MADURAI KAMARAJ UNIVERSITY

(University with Potential for Excellence)

DIRECTORATE OF DISTANCE EDUCATION

B.Sc., PHYSICS

FIRST YEAR

ANCILLARY MATHEMATICS

PAPER - II

UNIT : 6 - 10

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B.Sc.,

ANCILLARY MATHEMATICS - PAPER - II

Dear Student,

We welcome you as a Student of the B.Sc degree Course.

This paper deals with Algebra, Calculus, Trigonometry, Analytical Geometry of Three Dimensions, Vector Calculus, Differential Equations, Applications of Differential Equations. The learning material for this paper will be supplemented by Contact seminars.

Learning through the Distance Education mode, as you are all aware, involves self – learning and self – assessment and in this regard you are expected to put in disciplined and dedicated effort.

On our part, we assume of our guidance and support.

With best wishes.

Unit VI: (Formulation of Linear Programming Problem)

Mathematical formulation of LPP – Slack and Surplus Variables – Graphical solution of a linear programming problem (with two variables).

Unit VII: (Solution of Linear Programming Problem)

Simplex method of solving a LPP, Charne's Method of Penalties – the concept of duality – formation of the dual LPP when the variables of the primal are non-negative – the dual of the dual is the Primal.

Unit VIII: (Transportation Problem)

Finding feasible solution by

i. North West Corner method
ii. Vogel's Approximation method

Optimal Solution of Transportation Problem
(Balanced Transportation Problem only)

Unit IX: (The Assignment Problem)

Solution of Assignment problem – unbalanced assignment problem and its solution.

Unit X: (Sequencing)

Introduction – Proccessing n jobs in two machines – processing of n jobs in m machines.

Reference Books:

i. Ancillary Mathematics – Paper III (Revised) by Dr.S.Arumugam & Isaac
ii. Operations Research – Volume I by Dr.S.Arumugam & Isaac.

CONTENTS

UNIT – VI
FORMULATION OF LINEAR PROGRAMMING PROBLEM

6.0 Introduction:

Many business and economic situations are concerned with a problem of planning activity. In each case, there are limited resources at your disposal and your problem is to make such a use of these resources so as to yield the maximum production or to minimise the cost of production, or to give the maximum profit, etc. Such problems are referred to as the problems of constrained optimisation. Linear Programming is a technique for determining an optimum schedule of interdependent activities in view of the available resources. Programming is just another word for 'planning and refers to the process of determining a particular plan of action from amongst several alternatives. The word linear stands for indicating that all relationships involved in a particular problem are linear.

In the present chapter mathematical formulation of the linear programming problem (LPP) and Graphical solutions of linear programming problem are discussed.

Objectives:

After working through examples and exercises of this unit you will be able to

1. Formulate the L.P.P
2. Graph the feasible region for a L.P.P with two decision variables.
3. Understand how basic and non – basic variables relate to the graph of problem.

Structure:

Mathematical formulation of L.P.P

Slack and Surplus variables

Graphical solutions of a L.P.P

Keywords

Answers to check your progress questions

Model Questions

6.1 Mathematical formulation of L.P.P

If $x_j (j=1,2,\dots\dots\dots,n)$ are the n decision variables of the problem and if the system is subject to m constraints, the general mathematical model can be written in the form

Optimize $Z=f(x_1,x_2,\dots\dots,x_n)$

Subject to $g_i(x_1,x_2,\dots\dots\dots,x_n) \le, =, \ge\ b_i,\ (i=1,2,\dots\dots\dots,m)$

(called structural constraints)

and $x_1,x_2,\dots\dots\dots,x_n \ge 0,$

(called the non – negativity restrictions or constraints)

Procedure for Mathematical formulation of L.P.P

Step: 1

Study the given situation to find the key decisions to be made.

Step: 2

Identify the variables involved and designate them by symbols $x_j (j=1,2,\dots\dots\dots,)$.

Step: 3

State the feasible alternatives which generally are $x_j \ge 0$, for all j.

Step: 4

Identify the constraints in the problem and express them as linear inequalities or equations LHS of which are linear functions of the decision variables.

Step: 5

Identify the objective function and express it as a linear function of the decision variables.

Example: 6.1.1

A company has three operational departments (Waving, Processing and Packing) with capacity to produce three different types of clothes namely suiting's, shirting's and woollens yielding a profit of Rs. 2, Rs. 4 and Rs. 3 per metre respectively. One metre of suiting requires 3 minutes in weaving, 2 minutes in processing and 1 minute in packing. Similarly one metre of shirting requires 4 minutes in weaving, 1 minute in processing and 3 minutes in packing. One metre of woolen requires 3 minutes in each department. In a week, total run time of each department is 60, 40 and 80 hours for weaving, processing and packing respectively.

Formulate the linear programming problem to find the product mix to maximize the profit.

Solution:

The data of the problem is summarized below:

Step: 1

Department				
	Weaving (in minutes)	**Processing (in minutes)**	**Packing (in minutes)**	**Profit (Rs. Per meter)**
Suiting's	3	2	1	2
Shirting's	4	1	3	4
Woolens	3	3	3	3
Availability (min)	60 x 60	40 x 60	80 x 60	

The key decision is to determine the weekly rate of production for the three types of clothes.

Step: 2

Let us designate the weekly production of suiting's, shirting's and woolens by x_1 meters, x_2 meters and x_3 meters respectively.

Step: 3

Since it is not possible to product negative quantities, feasible alternatives are set of values of x_1, x_2 and x_3 satisfying $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

Step: 4

The constraints are the limited availability of three operational departments. One meter of suiting requires 3 minutes of weaving. The quantity being x_1 meters, the requirement for suiting alone will be $3x_1$ units. Similarly, x_2 meters of shirting and x_3 meters of woolen will require $4x_2$ and $3x_3$ minutes respectively. Thus the total requirement of weaving will be $3x_1 + 4x_2 + 3x_3$, which should not exceed the available 3600 minutes. So, the labour constraint becomes $3x_1 + 4x_2 + 3x_3 \leq 3600$.

Similarly the constraints for the processing department and packing departments are $2x_1 + x_2 + 3x_3 \leq 2400$ and $x_1 + 3x_2 + 3x_3 \leq 4800$ respectively.

Step: 5

The objective is to maximize the total profit from sales. Assuming that whatever is produced is sold in the market, the total profit is given by the linear relation $z = 2x_1 + 4x_2 + 3x_3$.

The linear programming problem can be put in the following mathematical format:

Maximize $Z = 2x_1 + 4x_2 + 3x_3$

Subject to the constraints

$$3x_1 + 4x_2 + 3x_3 \leq 3600$$

$$2x_1 + x_2 + 3x_3 \leq 2400$$

$$x_1 + 3x_2 + 3x_3 \leq 4800$$

$x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

Example: 6.1.2

The manager of an oil refinery must decide on the optimum mix of two possible blending processes of which the input and output production runs are as follows.

Process	Input		Output	
	Crude A	Crude B	Gasoline X	Gasoline Y
1	6	4	6	9
2	5	6	5	5

The maximum amounts available of crudes A and B are 250 units and 200 units respectively. Market demand shows that at least 150 units of gasoline X and 130 units of gasoline Y must be produced. The profits per production run from process 1 and process 2 are Rs. 4 and Rs. 5 respectively. Formulate the problem for maximizing the profit.

Solution:

Decision Variables:

x_1 = number of units of Gasoline from process 1.

x_2 = number of units of Gasoline from process 2.

Objective function : Maximize $Z = 4x_1 + 5x_2$

Constraints : $6x_1 + 5x_2 \leq 250$, $4x_1 + 6x_2 \leq 200$

(crude A)

$6x_1 + 5x_2 \geq 150$, $9x_1 + 5x_2 \geq 130$

(crude B)

$x_1 \geq 0$ and $x_2 \geq 0$.

Example: 6.1.3

The owner of metro sports wishes to determine how many advertisements to place in the selected three monthly maganizes A, B and C. His objective is to advertise in such a way that the total exposure to principal buyers of expensive sports good is maximized. Percentages of readers for each magazine are known. Exposure in any particular magazine is the number of advertisements placed multiplied by the number of principal buyers. The following data may be used.

	Magazine		
	A	**B**	**C**
Readers	1 Lakh	0.6 Lakh	0.4 Lakh
Principal Buyers	20 %	15 %	8 %
Cost per advertisement (Rs)	8000	6000	5000

The budgeted amount is at most Rs. 1 Lakh for the advertisement. The owner has already decided that magazine A should have no more than 15 advertisements and that B and C each have at least 80 advertisements. Formulate on LP model for the problem.

Solution:

Decision variables x_1 = number of insertions in Magazine A

x_2 = number of insertions in Magazine B and

x_3 = number of insertions in Magazine C.

Objective function:

Maximize (total exposure)

$$Z = (20\% \text{ of } 1{,}00{,}000)x_1 + (15\% \text{ of } 60{,}000)x_2 + (8\% \text{ of } 40{,}000)x_3$$

$$= 20{,}000x_1 + 9{,}000x_2 + 3{,}200x_3$$

Constraints

$$8{,}000x_1 + 6{,}000x_2 + 5{,}000x_3 \le 1{,}00{,}000$$

$$x_1 \le 15, \qquad x_2 \ge 8, \qquad x_3 \ge 8$$

$$x_1 \ge 0, \qquad x_2 \ge 0 \text{ and } x_3 \ge 0$$

Example: 6.1.4

A complete unit of a certain product consists of four units of component A and three units of component B. The two components (A and B) are manufactured from two different raw materials of which 100 units and 200 units, respectively are available. Three departments are engaged in the production process with each department using a different method for manufacturing the components per production run and the recounting units of each component are given below.

Department	Input per run (units)		Output per run (units)	
	Raw Material I	Raw Material II	Component A	Component B
1	7	5	6	4
2	4	8	5	8
3	2	7	7	3

Formulate this problem as a linear programming model. So as to determine the number of production runs for each department which will maximize the total number of complete units of the final product.

Solution:

Decision Variables:

x_1 = number of production runs for department 1

x_2 = number of production runs for department 2 and

x_3 = number of production runs for department 3.

Objective function:

Since each unit of the final product requires 4 units of component A and 3 units of component B, therefore, maximum number of units of the final product cannot exceed the smaller value of

$$\left\{\frac{\text{Total number of units A produced}}{4}, \frac{\text{Total number of units B produced}}{3}\right\}$$

(i.e.), Minimum of $\left\{\frac{6x_1+5x_2+7x_3}{4}, \frac{4x_1+8x_2+3x_3}{3}\right\}$

Constraints: (i)

If y is the number of component units of final product, then obviously, we have $\frac{6x_1+5x_2+7x_3}{4} \geq y$ and $\frac{4x_1+8x_2+3x_3}{3} \geq y$.

(i.e.) $6x_1+5x_2 7x_3-4y \geq 0$

$4x_1+8x_2+3x_3-3y \geq 0$

(ii) $7x_1+4x_2+2x_3 \leq 100$

$5x_1+8x_2+7x_3 \leq 200$

$x_1 \geq 0,\ x_2 \geq 0$ and $x_3 \geq 0$

Example: 6.1.5

Old hens can be bought at Rs. 2 each and young ones at Rs. 5 each. The old hens lay 3 eggs per week and the young ones lay 5 eggs per week, each egg being worth 30 paise. A hen costs Rs. 1 per week to feed. A person has only Rs. 80 to spend for hens. How many of each kind should he buy to give a profit of more than Rs. 6 per week assuming that he cannot house more than 20 hens. Formulate this as a L.P.P.

Solution:

The person decides to by x_1 old hens and x_2 young hens to maximize his profit.

Since he has only Rs. 80 to spend for hens and old hen costs Rs. 2 and young hen costs Rs. 5 each,

$2x_1+5x_2 \leq 80$.

Also, since he cannot house more than 20 hens,

$x_1+x_2 \leq 20$.

The total sale of eggs will be $=$ Rs. 0.3 $(3x_1+5x_2)$

Expenditure on feeding will be $=$ Rs. 1 (x_1+x_2)

$\therefore$ The net profit is $= \text{Rs. } [0.3(3x_1 + 5x_2) - 1(x_1 + x_2)]$

$= \text{Rs. } (0.5x_2 - 0.1x_1)$

$\therefore \quad 0.5x_2 - 0.1x_1 \geq 6.$

The constraints are

$$2x_1 + 5x_2 \leq 80$$

$$x_1 + x_2 \leq 20$$

$$0.5x_2 - 0.1x_1 \geq 6$$

and $x_1, x_2 \geq 0$.

$\therefore$ The complete formulation of the L.P.P is maximize $Z = 0.5x_2 - 0.1x_1$, subject to the constraints

$$2x_1 + 5x_2 \leq 80$$

$$x_1 + x_2 \leq 20$$

$0.5x_2 - 0.1x_1 \geq 6$ and $x_1, x_2 \geq 0$.

6.2 Slack and Surplus Variables

Definition: Slack Variables

Let the constraints of a general L.P.P be $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$, $i = 1,2,\ldots\ldots.,k$.

Then, the non – negative variables x_{n+i} which satisfy $\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i$, $i = 1,2,\ldots\ldots.,k$ are called slack variables.

Definition: Surplus Variables

Let the constraints of a general L.P.P be $\sum_{j=1}^{n} a_{ij}x_j \geq b_i, i = k+1, k+2, \ldots\ldots., \ell$.

Then the non – negative variables x_{n+i} which satisfy $\sum_{j=1}^{n} a_{ij}x_j - x_{n+i} = b_i$, $i = k+1, k+2, \ldots\ldots\ldots., \ell$ are called surplus variables.

6.3 Graphical Solution of a L.P.P

Linear programming problems involving only two variables can be effectively solved by a graphical method which provides a pictorial representation of the problems and its solutions and which gives the basic concepts used in solving general L.P.P which may involve any finite number of variables. This method is simple to understand and easy to use. Graphical method is not a powerful tool of linear programming as most of the practical situations do involve more than two variables. But the method is really useful to explain the basic concepts of L.P.P. to the persons who are not familiar with this. Though graphical method can deal with any number of constraints but since each constraint is shown as a line on a graph, a large number of lines make the graph difficult to read.

Working procedure for Graphical Method

Step: 1

Identify the problem – the decision variables, the objective and the restrictions.

Step: 2

Set up the mathematical formulation of the problem.

Step: 3

Plot a graph representing all the constraints of the problem and identify the feasible region (solution space). The feasible region is the intersection of all the regions represented by the constraints of the problem and is restricted to the first quadrant only.

Step: 4

The feasible region obtained in step 3 may be bounded or unbounded. Compute the coordinates of all the corner points of the feasible region..

Step: 5

Find out the value of the objective function at each corner (solution) point determined in step 4.

Step: 6

Select the corner point that optimize (maximizes or minimizes) the value of the objective function. It gives the optimum feasible solution.

Example: 6.3.1

A company makes two kinds of leather belts. Belt A is a high quality belt, and belt B is at low quantity. The respective profits are Rs. 4.00 and Rs. 3.00 per belt. Each belt of type A requires twice as much time as a belt of type B, and if all belts were of type B, the company could make 1000 per day. The supply of leather is sufficient for only 800 belts per day (both A and B combined). Belt A requires a fancy buckles and only 400 per day are available. There are only 700 buckles a day available for belt B. Determine the optimal product mix.

Solution:

Step: 1

The mathematical formulation of the given linear programming problem is

Maximize $Z = 4x_1 + 3x_2$

Subject to the constraints:

$$2x_1 + x_2 \leq 1{,}000$$

$$x_1 + x_2 \leq 800$$

$x_1 \leq 400$ and $x_2 \leq 700$

$x_1 \geq 0$ and $x_2 \geq 0$

Where $x_1 =$ number of belts of type A.

$x_2 =$ number of belts of type B.

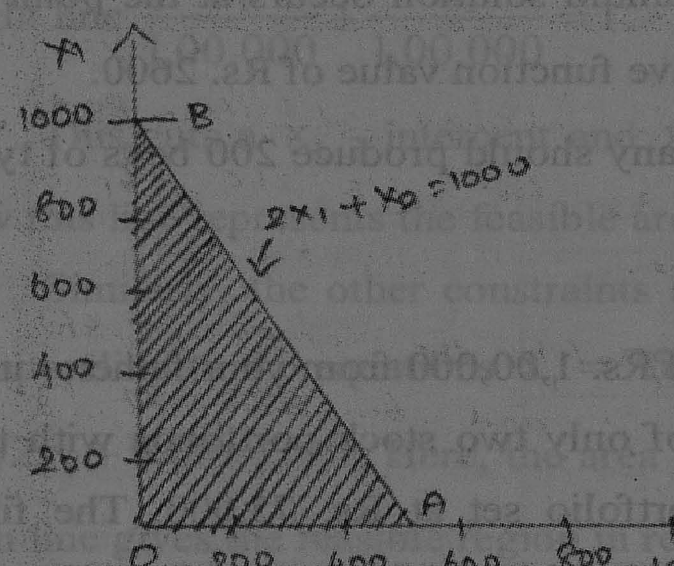
Step: 2

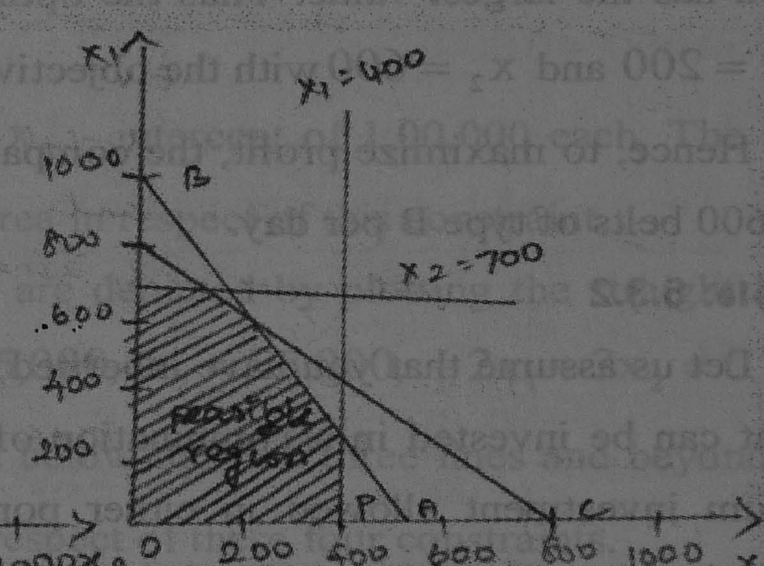
Next we construct the graph by considering the Cartesian rectangular axis $O X_1 X_2$ in the plane. As each point has the coordinates of the type (x_1, x_2); any point satisfying the conditions $x_1 \geq 0$ and $x_2 \geq 0$ lies in the first quadrant.

Now, the inequalities are graphed taking, them as equations, e.g., the first constraint $2x_1 + x_2 \leq 1000$. The equation is re – written as $\frac{x_1}{500} + \frac{x_2}{1000} = 1$.

This equation indicates that when it is plotted on the graph, it cuts an x_1- intercept of 500 and x_2- intercept of 1000. These two points are then connected by a straight line which is shown in fig. (i) as line AB. Any point (representing a combination of x_1 and x_2) that fall on this line or in the area below it, is acceptable in so far as this constraint is concerned. The region OAB formed by two axes and the line representing the equation $2x_1 + x_2 = 1000$ is the region containing accepted values of x_1 and x_2 in respect of this constraint.

Similarly, the constraint $x_1 + x_2 \leq 800$ can be plotted. The line CD in fig (ii) represents the equation $x_1 + x_2 = 800$. The region OCD, formed by the two axes and this line represents the area in which any point would satisfy this constraint of leather availability. Further, the constraints $x_1 \leq 400$ and $x_2 \leq 700$ are also plotted on the graph which represents the area between the two axes and the lines $x_1 = 400$ and $x_2 = 700$ as shown in fig (ii).





Now all the constraints have been graphed. The area bounded by all these constraints, called feasible region (or) solution space, as shown in fig (ii) by the shaded area OPQRST.

Step: 3

The optimum value of objective function occurs at one of the extreme (corner) points of the feasible region. The coordinates of the extreme points are

I = (0, 0), P = (400, 0), Q = (400, 200), R = (200, 600), S = (100, 700) and T = (0, 700).

Step: 4

We now compute the z – values corresponding to the extreme points

Extreme point	(x_1, x_2)	$z = 4x_1 + 3x_2$	
O	(0, 0)	0	
P	(400, 0)	1600	
Q	(400, 200)	2200	
R	(200, 600)	2600	← Maximum
S	(100, 700)	2500	
T	(0, 700)	2100	

Step: 5

The optimum solution is that extreme point for which the objective function has the largest value. Thus the optimum solution occurs at the point R, i.e., $x_1 = 200$ and $x_2 = 600$ with the objective function value of Rs. 2600.

Hence, to maximize profit, the company should produce 200 belts of type A and 600 belts of type B per day.

Example: 6.3.2

Let us assume that you have inherited Rs. 1,00,000 from your father – in – law that can be invested in a combination of only two stock portfolios with the maximum investment allowed in either portfolio set at Rs. 75,000. The first portfolio has an average rate of return of 10%, whereas the second has 20%. In terms of risk factors associated with these portfolios, the first has a risk rating of 4 (on a scale from 0 to 10), and the second has 9. Since you wish to maximize your return, you will not accept an average rate of return below 12% or a risk factor above 6. Hence, you then face the important question. How much should you invest in each portfolio?

Solution:

Step: 1

The mathematical formulation of the linear programming problem is

Maximize $Z = 0.10x_1 + 0.20x_2$

Subject to the constraints

$x_1 + x_2 \leq 1{,}00{,}000$, $x_1 \leq 75{,}000$, $x_2 \leq 75{,}000$

$0.10x_1 + 0.20x_2 \geq 0.12(x_1 + x_2)$ (or) $-0.02x_1 + 0.08x_2 \geq 0$

$4x_1 + 9x_2 \leq 6(x_1 + x_2)$ (or) $-2x_1 + 3x_2 \leq 0$

$x_1 \geq 0$ and $x_2 \geq 0$

Where x_1 = amount invested in portfolio 1, and

x_2 = amount invested in portfolio 2

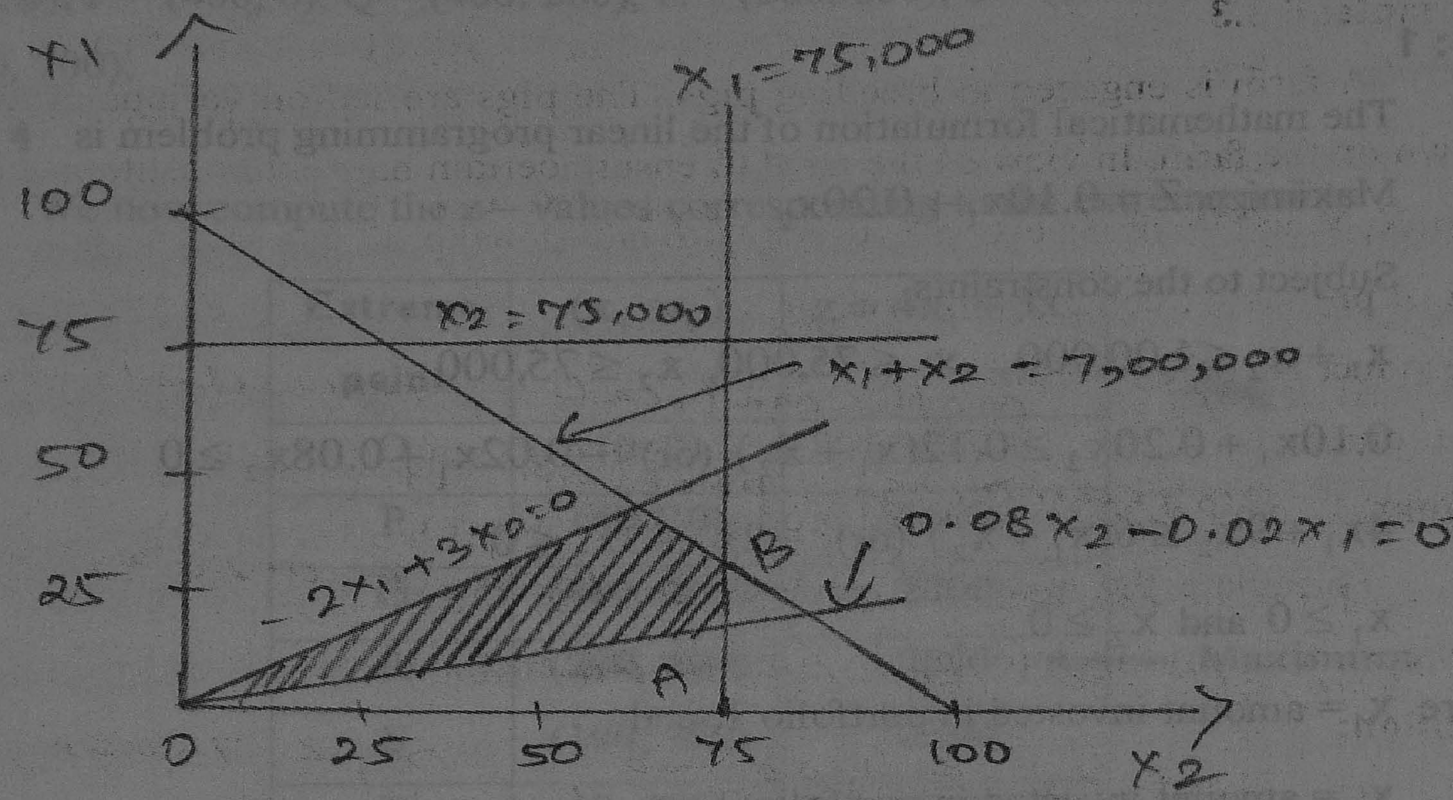
Step: 2

The first constraint $x_1 + x_2 \leq 1{,}00{,}000$ can be graphed by plotting the straight line $\dfrac{x_1}{1{,}00{,}000} + \dfrac{x_2}{1{,}00{,}000} = 1$.

This cuts a x_1 - intercept and x_2 - intercept of 1,00,000 each. The area below this line represents the feasible area in respect of this constraint.

Similarly, the other constraints are depicted by plotting the straight lines corresponding to the equations $x_1 = 75{,}000$, $x_2 = 75{,}000$, $-2x_1 + 3x_2 = 0$ and $-0.02x_1 + 0.08x_2 = 0$. Here, the area below the first three lines and beyond the fourth line gives the feasible region in respect of these four constraints.

Thus the feasible region in respect of the given problem is as shown in figure.



Step: 3

The coordinates of the extreme points are:

O (0,0), A = (75,000, 18,750) B = (75,000, 25,000) and

C = (60,000, 40,000)

Step: 4

The z – values corresponding to the extreme points are:

Extreme Point	(x_1, x_2)	$z = 0.10x_1 + 0.20x_2$
O	(0, 0)	0
A	(75,000, 18,750)	11,250
B	(75,000, 25,000)	12,500
C	(60,000, 40,000)	14,000 ← Maximum

Hence the optimal solution is $x_1 = 60{,}000$, $x_2 = 40{,}000$ and maximum return = Rs. 14,000.

Example: 6.3.3

A farm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. In view of the need to ensure certain nutrient constituents (call them X, Y and Z), it is necessary to buy two additional products say, A and B, one unit of product A contains 36 units of X, 3 units of Y and 20 units of Z. One unit of product B contains 6 units of X, 12 units of Y and 10 units of Z. The minimum requirement of X, Y and Z is 108 units, 36 units and 100 units respectively. Product A cost Rs. 20 per unit and product B Rs. 40 per unit.

Formulate the above as a linear programming problem to minimize the total cost, and solve the problem by using graphic method.

Solution:

The data of the given problem can be summarized as follows:

Nutrient Constituents	**Nutrient content in product**		**Minimum amount of nutrient**
	A	**B**	
X	36	06	108
Y	03	12	36
Z	20	10	100
Cost of Product	Rs. 20	Rs. 40	

Making use of above information, the appropriate mathematical formulation of the linear programming problem is

Minimize $Z = 20x_1 + 40x_2$

Subject to the constraints

$$36x_1 + 6x_2 \geq 108$$

$$3x_1 + 12x_2 \geq 36$$

$$20x_1 + 10x_2 \geq 100 \text{ and } x_1, x_2 \geq 0.$$

Consider now a set of Cartesian rectangular axis OX_1X_2 in the plane. As each point has the coordinates of the type (x_1, x_2) any point satisfying the conditions $x_1 \geq 0$ and $x_2 \geq 0$ lies in the first quadrant only.

