MADURAI KAMARAJ UNIVERSITY

(University with Potential for Excellence)

DISTANCE EDUCATION

B.Sc. (Mathematics)

THIRD YEAR

UNIT: 6 - 10 (Volume -2)

Paper - IV

LINEAR PROGRAMMING AND OPERATIONS RESEARCH

Recognised by D.E.C.

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CONTENTS

UNIT – 6

TRANSPORTATION PROBLEMS

Introduction:

The Transportation Problem (T.P) is one of the subclasses of L.P.P's in which the objective is to transport various quantities of a single homogeneous commodity, that are initially stored at various origins, to different destinations in such a way that the total transportation cost is minimum. To achieve this objective we must know the amount and location of available supplies and the quantities demanded. In addition, we must know the costs that result from transporting one unit of commodity from various origins to various destinations.

Objectives:

Students will be able to

1. Structure linear programming problems using the transportation models.
2. Use the North – West corner rule and VAM method.
3. Solve facility location and other application problems with transportation methods.

Structure:

6.1 Mathematical Formulation of a Transportation problem.

6.2 Methods for finding initial basic feasible solution.

6.3 Modified Distribution Method.

6.4 Degeneracy in Transportation problems.

6.5 Unbalanced Transportation problems.

6.6 Keywords

6.7 Answers to check your progress Questions.

6.8 Model Questions.

6.1 Mathematical Formulation of a Transportation Problem

Let us assume that there are m sources and n destinations.

Let a_i be the supply (capacity) at source i, b_j be the demand at destination j, c_{ij} be the unit transportation cost from source i to destination j and x_{ij} be the number of units shifted from source i to destination j.

Then the transportation problem can be expressed mathematically as

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \, x_{ij}$$

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1,2,3,\ldots\ldots, m.$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1,2,3,\ldots\ldots\ldots, n.$$

and $x_{ij} \geq 0$, for all i and j.

Note:

The two sets of constraints will be consistent if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j,$$

(total supply) (total demand)

which is the necessary and sufficient condition for a transportation problem to have a feasible solution. Problems satisfying this condition are called balanced transportation problems.

Note: 2

If $\sum a_i \neq \sum b_j$, then the transportation problem is said to be **unbalanced.**

Note: 3

For any transportation problem, the coefficients of all x_{ij} in the constraints are unity.

Note: 4

The objective function and the constraints being all linear, the transportation problem is a special class of linear programming problem. Therefore it can be solved by simplex method. But the number of variables being large, there will be too many calculations. So we can look for some other technique which would be simpler than the usual simplex method.

Standard transportation table:

Transportation problem is explicitly represented by the following transportation table

Destination

	D_1	D_2	D_3	...	D_j	...	D_n	Supply
S_1	c_{11}	c_{12}	c_{13}		c_{1j}		c_{1n}	a_1
S_2	c_{21}	c_{22}	c_{23}		c_{2j}		c_{2n}	a_2
Source								.
								.
S_i	c_{i1}	c_{i2}			c_{ij}		c_{in}	.
S_m	c_{m1}	c_{m2}			c_{mj}		c_{mn}	a_m
Demand	b_1	b_2	b_3	...	...	...	b_n	$\sum a_i = \sum b_j$

The mn squares are called **cells**. The unit transportation cost c_{ij} from the i^{th} source to the j^{th} destination is displayed in the **upper left side of the** $(i, j)^{th}$ **cell**. Any feasible solution is shown in the table by entering the value of x_{ij} **in the centre of the** $(i, j)^{th}$ cell. The various a's and b's are called **rim requirements.** The feasibility of a solution can be verified by summing the values of x_{ij} along the rows and down the columns.

Theorem: 1

(Existence of Feasible Solution). A necessary and sufficient condition for the existence of a feasible solution to the general transportation problem is that

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = \lambda \text{ (say).}$$

Proof:

The condition is necessary. Let there exist a feasible solution to the T.P. Then, we have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i \quad \text{and} \quad \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j$$

yielding

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = \lambda \text{ (say)}$$

Sufficiency. We assert that there exists a feasible solution given by $x_{ij} = a_i b_j / \lambda$ for all i and j. Clearly, $x_{ij} \geq 0$ since $a_i > 0,\ b_j > 0$ for all i and j.

Also $\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} (a_i b_j/\lambda) = \frac{a_i}{\lambda}\sum_{i=1}^{n} b_j = a_i, \qquad i = 1,2,\ldots\ldots\ldots, m$

and

$$\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} (a_i b_j/\lambda) = \frac{b_j}{\lambda}\sum_{i=1}^{m} a_i = b_j, \qquad j = 1,2,\ldots\ldots\ldots, n$$

Thus x_{ij} satisfies all the constraints of the T.P. and hence is a feasible solution.

Corollary. (Existence of an Optimum Solution). There always exists an optimum solution to a T.P.

Proof:

Let $\sum a_i = \sum b_j$ so that a feasible solution x_{ij} exists. It follows from the constraints of the problem that each x_{ij} is bounded, viz.,

$$0 \le x_{ij} \le \min.(a_i, b_j)$$

Thus the feasible region of the problem is closed, bounded and non – empty and hence there exists an optimum solution.

Theorem: 2

(Basic Feasible Solution). The number of basic (decision) variables of the general transportation problem at any stage of feasible solution must be $m + n - 1$.

Proof:

Consider the $m + n$ constraints of the transportation problem:

$$\sum_{i=1}^{m} x_{ij} = b_i \text{ and } \sum_{j=1}^{n} x_{ij} = a_i \qquad (j = 1,2,\ldots,n; i = 1,2,\ldots..m)$$

Taking summation on both sides, these yield

$$\sum_{j=1}^{n-1} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n-1} b_j \text{ and } \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_j$$

Subtracting the two, we have

$$\sum_{j=1}^{n-1} \sum_{i=1}^{m} x_{ij} - \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n-1} b_j - \sum_{i=1}^{m} a_i$$

or $$\sum_{i=1}^{m} \left(\sum_{j=1}^{n-1} x_{ij} - \sum_{j=1}^{n} x_{ij} \right) = \sum_{j=1}^{n-1} b_j - \sum_{j=1}^{n} b_j \quad \left(\because \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \right)$$

or $$\sum_{i=1}^{m} x_{in} = b_n$$

which happens to be the last constraint of the problem.

This indicates that if the first $m + n - 1$ constraints are satisfied, then $\sum a_i = \sum b_j$, ensures that the $(m + n)^{th}$ constraint will be automatically satisfied.

Thus, out of $m + n$ equations, we have only $(m+n-1)$ linearly independent equations. Therefore a basic feasible solution will consist of at most $(m+n-1)$positive variables the rest being zero. Further a feasible solution involving exactly $(m+n-1)$ positive variables is known as non – degenerate basic feasible solution, otherwise it is said to be degenerate basic feasible.

Remarks: 1

When the total demand is equal to total supply, the transportation problem is said to be balanced and otherwise unbalanced.

Remarks: 2

The allocated cells in the transportation table will be called occupied cells and empty cells will be called non – occupied cells.

Definition: 1

A set of non – negative values x_{ij}, $i = 1,2,........,m$; $j = 1,2,......n$. That satisfies the constraints (rim conditions and also the non – negativity restrictions) is called a **feasible solution** to the transportation problem.

Note:

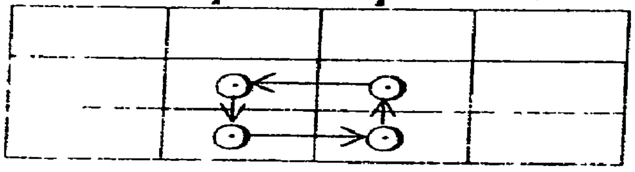
A balanced transportation problem will always have a feasible solution.

Definition: 2

A feasible solution to a $(m \text{ x } n)$ transportation problem that contains no more than $m + n - 1$ non – negative allocations is called a **basic feasible solution** (BFS) to the transportation problem.

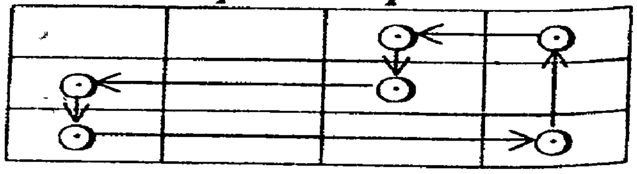
The allocations are said to be in **independent positions** if it is impossible to increase or decrease any allocation without either changing the position of the allocation or violating the rim requirements. A simple rule for allocations to be in independent positions is that it is impossible to rule for allocation, back to itself by a series of horizontal and vertical jumps from one occupied cell to another, without a direct reversal of the route. Example

Non–independent positions



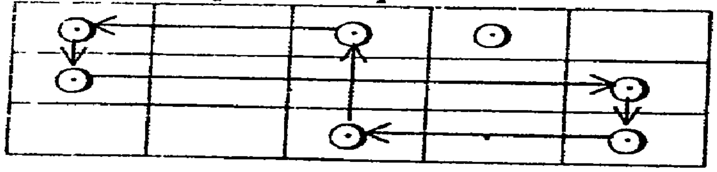
(i)

Non–independent positions



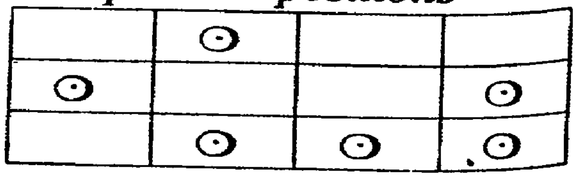
(ii)

Non–independent positions



(i)

Independent positions



(ii)

Definition: 3

A basic feasible solution to a (m x n) transportation problem is said to be a **non – degenerate basic feasible solution** if it contains exactly $m+n-1$ non – negative allocations in independent positions.

