclidean spaces. Algebra entered in Topology through the method of associating the 'Homology groups' for these surfaces by means of triangulations. It was a major result to prove that these groups were topological invariants. The numerical invariants associated with the groups had already been encountered in the Betti numbers. And for two dimensional surfaces in Euclidean spaces these groups provide a complete system of invariants, determining the surface up to homeomorphism (when the additional information is given whether the surface is orientable or not). More elaborate, and different homology and cohomology theories have been discussed and in the recent book by Cartan and Eilenberg a 'Homological Algebra' has been developed to give an algebraic background for these varied theories. Not only single groups, but families of groups or rings indexed by the integers and with homomorphisms between them corresponding to the boundary operator are treated.

At the other end of the scale, very general spaces, like the T-space of Kuratowski with a closure operator for subsets, can be treated by

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methods of lattice algebra. For a T-space S is fully determined by the structure of the (complete, atomic) Boolean algebra $B(S)$ of all subsets of S with the extra unary operation of closure defined on it. Denoting the closure of X by $\bar{X}$, this operation satisfies the four axioms (of Kuratowski) K_1 : X is contained in $\bar{X}$, for any X in $B(S)$, K_2 : $\overline{X u Y}$ is contained in $\bar{X} u \bar{Y}$, for any X, Y of $B(S)$; K_3 : $\bar{\bar{X}}$ is contained in $\bar{X}$, for any X of $B(S)$; and K_4 : the null set N, which is the least element of $B(S)$, has its closure $\bar{N}$ equal to N. Calling a Boolean algebra closed for a closure operator satisfying these four conditions a closure algebra, Mckinsey and Tarski show in their paper that such an algebra can be treated, up to isomorphism, as a subalgebra of the closure algebra determined by a T-space in the manner indicated above. They also characterize a universal algebra for finite closure algebras. In his book on '*Analytic topology*' Nöbeling also begins the study of partly ordered sets or lattices with a closure operator as above, but not necessarily satisfying the additivity condition K_2. Similarly he introduces the uniform structure by considering lattices or Boolean algebras with an indexed family of mappings U_i corresponding to taking the U_i neighbourhoods of subsets of a uniform space with the U_i as surroundings of the diagonal.

Topological methods in algebra. For a topological group a base of neighbourhoods for the neutral element, or as it is also called, a nuclear base or base of nuclei, determines the topology completely. If N_i, i in I, is such a base two uniform structures are determined for the group G : the left uniform structure is specified by the base of surroundings U_i, i in I, where, for any x, y, in C, (x, y) is in U_i or y is in $U_i(x)$ if and only if y is in $x.N_i$ (the left translate of N_i by x); the right uniform structure is similarly defined in terms of right translates. Both these uniformities are compatible with the topology of the topological group.

The situation becomes simpler, and in a sense more algebraic, if the nuclear base is a system of (normal) subgroups. The associated surroundings in the left or right uniform structure viewed as binary relations on G now become congruence relations on G, and their intersection is the indentity relation if the space is separated. Different aspects of this situation have been treated in papers by W. Hämisch and by H. Schöneborn.

In the paper of Hämisch, the general structure of algebras with congruence uniformities is considered. The process of completion with respect to such a uniformity is related to the structure of a certain product algebra whose terms are quotient structures of the original by the given congruences. Other types of products and lattice products are also defined and studied.

Schöneborn is interested in topological modules with topological operator rings of a special sort. Observing that the p-adic integers can be obtained from the module of integers by completion with respect to the congruence uniformity determined by the system of congruences modulo the ideals generated by the powers of p, the ring of all n-adic numbers is defined as the completion of the ring of integers for the congruence uniformity determined by the congruences modulo n, for all n; this last ring is the direct (Tychonoff) sum of the various p-adic number rings. Closed subrings of this ring are called n-adic rings if they have a unit; they are direct sums of selections of p-adic rings (up to bicontinuous isomorphisms, of course). The topological module, M, considered is to have a topological ring R as operator domain, with the additional condition, besides the usual algebraic ones, that for any element x of M and any subset S of R, $(\overline{x})$ is Rx, and $\bar{S}x$ is $\overline{Sx}$ where (x) is the submodule generated from x, and the closures are to be properly taken in M or R as the case may be. If further, the additive module of the operator ring R has such a (left) operator ring, then M is said to have R as a natural ring of operators. The principal results of M. Schöneborn are regarding modules with n-adic rings as natural operator domains. The general case is reduced then to that of certain 'primary' modules, and these are studied in detail.

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CLOSURE THEOREMS*

By M. V. SUBBA RAO

1. One of the important methods of approach to closure theorems in normed vector spaces E, i.e. theorems on the determination of dense linear subsets of E, is by an application of the Hahn Banach theorem, which, for our purposes, can be stated thus : If E is a normed vector space (briefly *n. v.* space) of which S is a linear subspace, and if x is an element of E, disjoint with $\bar{s}$ (closure of S) there exists a (non-null) linear continuous functional f orthogonal to S and $f(x) = 1$. Actually we know [1] that there exists such an f with the additional property $\|f\| = 1/d$, where d is the distance of x from $S = \underset{y \in S}{\text{glb}} \|x - y\|$.

1.1. We know that this result holds for a wider class of spaces, viz. locally convex topological vector spaces; such spaces, it is well known, are completely specified by a family of semi-norms, a semi-norm being the usual norm except that $\|x\| = 0$ need not necessarily imply that x is the null element θ. From the Hahn Banach theorem, we get at once the following result, which is basic for the construction of closure theorems. Throughout the paper, unless otherwise stated, E stands for a normed vector space, β any linear continuous functional of E, and E^* the dual space, i.e. the linear space of all such β's.

1.2. Fundamental Lemma. *Let $\alpha_1, \alpha_2, \dots,$ be a sequence of elements of a given space E. Then the linear manifold S spanned by the sequence (α_i) is dense in E, if and only if, every β orthogonal to each α_i, reduces to the null functional. In such a case, every element x of S can be expressed as the limit of finite linear combinations of α_i's.*

2. Using this approach, Mandelbrojt [6] obtained a number of closure theorems for the spaces L_p, and Ganapathy Iyer for the

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space of integral functions. In this paper I obtain some interesting closure theorems for the spaces l_p $(p \geqslant 1)$, for the space of power series satisfying certain properties, and for inner limiting spaces associated with these spaces.

3. The space l_p $(p \geqslant 1)$. This space, as is well known, consists of all sequences $\alpha = (a_1, a_2, \ldots, a_i, \ldots)$ such that $\Sigma\, |a_n|^p < \infty$. This is a Banach space under the norm $\|\alpha\| = (\Sigma\, |a_n|^p)^{1/p}$. Any linear continuous functional β of l_p is defined uniquely by a sequence $\beta = (b_1, b_2, \ldots, b_i, \ldots)$ such that $\sum_{i=1}^{\infty} |b_i|^q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, so that $\beta(\alpha) = \sum_{i=1}^{\infty} b_i\, a_i$. We will prove the following

THEOREM 1. *Let $\alpha = (a_1, a_2, \ldots, a_i, \ldots)$ be an element of l_p such that no a_i vanishes for any i. Let Z_n be a sequence of complex numbers, an infinity of them being distinct, and such that $\Sigma\, [1 - |z_n|]$ is divergent. Let $\alpha_n = (a_1 z_n,\ a_2 z_n^2 \ldots,\ a_i z_n^i, \ldots)$, $n = 1, 2, \ldots$. Then the linear manifold spanned by α_n's will be dense in l_p.*

PROOF. Let $\beta = (b_1, b_2, \ldots, b_i, \ldots)$ be a linear continuous functional of l_p which is orthogonal to α_n, for all n, so that,

$$\beta(\alpha_n) = \sum_{i=1}^{\infty} b_i\, a_i\, z_n^i = 0, \quad n = 1, 2, \ldots. \tag{3.1}$$

Consider the function $f(z)$ defined by

$$f(z) = \Sigma\, b_i\, a_i\, z_n^i.$$

This has a radius of convergence $\geqslant 1$ since $\Sigma\, |b_i\, a_i| < \infty$. Also, from (3.1) we see that z_n, $n = 1, 2, \ldots$ are all zeros of $f(z)$ all lying in $|z| \leqslant 1$. It follows now that $f(z) \equiv 0$ using Blasckhe's theorem that if a power series is bounded in the unit circle and has zeros (z_n), $n = 1, 2, \ldots$ in that circle, then $\Sigma\, (1 - |z_n|)$ must be convergent, or else the series is identically zero.

Hence we get here $b_i\, a_i = 0$, $i = 1, 2, \ldots$. But no a_i vanishes by hypothesis. Hence $b_i = 0$ for all i and β is the null functional, establishing the theorem.

Another closure theorem for the same space is

THEOREM 2. *Let* $\alpha = (a_1, \ldots a_i, \ldots)$ *be an element of* l_p, $p > 1$ *such that* $a_i \neq 0$ *for any* i. *Let* $\alpha_n = (a_1^n\, a_2^n, \ldots, a_i^n, \ldots)$, $n = 1, 2, \ldots$. *Then the linear manifold spanned by* (α_n)*'s is the whole space* l_p.

The proof depends upon

LEMMA 2. *If* $f(z) = \Sigma\, b_i\, e\, a^{a_i z}$ *is an integral function and vanishes identically, then each* b_i *is zero.*

A proof of this can be found in [4], [5].

PROOF OF THE THEOREM. Let $\beta = (b_1, b_2, \ldots, b_i \ldots)$ be a linear continuous functional of l_p, orthogonal to each α_n, so that

$$B(\alpha_n) = \sum_{1=i}^{\infty} b_i\, a_i^n = 0, \quad n = 1, 2, \ldots. \tag{3.2}$$

Now the function $f(z)$ defined by $f(z) = \sum_{i=1}^{\infty} b_i\, a_i\, e^{a_i z}$ is easily seen to be an integral function, since $\Sigma\,|\, b_i\, a_i \,| < \infty$ and $a_i \to 0$ as $i \to \infty$. Also from (3.2) we see that $f(z)$ and all its derivatives of all orders vanish at $z = 0$, and hence $f(z) \equiv 0$. Lemma 2 now gives $b_i a_i = 0$ for each i, or $b_i = 0$, since $a_i \neq 0$. Hence β is the null functional and the theorem follows.

4. We will next take the space $\Gamma(R)$ of power series $\alpha(z)$, $\alpha(z) = \Sigma\, a_n z_n$, such that $|\,\Sigma\,|\, a_n\, R^n < \infty$, R being a fixed positive number. This is easily seen to be a Banach space [3] with norm

$$N(\alpha\,;\, R) = \Sigma\,|\, a_n \,|\, R^n.$$

The dual space $\Gamma^*(R)$ consists of functionals $\beta = (b_1, b_2, \ldots)$ such that $|\, c_i/R^i \,| < \infty$ and $\beta(\alpha) = \Sigma\, b_i\, a_i$.

We can now prove two closure theorems for this space which are analogous to Theorems 1 and 2. It may be noted that convergence in $\Gamma(R)$ is uniform convergence over the circle $|\, z \,| = R$.

THEOREM 3. *Let* $\alpha(z) = \Sigma\, a_n z^n$ *be an element of* $\Gamma(R)$ *with no* a_n *being zero, and let* r *be its radius of convergence,* $r > R$. *Let*

$z_1, z_2, \ldots, z_n, \ldots,$ *be a sequence of complex numbers such that* $|z_n| < r/R$, *and let*

$$\text{(i) } \Sigma\,|a_n|\,r^n < \infty; \quad \text{(ii) } \Sigma\,|-1-(r/R)\,|z_n|| = \infty. \tag{4.1}$$

Then the closure of the linear manifold spanned by $L_n(z)$, where $\alpha_n(z) = \alpha(zz_n)$ is the whole space $\Gamma(R)$.

The proof of this is similar to that of Theorem 1 and is omitted.

It may be noted that the hypothesis (4.1)(ii) includes as a special case the hypothesis that the z_n's have a limit in $|z| = r/R$. Also, if this is used to replace (4.2)(ii), the theorem holds even without (4.1) (i).

ILLUSTRATION: Let us take $\alpha(z) = 1 + z/1^2 + z^2/2^2 + \ldots$ and $R = 1$. Now $\alpha(z)$ is bounded in $|z| \leqslant 1$. Take $z_n = 1 - 1/n$ so that (4.1) is satisfied. Thus the theorem gives the result that every power series $\Sigma a_n z^n$ such that Σa_n is convergent is the uniform limit over the unit circle of finite linear combinations of $\alpha(z - z_n)$, i.e.

$$1 + \sum_{p=1}^{\infty} z^p (1 - 1/n)^p / p^2, \quad n = 1, 2, \ldots.$$

Another closure theorem in a different direction is

THEOREM 4. *Let* $\alpha(z) = \Sigma\, a_i\, z^i$, $(a_i \neq 0,\ i = 0, 1, 2, \ldots)$ *have the radius of convergence* $r > R$. *Let* Λ *denote an infinite set of complex numbers* λ *such that* $|\lambda| \leqslant r - R$ *and* S_1 *the linear manifold spanned by the set of functions*

$$\alpha(z + \lambda), \quad \lambda \in \Lambda.$$

Let $\alpha^{(p)}$ *denote the p-th order derivative of* $\alpha(z)$ *and* S_2 *the linear manifold spanned by* $\alpha^p(z)$, $p = 0, 1, 2, \ldots$. *Then* $S_1 = S_2$.

PROOF. We have

$$\alpha^{(p)}(z) = \sum_{n=0}^{\infty} (n+1)(n+2)\ldots(n+p)\, a_{p+n}\, z^n.$$

If $f = (c_1, c_2, \ldots) \in \Gamma^*(R)$, and $f[\alpha^{(p)}(z)] = 0$, $p = 0, 1, 2, \ldots$ we get

$$A_p = 0, \quad p = 0, 1, 2, \ldots, \tag{4.2}$$

where

$$A_p^* = \sum_{n=1}^{\infty} (n+1)(n+2)\dots(n+p)\, c_n\, a_{p+n}.$$

Next, if $f[\alpha(z+\lambda)] = 0$, we have, using Taylor's theorem,

$$f[\alpha(\lambda) + z\,\alpha^1(\lambda) + \dots + \frac{z^p}{p!}\alpha^p(\lambda) + \dots] = 0$$

or

$$\sum_{p=0}^{\infty} c_p\, \alpha^{(p)}(\lambda)/p! = 0. \tag{4.3}$$

Using the value of $\alpha^{(p)}(z)$, (4.3) gives

$$\sum_{p=0}^{\infty} \frac{c_p}{p!} \sum_{n=0}^{\infty} (n+1)(n+2)\dots(n+p)\, a_{p+n}\, \lambda^n = 0,$$

i.e. $\sum_{p=0}^{\infty} A_p \lambda^p / p! = 0$ (on rearranging the double series which can easily be justified). This, by supposition, being true for all $\lambda \in \Lambda$, an infinite set, it follows that for every p, $A_p = 0$; just as before in (4.2). Hence the theorem follows.

5. We will now take up what may be called inner limiting spaces (what Bourbaki calls projective limit spaces) connected with l_p and $\Gamma(R)$.

Let $E_1 \supset E_2 \supset \dots \supset E_i \supset \dots$ be an infinite sequence of linear manifolds and N_i norm imposed on E_i, $i = 1, 2, \dots$ such that

$$N_1(x) \leqslant N_2(x) \leqslant \dots$$

for all $x \in E = \prod_i E_i$. Let D be a linear manifold belonging to each E_i. Let $[E_i, N_i]$ denote the topological space got by imposing on E_i the norm N_i and similarly (E, T) where T is the lattice product topology in the sense of [7] of the N_i topologies. If $\geqslant$ denotes 'topologically stronger than' we have evidently

$$[E, N_1] \geqslant [E, N_2] \geqslant \dots \geqslant [E, T].$$

We know that $[E, T]$ is matrizable by the metric

$$d(x, y) = \sum \frac{1}{2^i} \frac{N_i(x-y)}{1+N_i(x-y)}. \tag{5.1}$$

The space $[E, T]$ may be called the inner limiting space of the spaces $[E_i, N_i]$. We note the following results which are well known, or, easily proved.

If each $[E_i, N_i]$ is complete space, so also is $[E, T]$. (5.2)

If $\bar{D} = [E_i, N_i]$ for each i, D being linear subset of E, then $D = [E, T]$. (5.3)

This result holds even if norms are replaced by metrics.

In (5.3), $\bar{D}$, wherever it occurs, is the closure of D in the topology of the space on the right side.

The Hahn Banach theorem in § 1 holds in $[E, T]$, for $[E, T]$ is a locally convex space. (5.4)

But we can prove the following more precise result.

THEOREM 5. *If $x_0 \in E$ and $S \subset E$, a linear subset of E and $d(x, s) = \delta > 0$, then there exists an integer $i_0 > 0$ and a linear continuous functional β of $[E, T]$ such that*

$$\beta(x_0) = 1, \tag{5.5}$$

$$\beta(x) = 0 \textit{ for all } x \textit{ in } S, \tag{5.6}$$

$$|\beta(x)| \leqslant \frac{2-\delta}{\delta} N_{i_0}(x), \textit{ for all } x \textit{ in } S. \tag{5.7}$$

The proof depends upon

LEMMA 3. *If $d(x_0, S) = \delta > 0$, then for any $y \in S$,*

$N_i(x, y) \geqslant \delta_1 \left(= \frac{\delta}{2-\delta}\right)$ for all $i \geqslant i_0$, where i_0 is defermined by

$$\frac{1}{2^{i0}} < \frac{\delta}{2}. \tag{5.8}$$

For

$$d(x_0, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{N_i(x_0-y)}{1+N_i(x_0-y)} = \sum_{i=1}^{i_0} + \sum_{i=i_0+1}^{\infty}.$$

Hence if i_0 is chosen as in (5.8) we have $\sum_{i=i_0+1}^{\infty} < \frac{\delta}{2}$, so that

$$\sum_{i=1}^{i_0} > \frac{\delta}{2};$$

or since $N_i(x)$ increases with i,

$$\frac{N_{i_0}(y-x_0)}{1+N_{i_0}(y-x_0)} \sum_{i=1}^{i_0} \frac{1}{2^i} > \frac{\delta}{2},$$

so that

$$N_{i_0}(y-x_0) > \frac{\delta}{2-\delta} = \delta_1$$

and the lemma follows.

To prove Theorem 5, we have, by Lemma 3,

$$N_i(y-x_0) > \frac{\delta}{2-\delta}, \quad y \in S.$$

Hence by a well-known property of normed spaces, there is a $\beta \in (E, N_{i_0})^*$ having the properties (5.5)–(5.7). Finally we have

$$[E, T]^* = \sum_i [E, N_i]^*. \tag{5.9}$$

This result is by no means trivial for, while $\beta \in [E, N_i]^*$, for any i implies, obviously, $\beta \in (E, T]^*$ so that

$$\sum_i (E, N_i)^* \subset (E, T)^*.$$

Also (5.9) implies that the reverse relation is also true, which is a less obvious result and requires the use of the property that the topologies (E, N_i) steadily become weaker as i increases. The result (5.9) can be easily extended to the case when the topologies defined by the norms N_i's are replaced by any comparable family of topologies, i. e. a family of every two members of which one is weaker than the other.

Another remark regarding (5.9) can be made here. While for every given functional on (E_i, N_i) there corresponds a unique functional on (E, T) (the induced functional), the same functional on (E, T) may be obtained from more than one functional on (E_i, N_i), since there exists non-zero functionals on (E_i, N_i) which vanish over E. Thus we can state (5.9) in the following equivalent form. Let F denote the set of functionals f belonging to $(E_i, N_i)^*$ for any i and vanishing over E. Then $(E, T)^*$ is algebraically isomorphic to the factor space $\Sigma (E_i, N_i)^*/F$.

We will in conclusion give two illustrations of these inner limiting spaces.

6. Let us first consider the inner limiting spaces associated with the spaces l_p, $p \geqslant 1$.

Let S_p denote the linear manifold of sequences $\alpha = (a_1, a_2, \ldots)$ for which $\Sigma |a_i|^p < \infty$.

Let $N_p(\alpha)$ denote the norm of α given by

$$N_p(\alpha) = \left(\sum |a_i|^p \right)^{1/p}.$$

Consider the topological space obtained by imposing on S_p the norm N_p, i.e. space $[S_p, N_p]$, in the notation of (5). This is the same as the space usually denoted by l_p. We have obviously $S_i \subset S_j$ if $i < j$. Let $S_{p+0} = \prod_{i>p} S_i$ so that S_{p+0} consists of all sequences $\alpha = (a_i)$ such that $\sum_{i=1}^{\infty} |a_i|^{p+\epsilon} < \infty$ for every $\epsilon > 0$. Also using the notation of (5), we have, if $i < j$,

$$N_i > N_j,$$

$$(S_{p+0}; N_i) \leqslant (S_{p+0}; N_j).$$

Let T be the lattice product topology of the normed topologies defined by N_i on S_{p+0}, $i > p$. Then we get the inner limiting space associated with l_p spaces, viz. (S_{p+0}, T). Using the results of (5) we see that (S_{p+0}, T) is metrizable with the metric

$$d(x, y) = d(x-y, 0) = \sum \frac{1}{2^i} \frac{N_{p+\epsilon_i}(x-y)}{1 + N_{p+\epsilon_i}(x-y)},$$

where ϵ_i is any sequence of positive numbers tending to zero.

It is a complete space with this metric. The linear continuous functionals of this space consist of all sequences (b_i) for which

$$\sum |b_i|^{q-\epsilon} < \infty,$$

for *some* $\epsilon > 0$, where $1/p + 1/q = 1$.

Since the functionals are determined, closure theorems analogous to those for l_p can be obtained for this space also. For example, Theorem 1 holds here also provided that the element α there is assumed to belong to S_{p+0} instead of to S_p.

7. Lastly we can consider the inner limiting space $\Gamma(R-0)$ associated with the spaces $\Gamma(r)$, $0 < r < R$, introduced in § 4; it is evident that the sets $\Gamma(r)$, viz. the set of all power series $\alpha(z) = \Sigma\, a_n z^n$ for which $\Sigma\, |a_n|\, r^n < \infty$ decreases as r increases, while the norm $N(\alpha\,; r) = \Sigma\, |a_n|\, r^n$ increases with r. We have thus a situation analogous to that described in § 5. The inner limiting *set* $\Gamma(R-0)$ of the *set* $\Gamma(r)$ consists of all power series $\alpha(z) = \Sigma\, a_n z^n$ for which $\Sigma\, |a_n|\, r^n < \infty$ for every $r < R$. On this set $\Gamma(R-0)$, there are the various topologies defined by the norms $N(\alpha\,; r)$, $0 < r < R$, and their lattice product topology T is metrizable with the metric

$$\sum \frac{1}{2^i} \frac{N(\alpha\,; r-\epsilon_i)}{1+N(\alpha\,; r-\epsilon_i)},$$

where ϵ_i is any sequence of positive numbers tending to zero. It is of interest to note that this space can be proved to be non-normable (proof is to appear shortly elsewhere), and in fact that on the set $\Gamma(R-0)$ there can exist no normable topology weaker than the family of normed topologies defined by $N(L\,; r)$, $0 < r < R$.

Every functional β of $\Gamma(R-0)$ is of the form

$$\beta(\alpha) = \sum b_n a_n,$$

where $\alpha = \Sigma\, a_n z^n$ and $\dfrac{|b_n|}{r^n} < M < \infty$, for some $r < R$ and finite M.

It may also be noted that non-convergence in $\Gamma(R-0)$ is uniform convergence over every circle contained in $|z| = R$.

The closure Theorems 4 holds for this space also, while Theorem 3 holds if the hypothesis (4.1) is replaced by "the z_n's have a limit point in $|z| = r/R$".

If however (4.1) is to be retained as it is, some further (rather complicated) assumptions seem to be necessary. I conclude with mentioning the following problem for solution: If no a_i is zero, is it true that the linear manifold spanned by the sequence of elements $x_1, x_2, \ldots$ is dense in l_p, $p > 1$, where

$$x_1 = (a_1, a_2, \ldots) \in l\,p\,; \quad x_2 = (0, a_1, a_2, \ldots); x_3 = (0, 0, a_1, a_2, \ldots)?$$

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SYMPOSIUM ON FUNCTIONAL ANALYSIS

Chairman : Professor V. GANAPATHY IYER

THE term "*functional analysis*" has come to denote that discipline in mathematics in which the problems in classical analysis are sought to be generalized to situations presented by abstract algebra and topology and to the review of the problems in classical analysis from the general standpoint.

Dr. M. Venkataraman will lead the symposium by giving a survey of functions, topological vector spaces and function spaces in general. He will be followed by Dr. U. N. Singh who will give an account of some of the important special function spaces and closure theorems. Sri M. R. Parameswaran will speak on the applications of the general ideas of functional analysis to the problems in summability methods, giving a review of the recent work of K. Zeller in this field and his own efforts in this direction. Sri M. S. Ramanujan will give an account of moment problems in general function spaces giving an account of the work of Lorentz and others in this field. I shall give a few general remarks indicating the orientation to their talk.

Till a decade ago, the main problems investigated by scholars working in the theory of divergent series were concerned with special methods of summability, the limits to which sequences transformed by them converged and additional conditions on the sequences which along with their summability by a given method implied convergence (known collectively as Tauberian theorems). Rarely attempts were made to consider the class of sequences summable by a given method as a whole and the class of convergent sequences into which they were transformed. Similarly the summability methods were usually considered in their isolation and only isolated results on the inter-relations between different methods were known. Only recently, systematic attempts have been made to study the family of sequences summed by a process in relation to the range of the transformed sequences in the space of convergent sequences. This aspect has been systematically investigated recently by K. Zeller, and Sri M. R. Parameswaran will give an account of these investigations. The general idea is to convert the class of sequences summed by a process (known as the field of summability) into a locally convex topological vector space by using a suitable family of semi-norms so that the space obtained becomes a complete space in the topology thus obtained. Another aspect which will be presented by Sri Parameswaran will be the consideration of the class of all summability methods as a Banach algebra. Out of this characterization several consequences follow. For instance, those methods which are regular in this algebra (that is, have an inverse) cannot convert any non-convergent sequence into a convergent sequence. Such elements form an open set in the algebra, so that effective summability methods (that is, whose field of summability is wider than the space of convergent sequences) form a closed set. Again the family of all those processes summing a specified family of sequences form a left ideal in the algebra. This point of view enables one to study permutable methods of summability forming sub-algebras—for instance, the Hausdorff methods constitute one such family. I have said just enough as an introduction to the talk by the last two participants in the symposium.

ABSTRACT STRUCTURES IN THE THEORY OF FUNCTIONS

By M. VENKATARAMAN

1. Function spaces. The theory of fuctions can be said to have been properly founded only when the distinction was clearly made between a function and a formula or expression specifying a function. A function $f(x)$ is a correspondence or mapping which associates with every number x (real or complex) another number y, denoted by $f(x)$. It is obvious that there are numerous functions, other than those given by simple expressions like $y = a_0 + a_1 x + \ldots + a_n x^n$ or $f(x) = (a_0 + a_1 x + \ldots + a_m x^m)/(b_0 + b_1 x + \ldots + b_n x^n)$. The central problem of the theory of functions now arises, namely to be able to construct newer formula or expressions which would represent various wider classes of functions, specified otherwise.

The simplest and the most widely known representations are by means of infinite series, e.g. $f(x) = \sum_1^\infty a_n x^n$. For this expression to have any meaning we must have some topological structure by which we can say that $\sum_1^\infty a_n x^n$ converges (or not) to the function $f(x)$ as $x \rightarrow \infty$. Various classes of functions and topologies therein have been studied, typical of which is the space $L_p(S)$ of functions $f(x)$ which are measureable in a set S and whose pth power is Lebesgue-integrable. We say that $f_n \rightarrow f$ in L_p if the distance between them $d(f_n, f) = [\int |f_n^p(x) - f^p(x)|\, da]^{1/p}$ tends to 0 as $n \rightarrow \infty$. It is a classical result that a sequence of functions f_n in L_p converges in L_p to a function $f(x)$ if and only if f_n is a Cauchy sequence, i.e. $d(f_m, f_n) \rightarrow 0$ with $1/m + 1/n$. The common abstract structure of these classes of functions is the Banach space. It is defined to be a class R of elements which is (1) a commutative group with respect to addition, i.e. addition and subtraction (satisfying the usual rules) are possible, (2) allows multiplication by numbers (real or complex as the case may be) satisfying the usual associative and distributive laws, and (3) in which every element x has a magnitude $\|x\|$ such that $\|x+y\| \leqslant \|x\|+\|y\|$; $\|\lambda x\| = |\lambda|.\|x\|$; $\|x\| > 0$ if $x \neq 0$. Further (4) the distance function $d(f, g) = \|f-g\|$ is such that every Cauchy sequence of elements f_n is convergent, i.e. whenever $d(f_n, f_m) \to 0$ as $m, n \to \infty$, there exists an f such that $d(f_n, f) \to 0$.