The constraints of the given problem are plotted as described earlier by treating them as equations:

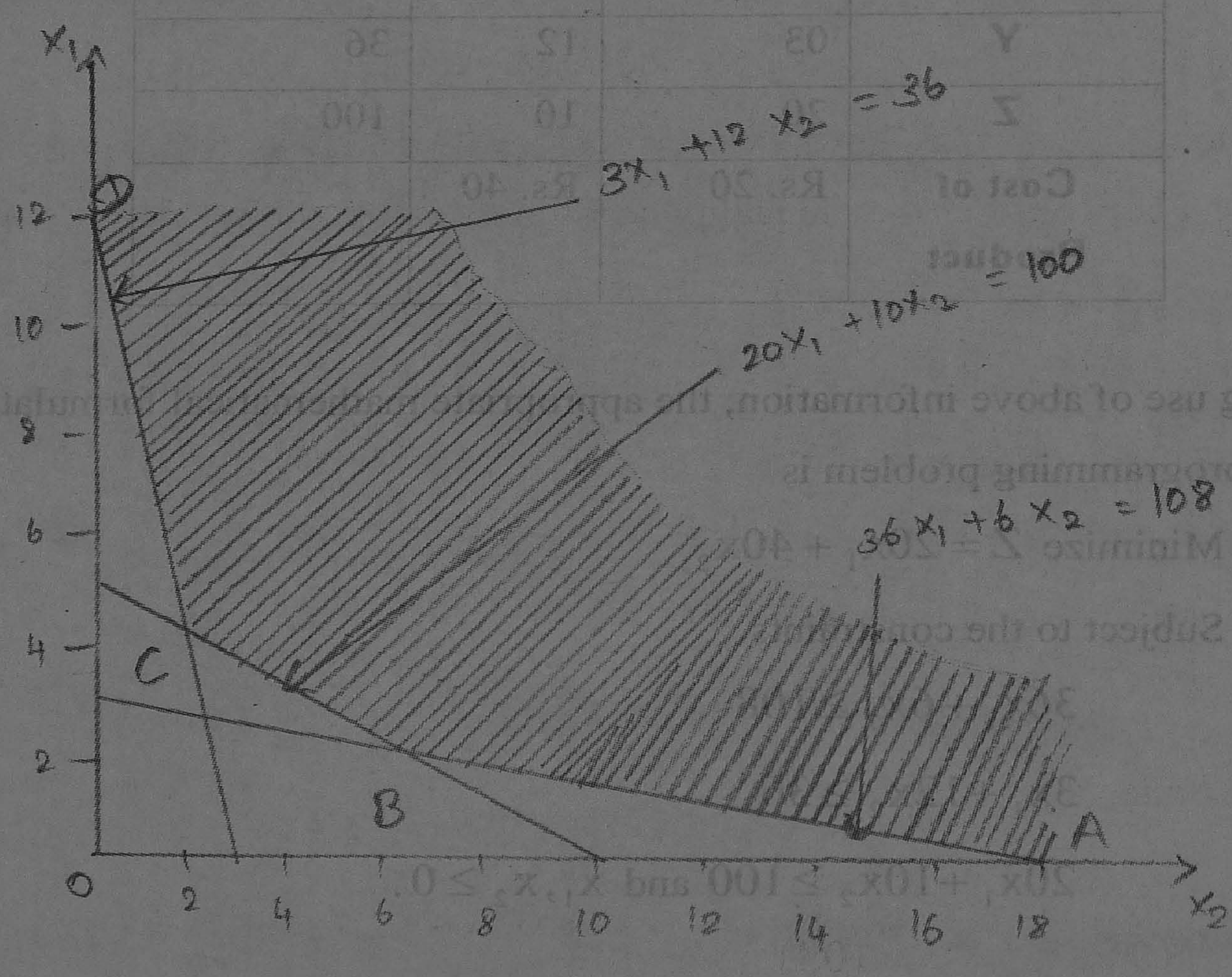
$36x_1 + 6x_2 = 108$

$3x_1 + 12x_2 = 36$ and

$20x_1 + 10x_2 = 100$

(or) $\frac{x_1}{3} + \frac{x_2}{18} = 1$, $\frac{x_1}{12} + \frac{x_2}{3} = 1$ and $\frac{x_1}{5} + \frac{x_2}{10} = 1.$

The area beyond these lines represents the feasible region in respect of these constraints; any point on the straight lines or in the region above these lines would satisfy the constraints. The feasible region of the problem is as shown in figure.



The coordinates of the extreme points of the feasible region are:

$A = (0,18)$, $B = (2,6)$, $C = (4,2)$ and $D = (12,0)$.

The value of the objective function at each of the extreme points can be evaluated as follows:

Extreme Point	(x_1, x_2)	$z = 20x_1 + 40x_2$
A	(0,18)	720
B	(2,6)	280
C	(4,2)	160 ← Minimum
D	(12,0)	240

Hence the optimum solution is to purchase 4 units of product A and 2 units of product B in order to maintain a minimum cost of Rs. 160.

Example: 6.3.4

Solve the following L.P.P by the graphical method

$$\text{Max } Z = 3x_1 + 2x_2$$

Subject to $-2x_1 + x_2 \le 1$

$$x_1 \le 2$$

$$x_1 + x_2 \le 3 \text{ and } x_1, x_2 \ge 0$$

Solution:

First consider the inequality constraints as equalities.

$$-2x_1 + x_2 = 1 \quad (1)$$

$$x_1 = 2 \quad (2)$$

$$x_1 + x_2 = 3 \quad (3)$$

and

$$x_1 = 0 \quad (4)$$

$$x_2 = 0 \quad (5)$$

For the line,

$$-2x_1 + x_2 = 1,$$

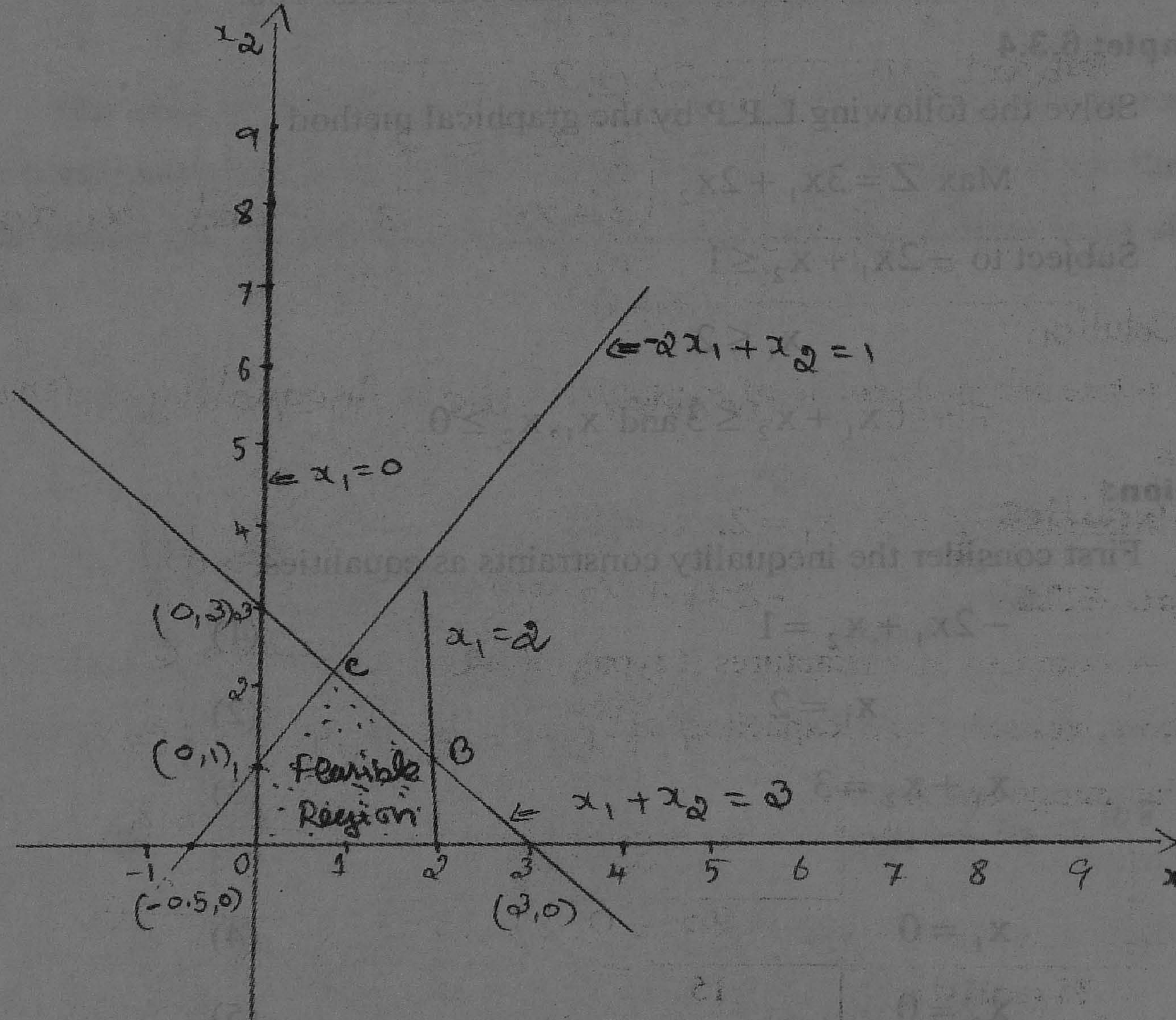
Put $x_1 = 0 \Rightarrow x_2 = 1 \Rightarrow (0,1)$

Put $x_2 = 0 \Rightarrow -2x_1 = 1 \Rightarrow x_1 = -0.5 \Rightarrow (-0.5,0)$

So, the line (1) passes through the points (0, 1) and (-.05, 0). The points on this line satisfy the equation $-2x_1 + x_2 = 1$. Now orgin (0, 0), on substitution, gives $-0+0=0<1$; hence it also satisfies the inequality $2x_1 + x_2 \leq 1$.

Thus all points on the origin side and on this line satisfy the inequality $-2x_1 + x_2 \leq 1$.

Similarly interpreting the other constraints we get the feasible region OABCD. The feasible region is also known as solution space of the L.P.P. Every point within this area satisfies all the constraints.



Now our aim is to find the vertices of the solution space. B is the point of intersection of $x_1 = 2$ and $x_1 + x_2 = 3$. Solving these two equations, we have $x_1 = 2, x_2 = 1$.

Therefore we have the vertex B (2, 1). Similarly, C is the intersection of $-2x_1 + x_2 = 1$ and $x_1 + x_2 = 3$. Solving these we have $C = \left(\frac{2}{3}, \frac{7}{3}\right)$.

$\therefore$ The vertices of the solution space are $0(0,0)$, $A(2,0)$, $B(2,1)$, $C\left(\frac{2}{3}, \frac{7}{3}\right)$ and $D(0,1)$.

The values of Z at these vertices are given by

Extreme Point	**Vertex** (x_1, x_2)	$Z = 3x_1 + 2x_2$
O	(0, 0)	0
A	(2, 0)	6
B	(2, 1)	8
C	$\left(\frac{2}{3}, \frac{7}{3}\right)$	$\frac{20}{3}$
D	(0, 1)	2

Since the problem is of maximization type, the optimum solution to the L.P.P is

Maximize $Z = 8,\ x_1 = 2,\ x_2 = 1$.

Example: 6.3.5

A company manufactures 2 types of printed circuits. The requirements of transistors, resistors and capacitors for each type of printed circuits along with other data are given below:

	Circuit		**Stock available**
	A	**B**	
Transistor	15	10	180
Resistor	10	20	200
Capacitor	15	20	210
Profit	Rs. 5	Rs. 8	

How many circuits of each type should the company produce from the stock to earn maximum profit.

Solution:

Let x_1 be the number of type A circuits and x_2 be the number of type B circuits to be produced.

To produce these units of type A and type B circuits, the company requires

Transistors = $15x_1 + 10x_2$

Resistor = $10x_1 + 20x_2$

Capacitors = $15x_1 + 20x_2$

Since the availability of these transistors, resistors and capacitors are 180, 200 and 210 respectively, the constraints are

$15x_1 + 10x_2 \leq 180$

$10x_1 + 20x_2 \leq 200$

$15x_1 + 20x_2 \leq 210$

and $x_1 \geq 0, x_2 \geq 0$.

Since the profit from type A is Rs. 5 and from type B is Rs. 8 per units the total profit is $5x_1 + 8x_2$.

$\therefore$ The complete formulation of the L.P.P is

$$\text{Maximize } Z = 5x_1 + 8x_2$$

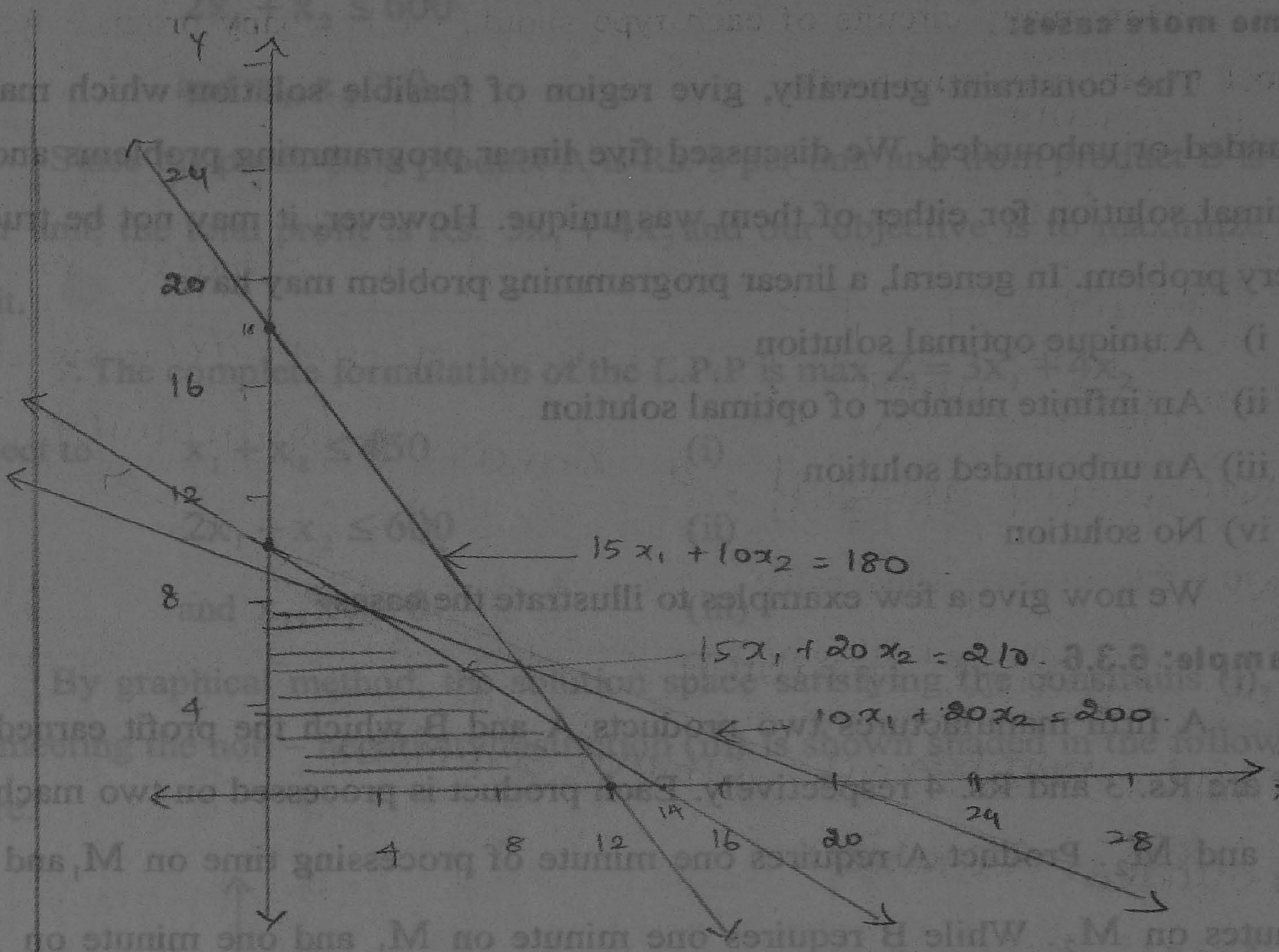
Subject to $15x_1 + 10x_2 \leq 180$

$10x_1 + 20x_2 \leq 200$

$15x_1 + 20x_2 \leq 210$

and $x_1, x_2 \geq 0$

By using graphical method, the solution space is given below with shaded area OABCD with vertices O(0,0), A(12,0), B(10,3), C(2,9) and D(0,10).



The value of Z of these vertices are given by

Extreme point	(x_1, x_2)	$Z = 5x_1 + 8x_2$
O	(0, 0)	0
A	(12, 0)	60
B	(10, 3)	74
C	(2, 9)	82
D	(0, 10)	80

Since the problem is manimization type, the optimum solution is maximum Z = 82, $x_1 = 2$, $x_2 = 9$.

Note:

From the above examples, for problems involving two variables and having a finite solution, we observed that the optimal solution existed at a vertex of the feasible region that is "if there exists an optimal solution of an L.P.P, it will be at one of the vertices of the feasible region".

Some more cases:

The constraint generally, give region of feasible solution which may be bounded or unbounded. We discussed five linear programming problems and the optimal solution for either of them was unique. However, it may not be true for every problem. In general, a linear programming problem may have:

i) A unique optimal solution

ii) An infinite number of optimal solution

iii) An unbounded solution

iv) No solution

We now give a few examples to illustrate the cases.

Example: 6.3.6

A firm manufactures two products A and B which the profit earned per unit are Rs. 3 and Rs. 4 respectively. Each product is processed on two machines M_1 and M_2. Product A requires one minute of processing time on M_1 and two minutes on M_2. While B requires one minute on M_1 and one minute on M_2. Machine M_1 is available for not more than 7 hours 30 minutes while machine M_2 is available for 10 hours during any working day. Find the number of units products A and B to be manufactured to get maximum profit. Formulate the above as a L.P.P and solve by graphical method.

Solution:

Let the firm decide to manufacture x_1 units of product A and x_2 units of product B.

To produce these units of products A and B, it requires

$x_1 + x_2$ hours of processing times on M_1.

$2x_1 + x_2$ hours of processing times on M_2.

But the availability of these two machines M_1 and M_2 are 450 minutes and 600 minutes respectively, the constraints are

$$x_1 + x_2 \leq 450$$

$2x_1 + x_2 \le 600$

and $x_1, x_2 \ge 0$.

Since the profit from product A is Rs. 3 per unit and from product B is Rs. 4 per unit, the total profit is Rs. $3x_1 + 4x_2$ and our objective is to maximize the profit.

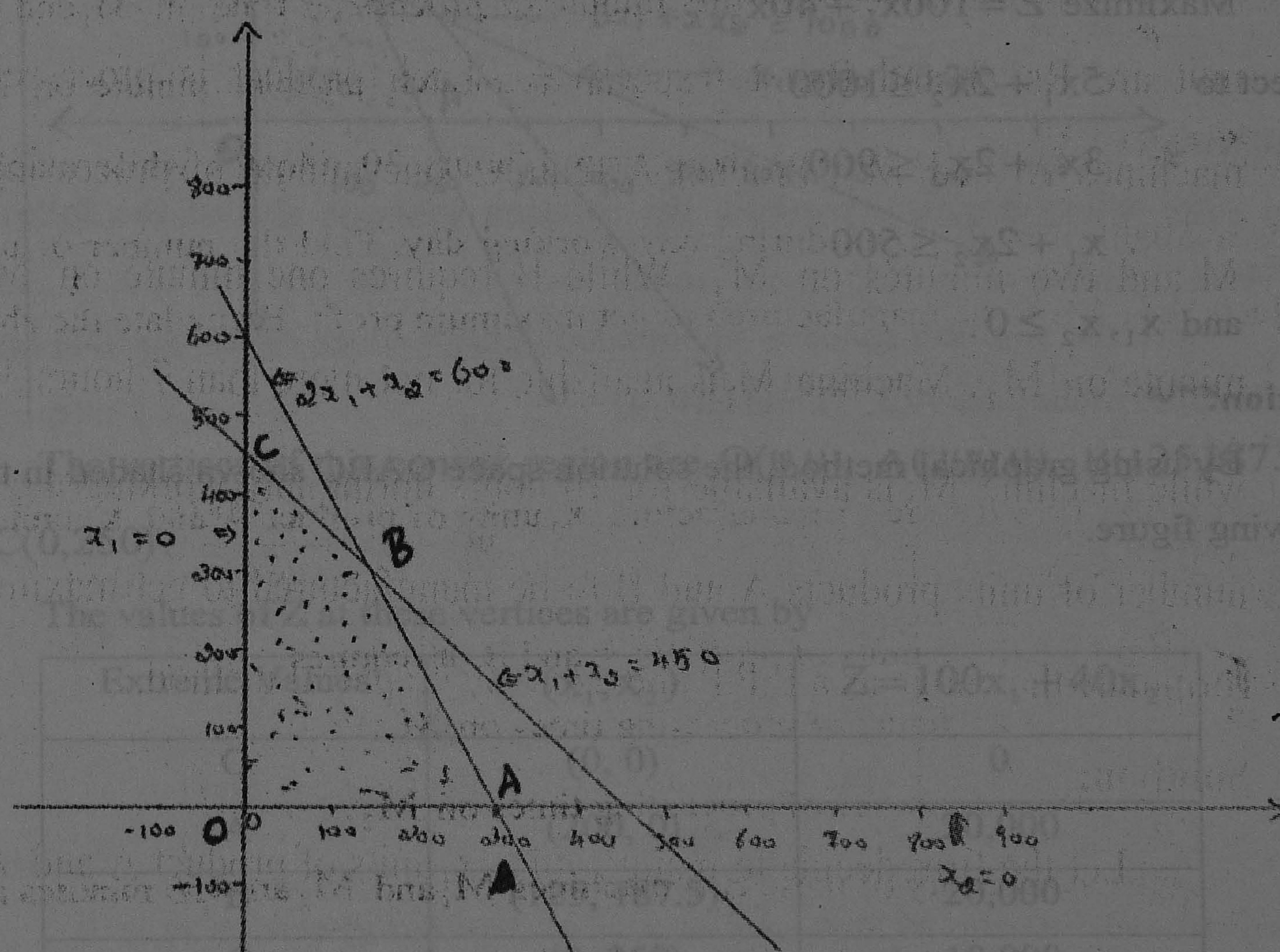
$\therefore$ The complete formulation of the L.P.P is max $Z = 3x_1 + 4x_2$

Subject to $x_1 + x_2 \le 450$ (i)

$2x_1 + x_2 \le 600$ (ii)

and $x_1, x_2 \ge 0$ (iii)

By graphical method, the solution space satisfying the constrains (i), (ii) and meeting the non – negativity restriction (iii) is shown shaded in the following figure.



The solution space is the region OABC. The vertices of this solution spaces are O(0,0), A(300,0), B(150,300) and C(0,450).

The values of Z at these vertices are given by

Extreme Point	(x_1, x_2)	$Z = 3x_1 + 4x_2$
O	(0, 0)	0
A	(300, 0)	900
B	(150, 300)	1650
C	(0, 450)	1800

Since the problems is of maximization type and the minimum value of Z is attained at a single vertex, this problem has a unique optimal solution.

$\therefore$ The optimal solution is

$$\text{Maximum } Z = 1800, \quad x_1 = 0, \quad x_2 = 450.$$

Example: 6.3.7

Solve the following L.P.P graphically

Maximize $Z = 100x_1 + 40x_2$

Subject to $\quad 5x_1 + 2x_2 \leq 1000$

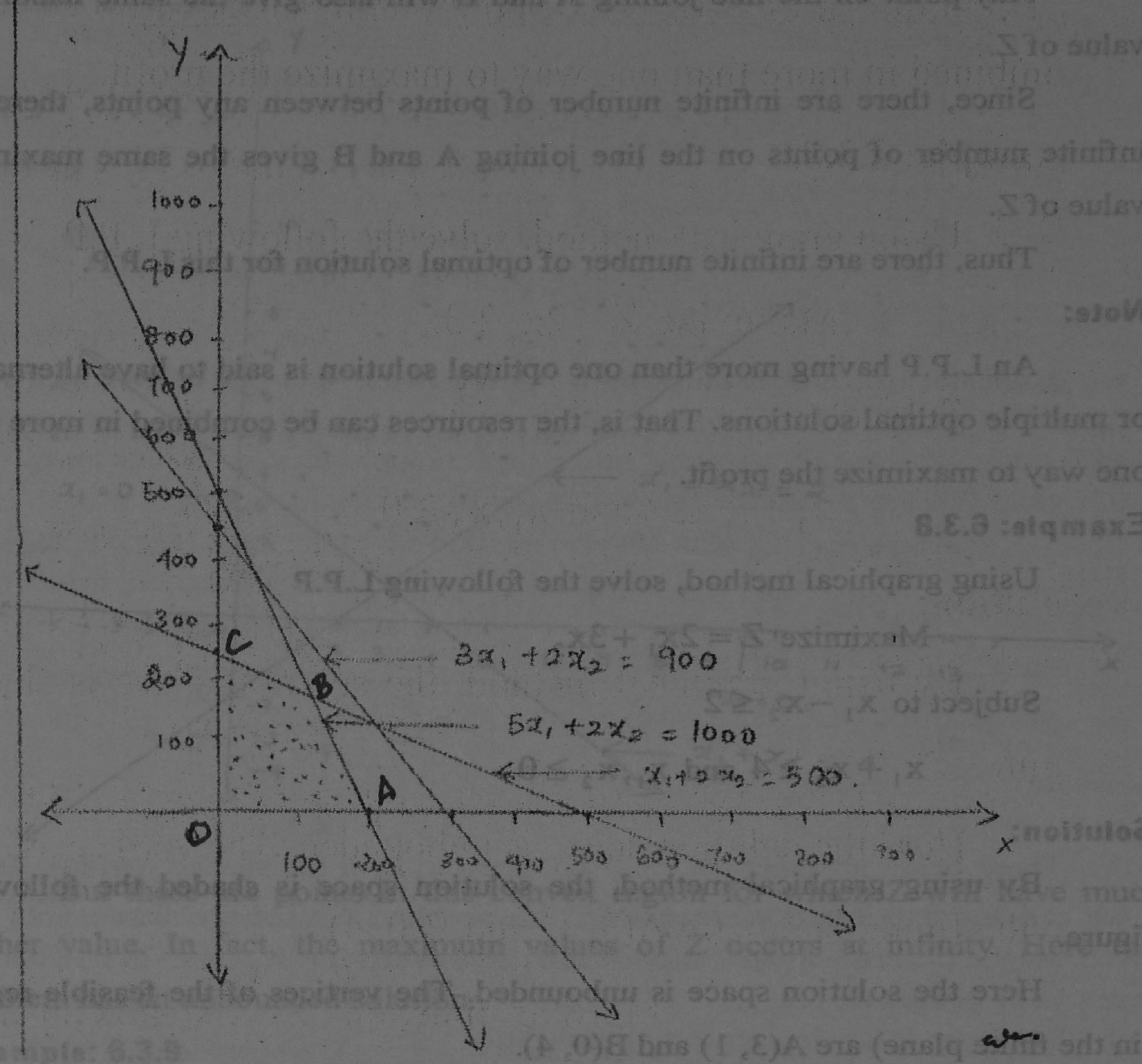
$3x_1 + 2x_2 \leq 900$

$x_1 + 2x_2 \leq 500$

and $x_1, x_2 \geq 0$.

Solution:

By using graphical method, the solution space OABC shown shaded in the following figure.



The vertices of this convex region are $O(0,0)$, $A(200,0)$, $B(125,187.5)$ and $C(0,250)$.

The values of Z at these vertices are given by

Extreme Values	(x_1, x_2)	$Z = 100x_1 + 40x_2$
O	(0, 0)	0
A	(200, 0)	20,000
B	(125, 187.5)	20,000
C	(0, 250)	10,000

Here the minimum value of Z occurs at two vertices A and B.

Any point on the line joining A and B will also give the same maximum value of Z.

Since, there are infinite number of points between any points, there are infinite number of points on the line joining A and B gives the same maximum value of Z.

Thus, there are infinite number of optimal solution for this L.P.P.

Note:

An L.P.P having more than one optimal solution is said to have alternative or multiple optimal solutions. That is, the resources can be combined in more than one way to maximize the profit.

Example: 6.3.8

Using graphical method, solve the following L.P.P

Maximize $Z = 2x_1 + 3x_2$

Subject to $x_1 - x_2 \leq 2$

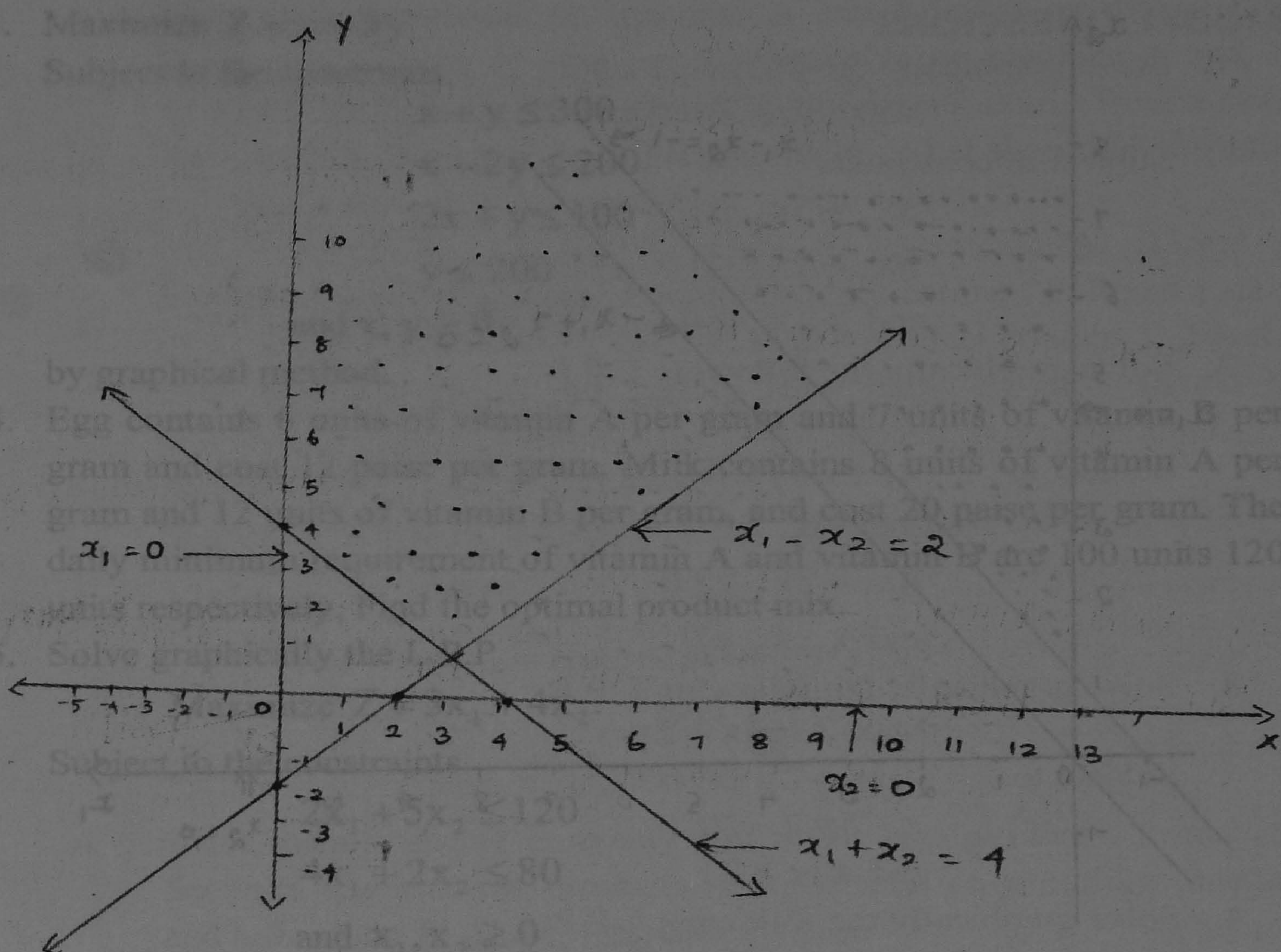
$x_1 + x_2 \geq 4$ and $x_1, x_2 \geq 0$

Solution:

By using graphical method, the solution space is shaded the following figure.