Definition: 4

A basic feasible solution that contains less than $m+n-1$ non – negative allocations is said to be a **degenerate basic** feasible solution.

Definition: 5

A feasible solution (not necessarily basic) is said to be **an optimal solution** if it minimizes the total transportation cost.

Note:

The number of basic variables in an m x n balanced transportation problem is atmost $m+n-1$.

Note:

The number of non – basic variables in an m x n balanced transportation problem is atleast $mn-(m+n-1)$.

6.2 Methods for finding Initial basic feasible solution

The transportation problem has a solution if and only if the problem is balanced. Therefore before starting to find the initial basic possible solution. Check whether the given transportation problem is balanced. If not one has to balance the transportation problem first. The way of doing this is discussed in section 6.5. In this section all the given transportation problems are balanced.

An initial feasible solution is obtained by any one of the following methods.

1. Row Minima Method.
2. Column Minima Method.
3. North West corner rule.
4. Least cost Method.
5. Vogel's Approximation Method.

Method 1: Row Minima Method

In this method we determine allotment row – wise. First allotment should be made to the least cost cell in that row. The allotment should be the minimum of availability and requirement corresponding to that row. Cut off the row or column where the availability or the requirement is exhausted. If still some availability is left out in the first row then choose the next least cost in that row for allotment. In this way, exhaust all the availability in that row.

Next choose the least cost unallotted cell in the second row and decide the allotment for that cell. In this way exhaust the availability for that cell. In this way exhaust the availability for the second row. Proceed in the same way till all the row availability are exhausted.

Thereby we get the initial feasible solution to the transportation problem.

Example: 6.2.1

Determine the initial feasible solution to the following transportation problem using row minima method.

	S_1	S_2	S_3	S_4	Availability
A	5	2	4	3	22
B	4	8	1	6	15
C	4	6	7	5	8
Demand	7	12	7	19	

Solution:

Since $\sum a_i = \sum b_j = 45$ then Transportation problem is balanced.

The initial feasible solution is under Row Minima Method.

5	2 12	4	3 10	22
4 7	8	1 7	6 1	15
4	6	7	5 8	8
7	12	7	19	

Transportation cost

$$= (2x12) + (3x10) + (4x7) + (1x7) + (6x1) + (5x8)$$

$$= 24 + 30 + 28 + 7 + 6 + 40 = 135$$

Method 2: Column Minima Method

The procedure for obtaining the initial solution by this method is the same as the above method except that allotments are made column – wise instead of row – wise.

Example: 6.2.2

Determine the initial feasible solution to the following transportation problem using column minima method.

	S_1	S_2	S_3	S_4	Availability
A	5	2	4	3	22
B	4	8	1	6	15
C	4	6	7	5	8
Demand	7	12	7	19	

Solution:

Since $\sum_{ai} = \sum b_j = 45$ then the Transportation problem is balanced.

The initial feasible solution is under column minima method.

	S1	S2	S3	S4	Availability
A	5	2 12	4	3 10	22
B	4	8	1 7	6 8	15
C	4 7	6	7	5 1	8
Demand	7	12	7	19	

Transportation cost

$$= (2x12) + (3x10) + (1x7) + (6x8) + (4x7) + (5x1)$$

$$= 24 + 30 + 7 + 48 + 28 + 5 = 142$$

Method 3: North West Corner Rule:

Step: 1

The first assignment is made in the cell occupying the upper left hand (north – west) corner of the transportation table. The maximum possible amount is allocated there. That is $x_{11} = \min\{a_1,\ b_1\}$.

Case (i) : If $\min\{a_1, b_1\} = a_1$, then put $x_{11} = a_1$, decrease b_1 by a_1 and move vertically to the 2nd row (i.e.,) to the cell (2,1) cross out of the first row.

Case (ii) : If $\min\{a_1, b_1\} = b_1$, then put $x_{11} = b_1$, and decrease a_1 by b_1 and move horizontally right (i.e.,) to the cell (1,2) cross out the first column.

Case (iii) : If $\min\{a_1, b_1\} = a_1 = b_1$ then put $x_{11} = a_1 = b_1$ and move diagonally to the cell (2,2) cross out the first row and the first column.

Step: 2

Repeat the procedure until all the rim requirements are satisfied.

Example: 6.2.3

Determine basic feasible solution to the following transportation problem using North West Corner Rule:

		A	B	C	D	E	Supply
	P	2	11	10	3	7	4
Origin	Q	1	4	7	2	1	8
	R	3	9	4	8	12	9
Demand		3	3	4	5	6	

Solution:

Since $a_i = b_j = 21$, the given problem is balanced. $\therefore$ There exists a feasible solution to the transportation problem.

2 3	11	10	3	7	4
1	4	7	2	1	8
3	9	4	8	12	9
3	3	4	5	6	

Following North West Corner rule, the first allocation is made in the cell (1,1).

Here $x_{11} = \min\{a_1, b_1\} = \min\{4,\ 3\} = 3$

$\therefore$ Allocate 3 to the cell (1,1) and decrease 4 by i.e., $4-3=1$

As the first column is satisfied, we cross out the first column and the resulting reduced Transportation table is

11 1	10	3	7	1
4	7	2	1	8
9	4	8	12	9
3	4	5	6	

Here the North West Corner cell is (1,2).

So Allocate $x_{12} = \min\{1,\ 3\} = 1$ to the cell (1, 2) and move vertically to cell (2, 2). The resulting reduced transportation table is

4 2	7	2	1	8
9	4	8	12	9
2	4	5	6	

Allocate $x_{22} = \min\{8, 2\} = 2$ to the cell (2, 2) and move horizontally to the cell (2, 3). The resulting transportation table is

7 4	2	1	6
4	8	12	9
4	5	6	

Allocate $x_{23} = \min\{6, 4\} = 4$ and move horizontally to the cell (2, 4). The resulting reduced transportation table is

2 2	1	2
8	12	9
5	6	

Allocate $x_{24} = \{2, 5\} = 2$ and move vertically to the cell (3, 4). The resulting reduced transportation table is

8 3	12	9
3	6	

Allocate $x_{34} = \min\{9, 3\} = 3$ and move horizontally to the cell (3, 5), which is

12 6	6
6	

Allocate $x_{35} = \min\{6, 6\} = 6$

Finally the initial basic feasible solution is as shown in the following table.

2 3	11 1	10	3	7
1	4 2	7 4	2 2	1
3	9	4	8 3	12 6

From this table we see that the number of positive independent allocations is equal to $m+n-1 = 3+5-1 = 7$. This ensures that the solution is not degenerate basic feasible.

∴ The initial transportation cost

$$= \text{Rs.}(2x3)+(11x1)+(4x2)+(7x4)+(2x2)+(8x3)+(12x6)$$

$$= \text{Rs.}153/-$$

Example: 6.2.4

Obtain an initial basic feasible solution to the following transportation problem using the north – west corner rule.

	D	E	F	G	Available
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Requirement	200	225	275	250	

Solution:

Since $\sum a_i = \sum b_j = 950$, there exists a feasible solution to the transportation problem. We obtain initial feasible solution as follows:

The transportation table of the given problem has 12 cells. Following north – west corner method, the first allocation is made in the cell (1,1), the magnitude being $x_{11} = \min.(250, 200) = 200$. The second allocation is made in the cell (1,2) and the magnitude of the allocation is given by

$$x_{12} = \min(250-200,\ 225) = 50.$$

The third allocation is made in the cell (2, 2), the magnitude being $x_{22} = \text{min.}(300,\ 225-50) = 175$. The magnitude of fourth allocation in the cell (2, 3) is given by $x_{23} = \text{min.}(300-175,\ 275) = 125$. The fifth allocation is made in the cell (3, 3), the magnitude being $x_{33} = \text{min.}(400,\ 275-125) = 150$ and the sixth (last) allocation is made in the cell (3, 4) with magnitude $x_{34} = \text{min.}(400-150,\ 250) = 250$. Hence an initial basic feasible solution to the given T.P. has been obtained and is displayed in Table.

The transportation cost according to the above route is given by

$$z = (200x11) + (50x13) + (175x18) + (125x14) + (150x13) + (250x10)$$

$$= 12{,}200.$$

$\boxed{200}$ 11	$\boxed{50}$ 13	17	14	250
16	$\boxed{175}$ 18	$\boxed{125}$ 14	10	300
21	24	$\boxed{150}$ 13	$\boxed{250}$ 10	400
200	225	275	250	

Method: 4

Least Cost method (or) Matrix minima method (or) Lowest cost entry method:

Step: 1

Identify the cell with smallest cost and allocate $x_{ij} = \text{Min}\{a_i, b_j\}$.

Case: (i)

If $\min\{a_i, b_j\} = a_i$, then put $x_{ij} = a_i$, cross out the i^{th} row and decrease b_j by a_i, Go to step (2).

Case: (ii)

If min $\{a_i, b_j\} = b_j$ then put $x_{ij} = b_j$ cross out the j^{th} column and decrease a_i by b_j Go to step (2).

Case: (iii)

If $\min\{a_i, b_j\} = a_i = b_j$, then put $x_{ij} = a_i = b_j$, cross out either i^{th} row or j^{th} column but not both, Go to step (2).

Step: 2

Repeat step (1) for the resulting reduced transportation table until all the rim requirements are satisfied.

Example: 6.2.5

Find the initial basic feasible solution for the following transportation problem by Least Cost Method.

			To		Supply
	1	2	1	4	30
From	3	3	2	1	50
	4	2	5	9	20
Demand	20	40	30	10	

Solution:

Since $\sum a_i = \sum b_j = 100$, the given TPP is balanced. $\therefore$ There exists a feasible solution to the transportation problem.

1 20	2	1	4	30
3	3	2	1	50
4	2	5	9	20
20	40	30	10	

By least cost method, min $c_{ij} = c_{11} = c_{13} = c_{24} = 1$

Since more than one cell having the same minimum c_{ij}, break the tie.