The spaces $L_p (p \geqslant 1)$ are all Banach spaces. Particularly interesting among the space L_p is the space L_2, wherein given any two elements f, g, we can define a number (f, g) called the scalar product, corresponding to the dot product of vectors in Euclidean 3 space. The interest in L_2 space stems out of the fact that it is a very natural generalization of Euclidean spaces (and that transformations in such spaces are of use in a mathematical formulation of quantum mechanics). We can talk about orthogonal vectors, and the Pythogoras theorem is true. Only instead of 3 or 'n' being the maximum number of mutually perpendicular unit vectors we may have an infinity of mutual orthogonal directions or vectors. A classical result is that the Hermite functions $H_n(x)$ constitute a complete set of unit, orthogonal vectors in $L_2(-\infty, \infty)$ such that every element $f(x)$ can be expanded in the form $\Sigma\, a_n H_n(x)$, where $a_n H_n(x)$ is the projection of $f(x)$ in the direction $H_n(x)$, i.e. $a_n = (f, H_n)$. Here $\Sigma\, a_n H_n(x)$ means the limit in the L_2 sense of the partial sums of the infinite series. Similarly every function $f(x)$ in $L_2(-\pi, \pi)$ can be expressed in the form $\Sigma\, a_n \dfrac{e^{inx}}{\sqrt{(2\pi)}}$, where $a_n = \int_{-\pi}^{\pi} f(x) \dfrac{e^{-inx}}{\sqrt{(2\pi)}}\, dx$ and the series converges in the L_2 sense. If these results indicate that every function of the class $L_2(-\infty, \infty)$ can be expressed as the sum of multiples of the functions $H_n(x)$, and that the functions of the class $L_2(-\pi, \pi)$ can be expressed as the sum of multiples of the functions e^{inx}, there is a generalization that every function $f(x)$ of the class $L_2(-\infty, \infty)$ can be expressed as the sum of multiples of the continuous infinity of functions e^{itx} $(\infty < t < \infty)$ in the form $f(x) = \int_{-\infty}^{\infty} \phi(t) \dfrac{e^{itx}}{\sqrt{(2\pi)}}\, dt$ the integral again being considered as the L_2 limit of $\int_{-n}^{n} \phi(t) \dfrac{e^{itx}}{\sqrt{(2\pi)}}\, dt$. Here $\phi(t)$ will be equal to $\int_{-\infty}^{\infty} f(x) \dfrac{e^{-itx}}{\sqrt{(2\pi)}}\, dx$, this integral again being an L_2 limit. It is to be noted that we have a continuous infinity of Fourier coefficients $\phi(t)$, which is called the Fourier-transform of $f(x)$. This transformation of $f(x)$ to $\phi(t)$ is a 'rotation' in Hilbert space, in as much as $\|f\| = \|\phi\|$.

2. Analysis in Banach spaces. In view of these results, naturally the study of Banach spaces received a large impetus. And one of the problems which has been tackled is to study possible representations of a mapping $f(x)$ from one Banach space X to another Banach space Y. The natural attempt has been to get generalizations of the best-developed* part of function theory—that of uniform analytic functions of complex variable. Here we know that the following classes of functions are equivalent : (1) those which can be expanded as power-series $\Sigma a_n x^n$; (2) those which have a differential coefficient at all points ; (3) those whose integral taken around any closed contour is zero. In attempting a generalization of such a theory to functions defined in a general Banach space, a certain amount of ingenuity is needed in defining the analogues of the notions of differentiation, polynomial, integration, etc.

A function $f(x)$ from B_1 to B_2 is defined to be a polynomial of degree 'n' if it satisfies the condition $f(x + hx_0) = f(x) + hP_1(x, x_0) + \dots + h^n P_n(x, x_0)$ are suitable functions depending on x and x_0 only, and with values in B_2, h being an arbitrary scalar. If further $f(hx) = h^n.f(x)$ we say that $f(x)$ is a power, or a homogeneous polynomial. An alternative definition which can be proved tó be equivalent to this is as follows : Defining the difference $\delta_x f(z)$ of this polynomial $f(z)$ to be $f(z + x) - f(z)$ and $\delta_{x_1 \dots x_m} f(z) = \delta_{x_m}(\delta_{x_1} \dots x_{m-1} f(z))$ it is easily verified that $\delta_{x_1 \dots x_n} f(z) = L(x_1 \dots x_n)$ is independent of x, is symmetric in $x_1 \dots x_n$ and is linear in each of the arguments x_n. Also $L(x \dots x) = f(x)$. This suggests the alternative definition of a polynomial $f(x)$ of degree 'n' as the value of an n-linear symmetric function $L(x_1 \dots x_n)$ when $x_1 = x_2 = \dots = x_n = x$.

* A more detailed exposition of allied developments are being given by Dr. Singh, Dr. Subba Rao and Sri Ramanujan.

It may be mentioned that we cannot assert that a polynomial is always a continuous function.

Next, we pass on to the definition of the notion of the differential coefficient of a function $f(x)$ from B_1 to B_2. We say that $f(x)$ is G-differentiable if $\frac{f(x+zh)-f(x)}{z}$ tends to a definite limit $\delta f(x, h)$ as the scalar z tends to zero, for all x and h in B_1. If this $\delta f(x, h)$ is also continuous in h not necessarily in x, we say that $f(x)$ is F-differentiable. In this case, it follows that $\lim \frac{1}{\|h\|}(f(x+h)-f(h)-\delta f(x, h))=0$ when $\|h\| \to 0$ and that $\delta f(x, h)$ is linear in h. It is to be remarked how $\delta f(x, h)$ corresponds to our normal conception of a differential rather than the differential co-efficient.

It is satisfying to find that a power is always G-differentiable and will be F-differentiable if it is continuous. We can prove that if B_1, B_2 are complex Banach spaces, $f(x)$ is G-differentiable if and only if $f(x+zh)$ for every x and h is an analytic function of the complex scalar z. The limit of a uniformly convergent sequence of G-differentiable functions will be G-differentiable, and its power-series is the limit of the power-series of the sequence.

Calling a function analytic if it is locally bounded and G-differentiable (and hence continuous and so F-differentiable) we have the following analogues of classical results. An analytic function vanishing in a sphere, however small it may be, is identically zero. The limit of a sequence of convergent and uniformly bounded analytic functions is again analytic. The sum of an F-power-series is analytic and conversely. Every analytic function can be expressed as the sum of an F-power-series. I have not come across any discussion relating the Cauchy integral and the analyticity, though the ordinary theory has valid analogues when B_1 is the usual complex number space. Analysis in Banach algebras has a rich theory quite analogous to that of functions of a complex variable (see E. Hille [1]).

3. **Integration.** Next, we shall pass on to the concept of the integral. Here, we might add that the theory of L_p and L_2 spaces holds only when we use the notion of the Lebesgue-integral and

is not so satisfactory with the notion of the Riemann-integral. This naturally focuses attention on a closer study of the Lebesgue integral and the Lebesgue-Stieltjes integral. We shall content ourselves here by mentioning that each such integration process (I) on the real axis associates with every continuous function $f(x)$ vanishing outside some bounded set, a number $\int f(x)\ (I)$ and this association is a linear mapping of the class of such functions and that it carries a sequence f_n converging uniformly on compact sets into a convergent sequence I_n of numbers. A converse of this is also valid, and has led to the study, by Schwarz [7] of the linear functionals over other allied function spaces, in the theory of distributions.

4. Locally convex spaces. Banach spaces are special instances of locally convex vector spaces. These are vector spaces together with a topology with respect to which the vector operations are continuous and which have a basis of convex neighbourhoods at the origin. Tychanoff [9] proved that a continuous image of any bicompact subset of such a space into itself always has a fixed point. The proof is based on the famous analogous theorem of Brouwer for finite dimensional Euclidean spaces. As simple applications we have the following : (1) Every polynomial $f(z)$ has a root. We have only to consider the mapping $z \rightarrow f(z) + z$ and to show that it takes a suitable circle into itself. (2) Let y_α be a family of unknown functions of the variable x in an interval about x_0. Let $f_\alpha(x, \ldots y_\alpha \ldots)$ be a function which is continuous in all the arguments. Then the set of equations $dy_\alpha/dx = f_\alpha;$, $y_\alpha(x_0) = y_\alpha^0$ has a solution. We may note this is a generalization to an infinite number of variables of the existence theorem for ordinary differential equations.

If we could prove the uniqueness of the solution, this gives a method of specifying the set of functions y_α. The simplest example is the one where we define e^x as the function which satisfies $dy/dx = y$; $y(0) = 1$. The uniqueness situation is handled more easily in the case of *linear operators* in function spaces, for, then it is equivalent to saying that the operator has an inverse.*

* Sri M. R. Parameswaran will be reporting on applications of locally convex spaces to summation process.

5. Analytic functions and Riemann surfaces. A great impetus to the study of topological properties to surfaces came with Riemann's memoir showing that a many-valued analytic function $w = f(z)$, say, the one given by a polynomial equation $p(w, z) = 0$ of degree greater than one in w, can be represented as a single valued continuous function defined on a suitable Riemann-surface, which is obtained by taking n sheets of the closed complex plane, cutting them along suitable lines and attaching different sheets suitably along the cut edges. The question arises whether we could find the characteristic properties of this surface. We notice that (a) to each point on this Riemann-surface there corresponds a projection P on the complex plane, identified with, say, the bottom sheet and that this projection is 1 - 1 and bi-continuous in the neighbourhood of each simple point on the surface and is like the mapping $z = p^n$ in a neighbourhood of a non-simple point where n sheets branch out. It is natural to call a topological space with the property (A), a covering space. Stoilow [8] has shown that a topological space is a Riemann-surface if and only if it is a two dimensional orientable covering space of the complex plane. Some of the early achievements of combinatorial topology were the classification of compact Riemann surface by their genus; and their canonical representations as a Riemann sphere with a number of circular holes punched out and filled with handles instead.

6. Topological methods in function theory. The next significant problem tackled in the topological nature of the theory of functions of complex variable is the following:

Let us define two functions $W = f(z)$, $W = g(z)$ from the complex z-plane into the complex w-plane as topologically equivalent if there exist homeomorphisms α, β of the z-plane and of the w-plane onto themselves such that $g(z) = \beta(f(\alpha(z))$. It is to be noted that $z \to \alpha(z)$ and $w \to \beta(w)$ can be considered as mere re-namings of the z and w-plane respectively. Then $g(z)$ is just the renamed function $f(z)$. For an extensive study of the theory of functions of a complex variable along allied points view, we may refer the reader to Morsten Morse's brochure [5], and to Whyburn's work [10]. We shall only state here

Stoilow's theorem that a continuous mapping of a two dimensional space into the complex numbers is equivalent to an analytic function on a Riemann-surface if and only if the mapping is open, i. e. carries open sets into open sets. It is to be noted that many properties of analytic functions, e. g. the maximum modulus theorem can be proved by means of these characterizations. See [8] and [11].

One may again consider two continuous functions $f(z)$, $g(z)$ of the complex variable z as (homotopic) equivalent when there exist a 1-parameter family of functions $f_t(z)$, $0 \leqslant t \leqslant 1$ continuous in t and z together such that $f_0(z) = f(z)$; $f_1(z) = g(z)$, (i. e. when they can be continuously deformed into each other). The investigations of Hopf [2] have shown that there exist only a countable number of non-equivalent mappings of S_2 (the surface of the unit sphere in 3-space) into itself—one of each integral degree n. Remembering that the closed complex plane is homeomorphic to S_2 and that a function meromorphic over the entire closed complex plane is a *continuous function of S_2 into itself*, the above result gives a complete classification, upto homotopic equivalence of the meromorphic functions of the closed complex plane, by means of their degree.

This raises the question of classification of complex functions meromorphic in the region $|z| < 1$. The answer has been given by Morse and Heins [6]. We start with the definition of the angular index $d(k)$ of a curve k from a to b in the z-plane. We plot on the unit circle, the following vectors: the directions of the vectors from any x to a when x varies from b to a on k; then we plot the tangential directions of the vectors from x to b when x varies from b to a. The total angular variation in this path on the unit circle is defined as the angular index $d(k)$ of the curve k. Suitable modifications can be made when curve is not differentiable, or when $b = a$. Next, let $f(z)$ be a meromorphic function with zeros and poles at $a_0 a_1 \dots a_r \dots a_n$ the first $r+1$ being the zeros and the rest poles. Let $b_1 \dots b_m$, be the zeros of the derivative — the 'branch point antecedents'. Let k_i be any curve which joins a to a_i and does not pass through any of $(a_0 \dots b_m)$ except a_1 and a_0. If the curve is deformed so as not to pass through any of these points, then $d(f(k_i))$ will maintain the same value. For other deformations the value may be affected. We shall now define $V(k_i)$ to be the angular-variations of $\frac{(z-b_0)\dots(z-b_m)\,(z-a_i)}{(z-a_1)\dots(z-a_n)}$ as z varies from a_0 to a_i. It can be shown that $df(k_i) - V(k_i) = J_i$ is the same for all curves k_i joining a_0 to a_1. The assertion is that two meromorphic functions in $|z| < 1$ can be deformed into each other by the functions of the same type and with the same zeros, poles and branch points antecedents if and only if the J's are the same for both the functions.

No similar characterization of equivalence classes of continuous real-valued functions of a real variable is known. Since however, continuous periodic functions of the real variable may be considered as functions of S_1 (the boundry of the unit-circle in the complex plane) into itself, Hopf's results show that these can be classified by their degree.

7. Vector lattices. Theorems which give a complete specification of the object of our study are always found to be of interest. Thus if we take the class of all continuous functions defined on a closed interval of any T_2 bi-compact space, these constitute with the usual upper bound norm, a Banach space. What is the peculiarity of this Banach space among other Banach spaces? Can every such space be abstractly identified with the Banach space of all continuous functions on a certain bicompact space X? The answer is in the negative, and the peculiarity is given as follows: (1) on the unit sphere, there exists a point P such that given any other point Q on the unit sphere, the segment joining P to Q or the segment joining P to $(-Q)$ must be completely on the sphere. (2) If E_p is the semi-cone with vertex P and base the unit sphere, then the intersection of any two translates of E_p is again a translate of E_p. This space can also be characterized as a Banach space in which there is partial order $f > g$ $(f(x) > g(x))$ for every x. The partial order is seen to be a lattice where the lattice operations are compatible with the vector and topological structures. The characterization of this Banach lattice among all others is that (1) if x, $y \geqslant 0$, then $\|x+y\| = \max(\|x\|, \|y\|)$ and (2) if $x \wedge y = 0$ then $\|x+y\| = \|x-y\|$. An interesting result stemming out of an analysis along similar lines is the following generalization by M. H. Stone of the famous theorem of Weierstrass on the approximation of continuous functions by polynomials: Let Γ be a class of continuous real functions defined on a compact space R such that whenever f, g are in Γ, so also are $af+bg$ (f, a, b real constants) and max (f, g). Also for every pair of point x, y in R, let there be a function f in Γ such that $f(x) \neq f(y)$. Then every real continuous function $h(x)$ in R can be expressed as the uniform limit of a sequence of functions in Γ.

8. Rings of functions. Kakutani [3] has shown that the class of functions analytic on an open set determines by their algebraic structure the nature of the domain D. He proves that the class of such functions constitutes with the usual laws of addition and multiplication a ring. It can be proved that two domains D_1, D_2 are conformally equivalent if and only if, the two corresponding rings are abstractly identical. If D is the sum of two such open sets D_1, D_2 then the associated ring of D will be the direct sum of those of D_1, D_2 (D_i is supposed to be such that there exists an analytic function in D_i with an essential singularity at any given point on the boundry of D_i). The status of these rings among abstract rings seems to be still an open question.

9. Conclusion. With this we shall conclude this brief resumé of the basic results in the region of contact of the classical theory of functions with the abstract structures of modern mathematics. For an idea of the various other developments, the reader might refer to the literature quoted in the various references and particularly in Kaplan [4].

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TOPOLOGICAL VECTOR SPACE OF ENTIRE FUNCTIONS

By U. N. SINGH

1. M. Fréchet was perhaps the first mathematician, who considered [3] the class of entire functions as a metric space by defining the distance $d(f, g)$ between any two entire functions $f(z)$ and $g(z)$ as

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{n!} \cdot \frac{(f, g)_n}{1 + (f, g)_n}, \text{where } (f, g)_n \max_{|z|=n} |f(z) - g(z)|.$$

He also showed that this metric topology was equivalent to uniform convergence of sequences of entire functions on compact sets. In a series of four papers ([4], [5], [6], [7]) V. G. Iyer has studied certain interesting structural properties of the class of entire functions by defining a metric for it which is simple and direct. The topology introduced by this metric is equivalent, according to a theorem proved by Iyer himself ([4], Th. 3) to uniform convergence on compact sets. This report deals with some of the important results relating to the space of entire functions obtained by V. G. Iyer and M. G. Arsove [1].

Let $\alpha \equiv \alpha(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. Following V. G. Iyer [4], we denote by $|\alpha|$† the max $[|a_0|, |a_1|, \ldots, |a_n|^{1/n}]$. It is easy to see that $|\alpha|$ is a finite positive real number and the following relations are satisfied:

(i) $|\alpha| \geqslant 0$ and $|\alpha| = 0$ if and only if $\alpha(z) \equiv 0$,

(ii) $|\alpha + \beta| \leqslant |\alpha| + |\beta|$ for any two entire functions $\alpha(z)$ and $\beta(z)$,

(iii) $|t\alpha| \leqslant A(t)\, |\alpha|$ for any complex number t, where $A(t) = \max(1, |t|)$.

† The context will make it clear as to whether the notation $|\alpha|$ is used in this sense or in the sense of absolute value of a complex number.

The relations (i) and (ii) clearly show that a metric can be defined for the class of all entire functions by setting $d(\alpha, \beta) = |\alpha - \beta|$, α and β being any two entire functions. The space of all entire functions endowed with this metric topology has been denoted by Γ by V. G. Iyer who has shown :

With this metric topology, Γ becomes a linear topological space which is complete and separable. The separability of Γ follows, indeed, from the consideration that the set of all polynomials with complex rational coefficients is dense in Γ. As the ordinary product of two entire functions is again an entire function, Γ can also be regarded as a linear topological ring. However, this topology of Γ cannot be derived from a norm, as Γ does not contain a bounded open subset.

The convergence in the topology of Γ of a sequence α_p of elements of Γ to an element $\alpha \in \Gamma$ is equivalent to uniform convergence in any finite circle of the corresponding sequence of functions $\alpha_p(z)$ to $\alpha(z)$.

There is one-to-one correspondence between the class of all linear continuous functionals defined on Γ and the class of all complex sequences $\{c_n\}$ such that $|c_n|^{1/n}$ is bounded. More precisely, corresponding to every linear continuous functional f on Γ there exists a unique sequence $\{c_n\}$ with the property that $|c_n|^{1/n}$ is bounded and is such that for $\alpha \equiv \alpha(z) = \sum_{n=0}^{\infty} a_n z^n$, $f(\alpha) = \sum_{n=0}^{\infty} c_n a_n$ and conversely, every such sequence $\{c_n\}$ determines a linear continuous functional on Γ. Hence the class of all linear continuous functionals on Γ can be identified with the class of all sequences $\{c_n\}$ for which $|c_n|^{1/n}$ is bounded or what comes to the same thing, it can be identified with the class of all power series $\Sigma\, c_n z^n$ with positive radii of convergence. It will be observed that this identification of linear continuous functionals on Γ with the class of all power series convergent in some neighbourhood of the origin enables one to regard it as a metric space, where the distance between the two functionals $f = \sum_{n=0}^{\infty} c_n z^n$, $g = \sum_{n=0}^{\infty} d_n z^n$ is defined to be

$d(f, g) = \text{l. u. b.} [|c_0 - d_0|, |c_1 - d_1|, |c_2 - d_2|^{1/2}, \ldots, |c_n - d_n|^{1/n} \ldots]$.
The set of all l.c.f. on Γ (i.e. the conjugate, or adjoint space of Γ) is denoted by Γ^*. Evidently Γ is an isometric subset of Γ^*.

Γ^* is also a complete metric space, but is not a linear metric space nor is it separable. In fact, every subset of Γ^* which is also a linear metric space is contained in Γ and hence Γ is the greatest linear metric subspace of Γ^*.

An important result proved by Iyer [4] is: In the space Γ the notions of strong convergence and weak convergence are equivalent, but this is not true for the space Γ^*. This result is remarkable since the only other known example of a topological vector space for which the notions of weak and strong convergence are equivalent is the normed space of absolutely convergent series.

2. Normed topologies. It has been mentioned earlier that the metric topology of Γ cannot be derived from a norm. However, normed topologies, which can be related to the topology of Γ can be defined on the set of all entire functions. The characterization of Γ in terms of these normed topologies has led Iyer to prove the Hahn-Banach extension theorem for the space Γ, which could also be inferred from the fact that Γ is a locally convex space. To state certain other results proved by him, we need the following definitions and notations.

For each $R > 0$ and for every entire function $\alpha = \alpha(z) = \sum_0^\infty a_n z^n$, let

$$|\alpha ; R| = \sum_0^\infty |a_n| R^n. \tag{2.1}$$

Ler $\bar{A}_\tau$ denote the closure of a set A with respect to the topology τ. It is easy to see that for every fixed $R > 0$, (2.1) defines a norm on the class of entire functions. Let $\Gamma(R)$ denote the normed vector space of all entire functions normed by (2.1) and let $\Gamma^*(R)$ denote its conjugate space. Since for $R_1 > R_2$, $|\alpha ; R_1| \geqslant |\alpha ; R_2|$, it follows that the topology $\Gamma(R_1)$ is stronger (in the sense of Bourbaki) than the topology $\Gamma(R_2)$ when $R_1 > R_2$. Also the topology Γ is stronger than every topology $\Gamma(R)$; $R > 0$ because $\lim_{p\to\infty} |\alpha_p| = 0$ always implies that $\lim_{p\to\infty} |\alpha_p ; R| = 0$ for every $R > 0$. The following results proved by Iyer [5] characterize Γ and Γ^* in terms of $\Gamma(R)$ and $\Gamma^*(R)$ respectively.

For any subset S of the set of entire functions

$$(\bar{S})_\Gamma = \bigcap_{R>0} (\bar{S})_{\Gamma(R)}, \tag{2.2}$$

$$\Gamma^* = \bigcup_{R>0} \Gamma^*(R). \tag{2.3}$$

These results lead to the following results.

Let $\alpha_0 \in \Gamma$ be at a distance $d > 0$ from a linear subspace S of Γ. Then for each $R > A(1/d)$, [for $t > 0$, $A(t)$ denotes max $(1, t)$] there is a functional $f \in \Gamma^*$ such that (i) $f(\alpha_0) = 1$, (ii) $f(\alpha) = 0$, $\alpha \in S$ and (iii) $|f(\alpha)| \leqslant |\alpha ; R|/d$ for all $\alpha \in \Gamma$, i. e. $f \in \Gamma^*(R)$ for $R > A(1/d)$. (2.4)

If $f(\alpha)$ be a linear continuous functional defined on a linear subspace S of Γ, then there exists a functional $F \in \Gamma^*$ such that $F(\alpha) = f(\alpha)$ for $\alpha \in S$. That is to say, F is the extension of f to the whole space Γ. (2.5)

These two theorems can clearly be recognized as analogues of known results for normed vector spaces (see [9]).

We shall now consider bases in Γ.

A sequence $\{\alpha_n\}$ of elements of Γ is said to be a *base* in Γ, if every element $\alpha \in \Gamma$ can be represented as

$$\alpha = \sum_{n=0}^{\infty} t_n \, \alpha_n, \tag{2.6}$$

where $\{t_n\}$ is a sequence of complex numbers which is determined uniquely by α and convergence in (2.6) is taken in the topology of Γ, e.g. $\{z^n\}$ is a base; for other examples see [5]. It is known that for a Banach space ([2] p. 111), each of the coefficients $t_n \equiv t_n(\alpha)$ in (2.6) determines a continuous linear functional. This result is also true for Γ. In fact Iyer has proved the following theorem [5].

Let $\{\alpha_n\}$ *be a base in* Γ *and let* $t_n \equiv t_n(\alpha)$ *be defined by* (2.6). *If we set* $f_n(\alpha) = t_n$ *for each* $\alpha \in \Gamma$, *then for each* n, f_n *is a linear continuous functional on* Γ *and* $\{f_n\}$ *is an orthonormal sequence to* $\{\alpha_n\}$ [*i.e.* $f_n(\alpha_m) = 0$ *for* $n \neq m$ *and* $f_n(\alpha_n) = 1$].

3. Closure theorems. For any subset $E \subset \Gamma$ let $L(E)$ denote the subspace of Γ generated (or spanned) by the elements of E. That is to say $L(E)$ is the closure (in the topology of Γ) of all finite linear combinations of elements of E. It is easily seen that $L(E)$ is a closed linear subspace of Γ. Also as an immediate consequence of the definition of $L(E)$ and the result stated in (2.4) we obtain the following :

An element $\alpha \in \Gamma$ will belong to $L(E)$ for any subset $E \subset \Gamma$, if and only if every $f \in \Gamma^*$ which is orthogonal to E [i.e. $f(p) = 0$ for every $p \in E$] is also orthogonal to α [i. e. $f(\alpha) = 0$]. (3.1).

Using Theorem (3.1), which is again the extension of a classical result known for normed vector spaces (see [2], p. 58), Iyer has proved [6] the following two interesting theorems leading to the construction of particular sequence $\{\alpha_n\}$ for which $L(\alpha_n, n \geqslant 1) = \Gamma$.

Let $\alpha = \alpha(z) = \sum_{0}^{\infty} a_n z^n$ be an entire function such that $a_n \neq 0$ for every n and suppose that $\{z_n\}_1^\infty$ is a sequence of distinct complex numbers. Let $\alpha_n = \alpha(z_n z)$. If either

(i) the sequence z_n has a finite limit point, or

(ii) α is of order p and finite type and $\limsup_{n\to\infty} \dfrac{n}{|z_n|^p} = \infty$, then

$L(\alpha_n, n \geqslant 1) = \Gamma$. (3.2)

For example, the sequences $\{e^{z/n}\}$, $\{e^{z\sqrt{n}}\}$ span the whole space Γ.

For any two entire functions $\alpha(z) = \sum_{0}^{\infty} a_n z^n$ and $\beta(z) = \sum_{0}^{\infty} b_n z^n$, let $\alpha \odot \beta$ denote the entire function $\sum a_n b_n z^n$ and let $(\alpha)_n = \alpha \odot \alpha \odot \ldots \odot \alpha$ (n times). (3.3)

If α be an entire function with distinct and non-zero coefficients, then $L\{(z)_n, n \geqslant 1\} = \Gamma$.

For example $\alpha(z)$ can be taken either as $e^z - \frac{1}{3}z$ or as $\cosh \sqrt{z}$.

The original form of the result (3.3) contained an extra condition on the coefficients of α. The theorem was proved in this form by Iyer in [8].

4. **Continuous linear transformations.** A continuous linear transformation of the normed vector space $\Gamma(R_1)$ into the normed vector space $\Gamma(R_2)$ will be denoted by $T(R_1 \to R_2)$ and the family of all such transformations by $F(R_1 \to R_2)$. Since the topology of Γ can be deemed, in certain sense, as the 'limit' of the normed topologies $\Gamma(R)$, the notation $T\ (\infty \to \infty)$ will be used to denote a linear continuous transformation of Γ into Γ, the family of all such transformations being denoted by $F(\infty \to \infty)$. The following result [6] exhibits the intimate connection between the family $F(\infty \to \infty)$ and $F(R_1 \to R_2)$:

$$F\ (\infty \to \infty) = \bigcap_{R_2 > 0} \Big[\bigcup_{R_1 > 0} F\ (R_1 \to R_2)\Big]. \tag{4.1}$$

That is, every $T(\infty \to \infty)$ is a $T(R_1 \to R_2)$ for each $R_2 > 0$ and a suitably chosen $R_1 > 0$.

For every value of n, z^n, is an entire function; thus (z^n) is a sequence of elements of Γ. We denote this sequence by $\{\delta_n\}$. It is obvious that $\{\delta_n\}$ is a base in Γ. Also $\{\delta_n\}$ is the simplest and most important base. Every $\alpha \in \Gamma$ can be represented as $\alpha = \sum_0^\infty t_n \delta_n$, such that

$$\lim_{n \to \infty} |t_n(\alpha)|^{1/n} = 0 \text{ for every } \alpha \in \Gamma. \tag{4.2}$$

This led Iyer to designate those bases as '*proper bases*' for which (4.2) holds for each α. He suspected that perhaps every base was a proper base, but (as was pointed out by Arosve [1]) the base $\{z^n/n!\}$ is not a proper base since for the function e^z, $t_n = 1$. The following theorem gives the relations between bases and automorphisms of Γ [6].