Here the solution space is unbounded. The vertices of the feasible region (in the finite plane) are A(3, 1) and B(0, 4).

Value of the objective function $Z = 2x_1 + 3x_2$ at these vertices are Z(A) = 9 and Z(B) = 12.



But these are points in this convex region for which Z will have much higher value. In fact, the maximum values of Z occurs at infinity. Here this problem has an unbounded solution.

Example: 6.3.9

Solve graphically the following L.P.P:

Maximize $Z = 4x_1 + 3x_2$

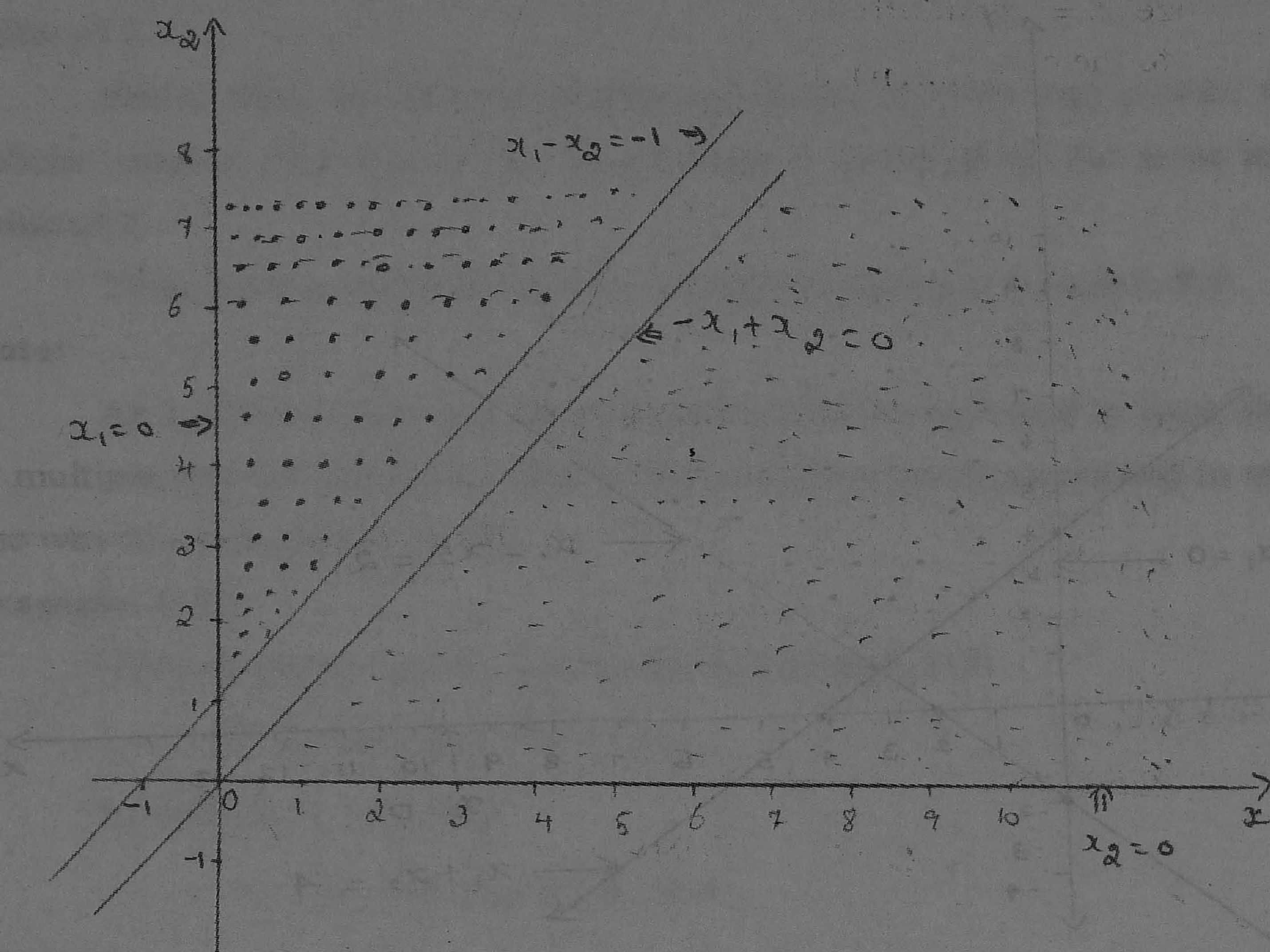
Subject to $x_1 - x_2 \leq -1$ (1)

$-x_1 + x_2 \leq 0$ (2)

and $x_1, x_2 \geq 0$ (3)

Solution:

Any point satisfying the non – negativity restrictions (iii) lies in the first quadrant only. The two solution spaces, one satisfying (i), and other satisfying (ii) are shown shaded in the following figure.



These being no point (x_1, x_2) common to both the shaded regions. That is, we can not find a convex region for this problem. So the problem cannot be solved. Hence the problem have no feasible solution.

Check your progress: 6.1

1. Solve the following problem graphically.

Maximize $\quad Z = 2x_1 + x_2$

Subject to $\quad 3x_1 + 2x_2 \leq 12.0$

$x_1 + 2x_2 \leq 7$

$x_1 + x_2 \leq 5$

and $\quad x_1, x_2 \geq 0$

2. Use graphical method to Maximize $Z = 6x_1 + 4x_2$

Subject to $\quad -2x_1 + x_2 \leq 2$

$x_1 - x_2 \leq 2$

$3x_1 + 2x_2 \leq 9$

$x_1 \geq 0, x_2 \geq 0$

3. Maximize $Z = x - 3y$
Subject to the constrains

$$x + y \leq 300$$
$$x - 2y \leq 200$$
$$2x + y \leq 100$$
$$y \leq 200$$
$$\text{and } x, y \geq 0$$

by graphical method.

4. Egg contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and cost 12 paise per gram. Milk contains 8 units of vitamin A per gram and 12 units of vitamin B per gram, and cost 20 paise per gram. The daily minimum requirement of vitamin A and vitamin B are 100 units 120 units respectively. Find the optimal product mix.

5. Solve graphically the L.P.P

Maximize $Z = 3x_1 + 4x_2$

Subject to the constraints

$$2x_1 + 5x_2 \leq 120$$
$$4x_1 + 2x_2 \leq 80$$
$$\text{and } x_1, x_2 \geq 0.$$

KEY WORDS

Linear Programming, Mathematical Formulation, Graphical Solution.

ANSWER TO CHECK YOUR PROGRES QUESTIONS

Check your progress:

1. Max $Z = 8,\ x_1 = 4,\ x_2 = 0$
2. An infinite number of solution with Z = 18.

(i) $x_1 = \frac{13}{5},\ x_2 = \frac{3}{5}$

(ii) $x_1 = \frac{5}{7},\ x_2 = \frac{24}{7}$ etc.

3. Min Z = -600, x = 0, y = 200.
4. Min $Z = 12x_1 + 20x_2$

Subject to $6x_1 + 8x_2 \geq 100$

$$7x_1 + 12x_2 \geq 120$$
$$\text{and } x_1\ x_2 \geq 0$$

Also min $Z = 205,\ x_1 = 15, x_2 = 1.25.$

5. Max $Z = 110,\ x_1 = 10,\ x_2 = 20.$

MODEL QUESTIONS

1. Solve graphically the following L.P.P:

   $$\text{Maximize } Z = 5x_1 + 3x_2$$

   Subject to constraints

   $$3x_1 + 5x_2 \le 15$$
   $$5x_1 + 2x_2 \le 10$$
   $$\text{and } x_1, x_2 \ge 0$$

2. Solve graphically the following L.P.P

   $$\text{Minimize } Z = 20x_1 + 10x_2$$

   Subject to $x_1 + 2x_2 \le 40$

   $$3x_1 + x_2 \ge 30$$
   $$4x_1 + 3x_2 \ge 60$$
   $$\text{and } x_1, x_2 \ge 0$$

3. Solve graphically the following L.P.P.

   $$\text{Max } Z = 3x + 2y$$

   Subject to $-2x + 3y \le 9$

   $$x - 5y \ge -20$$
   $$\text{and } x, y \ge 0$$

4. Solve graphically the following L.P.P:

   $$\text{Minimize } Z = -6x_1 - 4x_2$$

   Subject to $2x_1 + 3x_2 \ge 30$

   $$3x_1 + 2x_2 \le 24$$
   $$x_1 + x_2 \ge 3$$
   $$\text{and } x_1, x_2 \ge 0$$

5. Solve graphically the following L.P.P.

   $$\text{Maximize } Z = 3x_1 - 2x_2$$

   Subject to $x_1 + x_2 \le 1$

   $$2x_1 + 2x_2 \ge 4$$
   $$\text{and } x_1, x_2 \ge 0$$

6. Solve graphically the following L.P.P.

   $$\text{Maximize } Z = x_1 + x_2$$

   Subject to $x_1 - x_2 \ge 0$

   $$-3x_1 + x_2 \ge 3$$
   $$\text{and } x_1, x_2 \ge 0$$

7. A manufacture of furniture makes two products, chairs and tables. Processing of these products is done on two machines A and B. A chair requires 2 hours on machine A and 6 hours on machine B. A

table requires 5 hours on machine A and 6 hours on machine B. There are 16 hours of time per day available on machine A and 30 hours on machine B. Profit gained by the manufacturer from a chair and a table is Rs. 2 and Rs. 10 respectively. What should be the daily production of each of the products.

8. A person requires 10, 12 and 12 units of chemicals A, B and C respectively for his garden. A liquid product contains 5, 2 and 1 units of A, B and C respectively per Jar. A dry product contain 1, 2 and 4 units of A, B and C per carton. How many of each should be purchase in order to minimize the cost and meet the requirement.

9. Solve the following problem graphically,

Maximize $Z = 40x_1 + 100x_2$

Subject to $12x_1 + 6x_2 \leq 3000$

$$4x_1 + 10x_2 \leq 2000$$

$$2x_1 + 3x_2 \leq 900$$

$$\text{and } x_1, x_2 \geq 0$$

10. A garment manufacturing company can make two products, Prima and Seconda. Each of the products requires time on a cutting machine and a finishing Machine. Relevant data are

	Product	
	Prima	Seconda
Cutting hours (per unit)	2	1
Finishing hours (per unit)	3	3
Unit cost Rs.	128	120
Selling price Rs.	134	129
Maximum sales (units per week)	200	200

The number of cutting hours available per week is 390 and the number of finishing hours available per week is 810. How much should each product be produced in order to maximize the profit?

UNIT – VII
SOLUTION OF LINEAR PROGRAMMING PROBLEM

7.0 Introduction:

In the present chapter, we shall introduce and explain the computational procedure of the simplex method. The theory behind the method will be first developed and then the computational techniques explained and illustrated.

Objectives:

After working through examples and exercise of this unit you will be able to

1. Solve the L.P.P using simple method, Big – M method and two phase method.
2. Solve the industry problems using simplex method.

Structure:

The Simplex Method
Big – M Method
Two Phase Method
Degeneracy and Cycling in L.P.P.
Application of Simplex Method
Key words
Answer to check your progress questions
Model Questions

7.1 The Simplex Method

The simplex method also called the simplex technique or the simplex algorithm is an iterative procedure for solving a linear programming problem in a finite number of steps. The method provides an algorithm which consists in moving from one vertex of the region of feasible solutions to another in such a manner that the value of the objective function at the succeeding vertex is less (or more, as the case may be) than at the preceding vertex. This procedure of jumping from one vertex to another is then repeated. Since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps or indicates the existence of an unbounded solution.

Definition: 1

Given a system of m linear equations with n variables (m < n). The solution obtained by setting (n – m) variables equal to zero and solving for the remaining m variables is called a basic solution. The m variables are called basic variables and they form the basic solution. The (n – m) variables which are put to zero are called as non – basic variables.

Definition: 2

A basic solution is said to be a non – degenerate basic solution if none of the basic variables is zero.

Definition: 3

A basic solution is said to be a degenerate basic solution if one or more of the basic variables are zero.

Definition: 4

A feasible solution which is also basic is called a basic feasible solution.

Definition: 5

Let X_B be a basic feasible solution to the L.P.P.

Maximize Z = cX

Subject to AX = b

and $X \geq 0$.

Then the vector $C_B = (C_{B1}, C_{B2}, \ldots\ldots, C_{Bm})$ where C_{Bi} are components of C associated with the basic variables, is called the cost vector associated with the basic feasible solution X_B.

Note: 1

1. If a LPP has a feasible solution, then it also has a basic feasible solution.
2. There exists only finite number of basic feasible solutions to a L.P.P.
3. Let a L.P.P have a feasible solution. If we drop one of the basic variables and introduce another variable in the basis set, then the new solution obtained is also a basic feasible solution.

Definition: 6

Let $X_B = B^{-1}b$ be a basic feasible solution to the L.P.P."

Maximize $Z = CX$, where $AX = b$ and $X \geq 0$.

Let C_B be the cost vector corresponding to X_B. For each column vector a_j in A which is not a column vector of B, let

$$a_j = \sum_{i=1}^{m} a_{ij} b_i$$

Then the number $Z_j = \sum_{i=1}^{m} C_{Bi} a_{ij}$ is called the evaluation corresponding to a_j and the number $(Z_j - C_j)$ is called the net evaluation corresponding to a_j.

Note: 1

If $(Z_j - C_j) = 0$ for atleast one j for which $a_{ij} > 0$, $i = 1,2,\ldots\ldots,m$; then another basic feasible solution is obtained which gives an unchanged value of the objective function.

Note: 2 (Unbounded solution)

Let there exist a basic feasible solution to a given L.P.P. If for atleast one j, for which $a_{ij} \leq 0 (i = 1,2,3,\ldots\ldots m)$ and $(Z_j - C_j)$ is negative, then there does not exist any optimum solution to this LPP.

Note: 3

A necessary and sufficient condition for a basic feasible solution to a LPP to be an optimum (maximum) is that $(Z_j - C_j) \geq 0$ for all j, for which $a_j \notin B$.

Note: 4

The two fundamental conditions on which the simplex method is based are (i) **Feasibility Condition:** It ensures that if the initial (starting) solution is basic feasible then during computation only basic feasible solution will be obtained. (ii) **Optimality Condition:** It guarantees that only improved solutions will be obtained.

The simplex Algorithm:

For the solution of any LPP by simplex algorithm, the existence of an initial basic feasible solution is always assumed. The steps for the computation of an optimum solution are as follows.

Step: 1

Check whether the objective function is to be maximized or minimized. If it is to be minimized, then convert it into a problem of maximization, by

$$\text{Minimize } Z = -\text{ Maximize } (-Z)$$

Step: 2

Check whether all b_i's are positive. If any of the b_i's are negative, multiply both sides of that constraint by -1 so as to make its right hand side positive.

Step: 3

By introducing slack and/ or surplus variables, convert the inequality constraints into equations and express the given L.P.P into its standard form.

Step: 4

Find an initial basic feasible solution and express the above information conveniently in the following simplex table.

C_B	Y_B	X_B	x_1	x_2	x_3	..........S_1	S_2	S_3	....
C_{B1}	S_1	b_1	a_{11}	a_{12}	a_{13}	1	0	0	....
C_{B2}	S_2	b_2	a_{21}	a_{22}	a_{23}	0	1	0	....
C_{B3}	S_3	b_3	a_{31}	a_{32}	a_{33}	0	0	1	....
.	.	.	.	.	.	....			....
.	.	.	.	.	.	....			....
.	.	.	.	.	.	....			....
.	.	.		bodymatrix		....	unitmatrix		....
$(Z_j - C_j)$		Z_0	$Z_1 - C_1$	....	....				....

(where C_j - row denotes the coefficients of the variables in the objective function.

C_B - column denotes the coefficients of the basic variables in the objective function. Y_B- column denotes the basic variables. X_B- column denotes the values of the basic variables. The coefficients of the non – basic variables in the constraint equations constitute the body matrix which the coefficients of the basic variables constitute the unit matrix. The row $(Z_j - C_j)$ denotes the net evaluations (or) index for each column).

Step: 5

Compute the net evaluations $(Z_j - C_j)$ $(j = 1,2,.......,n)$ by using the relation $Z_j - C_j = C_B \; a_j - c_j$.

Examine the sign of $Z_j - C_j$.

(i) If all $(Z_j - C_j) \geq 0$ then the current basic feasible solution X_B is optimal.

(ii) If atleast one $(Z_j - C_j) < 0$ then the current basic feasible solution is not optimal, go to the next step.

Step: 6(To find the entering variable)

The entering variable is the non – basic variable corresponding to the most negative value of $(Z_j - C_j)$. Let it be x_r for some j = r. The entering variable column is known as the key column (or) pivot column which is shown marked with an arrow at the bottom. If more than one variable has the same most negative $(Z_j - C_j)$, any of these variables may be selected arbitrarily as the entering variable.

Step: 7(To find the leaving variable)

Compute the ratio $\theta = \text{Min}\left\{\dfrac{X_{Bi}}{a_{ir}},\; a_{ir} > 0\right\}$ (i.e., the ratio between the solution column and the entering variable column by considering only the positive denominators).

(i) If all $a_{ir} \leq 0$, then there is an unbounded solution to the given L.P.P.

(ii) If atleast one $a_{ir} > 0$, then the leaving variable is the basic variable corresponding to the minimum ratio θ. If $\theta = \frac{X_{BK}}{a_{kr}}$, then the basic variable x_k leaves the basis. The leaving variable row is called the key row (or) pivot equation and the element at the intersection of the pivot column and pivot row is called the pivot element or key element (or) leading element.

Step: 8

Drop the leaving variable and introduce the entering variable along with its associated value under C_B column. Convert the pivot element to unity by dividing the pivot equation by the pivot element and all other elements in its column to zero by making use of

(i) New pivot equation = Old pivot equation $\div$ Pivot element

(ii) New equation(all other rows including $(Z_j - C_j)$ = Old equation $-$

$$\begin{pmatrix}\text{Corresponding}\\ \text{Column}\\ \text{coefficient}\end{pmatrix} - \begin{pmatrix}\text{New pivot}\\ \text{equation}\end{pmatrix}$$

Step: 9

Go to step (5) and repeat the procedure until either an optimum solution is obtained or there is an indication of an unbounded solution.

Note: 1

For maximization problems:

(i) If all $(Z_j - C_j) \leq 0$, then the current basic feasible solution is optimal.

(ii) If atleast one $(Z_j - C_j) < 0$, then the current basic feasible solution is not optimal.

(iii)The entering variable is the non – basic variable corresponding to the most negative value of $(Z_j - C_j)$.

Note: 2

For minimization problems:

(i) If all $(Z_j - C_j) > 0$, then the current basic feasible solution is not optimal.

(ii) If atleast one $(Z_j - C_j) > 0$, then the current basic feasible solution is not optimal.

(iii) The entering variable is the non – basic variable corresponding to the most positive value of $(Z_j - C_j)$.

Note: 3

For both maximization and minimization problems, the leaving variable is the basic variable corresponding to the minimum ratio θ.

Example: 7.1.1

Use simplex method to solve the L.P.P

Maximize $Z = 4x_1 + 10x_2$

Subject to

$$2x_1 + x_2 \le 50$$

$$2x_1 + 5x_2 \le 100$$

$$2x_1 + 3x_2 \le 90 \text{ and } x_1, x_2 \ge 0.$$

Solution:

By introducing the slack variables S_1, S_2 and S_3 the problem in standard form becomes

Maximize $Z = 4x_1 + 10x_2 + 0.S_1 + 0.S_2 + 0.S_3$

Subject to

$$2x_1 + x_2 + S_1 + 0.S_2 + 0.S_3 = 50$$

$$2x_1 + 5x_2 + 0.S_1 + S_2 + 0.S_3 = 100$$

$$2x_1 + 3x_2 + 0.S_1 + 0.S_2 + S_3 = 90$$

and $\quad x_1, x_2, S_1, S_2, S_3 \ge 0$

Since there are 3 equations with 5 variables, the initial basic feasible solution is obtained by equating $(5-3)=2$ variables to zero.

$\therefore$ The initial basic feasible solution is $S_1 = 50,\ S_2 = 100,\ S_3 = 90$.

$(x_1 = 0,\ x_2 = 0$, non – basic),

The initial simplex table is given by

C_B	Y_B	X_B	X_1	X_2	S_1	S_2	S_3	$\theta = \min\left(\frac{X_{Bi}}{a_{ir}}\right)$
0	S_1	50	2	1	1	0	0	50
0	S_2	100	2	(5)	0	1	0	20*
0	S_3	90	2	3	0	0	1	30
Z_j	$-C_j$	0	-4	-10	0	0	0	

Here the net evaluations are calculated as

$$Z_j - C_j = C_B a_j - C_j,$$

$Z_1 - C_1 = C_B\ a_1 - C_1 = (0\quad 0\quad 0)[2\quad 2\quad 2]^T - 4$ [where T denotes transpose]

$$= (0\quad 0\quad 0)\begin{pmatrix}2\\2\\2\end{pmatrix} - 4 = -4.$$

$$Z_2 - C_2 = C_B\ a_2 - C_2 = (0\quad 0\quad 0)[1\quad 5\quad 3]^T - 10 = -10$$
$$Z_3 - C_3 = C_B\ a_3 - C_3 = (0\quad 0\quad 0)[1\quad 0\quad 0]^T - 10 = 0$$
$$Z_4 - C_4 = C_B\ a_4 - C_4 = (0\quad 0\quad 0)[0\quad 1\quad 0]^T - 0 = 0$$
$$Z_5 - C_5 = C_B\ a_5 - C_5 = (0\quad 0\quad 0)[0\quad 0\quad 1]^T - 0 = 0$$

Since these are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

To find the entering variable

Since $(Z_2 - C_2) = -10$ is the most negative, the corresponding non – basic variable x_2 enters the basis. The column corresponding to this x_2 is called the key column or pivot column.

To find the leaving variable:

Find the ratio $\theta = \min\left\{\frac{X_{B1}}{a_{1r}},\ a_{1r} > 0\right\}$

$$= \min\left\{\frac{X_{B1}}{a_{12}},\ a_{12} > 0\right\}$$

$$\theta = \min\left\{\frac{50}{1}, \frac{100}{5}, \frac{90}{3}\right\}$$

$= \min\{50,\ 20,\ 30\} = 20$ which corresponds to S_2 the minimum ratio $\theta = 20$. The leaving variable row is called the pivot – row or key row or pivot equation and 5 is the pivot element. Now pivot equation = old pivot equation ÷ pivot element.

$$= (100\quad 2\quad 5\quad 0\quad 1\quad 0) \div 5$$

$$= 20\quad \frac{2}{5}\quad 1\quad 0\quad \frac{1}{5}\quad 0$$

Now S_1 equation = old S_1 equation - $\begin{pmatrix}\text{Corresponding} \\ \text{column} \\ \text{coefficient}\end{pmatrix} - \begin{pmatrix}\text{New} \\ \text{pivot} \\ \text{equation}\end{pmatrix}$

$$\begin{array}{rcrrrrrr} & = & 50 & 2 & 1 & 1 & 0 & 0 \\ (-) & & 20 & \frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 \\ \hline & = & 30 & \frac{8}{5} & 0 & 1 & -\frac{1}{5} & 0 \\ \hline \end{array}$$

$$\begin{array}{lcrrrrrr} \text{Now } S_3 \text{ equation} & = & 90 & 2 & 3 & 0 & 0 & 1 \\ (-) & & 60 & \frac{6}{5} & 3 & 0 & \frac{3}{5} & 0 \\ \hline & = & 30 & \frac{4}{5} & 0 & 0 & -\frac{3}{5} & 1 \\ \hline \end{array}$$

$$\begin{array}{lcrrrrrr} \text{Now } (Z_j - C_j) \text{ eqn.} & = & 0 & -4 & -10 & 0 & 0 & 0 \\ (-) & & -200 & \frac{-20}{5} & -10 & 0 & \frac{-10}{5} & 0 \\ \hline & = & 200 & 0 & 0 & 0 & 2 & 0 \\ \hline \end{array}$$

∴ The improved basic feasible solution is given in the following simplex table.

First Iteration:

		C_J	4	10	0	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3
0	S_1	30	$\frac{8}{5}$	0	1	$\frac{-1}{5}$	0
10	x_2	20	$\frac{2}{5}$	1	0	$\frac{1}{5}$	0
0	S_3	30	$\frac{4}{5}$	0	0	$\frac{-3}{5}$	1
Z_J	$-C_J$	200	0	0	0	2	0

Since all $Z_j - C_j \geq 0$ the current basic feasible solution is optimal.

$\therefore$ The optimal solution is max $Z = 200$, $x_1 = 0$, $x_2 = 20$

Example: 7.1.2

Solve the following

$$\text{Maximize} \quad Z = 15x_1 + 6x_2 + 9x_3 + 2x_4$$

$$\text{Subject to} \quad 2x_1 + x_2 + 5x_3 + 6x_4 \leq 20$$

$$3x_1 + x_2 + 3x_3 + 25x_4 \leq 24$$

$$7x_1 + x_4 \leq 70$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

Solution:

By introducing non – negative slack variables S_1, S_2 and S_3, the standard form of L.P.P becomes

$$\text{Maximize} \quad Z = 15x_1 + 6x_2 + 9x_3 + 2x_4 + 0.S_1 + 0.S_2 + 0.S_3$$

$$\text{Subject to} \quad 2x_1 + x_2 + 5x_3 + 6x_4 + S_1 + 0.S_2 + 0.S_3 = 20$$

$$3x_1 + x_2 + 3x_3 + 25x_4 + 0.S_1 + S_2 + 0.S_3 = 24$$

$$7x_1 + 0x_2 + 0x_3 + x_4 + 0.S_1 + 0.S_2 + S_3 = 70$$

$$x_1, x_2, x_3, x_4, S_1, S_2, S_3 \geq 0.$$

$\therefore$ The initial basic feasible solution is $S_1 = 20$, $S_2 = 24$, $S_3 = 70$. ($x_1 = x_2 = x_3 = x_4 = 0$ non basic). The initial simplex table is given by

Initial Iteration:

C_B	Y_B	C_j / X_B	15 / x_1	6 / x_2	9 / x_3	2 / x_4	0 / S_1	0 / S_2	0 / S_3	θ
0	S_1	20	2	1	5	6	1	0	0	$\frac{20}{2}=10$
0	S_2	24	(3)	1	3	25	0	1	0	$\frac{24}{3}=8*$
0	S_3	70	7	0	0	1	0	0	1	$\frac{70}{7}=10$
Z_j	$-C_j$	0	−15	−6	−9	−2	0	0	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non – basic variable x_1 enters into the basis and the basic variable S_2 leaves the basis.

First Iteration:

C_B	Y_B	C_j / X_B	15 / x_1	6 / x_2	9 / x_3	2 / x_4	0 / S_1	0 / S_2	0 / S_3	θ
0	S_1	4	0	$\frac{1}{3}$	3	$\frac{-32}{3}$	1	$\frac{-2}{3}$	0	12 *
15	x_1	8	1	$\frac{1}{3}$	1	$\frac{25}{3}$	0	$\frac{1}{3}$	0	24
0	S_3	14	0	$\frac{-7}{3}$	−7	$\frac{-172}{3}$	0	$\frac{-7}{3}$	1	–
Z_j	$-C_j$	120	0	−1	6	123	0	5	0	

Since $(Z_2 - C_2) = -1 < 0$, the current basic feasible solution is not optimal. The non – basic variable x_2 enters into the basis and the basic variable S_1 leaves the basis.

Second Iteration:

C_B	Y_B	C_j / X_B	(15 / x_1	6 / x_2	9 / x_3	2 / x_4	0 / S_1	0 / S_2	0) / S_3
6	x_2	12	0	1	9	−32	3	−2	0
15	x_1	4	1	0	−2	$\frac{57}{3}$	−1	1	0
0	S_3	42	0	0	14	−132	7	−7	1
Z_j	$-C_j$	132	0	0	15	91	3	3	0

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is given by

$$\text{Max } Z = 132, \; x_1 = 4, \; x_2 = 12, \; x_3 = 0 \text{ and } x_4 = 0.$$

Example: 7.1.3

Solve the following L.P.P by simplex method

$$\text{Minimize } Z = 8x_1 - 2x_2$$

Subject to $\quad -4x_1 + 2x_2 \le 1$

$$5x_1 - 4x_2 \le 3$$

$$\text{and } x_1, \; x_2 \ge 0.$$

Solution:

Since the given objective function is of minimization type, we shall convert it into a maximization type as follows.

Maximize (-Z) = Maximize $Z^* = -8x_1 + 2x_2$

Subject to

$$-4x_1 + 2x_2 \le 1$$

$$5x_1 - 4x_2 \le 3$$

$$x_1, \; x_2 \ge 0.$$

By introducing non – negative slack variable S_1, S_2, the standard form of L.P.P becomes

$$\text{Maximize } Z^* = -8x_1 + 2x_2 + 0.S_1 + 0.S_2$$

Subject to the constraints

$$-4x_1 + 2x_2 + S_1 + 0.S_2 = 1$$

$$5x_1 - 4x_2 + 0.S_1 + S_2 = 3$$

$$\text{and } x_1, \; x_2, \; S_1, \; S_2 \ge 0.$$

$\therefore$ The initial basic feasible solution is given by $S_1 = 1, \; S_2 = 3, \; (x_1 = x_2 = 0, \text{ non-basic})$

Initial Iteration:

		C_j	-8	2	0	0	
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	θ
0	S_1	1	-4	2	1	0	$\frac{1}{2}$*
0	S_2	3	5	-4	0	1	–
Z_j^*	$-C_j$	0	8	-2	0	0	

Since $(Z_2^* - C_2) = -2 < 0$, the current basic feasible solution is not optimal.