Let us choose the cell (1, 1) and allocate $x_{11} = \min\{a_1, b_1\} = \min\{30,\ 20\} = 20$ and cross out the satisfied column and decrease 30 by 20.

The resulting reduced transportation table is

2	1 10	4	10
3	2	1	50
2	5	9	20
40	30	10	

Here min $c_{ij} = c_{13} = c_{24} = 1$

Choose the cell (1, 3) and allocate $x_{13} = \min\{a_1, b_3\} = \min\{10, 30\} = 10$ and cross out the satisfied row.

The resulting reduced transportation table is

3	2	1 10	50
2	5	9	20
40	20	10	

Here min $c_{ij} = c_{24} = 1$.

$\therefore$ Allocate $x_{24} = \min\{a_2, b_4\} = \min(50,\ 10) = 10$ and cross out the satisfied column.

The resulting transportation table is

3	2 20	40
2	5	20
40	20	

Here $\min c_{ij} = c_{23} = c_{32} = 2$. Choose the cell (2, 3) and allocate $x_{23} = \min\{a_2, b_3\} = \min(40,\ 20) = 20$ and cross out the satisfied column.

The resulting reduced transportation table is

3	20
2 20	20
40	

Here $\min c_{ij} = c_{32} = 2$. Choose the cell (3, 2) and allocate $x_{32} = \min\{a_3, b_2\} = \min(20,\ 40) = 20$ and cross out the satisfied row.

The resulting reduced transportation table is

3 20	20
20	

Finally the initial basic feasible solution is as shown in the following table.

1 20	2	1 10	4
3	3 20	2 20	1 10
4	2 20	5	9

From this table we see that the number of positive independent allocations is equal to $m+n-1=3+4-1=6$. This ensures that the solution is non degenerate basic feasible.

$\therefore$ The initial transportation cost

$$= \text{Rs.}(1x20)+(1x10)+(3x20)+(2x20)+(1x10)+(2x20)$$

$$=20+10+60+40+10+40$$

$$=\text{Rs.}180/-$$

Example: 6.2.6

Obtain an initial basic feasible solution to the following T.P. using the matrix minima method.

	D_1	D_2	D_3	D_4	Capacity
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	

Where O_i and D_j denote ith origin and jth destination respectively.

Solution:

Since $\sum a_i = \sum b_j = 24$ then TPP is balanced.

$\therefore$ There exists a feasible solution to the transportation problem.

1	2	3	4	6
4	3	2	0	8
0 4	2	2	1	10
4	6	8	6	

By least cost method, min $c_{ij} = c_{31} = c_{24} = 0$.

Since more than one cell having the same minimum c_{ij} break the tie.

Let us choose the cell (3, 1) and allocate $x_{31} = \min\{a_3, b_1\} = \min\{4, 10\} = 4$ and cross out the satisfied column and decrease 10 by 6. The resulting reduced transportation table is

2	3	4	6
3	2	0 6	8
2	2	1	6
6	8	6	

Here min $c_{ij} = c_{24} = 0$

Choose the (2, 3) and allocate $x_{23} = \min\{a_2, b_3\} = \min\{6, 8\} = 6$ and cross out the satisfied column. The resulting reduced transportation table is

2 6	3	6
3	2	2
2 0	2	6
6	8	

Here min $c_{ij} = c_{12} = c_{23} = c_{32} = c_{33} = 2$

Choose the cell c_{12} and allocate $x_{12} = \min\{a_1, b_2\} = \min\{6, 6\} = 6$ and cross out either the second column or the first row. We choose to cross out the first row of the table. The next allocation of magnitude $x_{32} = 0$ is made in the cell

c_{32} . Cross out the second column.

The resulting reduced transportation table is

2 2	2
2	6
8	

Here min $c_{ij} = c_{23} = c_{33} = 2$

Choose the cell c_{23} and allocate $x_{23} = \min\{a_2, b_3\} = \min\{2, 8\} = 2$ and cross out the satisfied row. The resulting reduced transportation table is

2 6	6
6	

Finally the initial basic feasible solution is as shown in the following task.

1	2 6	3	4
4	3	2 2	0 6
0 4	2 6	2 6	1

Now, all the rim requirements have been satisfied and hence an initial solution has been determined. This solution is displayed in transportation Table.

Since the cells do not form a loop, the solution is basic one.

Moreover the solution is degenerate also. The transportation cost according to the above route is given by

$$z=(6x2)+(2x2)+(6x0)+(4x0)+(2x\in)+(6x2)=28+2\in \;=\; 28 \;\text{ as} \in\rightarrow 0$$

Method: 5

Vogel's approximation method (VAM) (or) Unit cost penalty method:

Step: 1

Find the difference (penalty) between the smallest and next smallest costs in each row (column) and write them in brackets against the corresponding row (column).

Step: 2

Identify the row (or) column with largest penalty. If a tie occurs, break the tie arbitrarily. Choose the cell with smallest cost in that selected row or column and allocate as much as possible to this cell and cross out the satisfied row or column and go to step (3).

Step: 3

Again compute the column and row penalties for the reduced transportation table and then go to step (2). Repeat the procedure until all the rim requirements are satisfied.

Example: 6.2.7

Find the initial basic feasible solution for the following transportation problem by VAM.

Distribution Centres

		D_1	D_2	D_3	D_4	Availability
	S_1	11	13	17	14	250
Origin	S_2	16	18	14	10	300
	S_3	21	24	13	10	400
Requirements		200	225	275	250	

Solution:

Since $\sum a_i = \sum b_j = 950$, the given problem is balanced.

$\therefore$ There exists a feasible solution to this problem.

11 200	13	17	14	250(2)
16*	18	14	10	300 (4)
21	24	13	10	400 (3)
200	225	275	250	
(5)	(5)	(1)	(0)	

First let us find the difference (penalty) between the smallest and next smallest costs in each row and column and write them in brackets against the respective rows and columns.

The largest of these difference is (5) and is associated with the first two columns of the transportation table. We choose the first column arbitrarily.

In this selected column, the cell (1,1) has the minimum unit transportation cost $c_{11} = 11$.

$\therefore$ Allocate $x_{11} = \min\{250, 200\} = 200$ to this cell (1, 1) and decrease 250 by 200 and cross out the satisfied column.

The resulting reduced transportation table is

13 50	17	14	50 (1)
18	14	10	300 (4)
24	13	10	400 (3)
225	275	250	
(5)	(1)	(0)	

The row and column difference and now computed for this reduced transportation table. The largest of these is (5) which is associated with the second column. Since $c_{12} = 13$ is the minimum cost, we allocated $x_{12} = \min\{50, 225\} = 50$ to the cell (1,2) and decrease 225 by 50 and cross out the satisfied row.

Continuing in this manner, the subsequent reduced transportation tables and the differences for the surviving rows and columns are shown below:

18 175	14	10	300 (4)
24	13	10	400 (3)
175	275	250	
(6)	(1)	(0)	

14	10 125	125 (4)
13	10	400 (3)
275	250	
(1)	(0)	

13	10 125	400
275	125	

13 275	25
275	

Finally the initial basic feasible solution is as shown in the following table.

11 200	13 50	17	14
16	18 175	14	10 125
21	24	13 275	10 125

From this table we see that the number of positive independent allocations is equal to $m+n-1=3+4-1=6$. This ensures that the solution is non degenerate basic feasible.

$\therefore$ The initial transportation cost

$$= Rs.(11x200)+(13x50)+(18x175)+(10x125)+(13x275)+(10x125)$$

$$= Rs.12075/-$$

Example: 6.2.8

Find the starting solution of the following transportation model

1	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

Using

(i) North West Corner rule

(ii) Least Cost method

(iii) Vogel's approximation method.

Solution:

Since $\sum a_i = \sum b_j = 30$, the given Transportation problem is balanced. Hence there exists a basic feasible solution to this problem.

(i) North West Corner rule:

Using this method, the allocation are shown in the tables below:

1 7	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

0 3	4	2	12
3	1	5	11
3	10	10	

4 9	2	9
1	5	11
10	10	

1 5	5	11
1	10	

5 10	10
10	

The starting solution is as shown in the following table

1 7	2	6
0 3	4 9	2
3	1 1	5 10

∴ The initial transportation cost

$$= \text{Rs.}(1\text{x}7) + (0\text{x}3) + (4\text{x}9) + (1\text{x}1) + (5\text{x}10)$$

$$= \text{Rs.}94/-$$

(ii) Least Cost Method:

Using this method, the allocation are as shown in the table below:

1	2	6 7	7
0 10	4	2	12
3	1	5	11
10	10	10	

2	6	7
4	2	2
1 10	5	11
10	10	

6	7
2 2	2
5	1
10	

6	7
5 1	1
8	

6 7	7
7	

The starting solution is as shown in the following table:

1	2	6 7
0 10	4	2 2
3	1 10	5 1

The initial transportation cost

$$= \text{Rs.}6x7 + 0x10 + 2x2 + 1x10 + 5x1$$

$$= \text{Rs.}61/-$$

(iii) Vogel's Approximation Method:

Using this method, the allocations are shown in the tables below:

1	2	6	7	(1)
0	4	2 10	12	(2)
3	1	5	11	(2)
10	10	10		
(1)	(1)	(3)		

1	2	7	(1)
0 2	4	2	(4)
3	1	11	(2)
10	10		
(1)	(1)		

1	2	7	(1)
3	1 10	11	(2)
8	10		
(2)	(1)		

1 7	7
3	1
8	

3 1	1
1	

The starting solution is an shown in the following table:

1 7	2	6
0 2	4	2 10
3 1	1 10	5

$\therefore$ The initial transportation cost

$$= \text{Rs.}(1x7)+(0x2)+(2x10)+(3x1)+(1x10)$$

$$= \text{Rs.}40/-$$

Note:

For the above problem, the number of positive allocation in independent positions is $(m+n-1)$(i.e., $m+n-1=3+3-1=5$). This ensures that the given problem has a non – degenerate basic feasible solution by using all the three methods. This need not be the case in all the problems.