An automorphism T of Γ (i.e. a bi-uniform bi-continuous linear transformation of Γ onto Γ) transforms every base into a base. In particular $T(\delta_n)$ will be a base. (4.3)

On the other hand, if T be a transformation of $F(\infty \to \infty)$ such that $T(\delta_n)$ is a base then T is an automorphism if either

(i) T is a transformation of Γ onto Γ, or (ii) T transforms closed sets of Γ into closed sets of Γ or (iii) the base $T(\delta_n)$ is a proper base.

A linear transformation T of Γ into Γ will be called isometric if $|T(\alpha)| = |\alpha|$. Clearly, an isometric linear transformation is necessarily continuous. Isometric transformations of Γ into itself can be only of two types as is revealed by the following theorem [7].

Every isometric linear transformation T of Γ into Γ is of one or other of the following two types:

Type I. $T(\delta_n) = k_n\, \delta_n,\ n \geqslant 0$ and $T(\alpha) = \sum_0^\infty k_n\, a_n\, \delta_n$;

Type II. $T(\delta_0) = k_0 \delta_1,\ T(\delta_1) = k_1\, \delta_0,\ T(\delta_n) = k_n\, \delta,\ n \geqslant 2$

and

$$T(\alpha) = k_0\, a_0\, \delta_1 + k_1\, a_1\, \delta_0 + \sum_2^\infty k_n\, a_n\, \delta_n,$$

where $k_n,\ n \geqslant 0$ are complex numbers with $|k_n| = 1$ and $\alpha = \sum_0^\infty a_n\, \delta_n$; conversely, a transformation of either of the two types is an isometric linear transformation of Γ into Γ. (4.4)

5. Automorphisms and proper bases. A characteristic property of power series is that the circle of convergence is also the circle of uniform convergence and absolute convergence. Basing on this property of power series his approach to define a proper base in Γ, Arsove [1] has found interesting and satisfactory results regarding automorphisms and proper bases. This notion of proper base, which is a little different from Iyer's, will now be explained. We continue to denote z^n by δ_n.

A sequence $\{\alpha_n\}$ of elements of Γ will be said to be *linearly independent* if for every sequence $\{c_n\}$ of complex numbers uniform convergence to zero on compact sets of $\sum_0^\infty c_n \alpha$ implies that $c_n = 0$, for $n = 0, 1, 2, \ldots$.

A sequence $\{\alpha_n\}$ is said to span a subspace Γ_0 of Γ if Γ_0 consists of all linear combinations $\sum_0^\infty c_n \alpha_n$, where $\{c_n\}$ is any sequence of complex numbers for which the series converges uniformly on compact sets.

A sequence $\{\alpha_n\}$ is said to be a *basis in* Γ_0 if it is linearly independent and spans Γ_0.

$\{\alpha_n\}$ is said to be *absolutely linearly independent* if $\sum_0^\infty c_n \alpha_n = 0$ implies that $c_n = 0$ $(n = 0, 1, 2, \ldots)$ for every sequence $\{c_n\}$ for which $\sum_0^\infty |c_n \alpha_n|$ converges uniformly on compact sets.

$\{\alpha_n\}$ spans Γ_0 absolutely if Γ_0 consists of all linear combinations $\sum_0^\infty c_n \alpha_n$ such that $\sum_0^\infty |c_n \alpha_n|$ converges uniformly on compact sets. $\{\alpha_n\}$ will be called an absolute basis in Γ_0 if $\{\alpha_n\}$ is absolutely linearly independent and spans absolutely the subspace Γ_0.

A sequence $\{\alpha_n\}$ of entire functions will be said to be a *proper basis* for a subspace Γ_0 of Γ provided that (i) $\{\alpha_n\}$ is an absolute basis in Γ_0, (ii) for every sequence $\{c_n\}$ of complex numbers the series $\sum_0^\infty |c_n \alpha_n|$ converges uniformly on compact sets if and only if $|c_n|^{1/n} \to 0$.

For example, $\{\delta_n\}$ is a proper basis but $\alpha_n = n!\, z^n$ is not a proper basis.

The results of Arsove [1] can now be stated.

Let $\{\alpha_n\}$ be an absolute basis for a subspace Γ_0 of Γ, then $\{\alpha_n\}$ is a proper basis if and only if (i) $\limsup_{n\to\infty} [M_n(R)]^{1/n} < \infty$ for each $R > 0$,

$$M_n(R) = \max_{|z|=R} |\alpha_n(z)| \text{ and (ii) } \lim_{R\to\infty} \{\liminf_{n\to\infty} [M_n(R)]^{1/n}\} = \infty. \qquad (5.1)$$

If T be a linear homeomorphic mapping of Γ into Γ, then $\{T\delta_n\}$ is a proper basis in some closed subspace Γ_0 of Γ. (5.2)

Conversely, if $\{\alpha_n\}$ is a proper basis in a closed subspace Γ_0 of Γ, then there exists a linear homeomorphic mapping T of Γ into Γ_0 such that $T\delta_n = \alpha_n$ $(n = 0, 1, 2, \ldots)$.

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SOME APPLICATIONS OF FUNCTIONAL ANALYSIS IN SUMMABILITY

By M. R. PARAMESWARAN

LET $A = (a_{nk})$, $(n, k = 0, 1, 2, \ldots)$ be a matrix whose elements are real or complex numbers. Then a (real or complex) sequence $\mathfrak{x} = \{x_n\}$ is said to be summable by the method A to the limit l if the sums

$$y_n = \sum_{k=0}^{\infty} a_{nk}\, x_k$$

exist for each $n = 0, 1, \ldots$ and y_n tends to the limit l as n tends to infinity. The sequence $\mathfrak{y} = \{y_n\}$ is called the A-transform of $\mathfrak{x}$.

The martix A will transform every convergent sequence into a convergent sequence if and only if A satisfies the conditions:

$$\| A \| \equiv \sup_n \sum_{k=0}^{\infty} | a_{nk} | < \infty\,; \tag{1}$$

$$\rho_n \equiv \sum_{k=0}^{\infty} a_{nk} \to \rho \text{ as } n \to \infty\,; \tag{2}$$

and

$$\lim_{n\to\infty} a_{nk} = a_k \text{ exists for each } k. \tag{3}$$

A is called a *conservative* (*convergence-preserving* or K-), matrix if $\rho = 1$ and $a_k = 0$ for all k, then A-lim $x_n =$ lim x_n whenever the latter exists and A is called a *permanent* (*regular* or T-) matrix. [14, 27, 28, 9].

The set $\mathscr{E}$ of points (sequences) $\mathfrak{x}$ whose A-transforms lie in a given set $\mathscr{F}$ is donated by $\mathscr{E} = \mathscr{F} A$. The set $\mathfrak{S}_c A$ where $\mathfrak{S}_c$ denotes the set of convergent sequences, is called the summability field of A.

The set of matrices satisfying the condition (1) above form a Banach algebra $\mathfrak{A}$ under the usual operations of addition and multiplication and where the norm is defined by (1) itself and the unit element I is the matrix of the identity transformation, $\|I\| = 1$. The set of conservative matrices form a subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$. A matrix will transform every bounded sequence into a bounded sequence if and only if it belongs to $\mathfrak{A}$ [20, 21, 22].

The regular elements in these algebras, that is, matrices A whose inverses also belong to the algebra, form an open set. If A is a conservative matrix with $\|A\| < 1$, then $(I + A)^{-1}$ exists and is itself conservative. This is helpful in proving a number of theorems of the Mercerian type, for example the following theorem due to Agnew [1, 3, 20].

If A is conservative and

$$|a_{nn}| - \sum_{k \neq n} |a_{nk}| > \theta > 0,$$

then A sums no bounded divergent sequences.

The conservative matrices have the interesting property that if A is conservative and has an inverse, i.e. a two-sided reciprocal A^{-1}, which is of finite norm, then A^{-1} is itself conservative [21, 23, 36].

Matrix methods as topological spaces have not been studied much. In one of the early attempts at a study of classes of general summation methods V. Ganapathy Iyer [8] proved:

Let $\mathfrak{x}$ be a bounded sequence summable by each member of a set Λ of permanent methods. Then $\mathfrak{x}$ will be summable also by every method H belonging to the linear closed convex hull of Λ. (The closure is in terms of the norm defined by (1)). This is in essence the principle underlying Wiener's tauberian theorems [29].

The earliest attempts at a systematic application of functional analysis to summability were restricted to a consideration of normal, permanent matrices; reversible regular matrix methods came to be considered soon after. In these cases the summability fields of the methods are Banach spaces. However it was discovered as early as 1932 by Mazur and Orlicz that for a generalization to wider classes of methods, the Banach spaces were not sufficient since the fields of the summability methods constitute more general spaces, of the type they called B_0-spaces. The main results obtained by them then were announced [16] but the proofs were not published till 1955 [18] due to various reasons. Their work has been largely rediscovered by Zeller [38, 39, 40, 41, 43] who in recent years has given a methodical approach to the study of summability by functional analytic methods. His work brings into perspective the works of earlier authors like Mazur and Orlicz, Agnew, Hill, Wilansky and others [2, 4, 5, 6, 7, 11, 12, 15, 16, 30, 31, 32].

Let $[\mathscr{E}; p_j]$ be a complete linear co-ordinate space $\mathscr{E}$ in which a sequence of seminorms p_j are defined for which $p_j(\mathfrak{x}) = 0$ for all j implies $\mathfrak{x} = 0$. Let further $\mathfrak{x}^r \to \mathfrak{x}$ in $[\mathscr{E}; p_j]$ imply co-ordinatewise convergence, i.e. $x_k^r \to x_n$ for every k. Then $[\mathscr{E}; p_j]$ is called an FK space. An FK-space is not in general a Banach space, but is an example of the more general B_0-spaces studied by Mazur and Orlicz [17] and others. An FK-space, as may be seen from the definition, is a locally convex F-space; it is a linear co-ordinate space which is complete under a sequence of homogeneous seminorms, which are together equivalent to a nonhomogeneous norm, e.g. $p(\mathfrak{x}) = \sum_{j=0}^{\infty} \frac{1}{2^j} \frac{p_j(\mathfrak{x})}{1 + p_j(\mathfrak{x})}$ and where norm-convergence implies co-ordinatewise convergence.

Some examples of FK-spaces are, with the seminorms :

for $\mathfrak{S}_K$, the set of all number sequences: $|x_0|, |x_1|, \ldots$

for $\mathfrak{S}_B$, the set of bounded sequences: l.u.b. $|x_n|$

for $\mathfrak{S}_C$, the set of convergent sequences: l.u.b. $|x_n|$

for $\mathfrak{S}_N$, the set of null sequences : l.u.b. $|x_n|$.

If A is any matrix, then its summability field $\mathfrak{S}_c A$ is an FK-space, with the seminorms

$$\underset{0 \leqslant l < \infty}{\text{l.u.b.}} \left| \sum_{k=0}^{l} a_{0k} x_k \right|, \ \underset{0 \leqslant l < \infty}{\text{l.u.b.}} \left| \sum_{k=0}^{l} a_{1k} x_k \right|, \ldots, \ldots \tag{1}$$

$$|x_0|, |x_1|, \ldots, \ldots \tag{2}$$

$$\underset{0 \leqslant n < \infty}{\text{l.u.b.}} \left| \sum_{k=0}^{\infty} a_{nk} x_k \right| \tag{3}$$

Indeed; if $\mathscr{F}$ is an FK-space, then so is $\mathscr{E} = \mathscr{F} A$ [38].

Let $[\mathscr{E}]$ be an FK-space and let $\{y_n\} \equiv y = \phi(x) \equiv \{f_n(x)\}$ be a mapping of $\mathscr{E}$ into an FK-space $\mathscr{F}$, the f_n's being linear continuous functionals over $\mathscr{E}$. Then, the map ϕ will itself be linear and continuous. In particular, when $\mathscr{F} = \mathfrak{S}_c$ and ϕ is given by a matrix A, with $y_n = \sum_{k=0}^{\infty} a_{nk} x_k$, then A-lim $\mathfrak{x}$ will be linear and continous.

If now $\mathscr{E} \subseteq \mathscr{F}$, then taking ϕ as the identity transformation, we get that

$$\mathfrak{x}^r \to \mathfrak{x} \text{ in } [\mathscr{E}] \text{ implies } \mathfrak{x}^r \to \mathfrak{x} \text{ in } [\mathscr{F}],$$

that is the identity mapping is continuous and hence either $\mathscr{E} = \mathscr{F}$ or, if $\mathscr{E} \subset \mathscr{F}$, i.e. is a proper subset of $\mathscr{F}$, then $\mathscr{E}$ is of the first category in $\mathscr{F}$. But an FK-space is not of the first category in itself and hence the union of a finite or countable number of FK-spaces none of which is equal to the union itself, is *not* an FK-space. On the other hand the intersection of a finite or countable number of FK-space $[\mathscr{E}^{(i)}; p_{j(i)}]$ is an FK-space $[\mathscr{E}; p_{ji}]$.

The summability fields of matrix methods being FK-spaces the following "inequivalence" theorems follow [38]:

I. *If $A^{(i)}$ are a finite or countable number of matrix methods none of which is the strongest, then there is no matrix A with*

$$\mathfrak{S}_c A = \bigcup_i \mathfrak{S}_c A^{(i)}.$$

II. *If each matrix $A^{(i)}$ is weaker than a matrix A, then their sum is also weaker than A, that is, $\bigcup \mathfrak{S}_c A^{(i)}$ is a proper subset of $\mathfrak{S}_c A$.*

III. *In particular, the C_∞ method is not equivalent to a matrix method.* (A sequence is said to be C_∞ summable to l if

$$\lim_{r\to\infty} \limsup_{n\to\infty} c_n^{(r)}(x) = \lim_{r\to\infty} \liminf_{n\to\infty} c_n^{(r)}(x) = l$$

where $c_n^{(r)}(x)$ is the nth Cesàro mean, of order r, of the sequence x).

For, we have $\mathfrak{S}_c C_\infty = \bigcup \mathscr{C}^{(r)}$ where $\mathscr{C}^{(r)} = \mathfrak{S}_c C_\infty \cap \mathfrak{S}_B \dot{\mathscr{C}}^{(r)}$, so that if $\mathfrak{S}_c C_\infty$ is an FK-space, so is $\mathscr{C}^{(r)}$, being the intersection of two FK-spaces; but $\mathscr{C}^{(r)} \subset \mathscr{C}^{(r+1)}$ and hence $\bigcup \dot{\mathscr{C}}^{(r)}$ cannot be an FK-space and we come to a contradiction.

Similar statements hold good for the E_∞ and H_∞ methods corresponding to the Euler and Hölder transformations respectively.

Existence of bounded and unbounded sequences in the summability fields of matrix methods. Let A be the matrix of a conservative method and f a linear continuous functional in the FK-space $[\mathfrak{S}_c A]$. Then it can be shown [38] from the form of linear continuous functionals in the various FK-spaces, that

(i) if $\chi(A) \equiv \rho - \sum_{k=0}^{\infty} a_k \neq 0$ and f vanishes over $\mathfrak{S}_c$, then it vanishes over $\mathfrak{S}_B \cap \mathfrak{S}_c A$;

(ii) if $\chi(A) = 0$ and f vanishes over $\mathfrak{S}_N$, then it vanishes over $\mathfrak{S}_c \cap \mathfrak{S}_c A$.

Thus, if $\chi(A) \neq 0$, then $\mathfrak{S}_c$ is dense in the FK-space $[\mathfrak{S}_B \cap \mathfrak{S}_c A]$. If now there exists a bounded divergent A-summable sequence $\mathfrak{x}$, then $\mathfrak{x}$ is a contact point of $\mathfrak{S}_c$ in $[\mathfrak{S}_B \cap \mathfrak{S}_c A]$. But $\mathfrak{S}_c$ is a closed subspace of the FK-space $[\mathfrak{S}_B]$ and so $\mathfrak{x}$ is not a contact point of $\mathfrak{S}_c$ in $[\mathfrak{S}_B]$—which it would be if $\mathfrak{S}_c A \subseteq \mathfrak{S}_B$. If $\chi(A) = 0$ then it can be proved similarly that A must necessarily sum some unbounded sequence. Hence the theorem: *If a conservative matrix A sums a divergent sequence, then it sums an unbounded sequence* [cf. 16]; *a conservative A with $\chi(A) = 0$ always sums an unbounded sequence* [38].

A related question is: "*When can it be asserted that a conservative matrix A will sum some bounded divergent sequence?*". There exist conservative matrices whose summability fields contain some unbounded sequences but not any bounded divergent sequences. Two cases when the question has an affirmative answer are (1) when $\chi(A) = 0$ and (2) when the method A is "*perfect*", i.e. when the closure of $\mathfrak{S}_c$ in $[\mathfrak{S}_c A]$ is the whole of $\mathfrak{S}_c A$.

More generally, the "perfect part" $\mathfrak{A}_P =$ the closure of $\mathfrak{S}_c$ in $[\mathfrak{S}_c\ A]$ either contains a bounded divergent sequence or contains only convergent sequences, and the perfect part $\mathfrak{A}_P$ of the field of a permanent method A contains all A-summable bounded sequences [19]. It can also be proved that the perfect part of the field of a conservative method must contain an unbounded sequence if it contains a (bounded) divergent sequence [25].

Let us denote by $\mathfrak{A}$ the sequence-summability field $\mathfrak{S}_c A$ of A, and by $\tilde{\mathfrak{A}}$ the corresponding series-summability field of the method A. The relation $s \longleftrightarrow \tilde{s}$, where $s_k = \sum_{i=0}^{k} \tilde{s}_i$ is a (1, 1)-correspondence between $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ and the perfect part $\mathfrak{A}_P$ of $\mathfrak{A}$ corresponds to the perfect part $\tilde{\mathfrak{A}}_P$ of $\tilde{\mathfrak{A}}$. Obviously A is perfect means $\mathfrak{A}_P = \mathfrak{A}$, or equivalently, $\tilde{\mathfrak{A}}_P = \tilde{\mathfrak{A}}$.

A striking application of these ideas is in the proof of the famous High Indices Theorem and in Gap-Tauberian theorems [19]. The High Indices theorem, due to Hardy and Littlewood, states :

If the sequence $\{\lambda_k\}$, $(0 \leqslant \lambda_0 < \lambda_1 < \ldots)$ *satisfies the condition* $\lambda_{k+1} - \lambda_k > \theta \lambda_k$ *for some* $\theta > 0$ *and all* k, *and if* $\lim_{\mu \to +0} t(\mu)$ *exists, where* $t(\mu) = \sum_{k=0}^{\infty} e^{-\lambda_k \mu} u_k$ *exists for each* $\mu < 0$, *then the series* Σu_k *is convergent.*

This result is proved [19] by showing that the method D_λ given by the transformation $t(\mu)$ is perfect under the hypothesis on $\{\lambda_k\}$. Hence $\tilde{D}_\lambda$, the series summability field of D_λ either contains a divergent series with bounded partial sums or contains only convergent series, and it is enough to prove the theorem under the additional hypothesis that the partial sums of Σu_k are bounded. This however implies that

$$u_k = O\left(\frac{\lambda_k - \lambda_{k-1}}{\lambda_k}\right)$$

when the convergence of Σu_k follows from an easier classical theorem.

Gap-Tauberian theorems. A Gap-Tauberian theorem is an assertion of the form : "*Let* $\Sigma u_n = 0 + 0 + \dots + u_{n_0} + 0 + \dots + 0 + u_{n_1} + 0 + \dots$ *be a series summable by a method* M, *say, and if*

$$u_n = 0 \textit{ for } n \neq n_k \quad (k = 0, 1, 2, \dots)$$

where the gaps (n_k, n_{k+1}) *satisfy the condition* G_M, *depending on* M, *then* Σu_n *is convergent.*"

It is known that the gap-Tauberian theorem is true in the following cases : (1) $M \equiv C^1$, the Cesàro method $(C, 1)$ with G_c^1 given by $n_{k+1} - n_k > \theta n_k$ for some $\theta > 0$,

(2) M is Abel's method, with G_A same as G_c^1; and

(3) M is the Euler-Knopp method E_α, with G_E given by

$$n_{k+1} - n_k > \theta . \sqrt{(n_k)} \text{ for some } \theta > 0.$$

Suppose now that A is a matrix which sums the series Σu_n, with $u_n = 0$, $n \neq n_k$, i.e. has gaps G. Let the matrix A_G be formed by adding together, row by row, the elements of the first $k_1 - k_0$ columns of A, then the next $k_2 - k_1$ columns of A, and so on. Then A will sum a divergent series with gaps G if and only if the method A_G sums a divergent series. If now it can be shown that A_G is perfect, then the method A_G will sum a divergent series if and only if it sums one with bounded partial sums ; and A will sum divergent series with gaps G if and only if it sums a series with gaps G and with bounded partial sums. Therefore, to prove that the gap G is Tauberian for A-summability, it is enough to prove, when A_G is perfect, the same result for series with bounded partial sums. Methods A for which the A_G corresponding to every gap G is perfect, are called gap-perfect. The permanent method A will be gap-perfect if and only if every $u \in \tilde{\mathfrak{A}}$ can be approximated in $[\tilde{\mathfrak{A}}]$ by points which have co-ordinates zero at least whenever that is the case with u [19]. (u is said to be approximated by points of $\mathscr{E}$ if there exists a sequence $\{u^{(p)}\}$, $u^{(p)} \in \mathscr{E}$ such that $u^{(p)} \to u$ in $[\tilde{\mathfrak{A}}]$).

Now the methods C^α $(\alpha \geqslant 0)$, Abel's method, E_α $(0 < \alpha \leqslant 1)$ are gap-perfect [19]. For these methods therefore it is enough to prove the gap-Tauberian theorems under the additional hypothesis that $\Sigma\, u_n$ has bounded partial sums. The gap-Tauberian theorem for the method C^1 is proved quite easily by elementary methods. By the previous observation this gives the gap-Tauberian theorem for Abel-summability, for every Abel-summable bounded sequence is C^1-summable, indeed C^α-summable for all $\alpha > 0$; also the methods E_α $(0 < \alpha < 1)$ are all equivalent for bounded sequences and hence it is enough to prove the gap-Tauberian theorem for the Euler-Knopp methods for any one value of α and under the additional hypothesis of the boundedness of $\Sigma\, u_n$. This last case for $\alpha = \frac{1}{2}$ is a well-known result.

An interesting point about the gap-Tauberian conditions given above is that they are also necessary for convergence, in a sense. For example, if G is a gap which does not satisfy the condition G_c, then there exists a divergent series Σu_n which has gaps G and is summable C^1. This observation can be used in connection with an interesting result recently pointed out by Ramanujan [26] that every Borel-summable bounded sequence is also summable by a certain large class of methods, and in particular by the methods $C^\alpha(\alpha > 0)$. While particular examples are known [9, 10, 13] which show that the converse is not true, we can show that corresponding to every gap G which is Tauberian for the Euler-Knopp method but not for the Cesàro method, there exist (1) a series with gaps G and bounded partial sums which is C^α-summable for every $\alpha > 0$ but is not Borel-summable, and also (2) a series with gaps G and unbounded partial sums which is also C^α-summable for every $\alpha > 0$ but is not Borel-summable [25].

Other applications. Functional analysis methods have been applied to reversible methods by several authors, in particular by Hill and Wilansky. They have also been used in the definition and application of notions like section-convergence (Abschnitts-konvergenz [39], weak section-convergence, section-boundedness (Abschnittsbeschranktheit [35]) etc., which are of use in consistency theorems, summability factors and converse and Mercerian theorems [35, 40]. It has also been shown [35] that the general methods of functional analysis are not limited to ordinary summability only, but can be applied also to strong summability, absolute summability etc., Wlodarski [37] has extended the study by functional analysis to continuous methods of summation. However, the general problem of characterising the FK-spaces which are summability fields of matrix methods still remains open. Zeller has shown [42] in particular, that the summability field of the Abel method is neither contained in that of any row-finite permanent matrix method, nor is it identical with the summability field of any matrix method.

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GENERALIZED MOMENT PROBLEMS IN FUNCTION SPACES

By M. S. RAMANUJAN

We shall start with the Banach space L^p, $p \geqslant 1$, the elements of which are measurable functions $f(x)$, defined, say, on (0, 1) such that $|f(x)|^p$ is integrable. Now for an integrable function the existence of $\int |f|^p\,dx$ is a restriction on the magnitude of $|f(x)|$; a similar restriction is obtained by assuming the existence of $\int f\,dg$, where $g(x)$ is a positive function taking large values on (0, 1). Now the norm of the function $f(x)$ depends only on the magnitude of $|f(x)|$. Thus we could have various spaces and norms of various types for function spaces. In this account, we are concerned with more general spaces of integrable functions, under suitable norms constituting Banach spaces, and with the problem of moments for these spaces. It is but appropriate that we now give a short statement of the moment problem. To start with, we assume that the notions of functionals, linear functionals and continuous functionals are known. The term moment by itself is a much familiar one and is often met with in dynamics and statistics. Given a distribution function $F(x)$ the n-th moment of the distribution about the origin is defined by

$$\mu_n = \int x^n\,dF.$$

Now, the moment problem stated in an easily understandable way consists in finding whether a sequence $\{\mu_n\}$ can represent a sequence of moments, or what is the same as, in finding the necessary and sufficient conditions to ensure that the given sequence $\{\mu_n\}$ implies the existence of a distribution function $F(x)$ whose moments are the terms of the given sequence. Stated in a more precise and exact fashion, it runs as below. Let f_ν be a sequence of elements of a Banach space X and μ_ν, $\nu = 0, 1, \ldots$ a sequence of real numbers. Then does there exist a linear functional F, belonging to a certain class of functionals, such that $F(f_\nu) = \mu_\nu$?

The F above can be a continuous functional or $F(f)$ may be an integral, say, $\int F dg$, where $g(x)$ is of bounded variation. Also there may be a condition on $\| F \|$ of the linear functional $F(f)$. If the problem has a solution then μ_n is called a moment sequence. Thus the moment problem consists in finding necessary and sufficient conditions to ensure the existence of a functional F so that $F(f_\nu) = \mu_\nu$. But we shall be concerned with only one particular type of moment problem. In what follows we assume that X is a space of functions and $f_\nu = x^\nu$, $\nu = 0, 1, \ldots$.

Now, any linear continuous functional $F(f)$ defined on the space of functions, continuous on (0, 1) has the form

$$F(f) = \int_0^1 f(x)\, dg(x)$$

where $g(x)$ is a function of bounded variation on (0, 1). So the moment problem for the space of continuous functions above reduces to that of finding the necessary and sufficient conditions to ensure the existence of a function $g(x)$ of bounded variation in (0, 1) such that

$$\mu_n = \int_0^1 x^n\, dg(x), \quad n = 0, 1, \ldots.$$

The moment problem above has a solution if and only if

$$\sup_n \sum_K \binom{n}{k} | \Delta^{n-k} \mu_k | < \infty,$$

and in this case the solution can also be proved to be unique; in particular the problem has a solution in an increasing $g(x)$ if and only if $\Delta^k \mu_n \geqslant 0, (n, k = 0, 1, \ldots)$. The particular moment problem for the space C of continuous functions has a special interest and significance in the summability theory to which I shall revert later. But now, we shall define two particular function spaces, one of which includes the class L^p as a special class. We precede this discussion with the following definitions and preliminary ideas. In what follows measurability is for Lebesgue measure.

Two measurable functions $f(x)$ and $g(x)$, defined say on $(0, 1)$ are said to be equi-measurable or re-arrangements of each other if the sets $[f(x) \geqslant a)]$ and $[g(x) \geqslant a]$ have equal measure for all real a. If $f(x)$ and $g(x)$ are equi-measurable then so are the functions $|f|$ and $|g|$, f^* and g^* defined below. If one of the equi-measurable functions is integrable over $(0, 1)$ then so is the other. For any measurable function $f(x)$ on $(0, 1)$ there is a decreasing function $f^*(x)$ which is equi-measurable with $f(x)$ and is obtained by inverting the function $x = M(y)$, where $M(y)$ is the measure of the set of points x for which $f(x) \geqslant y$. In at most denumerable points where $M^{-1}(x)$ is not defined, we may put $f^*(x) = f^*(x-)$. In case $f(x)$ is not positive, $f^*(x)$ is the decreasing function equimeasurable with $|f(x)|$.