The non – basic variable x_2 enters the basis and the basic variable S_1 leaves the basis.

First Iteration:

		C_j	-8	2	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2
2	x_2	$\frac{1}{2}$	-2	1	$\frac{1}{2}$	0
0	S_2	5	-3	0	2	1
Z_j^*	$-C_j$	1	4	0	1	0

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is given by

$$\text{Maximize } Z^* = 1,\ x_1 = 0,\ x_2 = \frac{1}{2}.$$

But Minimize Z = - Maximize (-Z) = - Maximize Z*

$$= -1$$

$$\therefore \text{Min } Z = -1,\ x_1 = 0,\ x_2 = \frac{1}{2}$$

Aliter:

The above problem can be solved without converting the objective function in the maximization type.

Given Minimize $Z = 8x_1 - 2x_2$

Subject to the constraints

$$-4x_1 + 2x_2 \leq 1$$

$$5x_1 - 4x_2 \le 3$$

$$x_1, x_2 \ge 0.$$

$\therefore$ By introducing the non – negative slack variable S_1, S_2 the L.P.P becomes

Minimize $Z = 8x_1 - 2x_2 + 0.S_1 + 0.S_2$

Subject to the constraints

$$-4x_1 + 2x_2 + S_1 + 0.S_2 = 1$$

$$5x_1 - 4x_2 + 0.S_1 + S_2 = 3$$

$$x_1, x_2, S_1, S_2 \ge 0.$$

The initial basic feasible solution is given by $S_1 = 1$, $S_2 = 3$ (basic) ($x_1 = x_2 = 0$, non - basic).

Initial Iteration:

		C_j	(8	-2	0	0)	
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	θ
0	S_1	1	-4	2	1	0	$\frac{1}{2}$*
0	S_2	3	5	-4	0	1	–
Z_j	$-C_j$	0	-8	2	0	0	

Since $(Z_2 - C_2) = 2 > 0$, the current basic feasible solution is not optimal.

To find the entering variable:

Since $(Z_2 - C_2) = 2$ is most positive, the corresponding non – basic variable x_2 enters into the basis.

To find the leaving variable:

The leaving variable is the basic variable S_1 corresponding to the minimum ratio $\theta = \frac{1}{2}$.

First Iteration:

	C_j		8	−2	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2
−2	x_2	1/2	−2	1	1/2	0
0	S_2	5	−3	0	2	1
Z_j	$-C_j$	−1	−4	0	−1	0

Since all $(Z_j - C_j) \le 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Min $Z = -1$, $x_1 = 0, x_2 = \frac{1}{2}$.

Example: 7.1.4

Use simplex method to

$$\text{Min } Z = x_2 - 3x_3 + 2x_5$$

Subject to

$$3x_2 - x_3 + 2x_5 \le 7$$

$$-2x_2 + 4x_3 \le 12$$

$$-4x_2 + 3x_3 + 8x_5 \le 10$$

and $x_2, x_3, x_5 \ge 0$

Solution:

Since the given objective function is minimization type. We shall convert it into a maximization type as follows.

Maximize (-Z) = Maximize $Z^* = -x_2 + 3x_3 - 2x_5$

Subject to $\quad 3x_2 - x_3 + 2x_5 \le 7$

$$-2x_2 + 4x_3 \le 12$$

$$-4x_2 + 3x_3 + 8x_5 \le 10$$

$$x_2, x_3, x_5 \ge 0.$$

By introducing non – negative slack variables S_1, S_2 and S_3 the standard form of the L.P.P becomes

Maximize $Z^* = -x_2 + 3x_3 - 2x_5 + 0.S_1 + 0.S_2 + 0.S_3$

Subject to the constraints

$$3x_2 - x_3 + 2x_5 + S_1 + 0.S_2 + 0.S_3 = 7$$

$$-2x_2 + 4x_3 + 0.x_5 + 0.S_1 + S_2 + 0.S_3 = 12$$

$$-4x_2 + 3x_3 + 8x_5 + 0.S_1 + 0.S_2 + S_3 = 10$$

and $x_2, x_3, x_5, S_1, S_2, S_3 \geq 0$.

$\therefore$ The initial basic feasible solution is given by

$S_1 = 7,\ S_2 = 12,\ S_3 = 10 (x_2 = x_3 = x_5 = 0$, non – basic)

Initial Iteration:

		C_j	$(-1$	3	-2	0	0	$0)$	
C_B	Y_B	X_B	x_2	x_3	x_5	S_1	S_2	S_3	θ
0	S_1	7	3	-1	2	1	0	0	–
0	S_2	12	-2	(4)	0	0	1	0	$\frac{12}{4} = 3^*$
0	S_3	10	-4	3	8	0	0	1	$\frac{10}{3} = 3.33$
Z_j^*	$-C_J$	0	1	-3	2	0	0	1	

Since $(Z_2^* - C_2) < 0$, the current basic feasible solution is not optimal.

The non – basic variable x_3 enters into the basis and the basic variable S_2 leaves the basis.

First Iteration:

		C_J	$(-1$	3	-2	0	0	$0)$	
C_B	Y_B	X_B	x_2	x_3	x_5	S_1	S_2	S_3	θ
0	S_1	10	$5/2$	0	2	1	$1/4$	0	$\frac{20}{5} = 4^*$
3	x_3	3	$-1/2$	1	0	0	$1/4$	0	–
0	S_3	1	$-5/2$	0	8	0	$-3/4$	1	–
Z_J^*	$-C_J$	9	$-1/2$	0	2	0	$3/4$	0	

Since $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

The non – basic variable x_2 enters into the basis and the basic variable S_1 leaves the basis.

Second Iteration:

		C_j	(−1	3	2	0	0)	
C_B	Y_B	X_B	x_2	x_3	x_5	S_1	S_2	S_3
−1	x_2	4	1	0	4/5	2/5	1/10	0
3	x_3	5	0	1	2/5	1/5	3/10	0
0	S_3	11	0	0	10	1	−1/2	1
Z_j^*	$-C_j$	11	0	0	12/5	1/5	4/5	0

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal.

∴ The optimal solution is given by

Maximize $Z^* = 11,\ x_2 = 4,\ x_3 = 5, x_5 = 0$.

But

Minimize Z = - Maximize $Z^* = -11$

∴ Min Z = -11, $x_2 = 4,\ x_3 = 5,\ x_5 = 0$.

Check your progress 7.1

1. Using simplex method, find non – negative values of x_1, x_2, x_3 which

   maximize $Z = x_1 + 4x_2 + 5x_3$ subject to the constraints

   $3x_1 + 6x_2 + 3x_3 \leq 22,\ x_1 + 2x_2 + 3x_3 \leq 14$ and $3x_1 + 2x_2 \leq 14$.

2. Apply the simplex method to solve the problem.

   Maximize $Z = 100x_1 + 200x_2 + 50x_3$

   Subject to $5x_1 + 5x_2 + 10x_3 \leq 1000$

   $10x_1 + 8x_2 + 5x_3 \leq 2000$

   $10x_1 + 5x_2 \leq 500$

   $x_1, x_2, x_3 \geq 0$.

3. Using simplex method, find the non – negative values of x_1 and x_2 which maximize the objective function $Z = 2x_1 + x_2$ subject to the constraints

$$x_1 - 2x_2 \le 1$$
$$x_1 + x_2 \le 6$$
$$x_1 + 2x_2 \le 10$$

and $x_1 - x_2 \le 2$

4. Solve the following LPP using simplex method:

Maximize $Z = x_1 + x_2 + 3x_3$

Subject to $3x_1 + 2x_2 + x_3 \le 3$

$$2x_1 + x_2 + 2x_3 \le 2$$
$$x_1, x_2, x_3 \ge 0.$$

7.2 BIG – M Method (Charne's Method)

To solve a LPP by simplex method, we have to start with the initial basic feasible solution and construct the initial simplex table. In the previous problems, we see that the slack variables readily provided the initial basic feasible solution. However, in some problems, the slack variables cannot provide the initial basic feasible solution. In these problems atleast one of the constraints is of < or ≥ type. To solve such linear programming problems, there are two (closely related) methods available.

(i) The "**Big M – method**" or "**M – technique**" or the "**Method of penalties**".

(ii) The "**Two phase**" method

The Big M – Method:

Step: 1

Express the linear programming problem in the standard form by introducing slack and/or surplus variables, if necessary.

Step: 2

Introduce the non – negative artificial variables $R_1, R_2, \ldots$ to the left hand side of all the constraints of $\geq$ or $=$ type. The purpose of introducing artificial variables is just to obtain an initial basic feasible solution. However, addition of these artificial variables causes violation of the corresponding constraints. Therefore we would like to get rid of these variables and would not allow them to appear in the final solution. To achieve this we assign a very large penalty (-M for maximization problems and +M for minimization problems) as the coefficients of the artificial variables in the objective function.

Step: 3

Solve the modified linear programming problem by simplex method.

While making iterations, using simplex method, one of the following three cases may arise:

(i) If no artificial variable remains in the basis and the optimality condition is satisfied, then the current solution is an optimal basic feasible solution.

(ii) If atleast one artificial variable appears in the basis at zero level (with zero value of X_B column) and the optimality condition is satisfied, then the current solution is an optimal basic feasible (though degenerated) solution.

(iii) If atleast one artificial variable appears in the basis at non – zero level (with positive value in X_B column) and the optimality condition is satisfied, then the original problem has no feasible solution. The solution satisfies the constraints but does not optimize the objective function since it contains a very large penalty M and is called **pseudo – optimal solution.**

Note:

While applying simplex method, whenever an artificial variable happens to leave the basis, we drop that artificial variable and omit all the entries corresponding to its column from the simplex table.

Example: 7.2.1

Solve the following LPP by simplex method:

Maximize $Z = 3x_1 + 2x_2$

Subject to $2x_1 + x_2 \leq 2$

$3x_1 + 4x_2 \geq 12$

and $x_1, x_2 \geq 0$.

Solution:

By introducing the non – negative slack variable S_1 and surplus variable S_2, the standard form of the LPP becomes

$$\text{Maximize } Z = 3x_1 + 2x_2 + 0.S_1 + 0.S_2$$

Subject to
$$2x_1 + x_2 + S_1 + 0.S_2 = 2$$
$$3x_1 + 4x_2 + 0.S_1 - S_2 = 12$$
$$x_1, x_2, S_1, S_2 \geq 0.$$

But this will not yield a basic feasible solution. To get the basic feasible solution add the artificial variable R, to the left hand side of the constraint equation which does not possess the slack variable and assign – M to the artificial variable in the objective function. The LPP becomes

$$\text{Max } Z = 3x_1 + 2x_2 + 0.S_1 + 0.S_2 - MR_1$$

Subject to
$$2x_1 + x_2 + S_1 + 0.S_2 = 2$$
$$3x_1 + 4x_2 + 0.S_1 - S_2 + R_1 = 12$$
$$x_1, x_2, S_1, S_2, R_1 \geq 0$$

The initial basic solution is given by

$S_1 = 2$, $R_1 = 12$(basic) ($x_1 = x_2 = S_2 = 0$, non – basic).

Initial Iteration:

		C_j	(3	2	0	0	–M)	
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	R_1	θ
0	S_1	2	2	(1)	1	0	0	2/1=2
–M	R_1	12	3	4	0	–1	1	12/4=3
Z_j	$-C_j$	–12M	–M–3	–4M–2	0	M	0	

Since there are some $(Z_j - C_j) < 0$, the current basis feasible solution is not optimal.

The non – basic variable x_2 enters into the basis and the basic variable S_1 leaves the basis.

First Iteration:

C_B	Y_B	C_j X_B	(3 x_1	2 x_2	0 S_1	0 S_2	$-M$) R_1
2	x_2	2	2	1	1	0	0
$-M$	R_1	4	-5	0	-4	-1	1
Z_j	$-C_j$	$-4M+4$	$5M+1$	0	$4M+2$	M	0

Since all $(Z_j - C_j) \geq 0$ and an artificial variable R_1 appears in the basis at non – zero level, the given LPP does not possess any feasible solution. But the LPP possess a pseudo optimal solution.

Example: 7.2.2

Use Big – M method to solve

Minimize $Z = 4x_1 + 3x_2$

Subject to $-3x_1 + 2x_2 \geq 10$

$x_1 + x_2 \geq 6$

and $x_1, x_2 \geq 0$

Solution:

Given Min $Z = 4x_1 + 3x_2$

Subject to $2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6$

$x_1, x_2 \geq 0$

That is Max $Z^* = -4x_1 - 3x_2$

Subject to $2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6$

$x_1, x_2 \geq 0$

By introducing the non – negative slack, surplus and artificial variables, the standard form of LPP becomes

$$\text{Max } Z^* = -4x_1 - 3x_2 + 0.S_1 + 0.S_2 + 0.S_3 - MR_1 - MR_2$$

Subject to $\quad 2x_1 + x_2 - S_1 + 0.S_2 + 0.S_3 + R_1 = 10$

$$-3x_1 + 2x_2 + 0.S_1 + S_2 + 0.S_3 = 6$$

$$x_1 + x_2 + 0.S_1 + 0.S_2 - S_3 + R_2 = 6$$

and $\quad x_1, x_2, x_3, S_1, S_2, S_3, R_1, R_2 \geq 0$

The initial basic feasible solution is given by $R_1 = 10,\ S_2 = 6,\ R_2 = 6$ (basic) $(x_1 = x_2 = S_1 = S_3 = 0$, non – basic)

Initial Iteration:

		C_j	$(-4$	-3	0	0	0	$-M$	$-M)$	
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	R_1	R_2	θ
$-M$	R_1	10	(2)	1	-1	0	0	1	0	$10/2=5$
0	S_2	6	-3	2	0	1	0	0	0	–
$-M$	R_2	6	1	1	0	0	-1	0	1	$6/1=6$
Z_j^*	$-C_j$	$-16M$	$-3M+4$	$-2M+3$	M	0	M	0	0	

Since these are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

The non – basic variable x_1 enters into the basis and the basic variable R_1 leaves the basis.

First Iteration:

		C_j	$(-4$	-3	0	0	0	$-M)$	
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	R_2	θ
-4	x_1	5	1	$1/2$	$-1/2$	0	0	0	10
0	S_2	21	0	$7/2$	$-3/2$	1	0	0	$42/7$
$-M$	R_2	1	0	$(1/2)$	$1/2$	0	-1	1	2
Z_j^*	$-C_j$	$-M-20$	0	$\dfrac{-M+2}{2}$	$\dfrac{-M+4}{2}$	0	M	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

The non – basic variable x_2 enters into the basis and the basic variable R_2 leaves the basis.

Second Iteration:

		C_j	(−4	−3	0	0	0)
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3
−4	x_1	4	1	0	−1	0	1
0	S_2	14	0	0	−5	1	7
−3	x_2	2	0	1	1	0	−2
Z_j^*	$-C_j$	−22	0	0	1	0	2

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal

$$\therefore \text{Max } Z^* = -22, \; x_1 = 4, \; x_2 = 2$$

But Min $Z = -\text{Max}(-Z) = -\text{Max}Z^* = -(-22) = 22$

$\therefore$ The optimal solution is Min $Z = 22, \; x_1 = 4, \; x_2 = 2$

Example: 7.2.3

Use penalty method to

Maximize $z = 2x_1 + x_2 + x_3$

Subject to $4x_1 + 6x_2 + 3x_3 \leq 8$

$3x_1 - 6x_2 - 4x_3 \leq 1$

$2x_1 + 3x_2 - 5x_3 \geq 4$

and $x_1, x_2, x_3 \geq 0$

Solution:

By introducing the non-negative slack, surplus and artificial variables, the standard form of the LPP becomes.

Max $z = 2x_1 + x_2 + x_3 + 0.S_1 + 0.S_2 + 0.S_3 - MR_1$,

Subject to $4x_1 + 6x_2 + 3x_3 + S_1 + 0.S_2 + 0.S_3 = 8$

$$3x_1 - 6x_2 - 4x_3 + 0.S_1 + 0.S_2 + 0.S_3 = 1$$

$$2x_1 + 3x_2 - 5x_3 + 0.S_1 + 0.S_2 - S_3 + R_1 = 4$$

and $\quad x_1, x_2, x_3, s_1, s_2, s_3, R_1 \geq 0.$

(Here: S_1, S_2 – slack, S_3 – surplus, R_1,-artificial)

The initial basic feasible solution is given by

$S_1 = 8,\ S_2 = 1,\ R_1 = 4$ (basic) ($x_1 = x_2 = S_3 = 0$, non–basic)

Initial Iteration:

	C_j	(2	1	1	0	0	0	–M)	
C_B Y_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	R_1	Q
0 S_1	8	4	6	3	1	0	0	0	$\frac{8}{6}=1.33$
0 S_2	1	3	–6	–4	0	1	0	0	–
–M R_1	4	2	(3)	–5	0	0	–1	1	$\frac{4}{3}=1.33$
Z_j $-C_j$	–4M	–2M–2	–3M–1	5M–1	0	0	M	0	

Since there are some $(z_j - c_j) < 0$, the current base feasible solution is not optimal.

The non-basic variable x_2 enters into the basis and the basic variable R_1 leaves the basis.

First iteration:

	C_j	C_2	1	1	0	0	0	
C_B Y_B	Y_B	x_1	x_2	x_3	S_1	S_2	S_3	θ
0 S_1	0	0	0	(13)	1	0	2	0
0 S_1	9	7	0	–14	0	1	–2	–
1 x_2	$\frac{4}{3}$	$\frac{2}{3}$	1	$\frac{-5}{3}$	0	0	$\frac{-1}{3}$	–
$Z_j -$ C_j	$\frac{4}{3}$	$\frac{-4}{3}$	0	$\frac{-8}{3}$	0	0	$\frac{-1}{13}$	

Since there are some $(Z_j - C_j) < 0$, The current basic feasible solution is not optimal. The non – basic variable x_3 enters into the basis and the basic variable S_1 leaves the basis.

Second Iteration:

C_B	Y_B	C_j / X_B	2 / x_1	1 / x_2	1 / x_3	0 / S_1	0 / S_2	0 / S_3	θ
1	x_3	0	0	0	1	$\frac{1}{13}$	0	$\frac{2}{13}$	–
0	S_2	9	(7)	0	0	$\frac{14}{13}$	1	$\frac{2}{13}$	$\frac{9}{7}$
1	x_2	$\frac{4}{3}$	$\frac{2}{3}$	1	0	$\frac{5}{39}$	0	$\frac{-1}{13}$	2
$Z_j -$	C_j	$\frac{4}{3}$	$\frac{-4}{3}$	0	0	$\frac{8}{39}$	0	$\frac{1}{13}$	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal

The non-basic variable X_3 enters into the basis and the basic variable S_2 leaves the basis.

Third Iteration:

C_B	Y_B	C_j / X_B	2 / x_1	1 / x_2	1 / x_3	0 / S_1	0 / S_2	S_3
1	x_3	0	0	0	1	$\frac{1}{13}$	0	$\frac{2}{13}$
2	x_1	$\frac{9}{7}$	1	0	0	$\frac{1}{13}$	$\frac{1}{7}$	$\frac{2}{91}$
1	x_2	$\frac{10}{21}$	0	1	0	$\frac{1}{39}$	$\frac{-2}{21}$	$\frac{-25}{273}$
$Z_j -$	C_j	$\frac{64}{21}$	0	0	0	$\frac{16}{39}$	$\frac{4}{21}$	$\frac{29}{273}$

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

The optimal solution is

$$\text{Max } Z = \frac{64}{21}$$

$$x_1 = \frac{9}{7},\ x_2 = \frac{10}{21},\ x_3 = 0$$

Check your progress 7.2

1) Solve the following LPP Minimize $Z = 4x_1 + 2x_2$

   Subject to $3x_1 + x_2 \geq 27$

   $x_1 + x_2 \geq 21$

   $x_1 + 2x_2 \geq 30$

   $x_1, x_2 \geq 0.$

2) Solve the following LPP: Minimize $Z = 2x_1 + 3x_2$

   Subject to $x_1 + x_2 \geq 5$

   $x_1 + 2x_2 \geq 6$

   $x_1, x_2 \geq 0.$

7.3 DUALITY

Introduction:

For every linear programming problem there is a unique linear programming problem associated with it, involving the same data and closely related optimal solutions. The original (given) problem is then called the **Primal** problem while the other is called its **dual** problem. But in general, the two problems are said to be **duals** of each other.

The importance of the duality concept is due to two main reasons. Firstly, if the primal contains a large number of constraints and a smaller number of variables, the labour of computation can be considerably reduced by converting in to the dual variables from the cost of economic point of view proves extremely useful in making future decisions in the activities being programmed.

A linear programming problem in which all or some of the decision variable are constrained to assume non – negative integer values is called an Integer Programming Problem. This type of problem is of particular importance in business and industry, where quite often, the fractional solution are unrealistic because the units are not divisible. For example, it is absurd to speak of 2.3 men working on a project or 8.7 machines in a workshop. The integer solution to a problem can, however, be obtained by rounding off the optimum values of the variables to the nearest integer values. But, it is generally inaccurate to obtain an integer solution by rounding off in this manner, for there is no guarantee that the deviation from the 'exact' integer solution will not be too large to retain the feasibility.

The linear programming problem with the additional requirements that the variables can take on only, integer values may have the following mathematical form

Objectives:

After working through examples and exercise of this unit you will be able to

1. Formulate the dual problem from primal.
2. Solve the L.P.P using dual simplex method.
3. Solve the integer programming problem using culting plan method.

Structure:

Formulation of dual problems

Duality Theorems

Duality and Simplex Methods

Dual Simplex Method

Integer programming

Keywords

Answers to check your progress

Model Questions

7.4 Formulations of Dual Problems

Formulation of dual problems

There are two important forms of primal – dual pairs, namely **symmetric form** and **unsymmetric form.**

Symmetric Form:

Consider the following LPP.

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + c_3x_3 + \dots\dots + c_nx_n$$

Subject to the constraints

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n \le b_1 \\
&a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n \le b_2 \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \qquad (1) \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&a_{m1}x_1 + a_{m2}x_2 + \dots\dots\dots + a_{mn}x_n \le b_m
\end{aligned}
$$

and $x_1, x_2, \dots\dots\dots, x_n \ge 0$

i.e., $\quad$ Max $Z = CX$

subject to $\quad AX \le b$

and $\quad X \ge 0$

where $C = (c_1 c_2, \dots\dots\dots c_n)$

$$
A = \begin{pmatrix} a_{11} & a_{12} & \dots\dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots\dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots\dots\dots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{pmatrix}
$$

To construct the dual problem, we adopt the following guidelines:

(i) The maximization problem in the primal becomes the minimization problem in the dual and vice versa.

(ii) The maximization problem has $(\le)$ constraints while the minimization problems has $(\ge)$ constraints.

(iii)If the primal contains m constraints and n variables, then the dual will contain n constraints and m variables. i.e., the transpose of the body matrix of the primal problem gives the body matrix of the dual and vice versa.

(iv)The constants $c_1, c_2, c_3, \dots\dots\dots, c_n$ in the objective function of the primal appear in the constraints of the dual.

(v) The constants $b_1, b_2, \ldots\ldots, b_m$ in the constraints of the primal appear in the objective function of the dual.

(vi) The variables in both problems are non – negative.

The primal – dual relationships can be conveniently displayed as below:

Primal variables

$$\text{Dual Variables}\quad \begin{array}{c} \\ y_1 \\ y_2 \\ \cdots \\ \cdots \\ y_m \end{array}\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & \cdots & \cdots & x_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mn} \end{bmatrix} \\ \begin{array}{cccccc} \geq & \geq & \geq & \cdots & \cdots & \geq \\ c_1 & c_2 & c_3 & \cdots & \cdots & c_n \end{array} \end{array}\begin{array}{l} \\ \leq b \\ \leq b_2 \\ \cdots \\ \cdots \\ \leq b_m \\ \\ \\ \end{array}\ \text{R.H.S. of Primal constraints}$$

R.H.S of dual constraints

Then the dual problem of (1) will be $\left(\text{where } Y = \begin{pmatrix} y_1 \\ y \\ . \\ y_m \end{pmatrix} \right)$

Minimize $W = b^T Y$

Subject to the constraints $\quad A^T Y \geq C^T$

and $\quad Y \geq 0$

i.e., Minimize $W = b_1 y_1 + b_2 y_2 + \ldots\ldots\ldots\ldots + b_m y_m$

subject to the constraints

$$\left.\begin{array}{l} a_{11}y_1 + a_{21}y_2 + \ldots\ldots\ldots\ldots + a_{m1}y_m \geq c_1 \\ a_{12}y_1 + a_{22}y_2 + \ldots\ldots\ldots\ldots + a_{m2}y_m \geq c_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1n}y_1 + a_{2n}y_2 + \ldots\ldots\ldots\ldots + a_{mn}y_m \geq c_n \\ \quad \text{and } y_1, y_2, \ldots\ldots, y_m \geq 0 \end{array}\right. \qquad (2)$$

Equations (1) and (2) are called **symmetric primal – dual pairs.**

Example: 7.4.1

Write the dual of the following primal LPP.

Maximize $F = x_1 + 2x_2 + x_3$

Subject to $2x_1 + x_2 - x_3 \le 2$

$-2x_1 + x_2 - 5x_3 \ge -6$

$4x_1 + x_2 + x_3 \le 6$

$x_1, x_2, x_3 \ge 0$

Solution:

Given primal LPP Maximize $F = x_1 + 2x_2 + x_3$

Subject to $2x_1 + x_2 - x_3 \le 2$

$-2x_1 + x_2 - 5x_3 \ge -6$

$4x_1 + x_2 + x_3 \le 6$

and $x_1, x_2, x_3 \ge 0$.

That is, Maximize $F = x_1 + 2x_2 + x_3$

Subject to $2x_1 + x_2 - x_3 \le 2$

$2x_1 - x_2 + 5x_3 \le 6$

$4x_1 + x_2 + x_3 \le 6$

and $x_1, x_2, x_3 \ge 0$.

That is Max $F = CX$

Subject to $AX \le b$

and $X \ge 0$

Where $C = (1\ 2\ 1)$, $A = \begin{pmatrix} 2 & 1 & -1 \\ 2 & -1 & 5 \\ 4 & 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Since the primal problems is maximization type with $(\le)$ type constraints, with 3 constraints and 3 variables, the dual problem is minimization type with $(\ge)$ type constraints, with 3 constraints and 3 dual variables y_1, y_2, y_3.

The dual problem is Minimize $W = b^T Y$

subject to the constraints $A^T Y \geq C^T$

and $Y \geq 0$

$$\text{i.e., Min } W = \begin{pmatrix} 2 & 6 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\text{subject to } \begin{pmatrix} 2 & 2 & 4 \\ 1 & -1 & 1 \\ -1 & 5 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq 0$$

i.e., Min $W = 2y_1 + 6y_2 + 6y_3$

subject to

$$2y_1 + 2y_2 + 4y_3 \geq 1$$

$$y_1 - y_2 + y_3 \geq 2$$

$$-y_1 + 5y_2 + y_3 \geq 1$$

$$y_1, y_2, y_3 \geq 0$$

Example: 7.4.2

Find the dual of the LPP

$$\text{Max } Z = 3x_1 - x_2 + x_3$$

Subject to

$$4x_1 - x_2 \leq 8$$

$$8x_1 + x_2 + 3x_3 \geq 12$$

$$5x_1 - 6x_3 \leq 13$$

$$x_1, x_2, x_3 \geq 0$$

Solution:

Given primal LPP is

Max $Z = 3x_1 - x_2 + x_3$

Subject to $4x_1 - x_2 + 0x_3 \leq 8$

$$-8x_1 - x_2 - 3x_3 \leq -12$$

$$5x_1 + 0x_2 - 6x_3 \leq 13$$

and $x_1, x_2, x_3 \geq 0$.

That is Max $Z = CX$

Subject to $AX \le b$

and $X \ge 0$

$$\text{Where } C = (3 \quad -1 \quad 1),\ A = \begin{pmatrix} 4 & -1 & 0 \\ -8 & -1 & -3 \\ 5 & 0 & 6 \end{pmatrix},\quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},\quad b = \begin{pmatrix} 8 \\ -12 \\ 13 \end{pmatrix}$$

The dual problem is Minimize $W = b^T Y$

Subject to the constraints $A^T Y \ge C^T$

and $Y \ge 0$

$$\text{i.e., Min } W = (8 \quad -12 \quad 13)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\text{subject to } \begin{pmatrix} 4 & -8 & 5 \\ -1 & -1 & 0 \\ 0 & -3 & -6 \end{pmatrix} \ge \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \ge 0$$

i.e., Min $W = 8y_1 - 12y_2 + 13y_3$

subject to $4y_1 - 8y_2 + 5y_3 \ge 3$

$-y_1 - y_2 + 0y_3 \ge -1$

$0y_1 - 3y_2 - 6y_3 \ge 1$

$y_1, y_2, y_3 \ge 0$

i.e., The dual problem is

Min $W = 8y_1 - 12y_2 + 13y_3$

Subject to $4y_1 - 8y_2 + 5y_3 \ge 3$

$y_1 + y_2 \le 1$

$-3y_2 - 6y_3 \ge 1$

and $y_1, y_2, y_3 \ge 0$

Example: 7.4.3

Construct the dual of the LPP

$$\text{Min } Z = 4x_1 + 6x_2 + 18x_3$$

Subject to $x_1 + 3x_2 \geq 3$

$x_2 + 2x_3 \geq 5$

and $x_1, x_2, x_3 \geq 0$

Solution:

Given primal LPP is

$$\text{Min } Z = 4x_1 + 6x_2 + 18x_3$$

Subject to $x_1 + 3x_2 + 0x_3 \geq 3$

$0x_1 + x_2 + 2x_3 \geq 5$

and $x_1, x_2, x_3 \geq 0$.

That is Min $Z = CX$

Subject to $AX \geq b$

and $X \geq 0$.