Check your progress: 6.1

1. For the transportation problem given below, find a basic feasible solution by North – West Corner method:

	D_1	D_2	D_3	D_4	D_5	D_6	Supply
O_1	6	4	8	4	9	6	4
O_2	6	7	13	6	8	12	5
O_3	3	9	4	5	9	13	3
O_4	10	7	11	6	11	10	9
Demand	4	4	5	3	2	3	

2. Determine an initial basic feasible solution to the following transportation problem using North – West Corner method:

	D_1	D_2	D_3	D_4	Availability
O_1	5	3	6	2	19
O_2	4	7	9	1	37
O_3	3	4	7	5	34
Demand	16	18	31	25	

3. Consider the following transportation problem:

Source	Destination				Availability
	1	2	3	4	
1	20	22	17	4	120
2	24	37	9	7	70
3	32	37	20	15	50
Requirement	60	40	30	110	240

Determine an initial basic feasible solution using the (a) row minima method, and (b) Vogel's approximation method.

4. Obtain an initial basic feasible solution to the following T.P. using the Vogel's approximation method:

Warehouse	Stores				Availability
	I	II	III	IV	
A	5	1	3	3	34
B	3	3	5	4	15
C	6	4	4	3	12
D	4	-1	4	2	19
Requirement	21	25	17	17	80

6.3 Modified Distribution Method (MODI Method)

Transportation Algorithm (or) MODI Method (Test for optimal solution).

Step: 1

Find the initial basic feasible solution of the given problem by North West Corner rule (or) Least Cost method or VAM.

Step: 2

Check the number of occupied cells. If these are less than $m+n-1$, there exists degeneracy and we introduce a very small positive assignment of $\in (\approx 0)$ in suitable independent positions, so that the number of occupied cells is exactly equal to $m+n-1$.

Step: 3

Find the set of values $u_i, v_j (i=1,2,3,\ldots..m;\ j=1,2,3,\ldots\ldots.,n)$ from the relation $c_{ij}=u_i+v_j$ for each occupied cell (i,j), by starting initially with $u_i=0$ or $v_j=0$ preferably for which the corresponding row or column has maximum number of individual allocations.

Step: 4

Find u_i+v_j for each unoccupied cell (i,j) are enter at the upper right corner of the corresponding cell (i,j).

Step: 5

Find the cell evaluations $d_{ij}=c_{ij}-(u_i+v_j)(d_{ij}=$ upper left – upper right) for each unoccupied cell (i,j) and enter at the lower right corner of the corresponding cell (i,j).

Space for Hints

Step: 6

Examine the cell evaluations d_{ij} for all unoccupied cells (i,j) and conclude that

(i) If all $d_{ij} > 0$, then the solution under the test is optimal and unique.

(ii) If all $d_{ij} > 0$, with atleast one $d_{ij} = 0$, then the solution under the test is optimal and an alternative optimal solution exists.

(iii) If atleast one $d_{ij} < 0$, then the solution is not optimal. Go to the next step.

Step: 7

From a new B.F.S by giving maximum allocation to the cell for which d_{ij} is most negative by making an occupied cell empty. For that draw a closed path consisting of horizontal and vertical lines beginning and ending at the cells for which d_{ij} is most negative and having its **other corners at some allocated cells.** Along this closed loop indicate $+\theta$ and $-\theta$ alternatively at the corners. Choose minimum of the allocations from the cells having $-\theta$. Add this minimum allocation to the cells with $+\theta$ and subtract this minimum allocation from the allocation to the cells with $-\theta$.

Step: 8

Repeat steps (2) to (6) to test the optimality of this new basic feasible solution.

Step: 9

Continue the above procedure till an optimum solution is attained.

Note:

The Vogels approximation method (VAM) takes into account not only the least cost c_{ij} but also the costs that just exceed the least cost c_{ij} and therefore yields better initial solution than obtained from other methods in general. This can be justified by the above example (4). So to find the initial solution, give preference to VAM unless otherwise specified.

Example: 6.3.1

Solve the transportation problem:

	1	2	3	4	Supply
I	21	16	25	13	11
II	17	18	14	23	13
III	32	27	18	41	19
Demand	6	10	12	15	

Solution:

Since $\sum a_i = \sum b_j = 43$, the given transportation problem is balanced. ∴There exists a basic feasible solution to this problem.

By Vogel's approximation method, the initial solution is as shown in the following table:

21	16	25	13 11	(3)	-	-	-
17 6	18 3	14	23 4	(3)	(3)	(3)	(4)
32	27 7	18 12	41	(9)	(9)	(9)	(9)
(4)	(2)	(4)	**(10)**				
(15)	(9)	(4)	**(18)**				
(15)	(9)	(4)	-				
-	**(9)**	(4)	-				

That is

21	16	25	13 11
17 6	18 3	14	23 4
32	27 7	18 12	41

From this table, we see that the number of non – negative independent allocation is $(m+n-1)=(3+4-1)=6$. Hence the solution is non – degenerate basic feasible.

$\therefore$ The initial transportation cost

$= \text{Rs.}(13x11)+(17x6)+(18x3)+(23x4)+(27x7)+(18x12)$

$= \text{Rs.}796/-$

To find the optimal solution

Consider the above transportation table. Since $m+n-1=6$, we apply MODI method,

Now we determine a set of values u_i and v_j for each occupied cell (i,j) by using the relation $c_{ij}=u_i+v_j$. As the 2nd row contains maximum number of allocations, we choose $u_2=0$.

Therefore

$c_{21}=u_2+v_1 \Rightarrow 17=0+v_1 \Rightarrow v=17$

$c_{22}=u_2+v_2 \Rightarrow 18=0+v_2 \Rightarrow v_2=18$

$$c_{24} = u_2 + v_4 \Rightarrow 23 = 0 + v_4 \Rightarrow v_4 = 23$$

$$c_{14} = u_1 + v_4 \Rightarrow 13 = u_1 + 23 \Rightarrow u_1 = -10$$

$$c_{32} = u_3 + v_2 \Rightarrow 27 = u_3 + 18 \Rightarrow u_3 = 9$$

$$c_{33} = u_3 + v_3 \Rightarrow 18 = 9 + v_3 \Rightarrow v_3 = 9$$

Thus we have the following transportation table:

21	16	25	13 11	$u_1 = -10$
17 6	18 3	14	23 4	$u_2 = 0$
32	27 7	18 12	41	$u_3 = 9$
$v_1 = 17$	$v_2 = 18$	$v_3 = 9$	$v_4 = 23$	

Now we find $u_i + v_j$ for each unoccupied cell (i, j) and enter at the upper right corner of the corresponding unoccupied cell (i, j).

Then we find the cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ (i.e., upper left corner – upper right corner) for each unoccupied cell (i, j) and enter at the lower right corner of the corresponding unoccupied cell (i, j)

Thus we get the following table:

21 7 14	16 8 8	25 -1 26	13 11	$u_1 = -10$
17 6	18 3	14 9 5	23 4	$u_2 = 0$
32 26 6	27 7	18 12	41 32 9	$u_3 = 9$
$v_1 = 17$	$v_2 = 18$	$v_3 = 9$	$v_4 = 23$	

Since all $d_{ij} > 0$, the solution under the test is optimal and unique.

$\therefore$ The optimum allocation schedule is given by $x_{14} = 11$, $x_{21} = 6$, $x_{22} = 3$, $x_{24} = 4$, $x_{32} = 7$, $x_{33} = 12$ and the optimum (minimum) transportation cost

$$= Rs.(13x11) + (17x6) + (18x3) + (23x4) + (27x7) + (18x12)$$

$$= Rs.796/-$$

Example: 6.3.2

Obtain on optimum basic feasible solution to the following transportation problem:

	To			Available
	7	3	2	2
From	2	1	3	3
	3	4	6	5
Demand	4	1	5	10

Solution:

Since $\sum a_i = \sum b_j = 10$, the given transportation problem is balanced. $\therefore$ There exists a basic feasible solution to this problem.

By Vogel's approximation method, the initial solution is as shown in the following table:

7	3	2 2	(1)	**(5)**	-
2	1 1	3 2	(1)	(1)	(1)
3 4	4	6 1	(1)	(3)	**(3)**
(1)	(2)	(1)			
(1)	-	(1)			
(1)		(3)			

That is

7	3	2 2
2	1 1	3 2
3 4	4	6 1

From this table we see that the number of non – negative allocation is $m+n-1=(3+3-1)=5$.

Hence the solution is non – degenerate basic feasible

$\therefore$ The initial transportation cost

$$= \text{Rs.}(2x2)+(1x1)+(3x2)+(3x4)+(6x1)$$

$$= \text{Rs.}29/-$$

For Optimality:

Since the number of non – negative independent allocations is $m+n-1$ we apply MODI method.

Since the third column contains maximum number of allocations, we choose $v_3 = 0$.

Now we determine a set of values u_i and v_j by using the occupied cells and the relation $c_{ij} = u_i + v_j$.

That is,

7 -1	3 0	2 2	$u_1 = 2$
2	1 1	3 2	$u_2 = 3$
3 4	4	6 1	$u_3 = 6$
$v_1 = -3$	$v_2 = -2$	$v_3 = 0$	

Now we find $u_i + v_j$ for each unoccupied cell (i, j) and enter at the upper right corner of the corresponding unoccupied cell (i, j).

Then we find the cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ for each unoccupied cell (i, j) and enter at the lower right corner of the corresponding unoccupied cell (i, j).