We shall now define the two spaces of integrable functions.† Let $\phi(x) > 0$, be a decreasing integrable function in $(0, 1)$; let $p \geqslant 1$. Suppose also that we have normalized $\phi(x)$ by assuming that $\int_0^1 \phi\, dx = 1$. For each measurable function $f(x)$ on $(0, 1)$, we introduce the norm

$$\|f\| = \|f\|_{\Lambda(\phi,p)} = \left\{ \int_0^1 f^*(x)^p \, \phi(x)\, dx \right\}^{1/p}.$$

Now the space $\Lambda(\phi, p)$ consists of all those functions $f(x)$ for which the above norm is finite. Now if in particular $\phi(x) \equiv 1$ we get our usual L^p space. Another particular space of the above type is the space where $\phi(x) = \alpha x^{\alpha-1}$, which we shall denote by $\Lambda(\alpha, p)$. If $p = 1$, we denote this by $\Lambda(\alpha)$. Now, a norm equivalent to the one above is

$$\|f\| = \sup \left\{ \int_0^1 \phi_r(x) \; |f|^p \; dx \right\}^{1/p}$$

where the supremum is taken for all rearrangements $\phi_r(x)$ of $\phi(x)$. It may be easily verified that $\Lambda(\phi, p)$ is a Banach space under the above norm.

† See note added in proof.

The space $M(\phi, p)$ consists of all those functions $f(x)$, measurable on (0, 1), for which

$$\|f\|_{M(\phi,p)} = \sup_{e\subset(0,1)} \left\{ \Phi(me)^{-1} \int_e |f|^p \, dx \right\}^{1/p}$$

is finite, the function $\phi(x)$ being defined as earlier and $\Phi(x)$ denotes the integral of $\phi(t)$ over $(0, x)$. This space $M(\phi, p)$ is also a Banach space under the norm defined and we define as before, the particular spaces, $M(\alpha, p)$, $M(\alpha)$. It is now appropriate to point out that the space $\Lambda(\alpha)$ is separable while the space $M(\alpha)$ is not separable.

We shall now compare the spaces $\Lambda(\phi, p)$ and $M(\phi, p)$ with L^p in the following sense. Let X and Y be Banach spaces, whose elements are functions $f(x)$ measurable in (0, 1), and with norms $\|f\|_x$ and $\|f\|_y$. We assume that convergence in measure is implied by convergence in norm. Then $X \subset Y$ if $\|f\|_y \leqslant C\,\|f\|_x$. We shall hereafter say that $x \subset y$ if there exists a constant C of the above type. In this sense we have the following results : (i) $\Lambda(\alpha, p) \subset \Lambda(\beta, q)$ if $\alpha p^{-1} < \beta q^{-1}$ or $\alpha p^{-1} = \beta q^{-1}$ and $p \leqslant q$, and (ii) $L^{p\alpha^{-1}+\epsilon} \subset \Lambda(\alpha, p) \subset L^{p\alpha^{-1}}$. But when $M(\alpha, p)$ replaces $\Lambda(\alpha, p)$, the inequalities are reversed.

Consider now the series $\frac{1}{2} a_0 + \sum_n (a_n \cos nx + b_n \sin nx)$, and let $\sigma_n(x)$ denote the first arithmetic mean of the sequence of partial sums of the above series. Then if $\|\sigma_n\| \leqslant \eta$ in the metric of the space of $\Lambda(\alpha)$ or of that of $M(\alpha)$, then the series above will be the Fourier series of a function $f(x) \in \Lambda(\alpha)$ or $M(\alpha)$ and $\sigma_n \to f$ in the metric of the space.

We shall now pass on to a more general space of integrable functions which will include $\Lambda(\phi, p)$ and $M(\phi, p)$ as special classes. We start with a class of C of positive integrable functions $c(x)$ on (0, 1). Let C have the following properties:

(i) $1 \in C$;

(ii) C is normal in the sense that if $c(x) \in C$ and $c_1(x)$ is a measurable function such that $0 \leqslant c_1(x) \leqslant C(x)$, *a.e.* then $c_1(x) \in C$;

(iii) $\int_0^1 c(x)\, dx$, $c \in C$, are bounded.

Thus C contains necessarily all bounded functions. Let now $X(C)$ consist of all functions $f(x)$ for which

$$\|f\| = \sup_{c \in C} \int_0^1 c(x)\, |f(x)|\, dx$$

is finite. This space $X(C)$ consists of all measurable functions with $\|f\| < \infty$. Also all bounded functions necessarily belong to $X(C)$. Moreover the space $X(C)$ is normal and it can also be proved, using Fatou's lemma, that the space is a Banach space under the norm defined. Such spaces $X(C)$ are called Köthe-Toeplitz spaces. In particular if $c(x) = \phi_r(x)^{1/p}.\ g(x)$, where $\phi_r(x)$ are all the rearrangements of a positive decreasing integrable function $\phi(x)$ and $\int |g|^q\, dx \leqslant 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then $X(C)$ reduces to the space $\Lambda(\phi, p)$. Similarly we can arrive at $M(\phi, p)$. Also, the spaces considered above have all the property of being rearrangement invariant, i.e. for a function $f(x)$ of the space, $\|f\| = \|f_r\|$ for all rearrangements $f_r(x)$ of $f(x)$. For a space $X(C)$ this property is true if and only if

$$\|f\| = \sup_{c \in C} \int_0^1 c^*(x)\, f^*(x)\, dx.$$

We shall now derive the form of a linear continuous functional, in the spaces we have defined.

Let X be a Banach space of integrable functions in $(0, 1)$. Let the following properties hold in X:

(i) X contains the function 1;

(ii) X is normal;

(iii) If $f \in X$ and χ_e is the characterestic function of the measurable set e, then $\|f\chi_e\| \to 0$ with $m(e) \to 0$.

With this definition we have the following general result. Let X satisfy (i) — (iii) above and let Y consist of all those functions $g(x)$ measurable in (0, 1) and for which $\int_0^1 fg\,dx$, exist for all $f \in X$. Then $F(f) = \int_0^1 fg\,dx$, $g \in y$ is the general form of a linear continuous functional on X and

$$\|g\| = \sup_{\|f\| \leqslant 1} \int_0^1 fg\,dx < +\infty.$$

From the above result every function $g(x)$ in Y gives rise to a linear continuous functional over X and thus the space Y can be identified with the space X^* of linear continuous functionals of X. Also it is easily verified that Y is a Banach space. The space X^* is called the conjugate space of X and now we shall forget all the difference between X^* and Y and say that the functions $g(x)$ constitute X^*.

If now the space X has the additional property that

(iv) if $f_n(x) \to f(x)$ a.e., $f_n(x) \in X$ and $\|f_n\| \leqslant M$ then $f(x) \in X$,

then the existence of $\int fg\,dx$ for all $g \in Y$ will imply that $f \in X$. Also now the space Y satisfies (i) and (iii) when X satisfies (i)—(iv). That Y satisfies (ii) is evident. Thus we shall get $Y^* = X$; i.e. each linear continuous functional $F(g)$ on Y has the form $\int fg\,dx$, where $f(x) \in Y$.

We shall now pass on to the particular spaces $\Lambda(\phi, p)$ and $M(\phi, p)$. We need a few definitions.

For two functions $f(x)$ and $g(x)$ which are positive on (0, 1), we write $f \prec g$ if $\int f\,dx < \int g\,dx$. Let now $\phi(x)$ be a fixed integrable function on (0, 1) and let $\Phi(x)$ denote its integral over $(0, x)$. Now, a function $G(x)$ is said to be concave with respect to $\phi(x)$ on (0, 1), if for arbitrary a and b in (0, 1)

$$\frac{G(x) - G(a)}{\Phi(x) - \Phi(a)} \geqslant \frac{G(b) - G(a)}{\Phi(b) - \Phi(a)}, \; a < x \leqslant b.$$

Now each function $G(x)$ bounded on (0, 1) has a least ϕ-concave majorant $G^0(x)$; i.e. a smallest ϕ-concave $\bar{G}(x)$ satisfying $\bar{G}(x) \geqslant G(x)$. Hereafter we consider only functions of the type $G(x) = \int_0^x g(t)dt$, where $g(t)$ is integrable. A function $g(x)$ is said to be decreasing with respect to $\phi(x)$ if and only if $G(x) = \phi(x)\, D(x)$, where $D(x)$ is decreasing. For any function $G(x) = \int g(x)\, dx$, the function $G^0(X)$ is also of the form $\int g^0\, dx$ and for any function $g(x)$ the function $g^0(x)$ is called the level function of $g(x)$ w.r.t. $\phi(x)$. $g^0(x)$ is now the smallest (in the sense of $\prec$) $g_1(x)$ of the form $g_1 = \phi D$ which satisfies $g_1 \succ g$. With these definitions, we have

The general form of a linear continuous functional $F(f)$ on $\Lambda(\phi, p), p \geqslant 1$ is given by $\int_0^1 fg\, dx = F(f)$, where $g(x)$ is an arbitrary function of $\Lambda^*(\phi, p)$, where $\Lambda^*(\phi, p) = M(\phi)$, $p = 1$ and consists of all functions $g(x)$ which have a level function $g^0 = (g^*)^0 = \phi D^0$ with $\int_0^1 \phi D^0\, dx < \infty$, if $p > 1$; and the norm in the space $\Lambda^*(\phi, p)$, $(p > 1)$ is given by

$$\| g \|_{\Lambda^*} = \left\{ \int_0^1 \phi D^{0q}\, dx \right\}^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

The space $\Lambda^*(\phi, p)$ has the interesting property that the conjugate space of $\Lambda^*(\phi, p)$ is the space $\Lambda(\phi, p)$ if $p > 1$. But with the space $M(\phi, p)$ the situation is slightly different and is somewhat similar to the one that prevails for the space L for which we have $L^* = M$, but $M^* \supset L$, where M is the space of bounded functions. Now, let $\phi(x)$ be an integrable, positive function, decreasing in (0, 1). Let as usual, $\phi(x)$ denote the integral of $\phi(t)$ over $(0, x)$. Let also $\psi(x) = [\phi(x)]^{1/p}$. Then for a positive function $g(x)$ we say $g \underset{p}{\prec} \psi$ if

$$\int_0^a g^p\, dx \leqslant \int_0^a \psi^p\, dx = \Phi(a), \quad 0 \leqslant a \leqslant 1.$$

Let now $\Gamma(\phi, p)$ consist of all measurable functions $f(x)$ on (0, 1) for which

$$\|f\| = \sup \int_0^1 f^* g \, dx < \infty$$

the supremum being taken for all $g(x)$ with the property $g^* \underset{p}{\prec} \psi$. The space $\Gamma(\phi, p)$ is a space of the type $X(C)$ defined earlier and consequently a Banach space under the above norm. It is this space $\Gamma(\phi, p)$ which has the property that $\Gamma^*(\phi, p) = M(\phi, p)$.

We define the dual space $\tilde{X}$ of the space X of all functions $f(x)$ integrable in (0, 1) as the space of all functions $g(x)$ for which

$$\|g\| = \sup_{\|f\| \leqslant 1} \int_0^1 |f| \, |g| \, dx < +\infty,$$

since each $g(x)$ defines a linear continuous functional on X, $\tilde{X} \subset X^*$. If $\tilde{\tilde{X}} = X$, then X is said to be perfect. With this definition, we have that $\tilde{\Gamma} = \Gamma^* = M$, $\tilde{M} = \Gamma$ and thus Γ and M are perfect.

We shall next take up the solution of the moment problem for the spaces defined hitherto. We shall perhaps start best with the most general space $X(C)$. We need a few preliminaries.

Given a sequence $\{\mu_\nu\}$ we shall denote by $\{\mu_{n\nu}\}$, the sequence $\left\{\binom{n}{\nu} \Delta^{n-\nu} \mu_\nu\right.$; also let $f_n(x) = (n+1)\, \mu_{n\nu}$, $\dfrac{\nu}{n+1} \leqslant x < \dfrac{\nu+1}{n+1}$. For a function $f(x)$ belonging to the space $X(C)$ the indefinite integrals

$$F(e) = \int_e f(x) \, dx, \ f \in x, \ \|f\| \leqslant 1$$

are said to have the property of uniform absolute continuity if for each $\epsilon > 0$ there is a $\delta > 0$ such that $m(e) \leqslant \delta$ implies $|F(e)| \leqslant \epsilon$ for all the functions above.

Let the space $X(C)$ we start with, have the property of rearrangement invariant norm. Also, let the integrals defined just now have the property of being uniformly absolutely continuous. Then the moment problem has a bounded solution $f(x)$ in $X(C)$ if, and only if, the norms of the functions $\{f_n(x)\}$ satisfy the conditions $\|f_n\| \leqslant M$.

In the case of the particular spaces, we have a better picture of this result. For example, for the space L^p, the last condition is equivalent to the existence of a M such that

$$\sum_{\nu=0}^{n} | \mu_{n\nu} |^p \leqslant M (n + 1)^{-p+1}, \quad n = 0, 1, \ldots$$

and for the space of bounded functions, to

$$| \mu_{n\nu} | \leqslant M(n + 1)^{-1}.$$

For the space $\Lambda(\phi, p)$ the sequence μ_n is a moment sequence if and only if

$$\sum_{\nu=0}^{n} \Phi_{n\nu} \, \mu_{n\nu}^{*p} \leqslant M(n + 1)^{-p},$$

where

$$\Phi_{n\nu} = \sum_{\nu/n+1}^{(\nu+1)/(n+1)} \phi(x) \, dx$$

and $\mu^*_{n\nu}$ are the rearrangements of $| \mu_{n\nu} |$ in the decreasing order. In particular for the space $\Lambda(\alpha)$ this reduces to

$$\sum_{\nu=0}^{n} (\nu + 1)^{\alpha-1} \, \mu^*_{n\nu} \leqslant M \, (n + 1)^{\alpha-1}.$$

Also for the space $M(\phi, p)$ the condition could be similarly reduced.

I shall now recall the necessary and sufficient condition for the moment problem to have a solution in the space of functions continuous in (0, 1). The sequence μ_n is a moment sequence for this space C if and only if

$$\sup_n \sum_{\nu=0}^{n} \binom{n}{\nu} | \Delta^{n-\nu} \mu_\nu | < \infty.$$

The significance of this result in the general summability theory can now be easily brought out. We are all familiar with particular methods as the method of the arithmetic means called the $(H, 1)$ method or the $(C, 1)$ method, whose further generalizations are the Hölder and Cesàro methods and also with the Euler method of summability of sequences by transforming the sequences into convergent sequences. These methods are particular cases of a general class of summability methods involving a sequence-to-sequence transformation. Among such general class of methods we have what are called the Hausdorff methods of summability, which by themselves are quite general and include the classical method mentioned above. These methods were originally found out by Hurwitz and Silverman and were re-discovered by Hausdorff who systematically developed this class of methods. The Hausdorff methods are defined by the matrix $(H, \mu_n) \equiv H \equiv (h_{nk}) = \binom{n}{k} \Delta^{n-k} \mu_k \; (n \geqslant k)$ and $h_{nk} = 0, \; (n > k)$. For the matrix $H \equiv (H, \mu_n)$ to define a conservative or regular sequence-to-sequence transformation it must satisfy necessarily $\sup\limits_n \sum\limits_k |h_{nk}| < \infty$, i.e. $\sup \sum\limits_{k=0}^{n} \binom{n}{k} |\Delta^{n-k} \mu_k| < \infty$.

Now the solution of the moment problem for the space of continuous functions of $(0, 1)$ shows clearly that the condition above is satisfied if and only if the sequence μ_n is a moment sequence for the space C, i.e. if and only if there exists a function $\chi(u)$ of bounded variation in $(0, 1)$ such that

$$\mu_n = \int\limits_0^1 \mu^n \, d\chi(u).$$

When this condition is satisfied we have also

$$\lim_n h_{nk} \text{ exists for each } k \text{ and } \lim_n \sum_k h_{nk} \text{ exists,}$$

and thus the transformation by the Hausdorff matrix will be conservative (for sequence-to-sequence transformations) if and only if μ_n is a moment sequence. If in addition $\chi(+0) = \chi(0) = 0, \chi(1) = 1.$

then

$$\lim_n h_{nk} = 0 \text{ and } \lim_n \sum_k h_{nk} = 1,$$

and we prove that the transformation by (H, μ_n) is regular if and only if the above two additional conditions are satisfied, when we say that μ_n is a regular moment constant. Now associated with a moment sequence we have another interesting property. We start with the matrix (H^*, μ_n) the transpose of the matrix (H, μ_n) and called the quasi-Hausdorff matrix. Now it can be shown that the condition

$$\sup_n \sum_{k \geqslant n} \binom{k}{n} |\Delta^{k-n} \mu_{n+1}| < \infty$$

is satisfied if and only if μ_n is a moment constant (for C) and this enables us to prove that the series-to-series transformation defined by a quasi-Hausdorff matrix (H^*, μ_n) is conservative if and only if μ_n is a moment constant and it is regular transformation stronger than convergence, if and only if the sequence μ_n is a regular moment sequence. Thus if we start with a moment sequence or a regular moment sequence we have two matrices (H, μ_n) and (H^*, μ_n) such that the former defines a conservative or regular sequence-to-sequence transformation and the latter a conservative or regular series-to-series transformation; it is also known that the roles of the two matrices can be interchanged. Also the various properties of the two methods such as translativity, Borel property, strong regularity, absolute regularity of these methods can all be characterized in terms of the function $\chi(u)$ which generates the moment constants. Indeed these show that there is a close relationship between the Hausdorff methods defining the sequence-to-sequence transformation and the quasi-Hausdorff matrices defining the transformations of series into series. This naturally raises the question as to what we can say regarding the capacity of the two methods to sum a sequence $\{s_n\}$ which is the partial sum sequence of Σa_n? To this question we do not have a complete answer as yet; however when the sequences concerned are bounded and are Borel summable then the two methods (H, μ_n) and (H^*, μ_n) defined by regular moment sequences $\{\mu_n\}$ both include Borel's method if and only if the function $\chi(u)$ is such that $\chi(1) = \chi(1-0)$. But when the sequences are not subjected to the above two restrictions, i.e. for unrestricted sequences, the result that (H, μ_n) summability of $\{s_n\}$ implies the (H^*, μ_n) summability of $\Sigma\ a_n$ is false.

[NOTE ADDED IN PROOF: Only at the time of proof correction I found, to my dismay, that I have nowhere mentioned that this talk is mostly a coverage of Lorentz's work [4], [5]; the spaces $\Lambda(\phi, p)$ and $M(\phi, p)$ were defined by him and their properties mentioned here are his results.]

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SYMPOSIUM ON RESEARCH AND MATHEMATICAL DEVELOPMENT IN INDIA

Chairman : Professor V. GANAPATHY IYER

In connection with this symposium the Programme Committee had sent a request to the Universities and Research Institutes to prepare and submit to the Society a memorandum on the following points:

1. Research facilities (by way of library, research scholarships or fellowships) available in the Institution concerned.
2. Topics in pure and applied mathematics, in which advanced study or research is carried on.
3. Whether the majority of the teaching staff are engaged in research and whether research qualifications are insisted upon for teachers in institutions under the University where post-graduate teaching is done.
4. Opportunities for employment as teachers or research workers for those who have research aptitude.
5. Any general suggestion as to how to improve research work in the country as a whole and the concrete steps which the Indian Mathematical Society may take in this connection.

In response to the above request, the Society received the memoranda from the following Universities and Institutions: Aligarh, Allahabad, Annamalai, Delhi, Madras and Sri Venkateswara Universities, Ramanujan Institute of Mathematics, and the Indian Institute of Technology, Kharagpur. In addition, Dr. S. Minakshisundaram explained the position as regards the Andhra University. The memoranda were presented at the Conference by the delegates of the various institutions concerned.

Only in some of the Universities research qualifications are insisted upon for recruitment to teaching jobs at the post-graduate level. Also it was the expressed view that the opportunities for the research worker to get employed at his own place happen to be very few. This largely handicaps the development of a school in any particular branch. Moreover, when persons with research qualifications are recruited for the jobs, proper weightage is not given for the time they have spent in full-time research. To improve research work in this country as a whole and the concrete steps the Indian Mathematical Society may take up in this direction, several suggestions were put up and they centred mainly round the following :

(i) There must be an organised effort to bring together many scholars who have common or allied mathematical interests and also to build up schools of different branches of mathematics at different centres in India.

(ii) Those in charge of guiding research can insist on research workers equipping themselves with as wide a background of mathematical knowledge as possible. Also they must discover talent and stimulate a passion for mathematical research.

(iii) There should be a fairly free exchange of scholars, at all levels—professorial or that of research scholars—between various centres of mathematical activity. In particular, the student should be encouraged to spend as much time as he wants at any particular place, even though he may be financially assisted by a particular University or Institute. Also, the Universities should encourage research work at the level of the staff members by giving adequate and easy leave facilities for spending some time in full-time research.

(iv) For University teachers of M.A. or M.Sc. classes, a Doctorate degree or its equivalent must be insisted upon and at the time of appointment, the period spent in full-time regular research work must be taken into account in fixing the initial salary. In this direction the Indian Mathematical Society can suggest to the various Universities that research work should be insisted upon as an essential qualification for appointment and also that the Universities should give increased facilities for mathematical research. But it must be borne in mind that this should not lead to an "in-breeding" or taking in only those who have done work at that particular centre. This will ultimately greatly retard the progress of mathematical research.

(v) The Indian Mathematical Society should take up these facts and persuade the Central Government in particular :

(*a*) to give increased financial assistance to the Ramanujan Institute of Mathematics and develop it into a truly all-India Institute, by expanding its activities and also to continue to give its substantial aid to the Tata Institute of Fundamental Research ;

(*b*) to open at least two more research institutes for mathematical research ;

(*c*) to utilize the mathematical talent available in the country in industrial, social and commercial problems and to solve problems of national interest, such as flood control.

In winding up the discussions, the Chairman said that all the proposals put forward will be considered and placed also before the Council of the Society for necessary consideration and action.

ABSTRACTS

ALGEBRA AND THEORY OF NUMBERS

D. R. KARPEKAR, Deolali : *Criterion for a self-number.*

A computation is suggested for checking whether a number is a self-number [*Math. Student*, 1952, 22-23] or not, and some self-numbers within 5000 are tabulated.

S. MANZUR HUSSAIN, Pakistan : *Evaluation of an eleventh ordered determinant in the theory of partitions.*

R. SRIDHARAN, Bombay : *On some algebras of infinite cohomological dimension.*

G. Hochschild [*Duke Math. Jour.* 1947] has proved the following theorem : If Λ is a nilpotent algebra of finite rank over a field K, the cohomological dimension of Λ is greater than or equal to 3. It is proved here that the dimension is actually infinite. As a consequence, it is deduced that the cohomological dimension of the Grassmann ring on n-letters over a commutative semi-simple ring is infinite. This provides incidentally counter examples to certain questions in homological algebra. A simple proof of the result that the cohomological dimension of the ring of formal power series in one variable over a field is infinite, is also given. For this a theorem of Eilenberg-Rosenberg-Zelinsky [to appear] is needed. These authors prove the theorem using spectral sequences; an alternative elementary proof without using spectral sequences is given here.

M. V. SUBBA RAO, Tirupati : *Some properties of quadratic residues.*

In this note, the author obtains generalizations of the results of J. B. Kelly [*Proc. American Math. Soc.* 5 (1954), 38-46] and Hansraj Gupta [*Math. Student*, 23 (1955) 106-107]. Let p be an odd prime and R_i, L_i, $i = 0, 1, \ldots, n-1$, denote respectively the set of integers of the form $(p^i.q)$, $(p, q) = 1$, for which p is a quadratic residue or non-residue modulo p^{n-i}. Let t be an arbitrary integer and

S any set of integers and $t \oplus S$ denote the set of integers obtained by adding t to each of the elements of S. Let r_i, l_i be arbitrary members of R_i, L_i respectively. Let $j = p^{n-i-1}\left[\frac{p-1}{4}\right]$, $K = p^{-n-i-1}.\left[\frac{p-3}{4}\right]$, where $[x]$ stands for the greatest integer not exceeding x ; $m = p^{n-i-1}$ or 0 according as $p \equiv 1$ or $-1 \pmod 4$; $m_1 = p^{n-i-1} - m$. Then the author proves :

THEOREM 1. *Each of the sets* $l_i \oplus R_i$; $r_i \oplus L_i$ *gives* j *members of each of the sets* L_i, R_i *together with* m *members which are multiples of* p^{i+1}. *These* m *numbers are made up of* $\phi(p^{n-\alpha})$ *numbers of the form* $(p^\alpha . q)$, $(p, q) = 1$, $\alpha = i+1, i+2, \ldots, n$.

THEOREM 2. (*a*) *The set* $n_i \oplus R_i$ *gives* K *numbers of* R_i *and* $K + p^{n-i-1}$ *numbers of* L_i *together with* m_1 *numbers which are multiples of* p^{i+1}. (*b*). *The set* $l_i \oplus L_i$ *gives* K *numbers of* L_i *and* $K_i + p^{n-i-1}$ *numbers of* R_i *and* m_1 *numbers which are multiples of* p^{i+1}. *In either case these* m_1 *numbers are made up of* $\phi(p^{n-\alpha})$ *numbers of the form* $(p^\alpha . q)$, $(p,q) = 1$, $\alpha = i+1, \ldots, n$.

The author shows that these results themselves follow easily from Vaidyanathaswamy's class algebra of quadratic residues [*J. Indian Math. Soc.* (2) II, 239-248].

K. VARADARAJAN, Bombay : *On the rank of a reductive Lie algebra.*

It is proved that the rank of a reductive algebra $\mathscr{G}$ over a field of characteristic zero is equal to the number of linearly independent homogeneous primitive elements of the cohomology algebra $H^*(\mathscr{G})$ of $\mathscr{G}$.

ANALYSIS

S. H. DWIVEDI, Aligarah : *A note on an entire function of integral order.*

Let $f(z)$ be an entire function of integral order ρ. Let $n(r)$ and $N(r)$ have their usual meaning. Let $\phi(x)$ be any positive, continuous, function such that $\int^{\infty} \frac{dx}{x\,\phi(x)} < \infty$. The purpose of this note is to prove the following :

THEOREM 1. *If $f(z)$ is an entire function of integral order and of the same genus as that of its canonical product, then*

$\limsup_{r\to\infty} N(r)\,\phi(r)/\log M(r) = \infty$ *for any positive $\phi(r)$ such that* $\int^{\infty} \frac{dx}{x\,\phi(x)}$ *converges.*

We remark that the above theorem is still true if $f(z)$ instead of being a canonical product, is of the form $(z) = e^{g(z)}\,P(z)$ provided that $\rho = P(p \geqslant 0)$.

THEOREM 2. *If $f(z)$ is an entire function of zero order then*

$$\limsup_{r\to\infty} N(r)\,\phi(r)/\log M(r) = \infty.$$

Theorems 1 and 2 are analogous to a result of S. M. Shah [*Jour. L. M. S.* 15 (1940)] where he proves them for $n(r)$ instead of $N(r)$.

S. H. DWIVEDI and S. K. SINGH, Aligarh : *On proximate order and a-points of an entire function.*

Let $f(z)$ be an entire function of order $\rho (0 < \rho < \infty)$ and $\rho(r)$ be its proximate order L and let $n(r, a)$ and $N(r, a)$ have their usual meanings. The authors prove the following theorems.

THEOREM 1. *If $N(r, a)/r^{\rho(r)} \to 0$ as $r \to \infty$ then*

$$0 < \liminf_{r\to\infty} N(r, x)/r^{\rho(r)} \leqslant \limsup_{r\to\infty} N(r, x)/r^{\rho(r)} \leqslant 1 \text{ for all } x \neq a.$$

THEOREM 2. *If $f(z)$ be an entire function of finite non-zero order for which $n(r, a)/\log M(r, f) \to 0$ as $r \to \infty$, then*

$$\liminf_{r\to\infty} N(r, a)/\log M(r, f) = 0.$$

THEOREM 3. *If* $\liminf_{r\to\infty} N(r, a)/r^{\rho(r)}$ *exists then* $\lim_{r\to\infty} n(r, a)/r^{\rho(r)} = \rho \lim_{r\to\infty} N(r, a)/r^{\rho(r)}$.

HARI SHANKAR, Moradabad: *On the characteristic function of a meromorphic function.*

Let $w(z)$ be a meromorphic function of order ρ and lower order λ. The author has already established some inequalities elsewhere [*Tohoku Math. J.* to appear] between τ, t and μ, ν which are the limits of indetermination as $r \to \infty$ of the ratios $T(r)/t^{\rho}$ and $S(r)/r^{\rho}$ respectively, where $T(r)$ and $S(r)$ have their usual meanings. [R. Nevanlinna, *Eindeutige Analytische Funkctionen*, Chap. VI, § 3]. Here the following results are proved.

THEOREM 1. *If* $w(z)$ *is of perfectly regular growth and of order* p *then* $\nu = \mu$.