Where $C = (4 \quad 6 \quad 18)$, $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Since the primal contains 2 constraints and 3 variables, the dual will contain 3 constraints and 2 variables y_1, y_2

The dual LPP is Max $W = b^T Y$

Subject to the constraints $A^T Y \leq C^T$

and $Y \geq 0$.

i.e., Max $W = (3 \quad 5)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ subject to $\begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \leq \begin{pmatrix} 4 \\ 6 \\ 18 \end{pmatrix}$

and $y_1, y_2, y_3 \geq 0$

i.e., The dual problem is Max $W = 3y_1 + 5y_2$

subject to $y_1 \leq 4$

$$3y_1 + y_2 \le 6$$

$$2y_2 \le 18$$

and $$y_1, y_2 \ge 0.$$

Example: 7.4.4

Write the dual of the LPP

$$\text{Min } Z = 3x_1 - 2x_2 + 4x_3$$

Subject to
$$3x_1 + 5x_2 + 4x_3 \ge 7$$
$$6x_1 + x_2 + 3x_3 \ge 4$$
$$7x_1 - 2x_2 - x_3 \le 10$$
$$x_1 - 2x_2 + 5x_3 \ge 3$$
$$4x_1 + 7x_2 - 2x_3 \ge 2$$
and $x_1, x_2, x_3 \ge 0$.

Solution:

Given primal LPP is

$$\text{Min } Z = 3x_1 - 2x_2 + 4x_3$$

Subject to
$$3x_1 + 5x_2 + 4x_3 \ge 7$$
$$6x_1 + x_2 + 3x_3 \ge 4$$
$$-7x_1 + 2x_2 + x_3 \ge -10$$
$$x_1 - 2x_2 + 5x_3 \ge 3$$
$$4x_1 + 7x_2 - 2x_3 \ge 2$$
and $x_1, x_2, x_3 \ge 0$.

i.e.,
$$\text{Min } Z = CX$$
subject to $AX \ge b$

and $X \ge 0$.

Where $C = (3 \quad -2 \quad 4)$, $A = \begin{pmatrix} 3 & 5 & 4 \\ 6 & 1 & 3 \\ -7 & 2 & 1 \\ 1 & -2 & 5 \\ 4 & 7 & -2 \end{pmatrix}$, $b = \begin{pmatrix} 7 \\ 4 \\ -10 \\ 3 \\ 2 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Since the primal contains 5 constraints and 3 variables, the dual will contain 3 constraints and 5 variables y_1, y_2, y_3, y_4 and y_5.

The dual LPP is Max $W = b^T Y$

Subject to the constraints $A^T Y = C^T$

and $Y \geq 0$.

$$\text{Max } W = (7 \quad 4 \quad -10 \quad 3 \quad 2), \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}$$

$$\text{subject to} \begin{pmatrix} 3 & 6 & -7 & 1 & 4 \\ 5 & 1 & 2 & -2 & 7 \\ 4 & 3 & 1 & 5 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} \leq \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \text{ and } \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} \geq 0.$$

i.e., Max $W = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$

subject to
$$3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \leq 3$$
$$5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2$$
$$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4$$
$$y_1, y_2, y_3, y_4, y_5 \geq 0.$$

Unsymmetric Form

S. No	Primal / Dual	Dual / Primal
1	Max $Z = CX$ Subject to $AX = b$ $X \geq 0$	Min $W = b^T Y$ Subject to $A^T Y \geq C^T$ Variables are unrestricted
2	Min $Z = CX$ Subject to $AX = b$ $X \geq 0$	Max $W = b^T Y$ Subject to $A^T Y \leq C^T$ Variables are unrestricted

Remark: 1

If the k^{th} constraints of the primal problem is an equality, then the corresponding dual variable y_k is unrestricted in sign and vice versa.

Remark: 2 If any variable of the primal problem is unrestricted in sign, the corresponding constraint in the dual problem will be an equality and vice versa.

Example: 7.4.5

Write the dual of the LPP

$$\text{Max} \quad Z = 3x_1 + 10x_2 + 2x_3$$

Subject to

$$2x_1 + 3x_2 + 2x_3 \leq 7$$

$$3x_1 - 2x_2 + 4x_3 = 3$$

$$\text{and } x_1, x_2, x_3 \geq 0.$$

Solution:

Given primal LPP is

$$\text{Max } Z = 3x_1 + 10x_2 + 2x_3$$

Subject to

$$2x_1 + 3x_2 + 2x_3 \leq 7$$

$$3x_1 - 2x_2 + 4x_3 = 3$$

$$\text{and } x_1, x_2, x_3 \geq 0.$$

Since the primal problem contains 2 constraints and 3 variables, the dual problem will contain 3 constraints and 2 dual variables y_1, y_2.

Also, since the second constraint in the primal problem is an equality, the corresponding second dual variable y_2 is unrestricted in sign.

$\therefore$ The dual problem is

$$\text{Min} \quad W = 7y_1 + 3y_2$$

Subject to
$$2y_1 + 3y_2 \geq 3$$
$$3y_1 - 2y_2 \geq 10$$
$$2y_1 + 4y_2 \geq 2$$
and $y_1 \geq 0$, y_2 is unrestricted.

Example: 7.4.6

Write the dual of the LPP

$$\text{Min} \quad Z = x_2 + 3x_3$$

Subject to
$$2x_1 + x_2 \leq 3$$
$$x_1 + 2x_2 + 6x_3 \geq 5$$
$$-x_1 + x_2 + x_3 = 2$$
$$x_1, x_2, x_3 \geq 0.$$

Solution:

Given primal LPP is

$$\text{Min} \quad Z = 0x_1 + x_2 + 3x_3$$

Subject to
$$-2x_1 - x_2 + 0x_3 \geq -3$$
$$x_1 + 2x_2 + 6x_3 \geq 5$$
$$-x_1 + x_2 + x_3 = 2$$
and $x_1, x_2, x_3 \geq 0$.

Since the primal problem contains 3 constraints and 3 variables, the dual problem will contain 3 constraints and 3 dual variables y_1, y_2, y_3.

Also, since the third constraint in the primal problem is an equality the corresponding third dual variable y_3 is unrestricted in sign.

The dual LPP is Max $W = -3y_1 + 5y_2 + 2y_3$

Subject to $-2y_1 + y_2 - y_3 \leq 0$

$$-y_1 + 2y_2 + y_3 \leq 1$$

$$6y_2 + y_3 \leq 3$$

and $y_1, y_2 \geq 0$, y_3 is unrestricted.

Example: 7.4.7

Write the dual of the following primal LPP

$$\text{Min } Z = 4x_1 + 5x_2 - 3x_3$$

Subject to $x_1 + x_2 + x_3 = 22$

$$3x_1 + 5x_2 - 2x_3 \leq 65$$

$$x_1 + 7x_2 + 4x_3 \geq 120$$

$x_1 \geq 0$, $x_2 \geq 0$ and x_3 unrestricted.

Solution:

Given primal LPP is

$$\text{Min } Z = 4x_1 + 5x_2 - 3x_3$$

Subject to $x_1 + x_2 + x_3 = 22$

$$-3x_1 - 5x_2 + 2x_3 \geq -65$$

$$x_1 + 7x_2 + 4x_3 \geq 120$$

$x_1 \geq 0, x_2 \geq 0$ and x_3 unrestricted.

Since the primal problem contains 3 constraints and 3 variables, the dual problem will contain 3 constraints and 3 dual variables y_1, y_2, y_3. Since the first primal constraint is an equality, the corresponding first dual variable y_1 is unrestricted in sign. Also, since the third primal variable x_3 is unrestricted in sign, the corresponding third dual constraint will be an equality.

The dual LPP is $\quad \text{Max } W = 22y_1 - 65y_2 + 120y_3$

Subject to

$$y_1 - 3y_2 + y_3 \leq 4$$

$$y_1 - 5y_2 + 7y_3 \leq 5$$

$$y_1 + 2y_2 + 4y_3 = -3$$

and $y_2, y_3 \geq 0$, y_1 is unrestricted.

Example: 7.4.8

Write the dual of the primal:

$$\text{Max } Z = 6x_1 + 6x_2 + x_3 + 7x_4 + 5x_5$$

Subject to

$$3x_1 + 7x_2 + 8x_3 + 5x_4 + x_5 = 2$$

$$2x_1 + x_2 + 3x_4 + 9x_5 = 6$$

$x_1, x_2, x_3, x_4 \geq 0$ and x_5 unrestricted.

Solution:

Given primal LPP is

$$\text{Max } Z = 6x_1 + 6x_2 + x_3 + 7x_4 + 5x_5$$

Subject to

$$3x_1 + 7x_2 + 8x_3 + 5x_4 + x_5 = 2$$

$$2x_1 + x_2 + 0x_3 + 3x_4 + 9x_5 = 6$$

$x_1, x_2, x_3, x_4 \geq 0$ and x_5 unrestricted.

Since the primal problem contains 2 constraints and 5 variables, the dual problem will contain 5 constraints and 2 dual variables y_1, y_2. Since all the constraints in the primal are equality, all the dual variable are unrestricted in sign. Also, since the primal variable x_5 is unrestricted in sign, the corresponding fifth dual constraint is an equality.

The dual LPP is $\quad \text{Min } W = 2y_1 + 6y_2$

Subject to

$$3y_1 + 2y_2 \geq 6$$

$$7y_1 + y_2 \geq 6$$

$$8y_1 \geq 1$$

$$5y_1 + 3y_2 \geq 7$$

$$y_1 + 9y_2 = 5$$

and y_1, y_2 are unrestricted.

Check your progress 7.4

1. Obtain the dual problem of the following L.P.P:

   Maximize $f(x) = 2x_1 + 5x_2 + 6x_3$ subject to the constraints:

   $5x_1 + 6x_2 - x_3 \le 3, -2x_1 + x_2 + 4x_3 \le 4,\ x_1 - 5x_2 + 3x_3 \le 1.$

   $-3x_1 - 3x_2 + 7x_3 \le 6,\ x_1, x_2, x_3 \ge 0.$

2. Find the dual of the following L.P.P:

   Maximize $z = 2x_1 + x_2$ subject to the constraints:

   $x_1 + 5x_2 \le 10,\ x_1 + 3x_2 \ge 6,\ 2x_1 + 2x_2 \le 8;\ x_2 \ge 0$ and x_1 unrestricted.

3. Obtain the dual of the following linear programming problem:

   Maximize $z = 2x_1 + 3x_2 + x_3$ subject to the constraints:

   $4x_1 + 3x_2 + x_3 = 6,\ x_1 + 2x_2 + 5x_3 = 4,\ x_1, x_2, x_3 \ge 0.$

4. Obtain the dual problem of the following L.P.P:

   Maximize $z = x_1 - 2x_2 + 3x_3$ subject to the constraints:

   $-2x_1 + x_2 + 3x_3 = 2,\ 2x_1 + 3x_2 + 4x_3 = 1;\ x_1, x_2, x_3 \ge 0.$

5. Obtain the dual of the linear programming problem:

   Minimize $z = x_3 + x_4 + x_3$ subject to the constraints:

   $x_1 - x_3 + x_4 - x_5 = -2;\ x_2 - x_3 - x_4 + x_5 = 1,\ x_j \ge 0$ for $j = 1,2,3,4,5.$

6. Write the dual of the following linear programming problem:

   Minimize $z = 3x_1 - 2x_2 + 4x_3$ subject to the constraints:

   $3x_1 + 5x_2 + 4x_3 \ge 7,\ 6x_1 + x_2 + 3x_3 \ge 4$

   $7x_1 - 2x_2 - x_3 \le 10,\ x_1 - 2x_2 + 5x_3 \ge 3$

   $4x_1 + 7x_2 - 2x_3 \ge 2,\ x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$

7.5 Duality Theorem:

Theorem: 7.5.1

The dual of the dual is the primal.

Proof:

Let the primal L.P.P be to determine $x^T \in R^n$ so as to

$$\text{Maximize } f(x) = cx, \; c \in R^n \text{ subject to the constraints:}$$

$$Ax = b \text{ and } x \geq 0, \; b^T \in R^m,$$

where A is an m x n real matrix.

The dual of this primal is the L.P.P of determining $w^T \in R^m$ so as to

$$\text{Minimize } f(w) = b^T w, \quad b^T \in R^m \text{ subject to the constraints:}$$

$$A^T w \geq c^T, \text{ w unrestricted}, \; c \in R^n.$$

Now, introduce surplus variable $s \geq 0$ in the constraints of the dual and write $w = w_1 - w_2$, where $w_1 \geq 0$ and $w_2 \geq 0$.

The standard form of dual then is to

$$\text{Minimize } g(w) = b^T(w_1 - w_2) \;\; b^T \in R^m \text{ subject to the constraints:}$$

$$A^T(w_1 - w_2) - I_n s = c^T \quad c \in R^n$$

$$w_1, w_2 \text{ and } s \geq 0.$$

Considering this linear programming problem as our standard primal, the associated dual problem will be to

$$\text{Maximize } h(y) = cy, \; c \in R^n \text{ subject to the constraints:}$$

$$(A^T)^T y \leq (b^T)^T, \; -(A^T)^T y \leq -(b^T)^T,$$

$$-y \leq 0 \;\; (\Rightarrow y \geq 0) \text{ and y unrestricted.}$$

Eliminating redundancy, the dual problem may be re – written as:

$$\text{Maximize } h(y) = cy, \; c \in R^n \text{ subject to the constraints:}$$

$$\left.\begin{array}{l} Ay \leq b \\ Ay \geq b \end{array}\right\} \Rightarrow Ay = b, \; b^T \in R^m$$

$$y \geq 0.$$

This problem, which is the dual of the dual problem, is just the primal problem we had started with.

This completes the proof.

UNIT – VIII
TRANSPORTATION PROBLEMS

8.0 Introduction:

The Transportation Problem (T.P) is one of the subclasses of L.P.P's in which the objective is to transport various quantities of a single homogeneous commodity, that are initially stored at various origins, to different destinations in such a way that the total transportation cost is minimum. To achieve this objective we must know the amount and location of available supplies and the quantities demanded. In addition, we must know the costs that result from transporting one unit of commodity from various origins to various destinations.

Objectives:

Students will be able to

1. Structure linear programming problems using the transportation models.
2. Use the North – West corner rule and VAM method.
3. Solve facility location and other application problems with transportation methods.

Structure:

Mathematical Formulation of a Transportation problem.

Methods for finding initial basic feasible solution.

Modified Distribution Method.

Degeneracy in Transportation problems.

Unbalanced Transportation problems.

Keywords

Answers to check your progress Questions.

Model Questions.

8.1 Mathematical Formulation of a Transportation Problem

Let us assume that there are m sources and n destinations.

Let a_i be the supply (capacity) at source i, b_j be the demand at destination j, c_{ij} be the unit transportation cost from source i to destination j and x_{ij} be the number of units shifted from source i to destination j.

Then the transportation problem can be expressed mathematically as

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1,2,3,\ldots\ldots, m.$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1,2,3,\ldots\ldots\ldots, n.$$

and $x_{ij} \geq 0$, for all i and j.

Note:1

The two sets of constraints will be consistent if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j,$$

(total supply) (total demand)

which is the necessary and sufficient condition for a transportation problem to have a feasible solution. Problems satisfying this condition are called balanced transportation problems.

Note: 2

If $\sum a_i \neq \sum b_j$, then the transportation problem is said to be **unbalanced.**

Note: 3

For any transportation problem, the coefficients of all x_{ij} in the constraints are unity.

Note: 4

The objective function and the constraints being all linear, the transportation problem is a special class of linear programming problem. Therefore it can be solved by simplex method. But the number of variables being large, there will be too many calculations. So we can look for some other technique which would be simpler than the usual simplex method.

Standard transportation table:

Transportation problem is explicitly represented by the following transportation table

Destination

		D_1	D_2	D_3	...	D_j	...	D_n	Supply
	S_1	c_{11}	c_{12}	c_{13}		c_{1j}		c_{1n}	a_1
	S_2	c_{21}	c_{22}	c_{23}		c_{2j}		c_{2n}	a_2
Source									⋮
									⋮
	S_i	c_{i1}	c_{i2}			c_{ij}		c_{in}	⋮
	S_m	c_{m1}	c_{m2}			c_{mj}		c_{mn}	a_m
Demand		b_1	b_2	b_3	...	...	...	b_n	$\sum a_i = \sum b_j$

The mn squares are called **cells**. The unit transportation cost c_{ij} from the i^{th} source to the j^{th} destination is displayed in the **upper left side of the (i, j)th cell**. Any feasible solution is shown in the table by entering the value of x_{ij} **in the centre of the** (i, j)th cell. The various a's and b's are called **rim requirements.** The feasibility of a solution can be verified by summing the values of x_{ij} along the rows and down the columns.

Theorem: 1

(Existence of Feasible Solution). A necessary and sufficient condition for the existence of a feasible solution to the general transportation problem is that

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = \lambda \text{ (say)}.$$

Proof:

The condition is necessary. Let there exist a feasible solution to the T.P. Then, we have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i \quad \text{and} \quad \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j$$

yielding

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = \lambda \text{ (say)}$$

Sufficiency. We assert that there exists a feasible solution given by $x_{ij} = a_i b_j / \lambda$ for all i and j. Clearly, $x_{ij} \geq 0$ since $a_i > 0,\ b_j > 0$ for all i and j.

Also $\displaystyle \sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} (a_i b_j / \lambda) = \frac{a_i}{\lambda} \sum_{i=1}^{n} b_j = a_i, \qquad i = 1,2,\ldots\ldots,m$

and $$\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} (a_i b_j / \lambda) = \frac{b_j}{\lambda} \sum_{i=1}^{m} a_i = b_j, \qquad j = 1,2,\ldots\ldots,n$$

Thus x_{ij} satisfies all the constraints of the T.P. and hence is a feasible solution.

Corollary. (Existence of an Optimum Solution). There always exists an optimum solution to a T.P.

Proof:

Let $\sum a_i = \sum b_j$ so that a feasible solution x_{ij} exists. It follows from the constraints of the problem that each x_{ij} is bounded, viz.,

$$0 \le x_{ij} \le \min.(a_i, b_j)$$

Thus the feasible region of the problem is closed, bounded and non – empty and hence there exists an optimum solution.

Theorem: 2

(Basic Feasible Solution). The number of basic (decision) variables of the general transportation problem at any stage of feasible solution must be $m + n - 1$.

Proof:

Consider the $m + n$ constraints of the transportation problem:

$$\sum_{i=1}^{m} x_{ij} = b_i \text{ and } \sum_{j=1}^{n} x_{ij} = a_i \qquad (j = 1,2,\ldots,n; i = 1,2,\ldots..m)$$

Taking summation on both sides, these yield

$$\sum_{j=1}^{n-1} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n-1} b_j \text{ and } \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_j$$

Subtracting the two, we have

$$\sum_{j=1}^{n-1} \sum_{i=1}^{m} x_{ij} - \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n-1} b_j - \sum_{i=1}^{m} a_i$$

or $$\sum_{i=1}^{m} \left(\sum_{j=1}^{n-1} x_{ij} - \sum_{j=1}^{n} x_{ij} \right) = \sum_{j=1}^{n-1} b_j - \sum_{j=1}^{n} b_j \quad \left(\because \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \right)$$

or $$\sum_{i=1}^{m} x_{in} = b_n$$

which happens to be the last constraint of the problem.

This indicates that if the first $m+n-1$ constraints are satisfied, then $\sum a_i = \sum b_j$, ensures that the $(m+n)^{th}$ constraint will be automatically satisfied.

Thus, out of $m+n$ equations, we have only $(m+n-1)$ linearly independent equations. Therefore a basic feasible solution will consist of at most $(m+n-1)$ positive variables the rest being zero. Further a feasible solution involving exactly $(m+n-1)$ positive variables is known as non – degenerate basic feasible solution, otherwise it is said to be degenerate basic feasible.

Remarks: 1

When the total demand is equal to total supply, the transportation problem is said to be balanced and otherwise unbalanced.

Remarks: 2

The allocated cells in the transportation table will be called occupied cells and empty cells will be called non – occupied cells.

Definition: 1

A set of non – negative values x_{ij}, $i = 1,2,\dots\dots, m$; $j = 1,2,\dots\dots n$. That satisfies the constraints (rim conditions and also the non – negativity restrictions) is called a **feasible solution** to the transportation problem.

Note:

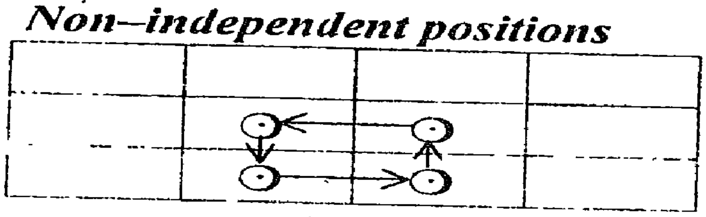
A balanced transportation problem will always have a feasible solution.

Definition: 2

A feasible solution to a $(m \times n)$ transportation problem that contains no more than $m + n - 1$ non – negative allocations is called a **basic feasible solution** (BFS) to the transportation problem.

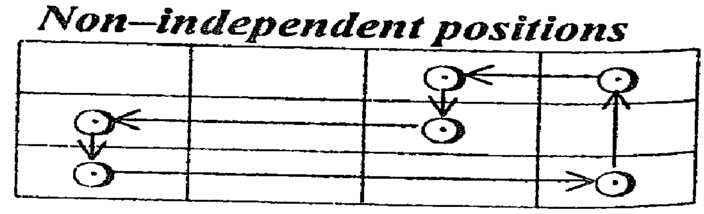
The allocations are said to be in **independent positions** if it is impossible to increase or decrease any allocation without either changing the position of the allocation or violating the rim requirements. A simple rule for allocations to be in independent positions is that it is impossible to rule for allocation, back to itself by a series of horizontal and vertical jumps from one occupied cell to another, without a direct reversal of the route. Example

Non–independent positions



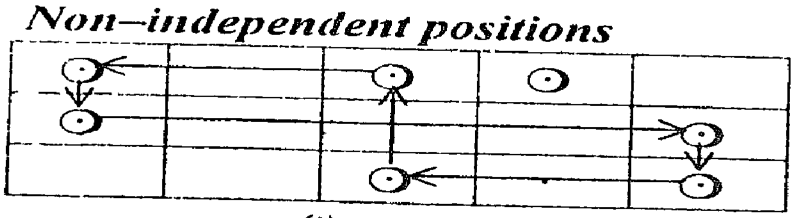
(i)

Non–independent positions



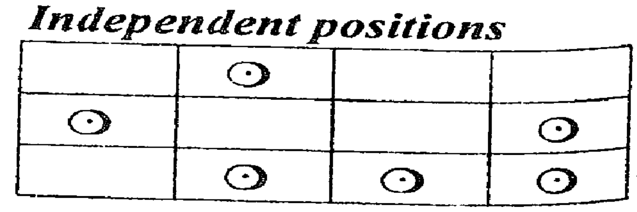
(ii)

Non–independent positions



(i)

Independent positions



(ii)

Definition: 3

A basic feasible solution to a $(m \times n)$ transportation problem is said to be a **non – degenerate basic feasible solution** if it contains exactly $m+n-1$ non – negative allocations in independent positions.

Definition: 4

A basic feasible solution that contains less than $m+n-1$ non – negative allocations is said to be a **degenerate basic** feasible solution.

Definition: 5

A feasible solution (not necessarily basic) is said to be **an optimal solution** if it minimizes the total transportation cost.

Note:

The number of basic variables in an $m \times n$ balanced transportation problem is atmost $m+n-1$.

Note:

The number of non – basic variables in an $m \times n$ balanced transportation problem is atleast $mn-(m+n-1)$.

8.2 Methods for finding Initial basic feasible solution

The transportation problem has a solution if and only if the problem is balanced. Therefore before starting to find the initial basic possible solution. Check whether the given transportation problem is balanced. If not one has to balance the transportation problem first. The way of doing this is discussed in section 6.5. In this section all the given transportation problems are balanced.

An initial feasible solution is obtained by any one of the following methods.

1. Row Minima Method.
2. Column Minima Method.
3. North West corner rule.
4. Least cost Method.
5. Vogel's Approximation Method.

Method 1: Row Minima Method

In this method we determine allotment row – wise. First allotment should be made to the least cost cell in that row. The allotment should be the minimum of availability and requirement corresponding to that row. Cut off the row or column where the availability or the requirement is exhausted. If still some availability is left out in the first row then choose the next least cost in that row for allotment. In this way, exhaust all the availability in that row.

Next choose the least cost unallotted cell in the second row and decide the allotment for that cell. In this way exhaust the availability for that cell. In this way exhaust the availability for the second row. Proceed in the same way till all the row availability are exhausted.

Thereby we get the initial feasible solution to the transportation problem.

Example: 8.2.1

Determine the initial feasible

	S_1	S_2	S_3	S_4	Availability
A	5	2	4	3	22
B	4	8	1	6	15
C	4	6	7	5	8
Demand	7	12	7	19	

solution to the following transportation problem using row minima method.

Solution:

Since $\sum a_i = \sum b_j = 45$ then Transportation problem is balanced.

The initial feasible solution is under Row Minima Method.

5	2 12	4	3 10	22
4 7	8	1 7	6 1	15
4	6	7	5 8	8
7	12	7	19	

Transportation cost

$$= (2x12) + (3x10) + (4x7) + (1x7) + (6x1) + (5x8)$$

$$= 24 + 30 + 28 + 7 + 6 + 40 = 135$$

Method 2: Column Minima Method

The procedure for obtaining the initial solution by this method is the same as the above method except that allotments are made column – wise instead of row – wise.

Example: 8.2.2

Determine the initial feasible solution to the following transportation problem using column minima method.

	S_1	S_2	S_3	S_4	Availability
A	5	2	4	3	22
B	4	8	1	6	15
C	4	6	7	5	8
Demand	7	12	7	19	

Solution:

Since $\sum a_i = \sum b_j = 45$ then the Transportation problem is balanced.

The initial feasible solution is under column minima method.

	S1	S2	S3	S4	Availability
A	5	2 12	4	3 10	22
B	4	8	1 7	6 8	15
C	4 7	6	7	5 1	8
Demand	7	12	7	19	

Transportation cost $= (2x12) + (3x10) + (1x7) + (6x8) + (4x7) + (5x1)$

$= 24 + 30 + 7 + 48 + 28 + 5 = 142$

Method 3: North West Corner Rule:

Step: 1

The first assignment is made in the cell occupying the upper left hand (north – west) corner of the transportation table. The maximum possible amount is allocated there. That is $x_{11} = \min\{a_1, b_1\}$.

Case (i) : If $\min\{a_1, b_1\} = a_1$, then put $x_{11} = a_1$, decrease b_1 by a_1 and move vertically to the 2nd row (i.e.,) to the cell (2,1) cross out of the first row.

Case (ii) : If $\min\{a_1, b_1\} = b_1$, then put $x_{11} = b_1$, and decrease a_1 by b_1 and move horizontally right (i.e.,) to the cell (1,2) cross out the first column.

Case (iii) : If $\min\{a_1, b_1\} = a_1 = b_1$ then put $x_{11} = a_1 = b_1$ and move diagonally to the cell (2,2) cross out the first row and the first column.

Step: 2

Repeat the procedure until all the rim requirements are satisfied.

Example: 8.2.3

Determine basic feasible solution to the following transportation problem using North West Corner Rule:

		A	B	C	D	E	Supply
	P	2	11	10	3	7	4
Origin	Q	1	4	7	2	1	8
	R	3	9	4	8	12	9
Demand		3	3	4	5	6	

Solution:

Since $a_i = b_j = 21$, the given problem is balanced. $\therefore$ There exists a feasible solution to the transportation problem.

2 3	11	10	3	7	4
1	4	7	2	1	8
3	9	4	8	12	9
3	3	4	5	6	

Following North West Corner rule, the first allocation is made in the cell (1,1).

Here $x_{11} = \min\{a_1, b_1\} = \min\{4, 3\} = 3$

$\therefore$ Allocate 3 to the cell (1,1) and decrease 4 by i.e., $4 - 3 = 1$

As the first column is satisfied, we cross out the first column and the resulting reduced Transportation table is

11 1	10	3	7	1
4	7	2	1	8
9	4	8	12	9
3	4	5	6	

Here the North West Corner cell is (1,2).

So Allocate $x_{12} = \min\{1, 3\} = 1$ to the cell (1, 2) and move vertically to cell (2, 2). The resulting reduced transportation table is

4 (2)	7	2	1	8
9	4	8	12	9
2	4	5	6	

Allocate $x_{22} = \min\{8, 2\} = 2$ to the cell (2, 2) and move horizontally to the cell (2, 3). The resulting transportation table is

7 (4)	2	1	6
4	8	12	9
4	5	6	

Allocate $x_{23} = \min\{6, 4\} = 4$ and move horizontally to the cell (2, 4). The resulting reduced transportation table is

2 (2)	1	2
8	12	9
5	6	

Allocate $x_{24} = \{2, 5\} = 2$ and move vertically to the cell (3, 4).

The resulting reduced transportation table is

8 (3)	12	9
3	6	

Allocate $x_{34} = \min\{9, 3\} = 3$ and move horizontally to the cell (3, 5), which is

12 (6)	6
6	

Allocate $x_{35} = \min\{6, 6\} = 6$

Finally the initial basic feasible solution is as shown in the following table.

2 3	11 1	10	3	7
1	4 2	7 4	2 2	1
3	9	4	8 3	12 6

From this table we see that the number of positive independent allocations is equal to $m+n-1 = 3+5-1 = 7$. This ensures that the solution is not degenerate basic feasible.

$\therefore$ The initial transportation cost

$$= \text{Rs.}(2x3)+(11x1)+(4x2)+(7x4)+(2x2)+(8x3)+(12x6)$$

$$= \text{Rs.}153/-$$

Example: 8.2.4

Obtain an initial basic feasible solution to the following transportation problem using the north – west corner rule.

Solution:

Since $\sum a_i = \sum b_j = 950$, there exists a feasible solution to the transportation problem. We obtain initial feasible solution as follows:

	D	E	F	G	Available
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Requirement	200	225	275	250	

The transportation table of the given problem has 12 cells. Following north – west corner method, the first allocation is made in the cell (1,1), the magnitude being x_{11} = min . (250, 200) = 200. The second allocation is made in the cell (1,2) and the magnitude of the allocation is given by

$$x_{12} = \min(250-200,\ 225) = 50.$$

The third allocation is made in the cell (2, 2), the magnitude being $x_{22} = \min.(300,\ 225-50) = 175$. The magnitude of fourth allocation in the cell (2, 3) is given by $x_{23} = \min.(300-175,\ 275) = 125$. The fifth allocation is made in the cell (3, 3), the magnitude being $x_{33} = \min.(400,\ 275-125) = 150$ and the sixth (last) allocation is made in the cell (3, 4) with magnitude $x_{34} = \min.(400-150,\ 250) = 250$. Hence an initial basic feasible solution to the given T.P. has been obtained and is displayed in Table.