Thus we get the following table

7 -1 8	3 0 3	2 2	$u_1 = 2$
2 0 2	1 1	3 2	$u_2 = 3$
3 4	4 4 0	6 1	$u_3 = 6$
$v_1 = -3$	$v_2 = -2$	$v_3 = 0$	

Since all $d_{ij} > 0$, with $d_{32} = 0$, the current solution is optimal and their exists an alternative optimal solution.

$\therefore$ The optimum allocation schedule is given by $x_{13} = 2,\ x_{22} = 1,\ x_{23} = 3,\ x_{31} = 4,\ x_{33} = 1$ and the optimum (minimum) transportation cost

$$= \text{Rs.}(2x2) + (1x1) + (3x2) + (3x4) + (6x1) = \text{Rs.}29/-$$

Example: 6.3.3

Find the optimal transportation cost of the following matrix using least cost method for finding the critical solution.

Market

		A	B	C	D	E	
							Available
	P	4	1	2	6	9	100
Factory	Q	6	4	3	5	7	120
	R	5	2	6	4	8	120
Demand		40	50	70	90	90	

Solution:

Since $\sum a_i = \sum b_j = 340$, the given transportation problem is balanced. $\therefore$There exists a basic feasible solution to this problem.

By using Least cost method, the initial solution is as shown in the following table:

4	1 50	2 50	6	9
6 10	4	3 20	5	7 90
5 30	2	6	4 90	8

$\therefore$The initial transportation cost

$$= \text{Rs.}(1x50)+(2x50)+(6x10)+(3x20)+(7x90)+(5x30)+(4x90)$$

$$= \text{Rs.}1410/-$$

For optimality:

Since the number of non – negative independent allocations is $(m+n-1)$, we apply MODI method:

That is

4 5 -1	1 50	2 50	6 4 2	9 6 3	$u_1 = -1$
6 10	4 2 2	3 20	5 5 0	7 90	$u_2 = 0$
5 30	2 1 1	6 2 4	4 90	8 6 2	$u_3 = -1$
$v_1 = 6$	$v_2 = 2$	$v_3 = 3$	$v_4 = 5$	$v_5 = 7$	

Since $d_{11} = -1 < 0$, the current solution is not optimal.

Now let us form a new basic feasible solution by giving maximum allocation to the cell (i, j) for which d_{ij} is most negative by making an occupied cell empty. Here the cell (1, 1) having the negative value $d_{11} = -1$. We draw a closed loop consisting of horizontal and vertical lines beginning and ending at this cell (1, 1) and having its other corners at some occupied cells. Along this closed loop indicate $+\theta$ and $-\theta$ alternatively at the corners. We have

4 $+\theta$	1 50	2 $50-\theta$	6	9
6 10 $-\theta$	4	3 20 $+\theta$	5	7 90
5 30	2	6	4 90	8

From the two cells (1, 3), (2, 1) having $-\theta$, we find that the minimum of the allocations 50, 10 is 10. Add this 10 to the cells with $+\theta$ and subtract this 10 to the cells with $-\theta$.

Hence the new basic feasible solution is displayed in the following table:

4 10	1 50	2 40	6	9
6	4	3 30	5	7 90
5 30	2	6	4 90	8

We see that the above table satisfies the rim conditions with $(m+n-1)$non – negative allocations at independent positions. So we apply MODI method.

4 10	1 50	2 40	6 3 3	9 6 3	$u_1 = 0$
6 5 1	4 2 2	2 30	5 4 1	7 90	$u_2 = 1$
5 30	2 2 0	6 3 3	4 90	8 7 1	$u_3 = 1$
$v_1 = 4$	$v_2 = 1$	$v_3 = 2$	$v_4 = 3$	$v_5 = 6$	

Since all $d_{ij} > 0$, with $d_{32} = 0$, the current solution is optimal and there exists an alternative optimal solution.

$\therefore$ The optimum allocation schedule is given by $x_{11} = 10,\ x_{12} = 50,\ x_{13} = 40,\ x_{23} = 30,\ x_{25} = 90,\ x_{31} = 30,\ x_{34} = 90$ and the optimum (minimum) transportation cost

$$= \text{Rs.}(4x10) + (1x50) + (2x40) + (3x30) + (7x90) + (5x30) + (4x90)$$

$$= \text{Rs.}1400/-$$

6.4 Degeneracy in Transportation Problems

In a transportation problem, whenever the number of non – negative independent allocations is less than $m+n-1$, the transportation problem is said to be a **degenerate** one. Degeneracy may occur either at the initial stage or at an intermediate stage at some subsequent iteration.

To resolve degeneracy, we allocate an extremely small amount (close to zero) to one or more empty cells of the transportation table (generally minimum cost cells if possible), so that the total number of occupied cells becomes $(m+n-1)$ at independent positions. We denote this small amount by $\in$ (epsilon) satisfying the following conditions:

(i) $0 < \in < x_{ij}$, for all $x_{ij} > 0$

(ii) $x_{ij} \pm \in = x_{ij}$, for all $x_{ij} > 0$

The cells containing $\in$ are then treated like other occupied cells and the problem is solved in the usual way. The $\in$'s are kept till the optimum solution is attained. Then we let each $\in \rightarrow 0$.

Example: 6.4.1

Find the non – degenerate basic feasible solution for the following transportation problem using

(i) North West corner rule

(ii) Least cost method

(iii) Vogel's approximation method.

	To				Supply
From	10	20	5	7	10
	13	9	12	8	20
	4	5	7	9	30
	14	7	1	0	40
	3	12	5	19	50
Demand	60	60	20	10	

Solution:

Since $\sum a_i = \sum b_j = 150$, the given transportation problem is balanced.

There exists a basic feasible solution to this problem.

(i) The starting solution by NWC rule is as shown in the following table.

10 10	20	5	7
13 20	9	12	8
4 30	5	7	9
14	7 40	1	0
3	12 20	5 20	19 10

Since the number of non – negative allocations at independent positions is 7 which is less than $(m+n-1)=(5+4-1)=8$, this basic feasible solution is a degenerate one.

To resolve this degeneracy, we allocate a very small quantity $\in$ to the unoccupied cell (5, 1) so that the number of occupied cells becomes $(m+n-1)$. Hence the non – degenerate basic feasible solution is as shown in the following table.

10 10	20	5	7
13 20	9	12	8
4 30	5	7	9
14	7 40	1	0
3 $\in$	12 20	5 20	19 10

The initial transportation cost

$$= \text{Rs.}(10x10)+(13x20)+(4x30)+(7x40)+(3x\in)+(12x20)+(5x20)+(19x10)$$

$$= \text{Rs.}(1290+3\in)$$

$$= \text{Rs.}1290/-, \text{ as } \in \rightarrow 0.$$

(ii) Least Cost Method:

Using this method the starting solution is as shown in the following table:

10	20 10	5	7
13	9 20	12	8
4 10	5 20	7	9
14	7 10	1 20	0 10
3 50	12	5	19

Since the number of non – negative allocations at independent positions is $(m+n-1)=8$, the solution is non – degenerate basic feasible.

The initial transportation cost

$$= \text{Rs.}(20x10)+(9x20)+(4x10)+(5x20)+(7x10)+(1x20)+(0x10)+(3x50)$$

$$= \text{Rs.}760/-$$

(iii) Vogel's approximation Method:

The starting solution by this method is as shown in the following table:

10 10	20	5	7
13	9 20	12	8
4	5 30	7	9
14	7 10	1 20	0 10
3 50	12	5	19

Since the number of non – negative allocations is 7 which is less than $(m+n-1)=(5+4-1)=8$, this basic solution is a degenerate one.

To resolve this degeneracy, we allocate a very small quantity $\in$ to the unoccupied cell (5, 2) so that the number of occupied cells becomes $(m+n-1)$Hence the non – degenerate basic feasible solution is as shown in the following table.

10 10	20	5	7
13	9 20	12	8
4	5 30	7	9
14	7 10	1 20	0 10
3 50	12 $\in$	5	19

$\therefore$The initial transportation cost

$= \text{Rs.}(10x10)+(9x20)+(5x30)+(7x10)+(1x20)+(0x10)+(3x50)+(12x\in)$

$= \text{Rs.}(670+12\in)$

$= \text{Rs.}670/-= \text{as } \in \rightarrow 0$.

Example: 6.4.2

Solve the following transportation problem using Vogel's method.

Warehouse

		A	B	C	D	E	F	Available
Factory	1	9	12	9	6	9	10	5
	2	7	3	7	7	5	5	6
	3	6	5	9	11	3	11	2
	4	6	8	11	2	2	10	9
Requirement		4	4	6	2	4	2	

Solution:

Since $\sum a_i = \sum b_j = 22$, the given transportation problem is balanced. ∴ There exists a basic feasible solution to this problem. By Vogel's approximation method, the initial solution is as shown in the following table:

9	12	9 5	6	9	10
7	3 4	7	7	5	5 2
6 1	5	9 1	11	3	11
6 3	8	11	2 2	2 4	10

The initial transportation cost

$$= \text{Rs.}(9x5) + (3x4) + (5x2) + (6x1) + (5x \in) + (9x1) + (6x3) + (2x2) + (2x4)$$

$$= \text{Rs.}(122 + 5 \in) = \text{Rs.}112/-, \ \in \to 0.$$

To find the optimal solution

Now the number of non – negative allocations at independent position is $(m+n-1)$. We apply the MODI method.

9 6 3	12 5 7	9 5	6 2 4	9 2 7	10 7 3	$u_1 = 0$
7 4 3	3 4	7 7 0	7 0 7	5 0 5	5 2	$u_2 = -2$
6 1	5 ∈	9 1	11 2 9	3 2 1	11 7 4	$u_3 = 0$
6 3	8 5 3	11 9 2	2 2	2 4	10 7 3	$u_4 = 0$
$v_1 = 6$	$v_2 = 5$	$v_3 = 9$	$v_4 = 2$	$v_5 = 2$	$v_6 = 7$	

Since all $d_{ij} > 0$ with $d_{23} = 0$, the solution under the test is optimal and an alternative optimal solution is also exists.