THEOREM 2. *If* $w(z)$ *be of order* ρ $(0 \leqslant \rho \leqslant \infty)$ *then*

$$\liminf_{r\to\infty} S(r)/T(r) \leqslant \lambda \leqslant \rho \leqslant \limsup_{r\to\infty} S(r)/T(r).$$

COROLLARY. *A sufficient condition that* $w(z)$ *be of regular growth and of order* ρ *is that* $S(r)/T(r) \to \rho$ *as* $r \to \infty$.

THEOREMS 3. *If* $w(z)$ *be of regular growth and of order* ρ *then* $\lim_{r\to\infty} (T(r))^k/S(r) = \infty$ *if* $k > 1$ *and* $\limsup_{r\to\infty} T(r))^k/S(r) < \infty$ if $k < 1$.

M. ISHAQ, Lucknow: *On the eigen-solutions of a sequence of bounded Hermitian operators.*

Let $\{A_p\}$, $(p = 0, 1, 2, \ldots)$, be a strongly convergent sequence of Hermitian operators A_p bounded by a positive constant M. Let x_0, $(\|x_0\| = 1)$, be an element of the Hilbert space; the elements x_q^p, $(q = 0, 1, 2, \ldots)$, are called *consequents* of x_0 by A_p if these satisfy the relations: $A_p x_0 = l_1^p x_1^p$, $A_p x_1^p = l_2^p x_2^p$, $\ldots$, $A_p x_{q-1}^p \ldots = l_q^p x_q^p$, where $\| A_p x_{q-1} \| = l_q^p$ and $\|x_q^p\| = 1$, $(q = 0, 1, 2, \ldots)$. These give $A_q^p x_0 = l_1^p . l_2^p \ldots l_q^p x_q^p$. The expression on the left hand side is called the *iterate* of x_0 of rank q and x_q^p the *consequent* of rank q. Define $\omega^p(x_0)$ and $\omega(x_0)$ from a sequence of the consequents of x_0 as

$\omega^p(x_0) = \frac{l_1^p}{l^p} \cdot \frac{l_2^p}{l_2^p} \ldots \omega(x_0) = \frac{l_1}{l} \cdot \frac{l_2}{l} \ldots$. If $l_1^p = 0$ or $l_1 = 0$, we do not define $\omega^p(x_0)$ and $\omega(x_0)$. We prove

THEOREM 1. *Let* $l_1^p \neq 0, l_1 \neq 0, \lim_{p \to +\infty} \omega^p(\chi_0) = 0$ *and* $\lim_{p \to +\infty} \omega^p(x_0) = \omega(x_0)$, *where* $\omega(x_0) \neq 0$. *The consequents* x_{2p}^p *converge strongly to an eigensolution of the operator* A_p^n, *where* $n \geqslant 2$ *is a positive integer.*

THEOREM 2. *If an element* x_0 *is orthogonal to an eigen-solution of the operator* A_p^q, *subject to the conditions of Theorem* 1, *then so are all its consequents.*

B. N. PRASAD and T. PATI, Allahabad : *On the theorems of consistency in the theory of absolute Riesz summability (Preliminary announcement).*

The direct analogue for absolute summability of Hardy's well-known second theorem of consistency for Riesz summability (*PLMS*, 1916) was demonstrated for integral orders of summability by Chandrasekharan (*JIMS*, 1942). Naturally, in this theorem the relation between the two types was logarithmico-exponential. In 1954 (*QJM*, Oxford) Pati extended the scope of applicability of the second theorem of consistency for absolute Riesz summability for positive integral orders. The following extension of Pati's theorem for the more abstruse case in which the order of summability is positive and non-integral, has recently been established by Prasad and Pati (*TAMS*, 1957).

THEOREM. *If* $\phi(t)$ *is a non-negative and monotonic increasing function of* t *for* $t \geqslant 0$, *steadily tending to infinity as* t *tends to infinity, such that* $\phi^{(1)}(t)$ *is monotonic nondecreasing for* $t \geqslant 0$, $\phi(t)$ *is a* $(k+2)$*th indefinite integral for* $t \geqslant 0$, *where* k *is the integral part of* K, *and* $t^r \phi^r(t)/\phi(t) \in \mathrm{BV}(h, \infty)$ $(r = 1, 2, \ldots, k+1)$, *where* h *is a finite positive number, then any infinite series which is summable* $|R, \lambda_n K|$, *is also summable* $|R, \phi(\lambda_n), K|$.

In the present paper this theorem of Prasad and Pati is proved under the less restrictive hypothesis $t^r \phi^{(r)}(t)/\phi(t) \in \mathrm{B}(h, \infty)$ $(r = 1, 2, \ldots, k+1)$, where h is a finite positive number.

M. S. Ramanujan, Madras : *The 'Translativity' problem for quasi-Hausdorff methods of summability.*

The 'translativity' problem for quasi-Hausdorff methods defining conservative series-to-series transformations is discussed in the paper and the following theorem is proved.

Theorem : *Let* (H^*, μ_n) *and* $(H^*, \mu_{n+1}/\mu_n)$ *be both* δ_0*-matrics. Then for the class* $\mathscr{B}$ *of series with bounded partial sums* (H^*, μ_n) *is translative to the left; if in addition the limit constant associated with* $\{\mu_{n+1}/\mu_n\}$ *is not* $\frac{1}{2}$, *then* (H^*, μ_n) *is translative, again for the class* $\mathscr{B}$.

The proof of the theorem indicates that the relaxation of boundedness condition on the sequence of partial sums is possible in certain cases where the matrix (H^*, μ_n) satisfies suitable additional restrictions. The results of Vermes [*American J. Math.* 71, 541-562, (1949), Theorems 3 III and 3 IV] on the translativity of the method $A(p)$ of Taylor series continuation are deduced. These results are true without any restriction on the sequence of partial sums.

C. S. Seshadri, Bombay : *Generalized multiplicative mermorphic functions on a Riemann surface.*

It is proved that every divisor (in the sense of A. Seil : *Comment. Math. Helve.* 1947) on an open Riemann surface is the divisor of a "generalized" multiplicative meromorphic function. From this it follows that every analytic vector-bundle on the unit disc or the plane is analytically trivial. This, in combination with a result of G. D. Birkhoff [*Collected Works*, I, pp. 240-251] on matrices of analytic functions, allows us to prove that every vector-bundle on the Riemann sphere is a direct sum of one-dimensional vector-bundles.

H. M. Sengupta and P. L. Ganguli, Calcutta : *On a class of steadily increasing continuous functions which are not absolutely continuous.*

In the present paper, the authors intend to build a class of Cantor functions, each continuous and steadily increasing in $0 \leqslant x \leqslant 1$, none of which is an integral. This class has the power of the continuum and the classical example of Cantor function appears as a degenerate case.

S. K. SINGH, Aligarh: *On the minimum modulus of an entire function.*

Let $f(z)$ be an entire function of order $\rho(0 < \rho < \infty)$. Let $m(r) = m(r, f) = \min \ |f(z)|$ on $|z| = r$. The purpose of this note is to explore a class of entire functions for which $m(r) \to 0$ as $r \to \infty$. In this note the following theorems are proved.

THEOREM 1: *If $f(z)$ be an entire function having '0' as an e.v. E. then $m(r) \to 0$ as $r \to \infty$.*

The result automatically carries over to e.v. P. and e.v. B. because e. v. P. or e.v. B $\to$ e.v. E.

THEOREM 2: *If $f(z)$ be an entire function having '0' as an e.v. N. then $m(r) = O(1)$.*

For the definition of e.v.E etc. see S. M. Shah: *Compositio Mathematica* VI. 9 (1951) pp. 227-238.

U. N. SINGH, Aligarh: *Fourier-Carleman transform and limits of a class of analytic functions.*

Let $C(k)$ denote the class of functions f which are integrable in the sense of Lebesgue in every finite interval and which satisfy the condition $\int_0^x |f(t)|\, dt = O(|x|^k)$, as $|x| \to \infty$, where $k \geqslant 0$. (g_1, g_2) denote the couple of analytic functions $g_1(z)$ and $g_2(z)$, regular for $I(z) > 0$ and $I(z) < 0$ respectively, which represent the Carleman-Fourier transform of the couple (f_1, f_2), or of the function $f \in C(k)$, in the sense of Carleman [see his book *L'Integrale de Fourier*]. The author has proved the following

THEOREM. *Let $f \in C(k) = (0 \leqslant k \leqslant \infty)$. A necessary and sufficient condition that $f(x)$ be the limit function of an analytic function $f(z)$, regular for $I(z) > 0$ which belongs to the class $(\gamma, k-1)$ if k is non-integral, or to the class $(\gamma, k-1+\epsilon)$ if k is integral, where*

$0 < \epsilon < 1$, $1 < \gamma < 2$, *is that* $\lim \{g_1(x+iy) - g_2(x-iy)\} = 0$, *uniformly in every closed finite interval of the negative real axis.*

This theorem generalizes a theorem of the author [*C.R.* (Paris) 236 (1953)] which in its turn is a generalization of a theorem of Hille and Tamarkin [*Ann. Math.* (2), 24, 1933].

S. R. SINHA, Allahabad : *The summability factors of infinite series.*

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. It may be assumed without loss of generality, that the constant term in the Fourier series of $f(t)$ is zero and $f(t) \sim \Sigma\, (a_n \cos nt + b_n \sin nt) = \Sigma\, c_n(t)$, $c_0 = 0$. Also let $\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}$. Cheng (*Duke Math. Jour.* 15, 1948, 17-27) obtained the following result : If $\{\lambda_n\}$ is any one of the sequences $\{1/(\log n)^{1+\epsilon}\}$, $\{1/\log n\ (\log\log n)^{1+\epsilon}\}, \ldots$

$$\{1/\log n \ldots \log\log \ldots \log_{p-1} n\, (\log_p n)^{1+\epsilon}\}, \ (\epsilon > 0).$$

and $\int_0^t |\phi(u)|\, du = o\,(t)$, as $t \to 0$, then the series $\Sigma \lambda_n c_n(t)$, at $t = x$, is summable $|C, \alpha|$ for every $\alpha > 1$. Pati (*Duke Math. Jour.*, 21, 1953, 271-284) extended this theorem to the case in which $\{\lambda_n\}$ is a convex sequence, such that the series $\Sigma\, n^{-1} \lambda_n$ is convergent. The object of the present paper is to extend Pati's theorem and to prove :

THEOREM. *If* $\{\lambda_n\}$ *is a convex sequence such that the series* $\Sigma\, n^{-1}\lambda_n$ *is convergent, then the series* $\Sigma\, \lambda_n\, c_n\, (t)$, *at* $t = x$, *is summable* $|C, \alpha|$ *for every* $\alpha > 1$, *provided that*

$$\int_0^t |\phi_1(u)|\, du = o\,(t), \text{ as } t \to 0, \text{ where } \phi_1(t) = t^{-1} \int_0^t \phi(u)\, du.$$

SULAXANA KUMARI, Allahabad: *On the logarithmic summability of the successively derived series of a Fourier series and its conjugate series.*

Suppose that $f(x)$ is integrable (L) over $(-\pi, \pi)$ and periodic outside this range with period 2π. Let the rth successively derived series of the Fourier series of $f(x)$ and of the conjugate series of the

Fourier series of $f(x)$ be (i) $\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ and (ii) $\sum_{n=1}^{\infty} \frac{d^r}{dx^r} (b_n \cos nx - a_n \sin nx)$, respectively. Suppose that $P(t)$ is a polynomial of $(r-1)$th degree in t, such that

$$g(t) = (1/2t^r) [\{f(x+t) - P(t)\} + (-1)^r \{f(x-t) - P(-t)\}]$$

and

$$h(t) = (1/2t^r) [\{f(x+1) - P(t)\} - (t)^r \{f(x-t) - P(-t)\}]$$

are integrable (L) over $(-\pi, \pi)$ and periodic outside this range with period 2π. Then the following theorems regarding the logarithmic summability of (i) and (ii) at a point x have been obtained:

THEOREM 1. *If the Fourier series of $g(t)$, at $t = 0$, be summable $(R, \log w, r + \delta)$, $\delta \geqslant 0$, to sum s, and if its $(R, \log w, \delta)$ mean be of order $o\{(\log w)^r\}$, then the series* (i) *is summable $(R, \log w, r + \delta)$, to sum $r!s$, at the point x.*

THEOREM 2. *If the conjugate series corresponding to $h(t)$, at $t = 0$, be summable $(R, \log w, r + \delta)$, $\delta \geqslant 0$, to sum s, and if its $(R, \log w, \delta)$ mean be of order $o\{(\log w)^r\}$, then the series* (ii) *is summable $(R, \log w, r + \delta)$, to sum $r!s$, at the point x.*

S. SWETHARANYAM, Annamalainagar : *On the function $\sigma_k(n)$.*

Let, for $k > 1$, $\sigma_k(n)$ denote the sum of the kth powers of the positive divisors of n. Then, in this paper, it is shown that the ratio $\sigma_k(n-1)/\sigma_k(n)$ lies between $\alpha\beta/2^k\delta$ and $\delta/\alpha\beta$, and that the ratio $\sigma_k(n+1)/\sigma_k(n)$ lies between $\alpha\beta/\delta$ and $3^k\delta/2^k\alpha\beta$, where $\alpha = 1/\zeta(2)$, $\beta = 1/\zeta(2).\zeta(2k)$ and $\delta = \zeta(k)$, $\zeta(s)$ being the Riemann Zeta function. When $k = 1$, the function becomes $\sigma_1(n)$ which is the same as $\sigma(n)$, the sum of the positive divisors of n. In this case, Schinzel and Sierpinski [*Bull. Acad. Polon. Sci.* (III), 2 (1954), 463-466, Theorems 3 and 4] have proved that the maximum limit of the corresponding ratios is $+\infty$ and the minimum limit is 0. Analogous results for the Euler function have been established by the same authors [ibid. Theorems 1 and 2] and for the divisor function $d(n)$ by Schinzel [*Publ. Math. Debrecen*, 3 (1954), 261-262].

V. Lakshmikanth, Hyderabad : *On the asymptotic problems between the solutions of the differential systems.*

This is a continuation of the previous paper ' On the boundedness of solutions of non-linear differential systems ' submitted last year. Here some more asymptotic problems of solutions and their derivates have been considered. Sufficient conditions, for the existence of an order relation between the solution of a given system and that of its approximate system have been obtained.

V. Lakshmikanth, Hyderabad : *On the asymptotic correspondence between the solutions of the differential systems.*

In this paper we have considered the problem of asymptotic correspondence between the solutions of the system $x' = ax + by + f(x, y, t)$; $y' = cx + dy + f'(x, y, t)$ and that of its trivial system $x' = ax + by$; $y' = cx + dy$ where, a, b, c, d are real constants, with a determinant $ad - bc \neq 0$ and f, f' are real functions defined in the region $0 \leqslant t < \infty, r \leqslant \alpha$ where $x = r \cos \theta, y = r \sin \theta$. The primes denote the differentiation with respect to t.

APPLIED MATHEMATICS

C. D. Ghildyal, Lucknow : *On steady axially symmetric superposable flows.*

This paper is the extension of the paper of Ballabh on "Steady superposable flows with cylindrical symmetry" [*Ganita*, 6, (1955)]. Ballabh took the case of non-viscous fluids only. The author of this paper studies superposability defined in the sense of Ballabh [*Proc. Benaras Math.* 2, (1940)] in the case of steady axially symmetric flows of viscous and non-viscous homogeneous incompressible fluids. He obtains the general expression $\psi = 4\,vx + vF\,(\psi) \int \omega^2\, d\phi + G(\psi)$ for the current function of the rotational flow superposable on a given irrotational flow of which the current function is ψ. The other symbols used have their usual meanings. When the liquid is non-viscous it is proved that the current function of the irrotational flow on which a steady rotational flow is superposable, is necessarily independent of x. The stream lines of the irrotational flow are curves of zero curvature. Two alternative proofs are given.

P. C. JAIN, Delhi : *Density fluctuations in turbulence in an inviscid compressible fluid.*

The new theory of turbulence as presented by S. Chandrasekhar [*Proc. Royal Soc.* A, 238, 1955], has been applied to the problem of density fluctuations in stationary homogeneous turbulence in an inviscid compressible fluid. On the basis of the assumptions that the fourth order correlation is related to the second-order correlations in the same manner as in a joint Gaussian distribution, and that the variations in density and pressure are adiabatic, a differential equation in the density correlation is obtained and solved. It is found that each scale of the density fluctuation varies periodically with time independently of the others and is propogated through the medium with velocity $\sqrt{(c^2 + 1/3\, u^2)}$. An invariant of the type of Loitsiansky invariant is also deduced from the equation of continuity.

J. N. KAPUR, Delhi : *The evaluation of co-volume function in Hunt-Hinds and Goldie's methods of internal ballistics by the use of Russian tables.*

The evaluation of co-volume function by methods used in Britain and France requires extensive calculations. No tables are available and generally approximation formulae have been suggested. It has been shown earlier by the author that these formulae are not in general satisfactory. In the present paper it is shown how this co-volume function can be expressed in terms of ballistic functions tabulated in Russian literature and thus a great deal of labour can be avoided in tabulating the co-volume function by the methods which are in use in India.

J. N. KAPUR, Delhi : *Superposability in magneto-hydrodynamics.*

In the present paper, the concept of superposability defined earlier by Ram Ballabh for ordinary fluid flows has been extended to the case of hydromagnetic flows. The conditions of superposability and the equation for determining the adjusted pressure have been obtained. Self-superposable flows have been studied and in particular it has been shown that for magneto-hydrostatic situations, these are the same as the 'force-free' fields, studied by Chandrasekhar (*Proc. Nat. Acad. Sci.*, U.S.A., 1956). For axially symmetric force-free fields, Chandrasekhar has established three conditions for the defining scalars. It has been shown here that the third is not an independent condition, but follows at once from the first. The equation characterizing the general force-free axially symmetric fields has also been obtained.

In another paper (*Astroph J.* 1956) Chandrasekhar has obtained four differential equations for the four defining scalars for the vectors $\vec{q}$ and $\vec{H}$ for an inviscid liquid. We have obtained here the corresponding equations for viscous liquids and find that most of the general results esablished by Chandrasekhar for non-viscous case (including those giving the relations between poloidal and toroidal components) do not hold for viscous fluids.

It has also been shown that the equation giving the decay of magnetic field for magneto-hydrostatics situations remains unchanged for the more general case of self-superposable hydromagnetic situations.

J. N. KAPUR, Delhi : *Some results for motion of non-Newtonian liquids with variable coefficient of cross viscosity.*

In a recent paper, Bhatnagar and Lakshmana Rao (*Proc. Indian Acad. Sc.* 1957) have given some general results about non-Newtonian liquids satisfying the relationship $p_{ij} = -p\,\delta_{ij} + 2\,\mu\,d_{ij} + 2\,\mu_c\,d_{i\alpha}\,d_{\alpha j}$ between the stress tensor p_{ij} and the rate-of-deformation tensor d_{ij}, with coefficient of viscosity μ and coefficient of cross-viscocity μ_c as constant. In the present paper are examined how far these results remain true for the most general non-Newtonian liquids in which μ and μ_c are functions of invariants

$$I_2 = d_{i\alpha}\, d_{\alpha i}\ , \text{ and } I_3 = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix}$$

It is found that their results for two-dimensional flows, viz. that the conditions for superposability and conservation of circulation, the equation of vorticity, and the possibility of the use of inverse and semi-inverse methods remain unaffected by the presence of μ are true even if μ_c is variable and μ is constant; but if μ and μ_c are both variable, none of these results is true.

M. K. Jain (*Zeit. Math. and Mech.* 1955) has shown that Taylor's and Dean's results for the motion of an infinite rigid cylinder in a rotating liquid remain true even if we take μ_c (assumed constant) into account. In this paper is shown that these results remain true for the most general non-Newtonian fluids, i.e. even if we take the variation of both μ and μ_c into account.

PREM PARKASH, Lucknow: *Image of a source with regard to a circle.*

It is well known that the image of a source with regard to a circle consists of an equal source at the inverse point together with an equal sink at the centre. Since there is a singularity at the location of the source, Green's uniqueness theorem may not hold; consequently there may be more than one pattern of flow due to the source with the given circle as the inner boundary. This paper gives an image system different from that given in text books on the subject and discusses it in some detail. The circular boundary is first transformed into a linear one with the help of a conformal transformation so that the source transforms into an equal source. The image system is known for the linear boundary. When retransformed into the original coordinates this gives a kind of general image system due to the source with regard to the circle. It consists of an equal source at the inverse point and a number of sinks distributed over the circle such that the sum of their strengths is twice the strength of the given source. The solution is valid for $r > a$, where a is the radius of the circle.

P. C. VAIDYA and K. B. SHAH, Ahmedabad : *A radiating mass particle in an expanding universe.*

The Einstein-Straus view [*Rev. Mod. Phys*. 17, 120, 1945] that individual stars can be regarded as placed at the centres of spherical empty holes in the sorrounding cosmological field is not applicable to the case when the stars are radiating. This is seen by a simple application of O'Brien-Synge jump conditions at the boundary of the hole which will now be filled with flowing radiation. However, it is found that the McVittie view [*Monthly Not. Roy. Astron. Soc.* 93, 1933] of regarding these holes not as empty but filled by spherically symmetric distribution of matter of non-zero density gradually tending to the homogeneous cosmic fluid-distribution at large distances, continues to hold good even when the effect of radiation emerging from the star is taken into account. Details of the effect of the radiation emerging from a McVittie star are calculated.

GEOMETRY

M. K. SINGAL and RAM BEHARI, Delhi : *Sub-spaces of a generalized Riemann space.*

The present paper deals with sub-spaces of a generalized Riemannian space. Generalized Riemann spaces were defined and first studied by Eisenhart (*Proc. Nat. Acad. Sc.* 37 (1951), 311-5). In this paper Gauss and Codazzi equations for a sub-space of a generalized Riemann space have been obtained. Curvature of a curve in a sub-space, and the angular spread vector and normal curvature of a vector-field along a curve in the sub-space have been studied. The equations of lines of curvature and generalizations of Rodrigue's formula and Euler's formula have been obtained. The relative curvature of a sub-space for the orientation determined by a pair of orthogonal directions in the sub-space has been studied and Failkow's theorem (*Bull. American Math. Soc.* 24, (1938), 253-7) has been extended to generalized Riemann spaces. Congruences canonical to a normal congruence have also been studied. Geodesics, asymptotic lines, totally geodesic sub-spaces and hyperplanes have also been defined and studied.

STATISTICS AND PROBABILITY

V. S. HUZURBAZAR, Poona: *Inverse probability and confidence intervals.*

It is shown that for distributions admitting a sufficient statistic when the range depends on the parameter, the method of inverse probability as developed by Jeffreys leads to the same results arrived at by the method of confidence intervals by Neyman.

G. SANKARANARAYANAN, Annamalainagar: *A note on the equidistribution of the sums of independent random variables.*

Let X_1, X_2, ... be a sequence of independent real random variables with a common distribution function $F(x)$. Let $S_1 = X_1 + \dots + X_{p_1}$; $S_2 = X_{p_1+1} + \dots + X_{p_1+p_2} + \dots$; .... We have shown that the sequence S_n is equidistributed with respect to a certain class H of functions $h(x)$ — which contains all almost periodic functions — in the sense that for any $h(x)$ in H, $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} h(S_j) = M(h)$, with probability one, where the constant $M(h)$ is given by $M(h) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} h(x)\, d(x)$ (when the limit exists), as defined in the theory of almost-periodic functions. This result will be true only if the sequence $\{p_j\}$ is an increasing sequence of integers which tend to infinity. We have also shown that the result may not be valid when the p_j's are bounded. For instance we have shown that $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \exp\ (it\, X_j) = \phi(t)$, with probability one, where $\phi(t)$ is the characteristic function of the X_j's. This shows that the first equality fails to hold when $p_j = 1$, $(j = 1,\ldots)$. The application of this result to a stochastic process with stationary increments is immediate. The above problem was suggested by the results obtained by Herbert Robbins in the *Proceedings of the American Mathematical Society* [4 (1953), 786-799] where he proves the result quoted when $S_n = X_1 + X_2 + \ldots + X_n$.

TOPOLOGY

R. Ranga Rao, Calcutta : *On a theorem of Yosida and Hewitt.*

The object of the paper is to present an alternative proof, based on Zorn's lemma, of the following theorem, due to Yosida and Hewitt (*Trans. American Math. Soc.* 72 (1952), 46-66) : Let X be an abstract space, R a ring of subsets of X and ν a finitely additive measure on R. Then there exists a countably additive measure m and a purely finitely additive measure p, such that $\nu = m + p$. The main object is to prove the existence of m and p, the uniqueness of the same following as in the earlier work of Yosida and Hewitt. Also it suffices to consider only non-negative set functions as the general case follows from it.

V. S. Varadarajan, Calcutta : *On a class of topological spaces.*

Given any completely regular Hausdorff space x it was proved by Hewitt [*Trans. American Math. Soc.* 64, (1948)] that there exists a unique Q-space x_1, containing x as a dense subset such that every continuous function on x can be uniquely and continuously extended to x_1. It was also shown by Shirota (*Osaka Math. J.* 1952) that Q-spaces are identical with closed subsets of products of real lines. It is shown here that the above results can be viewed from the standpoint of weak topologies on spaces of measures. Explicitly a Q-space is defined to be a completely regular Hausdorff space on which every two-valued measure is degenerate, and the Shirota characterization obtained. It is then shown that for any completely regular Hausdorff space x, the space x_1, of all two-valued probability measures on x, when weakly topologised, is the unique Q-extension of x.

MEMBERS OF THE CONFERENCE

S. K. Abhyankar, M. Agarwala, Bhagwan Das Agrawal, H. S. Ahluwalia, A. S. Apte, T. V. Avadhani, B. B. Bagi, V. K. Balachandran, S. Bandyopadhya, Ram Behari, Mrs. Ram Behari, M. S. Cheema, S. S. Cheema, S. D. Chopra, A. C. Choudhury, P. C. Counsul, B. C. Das, S. C. Das, Sadu Charan Das, S. M. Das, Miss R. De, K. K. Deshpande, Miss Prasanna Kumari Devi, S. S. Dubey, V. Ganapathy Iyer, S. Girdharilal, K. R. Gunjikar, Hansraj Gupta, Harishankar, S. K. Hindi, S. Manzur Hussain, V. S. Huzurbazar, M. E. Iyer, P. C. Jain, M. V. Jambunathan, Ramakant Jha, L. S. Kamat, V. L. Kantam, D. R. Kaprekar, J. N. Kapur, Mrs. Kapur, Miss Kapur, V. S. Krishnan, B. K. Lahiri, S. Mahadevan, Mrs. Mahadevan, Sahib Ram Mandan, S. Minakshisundaram, P. C. Misra, S. P. Misra, Sukadev Prasad Misra, P. C. Mohanty, R. Mohanty, S. Mohapatra, V. Mohapatra, S. K. Mukerjee, P. V. L. Narasinga Rao, Shanti Narayan, Mrs. Shanti Narayan, B. Y. Oke, J. Panda, J. N. Panda, K. C. Panda, S. K. Panda, P. B. Patel, Mrs. Sunanda Patel, M. Parameswara Iyer, M. R. Parameswaran, Mahendra Pratap, M. Raghavacharyalu, S. Ramakrishna, M. S. Ramanujan, P. C. Rath, R. Ranga Rao, M. N. Roy, Sakuntala Devi, G. Samal, G. Sankaranarayanan, D. K. Sen, H. M. Sen Gupta, Mrs. Sen Gupta, Miss Sen Gupta, C. C. Shah, K. B. Shah, D. L. Sharma, H. G. S. Sharma, P. L. Sharma, Mrs. Sharma, M. K. Singal, S. K. Singh, V. N. Singh, Vikramaditya Singh, S. R. Sinha, K. Sitaram, B. S. K. R. Somayajulu, M. V. Subba Rao, P. S. Subramanian, K. N. Sundaresan, M. Suryanarayana, S. Swetharanyam, P. Tiwari, R. R. Umarji, P. C. Vaidya, V. S. Varadarajan, M. Venkataraman, T. S. Venkataraman, H. K. Verma, M. K. Wadikhaye, N. Yegyanarayan.

Reception Committee

P. Parija (*Chairman*), B. C. Das (*Vice-Chairman*); R. Mohanty (*Secretary*), J. N. Panda (*Asst. Secretary*), G. C. Raṭh (*Treasurer*), S. C. Das, B. Misra, D. Misra, H. Mohapatra, G. C. Pattanayaka, K. Ramamurti, Surajmal Saha, B. Samantary (*Members*).

NEWS AND NOTICES

SHRI S. V. Keshava Hegde has been admitted to the life-membership of the Society.

The following persons have been admitted to membership in the Society : M. K. Agrawala, B. K. Lahiri, V. S. Lakshminarayan, V. Narayanaswami, Sahib Ram and A. P. Stone.