The transportation cost according to the above route is given by

$$z = (200 \times 11) + (50 \times 13) + (175 \times 18) + (125 \times 14) + (150 \times 13) + (250 \times 10)$$

$$= 12{,}200.$$

[200] 11	[50] 13	17	14	250
16	[175] 18	[125] 14	10	300
21	24	[150] 13	[250] 10	400
200	225	275	250	

Method: 4

Least Cost method (or) Matrix minima method (or) Lowest cost entry method:

Step: 1

Identify the cell with smallest cost and allocate $x_{ij} = \text{Min}\{a_i, b_j\}$.

Case: (i)

If $\min\{a_i, b_j\} = a_i$, then put $x_{ij} = a_i$, cross out the i^{th} row and decrease b_j by a_i, Go to step (2).

Case: (ii)

If min $\{a_i, b_j\} = b_j$ then put $x_{ij} = b_j$ cross out the j^{th} column and decrease a_i by b_j Go to step (2).

Case: (iii)

If $\min\{a_i, b_j\} = a_i = b_j$, then put $x_{ij} = a_i = b_j$, cross out either i^{th} row or j^{th} column but not both, Go to step (2).

Step: 2

Repeat step (1) for the resulting reduced transportation table until all the rim requirements are satisfied.

Example: 8.2.5

	To				Supply
From	1	2	1	4	30
	3	3	2	1	50
	4	2	5	9	20
Demand	20	40	30	10	

Find the initial basic feasible solution for the following transportation problem by Least Cost Method.

Solution:

Since $\sum a_i = \sum b_j = 100$, the given TPP is balanced. ∴ There exists a feasible solution to the transportation problem.

1 (20)	2	1	4	30
3	3	2	1	50
4	2	5	9	20
20	40	30	10	

By least cost method, min $c_{ij} = c_{11} = c_{13} = c_{24} = 1$

Since more than one cell having the same minimum c_{ij}, break the tie.

Let us choose the cell (1, 1) and allocate $x_{11} = \min\{a_1, b_1\} = \min\{30,\ 20\} = 20$ and cross out the satisfied column and decrease 30 by 20.

The resulting reduced transportation table is

2	1 (10)	4	10
3	2	1	50
2	5	9	20
40	30	10	

Here min $c_{ij} = c_{13} = c_{24} = 1$

Choose the cell (1, 3) and allocate $x_{13} = \min\{a_1, b_3\} = \min\{10{,}30\} = 10$ and cross out the satisfied row.

The resulting reduced transportation table is

3	2	1 10	50
2	5	9	20
40	20	10	

Here $\min c_{ij} = c_{24} = 1$.

$\therefore$ Allocate $x_{24} = \min\{a_2, b_4\} = \min(50, 10) = 10$ and cross out the satisfied column.

The resulting transportation table is

3	2 20	40
2	5	20
40	20	

Here $\min c_{ij} = c_{23} = c_{32} = 2$. Choose the cell (2, 3) and allocate $x_{23} = \min\{a_2, b_3\} = \min(40, 20) = 20$ and cross out the satisfied column.

The resulting reduced transportation table is

3	20
2 20	20
40	

Here $\min c_{ij} = c_{32} = 2$. Choose the cell (3, 2) and allocate $x_{32} = \min\{a_3, b_2\} = \min(20, 40) = 20$ and cross out the satisfied row.

The resulting reduced transportation table is

3 20	20
20	

Finally the initial basic feasible solution is as shown in the following table.

1 20	2	1 10	4
3	3 20	2 20	1 10
4	2 20	5	9

From this table we see that the number of positive independent allocations is equal to $m+n-1=3+4-1=6$. This ensures that the solution is non degenerate basic feasible.

$\therefore$ The initial transportation cost

$$= \text{Rs.}(1x20)+(1x10)+(3x20)+(2x20)+(1x10)+(2x20)$$

$$=20+10+60+40+10+40$$

$$=\text{Rs.}180/-$$

Example: 8.2.6

Obtain an initial basic feasible solution to the following T.P. using the matrix minima method.

	D_1	D_2	D_3	D_4	Capacity
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	

Where O_i and D_j denote ith origin and jth destination respectively.

Solution:

Since $\sum a_i = \sum b_j = 24$ then TPP is balanced.

$\therefore$ There exists a feasible solution to the transportation problem.

1	2	3	4	6
4	3	2	0	8
0 4	2	2	1	10
4	6	8	6	

By least cost method, min $c_{ij} = c_{31} = c_{24} = 0$.

Since more than one cell having the same minimum c_{ij} break the tie.

Let us choose the cell (3, 1) and allocate $x_{31} = \min\{a_3, b_1\} = \min\{4,\ 10\} = 4$ and cross out the satisfied column and decrease 10 by 6. The resulting reduced transportation table is

2	3	4	6
3	2	0 6	8
2	2	1	6
6	8	6	

Here min $c_{ij} = c_{24} = 0$

Choose the (2, 3) and allocate $x_{23} = \min\{a_2, b_3\} = \min\{6,\ 8\} = 6$ and cross out the satisfied column. The resulting reduced transportation table is

2 6	3	6
3	2	2
2 0	2	6
6	8	

Here min $c_{ij} = c_{12} = c_{23} = c_{32} = c_{33} = 2$

Choose the cell c_{12} and allocate $x_{12} = \min\{a_1, b_2\} = \min\{6, 6\} = 6$ and cross out either the second column or the first row. We choose to cross out the first row of the table. The next allocation of magnitude $x_{32} = 0$ is made in the cell c_{32}. Cross out the second column.

The resulting reduced transportation table is

2 2	2
2	6
8	

Here $\min c_{ij} = c_{23} = c_{33} = 2$

Choose the cell c_{23} and allocate $x_{23} = \min\{a_2, b_3\} = \min\{2, 8\} = 2$ and cross out the satisfied row. The resulting reduced transportation table is

2 6	6
6	

Finally the initial basic feasible solution is as shown in the following task.

1	2 6	3	4
4	3	2 2	0 6
0 4	2 6	2 6	1

Now, all the rim requirements have been satisfied and hence an initial solution has been determined. This solution is displayed in transportation Table.

Since the cells do not form a loop, the solution is basic one. Moreover the solution is degenerate also. The transportation cost according to the above route is given by

$$z = (6x2) + (2x2) + (6x0) + (4x0) + (2x \in) + (6x2) = 28 + 2\in = 28 \text{ as } \in \to 0$$

Method: 5

Vogel's approximation method (VAM) (or) Unit cost penalty method:

Step: 1

Find the difference (penalty) between the smallest and next smallest costs in each row (column) and write them in brackets against the corresponding row (column).

Step: 2

Identify the row (or) column with largest penalty. If a tie occurs, break the tie arbitrarily. Choose the cell with smallest cost in that selected row or column and allocate as much as possible to this cell and cross out the satisfied row or column and go to step (3).

Step: 3

Again compute the column and row penalties for the reduced transportation table and then go to step (2). Repeat the procedure until all the rim requirements are satisfied.

Example: 8.2.7

		D_1	D_2	D_3	D_4	Availability
	S_1	11	13	17	14	250
Origin	S_2	16	18	14	10	300
	S_3	21	24	13	10	400
Requirements		200	225	275	250	

Find the initial basic feasible solution for the following transportation problem by VAM.

Distribution Centres

Solution:

Since $\sum a_i = \sum b_j = 950$, the given problem is balanced.

$\therefore$ There exists a feasible solution to this problem.

11 200	13	17	14	250(2)
16*	18	14	10	300 (4)
21	24	13	10	400 (3)
200	225	275	250	
(5)	(5)	(1)	(0)	

First let us find the difference (penalty) between the smallest and next smallest costs in each row and column and write them in brackets against the respective rows and columns.

The largest of these difference is (5) and is associated with the first two columns of the transportation table. We choose the first column arbitrarily.

In this selected column, the cell (1,1) has the minimum unit transportation cost $c_{11} = 11$.

$\therefore$ Allocate $x_{11} = \min\{250,\ 200\} = 200$ to this cell (1, 1) and decrease 250 by 200 and cross out the satisfied column.

The resulting reduced transportation table is

13 50	17	14	50 (1)
18	14	10	300 (4)
24	13	10	400 (3)
225	275	250	
(5)	(1)	(0)	

The row and column difference and now computed for this reduced transportation table. The largest of these is (5) which is associated with the second column. Since $c_{12} = 13$ is the minimum cost, we allocated

$x_{12} = \min\{50, 225\} = 50$ to the cell (1,2) and decrease 225 by 50 and cross out the satisfied row.

Continuing in this manner, the subsequent reduced transportation tables and the differences for the surviving rows and columns are shown below:

18 175	14	10	300 (4)
24	13	10	400 (3)
175	275	250	
(6)	(1)	(0)	

14	10 125	125 (4)
13	10	400 (3)
275	250	
(1)	(0)	

13	10 125	400
275	125	

13 275	25
275	

Finally the initial basic feasible solution is as shown in the following table.

11 200	13 50	17	14
16	18 175	14	10 125
21	24	13 275	10 125

From this table we see that the number of positive independent allocations is equal to $m+n-1=3+4-1=6$. This ensures that the solution is non degenerate basic feasible.

$\therefore$ The initial transportation cost

$$= \text{Rs.}(11x200)+(13x50)+(18x175)+(10x125)+(13x275)+(10x125)$$

$$= \text{Rs.}12075/-$$

Example: 8.2.8

Find the starting solution of the following transportation model

1	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

Using

(i) North West Corner rule

(ii) Least Cost method

(iii) Vogel's approximation method.

Solution:

Since $\sum a_i = \sum b_j = 30$, the given Transportation problem is balanced. Hence there exists a basic feasible solution to this problem.

(i) North West Corner rule:

Using this method, the allocation are shown in the tables below:

1 7	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

0 3	4	2	12
3	1	5	11
3	10	10	

4 9	2	9
1	5	11
10	10	

1 5	5	11
1	10	

5 10	10
10	

The starting solution is as shown in the following table

1 7	2	6
0 3	4 9	2
3	1 1	5 10

$\therefore$ The initial transportation cost

$$= \text{Rs.}(1\text{x}7) + (0\text{x}3) + (4\text{x}9) + (1\text{x}1) + (5\text{x}10)$$

$$= \text{Rs.}94/-$$

(ii) Least Cost Method:

Using this method, the allocation are as shown in the table below:

1	2	6 7	7
0 10	4	2	12
3	1	5	11
10	10	10	

2	6	7
4	2	2
1 10	5	11
10	10	

6	7
2 2	2
5	1
10	

6	7
5 1	1
8	

6 7	7
7	

The starting solution is as shown in the following table:

1	2	6 7
0 10	4	2 2
3	1 10	5 1

$\therefore$ The initial transportation cost $= Rs.6x7 + 0x10 + 2x2 + 1x10 + 5x1$

$= Rs.61/-$

(iii) Vogel's Approximation Method:

Using this method, the allocations are shown in the tables below:

1	2	6	7	(1)
0	4	2 10	12	(2)
3	1	5	11	(2)
10	10	10		
(1)	(1)	(3)		

1	2	7	(1)
0 2	4	2	(4)
3	1	11	(2)
10	10		
(1)	(1)		

1	2	7	(1)
3	1 10	11	(2)
8	10		
(2)	(1)		

1 7	7
3	1
8	

3 1	1
1	

The starting solution is an shown in the following table:

1 7	2	6
0 2	4	2 10
3 1	1 10	5

$\therefore$ The initial transportation cost

$$= \text{Rs.}(1\text{x}7) + (0\text{x}2) + (2\text{x}10) + (3\text{x}1) + (1\text{x}10)$$
$$= \text{Rs.}40/-$$

Note:

For the above problem, the number of positive allocation in independent positions is $(m+n-1)$(i.e., $m+n-1=3+3-1=5$). This ensures that the given problem has a non – degenerate basic feasible solution by using all the three methods. This need not be the case in all the problems.

Check your progress: 6.1

1. For the transportation problem given below, find a basic feasible solution by North – West Corner method:

	D_1	D_2	D_3	D_4	D_5	D_6	Supply
O_1	6	4	8	4	9	6	4
O_2	6	7	13	6	8	12	5
O_3	3	9	4	5	9	13	3
O_4	10	7	11	6	11	10	9
Demand	4	4	5	3	2	3	

Warehouse	Stores				Availability
	I	II	III	IV	
A	5	1	3	3	34
B	3	3	5	4	15
C	6	4	4	3	12
D	4	-1	4	2	19
Requirement	21	25	17	17	80

Determine an initial basic feasible solution to the following transportation problem using North – West Corner method:

	D_1	D_2	D_3	D_4	Availability
O_1	5	3	6	2	19
O_2	4	7	9	1	37
O_3	3	4	7	5	34
Demand	16	18	31	25	

1. Consider the following transportation problem:

Source	Destination				Availability
	1	2	3	4	
1	20	22	17	4	120
2	24	37	9	7	70
3	32	37	20	15	50
Requirement	60	40	30	110	240

Determine an initial basic feasible solution using the (a) row minima method, and (b) Vogel's approximation method.

2. Obtain an initial basic feasible solution to the following T.P. using the Vogel's approximation method:

8.3 Modified Distribution Method (MODI Method)

Transportation Algorithm (or) MODI Method (Test for optimal solution).

Step: 1

Find the initial basic feasible solution of the given problem by North West Corner rule (or) Least Cost method or VAM.

Step: 2

Check the number of occupied cells. If these are less than $m+n-1$, there exists degeneracy and we introduce a very small positive assignment of $\in (\approx 0)$ in suitable independent positions, so that the number of occupied cells is exactly equal to $m+n-1$.

Step: 3

Find the set of values $u_i, v_j (i = 1,2,3,.....m;\ j = 1,2,3,.......,n)$ from the relation $c_{ij} = u_i + v_j$ for each occupied cell (i, j), by starting initially with $u_i = 0$ or $v_j = 0$ preferably for which the corresponding row or column has maximum number of individual allocations.

Step: 4

Find $u_i + v_j$ for each unoccupied cell (i, j) are enter at the upper right corner of the corresponding cell (i, j).

Step: 5

Find the cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$($d_{ij}$ = upper left – upper right) for each unoccupied cell (i, j) and enter at the lower right corner of the corresponding cell (i, j).

Step: 6

Examine the cell evaluations d_{ij} for all unoccupied cells (i, j) and conclude that

(i) If all $d_{ij} > 0$, then the solution under the test is optimal and unique.

(ii) If all $d_{ij} > 0$, with atleast one $d_{ij} = 0$, then the solution under the test is optimal and an alternative optimal solution exists.

(iii)If atleast one $d_{ij} < 0$, then the solution is not optimal. Go to the next step.

Step: 7

From a new B.F.S by giving maximum allocation to the cell for which d_{ij} is most negative by making an occupied cell empty. For that draw a closed path consisting of horizontal and vertical lines beginning and ending at the cells for which d_{ij} is most negative and having its **other corners at some allocated cells.** Along this closed loop indicate $+\theta$ and $-\theta$ alternatively at the corners. Choose minimum of the allocations from the cells having $-\theta$. Add this minimum allocation to the cells with $+\theta$ and subtract this minimum allocation from the allocation to the cells with $-\theta$.

Step: 8

Repeat steps (2) to (6) to test the optimality of this new basic feasible solution.

Step: 9

Continue the above procedure till an optimum solution is attained.

Note:

The Vogels approximation method (VAM) takes into account not only the least cost c_{ij} but also the costs that just exceed the least cost c_{ij} and therefore yields better initial solution than obtained from other methods in general. This can be justified by the above example (4). So to find the initial solution, give preference to VAM unless otherwise specified.

Example: 8.3.1

	1	2	3	4	Supply
I	21	16	25	13	11
II	17	18	14	23	13
III	32	27	18	41	19
Demand	6	10	12	15	

Solve the transportation problem:

Solution:

Since $\sum a_i = \sum b_j = 43$, the given transportation problem is balanced.

$\therefore$ There exists a basic feasible solution to this problem.

By Vogel's approximation method, the initial solution is as shown in the following table:

21	16	25	13 11	(3)	-	-	-
17 6	18 3	14	23 4	(3)	(3)	(3)	(4)
32	27 7	18 12	41	(9)	(9)	(9)	(9)
(4)	(2)	(4)	**(10)**				
(15)	(9)	(4)	**(18)**				
(15)	(9)	(4)	-				
-	**(9)**	(4)	-				

That is

21	16	25	13 11
17 6	18 3	14	23 4
32	27 7	18 12	41

From this table, we see that the number of non – negative independent allocation is $(m+n-1)=(3+4-1)=6$. Hence the solution is non – degenerate basic feasible.

$\therefore$ The initial transportation cost

$= Rs.(13x11)+(17x6)+(18x3)+(23x4)+(27x7)+(18x12)$

$= Rs.796/-$

To find the optimal solution

Consider the above transportation table. Since $m+n-1=6$, we apply MODI method,

Now we determine a set of values u_i and v_j for each occupied cell (i,j) by using the relation $c_{ij}=u_i+v_j$. As the 2nd row contains maximum number of allocations, we choose $u_2=0$.

Therefore

$$c_{21}=u_2+v_1 \Rightarrow 17=0+v_1 \Rightarrow v=17$$

$$c_{22}=u_2+v_2 \Rightarrow 18=0+v_2 \Rightarrow v_2=18$$

$$c_{24}=u_2+v_4 \Rightarrow 23=0+v_4 \Rightarrow v_4=23$$

$$c_{14}=u_1+v_4 \Rightarrow 13=u_1+23 \Rightarrow u_1=-10$$

$$c_{32}=u_3+v_2 \Rightarrow 27=u_3+18 \Rightarrow u_3=9$$

$$c_{33}=u_3+v_3 \Rightarrow 18=9+v_3 \Rightarrow v_3=9$$

Thus we have the following transportation table:

21	16	25	13 11	$u_1=-10$
17 6	18 3	14	23 4	$u_2=0$
32	27 7	18 12	41	$u_3=9$
$v_1=17$	$v_2=18$	$v_3=9$	$v_4=23$	

Now we find $u_i + v_j$ for each unoccupied cell (i, j) and enter at the upper right corner of the corresponding unoccupied cell (i, j).

Then we find the cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ (i.e., upper left corner – upper right corner) for each unoccupied cell (i, j) and enter at the lower right corner of the corresponding unoccupied cell (i, j)

Thus we get the following table:

21 7 14	16 8 8	25 -1 26	13 11	$u_1 = -10$
17 6	18 3	14 9 5	23 4	$u_2 = 0$
32 26 6	27 7	18 12	41 32 9	$u_3 = 9$
$v_1 = 17$	$v_2 = 18$	$v_3 = 9$	$v_4 = 23$	

Since all $d_{ij} > 0$, the solution under the test is optimal and unique.

$\therefore$ The optimum allocation schedule is given by $x_{14} = 11,\ x_{21} = 6,\ x_{22} = 3,\ x_{24} = 4,\ x_{32} = 7,\ x_{33} = 12$ and the optimum (minimum) transportation cost

$$= \text{Rs.}(13x11) + (17x6) + (18x3) + (23x4) + (27x7) + (18x12)$$

$$= \text{Rs.}796/-$$

Example: 8.3.2

Obtain on optimum basic feasible solution to the following transportation problem:

	To			Available
	7	3	2	2
From	2	1	3	3
	3	4	6	5
Demand	4	1	5	10

Solution:

Since $\sum a_i = \sum b_j = 10$, the given transportation problem is balanced.

$\therefore$ There exists a basic feasible solution to this problem.

By Vogel's approximation method, the initial solution is as shown in the following table:

7	3	2 2	(1)	**(5)**	-
2	1 1	3 2	(1)	(1)	(1)
3 4	4	6 1	(1)	(3)	**(3)**
(1)	(2)	(1)			
(1)		(1)			
(1)		(3)			

That is

7	3	2 2
2	1 1	3 2
3 4	4	6 1

From this table we see that the number of non – negative allocation is $m+n-1=(3+3-1)=5$.

Hence the solution is non – degenerate basic feasible

$\therefore$ The initial transportation cost

$$= \text{Rs.}(2x2)+(1x1)+(3x2)+(3x4)+(6x1)$$

$$= \text{Rs.}29/-$$

For Optimality:

Since the number of non – negative independent allocations is $m+n-1$ we apply MODI method.

Since the third column contains maximum number of allocations, we choose $v_3 = 0$.

Now we determine a set of values u_i and v_j by using the occupied cells and the relation $c_{ij} = u_i + v_j$.

That is,

7 -1	3 0	2 2	$u_1 = 2$
2	1 1	3 2	$u_2 = 3$
3 4	4	6 1	$u_3 = 6$
$v_1 = -3$	$v_2 = -2$	$v_3 = 0$	

Now we find $u_i + v_j$ for each unoccupied cell (i, j) and enter at the upper right corner of the corresponding unoccupied cell (i, j).

Then we find the cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ for each unoccupied cell (i, j) and enter at the lower right corner of the corresponding unoccupied cell (i, j).

Thus we get the following table

7 - 1 8	3 0 3	2 2	$u_1 = 2$
2 0 2	1 1	3 2	$u_2 = 3$
3 4	4 4 0	6 1	$u_3 = 6$
$v_1 = -3$	$v_2 = -2$	$v_3 = 0$	

Since all $d_{ij} > 0$, with $d_{32} = 0$, the current solution is optimal and their exists an alternative optimal solution.

$\therefore$ The optimum allocation schedule is given by $x_{13} = 2$, $x_{22} = 1$, $x_{23} = 3$, $x_{31} = 4$, $x_{33} = 1$ and the optimum (minimum) transportation cost

$$= \text{Rs.}(2x2) + (1x1) + (3x2) + (3x4) + (6x1) = \text{Rs.}29/-$$

Example: 8.3.3

Find the optimal transportation cost of the following matrix using least cost method for finding the critical solution.

Market

		A	B	C	D	E	Available
Factory	P	4	1	2	6	9	100
	Q	6	4	3	5	7	120
	R	5	2	6	4	8	120
Demand		40	50	70	90	90	

Solution:

Since $\sum a_i = \sum b_j = 340$, the given transportation problem is balanced.
$\therefore$ There exists a basic feasible solution to this problem.

By using Least cost method, the initial solution is as shown in the following table:

4	1 50	2 50	6	9
6 10	4	3 20	5	7 90
5 30	2	6	4 90	8

$\therefore$ The initial transportation cost

$= \text{Rs.}(1\text{x}50) + (2\text{x}50) + (6\text{x}10) + (3\text{x}20) + (7\text{x}90) + (5\text{x}30) + (4\text{x}90)$

$= \text{Rs.}1410/-$

For optimality:

Since the number of non – negative independent allocations is $(m+n-1)$, we apply MODI method:

That is

4 5 -1	1 50	2 50	6 4 2	9 6 3	$u_1 = -1$
6 10	4 2 2	3 20	5 5 0	7 90	$u_2 = 0$
5 30	2 1 1	6 2 4	4 90	8 6 2	$u_3 = -1$
$v_1 = 6$	$v_2 = 2$	$v_3 = 3$	$v_4 = 5$	$v_5 = 7$	

Since $d_{11} = -1 < 0$, the current solution is not optimal.

Now let us form a new basic feasible solution by giving maximum allocation to the cell (i, j) for which d_{ij} is most negative by making an occupied cell empty. Here the cell (1, 1) having the negative value $d_{11} = -1$. We draw a closed loop consisting of horizontal and vertical lines beginning and ending at this cell (1, 1) and having its other corners at some occupied cells. Along this closed loop indicate $+\theta$ and $-\theta$ alternatively at the corners. We have

4 $+\theta$	1 50	2 $50-\theta$	6	9
6 10 $-\theta$	4	3 20 $+\theta$	5	7 90
5 30	2	6	4 90	8

From the two cells (1, 3), (2, 1) having $-\theta$, we find that the minimum of the allocations 50, 10 is 10. Add this 10 to the cells with $+\theta$ and subtract this 10 to the cells with $-\theta$.

Hence the new basic feasible solution is displayed in the following table:

4 10	1 50	2 40	6	9
6	4	3 30	5	7 90
5 30	2	6	4 90	8

We see that the above table satisfies the rim conditions with $(m+n-1)$ non – negative allocations at independent positions. So we apply MODI method.

4 10	1 50	2 40	6 3 3	9 6 3	$u_1 = 0$
6 5 1	4 2 2	2 30	5 4 1	7 90	$u_2 = 1$

5 30	2 2 0	6 3 3	4 90	8 7 1	$u_3 = 1$
$v_1 = 4$	$v_2 = 1$	$v_3 = 2$	$v_4 = 3$	$v_5 = 6$	

Since all $d_{ij} > 0$, with $d_{32} = 0$, the current solution is optimal and there exists an alternative optimal solution.

$\therefore$ The optimum allocation schedule is given by $x_{11} = 10,\ x_{12} = 50,\ x_{13} = 40,\ x_{23} = 30,\ x_{25} = 90,\ x_{31} = 30,\ x_{34} = 90$ and the optimum (minimum) transportation cost

$$= \text{Rs.}(4x10) + (1x50) + (2x40) + (3x30) + (7x90) + (5x30) + (4x90)$$

$$= \text{Rs.}1400/-$$

UNIT – IX

ASSIGNMENT PROBLEMS

9.0 Introduction

The assignment problem is a particular case of the transportation problem in which the objective is to assign a number of tasks (Jobs or origins or sources) to an equal number of facilities (machines or persons or destinations) at a minimum cost (or maximum profit).

Suppose that we have 'n' jobs to be performed on 'm' machines (one Job to one machine) and our objective is to assign the jobs to the machines at the minimum cost (or maximum profit) under the assumption that each machine can perform each job but with varying degree of efficiencies.

The assignment problem can be stated in the form of $m \times n$ matrix (c_{ij}) called a **Cost matrix** (or) **Effectiveness matrix** where c_{ij} is the cost of assigning i^{th} machine to the j^{th} job.

$$\begin{array}{cc} & \text{Jobs} \\ & \begin{array}{cccccc} & 1 & 2 & 3 & \cdots\cdots & n \end{array} \\ \text{Machines} & \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ \vdots \\ \vdots \\ m \end{array} \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots\cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots\cdots & c_{2n} \\ c_{31} & c_{32} & c_{33} & \cdots\cdots & c_{3n} \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ c_{m1} & c_{m2} & c_{m3} & \cdots\cdots & c_{mn} \end{bmatrix} \end{array}$$

OBJECTIVES

After studying this unit you will be able to.

1. Structure special L.P.P using assignment models
2. Solve assignment problems with the Hungarian method.

9.1 MATHEMATICAL FORMULATION OF AN ASSIGNMENT PROBLEM

Consider an assignment problem of assigning *n* jobs to *n* machines (one job to one machine). Let c_{ij} be the unit cost of assigning i^{th} machine to the j^{th} job and

$$\text{Let } x_{ij} = \begin{cases} 1, \text{ if } j^{th} \text{ job is assigned to } i^{th} \text{ machine} \\ 0, \text{ if } j^{th} \text{ is not assigned to } i^{th} \text{ machine} \end{cases}$$

The assignment model is then given by the following LPP

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

Subject to the constraints

$$\sum_{i=1}^{n} x_{ij} = 1, \ j = 1, 2, \ldots n$$

$$\sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, \ldots n$$

$$\text{and } x_{ij} = 0 \text{ (or) } 1.$$

Theorem 9.1.1 (Reduction Theorem). In an assignment problem, if we add or subtract a constant to every element of any row (or column) of the cost matrix $[c_{ij}]$, then an assignment that minimizes the total cost on one matrix also minimizes the total cost on the other matrix. In other words, if $x_{ij} = x_{ij}$ minimizes

$$z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad \sum_{i=1}^{n} x_{ij} = 1, \sum_{j=1}^{n} x_{ij} = 1; x_{ij} = 0 \text{ or } 1;$$

then x_{ij}^* also minimizes $z^* = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}^* x_{ij}$, where $c_{ij}^* = c_{ij} - u_i - v_j$ for all $i, j = 1, 2, \ldots, n$ and u_i, v_j are some real numbers.

Proof

We write

$$z^* = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}^* x_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (c_{ij} - u_i - v_j) x_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - \sum_{i=1}^{n} u_i \sum_{j=1}^{n} x_{ij} - \sum_{i=1}^{n} x_{ij} \sum_{j=1}^{n} v_j$$

$$= z - \sum_{i=1}^{n} u_i - \sum_{j=1}^{n} v_j;$$

Since $\sum_{i=1}^{n} x_{ij} = \sum_{j=1}^{n} x_{ij} = 1$.

This shows that the minimization of the new objective function z^* yields the same solution as the minimization of original objective function z, because $\sum u_i$ and $\sum v_j$ are independent of x_{ij}.

Theorem 9.1.2

If $c_{ij} \geq 0$ such that minimum $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} = 0$, then x_{ij} provides an optimum solution (assignment)

The above two theorems form the basis of *Assignment Algorithm.* By selecting suitable constants to be added to or subtracted from the elements of the cost matrix we can ensure that $c_{ij}^* \geq 0$ and can produce at least one $c_{ij}^* = 0$ in each row and each column and try assignments from among these 0 positions. The assignment schedule will be optimal if there is exactly one assignment (i.e., exactly one assigned 0) in each row and each column.

Remarks:

It may be noted that assignment problem is a variation of transportation problem with two characteristics (i) the cost matrix is a square matrix, and (ii) the optimum solution for the problem would always be such that there would be only one assignment in a given row or column of the cost matrix.

9.2 ASSIGNMENT ALGORITHM OR HUNGARIAN METHOD

An efficient method for solving an assignment problem as developed by the Hungarian mathematician D. Konig is summarized below.