$\therefore$ The optimum allocation schedule is given by
$x_{13} = 5,\ x_{22} = 4,\ x_{26} = 2,$
$x_{32} = \in,\ x_{33} = 1,\ x_{41} = 3,\ x_{44} = 2,\ x_{45} = 4$ and the optimum (minimum) transportation cost is

$$= Rs.(9x5) + (3x4) + (5x2) + (6x1) + (5x\in) + (9x1) + (6x3) + (2x2) + (2x4)$$

$$= Rs.(112 + 5\in)$$

$$= Rs.112 \text{ as } \in \rightarrow 0.$$

Example: 6.4.3

Solve the transportation problem:

To

					Supply
	1	2	3	4	6
From	4	3	2	0	8
	0	2	2	1	10
Demand	4	6	8	6	

Solution:

Since $\sum a_i = \sum b_j = 24$, the given transportation problem is balanced. $\therefore$ There exists a basic feasible solution.

By using Vogel's approximation method, the initial solution is as shown in the following table:

1	2 6	3	4
4	3	2 2	0 6
0 4	2	2 6	1

Since the number of non – negative allocations at independent positions, is 5, which is less than $(m+n-1)=(3+4-1)=6$, this basic feasible solution is degenerate.

To resolve degeneracy, we allocate a very small quantity $\in$ to the cell $(3, 2)$ so that the number of occupied cells becomes $(m+n-1)$. Hence the non – degenerate initial basic feasible solution is given by

1	2 6	3	4
4	3	2 2	0 6
0 4	2 $\in$	2 6	1

The initial transportation cost

$$= \text{Rs.}(2x6)+(2x2)+(0x6)+(0x4)+(2x\in)+(2x6)$$

$$= \text{Rs.}(28+2\in)$$

$$= \text{Rs.}28/-, \text{ as } \in \to 0.$$

To find the optimal solution

Now the number non – negative allocations at independent positions is $(m+n-1)$. We apply the MODI method.

1 0 1	2 6	3 2 1	4 0 4	$u_1 = 0$
4 0 4	3 2 1	2 2	0 6	$u_2 = 0$
0 4	2 $\in$	2 6	1 0 1	$u_3 = 0$
$v_1 = 0$	$v_2 = 2$	$v_3 = 2$	$v_4 = 0$	

Since all $d_{ij} > 0$ the solution under the test is optimal and unique.

$\therefore$ The optimum allocation schedule is given by $x_{12} = 6$, $x_{23} = 2$, $x_{24} = 6$, $x_{31} = 4$, $x_{32} = \in$, $x_{33} = 6$ and the optimum (minimum) transportation cost.

$$= \text{Rs.}(2x6)+(2x2)+(0x6)+(0x4)+(2x\in)+(2x6)$$

$$= \text{Rs.}(28+2\in) = \text{Rs.}28, \text{ as } \in \to 0.$$

Example: 6.4.4

Find the optimal solution of the following problem

Destination

		X	Y	Z	Supply
	P	1	2	0	30
Origin	Q	2	3	4	35
	R	1	5	6	35
Demand		30	40	30	

Solution:

Since $\sum a_i = \sum b_j = 100$, the given transportation problem is balanced.

By using the Vogel's approximation method, the basic feasible solution is displayed in the following table.

1	2	0 30
2	3 35	4
1 30	5 5	6

Since the number of non – negative allocations at independent positions is 4 which is less than $(m+n-1)=3+3-1=5$, this initial solution is degenerate.

To resolve degeneracy we allocate a very small quantity $\in$ to the cell (3, 3), so that the number of occupied cells becomes $(m+n-1)$. Hence the non – degenerate basic feasible solution is given by

1	2	0 30
2	3 35	4
1 30	5 5	6 $\in$

Space for Hints

Now the number of non – negative allocations at independent positions is $(m+n-1)=5$. We apply MODI method.

1 -5 6	2 -1 3	0 30	$u_1 = -6$
2 -1 3	3 35	4 4 0	$u_2 = -2$
1 30	5 5	6 $\in$	$u_3 = 0$
$v_1 = 1$	$v_2 = 5$	$v_3 = 6$	

Since all $d_{ij} > 0$ with $d_{23} = 0$, the solution under the test is optimal and there exists an alternative optimal solution.

$\therefore$ The optimal allocation schedule is given by $x_{13} = 30$, $x_{22} = 35$, $x_{31} = 30$, $x_{32} = 5$, $x_{33} = \in$ and the optimum (minimum) transportation cost.

$$= \text{Rs.}(0x30) + (3x35) + (1x30) + (5x5) + (6x\in)$$

$$= \text{Rs.}(160 + 6\in)$$

$$= \text{Rs.}160/- \text{ as } \in \rightarrow 0.$$

Example: 6.4.5

Solve the following transportation problem to minimize the total cost of transportation.

		Destination				
		1	2	3	4	Supply
	1	14	56	48	27	70
Origin	2	82	35	21	81	47
	3	99	31	71	63	93
Demand		70	35	45	60	210

Space for Hints

Solution:

Since $\sum a_i = \sum b_j = 210$, the given transportation problem is balanced. ∴ There exists a basic feasible solution to this problem.

By using Vogel's approximation method, the initial solution is as shown in the following table:

14 70	56	48	27
82	35	21 45	81 2
99	31 35	71	63 58

Since the number of non – negative allocations is 5, which is less than $(m+n-1) = (3+4-1) = 6$, this basic feasible solution is degenerate.

To resolve degeneracy, we allocate a very small quantity $\in$ to the cell (1, 4), so that the number of occupied cells becomes $(m+n-1)$. Hence the non – degenerate basic feasible solution is given in the following table.

14 70	56	48	27 $\in$
82	35	21 45	81 2
99	31 35	71	63 58

To find the optimal solution:

Now the number of non – negative allocations at independent positions is $(m+n-1)=6$. We apply MODI method.

14 70	56 -5 61	48 -33 81	27 ∈	$u_1 = 27$
82 68 14	35 49 -14	21 45	81 2	$u_2 = 81$
99 50 49	31 35	71 3 68	63 58	$u_3 = 63$
$v_1 = -13$	$v_2 = -32$	$v_3 = -60$	$v_4 = 0$	

Since $d_{22} = -14 < 0$, the solution under the test is not optimal.

Now let us form a new basic feasible solution by giving maximum allocation to the cell (2, 2) by making an occupied cell empty. We draw a closed loop consisting of horizontal and vertical lines beginning and ending at this cell (2, 2) and having its other corners at some occupied cells. Along the closed loop, indicate $+\theta$ and $-\theta$ alternatively at the corners.

14 70	56	48	27 ∈
82	35 $+\theta$	21 45	81 2
99	31 35 $-\theta$	71	63 $+\theta$ 58

From the two cells (2, 4), (3, 2) having $-\theta$ we find that the minimum of the allocations 2, 35 is 2. Add this 2 to the cells with $+\theta$ and subtract this 2 to the cells with $-\theta$. Hence the new basic feasible solution is given by

14 70	56	48	27 $\in$
82	35 2	21 45	81
99	31 33	71	63 60

We see that the above table satisfies the rim conditions with $(m+n-1)$non – negative allocations at independent positions. We apply MODI method for optimality.

14 70	56 -5 61	48 -19 67	27 $\in$	$u_1 = -40$
82 54 28	35 2	21 45	81 67 14	$u_2 = 0$
99 50 49	31 33	71 17 54	63 60	$u_3 = -4$
$v_1 = 54$	$v_2 = 35$	$v_3 = 21$	$v_4 = 67$	

Since all $d_{ij} > 0$, the solution under the test is optimal.

$\therefore$The optimal allocation schedule is given by $x_{11} = 70$, $x_{14} = \in$, $x_{22} = 2$, $x_{32} = 33$, $x_{34} = 30$ and the optimum (minimum) transportation cost.

$$= Rs.(14x70)+(27x\in)+(35x2)+(21x45)+(31x33)+(63x60)$$

$$= Rs.6798/- \text{ as } \in \to 0.$$

Example: 6.4.6

Solve the following transportation problem, in which a_i is the availability at origin O_i and b_j is the requirement at the destination D_j and cell entries are unit costs of transportation from any origin to any destination:

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	4	7	3	8	2	4
O_2	1	4	7	3	8	7
O_3	7	2	4	7	7	9
O_4	4	8	2	4	7	2
b_j	8	3	7	2	2	

Solution:

Since $\sum a_i = \sum b_j = 22$, the given problem is balanced. ∴ There exists a basic feasible solution to this problem.

By using Vogel's approximation method, the initial solution is as shown in the following table:

4 1	7	3 1	8	2 2
1 7	4	7	3	8
7	2 3	4 6	7	7
4	8	2 $\in$	4 2	7

Since the number of non – negative allocations is 7, which is less than $(m+n-1)=(4+5-1)=8$, this basic feasible solution is degenerate.

To resolve this degeneracy, we allocate a very small quantity $\in$ to the cell (4, 3), so that the number of occupied cells becomes $(m+n-1)$. Hence the non – degenerate basic feasible solution is given in the following table.

4 1	7	3 1	8	2 2
1 7	4	7	3	8
7	2 3	4 6	7	7
4	8	2 $\in$	4 2	7

To find the optimal solution:

Now the number of non – negative allocations at independent positions is $(m+n-1)=8$. We apply MODI method.

4 1	7 1 6	3 1	8 5 3	2 2	$u_1=0$
1 7	4 -2 6	7 0 7	3 2 1	8 -1 9	$u_2=-3$
7 5 2	2 3	4 6	7 6 1	7 3 4	$u_3=1$
4 3 1	8 0 8	2 $\in$	4 2	7 1 6	$u_4=-1$
$v_1=4$	$v_2=1$	$v_3=3$	$v_4=5$	$v_5=2$	

Since all $d_{ij}>0$, the solution under the test is optimal.