Professor K. Chandrasekharan and Professor Ram Behari are the delegates to the International Congress of Mathematicians in Edinburgh.

Dr. M. Venkataraman and Dr. Alladi Ramakrishnan have been appointed respectively Professors of Mathematics and Physics of Madras University.

Messrs. M. S. Ramanujan, Md. Mohsin and Qazi Ibadur Rahman have been appointed as lecturers in Mathematics, Muslim University, Aligarh.

Dr. S. M. Shah has been appointed as visiting Professor for one semester (September 1958 — January 1959) in the University of Wisconsin, Madison, U.S.A.

Mr. C. R. Marathe has been awarded the degree of Ph. D. by the University of Aligarh, for his thesis on 'Distribution of eigenvalues of matrices'.

Dr. S. K. Singh (Aligarh) has been awarded a research assistantship by the University of Kansas.

The World Directory of Mathematicians has been issued.

The summer school of Mathematics organised by the Mathematics Seminar, Delhi University was held in Hansraj College, Delhi, from the 5th May to 1st June 1958. Lectures were given on various aspects of Mechanics of continuous media such as, Boundary layer theory, Compressible fluids, Turbulance, Magnetohydrodynamics, Electricity and Elastic waves. Expository lectures were given on

Mathematical logic, Theory of sets, Abstract algebra, Topology, Homotopy and Banach spaces. Applications of mathematics to social sciences, industry and defence were also emphasised. Discussions were held on various topics including Astronomy, Statistics and Theoretical physics.

The lecturers were drawn from Defence Science Organization and Departments of Mathematics of Panjab, Aligarh, Delhi and Rajasthan Universities. Financial help was rendered by the Universities Grants Commission and Messrs S. Chand and Co.

At the conclusion of the session resolutions regarding the setting of a Mathematics Research Centre, a Fluid dynamics laboratory at Delhi and the installation of Electronic Computors at various research centres in the country were passed.

The Summer School of Mathematics and Statistics organized by the Departments of Mathematics of the Universities of Aligarh and Lucknow was held in Nainital from May 22nd to June 20th. Lectures were delivered on 'Set Theory and Convergence' by Dr. D. N. Misra (Lucknow), on 'Lebesgue Integral and Hilbert space' by Dr. A. Sharma (Lucknow), on 'Complex Analysis and Fundamental Concepts' by Dr. V. Singh (Kanpur), on 'Meromorphic Function' by Dr. S. M. Shah (Aligarh), on 'Linear Algebra' by Dr. M. Ishaq (Lucknow), on 'Semi-Groups and Groups' by Dr. J. A. Siddiqi (Aligarh), on 'Basic Concepts and probability Theory and Introduction to Stochastic Processes' by Dr. B. P. Adhikari (Lucknow), and on 'Measures on Topological Spaces' by Dr. D. D. Joshi (Calcutta).

Free accommodation and a grant for cyclostyling lecture notes were kindly provided by the Government of Uttar Pradesh.

THE INDIAN MATHEMATICAL SOCIETY

Statement of Account for the
year ending 31st March 1958

THE INDIAN

Receipt and Payment Account for the

RECEIPTS	Rs. nP.		Rs. nP.	
To **Life Composition Fees**			400	00
,, **Membership & Subscription**			6,548	91
,, **Associate Society Membership**			374	25
,, **Sale of Publications**			950	68
,, **Grant-in-Aid**				
Bombay University, Bombay	200	00		
Madras University, Madras	150	00		
Osmania University, Hyderabad	100	00		
Central Government, Delhi, Education Dept....	2,000	00		
Tata Institute of Fundamental Research Bombay	1,000	00		
Government of India, New Delhi	2,000	00		
			5,450	00
,, **Interest on:**				
National Savings Certificates	3,600	00		
National Savings Certificates	400	00		
Bank Balances	34	54		
Investments	450	89		
			4,485	43
,, **Opening Balances** :				
Cash on hand	1	52		
C/A with the Indian Bank Ltd., Mylapore, Madras	645	66		
Savings A/C with Indian Bank Ltd., Mylapore, Madras	926	86		
Fixed Deposit with the Indian Bank Ltd., Mylapore, Madras	12,854	70		
Sangli Bank Ltd., C/A. Wellingdon College Branch	2,428	98		
			16,857	72
,, **National Savings Certificates: (Face Value)** ...			9,000	00
,, **A. Narsing Rao Gold Medal Fund :** (in the form of N. S. Certificate)			1,000	00
,, **Advance :—**				
With the Editor	485	03		
With Oxford University Press...	10	40		
			495	43
,, **Bills Payable**			217	99
Total Rs...			45,780	41

Bombay,

Dated : 18th August, 1958.

MATHEMATICAL SOCIETY

year ending 31-3-1958

PAYMENTS	Rs.	nP.	Rs.	nP.
By Printing of Journals			7,914	96
„ **Management Expenses** :-				
Misc. Printing & Stationery	497	81		
Postage & Rly. Freight for Journals	875	78		
Office Postage	376	37		
Office Expenses	27	62		
Honorarium & Remuneration	160	00		
Bank Commission	27	56		
Conveyance & Travelling	95	78		
Book-Binding Charges	314	50	2,375	42
„ **Audit Fees**			25	00
„ **Library Books**			2,152	69
„ **Furniture & Dead Stock**			361	13
„ **Closing Balance** :-				
Cash on hand	10	54		
Current Account with the Indian Bank Ltd., Mylapore, Madras 4	579	89		
Savings Account with the Indian Bank Ltd., Mylapore, Madras 4	947	68		
Current Account with the Sangli Bank Ltd., Willingdon College Branch	371	44	1,909	55
„ **Reserve for Building Fund** :-				
Invested in :				
National Savings Certificates	12,000	00		
Fixed Deposit with the Indian Bank Ltd., Madras	13,304	59	25,304	59
„ **Interest** Accrued on the National Savings Certificates of Rs. 9,000/-			3,600	00
„ **A. Narsing Rao Gold Medal Fund** (In the form of N. S. Certificates)	1,000	00		
Interest accrued on the same	400	00	1,400	00
„ **Advances** (To be adjusted against postage expenses).				
With the Editor of the Journal	522	02		
With the Librarian	215	05	737	07
Total Rs. ...			45,780	41

Examined and found correct.

P. G. BHAGWAT
Chartered Accountants.

A FEW USEFUL MODIFICATIONS OF NEWTON'S APPROXIMATION METHOD OF SOLVING REAL EQUATIONS

By L. C. HSU

1. Introduction. In this paper we shall give some convergence theorems concerning a certain type of modification of Newton's method for the approximate solution of a real equation. Moreover we shall illustrate that our modified method can be still extended to solving simultaneous equations. Finally, other types of modifications will also be sketched.

Let $f(x)$ be a single-valued real function, continuous together with its first two derivatives $f'(x)$ and $f''(x)$ within a certain interval. We shall consider the approximate solution of the equation

$$f(x) = 0. \tag{1}$$

Let x_0 and x_1 be two different initial approximate solutions of (1), and let the successive approximations x_2, x_3, etc. be constructed by the following iteration method:

$$x_{n+1} = x_n - \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) f(x_n), \quad (n = 1, 2, \ldots). \tag{2}$$

This is one of our proposed modifications of Newton's process, in which we are now chiefly interested. As regards some well-known modifications and extensions, see [1]—[9].

Evidently the process (2) is more advantageous than the ordinary method of chord (the so-called *regula falsi*), since it is unnecessary to adjust the points x_{n-1} and x_n in such a way that $f(x_{n-1})f(x_n) < 0$. Moreover, we see that it is also more convenient and available than the classical Newton method

$$x^*_{n+1} = x^*_n - \frac{f(x^*_n)}{f'(x^*_n)}, \quad (n = 0, 1, 2, \ldots; x^*_0 = x_0) \tag{3}$$

in case the values $f'(x^*_n)$ are not easily calculated (e.g. both $f(x)$ and $f'(x)$ may be quite complicated algebraic or transcendental functions).

Basing upon the asymptotic relation of the following

$$\left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) \sim \frac{1}{f'(x_n)} \quad \text{for } (x_n - x_{n-1}) \to 0, \tag{4}$$

we may predict naturally that the convergence speed of (2) is comparable with that of (3). That this statement is true will be justified in the next section.

2. Some convergence theorems. In what follows we always assume x_1 to be determined by the ordinary Newton method, viz.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

By means of certain geometrical considerations we may now prove the following (cf. the lemma of [13])

THEOREM 1. *Let I be a closed interval within which the equation* (1) *has at least one solution. Let* x_0 *be a point interior to I and let*

$$f(x_0)\, f''(x) > 0, \quad (x \in I). \tag{5}$$

Then the sequence $\{x_n\}$ *given by* (2) *converges monotonically to a solution* x^* *of* (1) *with* $x^* \in I$. *Moreover, the convergence speed is comparable with that of* $\{x_n^*\}$ *of the Newton process* (3), *viz.*

$$|x_{2n-1} - x^*| < |x_n^* - x^*|, \quad (n = 2, 3, \ldots). \tag{6}$$

PROOF. It is no real restriction to suppose that $f(x_0) > 0, f''(x) > 0$, $(x \in I)$, since otherwise we can consider $-f(x)$ instead of $f(x)$. Clearly the condition $f''(x) > 0$ just means that the curve $y = f(x)$ is strictly convex throughout I. This fact actually implies that $f'(x_0) \neq 0$. For proving this, suppose on the contrary that $f'(x_0) = 0$. Then $f(x)$ must attain an absolute minimum at $x = x_0$, and consequently $f(x) \geqslant f(x_0) > 0$ $(x \in I)$, contradicting the hypothesis that $f(x)$ has at least one zero within I.

Consider now the case $f'(x_0) < 0$. Let x^* denote a zero of $f(x)$ (in I), which is nearest to x_0. Then by the convexity of $y = f(x)$ we see that the Newton sequence $\{x_n^*\}$ $(x_0^* = x_0)$ is monotonically increasing to the upper limit x^*. Actually the relation $\lim_n x_n^* = x^*$ follows at once by letting $n \to \infty$ in (3 bis): $f'(x_n^*)\,(x_{n+1}^* - x_n^*) = -f(x_n^*)$.

In order to prove $\lim x_n = x^*$ and (6), we need to compare the convergence speed of $\{x_n\}$ with that of $\{x_n^*\}$. Geometrically the relation (2) means that x_{n+1} is just the abscissa of the x-axis determined by the intersection of the line through the pair of points $(x_{n-1}, f(x_{n-1}))$ and $(x_n, f(x_n))$. On the other hand, the abscissa x_{n+1}^* is determined by the intersecting line tangent to the curve $y = f(x)$ at $x = x_n^*$. Thus from the convexity of $y = f(x)$ we may infer inductively that $x_n < x_{n+1} < x^*$ (all n) and that

$$x_0 = x_0^* < x_1 = x_1^* < x_2 < x_2^* < x_3.$$

Assume now for instance $x_\nu^* < x_{\mu-1} < x_\mu$. Evidently it must follow that $x_{\nu+1}^* < x_{\mu+1}$, using the convexity again. In other words, there can never exist three consecutive points $x_{\mu-1}, x_\mu, x_{\mu+1}$ within the semi-open interval $x_\nu^* < x \leqslant x_{\nu+1}^*$. Hence the worst case regarding the convergence speed of $\{x_n\}$ is seen to be

$$x_0 < x_1^* < x_2 < x_2^* < x_3 < x_4 \leqslant x_3^* < x_5 < x_6 \leqslant x_4^* < \dots <$$
$$< x_\nu^* < x_{2\nu-1} < x_{2\nu} \leqslant x_{\nu+1}^* < \dots, \quad (x_n^* \uparrow x^*).$$

Consequently we obtain $\lim x_n = x^*$ and

$$|x_{2\nu-1} - x^*| < |x_\nu^* - x^*|, \quad (\nu = 2, 3, \dots).$$

The other case $f'(x_0) > 0$ can be treated in exactly the same way. Hence our theorem is proved.

For the case without assuming the existence of a solution, we have the

THEOREM 2. *Let x_0 be a point interior to a closed interval I and let*

$$f(x_0)\, f''(x) > 0, \quad f(x_0)\, f(\alpha) < 0, \quad (x \in I), \tag{7}$$

where α is an end-point of I. Then the sequence $\{x_n\}$ given by (2) *converges monotonically to a unique solution x^* of* (1) *with $x^* \in I$.*

PROOF. As before we may assume that $f(x_0) > 0$, $f''(x) > 0$. By (7) we have $f(\alpha) < 0$. Thus the function $f(x)$ has at least one zero in the open interval (α, x_0) [or (x_0, α)]. On the other hand, in case there are more than one zeros in the interval, it is seen that $y = f(x)$ must attain a maximum and a minimum between α and x_0.

This clearly contradicts the supposition that $f''(x) > 0$ $(x \in I)$. Hence in conclusion the solution of $f(x) = 0$ is unique; and our theorem follows from Theorem 1.

This theorem is actually an analogue of Ostrowski's convergence theorem for the Newton method ([10] or [1; Chap. 1, §5]). It is known that Ostrowski obtained also the following result:

Suppose that $f(x)$ is twice differentiable in the interval $I = [x_0, x_0 + 2h]$, where

$$h = -\frac{f(x_0)}{f'(x_0)} \neq 0, \sup_{x \in I} |f''(x)| < M, \quad 2M\,|h| \leqslant |f'(x_0)|. \qquad (8)$$

Then the equation (1) *has a unique solution x^* (in I), toward which the Newton process* (3) *converges.*

This result was afterwards extended by L. V. Kantorovich ([8] or [1, Ch. 1]) to the case of general functional equations according to which the convergence speed of (3) is given by

$$|x_n^* - x^*| \leqslant \left(\frac{1}{2}\right)^{n-1} \left(\frac{2M\,|h|}{|f'(x_0)|}\right)^{2^n-1} |h|, \ (n = 1, 2, \dots).$$

Thus, making use of both Theorem 1 and Ostrowski's theorem, we may state the following

THEOREM 3. *Let the conditions* (5) *and* (8) *be fulfilled for the closed interval $I = [x_0, x_0 + 2h]$. Then the sequence $\{x_n\}$ given by* (2) *converges to a unique solution x^* of* (1) *within I. Moreover, for the case I being defined by $I = [x_0 - 2h, x_0 + 2h]$ we have*

$$|x_{2n} - x^*| < |x_{2n-1} - x^*| \leqslant \left(\frac{1}{2}\right)^{n-1} \left(\frac{2M\,|h|}{|f'(x_0)|}\right)^{2^n-1} |h|. \qquad (9)$$

Clearly, conditions (5) and (8) are easily satisfied whenever x^* is neither a stationary point nor a point of inflection of $f(x)$, and x_0 is sufficiently close to x^* (i.e. I is a sufficiently small interval). The inequality (9) indicates that the convergence of $\{x_n\}$ is quite rapid especially for the case $2M|h| < |f'(x_0)|$.

A general remark worthy of mentioning is that almost all known theorems concerning the convergence of the Newton method can be extended to the case of (2). Thus for instance, corresponding to a theorem of I. P. Misovskih ([12] or [1 ; Ch. 1, §6]), we have the following

THEOREM 4. *Let the equation* (1) *have a solution* x^* *to which* x_0 *is an initial approximation, and let the following conditions be fulfilled* :

(i) $|x^* - x_0| \leqslant r$;

(ii) $f(x_0)f'(x_0) > 0$ (*or* $f(x_0)f'(x_0) < 0$) ;

(iii) $|f'(x)^{-1}| \leqslant B$, $|f''(x)| \leqslant K$ *and* $f(x_0)f''(x) > 0$ *hold in the interval*

$$I = [x_0 - (1 + \tfrac{1}{2}l)\, r, x_0] \quad (\textit{or}\; I = [x_0, x_0 + (1 + \tfrac{1}{2}l)\, r]);$$

(iv) $l = B\,Kr \leqslant 2$.

Then the process (2) *converges to the unique solution* x^* *of* (1) *within the interval* I.

In practical applications, of course, we do not know the precise value of x^*. But the number r as defined by (i) can usually be estimated from the function $f(x)$ itself.

Misovskih proved also the following theorem : *Suppose that*

$$f(x_0)f'(x_0) > 0 \quad (\textit{or}\; f(x_0)f'(x_0) < 0). \tag{10}$$

Let the condition

$$|f'(x)^{-1}| \leqslant B, \quad |f(x_0)| \leqslant \eta, \quad |f''(x)| \leqslant K \tag{11}$$

be fulfilled in a closed interval $I = [x_0 - \lambda,\ x_0]$ (*or* $I = [x_0,\ x_0 + \lambda]$), *where*

$$h = B^2 K \eta \leqslant 4, \quad \lambda \geqslant B\eta. \tag{12}$$

Then the Newton process (3) *converges to a unique solution* x^* *of* (1) *within the interval* I.

Introduce now the following condition instead of (11) :

$$|f'(x)^{-1}| \leqslant B, \quad |f(x_0)| \leqslant \eta, \quad |f''(x)| \leqslant K, \quad f(x_0)f''(x) > 0. \tag{11'}$$

Thus a combined application of Misovskih's theorem and our Theorem 1 will yield the following result :

THEOREM 5. *The process* (2) *converges monotonically to a unique solution* x^* *of* (1) *with* $x^* \in I$, *provided that the conditions* (10), (11′) *and* (12) *are fulfilled.*

H. S. Sun has worked out a few numerical examples of algebraic equations, showing that the whole labour of computations involved in using process (2) is relatively smaller than that of using Newton's method, in case the approximate solutions are required to be precise in six decimal places.

3. An extension. We shall now formulate an extension of the modified process (2) to the case of simultaneous equations involving two unknowns.

Given simultaneous equations of the following

$$f_1(x, y) = 0, \quad f_2(x, y) = 0, \tag{13}$$

the functions f_1 and f_2 being assumed to have continuous partial derivatives of the first order. We may of course regard the pair of functions f_1 and f_2 as defining an operation P transforming the vector (x, y) to the vector (f_1, f_2). The equation (13) may be therefore written in the form $P(X) = 0$ with $X = (x, y)$. In accordance with the functional analysis we know that the Fréchet derivative of P is actually represented by the matrix operation of the linear algebra:

$$P'(X) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{bmatrix}. \tag{14}$$

Accordingly we have

$$P'(X)^{-1} = \begin{bmatrix} \dfrac{1}{\Delta}\left(\dfrac{\partial f_2}{\partial y}\right) & \dfrac{-1}{\Delta}\left(\dfrac{\partial f_1}{\partial y}\right) \\ \dfrac{-1}{\Delta}\left(\dfrac{\partial f_2}{\partial x}\right) & \dfrac{1}{\Delta}\left(\dfrac{\partial f_1}{\partial x}\right) \end{bmatrix}, \tag{15}$$

where $\Delta = \dfrac{\partial(f_1, f_2)}{\partial(x, y)}$ is the Jacobian determinant. Thus the Newton process for the functional equation $P(X) = 0$ may be expressed in the matrix operational form

$$X_{n+1} = X_n - P'(X_n)^{-1} P(X_n), \quad (n = 0, 1, 2, \ldots) \tag{16}$$

where $X_n = (x_n, y_n)$ denotes a column vector and $X_0 = (x_0, y_0)$ is an initial approximate solution of (13).

In applying the process (16) to numerical equations, it usually needs to compute the values $f_i(x_\nu, y_\nu)$, $\left(\frac{\partial f_i}{\partial x}\right)_{X_\nu}$, $\left(\frac{\partial f_i}{\partial y}\right)_{X_\nu}$ $(i = 1, 2; 0 \leqslant \nu \leqslant n)$. This may be somewhat troublesome in case the f_i are complicated transcendental functions. Now in order to simplify the labour of calculations involved, we may devise a new process as follows.

Let $X_0 = (x_0, y_0)$ and $X_1 = (x_1, y_1)$ be determined by (16). Define

$$\begin{cases} \left(\dfrac{\Delta f_i}{\Delta x}\right)_n = \dfrac{f_i(x_n, y_n) - f_i(x_{n-1}, y_n)}{x_n - x_{n-1}}, \\ \left(\dfrac{\Delta f_i}{\Delta y}\right)_n = \dfrac{f_i(x_{n-1}, y_n) - f_i(x_{n-1}, y_{n-1})}{y_n - y_{n-1}}, \quad (i = 1, 2) \end{cases} \tag{17}$$

where $n = 1, 2, \ldots$, and let

$$D_n = \begin{vmatrix} \left(\dfrac{\Delta f_1}{\Delta x}\right)_n & \left(\dfrac{\Delta f_1}{\Delta y}\right)_n \\ \left(\dfrac{\Delta f_2}{\Delta x}\right)_n & \left(\dfrac{\Delta f_2}{\Delta y}\right)_n \end{vmatrix} \tag{18}$$

Then (16) may be modified to the form

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} \dfrac{1}{D_n}\left(\dfrac{\Delta f_2}{\Delta y}\right)_n & \dfrac{-1}{D_n}\left(\dfrac{\Delta f_1}{\Delta y}\right)_n \\ \dfrac{-1}{D_n}\left(\dfrac{\Delta f_2}{\Delta x}\right)_n & \dfrac{1}{D_n}\left(\dfrac{\Delta f_1}{\Delta x}\right)_n \end{bmatrix} \begin{bmatrix} (f_1)_n \\ (f_2)_n \end{bmatrix}, \quad (n = 1, 2, \ldots) \tag{19}$$

where $(f_i)_n = f_i(x_n, y_n)$. This is equivalent to

$$\left.\begin{aligned} x_{n+1} &= x_n - \frac{1}{D_n}\left\{\left(\frac{\Delta f_2}{\Delta y}\right)_n (f_1)_n - \left(\frac{\Delta f_1}{\Delta y}\right)_n (f_2)_n\right\}, \\ y_{n+1} &= y_n + \frac{1}{D_n}\left\{\left(\frac{\Delta f_2}{\Delta x}\right)_n (f_1)_n - \left(\frac{\Delta f_1}{\Delta x}\right)_n (f_2)_n\right\}. \end{aligned}\right\} \tag{20}$$

In obtaining the $(n+1)$th approximation, we see that the new process requires merely $4n+2$ quantities $f_i(x_0, y_0)$, $f_i(x_\nu, y_\nu)$, $f_i(x_{\nu-1}, y_\nu)$ $(1 \leqslant \nu \leqslant n)$; but the Newton process (16) has to evaluate $6n+6$ quantities in advance.

4. A modification of Tchebychef's process. One of the effective iteration processes much more precise than Newton's method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\left(\frac{f(x_n)}{f'(x_n)}\right)^2\left(\frac{f''(x_n)}{f'(x_n)}\right), \; (n = 0, 1, 2, \ldots). \tag{21}$$

This is the well-known process of Tchebychef for solving algebraic or transcendental equations. In view of the asymptotic relation between (2) and (3) we may naturally modify (21) to a form of the following

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\left(\frac{f(x_n)}{f'(x_n)}\right)^2\left(\frac{f'(x_n) - f'(x_{n-1})}{f'(x_n)(x_n - x_{n-1})}\right), \; (n = 1, 2, \ldots), \tag{22}$$

where x_0 and x_1 are assumed to be given as in the ordinary Newton method. Apparently this modification has the advantage that it uses only those quantities which have appeared already in the Newton process. As may be expected, the convergence speed of (22) may under some general conditions be compared with that of (21). However we have not yet got any convergence theorems for this modified process.

Other type of convenient modification may be given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\left(\frac{f(x_n)}{f'(x_n)}\right)^2\left(\frac{f''(x_0)}{f'(x_0)}\right), \quad (n = 0, 1, 2, \ldots). \tag{23}$$

But the convergence speed of this process is surely inferior if compared with (22).

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North-East People's University
Chang-chun, China

THE UNIFORM APPROXIMATION TO THE LIPSCHITZ CLASS OF FUNCTIONS BY A KIND OF TRIGONOMETRICAL POLYNOMIALS

By L. C. HSU

1. In proving the celebrated approximation theorem of Weierstrass for continuous periodic functions, it is known that de la Vallée-Poussin has introduced an elegant singular integral of the form

$$V_n(f;x) = \frac{(2n)!!}{(2n-1)!!}\,\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cos^{2n}\left(\frac{t-x}{2}\right)dt, \qquad (1)$$

where $n!!$ denotes the double paced factorial, e.g. $5!! = 1.3.5$; $8!! = 2.4.6.8$.

By $f \in \mathrm{Lip}_M\,\alpha$ $(0 < \alpha \leqslant 1)$ we shall mean that $f(x)$ is a function of the Lipschitz class with index α, i.e. $f(x)$ satisfies the condition of the type

$$|f(x') - f(x'')| \leqslant M\,|x' - x''|^{\alpha}.$$

I. P. Natanson ([1], [2]) has investigated the uniform approximation to the Lipschitz class $\mathrm{Lip}_1\,\alpha$ of periodic functions by the Vallée-Poussin polynomials defined by (1), and obtained the following precise result:

If we define

$$U_n(\alpha) = \sup_f \left\{ \max_x |\,V_n(f;x) - f(x)\,| \right\}, \qquad (2)$$

where the supremum is taken over all functions of period 2π *with* $f \in \mathrm{Lip}_1\,\alpha$, *then for n large we have*

$$U_n(\alpha) = \Gamma\left(\frac{1+\alpha}{2}\right)\frac{2^{\alpha}}{\sqrt{(\pi n^{\alpha})}} + o\left(\frac{1}{\sqrt{n^{\alpha}}}\right). \qquad (3)$$

The purpose of this note is to introduce a kind of trigonometrical polynomials which may be regarded as a generalization of Vallée-Poussin's. We shall show that our newly defined polynomials can in fact provide closer approximations to the functions of $\mathrm{Lip}_1\alpha$.

2. In our discussion use will be made of the following

LEMMA. *Let $\phi(x, t)$ be a continuous function defined on the square region $a \leqslant x \leqslant b$, $a \leqslant t \leqslant b$ and satisfying the conditions :*

1°. *the partial derivative $\phi''_{tt}(x, t)$ exists and is continuous ;*

2°. *the relation $|\phi(x, t)| < \phi(x, x) = 1$ holds for $t \neq x$.*

Then for every Lebesgue integrable function $f(t)$ $(a \leqslant t \leqslant b)$ and for $\alpha > -1$ we have

$$\int_a^b [\phi(x,t)]^n \, |t-x|^\alpha f(t) \, dt \sim f(x) \, \Gamma\left(\frac{1+\alpha}{2}\right) \left(\frac{-2}{n \, \phi''_{tt}(x, x)}\right)^{(1+\alpha)/2}, \quad (n \to \infty) \quad (4)$$

whenever x $(a < x < b)$ is a point belonging to the Lebesgue set of $f(t)$ with $f(x) \neq 0$.

This is a consequence of a more general result proved previously (cf. [3], [4]). The formula (4) was also obtained by P. G. Roney [5] (cf. [6], [7]). Clearly the lemma itself is an extension of the classical Laplace theorem for the asymptotic integration.

3. In what follows μ always denotes a positive odd integer. Let us now introduce the following generalization of Vallée-Poussin's singular integral :

$$V_n^\mu(f;\, x) = \frac{1}{K_n} \int_{-\pi}^{\pi} \left\{ \cos\left(\frac{t-x}{2}\right) \cos \mu \left(\frac{t-x}{2}\right) \right\}^n f(t) \, dt, \quad (5)$$

the number K_n being defined by

$$K_n = 4 \int_0^{\pi/2} (\cos \theta \cos \mu \, \theta)^n \, d\theta. \quad (6)$$

It is easily found that $V_n^\mu\,(1, x) = 1$ and that

$$K_n \sim \left(\frac{8\,\pi}{(1+\mu^2)\, n}\right)^{1/2}, \quad (n \to \infty) \quad (7)$$

by making use of the asymptotic formula (4).

Evidently the integral (5) has the characteristic that its integrand (kernel) will have violent oscillations when μ is large. Moreover, we see that (5) represents a trigonometrical polynominal of order $\frac{1}{2}(\mu+1)\,n$.