Step: 1

Determine the cost table from the given problem

I. If the number of sources is equal to the number of destination, go to step 3.

II. If the number of sources is not equal to the number of destinations, go to step 2.

Step: 2

Add a dummy source or dummy destination, so that the cost table becomes a square matrix. The cost entries of dummy source/destinations are always zero.

Step: 3

Locate the smallest element in each row of the given cost matrix and then subtract the same from each element of that row.

Step: 4

In the reduced matrix obtained in step 3, locate the smallest element of each column and then subtract the same from each element of that column. Each column and row now have at least one zero.

Step: 5

In the modified matrix obtained in step 4, search for an optimal assignment as follows:

a) Examine the rows successively until a row with a single zero is found. Enrectangle this zero ( ) and cross off ( ) all other zeros in its column. Continue in this manner until all the rows have been taken care of.

b) Repeat the procedure for each column of the reduced matrix.

c) If a row and / or column has two or more zeros and one cannot be chosen by inspection then assign arbitrary any one of the zeros and cross off all other zeros of that row/ column

d) Repeat (a) through (c) above successfully until the chain of assigning ( ) or cross (X) ends.

Step: 6

If the number of assignments (□) is equal to n (the order of the cost matrix), an optimal solution is reached.

If the numbers of assignments is less than n (the order of the matrix), go to the next step.

Step: 7

Draw the minimum number of horizontal and / or vertical lines to cover all the zeros of the reduced matrix. This can be conveniently alone by using a simple procedure:

a) Mark (✓) rows that do not have any assigned zero.

b) Mark (✓) columns that have zeros in the marked rows.

c) Mark (✓) rows that have assigned zeros in the marked columns.

d) Repeat (b) and (c) above until the chain of marking is completed.

e) Draw lines through all the unmarked rows and marked columns. This given us the desired minimum number of lines.

Step: 8

Develop the new revised cost matrices as follows:

a) Find the smallest element of the reduced matrix not covered by any of the lines.

b) Subtract this element from all the uncovered elements and add the same to all the element lying at the intersection of any two lines.

Step: 9

Go to step 6 and respect the procedure until an optimum solution is attained.

Example: 9.2.1

A departmental head has four subordinates, and four tasks to be performed. The subordinates differ in efficiency, and the tasks differ in their intrinsic difficulty. His estimate, of the time each man would take to perform each task, is given in the matrix below:

Tasks	Men			
	E	F	G	H
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

How should the tasks be allocated, one to a man, so as to minimize the total man – hours?

Solution:

Step: 3

Subtracting the smallest element of each row from every element of the corresponding row, we get the reduced matrix

7	15	6	0
0	15	1	13
23	4	3	0
9	16	14	0

Step: 4

Subtracting the smallest element of each column of the reduced matrix from every element of the corresponding column. We get the following reduced matrix:

7	11	5	0
0	11	0	13
23	0	2	0
9	12	13	0

Step: 5

Starting with row 1, we enrectangle ($\square$ i.e., make assignment 1 a single zero, if any, and cross (X) all other zeros in the column so marked. Thus, we get

7	11	5	$\boxed{0}$
$\boxed{0}$	11	$\cancel{0}$	13
23	$\boxed{0}$	2	$\cancel{0}$
9	12	13	$\cancel{0}$

In the above matrix, we arbitrarily enrectangled a zero in column 1, because row 2 had two zeros.

It may be noted that column 3 and row 4 do not have any assignment. So, we move on to the next step.

Step: 7

I) Since row 4 does not have any assignment, we mark this row (✓).

II) Now there is a zero in the fourth column of the marked row. So, mark four in column (✓).

III) Further there is an assignment in the first row of the ticked column. So we mark first row (✓).

IV) Draw straight lines through all unmarked rows and marked columns. Thus, we have

7	11	5	$\boxed{0}$	✓
$\boxed{0}$	11	$\cancel{0}$	13	
23	$\boxed{0}$	2	$\cancel{0}$	
9	12	13	$\cancel{0}$	✓
			✓	

Step: 8

In step 7, we observe that the minimum numbers of lines so drawn in 3, which is less than the orders of the cost matrix, indicating that the current assignment is not optimum.

To increase the minimum numbers of lines, we generate new zeros in the modified matrix.

The smallest element not covered by the lines is 5. Subtracting this element from all the obtain the following new reduced cost matrix:

2	6	0	0
0	11	0	18
23	0	2	5
4	7	8	0

Step: 9

Repeating step 5 on the reduced matrix, we get

2	6	$\boxed{0}$	$\cancel{0}$
$\boxed{0}$	11	$\cancel{0}$	18
23	$\boxed{0}$	2	5
4	7	8	$\boxed{0}$

Now, since each row and each column has one and only one assignment, an optimal solution is reached. The optimum assignment is:

$A \rightarrow G,\ \ B \rightarrow E,\ \ C \rightarrow F$ and $D \rightarrow H$.

The minimum total time for this assignment scheduled is 17+13+19+10 or 59 man – hours.

Example: 9.2.2

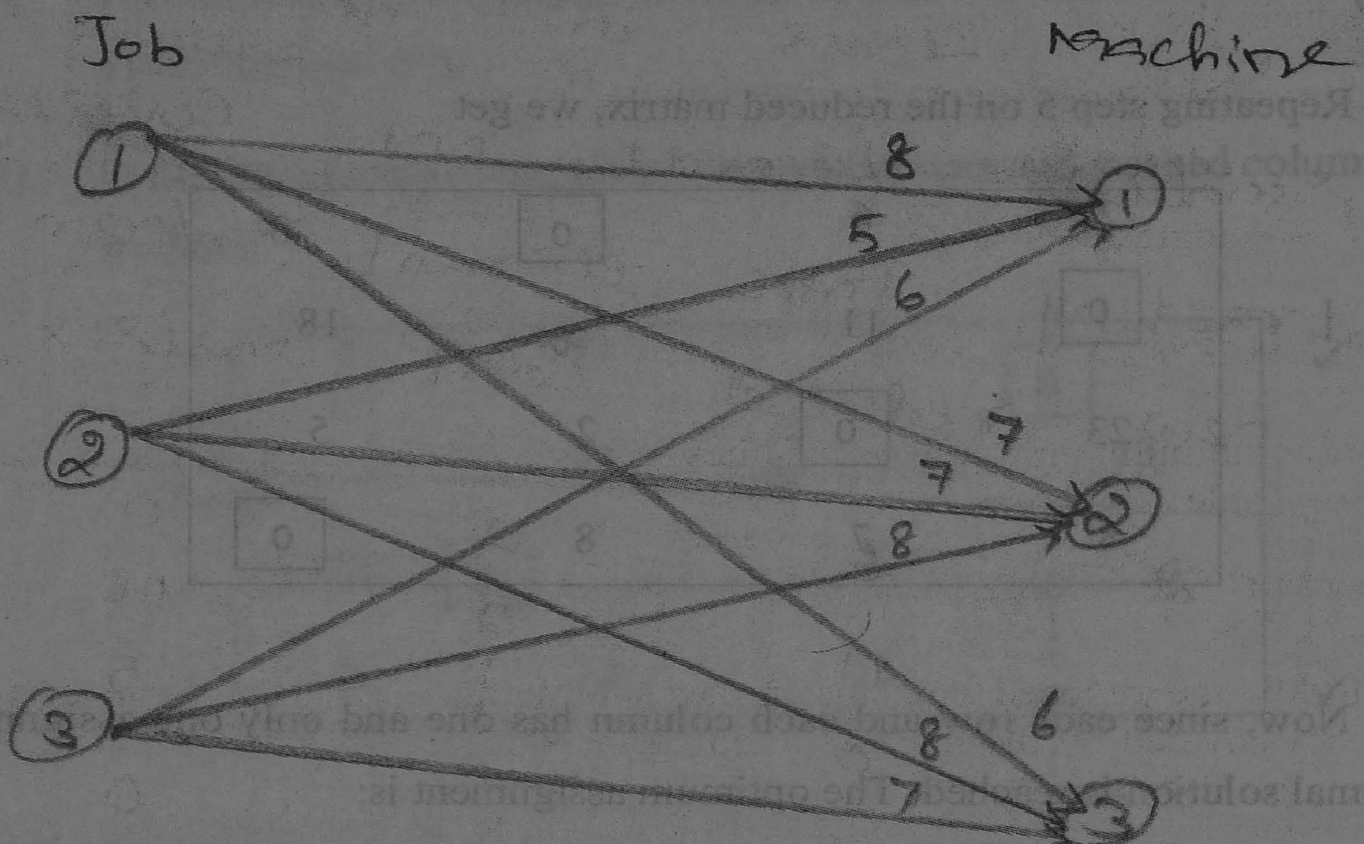
A Company wishes to assign 3 jobs to 3 machines in such a way that each job is assigned to some machine and no machine works on more than one job. The cost of assigning job i to machine j is given by the matrix below (ijth entry):

$$\text{Cost Matrix: } \begin{bmatrix} 8 & 7 & 6 \\ 5 & 7 & 8 \\ 6 & 8 & 7 \end{bmatrix}$$

Draw the associated network. Formulate the network LPP and find the minimum cost of making the assignment.

Solution:

a) Network formulation of the given problem is given as under:



b) Linear programming formulation of the given problem is minimize the total cost involved, i.e.,

Minimize Z =

$$(8x_{11} + 7x_{12} + 6x_{13}) + (5x_{21} + 7x_{22} + 8x_{23}) + (6x_{31} + 8x_{32} + 7x_{33})$$

Subject to the constraints:

$$x_{i1} + x_{i2} + x_{i3} = 1; \qquad i = 1, 2, 3.$$

$$x_{1j} + x_{2j} + x_{3j} = 1; \qquad j = 1, 2, 3.$$

$x_{ij} = 0$ or 1, for all i and j.

c) Reduce the cost matrix by subtracting smallest element of each row (column) from the corresponding row (column) elements. In the reduced matrix, make assignment in rows and columns having single zeros.

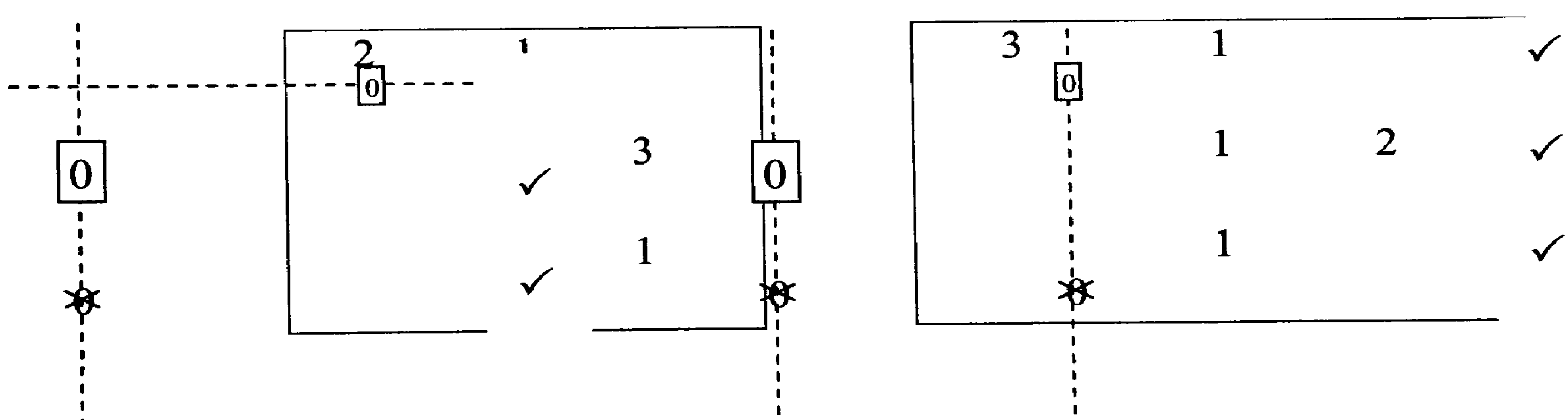
Initial Iteration:

Draw the minimum number of lines to cover all the zeros of the reduced matrix. See Table 1.

First Iteration:

Modify the reduced cost matrix by subtracting element 1 from all the elements not covered by the lines and adding the same at the intersection of two lines. See table 2.

Draw the minimum numbers of lines again.



Final Iteration:

Modify further the reduced table 2 by subtracting element 1 from all the elements not covered by the lines adding the same at the intersection of two lines. Thus, we get table 3

$$\begin{array}{|ccc|} 4 & \cancel{0} & \boxed{0} \\ \cancel{0} & \boxed{0} & 1 \\ \boxed{0} & \cancel{0} & \cancel{0} \end{array}$$

Or

$$\begin{array}{|ccc|} 4 & 1 & \boxed{0} \\ \boxed{0} & \cancel{0} & 1 \\ \cancel{0} & \boxed{0} & \cancel{0} \end{array}$$

Or

$$\begin{array}{|ccc|} 4 & \boxed{0} & \cancel{0} \\ \boxed{0} & \cancel{0} & 1 \\ \cancel{0} & \cancel{0} & \boxed{0} \end{array}$$

Table 3

Since the number of assignments is equal to the order of the matrix, we have the optimum assignment schedule:

Job 1 →Machine 3, Job 2 →Machine 2, Job 3→Machine 1; or

Job 1 →Machine 3, Job 2 →Machine 1, Job 3 →Machine 2; or

Job 1 →Machine 2, Job 2 →Machine 1, Job 3 →Machine 3

Total minimum cost is both the cases will be 19.

Example 9.2.3

A department head has four tasks to be performed and three subordinates, the subordinates differ in efficiency. The estimates of the time, each subordinate would take to perform, is given below in the matrix.

How should he allocate the tasks one to each man, so as to minimize the total man-hours?

Task	Men		
	1	2	3
I	9	26	15
II	13	27	6
III	35	20	15
IV	18	30	20

Solution:

Since the problem is unbalanced, we add a dummy column with all the entries as zero and use assignment methods for optimum solution.

Now reduce the balanced cost matrix and make assignments in rows and columns having single zeros. Thus, we have:

$\boxed{0}$	6	9	$\cancel{0}$
4	7	$\boxed{0}$	$\cancel{0}$
26	$\boxed{0}$	9	$\cancel{0}$
9	10	14	$\boxed{0}$

The optimum assignment is

I $\rightarrow$ 1, II $\rightarrow$ 3 and III $\rightarrow$ 2; while task IV should be assigned to a dummy man, i.e., it remains to be done.

The minimum time is 35 hours.

Example 9.1.4

The assignment cost of assigning any one operator to any one machine is given in the following table

		Operators			
		I	II	III	IV
	A	10	5	13	15
Machine	B	3	9	18	3
	C	10	7	3	2
	D	5	11	9	7

Find the optimal assignment by Hungarian method.

Solution: The cost matrix of the assignment problem is

$$\begin{pmatrix} 10 & 5 & 13 & 15 \\ 3 & 9 & 18 & 3 \\ 10 & 7 & 3 & 2 \\ 5 & 11 & 9 & 7 \end{pmatrix}$$

Since the number of rows is equal to the number of columns in the cost matrix, the given assignment problem is balanced.

Select the smallest cost element in each row and subtract this from all the elements of the corresponding row, we get the reduced matrix.

$$\begin{pmatrix} 5 & 0 & 8 & 10 \\ 0 & 6 & 15 & 0 \\ 8 & 5 & 1 & 0 \\ 0 & 6 & 4 & 2 \end{pmatrix}$$

Select the smallest cost element in each column and subtract this from all the element of the corresponding column, we get the reduced matrix

$$\begin{pmatrix} 5 & 0 & 7 & 10 \\ 10 & 6 & 14 & 0 \\ 8 & 5 & 0 & 0 \\ 0 & 6 & 3 & 2 \end{pmatrix}$$

Since each row and each column contain exactly one zero, we shall make the assignment in rows and columns of this reduced cost matrix.

5	$\boxed{0}$	7	10
$\cancel{0}$	6	14	$\boxed{0}$
8	5	$\boxed{0}$	$\cancel{0}$
$\boxed{0}$	6	3	2

Since each row and each column contain exactly one assignment (i.e., exactly one encircled zero), the current assignment is optimal.

The optimum assignment schedule is

$A \to II,\ \ B \to IV,\ \ C \to III,\ \ D \to I$

and the optimum (minimum) assignment cost

$= \text{Rs.}(5+3+3+5) = \text{Rs.}16/-$

Check your progress 9.1

1) Four professors are each capable of teaching any one of four different courses. Class preparation time in hours for different topics varies form professor to professor and is given in the table below. Each professor is assigned only one course. Determine an assignment schedule so as to minimize the total course preparation time for all' courses:

Professor	Linear programmes	Queueing theory	Dynamic programme	Regression analysis
A	2	10	9	7
B	15	4	14	8
C	13	14	16	11
D	4	15	13	9

2) An automobile dealer wishes to put four repairmen to four different jobs. The repairmen have somewhat different kinds of skills and they exhibit different levels of efficiency from one job to another. The dealer has estimated the number of man-hours that would be required for each job-man combination. This is given in the matrix form in the following table

Man	Job			
	A	B	C	D
1	5	3	2	8
2	7	9	2	6
3	6	4	5	7
4	5	7	7	8

Find the optimum assignment that will result in minimum man-hours needed.

3) Solve the following assignment problems:

(a)
$$\begin{array}{c|cccc} & A & B & C & D \\ \hline I & 1 & 4 & 6 & 3 \\ II & 9 & 7 & 10 & 9 \\ III & 4 & 5 & 11 & 7 \\ IV & 8 & 7 & 8 & 5 \end{array}$$

(b)
$$\begin{array}{c|cccc} & A & B & C & D \\ \hline I & 10 & 25 & 15 & 20 \\ II & 15 & 10 & 5 & 15 \\ III & 35 & 20 & 12 & 24 \\ IV & 17 & 25 & 24 & 20 \end{array}$$

$$\text{(c)}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 10 & 12 & 19 & 11 \\ 5 & 10 & 7 & 8 \\ 12 & 14 & 13 & 11 \\ 8 & 15 & 11 & 9 \end{bmatrix} \end{array} \qquad \text{(d)}\quad \begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array}\begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} 5 & 8 & 3 & 2 \\ 10 & 7 & 5 & 8 \\ 4 & 10 & 12 & 10 \\ 8 & 6 & 9 & 4 \end{bmatrix} \end{array}$$

9.3 Maximization case in Assignment problem

In an assignment problem, we may have to deal with maximization of an objective function. For example, we may have to assign persons to jobs in such a way that the total profit is maximized. The maximization problem has to be converted into an equivalent minimization problem and then solved by the usual Hungarian Method.

The conversion of the maximization problem into an equivalent minimization problem can be done by any one of the following methods:

(i) Since max Z = - min (-Z), multiply all the cost elements c_{ij} of the cost matrix by -1.

(ii) Subtract all the cost elements c_{ij} of the cost matrix from the highest cost element in that cost matrix.

Example 9.3.1

A company has a team of four salesmen and there are four districts where the company wants to start its business. After taking into account the capabilities of salesman and the nature of districts, the company estimates that the profit per day in rupees for each salesman in each district is as below:

	Districts			
	1	2	3	4
A	16	10	14	11

Salesmen	B	14	11	15	15
	C	15	15	13	12
	D	13	12	14	15

Find the assignment of salesmen to various districts which will yield maximum profit.

Solution: The cost matrix of the given assignment problem is

$$\begin{pmatrix} 16 & 10 & 14 & 11 \\ 14 & 11 & 15 & 15 \\ 15 & 15 & 13 & 12 \\ 13 & 12 & 14 & 15 \end{pmatrix}$$

Since this is a maximization problem, it can be converted into an equivalent minimization problem by subtracting all the cost elements in the cost matrix from the highest cost element 16 of this cost matrix. Thus the cost matrix of the equivalent minimization problem is

$$\begin{pmatrix} 0 & 6 & 2 & 5 \\ 2 & 5 & 1 & 1 \\ 1 & 1 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

Select the smallest cost element in each row (column) and subtract this from all the cost elements of the corresponding row (column). We get the reduced cost matrix

$$\begin{pmatrix} 0 & 6 & 2 & 5 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 2 & 3 & 1 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} 0 & 6 & 2 & 5 \\ 1 & 4 & 0 & 0 \\ 0 & (0) & 2 & 3 \\ 2 & 3 & 1 & 0 \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by

$A \to 1,\ B \to 3,\ C \to 2,\ D \to 4$ and the optimum (maximum) profit

$$= \text{Rs. } (16+15+15+15)$$

$$= \text{Rs. } 61/-$$

Example 9.3.2

Solve the assignment problem for maximization given the profit matrix (Profit in rupees).

Machines

		P	Q	R	S
	A	51	53	54	50
	B	47	50	48	50
Job	C	49	50	60	61
	D	63	64	60	60

Solution: The profit matrix of the given assignment problem is

$$\begin{pmatrix} 51 & 53 & 54 & 50 \\ 47 & 50 & 48 & 50 \\ 49 & 50 & 60 & 61 \\ 63 & (64) & 60 & 60 \end{pmatrix}$$

Since this is a maximization problem, it can be converted into an equivalent minimization problem by subtracting all the profit elements in the profit matrix from the highest profit element 64 of this profit matrix. Thus the cost matrix of the equivalent minimization problem is

$$\begin{pmatrix} 13 & 11 & 10 & 14 \\ 17 & 14 & 16 & 14 \\ 15 & 14 & 4 & 3 \\ 1 & 0 & 4 & 4 \end{pmatrix}$$

Select the smallest cost in each row and subtract this from all the cost elements of the corresponding row. We get

$$\begin{pmatrix} 3 & 1 & 0 & 4 \\ 3 & 0 & 2 & 0 \\ 12 & 11 & 1 & 0 \\ 1 & 0 & 4 & 4 \end{pmatrix}.$$

Select the smallest cost element in each column and subtract this from all the cost elements of the corresponding column. We get

$$\begin{pmatrix} 2 & 1 & 0 & 4 \\ 2 & 0 & 2 & 0 \\ 11 & 11 & 1 & 0 \\ 0 & 0 & 4 & 4 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} 2 & 1 & (0) & 4 \\ 2 & \boxed{0} & 2 & \cancel{0} \\ 11 & 11 & 1 & \boxed{0} \\ (0) & \cancel{0} & 4 & 4 \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by $A \rightarrow R,\ B \rightarrow Q,\ C \rightarrow S,\ D \rightarrow P$ and the optimum (maximum) profit

=Rs. (54+50+61+63)

=Rs. 228/-

Example 9.3.3

A company is faced with the problem of assigning four different salesman to four territories for promoting its sales. Territories are not equally rich in their sales potential and the salesman also differ in their ability to promote sales. The following table gives the expected annual sales (in thousands of Rs) for each salesman if assigned to various territories. Find the assignment of salesman so as to maximize the annual sales.

Territories

		1	2	3	4
	1	60	50	40	30
Salesmen	2	40	30	20	15
	3	40	20	35	10
	4	30	30	25	20

Solution: The cost matrix of the given assignment problem is

$$\begin{pmatrix} (60) & 50 & 40 & 30 \\ 40 & 30 & 20 & 15 \\ 40 & 20 & 35 & 10 \\ 30 & 30 & 25 & 20 \end{pmatrix}$$

Since this is a maximization problem, it can be converted into an equivalent minimization problem by subtracting all the cost elements in the cost matrix from the highest cost element 60 of this cost matrix. Thus the cost matrix of the equivalent minimization problem is

$$\begin{pmatrix} 0 & 10 & 20 & 30 \\ 20 & 30 & 40 & 45 \\ 20 & 40 & 25 & 50 \\ 30 & 30 & 35 & 40 \end{pmatrix}$$

Select the smallest cost element in each row (column) and subtract this from all the cost elements of the corresponding row (column). We get the reduced cost matrix

$$\begin{pmatrix} 0 & 10 & 15 & 20 \\ 0 & 10 & 15 & 15 \\ 0 & 20 & 0 & 20 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we make the assignment in rows and columns of this reduced cost matrix

$$\begin{pmatrix} \boxed{0} & 10 & 15 & 20 \\ \cancel{0} & 10 & 15 & 15 \\ \cancel{0} & 20 & \boxed{0} & 20 \\ \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since there are some rows and columns without assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines

$$\begin{array}{cc} \begin{pmatrix} 0 & 10 & 15 & 20 \\ 0 & \boxed{10} & 15 & 15 \\ 0 & 20 & 0 & 20 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} \checkmark \\ \checkmark \\ \\ \\ \end{array} \end{array}$$

Here 10 is the smallest cost element not covered by these straight lines. Subtract this 10 from all the uncovered elements, add this 10 to those elements which lie in the intersection of these straight lines and do not change the remaining elements which lie on these straight lines. We get

$$\begin{pmatrix} 0 & 0 & 5 & 10 \\ 0 & 0 & 5 & 5 \\ 10 & 20 & 0 & 20 \\ 10 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we make the assignment in rows and columns of this reduced cost matrix.

$$
\begin{array}{|cccc|}
\boxed{0} & \cancel{0} & 5 & 10 \\
\cancel{0} & \boxed{0} & 5 & 5 \\
10 & 20 & \boxed{0} & 20 \\
10 & \cancel{0} & \cancel{0} & \boxed{0}
\end{array}
$$

Since each row and each column contains exactly one assignment (i.e., exactly one encircled zero), the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by Salesman 1 $\rightarrow$ Territory 1, Salesman 2 $\rightarrow$ Territory 2, Salesman 3 $\rightarrow$ Territory 3, Salesman 4 $\rightarrow$ Territory 4.

The optimum (maximum) annual sales

= 60+30+35+20 (in thousand of rupees)

= 145 (in thousand of rupees)

= Rs. 1,45,000/-.

Note: For this problem, there exists alternative optimal assignment schedule with the same maximum sales Rs. 1,45,000/-.

Check your progress 9.2

1) MCS Inc is a software company that has three projects of Y2K with the department of health, education, and housing of Maharashtra Government. Based on the background and experiences of the project leaders, they differ in terms of their performance at various projects. The performance score matrix is given below:

Project Leaders	**Projects**		
	Health	**Education**	**Housing**
P_1	20	26	42
P_2	24	32	50
P_3	32	34	44

Help the management by determining the optimal assignment that maximises the total performance score.

2) (a) Suggest the optimal assignment schedule for the following assignment problem :

Sales (Rs. in lakh)

Salesmen	Markets			
	I	II	III	IV
A	80	70	75	72
B	75	75	80	85
C	78	78	82	78

What will be the total maximum sale?

3) (b) Solve the following assignment problem to find the maximum total expected sale

$$\text{Salesman}\quad \begin{array}{c c} \text{Area} & \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 42 & 35 & 28 & 21 \\ 30 & 25 & 20 & 15 \\ 30 & 25 & 20 & 15 \\ 24 & 20 & 16 & 12 \end{bmatrix} \end{array}$$

9.4 UNBALANCED ASSIGNMENT MODELS

If the number of rows is not equal to the number columns in the cost matrix of the given assignment problem, then the given assignment problem said to be unbalanced.

First convert the unbalanced assignment problem in to a balanced one by adding dummy rows or dummy columns with zero cost elements in the cost matrix depending upon whether $m < n$ or $m > n$ and then solve by the usual method.

Example 9.4.1:

A company has four machines to do three jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table.

Machines

Jobs		1	2	3	4
	A	18	24	28	32
	B	8	13	17	19
	C	10	15	19	22

What are job assignments which will minimize the cost?

Solution:

The cost matrix of the given assignment problem is

$$\begin{pmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 19 \\ 10 & 15 & 19 & 22 \end{pmatrix}$$

Since the number of rows is less than the number of columns in the cost matrix, the given assignment problem is unbalanced.

To make it a balanced one, add a dummy job D (row) with zero cost elements. The balanced cost matrix is given by

$$\begin{pmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 19 \\ 10 & 15 & 19 & 22 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now select the smallest cost element in each row (column) and subtract this from all elements of the corresponding row (column), we get the reduced matrix

$$\begin{pmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & 5 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In this reduced matrix, we shall make the assignment in rows and columns having single zero. We have

$\boxed{0}$	6	10	14
$\cancel{0}$	5	9	11
$\cancel{0}$	5	9	12
$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

Since there are some rows and columns without assignment, the current assignment is not optimal.

Cover the all zeros by drawing a minimum number of straight lines. Choose the smallest cost element not covered by these straight lines.

$$\begin{pmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & 5 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Here 5 is the smallest cost element not covered by these straight lines. Subtract this 5 from all the uncovered elements, add this 5 to those elements which lie in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines. We get

$$\begin{pmatrix} 0 & 1 & 5 & 9 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 4 & 7 \\ 5 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make assignment in the rows and columns having single zero. We get

$$\begin{array}{|cccc|} \hline \boxed{0} & 1 & 5 & 9 \\ \cancel{0} & (0) & 4 & 6 \\ \cancel{0} & \cancel{0} & 4 & 7 \\ 5 & \cancel{0} & \boxed{0} & \cancel{0} \\ \hline \end{array}$$

Since there are some rows and columns without assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

$$\begin{pmatrix} 0 & 1 & 5 & 9 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & (4) & 7 \\ 5 & 0 & 0 & 0 \end{pmatrix}$$

Choose the smallest cost element not covered by these straight lines, subtract this from all the uncovered elements, add this to those elements which are in the intersection of the lines and do not change the remaining elements which lie on these straight lines. Thus we get

$$\begin{pmatrix} 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 9 & 4 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in the rows and columns having single zero. We get

$\boxed{0}$	1	1	5
$\cancel{0}$	(0)	$\cancel{0}$	2
$\cancel{0}$	$\cancel{0}$	(0)	3
9	4	$\cancel{0}$	$\boxed{0}$

Since each row and contains exactly one assignment (i.e., exactly one encircled zero) the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by $A \rightarrow 1,\ B \rightarrow 2,\ C \rightarrow 3,\ D \rightarrow 4$ and the optimum (minimum) assignment cost $=(18+17+15+0)= 50/-$, units of cost.

Note: 1

For this problem, the alternative optimum schedule is $A \rightarrow 1,\ B \rightarrow 3,$ $C \rightarrow 2,\quad D \rightarrow 4$ with the same optimum assignment cost $= \text{Rs.}(18+17+15+0) = 50/-$units of cost.

Note: 2

Here the assignment $D \rightarrow 4$ means that the dummy Job D is assigned to the 4^{th} Machine. It means that machine 4 is left without any assignment.