$\therefore$ The optimal allocation schedule is given by $x_{11}=1,\ x_{13}=1,\ x_{15}=2,\ \ x_{21}=7,\ x_{32}=3,\ x_{33}=6,\ \cancel{x_{43}}=\in,\ x_{44}=2$ and the optimum (minimum) transportation cost

$$= Rs.(4x1)+(3x1)+(2x2)\cancel{+(1x7)}+(2x3)+(4x6)+(2x\in)+(4x2)$$

$$= Rs.(56+2\in)$$

$$= Rs.56/- \text{ as } \in \rightarrow 0.$$

Check your progress: 6.2

1 A company has three factories $F_i(i=1,2,3)$ from which it transports the product to four warehouses $W_j(j=1,2,3,4)$. The unit cost of production at the three factories are Rs. 4, 3, 5 respectively. Given the following information on unit costs of transportation, capacities at the three factories and of the requirement at the four warehouses, find the optimum allocation.

Factory	Unit cost. of production Rs./unit	Transportation cost Rs./unit				Capacity
		W_1	W_2	W_3	W_4	
F_1	4	5	7	3	8	300
F_2	3	4	6	9	5	500
F_3	5	2	6	4	5	200
Requirements		200	300	400	100	1,000

2. A company has three cement factories A, B and C, and four area distributors W, X, Y and Z. With identical costs of production at the three factories, the only variable cost involved is the transportation cost. The monthly production capacity (in tone) of the three factories, monthly demand of the four distributors, and the transportation cost per tone (in rupees) from the different factories to different distribution centers, are given below:

Factory	Distributor				Monthly availability
	W	X	Y	Z	
A	20	25	50	10	45,000
B	45	50	15	40	50,000
C	22	10	45	35	55,000
Monthly demand	50,000	40,000	30,000	30,000	

Suggest an optimal transport schedule and find the minimum transportation cost.

6.5 Unbalanced Transportation Problems

If the given transportation problem is unbalanced one, i.e., if $\sum a_i \neq \sum b_j$, then convert this into a balanced one by introducing a dummy source or dummy destination with zero cost vectors (zero unit transportation costs) as the case may be and then solve by usual method.

When the total supply is greater than the total demand, a dummy destination is included in the matrix with zero cost vectors. The excess supply is entered as a rim requirement for the dummy destination.

When the total demand is greater than the total supply, a dummy source is included in the matrix with zero cost vectors. The excess demand is entered as a rim requirement for the dummy source.

Example: 6.5.1

Solve the transportation problem

		Destination				
		A	B	C	D	Supply
	1	11	20	7	8	50
Source	2	21	16	20	12	40
	3	8	12	18	9	70
Demand		30	25	35	40	

Solution:

Since the total supply ($\sum a_i = 160$) is greater than the total demand ($\sum b_j = 130$), the given problem is an unbalanced transportation problem. To convert this into a balanced one, we introduce a dummy destination E with zero unit transportation costs and having demand equal to 160 – 130 = 30 units.

∴ The given problem becomes

Destination

		A	B	C	D	E	Supply
	1	11	20	7	8	0	50
Source	2	21	16	20	12	0	40
	3	8	12	18	9	0	70
		30	25	35	40	30	160

By using VAM the initial solution is as shown in the following table

11	20	7 35	8 15	0
21	16	20	12 10	0 30
8 30	12 25	18	9 15	0

The initial transportation cost

$$= Rs.(7x35)+(8x15)+(12x10)+(0x30)+(8x30)+(12x25)+(9x15)$$

$$= Rs.1160/-$$

$= Rs.(7x35)+(8x15)+(12x10)+(0x30)+(8x30)+(12x25)+(9x15)$

$= Rs.1160/-$

For optimality:

Since the number non – negative allocations at independent positions is $(m+n-1)$, we apply the MODI method.

11 7 4	20 11 9	7 35	8 15	0 -4 4	$u_1 = 8$
21 11 10	16 15 1	20 11 9	12 10	0 30	$u_2 = 12$
8 30	12 25	18 8 10	9 15	0 -3 3	$u_3 = 9$
$v_1 = -1$	$v_2 = 3$	$v_3 = -1$	$v_4 = 0$	$v_5 = -12$	

Since all $d_{ij} > 0$, the solution under the test is optimal and unique.

$\therefore$ The optimum allocation schedule is

$x_{13} = 35,\ x_{14} = 15,\ x_{24} = 10,\ x_{25} = 30,\ x_{31} = 30,\ x_{32} = 25,\ x_{34} = 15$

It can be noted that $x_{25} = 30$ means that 30 units are dispatched from source 2 to the dummy destination E. In other words, 30 units are left un dispatched from source 2.

The optimum (minimum) transportation cost

$= Rs.(7x35)+(8x15)+(12x10)+(0x30)+(8x30)+(12x25)+(9x15)$

$= Rs.1160/-$

Example: 6.25

Solve the transportation problem with unit transportation costs, demands and supplies as given below:

		Destination				
		D_1	D_2	D_3	D_4	Supply
	S_1	6	1	9	3	70
Source	S_2	11	5	2	8	55
	S_3	10	12	4	7	70
Demand		85	35	50	45	

Solution:

Since the total demand $\left(\sum b_i = 215\right)$ is greater than the total supply $\left(\sum a_j = 195\right)$, the given problem is unbalanced transportation problem. To convert this into a balanced one, we introduce a dummy source S_4 with zero unit transportation costs and having supply equal to $215-195=20$units. $\therefore$ The given problem becomes

		Destination				
		D_1	D_2	D_3	D_4	Supply
	S_1	6	1	9	3	70
Source	S_2	11	5	2	8	55
	S_3	10	12	4	7	70
	S_4	0	0	0	0	20
		85	35	50	45	215

As this problem is balanced, there exists a basic feasible solution to this problem. By using Vogel's approximation method, the initial solution is as shown in the following table.

6 65	1 5	9	3
11	5 30	2 25	8
10	12	4 25	7 45
0 20	0	0	0

$\therefore$ The initial transportation cost

$$= \text{Rs.}(6\text{x}65)+(1\text{x}5)+(5\text{x}30)+(2\text{x}25)+(4\text{x}25)+(7\text{x}45)+(0\text{x}20)$$

$$= \text{Rs.}1010/-$$

For optimality:

Since number of non – negative allocations at independent positions is $(m+n-1)$, we apply the MODI method:

6 65	1 5	9 -2 11	3 1 2	$u_1 = 6$
11 10 1	5 30	2 25	8 5 3	$u_2 = 10$
10 12 -2	12 7 5	4 25	7 45	$u_3 = 12$
0 20	0 -5 5	0 -8 8	0 -5 5	$u_4 = 0$
$v_1 = 0$	$v_2 = -5$	$v_3 = -8$	$v_4 = -5$	

Since $d_{31} = -2 < 0$, the solution under the test is not optimal.

Now let us form a new basic feasible solution by giving maximum allocation to the cell (3, 1) (since d_{31} is –ve) by making an occupied cell empty. For this, we draw a closed path consisting of horizontal and vertical lines beginning and ending at this cell (3, 1) and having its other corners at some occupied cells. Along this closed loop indicate $+\theta$ and $-\theta$ alternatively at the corners.

We have

6 65 $-\theta$	1 $5+\theta$	9	3
11	5 30 $-\theta$ 5	2 25 $+\theta$	8
10 $+\theta$	12	4 25 $-\theta$	7 45
0 20	0	0	0

From the three cells (1, 1), (2, 2), (3, 3) having $-\theta$, we find the minimum of the allocations 65, 30, 25 is 25. Add this 25 to the cells with $+\theta$and subtract this 25 to the cells with $-\theta$. Finally, the new basic feasible solution is displayed in the following table.

6 40	1 30	9	3
11	5 5	2 50	8
10 25	12	4	7 45
0 20	0	0	0

We see that that the above table satisfies the rim conditions with $(m+n-1)$ non – negative allocations at independent positions. Now we check for optimality.

6 40	1 30	9 -2 11	3 3 0	$u_1 = 6$
11 10 1	5 5	2 50	8 7 1	$u_2 = 10$
10 25	12 7 5	4 2 2	7 45	$u_3 = 10$
0 20	0 -5 5	0 -8 8	0 -3 3	$u_4 = 0$
$v_1 = 0$	$v_2 = -5$	$v_3 = -8$	$v_4 = -3$	

Since all $d_{ij} > 0$ with $d_{14} = 0$, the solution under the test is optimal and an alternative optimal solution exists.

$\therefore$ The optimal allocation schedule is given by

$x_{11} = 40,\ x_{12} = 30,\ x_{22} = 5,\ x_{23} = 50,\ x_{31} = 25,\ x_{34} = 45,\ x_{41} = 20.$

It can be noted that $x_{41} = 20$ means that 20 units are dispatched from the dummy source S_4 to the destination D_1. In other words, 20 units are not fulfilled for the destination D_1.

The optimum (minimum) transportation cost

$= \text{Rs.}(6x40) + (1x30) + (5x5) + (2x50) + (10x25) + (7x45) + (0x20)$

$= \text{Rs.}960/-$

Example: 6.5.3

Solve the transportation problem with unit transportation costs in rupees, demands and supplies as given below:

Destination

		D_1	D_2	D_3	Supply (units)
	A	5	6	9	100
Origin	B	3	5	10	75
	C	6	7	6	50
	D	6	4	10	75
Demand (units)		70	80	120	

Solution:

Since the total supply $(\sum a_i = 300)$ is greater than the total demand $(\sum b_j = 270)$, the given transportation problem is unbalanced.