Consider the class $\mathrm{Lip}_1\ \alpha$. We may prove the following

THEOREM. *If n denotes a positive even integer and if*

$$U_n^\mu(x) = \sup_f \left\{ \max_x |V_n^\mu(f;x) - f(x)| \right\}, \tag{8}$$

the supremum being taken over all functions of period 2π with $f \in \mathrm{Lip}_1\ \alpha$, then for n large we have

$$U_n^\mu(\alpha) = \Gamma\left(\frac{1+\alpha}{2}\right)\left(\frac{8}{1+\mu^2}\right)^{\alpha/2} \frac{1}{\sqrt{(\pi\, n^\alpha)}} + o\left(\frac{1}{\sqrt{n^\alpha}}\right). \tag{9}$$

PROOF. Notice that for each fixed x the function

$$\left\{\cos\left(\frac{t-x}{2}\right)\cos\mu\left(\frac{t-x}{2}\right)\right\}^n$$

is of period 2π in t. Thus for any function $f(t)$ of period 2π with $f \in \mathrm{Lip}_1\ \alpha$ we have

$$| V_n^\mu(f,x) - f(x) |$$

$$= \left| K_n^{-1} \int_{x-}^{x+\pi} \left\{\cos\left(\frac{t-x}{2}\right)\cos\mu\left(\frac{t-x}{2}\right)\right\}^n [f(t)-f(x)]dt \right|$$

$$\leqslant K_n^{-1} \int_{x-\pi}^{x+\pi} \left\{\cos\left(\frac{t-x}{2}\right)\cos\mu\left(\frac{t-x}{2}\right)\right\}^n |t-x|^\alpha\, dt$$

$$= K_n^{-1} \int_{-\pi}^{\pi} \left\{\cos\left(\frac{t}{2}\right)\cos\mu\left(\frac{t}{2}\right)\right\}^n |t|^\alpha\, dt.$$

Consequently we get

$$U_n^\mu(\alpha) \leqslant K_n^{-1} \int_{-\pi}^{\pi} [\psi(0,t)]^n\, |t|^\alpha\, dt, \tag{10}$$

where

$$\psi(x,t) = \cos\left(\frac{t-x}{2}\right) \cos\, \mu\left(\frac{t-x}{2}\right).$$

It is clear that the function $f_0(t) = |t|^\alpha \ (-\pi \leqslant t \leqslant \pi)$ can be extended as a continuous function of period 2π in the whole interval $(-\infty, \infty)$. Moreover, we see $f_0(t) \in \mathrm{Lip}_1\, \alpha$ inasmuch as

$$|f_0(t') - f_0(t'')| = ||t'|^\alpha - |t''|^\alpha| \leqslant |t' - t''|^\alpha, \quad (|t'| \leqslant \pi, |t''| \leqslant \pi)$$

and generally

$$|f_0(t' \pm 2k\pi) - f_0(t'' \pm 2l\pi)| = |f_0(t') - f_0(t'')| \leqslant |t' - t'' \pm 2\pi(k-l)|^\alpha.$$

Thus we also have

$$U_n^\mu(\alpha) \geqslant V_n^\mu(f_0;\, 0) - f_0(0) = K_n^{-1} \int_{-\pi}^{\pi} [\psi(0,t)]^n\, |t|^\alpha\, dt. \tag{11}$$

An easy calculation gives $\psi''_{tt}(0,0) = -\frac{1}{4}(1+\mu^2)$. Hence by comparison of (10) and (11) and by means of (4) we obtain

$$U_n^\mu(\alpha) = K_n^{-1} \int_{-\pi}^{\pi} [\psi(0,t)]^n\, |t|^\alpha\, dt \sim \Gamma\left(\frac{1+\alpha}{2}\right) \left(\frac{8}{1+\mu^2}\right)^{\alpha/2} \frac{1}{\sqrt{(\pi n^\alpha)}}.$$

The theorem is proved.

It is now easy to verify that (5) generally provides better approximations than (1) can. For instance, for $m = \frac{1}{2}(\mu+1)\,n$ (μ : odd) the integrals $V_m(f;\, x)$ and $V_n^\mu(f;\, x)$ represent trigonometrical polynomials of the same order m. But the ratio between their degrees of approximation is found to be

$$\frac{U_n^\mu(\alpha)}{U_m(\alpha)} \sim \left(\frac{1+\mu}{1+\mu^2}\right)^{\alpha/2},$$

which is asymptotic to $(1/\mu)^{\alpha/2}$ for μ being large.

4. Finally suppose that $f(t) \in L\,(a \leqslant t \leqslant b)$. As a consequence of our lemma (with $\alpha = 0$) we see that

$$\lim_{n\to\infty} \frac{1}{C_n} \int_a^b [\phi(x, t)]^n f(t)\, dt = f(x) \tag{12}$$

holds for every point x $(a < x < b)$ belonging to the Lebesgue set of $f(t)$, provided that C_n is defined by

$$\int_a^b [\phi(x, t)]^n\, dt \sim C_n \text{ (or } = C_n). \tag{13}$$

In particular we have

$$\lim_{n\to\infty} \left\{ \frac{(1+\mu^2)\, n}{8\pi} \right\}^{1/2} \int_{-\pi}^{\pi} \left\{ \cos\left(\frac{t-x}{2}\right) \cos \mu\left(\frac{t-x}{2}\right) \right\}^n f(t)\, dt = f(x), \tag{14}$$

where $f \in L\,(-\pi \leqslant t \leqslant \pi)$ and x is an interior point contained in the Lebesgue set of $f(t)$.

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North-East People's University
Chang-chun, China

ON REPRESENTATION OF NUMBERS AS SUM OF TWO SQUARES

By M. V. SUBBA RAO

1. The classical result that $r_2(n)$, the number of unrestricted representations of n as the sum of two integral squares, is given by

$$r_2(n) = 4 \, [d_1(n) - d_3(n)], \tag{1.1}$$

where $d_i(n)$ denotes the number of divisors of n of the form i (mod 4), admits of several proofs. In this paper I propose to sketch a proof based on the properties of multiplicative functions†, which, it is hoped, will be of some interest, and does not seem to have been attempted before.

In order to do this, we shall not consider the function $r_2(n)$ as it is (this is not a multiplicative function of n), but a closely allied, and in some ways, a more natural function $r(n)$ which happens to be multiplicative. This function $r(n)$ stands for the number of representations of n in the form $n = x^2 + y^2$, $x \geqslant 0$, $y \geqslant 0$ (unlike $r_2(n)$ where we admit negative values of x and y also), with the convention that if $x > 0$, $y > 0$, the representations $x^2 + y^2$ and $y^2 + x^2$ are to be treated as distinct, but if one of them, say y is zero, then $x^2 + 0^2$ and $0^2 + x^2$ are to be treated as one and the same and counted only once. If $x = y$, the resulting representation $x^2 + x^2$ will be counted only once in any case, since x is restricted to be non-negative.‡

Clearly $r_2(n) = 4r(n)$, since corresponding to a single representation in $r(n)$ there are four distinct representations in $r_2(n)$ and vice versa as shown under

†An arithmetic function $f(N)$ is said to be multiplicative if $f(M\,N) = f(M).\,f(N)$ whenever $(M, N) = 1$.

‡ $r(n)$ is the number of lattice points on the circle

$$x^2 + y^2 = n$$

in the first quadrant, x-axis being included but y-axis excluded.

$$x^2 + y^2 \rightleftarrows (\pm x)^2 + (\pm y)^2, x > 0, y > 0,$$

$$x^2 + 0^2 \rightleftarrows (\pm x)^2 + 0^2;\ 0^2 + (\pm x)^2;\ \text{and } x^2 + x^2 \rightleftarrows (\pm x)^2 + (\pm x)^2.$$

Thus the problem reduces to proving that

$$r(n) = d_1(n) - d_3(n). \tag{1.2}$$

We will establish this result by showing that each side of this equation is a multiplicative function of n and that these functions have equal values when n is the power of any prime; it then follows that the functions are identical for all values of n.

2. The remarkable fact that $d_1(n) - d_3(n)$ is a multiplicative function of n seems to have been first noticed by Vaidyanathaswamy [1] and can be easily verified directly. It can also be obtained as an immediate consequence of the following important and well-known

LEMMA. *If $f(n)$ and $g(n)$ are two multiplicative functions, so also is their composite defined as $\Sigma f(d)\, g(n/d)$, d/n.*

If in this we put $f(n) = 1$; and $g(n) = \pm 1$ according as n is $\pm 1 \pmod 4$, and $g(n) = 0$ if n is even, we see that both $f(n)$ and $g(n)$ are multiplicative, and hence also $d_3(n) - d_1(n)$, which is their composite.

Next to show that $r(n)$ is multiplicative, we introduce the function $\psi(n)$ which stands for the number of "primitive" representations of n as sum of two squares, where we define a representation as primitive if it has any one of the following forms: (i) $1^2 + 1^2$, (ii) $1^2 + 0^2$, (iii) $x^2 + y^2$, $(x, y) = 1, x, y > 0$. Thus $\psi(1) = \psi(2) = 1$; $\psi(3) = \psi(4) = 0$; $\psi(5) = 2$; .... Now that $\psi(n)$ is a multiplicative function of n is a consequence of the following two well-known results.

(i) If $f(x)$ is a polynomial of x with integral coefficients, the number of incongruent solutions of $f(x) = 0 \pmod n$ is a multiplicative function of n.

(ii) $\psi(n)$ is equal to the number of incongruent solutions of the congruence $t^2 + 1 \equiv 0 \pmod n$. (When $n = 1$ or 2, it has just one solution, viz. $t=0$ or 1 respectively). [Vide for ex. [2] p. 98, ex. 9.a].

Since every representation of n as $x^2 + y^2$, $(x, y) = d$, gives rise to a primitive representation of n/d^2 as $(x/d)^2 + (y/d)^2$, we get

$$r(n) = \sum \psi(n/d^2),\ d^2/n. \tag{2.1}$$

This shows that $r(n)$ is the composite of the multiplicative functions $\psi(n)$ and $h(n)$, where $h(n) = 1$ if n is a square number, and 0 otherwise. Hence, by the lemma, $r(n)$ is multiplicative.

3. It remains to show that $r(n)$ and $d_1(n) - d_3(n)$ are equal when $n = p^k$, p being a prime. It is easily verified directly that when $n = p^k$, $[d_1(n) - d_3(n)]$ gives 1 if $p = 2$, $k \geqslant 0$; $k+1$ if $p = 1$ (mod 4) and 1 or 0 if $p = 3$ (mod 4) according as k is even or odd. We get exactly the same values for $r(n)$ when we evaluate it for $n = p^k$, using (2.1), and the fact that $\psi(p) =$ the number of incongruent solutions of $t^2 + 1 \equiv 0 \pmod{p}$, which is $= 2$ or 0 according as $p \equiv \pm 1$ (mod 4) and $k > 0$, and which, when $p = 2$, gives 1 if $k = 0$ or 1, and 0 otherwise. Hence $r(n) = d_1(n) - d_3(n)$ for all values of n which are powers of primes, and hence for all n.

My thanks are due to Prof. R. Vaidyanathaswamy for helpful suggestions and encouragement.

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S. V. University
Tirupati

ON POSITIVE DEFINITE QUADRATIC FORMS†

By P. S. RAU

We prove from elementary determinant theory, the following well-known

THEOREM. *A necessary and sufficient condition that a homogeneous quadratic function in n-variables may represent a positive definite form is that its discriminant and the principal cofactors of every order of its discriminant are positive.*

Proofs of this theorem are given in standard treatises by matrix theory; the present proof is of interest in that the result on definite quadratic forms involving determinants is proved by determinant theory only by the method of induction from elementary theorems.

PROOF. We verify the theorem in the case of one, two and three variables and then deduce the general case by the method of induction.

The theorem is true in the case of one variable, since

$$ax^2 > 0 \longleftrightarrow a > 0, \text{ for every } x \neq 0.$$

In the case of two variables, the homogeneous quadratic function may be represented by $S \equiv ax^2 + 2hxy + by^2$. S reduces to ax^2 when $y = 0$, and $x \neq 0$, and to by^2 when $x = 0$, and $y \neq 0$.

So, for S to represent a positive definite form it is necessary that $a > 0, b > 0$. To obtain the sufficient conditions we use the identity

$$aS \equiv (ax + hy)^2 + (ab - h^2)y^2.$$

This gives $aS > 0 \longleftrightarrow (ab - h^2) > 0$, for every $x, y \neq 0$. But $a > 0$ is a necessary condition. So $aS > 0 \longleftrightarrow S > 0$. Thus we have

$$S > 0 \longleftrightarrow a > 0,\ b > 0,\ (ab - h^2) > 0, \text{ for every } x \text{ and } y.$$

In the case of three variables, the homogeneous quadratic function may be represented by

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy.$$

† Paper read at the Twenty-first Conference of the Indian Mathematical Society Banaras, 1955

When one of the variables takes the value zero, S becomes a function of the other two variables, and from the preceding case we have

$$a > 0, b > 0, c > 0; \quad ab - h^2 > 0, bc - f^2 > 0, ca - g^2 > 0,$$

as the necessary conditions for S to represent a positive definite form. To obtain the sufficient conditions, we introduce

$$\xi \equiv ax + hy + gz, \quad \eta \equiv hx + by + fz.$$

Then we have

$$ax + hy + (gz - \xi) \equiv 0, \quad hx + by + (fz - \eta) \equiv 0,$$

$$(gz + \xi)x + (fz + \eta)y + (cz^2 - S) \equiv 0,$$

from which we obtain

$$\left\{ \begin{array}{ccc} a & h & (gz - \xi) \\ h & b & (fz - \eta) \\ (gz + \xi) & (fz + \eta) & (cz^2 - S) \end{array} \right\} \equiv 0.$$

This gives the identity

$$(ab - h^2) S \equiv (a\,\eta^2 - 2h\,\xi\,\eta + b\,\xi^2) + \Delta\, z^2,$$

where

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

From the necessary conditions, $a > 0$, $b > 0$, $(ab - h^2) > 0$, we see that $a\,\eta^2 - 2h\,\xi\,\eta + b\,\xi^2$ is a positive definite form, and hence we have $(ab - h^2) S > 0 \longleftrightarrow \Delta > 0$.

That is to say,

$$S > 0 \longleftrightarrow \left\{ \begin{array}{l} a > 0, \quad b > 0, \quad c > 0, \\ ab - h^2 > 0, bc - f^2 > 0, ca - g^2 > 0 \\ \quad \text{and} \quad \Delta > 0 \end{array} \right\} \text{for all } x, y, z.$$

It is remarkable that the method employed in deducing the case of three variables from that of the two variables yields to generalization. Let us suppose that the theorem is true in the case of $(n - 1)$ variables. A homogeneous quadratic function of the n variables $x_1, x_2, \ldots, x_n$ may be represented by

$$S \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i\, x_j, \; (a_{ij} = a_{ji}).$$

When one of the variables, say, x_n takes the value zero, S becomes a function of the $(n-1)$ variables $x_1, x_2, \ldots, x_{n-1}$ and we have that the discriminant

$$\Delta_{n-1} \equiv \begin{vmatrix} a_{11} & \cdots & \cdots & a_{1(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{(n-1)1} & \cdots & \cdots & a_{(n-1)(n-1)} \end{vmatrix}.$$

and the principal cofactors of every order of Δ_{n-1} are > 0, as the necessary conditions for S to represent a positive definite form. To obtain the sufficient conditions, we introduce

$$\xi_i \equiv a_{i1}x_1 + \ldots + a_{in}x_n; \quad i = 1, 2, \ldots, n-1.$$

Then we have

$$a_{i1}x_1 + \ldots + a_{in-1}x_{n-1} + (a_{in}x_n - \xi_i) \equiv 0; \; i = 1, 2, \ldots, n-1;$$

and

$$(a_{1n}x_n + \xi_1)x_1 + \ldots + (a_{n-1n}x_n + \xi_{n-1})x_{n-1} + (a_{nn}x_n^2 - S) = 0,$$

from which we obtain

$$\begin{vmatrix} a_{11} & \cdots & \cdots & a_{1\,n-1} & (a_{1n}x_n - \xi_1) \\ a_{21} & \cdots & \cdots & a_{2\,n-1} & (a_{2n}x_n - \xi_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1\,1} & \cdots & \cdots & a_{n-1\,n-1} & (a_{n-1\,n}x_n - \xi_{n-1}) \\ (a_{1n}x_n + \xi_1) & \cdots & \cdots & (a_{n-1\,n}x_n + \xi_{n-1}) & (a_{nn}x_n^2 - S) \end{vmatrix} \equiv 0.$$

This gives the identity

$$S.\,\Delta_{n-1} \equiv \Delta_n . \, x_n^2 - \delta,$$

where

$$\delta \equiv \begin{vmatrix} a_{11} & \cdots & \cdots & a_{1n-1} & \xi_1 \\ \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \\ a_{n-11} & \cdots & \cdots & a_{n-1\,n-1} & \xi_{n-1} \\ \xi_1 & \cdots & \cdots & \xi_{n-1} & 0 \end{vmatrix}$$

and

$$'\Delta_n \equiv \begin{vmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix}$$

Expanding the determinant δ in products in pairs of the constituents of the last row and the last column, we obtain

$$S.\Delta_{n-1} \equiv \Delta_n . x_n^2 + \sum_{r=1}^{n-1} \sum_{s=1}^{n-1} A_{rs}\, \xi_r\, \xi_s,$$

where A_{rs} is the cofactor of a_{rs} in the expansion of Δ_{n-1}. We observe that the discriminant and the principal cofactors of every order of the discriminant of $\sum_{r=1}^{n-1} \sum_{s=1}^{n-1} A_{rs}\, \xi_r\, \xi_s$ are positive in view of the necessary conditions if we make use of the following

LEMMA. *If* Δ *is a determinant of order* n, $\mathscr{M}\begin{pmatrix} r_1, \dots, r_p \\ i_1, \dots, i_p \end{pmatrix}$ *is a minor of order* p *formed from the constituents of* $i_1, i_2, \dots, i_p$ *rows and* $r_1, r_2, \dots, r_p$ *columns of the reciprocal of* Δ, $M^c\begin{pmatrix} r_1, \dots, r_p \\ i_1 \dots, i_p \end{pmatrix}$ *is the complementary of the minor of order* p *formed from the constituents of the* $i_1, \dots, i_p$ *rows and* $r_1, \dots, r_p$ *columns of* Δ, *then*

$$\mathscr{M}\begin{pmatrix} r_1, r_2, \dots, r_p \\ i_1, i_2, \dots, i_p \end{pmatrix} = (-1)^{\sum_{t=1}^{p}(i_t + x_t)} M^c \begin{pmatrix} r_1, r_2, \dots, r_p \\ i_1, i_2, \dots, i_p \end{pmatrix} \Delta^{p-1}.$$

Thus $\sum_{r=1}^{n=1} \sum_{s=1}^{n-1} A_{rp}\, \xi_r\, \xi_s$ is a positive definite form. Hence we have

$$\Delta_{n-1} . S > 0 \longleftrightarrow \Delta_n > 0.$$

Hence $S > 0 \longleftrightarrow \Delta_n$ and the principal cofactors of every order of Δ_n are > 0.

This completes the proof of the theorem.

S. V. University
Tirupati

ON ENTIRE FUNCTIONS OF FINITE ORDER

By S. H. DWIVEDI

1. Let $f(z)$ be an entire function of integral order ρ. Let $n(r)$ be the number of zeros of $f(z)$ in $|z| \leqslant r$ and

$$N(r) = A + \int_{r_0}^{r} \frac{n(t)}{t}\, dt.$$

Let $\phi(r)$ be any positive, continuous function such that

$$\int_{A}^{\infty} \frac{dx}{x\,\phi(x)} < \infty.$$

S. M. Shah [2] has proved that for an entire function $f(z)$ of intergal order $\rho = p$ (p being its genus)

$$\limsup_{r\to\infty} \frac{n(r)\,\phi(r)}{\log M(r)} = \infty, \tag{1}$$

where $M(r) = \max_{|z|=r} |f(z)|$.

2. We prove in this note that (1) holds if we replace $n(r)$ by $N(r)$.

THEOREM. *If $f(z)$ is a canonical product entire function of integral order and of the same genus then*

$$\limsup_{r\to\infty} \frac{N(r)\,\phi(r)}{\log\, M(r)} = \infty.$$

3. For the proof of the theorem we require the following

LEMMA. *For a canonical product $f(z)$ we have*

$$\log M(r) < K \int_{0}^{\infty} \frac{N(t)\, r^{p+1}}{t^{p+1}\,(t+r)}\, dt,$$

where p is the genus of $f(z)$.

PROOF. We have

$$\log M(r) < K' \int_0^\infty \frac{n(t)\, r^{p+1}}{t^{p+1}\,(t+r)}\, dt$$

$$= K' \int_0^\infty \frac{r^{p+1}\, t\, d\, N(t)}{t^{p+1}\,(t+r)}$$

$$\leqslant K'\, r^{p+1} \int_0^\infty \frac{(pt + pr + t) N(t)\, dt}{(t+r)^2\, t^{p+1}}$$

Hence

$$\log M(r) < K' \int_0^\infty \frac{(p+1)\, N(t)\, r^{p+1}}{t^{p+1}\,(t+r)}\, dt$$

$$= K \int_0^\infty \frac{N(t)\, r^{p+1}\, dt}{t^{p+1}\,(t+r)}.$$

PROOF OF THEOREM. Since $f(z)$ is an entire function of order and genus p,

$$\int_A^\infty \frac{N(t)}{t^m}\, dt$$

is convergent for $m > p + 1$ and divergent for $m = p + 1$. Now we have by the lemma,

$$\log M(r) < K \int_0^\infty \frac{N(t)\, r^{p+1}}{t^{p+1}\,(t+r)}\, dt$$

$$< K \left\{ r^p \int_0^r \frac{N(t)}{t^{p+1}}\, dt + r^{p+1} \int_r^\infty \frac{N(t)}{t^{p+2}}\, dt \right\}. \quad (2)$$

We shall now prove that it is impossible for any finite C to have

$$N(r) \leqslant \frac{C}{\phi(r)} \left\{ r^p \int_1^r \frac{N(t)}{t^{p+1}}\, dt + r^{p+1} \int_r^\infty \frac{N(t)}{t^{p+2}}\, dt \right\} \quad (3)$$

for all $r \geqslant r_0$ provided r_0 is taken sufficiently large.

The proof from here onward is on the lines of R. P. Boas [1]. For the sake of completeness we reproduce it here.

Let

$$F(r_0) = \int_{r_0}^{\infty} \frac{dt}{t\,\phi(t)}$$

and suppose that (3) is true for $r \geqslant r_0$, where r_0 is so large that $C\,F(r_0) < 1$.

Let $p+1 < m < p+2$, then

$$\int_{r_0}^{\infty} \frac{N(u)}{u^m}\,du \leqslant C \int_{r_0}^{\infty} \frac{u^{p-m}}{\phi(u)}\,du \int_{1}^{u} \frac{N(t)}{t^{p+1}}\,dt + C \int_{r_0}^{\infty} \frac{u^{p-m+1}}{\phi(u)}\,du \int_{u}^{\infty} \frac{N(t)}{t^{p+2}}\,dt.$$

By changing the order of integration we get

$$\int_{r_0}^{\infty} \frac{N(u)}{u^m}\,du \leqslant C \int_{1}^{r_0} \frac{N(t)}{t^{p+1}}\,dt \int_{r_0}^{\infty} \frac{u^{p-m}}{\phi(u)}\,du + C \int_{r_0}^{\infty} \frac{N(t)}{t^{p+1}}\,dt \int_{t}^{\infty} \frac{u^{p-m}}{\phi(u)}\,du +$$

$$+ C \int_{r_0}^{\infty} \frac{N(t)}{t^{p+2}}\,dt \int_{r_0}^{t} \frac{u^{p-m+1}}{\phi(u)}\,du$$

$$\leqslant C\,r_0^{p-m+1}\,F(r_0) \int_{1}^{r_0} \frac{N(t)}{t^{p+1}}\,dt + C \int_{r_0}^{\infty} \frac{N(t)}{t^m}\,dt \int_{t}^{\infty} \frac{du}{u\,\phi(u)} +$$

$$+ C \int_{r_0}^{\infty} \frac{N(t)}{t^m}\,dt \int_{r_0}^{t} \frac{du}{u\,\phi(u)}$$

$$= C\,r^{p-m+1}\,F(r_0) \int_{1}^{r_0} \frac{N(t)}{t^{p+1}}\,dt + C\,F(r_0) \int_{r_0}^{\infty} \frac{N(t)}{t^m}\,dt.$$

Hence

$$\{1 - C\,F(r_0)\}\,r_0^m \int_{r_0}^{\infty} \frac{N(u)}{u^m}\,du \leqslant C\,r_0^{p+1}\,F(r_0) \int_{1}^{r_0} \frac{N(t)}{t^{p+1}}\,dt.$$

Holding r_0 fixed let $m \to p+1$. The right side is independent of m and finite and the left side $\to \infty$, since

$$\int_{r_0}^{\infty} \frac{N(t)}{t^{p+1}}\, dt$$

is divergent. So (3) leads to contradiction.

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N. R. E. C. College
Khurja

ADVANCES IN MATHEMATICS IN THE U.S.S.R.

By L. BRUTYAN

MATHEMATICS is one of the most ancient of sciences in the world. At all stages of development of human society starting with antiquity, it has played a highly important role in moulding the spiritual and material culture of peoples both as an instrument for apprehending the regularities of the ambient world and as a means of subjugating the forces of nature. While fulfilling this role, it has been developed on an ever broader scale in many countries of the world and further perfected.

What advances have been made in mathematics in the Soviet Union ? That was the question I put to Corresponding Member of the USSR Academy of Sciences I. N. Vekua, Assistant Director of the Academy's Institute of Mathematics named after V. A. Steklov, and this is what he told me.

The study of mathematics in Russia began back in olden days. Russia stepped out into the world mathematical arena with the brilliant works of the Petersburg Academician Euler and the Kazan Geometrician Lobatschevsky. The researches of Russian scientists greatly influenced the development of mathematics throughout the world ; however, in pre-revolutionary Russia, few were privileged to indulge in the study of this science.

Scientific research in this field was then concentrated around the mathematics departments of several universities. There were no mathematical research institutions at all.

In Soviet times, when the doors of secondary schools and institutions of higher learning were thrown wide open to young workers and farmers, the mathematics departments increased the number of graduates. Training of young researchers in mathematics forged ahead at post-graduate institutions.

Of great importance for the development of Soviet mathematics was the foundation of the Institute of Physics and Mathematics of the USSR Academy of Sciences, from which the Academy's Steklov Institute of Mathematics subsequently originated.

The Institute's sections devoted to the theory of numbers, algebra, differential equations, theory of functions, functional analysis, theory of probabilities and topology, among other fields, have made a big contribution to world mathematics.

During the past few years, Soviet scientists working in the field of differential equations have developed methods that mathematicians in all countries have made extensive use of in their research. Much that is new has been contributed to geometry. Soviet mathematicians have developed general methods of investigating surfaces in the large on minimum assumptions as regards their degree of smoothness, and also methods in the domain of differential geometry intimately connected with the theory of group representations.

A number of problems connected with the flexibility of surfaces have been solved. Soviet scientists have worked out a theory of algebraic systems that embraces the most general algebraic forms; they have solved the classical converse problem of Galois's theory for solvable groups.

Perhaps one of the biggest achievements of all in contemporary mathematics was the brilliant solution of the famous 200-year old Goldbach problem for odd numbers by Academy Member I. M. Vinogradov, the eminent Soviet mathematician, who did it with the help of an original method he devised.

Today there are mathematical centres not only in Moscow and Leningrad, but also in nearly all the Union republics. In the Ukraine, for instance, extensive research is under way on the theory of probabilities and functional analysis; in Armenia, on approximations in the complex region ; in Georgia, on the theory of numbers, the theory of functions, topology and elasticity ; and in Central Asia, on pressing problems in the fieds of the theory of probabilities and mathematical statistics.

In recent years, Soviet mathematicians have been ever more eagerly tackling problems of great practical importance (in the fields of hydrodynamics, aerodynamics, theory of oscillations, automatic control of production processes, utilization of nuclear energy, launching of ballistic rockets, artificial earth satellites, etc.).

Of tremendous importance for the advancement of mathematics is the development of rapid electronic calculating machines. With their help, mathematicians can quicky solve a number of highly complicated and pressing practical probems. At the same time, electronic calculators will evidently have a revolutionizing effect on the development of mathematics as a whole. They have already given rise, for instance, to a new branch of mathematics known as the theory of information.

The creation of modern calculating machines and their extensive utilization are a striking example of the successful joint effort of big staff of mathematicians, technologists and physicists. What is taking place is the interpenetration of sciences and methods. Mathematicians are carrying out the calculations for highly complicated engineering structures; and engineers are putting new means into the hands of mathematicians.

A striking example of the co-operation of mathematicians and physicists is the consummation of the work on the theory of superconductivity. The penetration of mathematics into physics facilitated the launching of an artificial earth satellite and fulfillment of all the highly complicated calculations involved in its flight.

MATHEMATICAL NOTES

A note on Fermat and Mersenne's numbers

By M. Satyanarayana, *S. V. University, Tirupati*

In this note it is proved that no Fermat number is a triangular number and also numbers of the form $2^n - 1$, where n is an odd integer > 1 are not triangular numbers.

Any positive integer is said to be a *triangular number* if it is of the form $n(n+1)/2$, where n is any positive integral number.

It is well known that in order that any positive number a be a triangular number it is necessary that $1 + 8a$ should be a perfect square. (See for example [Dickson: *History of theory of numbers*, Vol. 2, p. 3.]). It may also be proved without difficulty that the condition is sufficient.