Example 9.4.2

Assign four trucks 1, 2, 3 and 4 to vacant spaces A, B, C, D, E and F so that the distance travelled is minimized. The matrix below shows the distance.

	1	2	3	4
A	4	7	3	7
B	8	2	5	5
C	4	9	6	9
D	7	5	4	8
E	6	3	5	4
F	6	8	7	3

Solution

The matrix of the assignment problem is

$$\begin{pmatrix} 4 & 7 & 3 & 7 \\ 8 & 2 & 5 & 5 \\ 4 & 9 & 6 & 9 \\ 7 & 5 & 4 & 8 \\ 6 & 3 & 5 & 4 \\ 6 & 8 & 7 & 3 \end{pmatrix}$$

Since the number of rows is more than the number of columns, the given assignment problem is unbalanced. To make it balanced, let us introduce two dummy trucks (columns) with zero costs. We get

$$\begin{pmatrix} 4 & 7 & 3 & 7 & 0 & 0 \\ 8 & 2 & 5 & 5 & 0 & 0 \\ 4 & 9 & 6 & 9 & 0 & 0 \\ 7 & 5 & 4 & 8 & 0 & 0 \\ 6 & 3 & 5 & 4 & 0 & 0 \\ 6 & 8 & 7 & 3 & 0 & 0 \end{pmatrix}$$

Select the smallest cost element in each row (column) and subtract this from all the elements of the corresponding row (column). We get

$$\begin{pmatrix} 0 & 5 & 0 & 4 & 0 & 0 \\ 4 & 0 & 2 & 2 & 0 & 0 \\ 0 & 7 & 3 & 6 & 0 & 0 \\ 3 & 3 & 1 & 5 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 0 \\ 2 & 6 & 4 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we make the assignment in rows and columns having single zero. We get

$$\begin{array}{|cccccc|} \hline \boxed{0} & 5 & \boxed{0} & 4 & \boxed{0} & \boxed{0} \\ 4 & & 2 & 2 & & \\ & \boxed{0} & 3 & 6 & \cancel{0} & \cancel{0} \\ \boxed{0} & 3 & 1 & 5 & \cancel{0} & \cancel{0} \\ 2 & 1 & 2 & 1 & & \\ 2 & 6 & 4 & & \boxed{0} & \cancel{0} \\ & & & & \cancel{0} & \boxed{0} \\ \hline & & & \boxed{0} & \cancel{0} & \end{array}$$

Since each row and each column contains exactly one assignment (i.e., exactly one encircled zero), the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by $A \to 3,\ B \to 2,\ C \to 1,\ D \to 5,\ E \to 6,\ F \to 4,$ and the optimum (minimum) distance.

$$= (3 + 2 + 4 + 0 + 0 + 3)$$

Units of distance $= 12/-$ units of distance.

Example: 9.4.3

A batch of 4 jobs can be assigned to 5 different machines. The set up time (in hours) for each job on various machines is given below:

Machine Job

	1	2	3	4	5
1	10	11	4	2	8
2	7	11	10	14	12
3	5	6	9	12	14
4	13	15	11	10	7

Find an optimal assignment of jobs to machines which will minimize the total set up time.

Solution:

The matrix of the given assignment problem is

$$\begin{pmatrix} 10 & 11 & 4 & 2 & 8 \\ 7 & 11 & 10 & 14 & 12 \\ 5 & 6 & 9 & 12 & 14 \\ 13 & 15 & 11 & 10 & 7 \end{pmatrix}$$

Since the number of rows is less than the number of columns in the cost matrix, the given assignment problem is unbalanced.

To make it a balanced one, add a dummy job 5 (row) with zero cost elements. The balanced cost matrix is given by

$$\begin{pmatrix} 10 & 11 & 4 & 2 & 8 \\ 7 & 11 & 10 & 14 & 12 \\ 5 & 6 & 9 & 12 & 14 \\ 13 & 15 & 11 & 10 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now select the smallest cost element in each row (column) and subtract this from all the elements of the corresponding row (column). We get the reduced cost matrix.

$$\begin{pmatrix} 8 & 9 & 2 & 0 & 6 \\ 0 & 4 & 3 & 7 & 5 \\ 0 & 1 & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each columns contains atleast one zero, we shall make the assignment in rows and columns of this reduced cost matrix

$$\begin{array}{|ccccc|} \hline 8 & 9 & 2 & \boxed{0} & 6 \\ \boxed{0} & 4 & 3 & 7 & 5 \\ \cancel{0} & 1 & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & \boxed{0} \\ \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \hline \end{array}$$

Since there are some rows and columns with out assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

$$
\begin{array}{c c}
\begin{pmatrix}
8 & 9 & 2 & 0 & 6 \\
0 & 4 & 3 & 7 & 5 \\
0 & (1) & 4 & 7 & 9 \\
6 & 8 & 4 & 3 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
&
\begin{array}{c}
\\
\checkmark \\
\checkmark \\
\\
\\
\end{array}
\end{array}
$$

Here 1 is the smallest cost element not covered by these straight lines. Subtract this 1 from all the uncovered elements, and this 1 to those elements which lie in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines. We get

$$
\begin{pmatrix}
9 & 9 & 2 & 0 & 6 \\
0 & 3 & 2 & 6 & 4 \\
0 & 0 & 3 & 6 & 8 \\
7 & 8 & 4 & 3 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns of this reduced cost matrix

$$
\begin{array}{|ccccc|}
\hline
9 & 9 & 2 & \boxed{0} & 6 \\
\boxed{0} & 3 & 2 & 6 & 4 \\
\cancel{0} & \boxed{0} & 3 & 6 & 8 \\
7 & 8 & 4 & 3 & \boxed{0} \\
1 & \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} \\
\hline
\end{array}
$$

Since each row and each column contains exactly one assignment (i.e., exactly one encircled zero), the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by Job $1 \to M/c\ 4$, Job $2 \to M/c\ 1$, Job $3 \to M/c\ 2$, Job $4 \to M/c3$ is left without any assignment.

The optimum (minimum) total set up time

$$= 2 + 7 + 6 + 7 \text{ hours}$$

$$= 22 \text{ hours}.$$

Check your progress 9.3

1) A department head has four tasks to be performed by three subordinates, the subordinates differing in efficiency. The estimates of the time, each subordinate would take to perform, is given below in the matrix. How should he allocate the tasks one to each to each man, so as to minimize the total man-hour?

		Men		
		1	2	3
	I	9	26	15
Tasks	II	13	27	6
	III	35	20	15
	IV	18	30	20

2) A company has four machines on which to do three jobs. Each job can be assigned to one and only machine. The cast of each job on each machine is given in the following table:

		Machine			
		P	Q	R	S
Job	A	18	24	28	32
	B	8	13	17	19
	C	10	15	19	22

What are the job assignments which will minimize the cost?

3) Solve the following unbalanced problem of assigning four jobs to three different men (only one job to each man). The time to perform the job by different men is given in the following table:

		Job			
		J_1	J_2	J_3	J_4
Men	M_1	7	5	8	4
	M_2	5	6	7	4
	M_3	8	7	9	8

		I	II	III	IV
	A	42	35	28	21
lesman	B	30	25	20	15
	C	30	25	20	15
	D	24	20	16	12

1) A Sales manager has to assign salesman to four territories. He has 4 candidates of varying experience and capabilities and assesses the possible profit for each salesman in each territory as given below. Find the assignment which maximizes the profit.

		Territory			
		A	B	C	D
Salesman	1	35	27	28	37
	2	28	34	29	40
	3	35	24	32	28
	4	24	32	25	28

2) Solve the following assignment problem

		Task				
		A	B	C	D	E
	M_1	4	6	10	5	6
Machine	M_2	7	4	Not Suitable	5	4
	M_3	Not suitable	6	9	6	2
	M_4	9	3	7	2	3

3) Solve the following assignment problem

		Machine				
		1	2	3	4	5
	A	7	7	∞	4	8
	B	9	6	4	5	6
Task	C	11	5	7	∞	5
	D	9	4	8	9	4
	E	8	7	9	11	3

4) Given the following matrix of setup costs show how to sequence production so as to minimize setup cost per cycle.

		To				
		A	B	C	D	E
	A	–	2	5	7	1
	B	6	–	3	8	2
From	C	8	7	–	4	7
	D	12	4	6	–	5
	E	1	3	2	8	–

5) Solve the following travelling salesman problem:

	A	B	C	D	E	F
A	∞	5	12	6	4	8
B	6	∞	10	5	4	3
C	8	7	∞	6	3	11
D	5	4	11	∞	5	8
E	5	2	7	8	∞	4
F	6	3	11	5	4	∞

6) Solve the travelling salesman problem given by the following data.

$$c_{12} = 20,\ c_{13} = 4,\ c_{14} = 10,\ c_{23} = 6,$$

$$c_{25} = 10,\ c_{35} = 6,\ c_{45} = 20,\ \text{where}\, c_{ij} = c_{ji}$$

and there is not route between curies i and j if a value for c_{ij} is not shown above.

7) A company has five jobs to be done one five machines; any job can be done on any machine. The costs of doing the jobs in different machines are given below. Assign the jobs for different machines so as to minimize the total cost.

Machines

		A	B	C	D	E
Jobs	I	13	8	16	18	19
	II	9	15	24	9	12
	III	12	9	4	4	4
	IV	6	12	10	8	13
	V	15	17	18	12	20

8) The owner of a small machine shop has four machinists available to assign to jobs for the day. Five jobs are offered with expected profit for each machinist on each job as follows:

Jobs

		A	B	C	D	E
Machinist	M_1	12	28	0	51	32
	M_2	12	34	11	23	9
	M_3	37	42	61	21	31
	M_4	0	14	37	27	31

Assign machinists to jobs which results in overall maximum profit. Which job should be declined?

UNIT X

SEQUENCING PROBLEMS

10.0 Introduction:

The selection of an appropriate order for a series of jobs to be done on a finite number of service facilities, in some pre – assigned order, is called sequencing. A practical situation may correspond to an industry producing a number of products, each of which is to be processed through different machines, of course, finite in number.

Problem of Sequencing:

Consider a problem of machine operator who has to perform three operations, namely (i) turning, (ii) threading, and (iii) knurling on a finite number of different jobs. Let there be six jobs and the time required to perform these operations/(in minutes) for each job is known. Also it is given that each job first goes for turning, then for threading, and lastly for knurling. The problem of the machine operator is to decide which job should be for knurling. The problem of the machine operator is to decide which job should be processed first, which to process next, and so on, i.e., the order (sequence) of the jobs for the above – mentioned operations in order to minimize the total time required to turn out all the jobs. This is an example of six – job and three – machine sequencing problem. We now consider the general case.

The general sequencing problem may be defined as: Let there be n jobs to be performed one at a time on each of m machines. The sequence (order) of the machines in which each job should be performed is given. The actual or expected time required by the jobs on each of the machine is also given. The general sequencing problem, therefore, is to find the sequence out of $(n!)^m$ possible sequences which minimize the total elapsed time between the start of the job in the first machine and the completion of the last job on the last machine.

Given below are the assumptions underlying a sequencing problem:

1. Each job, once started on a machine, is to be performed up to completion on that machine.
2. The processing time on each machine is known. Such a time is independent of the order of the jobs in which they are to be processed.
3. The time taken by each job in changing over from one machine to another is negligible.
4. A job starts on the machines as soon as the job and the machine both are idle and job is next to the machine and the machine is also next to the job.
5. No machine may process more than one job simultaneously.
6. The order of completion of job has no significance, i.e., no job is to be given priority. The order of completion of jobs is independent of sequence of jobs.

Basic Terms used in Sequencing: Some of the basic terms in sequencing are:

1. **Number of machines:** It refers to the number of service facilities through which a job must pass before it is assumed to be completed.
2. **Processing order:** It refers to the order (sequence) in which given machines are required for completing the job.
3. **Processing time:** It indicates the time required by a job on each machine.
4. **Total elapsed time:** It is the time interval between starting the first job and completing the last job including the idle time (if any) in a particular order by the given set of machines.
5. **Idle time on a machine:** It is the time for which a machine does not have a job to process, i.e., idle time from the end of job $(i-1)$ to the start of job i.
6. **No passing rule:** It refers to the rule of maintaining the order in which jobs are to be processed on given machines. For example, if n jobs are to be processed through three machines M_1, M_2 and M_3 in the order M_1, M_2 and M_3, then this rule will mean that each job will go to machine M_1 first, then to M_2 and lastly to M_3.

10.1 Processing n jobs on two machines

Step – wise procedure

Let $A_{11}, A_{21}, A_{31}, \ldots\ldots, A_{n1}$ be processing times of n jobs on Machine 1 and let $A_{12}, A_{22}, A_{32}, \ldots\ldots, A_{n2}$ be the processing times of n jobs on Machine 2.

Step: 1

Find Min (A_{i1}, A_{i2}) $i = 1, 2, \ldots\ldots, n$.

Step: 2

(a) If this minimum be A_{l1} for $i = \ell$ process the ℓ^{th} job first

(b) If this minimum be A_{m2} for some $i = m$ process the m^{th} job last of all.

Step: 3

(a) If there is a tie, i.e., $A_{l1} = A_{m2}$ process the ℓ^{th} job first and m^{th} job in the last.

(b) If the tie for the minimum occurs among the $A_{i1} - s$, choose the job corresponding to the minimum of $A_{i2} - s$ and process it first of all

(c) If the tie for minimum occurs among the $A_{i2} - s$ choose the job corresponding to the minimum of $A_{i1} - s$ and process it in the last.

Step: 4

Cancel the jobs already assigned and repeat steps 1 to 3 until all the jobs have been assigned.

Example: 10.1

There are five jobs, each of which is to be processed through two machines M_1, M_2 in the order $M_1 M_2$. Processing hours are as follows:

Job	1	2	3	4	5
M_1	3	8	5	7	4
M_2	4	10	6	5	8

Determine the optimum sequence for the five jobs, and minimum total elapsed time. Find also the idle time of machines M_1 and M_2.

Solution:

Machine/Job No.	M_1	M_2	Order of cancellation
1	3	4	(1)
2	8	10	(5)
3	5	6	(3)
4	7	5	(4)
5	4	8	(2)

Optimal sequence

1	5	3	2	4

Explanation:

The smallest entry is 3 and it is against job 1 occurring under machine 1. $\therefore$ Process job 1 first

Next smallest entry is 4 against the job 5 under machine M_1, so next do the job 5. Now the smallest entry is 5. But there is a tie. 5 occurs against job 3 and 4. But job 3 is chosen since the entry corresponding to 5 under machine 2 is 6 smaller than 7 found for job 4 under Machine 1.

Obviously job 4 is chosen next and lastly the job 2 is chosen.

To find the minimum total lapsed time

Job	M_1		M_2	
No	Time in	Time out	Time in	Time out
1	0	3	3	7
5	3	7	7	15
3	7	12	15	21
2	12	20	21	31
4	20	27	31	36

$\therefore$ Minimum total elapsed time $= 36$ hrs.

Idle time for machine 1 $= 36 - 27 = 9$ hrs.

Idle time for machine 2 $= 3$ hrs.

Example: 10.2

Find the sequence that minimizes the total elapsed time required to complete the following tasks on machine M_1 and M_2 in the order M_2 in the order M_1, M_2. Also, find the minimum total elapsed time.

Optimal sequenc

Task	A	B	C	D	E	F	G	H	I
M_1	2	5	4	9	6	8	7	5	4
M_2	6	8	7	4	3	9	3	8	1

Solution:

Machine/Task	M_1	M_2	Order of cancellation
A	2	6	(1)
B	5	8	(7)
C	4	7	(4)
D	9	4	(6)
E	6	3	(2)
F	8	9	(9)
G	7	3	(3)
H	5	8	(8)
I	4	11	(5)

A	C	I	B	H	F	D	G	E

To find the minimum total lapsed time.

Job	M_1		M_2	
No	Time in	Time out	Time in	Time out
A	0	2	2	8
C	2	6	8	15
I	6	10	15	26
B	10	15	26	34
H	15	20	34	42
F	20	28	42	51
D	28	37	51	55
G	37	44	55	58
E	44	50	58	61

$\therefore$ Minimum total elapsed time $= 61$ time units.

Note: Idle time machine M_1 $= 11$ time units.

Idle time of machine M_2 $= 2$ time units.

Is the sequence ACIHBFDGE Optimal?

10.2 Processing n jobs on three machines

Let A_1, A_2, A_3 be the three machines. Let the order of operations be $A_1\ A_2\ A_3$. This problem can be converted into a two machine problem if any one of the following conditions is satisfied.

Condition (1) $\underset{i}{\text{Min}}\ A_{i1} \geq \underset{i}{\text{Max}}\ A_{i2}$ or (2) $\underset{i}{\text{Min}}\ A_{i3} \geq \underset{i}{\text{Max}}\ A_{i2}$

The method fails if none of these conditions is satisfied.

If one of the conditions is satisfied, we introduce two fictitious machines H and K in such a way that the processing times on H and K are given by

$$H_i = A_{i1} + A_{i2}, \qquad i = 1,2,3,\ldots\ldots\ldots,n$$

$$K_i = A_{i2} + A_{i3}, \qquad i = 1,2,3,\ldots\ldots\ldots,n$$

Now proceed to determine the optimal sequence using Johnson's Method.

Example: 10.3

Find the sequence that minimizes the total elapsed time required to complete the following tasks on the machines in the order 1 – 2 – 3. Find also the minimum total elapsed time (hours) and the idle times on the machines.

Task	A	B	C	D	E	F	G
Time on							
Machine 1	3	8	7	4	9	8	7
Machine 2	4	3	2	5	1	4	3
Machine 3	6	7	5	11	5	6	12

Solution:

Min. time on Machine 3 is 5.

Max time on Machine 2 is 5.

Min. time on machine 3 $\geq$ max. time on machine 2.

$\therefore$ The required condition to convert this into a two machine problem is satisfied.

Let H and K be the two fictitious machines. Then

Machine/Task	H	K	Order of cancellation
A	7	10	(2)
B	11	10	(6)
C	9	7	(3)
D	9	16	(4)
E	10	6	(1)
F	12	10	(7)
G	10	15	(5)

Optimal sequence

A	D	G	F	B	C	E

To find the total elapsed time

	Machine 1		Machine 2		Machine 3	
Job	Time in	Time out	Time in	Time out	Time in	Time out
A	0	3	3	7	7	13
D	3	7	7	12	13	24
G	7	14	14	17	24	36
F	14	22	22	26	36	42
B	22	30	30	33	42	49
C	30	37	37	39	49	54
E	37	46	46	47	54	$\boxed{59}$

The minimum total Elapsed Time $= 59$ hrs.

Idle time on Machine 1 $= 59 - 46 = 13$ hrs.

Idle time on Machine 2 $= 3 + 2 + 5 + 4 + 4 + 7 + 12 = 37$ hrs.

Idle time on Machine 3 $= 7$ hrs.

Example: 10.4

Find the sequence that minimizes the total elapsed time required to complete the following jobs on machines M_1, M_2 and M_3 in the order M_1, M_2, M_3.

Task	A	B	C	D	E	F
M_1	8	3	7	2	5	1
M_2	3	4	5	2	1	6
M_3	8	7	6	9	10	9

Solution:

Min Processing time on $M_1 = 1$

Max Processing time on $M_2 = 6$

Min Processing time on $M_3 = 6$

Min Processing time on $M_3 \geq$ Max Processing time on M_2

∴ The required condition to reduce this three machine problem to a two machine problem is satisfied.

Let A and K be two fictitious machines such that

$$H_i = M_{i1} + M_{i2}$$

$$K_i = M_{i2} + M_{i3}$$

Where $i = A, B, C, D, E$ and F and $H_i, M_{i1}, M_i, K_i, M_{i2}, M_{i3}$ are all processing times on machines H, M_1, M_2, K_3 and M_3:

	Machines	
Job	$H_i = M_{i1} + M_{i2}$	$K_i = M_{i2} + M_{i3}$
A	11	11
B	7	11
C	12	11
D	4	11
E	6	11
F	7	15

Optimal Sequence

D	E	B	F	C	A

10.3 Processing n jobs on m machines

Let there be m machines $A_1, A_2, \ldots\ldots, A_m$. This problem can be converted to a two machine problem if one of the following conditions is satisfied.

Let $A_{i1}, A_{i2}, \ldots\ldots\ldots, A_{im}$ be the processing times on machines

$A_1, A_2, \ldots\ldots, A_m$ respectively.

$$\text{Then if } \min_i A_{i1} \geq \max_i A_{ij}, \; j = 2, 3, \ldots\ldots, m-1$$

$$\text{or} \quad \min_i A_{im} \geq \max_i A_{ij}, \; j = 2, 3, \ldots\ldots, m-1$$

Then this problem can be converted to a two machine problem.

Introduce two fictitious machines H and K such that

$$H_i = A_{i1} + A_{i3} + \ldots\ldots.. + A_{im-1}$$

$$K_i = A_{i2} + A_{i3} + \ldots\ldots.. + A_{im} \qquad i = 1,2,3,\ldots.,n$$

Where H_i and K_i are the processing times for job i on machines H and K respectively.

Example: 10.5

Solve the following sequencing problem giving an optimal solution if passing is not allowed.

		Machines			
		M_1	M_2	M_3	M_4
	A	13	8	7	14
Jobs	B	12	6	8	19
	C	9	7	8	15
	D	8	5	6	15

Solution:

$$\text{Min } M_{i1} \geq \text{Max } M_{ij},\ j = 2,3$$

Even $\text{Min } M_{i4} \geq \text{Max } M_{ij},\ j = 2,3.$

Hence the required condition is satisfied.

$$H_i = M_{i1} + M_{i2} + M_{i3}$$

$$K_i = M_{i2} + M_{i3} + M_{i4}$$

Machine/Task	H	K	Order of cancellation
A	28	29	(4)
B	26	33	(3)
C	24	30	(2)
D	19	26	(1)

	M_1		M_2		M_3		M_4	
Job	Time in	Time out	Time in	Time out	Time in	Time out	Time in	Time out
D	0	8	8	13	**13**	19	**19**	34
C	8	17	**17**	24	**24**	32	34	49
B	17	29	**29**	35	**35**	43	49	68
A	29	**42**	**42**	**50**	**50**	**57**	68	$\boxed{82}$

Optimal sequence

D	C	B	A

Minimum total elapsed Time $= 82$ time units

Idle time on $M_1 = 82 - 42 = 40$ time units

Idle time on $M_2 = 8 + 4 + 5 + 7 + 32 = 56$ time units

Idle time on $M_3 = 13 + 5 + 3 + 7 + 25 = 53$ time units

Idle time on $M_4 = 19$ time units

Example: 10.6

Solve the following sequencing problem of 4 jobs on 6 machines (Processing time in hrs) Machines:

	Machines					
Job	M_1	M_2	M_3	M_4	M_5	M_6
A	19	8	8	3	11	24
B	18	6	9	6	9	18
C	12	5	8	5	7	15
D	20	5	3	4	8	11

Solution:

Solving a sequencing problem means determining (i) the optimal sequence (ii) Minimum total time and (iii) idle times of all the machines.

$$\text{Min} \quad M_{i1} = 12, \qquad \text{Min } M_{i6} = 11$$

$$\text{Max of } M_{i2} = 8$$

$$\text{Max} \quad M_{i3} = 9$$

$$\text{Max} \quad M_{i4} = 6$$

$$M_{i5} = 11$$

The required condition to reduce this problem to a two machine problem is satisfied.

Let H and K be two fictitious machines.

$$H_i = M_{i1} + M_{i2} + M_{i3} + M_{i4} + M_{i5}$$

$$K_i = M_{i2} + M_{i3} + M_{i4} + M_{i5} + M_{i6}$$

Job	H	K	Order of cancellation
A	49	54	(4)
B	48	48	(3)
C	37	40	(2)
D	40	31	(1)

Optimal sequence

C	B	A	D

To find Total elapsed time

	M_1		M_2		M_3		M_4		M_5		M_6	
Job	T1	To	T1	To	T1	To	T1	To	T1	To	T1	To
C	0	12	12	17	17	25	25	30	30	37	37	52
B	12	30	30	36	36	45	45	57	57	60	60	78
A	30	49	49	57	57	65	65	68	68	79	79	103
D	49	69	69	74	74	77	77	81	81	89	103	**114**

Minimum Total elapsed time $= 114$ hrs.

T_1 = Time in

To = Time out

Idle time:

M_1 : 45hrs; M_2 : 90hrs; M_3 : 86hrs; M_4 : 96hrs; M_5 : 79hrs; M_6 : 46hrs;
Is CABD an optimal sequence.

Check your progress:

1. We have five jobs, each of which must go through the two machines A and B in the order AB. Processing times in hours are given in the table below:

Job (i)	1	2	3	4	5
Machine A (A_i)	5	1	9	3	10
Machine B (B_i)	2	6	7	8	4

Determine a sequence for the five jobs that will minimize the elapsed time.

2. Six jobs go first over machine I and then over machine II. The order of the completion of jobs has no significance. The following table gives the machine time in hours for six jobs and the two machines:

Job No.	1	2	3	4	5	6
Time on Machine 1 (A_i)	5	9	4	7	8	6
Time on Machine II (B_i)	7	4	8	3	9	5

Find the sequence of jobs that minimizes the total elapsed time to complete the jobs. Find the minimum time by using Gnatt's chart or by any other method. Also compute the idle time for both the machines.

3. Determine the optimal sequence of jobs that minimizes the total elapsed time based on the following information processing time on machines given in hours and passing is not allowed:

Job	1	2	3	4	5
Machine A	3	8	7	5	2
Machine B	3	4	2	1	5
Machine C	5	8	10	7	6

4. Find the sequence that minimizes the total time required in performing the following jobs on three machines in the order ABC:

Processing time (in hours) on	Jobs					
	1	2	3	4	5	6
Machine A	8	3	7	2	5	1
Machine B	3	4	5	2	1	6
Machine C	8	7	6	9	10	9

Model Questions:

1. The cost of a machine is Rs. 6,100 and its scrap value is Rs. 100. The maintenance costs found from experience are as follows:

Year	1	2	3	4	5	6	7	8
Maintenance cost (Rs.)	100	250	400	600	900	1,200	1,600	2,000

When should the machine be replaced?

2. A firm is considering replacement of a machine whose cost price is Rs. 17,500 and the scrap value is Rs. 500. The maintenance costs (in Rs.) are found from experience to be as follows:

Year	1	2	3	4	5	6	7	8
Maintenance cost (Rs.)	200	300	3,500	1,200	1,800	2,400	3,300	4,500

3. A firm is using a machine whose purchase price is Rs. 13,000. The installation charges amount to Rs. 3,600 and the machine has a scrap value of only Rs. 1,600, because the firm has a monopoly of this type of work.

The maintenance cost in various years is given in the following table:

Year	1	2	3	4	5	6	7
Cost (Rs.)	250	750	1,000	1,500	2,100	2,900	4,000

4. Following table gives the running costs per year and resale price of a certain equipment whose purchase price is Rs. 5,000.

Year	1	2	3	4	5	6	7	8
Running cost (Rs.)	1,500	1,600	1,800	2,100	2,500	2,900	3,400	4,000
Resale value (Rs.)	3,500	2,500	1,700	1,200	800	500	500	500

At what year is the replacement due?

5. A manufacturer is offered two machines A and B. A is priced at Rs. 5,000 and running cost are estimated at Rs. 800 for each of the first five years, increasing by Rs. 200 per year in the sixth subsequent years. Machine B, which has the same capacity as A, costs Rs. 2,500 but will have running costs of Rs. 1,200 per year for six years, increasing by Rs. 200 per year thereafter.

If money is worth 10% per year, which machine should be purchased? (Assume that the machines will eventually be sold for scrap at a negligible price.)

6. An engineering company is offered two types of material handling equipment A and B. A is priced at Rs. 60,000 including cost of installation, and the costs

for operation and maintenance are estimated to be Rs. 10,000 for each of the first five years, increasing by Rs. 3,000 per year in the sixth Rs. 30,000 but in terms of operation and maintenance costs more than A. these costs more than A. these costs for B are estimated to be Rs. 13,000 per year for the first six years, increasing by Rs. 4,000 per year for each year from the 7th year onwards. The company expects a return of 10 per cent on all its investments. Neglecting the scrap value of the equipment at the end of its economic life, determine which equipment the company should buy.

7. An individual is planning to purchase a car. A new car will cost Rs. 1,20,000. The resale value of the car at the end of the year is 85% of the previous year value. Maintenance and operation value of car can be Rs. 40,000.

(i) When should the car be replaced to minimize average annual cost (ignore interest)?

(ii) If interest of 12% is assumed, when should the car be replaced?

8. A company has six jobs on hand coded 'A' to 'F'. All the jobs have to go through two machines 'M I' and 'M II'. The time required for each job on each machine, in hours, is given below:

	A	B	C	D	E	F
M I	3	12	18	9	15	6
M II	9	18	24	24	3	15

Draw a sequence table scheduling the six jobs on the two machines.

9. (a) A book binder has one printing press, one binding machine and the manuscripts of a number of different books. The times required to perform the printing and binding operation for each the total time required to process all the books. Find also the total time required. (Clearly state any algorithm you might use.)

Processing time (in minutes)

Book	1	2	3	4	5
Printing time	40	90	80	60	50
Binding time	50	60	20	30	40

(b) Suppose that an additional operation is added to the process described in (a), viz., finishing. The times required for operations are given below:

Finishing time (in minutes)

Book	1	2	3	4	5
Finishing time	80	100	60	70	100

What is the order in which the books should be processed? Find also the minimal total elapsed time.

10. A readymade garments manufacturer has to process 7 items through two stages of production, viz., cutting and sewing. The time taken for each of these at the different stages are given below in appropriate units:

	Item	1	2	3	4	5	6	7
Processing time	Cutting	5	7	3	4	6	7	12
	Sewing	2	6	7	5	9	5	8

(a) Find an order in which these items are to be processed though these stages so as to minimize the total processing time.

(b) Suppose a third stage of production is added, viz., pressing and packing, with processing time for these items as follows:

Item	1	2	3	4	5	6	7
Processing time	10	12	11	13	12	10	11

(Pressing and packing)

Find an order in which these seven items are to be processed so as to minimize the time taken to process all the items through all the three stages.

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