We can show now that *no Fermat number is a triangular number.* Fermat's numbers [Hardy and Wright: *An introduction to the theory of numbers*] are defined by $2^{2^n} + 1$, where $n \geqslant 1$.

$2^{2^n} + 1$ is a triangular number if $1 + 8(2^{2^n} + 1)$, i.e. $9 + 2^{2^n+3}$ is a perfect square.

If possible let $9 + 2^{2^n+3} = M^2$. Put $2^n + 3 = t$. Hence

$$2^t = M^2 - 9 = (M+3)(M-3).$$

It follows that

$$M + 3 = 2^s,\ M - 3 = 2^l, \tag{A}$$

where s and l are positive integers and $s + l = t$.

l cannot be equal to 0, since, if $l = 0$, $2^s = 7$, which is absurd since s is an integer. So it is clear that $s > l \geqslant 1$.

Eliminating M from (A) we get

$$6 = 2^s - 2^l = 2^l(2^{s-l} - 1).$$

i.e. $2^l = 2$, or $l = 1$, $2^{s-l} - 1 = 3$. Hence $s = 3$ and $t = 4$.

Hence $2^n + 3 = 4$ and $n = 0$.

Hence n cannot have any value other than zero. Hence the result.

Numbers of the form $2^n - 1$ are known to be *Mersenne's numbers*. We can establish now that *all Mersenne's numbers for which n is odd and greater than one cannot be triangular numbers.*

$2^n - 1$ can be a triangular number if $1 + 8(2^n - 1)$, i.e. $2^{n+3} - 7$ is a perfect square.

If possible let $2^{n+3} - 7 = M^2$. If n is odd, $n + 3$ is even. Let $n + 3 = 2k$.

$$2^{2k} - M^2 = 7, \text{ i.e. } (2^k + M)(2^k - M) = 7$$

$$2^k - M = 1 \quad \text{and } 2^k + M = 7.$$

On eliminating M we get $2^k + 2^k - 1 = 7$ or $2^k = 4$, i.e. $K = 2, n = 1$.

Hence all Mersenne's numbers for which n is odd and greater than one are not triangular numbers.

The general conic and the conicoid

By P. S. RAU, *Tirupati*

IT is well known in the theory of the linear homogenous equations that n equations in $n + 1$ unknowns always have a solution, which is unique if (and only if) the equations are linearly independent. We shall state it in the following form; though an immediate consequence of the theory it is not perhaps expressly stated so in standard works.

THEOREM 1. *n equations in $n + 1$ unknowns always have a (non-zero) solution. The solution is unique (up to the ratios) if and only if every $(n - 1)$ of these equations have a solution not satisfied by the other.*

As an easy consequence one may mention the following fundamental result in the theory of conics in the plane :

THEOREM 2. *Through any 5 points in the plane there is a conic. This conic is unique if and only if through every 4 of these points there is a conic not passing through the fifth.*

This follows from Theorem 1, on noting that a conic is specified by the ratios of the coefficients in its equation

$$ax^2 + by^2 + cz^2 + 2\,fyz + 2gzx + 2hxy = 0.$$

Also the points $(x_i, y_i, z_i)_{i=1\ldots5}$ lie on the conic if and only if the coefficients a, b, c, f, g, h satisfy the linear homogeneous equations $(x_i^2).\ a + (y_i)^2.\ b + (z_i)^2\ c + (2\,y_i\,z_i)\,f + (2\,z_i\,x_i)\ g + (2\,x_i\,y_i)\ h = 0.$

Likewise we may state the result :

THEOREM 3. *Through 9 points in S there always passes one quadric. This quadric is unique if and only if every 8 of these points lie on a conicoid not passing through the other.*

Clearly the result could also be stated for the Euclidean plane or space instead of the projective spaces; and also for higher dimensional spaces.

I wish to thank Prof. V. Ramaswami for help in the preparation of this note.

A note on odd perfect numbers

By M. PERISASTRI, *Vizianagram*

INTRODUCTION. A positive integer is called a perfect number, if the sum of all its divisors equals twice itself. Something is known about even perfect numbers, but it is not known whether an odd perfect number exists. In this direction Euler proved that if there exists an odd perfect number, it must be of the form $p^{4n+1}\,s^2$, where p is a prime of the form $4k + 1$ and s is odd. Further it was shown that if there exists an odd perfect number it should have at least six different prime factors.

In this note I give bounds for (1) the sum of the reciprocals of the prime factors, and (2) the least prime factor, of an odd perfect number, if it exists.

If an odd perfect number $n = \prod_{r=1}^{k} p_r^{\alpha_r}$ exists then

(a) $\frac{1}{2} < \sum_{r=1}^{k} \frac{1}{p_r} < 2 \log \frac{\pi}{2}$

(b) $p_1 < \frac{2}{3} k + 2,$

where we assume without loss of generality $3 \leqslant p_1 < p_2 < \ldots < p_k$.

PROOF OF (a). The first part of the inequality in (a) follows quickly, for, if $\sigma(n)$ denotes the sum of the divisor of n, then

$$\frac{\sigma(n)}{n} = \prod_{r=1}^{k} \left(1 - \frac{1}{p_r^{\alpha_r+1}}\right) \Big/ \left(1 - \frac{1}{p_r}\right). \tag{1}$$

Since n is a perfect number, $\sigma(n) = 2n$, so that we get

$$2 \prod_{r=1}^{k} \left(1 - \frac{1}{p_r}\right) < 1.$$

But

$$\prod_{r=1}^{k} \left(1 - \frac{1}{p_r}\right) > 1 - \sum_{r=1}^{k} \frac{1}{p_r}, \text{ and so } \sum_{r=1}^{k} \frac{1}{p_r} > \tfrac{1}{2}.$$

To prove the second part of the inequality in (a) we observe that $p_1 \geqslant 3, p_2 \geqslant 5, \ldots, p_k \geqslant q_k$, where q_k stands for the k^{th} prime.

Then

$$\begin{aligned}\prod_{r=1}^{k} \left(1 - \frac{1}{p_r^{\alpha_r+1}}\right) &\geqslant \left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right) \cdots \left(1 - \frac{1}{q_k^2}\right) \\ &\geqslant \frac{4}{3}\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{q_k^2}\right) \\ &> \frac{4}{3} \prod_{p} \left(1 - \frac{1}{p^2}\right),\end{aligned}$$

where p runs through all primes.

Since

$$\prod_p \left(1-\frac{1}{p^2}\right)=\frac{6}{\pi^2},$$

we get

$$\prod_{r=1}^{k}\left(1-\frac{1}{p_r^{\alpha_r+1}}\right)>\frac{8}{\pi^2}.$$

Hence from (1) we get,

$$2\prod_{r=1}^{k}\left(1-\frac{1}{p_r}\right)>\frac{8}{\pi^2}.$$

But $1-x<e^{-x}$, if $0<x<1$, and so

$$2\exp\left\{-\sum_{r=1}^{k}\frac{1}{p_r}\right\}>\frac{8}{\pi^2} \text{ which yields } \sum_{r=1}^{k}\frac{1}{p_r}<2\log\frac{\pi}{2}.$$

PROOF OF (b). (1) gives $2<\prod_{r=1}^{k}\frac{p_r}{p_r-1}$.

But it can be seen that

$$\frac{p_2}{p_2-1}\leqslant\frac{p_1+2}{p_1+1},\ \frac{p_3}{p_3-1}\leqslant\frac{p_2+2}{p_2+1}\leqslant\frac{p_1+4}{p_1+3},\ \ldots$$

and $\frac{p_k}{p_k-1}\leqslant\frac{p_1+2k-2}{p_1+2k-3}$; and so

$$2<\frac{p_1}{p_1-1}\cdot\frac{p_1+2}{p_1+1}\cdot\frac{p_1+4}{p_1+3}\cdots\frac{p_1+2k-2}{p_1+2k-3}. \tag{2}$$

But $\frac{p_1+2r-2}{p_1+2r-3}<\frac{p_1+2r-3}{p_1+2r-4}$; giving $r=1, 2, \ldots, k$ in this inequality and multiplying the results thus obtained, we get

$$\prod_{r=1}^{k}\left(\frac{p_1+2r-2}{p_1+2r-3}\right)^2<\frac{p_1+2k-2}{p_1-2}.$$

Hence from (2)

$$4<\frac{p_1+2k-2}{p_1-2}$$

and (b) follows.

Concurrent θ-normals

By A. A. GNANADOSS, *Madras Christian College*

VAIDYANATHASWAMY's theorem on concurrent θ-normals at any three points of a conic [R. VAIDYNATHASWAMY: 'On the θ-normals of a conic' *Math. Student.* 2(1933), 121-130; C. T. RAJAGOPAL: 'On the intersections of a central conic and its principal hyperbolas, *Math. Gazette*, 35 (1951), 97-104] which is really a theorem about triangles related to the Brocard-point theorem, is capable of some generalization and easy proof by elementary pure geometry.

THEOREM. *If P, Q, R are ordinary points on a plane analytic curve, then there exist one non-trivial and two trivial sets of concurrent θ-normals at P, Q, R.*

PROOF. Let the tangents at P, Q, R be BC, CA, AB. Then the non-trivial point of concurrence of θ-normals at P, Q and R is Ω, the second real intersection of the circles BPR and CPQ, and the trivial points of concurrence are the remaining intersections of the two circles, namely, the circular points at infinity.

NOTE 1. If the points Q and R are fixed but P is variable, then Ω lies on the circle AQR.

NOTE 2. The two Brocard points are the positions of Ω obtained by making P, Q, R coincide with B, C, A or C, A, B.

CLASSROOM NOTES

Concurrence of the normals

By R. RAGHAVENDRAN, *A. V. C. College, Mayuram*

1. Let the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (in rectangular Cartesians) at the point P on it, whose eccentric angle is θ, pass through any given point $Q\,(h, k)$.

Then we have

$$ah \sin\theta - bk \cos\theta = c^2 \sin\theta \cos\theta, \ (c^2 = a^2 - b^2). \tag{1}$$

If we substitute e for $e^{i\theta} \equiv \cos\theta + i \sin\theta$, then equation (1) becomes

$$c^2 e^4 + 2\,(ibk - ah)\, e^3 + 2\,(ibk + ah)\, e - c^2 = 0. \tag{2}$$

This last equation in e, has four roots e_1, e_2, e_3, e_4, say, and consequently, in general, the normals at the four points P_r, whose eccentric angles are θ_r, such that $e^{i\theta_r} = e_r$, pass through the point Q.

From (2), we have

$$e_1\, e_2\, e_3\, e_4 = -1, \tag{3}$$

and

$$\Sigma\, e_1\, e_2 = 0. \tag{4}$$

Now, (3) is equivalent to the equation

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = (2n + 1)\pi, \tag{5}$$

where n is an integer.

On using (3), $e_1\, e_2 + e_3\, e_4 = e_1\, e_2 - e_1^{-1}\, e_2^{-1} = 2i \sin\,(\theta_1 + \theta_2)$.

So (4) becomes,

$$\sin\,(\theta_1 + \theta_2) + \sin\,(\theta_2 + \theta_3) + \sin\,(\theta_3 + \theta_1) = 0. \tag{6}$$

2. We shall now prove the converse, viz.

If the eccentric angles θ_1, θ_2, θ_3 of three points on the ellipse satisfy equation (6), then the normals at these points are concurrent.

Given the three points P_r, whose eccentric angles are θ_r, find a fourth point P_4, whose eccentric angle θ_4 satisfies equation (5).

If $e_r = e^{i\theta_r}$, then (5) and (6) are equivalent to (3) and (4).

If (u, v) be the centre of mean position of the four points P_r, then we see that e_r are the roots of the equation

$$e^4 - 4\left(\frac{u}{a} + i.\frac{v}{b}\right)e^3 + 4\left(\frac{u}{a} - i\frac{v}{b}\right)e - 1 = 0. \tag{7}$$

Substituting $e^{i\theta}$ for e, and simplifying we find that $\theta_1, \theta_2, \theta_3, \theta_4$ are the roots of the equation

$$\frac{2u}{a}\sin\theta + \frac{2v}{b}\cos\theta = \sin\theta\cos\theta, \tag{8}$$

which when compared with (1), shows that the normals at the points P_r pass through the point $\left(\frac{2\,c^2\,u}{a^2}, -\frac{2\,c^2\,v}{b^2}\right)$.

Hence the result.

3. The above seems to be the *natural* method of proving results such as (5) and (6), and can be used, with advantage, in proving the following results also.

(a) The necessary and sufficient conditions that the normals at the four points (a cosec θ_r, b cot θ_r) on the hyperbola,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

should be concurrent, are (5) and (6).

(b) The necessary and sufficient conditions that the *α-normals* at the four points ($a\cos\theta_r$, $b\sin\theta_r$) on the ellipse, should be concurrent, are (5), and

$$\sin(\theta_1 + \theta_2) + \sin(\theta_2 + \theta_3) + \sin(\theta_3 + \theta_1) = -\frac{2ab\cot\alpha}{c^2}.$$

NOTE. If the tangent at a point P of a curve, makes an angle ϕ with the positive direction of the initial line, then the line PQ which makes the angle $\alpha + \phi$ with the positive direction of the initial line, is called the α-normal at P.

(c) The theorem on the eccentric angles of four concyclic points of an ellipse.

(d) Given an ellipse, and a circle, eight normals of the ellipse, in general, touch the circle ; the sum of the eccentric angles of the feet of these normals, is an even multiple of π radians.

4. All the above results, except (d), are well known. The above method of proof seems too simple to be new, but, it is not found in any of the standard text-books of analytical geometry.

An extension of the operator formula

By VAKRAGL, *Bangalore*

WE establish the following formula

$$D^n[e^{ax^s}v] = e^{+ax^s}(D + as\,x^{s-1})^n\,v\,[s \text{ may be integer or not}]$$

which generalizes the classical formula $D^n[e^{ax}v] = e^{ax}(D+a)^n v$.

We prove it by induction.

$$n = 1: \quad D[e^{ax^s}v] = e^{ax^s}Dv + asx^{s-1}e^{ax^s}v = e^{ax^s}[D + asx^{s-1}]\,v$$

$$n = 2: \quad D^2[e^{ax^s}v] = D[e^{ax^s}(D + asx^{s-1})\,v] = e^{ax^s}(D + asx^{s-1})^2\,v.$$

Suppose now that the formula is true for some n say $n = m$.

Then $$D^m[e^{ax^s}v] = e^{ax^s}[D + asx^{s-1}]^m\,v.$$

Operating on this by $D(= d/dx)$, we have

$$D^{m+1}[e^{ax^s}v] = e^{ax^s}[D + asx^{s-1}]\,[D + asx^{s-1}]^m\,v.$$

Thus the formula is true for $n = m + 1$. But it is true for $n = 1, 2$ etc.

Hence it is true for all n.

This formula was a consequence of an attempt to generalize the Hermite polynomial which will be published elsewhere.

Invariants

By V. BALASUBRAMANIA SARMA, *Vivekananda College, Madras*

WE show that the *necessary and sufficient condition* for a quadrilateral to be inscribed in a conic S' and for its four sides to touch another conic S is

$$\theta^3 - 4\,\Delta\,\theta\,\theta' + 8\,\Delta^2\,\Delta' = 0.$$

[In the usual text-books this condition is shown to be necessary only but not sufficient, see E. H. Askwith, *Analytical Geometry of the Conic Sections*, § 363.]

PROOF. Let $S = (abcfgh)(xyz)^2 = 0$, $S' = (a'b'c'f'g'h')(xyz)^2 = 0$. Let P be any point on S' and the tangent from P to S meet S' in B. Let the other tangent from B to S meet S' in C and the tangent from C to S meet S' in Q, and BP in A. Take ABC as the triangle of reference. Let the transformed equations be written as $S = (abcfgh)(xyz)^2 = 0$; $S' = a'x^2 + 2f'yz + 2g'zx + 2h'xy = 0$, where x, y, z refer to new coordinates and $a, b, \ldots a', b' \ldots$ are different from the given ones. Now the equation to PQ is $a'x + 2h'y + 2g'z = 0$. This will touch S if $2Fg'h' + Gg'a' + Ha'h' = 0$, where $A, B, \ldots$ are the minors of a, b in

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \quad \theta = 2(Ff' + Gg' + Hh') \text{ since } BC - F^2 = a\,\Delta, \text{etc.}$$

and $A = B = C = 0$;

$$\Delta\,\theta' = F^2 f'^2 + G^2 g'^2 + H^2 h'^2 + 2\,GH(g'h' - a'f') + $$
$$+ 2\,HF\,h'f' + 2\,FG\,f'g'.$$
$$= \Sigma(Ff')^2 - 2\,GH\,a'f'.$$

$$\Delta^2 = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 2\,FGH;\ \Delta' = 2f'g'h' - a'f'^2.$$

The relation $\theta^3 - 4\,\Delta\,\theta\,\theta' + 8\,\Delta^2\,\Delta'$ becomes

$$16\,GHf'(2\,Fg'h' + Gg'a' + Ha'h').$$

If PQ touches S we get the necessary condition.

Conversely, if $\theta^3 - 4\,\Delta\,\theta\,\theta' + 8\,\Delta^2\,\Delta' = \theta$, we get

$$16\,GHf'(2\,Fg'h' + Gg'a' + Ha'h') = 0.$$

Either $GHf' = 0$ or $2\,Fg'h' + Gg'a' + Ha'h' = 0$.

In the former case the conics degenerate and in the latter PQ touches S.

A note on invariants

By V. BALASUBRAMANIA SARMA, *Vivekanada College, Madras.*

The note by K. A. Viswanathan [*Math. Student* 25 (1957) p. 43] on invaniants assumes that $ab - h^2 \neq 0$. It fails when $ab - h^2 = 0$, Here is the general case.

If $S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ and S' after transformation, with the same origin, then $S + \lambda\,(x^2 + 2\,xy \cos\omega + y^2)$ becomes $S' + \lambda\,(x'^2 + 2x'y' \cos\omega + y'^2)$.

The values of λ which make the first expression equated to zero a line-pair, make the second equated to zero also a line-pair. Hence we get $\dfrac{\Delta}{\Delta'} = \dfrac{c \sin^2\omega}{c' \sin^2\omega'}$ and since $c = c'$ we get the result. It is easy to see that we get the same for a change of origin also.

BOOK REVIEWS

Algebra and trigonometry (for students of pre-university course) By Veppa Annapurniah, 5/1063, Assam gardens, Vishakapatnam (1957), pp. xi + 160, Rs. 3.50.

THIS book is written for students of pre-university course of one year after the matriculation and follows the usual syllabus of elementary algebra and trigonometry. The treatment of the subject matter although generally satisfactory is marred at a number of places by mathematically inaccurate statements and by a wrong use of accepted terminology. Each chapter contains a number of illustrative examples and exercises. A consistent inclusion of graded straightforward exercises in each chapter would enhance the utility of this book.

J. A. SIDDIQI

Modern trigonometry. By C. Brixly and R. V. Andree, Henry Holt and Co, New York (1955), pp. xii + 209.

THE book consists of nine chapters, the first six dealing with trigonometric functions of angles, logarithms, trigonometric tables, trigonometric identities and equations. The next two deal with polar coordinates and complex numbers. The last chapter introduces the reader to more advanced topics like matrix multiplication, möbius strip, four colour problem, quaternions and finite geometry. The book ends with a reading list, answers and tables.

The choice of the subject as also of problems has been made keeping in view the requirements of not only pure mathematicians, but of engineers and scientists.

The authors have rightly laid emphasis on principles rather than rules and have tried to make the subject interesting by frequent incursions in the domain of higher mathematics. They deserve congratulations for having produced a good book which can be profitably used by the junior students in colleges.

J. A. SIDDIQI

The University teaching of social sciences : Statistics: Survey prepared by P. C. Mahalanobis, Unesco (1957), pp. 209, $ 2-75.

THIS survey has been prepared on behalf of the International Statistical Institute, Hague, with the help of reports by leading statisticians of twenty-five countries. This is divided into two parts : the general survey and the survey of individual countries. Part One deals, among other things, with objectives in teaching statistics, education and training in statistics, organization, curricula, research and priorities. Part Two treats about facilities for teaching, degrees and diplomas, careers, teaching methods, research and syllabi in each of the countries. The survey points out the importance of statistics in various branches of science, industry, trade, agriculture etc.

Though statistics is taught in all universities and institutes there is very little co-ordination among them, some emphasizing the theoretical aspect and others the applied aspect only. This survey helps to co-ordinate the two aspects and contains in such a small compass a lot of details supported with figures for all the countries. The credit goes to the patience and industry of several contributors and more so to Professor Mahalanobis who has tried to keep up the uniformity of treatment while preserving the originality of the individual reports which are basically divergent. This book should be in the hands of everybody interested in statistics.

A. R. KOKAN

Tiloyapannatti ka Ganita (in Hindi), *By* Lakshmichandra Jain, M.Sc. Published by Jain Sanskriti Sanrakshak Sangh, Sholapur.

THE title of the book means "the mathematics of the Tiloyapannatti". As the author states, the Jain religious book entitled Tiloyapannatti deals with cosmogony and not primarily with mathematics. Yet the book deals mathematically with many problems and so it is important from the point of view of the history of mathematics. The date of the book has not been discussed, but from the brief introduction it may be guessed that the book was written after the time of Mahavira (850 A.D.).

The original book deals with the concept of the infinite at great length. One ingenious way (in a simplified form) is as follows: Assume vessels of huge dimensions, of bottoms equal to India itself, and imagine one of these filled with mustard seeds. Imagine now the number of vessels to be equal to the number of the mustard seeds in the first vessel, and each vessel filled with mustard seeds. The process is to be repeated a large number of times, and then you would get a "jaghanya paritasankhyata" (transfinite number).

On the other hand extremely minute quantities are also taken into account. For example, one finger's breadth (*angula*) is successively related to smaller units, each being one-eigth of the preceding, the smallest unit being 8^{-12} of one finger's breadth.

The method of using a sort of logarithm is of great interest. The number of times a given number of the form 2^n, where n is an integer, can be divided by 2 is utilised to simplify calculations, and it should be noted that the former is simply the logarithm of the given number to the base 2.

Various geometric shapes of huge dimensions are assumed as models of the universe and their volumes are evaluated, often in terms of living beings who can be contained.

Arithmetical and geometrical progressions and many astronomical problems have also been considered.

The author of the volume under review has explained, with the help of modern mathematics, most of the problems considered in the original book. He has frankly admitted his difficulty in explaining the remaining ones. He has cited in the footnotes parallel passages from various histories of mathematics and mathematical works. The author is to be congratulated on his successfully completing a difficult task.

At the same time it is to be regretted that there are many misprints in the book. Many technical words of the original text have been used without any explanation of their meanings. The glossary at the end of the book giving English equivalents of the Sanskrit technical words used is inadequate. The lettering on the figures is shabby and occasionally illegible. While the numerals in most figures are in Devanagari, in some they are in Roman. Devanagari letters have often been used in the order of the corresponding letters of the English alphabet, and not in their own natural order. At times the construction of the Hindi sentences is also faulty. It is hoped that these minor defects would be removed in the next edition.

GORAKH PRASAD

NEWS AND NOTICES

THE following have been admitted to the life-membership in the Society : G. L. Chandratreya, D. P. Gupta, D. R. Kaprekar, Mrs. Prakash, and R. Manohar.

The following persons have been admitted to membership in the Society.

Miss N. Buragohain, A. K. Bhattachari, O. P. Bhutani, G. Bandyopadyay, N. Das, D. K. Dhaon, M. M. Gaind, J. M. Gandhi, A. A. Gnanados, M. R. Gopal, N. D. Gupta, R. K. Jaggi, A. C. Jain, P. C. Jain (Alwar), P. C. Jain (Delhi), B. G. Jogal, L. N. Kaul, Krishna Rao, S. Lelamma, A. G. Lele, I. S. Luthar, M. Markandeshwara Rao, K. P. Mathew, S. S. Murdeshwar, R. S. Nanda, T. V. L. Narasimha-Sastry, R. Parthasarathi, J. M. Patnaik, V. Ramachandra Rao, P. Ramamurti, S. L. Rathna, M. Ray, J. S. Rustagi, Sampat Kumarachar, N. Sankaran, V. Seshagiri, N. Sethuraman, S. K. Sharma, Avatar Singh, V. P. Singh, Viswanath Singh, I. Sinha, P. K. Srinivasan, A. B. L. Srivastava, A. C. Srivastava, I. Unnisa, W. Unnisa, A. M. Vaidya, T. Venkareddy, B. G. Verma, D. N. Verma, P. D. S. Verma.

The following members of American Mathematical Society have been admitted to membership in the Society under reciprocity agreement : C. E. Aull, J. C. Bradford, C. J. Cillary, H. Kurss, Y. Lehner, Sister R. M. Mulligan, Chung Ki Pank, J. B. O'Toole.

We regret to report the death of Sir V. Ramesam, retired High Court Judge, Madras, and one of the oldest members of the Society.

The Narasinga Rao medals awarded by the Society for 1957 and 1958 have been given to Dr. C. S. Seshadri of Tata Institute, Bombay, for his paper on '*Multiplicative meromorphic functions*' and to Dr. V. Venugopal Rao of Baroda University for his paper on '*The lattice point problem.*'

It is proposed to award a cash prize of Rs. 1,000/- called the 'Racine Prize for Mathematics' for the best research paper in mathematics published before 31 December 1961 by an Indian under 30 years on that date. Further particulars can be had from Dr. K. G. Ramanathan, School of Mathematics, Tata Institute of Fundamental Research, Apollo Pier Road, Bombay-1.

The fourth Congress on Theoretical and applied mechanics was held at the Bengal Engineering College, Howrah, from December 28 to 31, 1958. The President was Dr. S. R. Sen Gupta, Director, Indian Institute of Technology, Kharagpur. About 250 delegates including those from different parts of the world attended the session. Besides the reading of papers, there were two half-hour addresses and two popular lectures. The next meeting will be in Roorkee in December 1959.

The Golden Jubilee of the Calcutta Mathematical Society was celebrated in Calcutta in the last week of December 1958 under the presidentship of Prof. S. N. Bose.

The forty-sixth Session of the Indian Science Congress was held in Delhi from January 20-27, 1959, under the auspices of the Delhi University. The President of the Mathematics Section was Professor M. Ray of Agra University.

Dr. B. N. Prasad has been appointed the Head of the Department of Mathematics at the Allahabad University.

The Allahabad Mathematical Society, with Prof. B. N. Prasad as its President, has started a half-yearly periodical, *The Indian Journal of Mathematics*. The first number has come out and we welcome this Journal.

Dr. V. K. Balachandran has been appointed Reader in Mathematics at the University of Madras and Dr. R. Manohar has been appointed Associated Professor of Mathematics at the Panjab Engineering College, Chandigarh.

Shri M. S. Ramanujan has been awarded the D. Sc. degree by the Annamalai University, for his thesis "*Contributions to the study of general matrix methods of summability with special reference to Hausdorff and quasi-Hausdorff methods*".

Shri R. Ramachandran has been awarded the Ph. D. degree by the University of Madras.

The Society has accepted the invitations of the Allahabad University to hold its annual conference at Allahabad in December 1959 and that of the University of Nagpur to hold the 1960 session at Nagpur.

Professor V. Ganapathy Iyer has been elected President of the Mathematics Section of the Indian Science Congress, January 1960 session in Bombay.

Professor K. Chandrasekharan has been awarded the title of "Padma Shri" by the President of India; we have pleasure in congratulating him.

The Mathematics Seminar, Delhi, will hold the Second Summer School of Mathematics at Hans Raj College, University of Delhi from 11th May 1959 for about a month. The following three main series of lectures are proposed.

i. In Pure Mathematics it has been decided to concentrate on basic concepts of Set theory, Modern Algebra and Topology.

ii. In Applied Mathematics the basic subject of study would be Magneto-Hydrodynamics.

iii. General lectures on modern applications of mathematics, History, Teaching and Research in mathematics.

Dr. Ram Behari, of University of Delhi, was invited to act as Chairman of a Session of the Section of Differential Geometry at the International Congress of Mathematicians held in Edinburgh from August 14-21, 1958. He read a paper on "*Some properties and applicatons of Eisenhart's generalized Riemann space*". He was also invited to deliver lectures at the United States National Science Foundation Summer Institutes held at the University of Notre Dame, and Oberline College, Ohio, U.S.A. Prof. R. L. Wilder, President of the American Mathematical Society for 1955-56, addressed a welcome to him on behalf of the Mathematical Community of his country. While at Oberlin, he was invited to a meeting of the City Council and was accorded a welcome by the Mayor.

Dr. B. N. Prasad represented the Indian Science Congress at the recent meeting of the British Association for the advancement of science in England. He also attended the International Congress and read a paper.