To convert this into a balanced one, we introduce a dummy source D_4 with zero unit transportation costs and having demand equal to $300 - 270 = 30$units. ∴ The given problem becomes

Destination

		D_1	D_2	D_3	D_4	Supply
	A	5	6	9	0	100
Origin	B	3	5	10	0	75
	C	6	7	6	0	50
	D	6	4	10	0	75
Demand		70	80	120	30	300

By using VAM the initial solution is given by

5	6	9 100	0
3 70	5 5	10	0
6	7	6 20	0 30
6	4 75	10	0

Since the number of non – negative allocations is 6, which is less than $(m+n-1)=4+4-1=7$, this basic feasible solution is degenerate.

To resolve this degeneracy, we allocate a very small quantity $\in$ to the cell (2, 4) so that the number of occupied cells becomes $(m+n-1)$. Hence the non – degenerate basic feasible solution is given in the following table

5	6	9 100	0
3 70	5 5	10	0 $\in$
6	7	6 20	0 30
6	4 75	10	0

Now the number of non – negative allocations at independent positions is $(m+n-1)$. We apply MODI method.

5 6 -1	6 8 -2	9 100	0 3 -3	$u_1=3$
3 70	5 5	10 6 4	0 $\in$	$u_2=0$
6 3 3	7 5 2	6 20	0 30	$u_3=0$
6 2 4	4 75	10 5 5	0 -1 1	$u_4=-1$
$v_1=3$	$v_2=5$	$v_3=6$	$v_4=0$	

Since there are some $d_{ij} < 0$, the current solution is not optimal.

Since $d_{14} = -3$ is the most negative, let us form a new basic feasible solution by giving maximum allocation to the corresponding cell (1, 4) by making an occupied cell empty. We draw a closed loop consisting of horizontal and vertical lines beginning and ending at this cell (1, 4) and having its other corners at some occupied cells. Along this closed loop indicate $+\theta$ and $-\theta$ alternatively at the corners.

We have

5	6	9 $-\theta$ 100	0 $+\theta$
3 70	5 5	10	0 $\in$
6	7	6 20 $+\theta$	0 30 $-\theta$
6	4 75	10	0

From the two cells (1, 3), (3, 4) having $-\theta$, we find the minimum of the allocations 100, 30 is 30. Add this 30 to the cells with $+\theta$ and subtract this 30 to the cells with $-\theta$. Hence the new basic feasible solution is given in the following table.

5	6	9 70	0 30
3 70	5 5	10	0 $\in$
6	7	6 50	0
6	4 75	10	0

We see that the above table satisfies the rim conditions with

$(m+n-1)$ non – negative allocations at independent positions. So we apply MODI method.

5 3 2	6 5 1	9 70	0 30	$u_1 = 0$
3 70	5 5	10 9 1	0 $\in$	$u_2 = 0$
6 0 6	7 2 5	6 50	0 -3 3	$u_3 = -3$
6 2 4	4 75	10 8 2	0 -1 1	$u_4 = -1$
$v_1 = 3$	$v_2 = 5$	$v_3 = 9$	$v_4 = 0$	

Since all $d_{ij} > 0$, the current solution is optimal and unique.

The optimum allocation schedule is given by

$x_{13} = 70,\ x_{14} = 30,\ x_{21} = 70,\ x_{22} = 55,\ x_{24} = \in,\ x_{33} = 50,\ x_{42} = 75$

and the optimum (minimum) transportation cost

$= \text{Rs.}(9x70) + (0x30) + (3x70) + (5x5) + (0x\in) + (6x50) + (4x75)$

$= \text{Rs.}1465/-$

Check your progress: 6.3

1. A company has three warehouses in cities A, B and C. These warehouse have the following quantities of the product in stock.

Distribution ships:	A	B	C
Capacity:	100 units	80 units	80 units

The four customers have the demand as follows:

(1) 60 units (2) 120 units (3) 50 units (4) 40 units

If cost is 20 paise per km to transport this product from a distributor to customer. The distance between warehouses and customers can be obtained from a map or suitable route tables as follows:

$$\begin{array}{ccc}
 & & \text{Customer} \\
 & & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\text{Warehouse} & \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 270 & 230 & 310 & 690 \\ 100 & 450 & 400 & 320 \\ 300 & 540 & 350 & 570 \end{bmatrix}
\end{array}$$

Which warehouse should deliver how much product to which customer so that the total transportation cost becomes a minimum?

2. A company has three plants at locations A, B and C which supply to warehouse located at D, E, F, G and H. Monthly plant capacities are 800, 500 and 900 units respectively. Monthly warehouse requirements are 400, 400, 500, 400 and 800 units respectively. Unit transportation costs (in rupees) are given below:

$$\begin{array}{ccc}
 & & \text{To} \\
 & & \begin{array}{ccccc} D & E & F & G & H \end{array} \\
\text{From} & \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 5 & 8 & 6 & 6 & 3 \\ 4 & 7 & 7 & 6 & 5 \\ 8 & 4 & 6 & 6 & 4 \end{bmatrix}
\end{array}$$

Determine an optimum distribution for the company in order to minimize the total transportation.

6.6 Keywords:

Transportation problem, North West Corner rule, Vozel approximation Method, Modi Method, Least Cost Method.

6.7 Answers to check your progress questions

Check your progress: 6.1

1. $x_{11} = 6, x_{22} = 4, x_{23} = 1, x_{33} = 3, x_{43} = 1, x_{44} = 3, x_{45} = 2, x_{46} = 3$

2. $x_{11} = 16, x_{12} = 3, x_{22} = 15, x_{23} = 22, x_{33} = 9$ and $x_{34} = 25$.

3. (a)
$x_{11} = 250, x_{12} = 50, x_{13} = 200, x_{23} = 550, x_{32} = 300, x_{44} = 200$
.(b)
$x_{11} = 250, x_{12} = 50, x_{13} = 200, x_{23} = 700, x_{32} = 300, x_{43} = 150,$
$x_{44} = 200, x_{45} = 150$; Minimum cost = 13,650

4. $x_{11} = 6, x_{12} = 6, x_{13} = 17, x_{14} = 5, x_{21} = 15, x_{34} = 12, x_{43} = 19$.

Check your progress: 6.2

1. $x_{13} = 300, x_{21} = 100, x_{22} = 300, x_{24} = 100, x_{31} = 100$ and $x_{33} = 100$; Minimum cost = Rs. 14,700.
2. AW=15,000, AZ=30,000, BW=20,000, BY=30,000, CW=15,000 CX = 40,000, Minimum cost = Rs. 26,80,000

Check your progress: 6.3

1. $x_{12} = 100, x_{21} = 40, x_{24} = 40, x_{31} = 20, x_{32} = 10, x_{33} = 50$
Total cost = Rs. 14,700.
2. $x_{15} = 800, x_{21} = 400, x_{24} = 100, x_{32} = 400, x_{33} = 200, x_{34} = 300$
Total cost = Rs. 9,200.

6.8 Model Questions:

1. National Oil Company (NOC) has three refineries and four deposits. Transportation cost per ton, capacities, and requirements are given below:

	D_1	D_2	D_3	D_4	Capacity (tons)
R_1	5	7	13	10	700
R_2	8	6	14	13	400
R_3	12	10	9	11	800
Requirement	200	600	700	400	

Space for Hints

Determine optimum allocation of output.

2. Four gasoline dealers A, B, C and D require 50, 40, 60 and 40 KL of gasoline respectively. It is possible to supply these from locations 1, 2 and 3 which have 80, 100 and 50 KL respectively. The cost (in Rs.) for shipping every KL is shown in the table below:

		A	B	C	D
	1	7	6	6	6
Location	2	5	7	6	7
	3	8	5	8	6

Determine the most economical supply pattern.

3. The following data is given

		Destinations			
		1	2	3	Capacities
	1	2	2	3	10
Sources	2	4	1	2	15
	3	1	3	X	40
Demands		20	15	30	

The cost of shipment from third source to the third destination is not known. How many units should be transported from sources to destinations so that the total cost of transporting all the units to their destinations is a minimum?

4. Solve the following transportation problem:

From	To			Available
	A	B	C	
I	6	8	4	14
II	4	9	8	12
III	1	2	6	5
Demand	6	10	15	

5. Three fertilizer factories X, Y and Z located at different places of the country produce 6, 4 and 5 lakh tones of urea respectively. Under the directive of the Central Government, they are to be distributed to 3 States A, B and C as 5, 3 and 7 lakh tones respectively. The transportation cost per tone in rupees is given below:

	A	B	C
X	11	17	16
Y	15	12	14
Z	20	12	15

Find out suitable transportation pattern at minimum cost.

6. A company has three warehouse W_1, W_2 and W_3. It is required to deliver a product from these warehouse to three customers A, B and C. The warehouses have the following units in stock.

Warehouse	W_1	W_2	W_3
No. of units	65	42	43

and customer requirements are:

Customer	A	B	C
No. of units	70	30	50

The table below shows the costs of transporting one unit from warehouse to the customer:

		Warehouse		
		W_1	W_2	W_3
Customer	A	5	7	8
	B	4	4	6
	C	6	7	7

Find the optimum transportation route.

7. Consider a transportation problem with $m = 3$ and $n = 4$, where:

$c_{11} = 2, \quad c_{12} = 3, \quad c_{13} = 11, \quad c_{14} = 7$

$c_{21} = 1, \quad c_{22} = 0, \quad c_{23} = 6, \quad c_{24} = 1$

$c_{31} = 5, \quad c_{32} = 8, \quad c_{33} = 15, \quad c_{34} = 9$

Suppose $S_1 = 6$, $S_2 = 1$ and $S_3 = 10$ whereas $D_1 = 7$, $D_2 = 5$, $D_3 = 3$ and $D_4 = 2$. Apply the transportation simplex method to find out an optimal solution.

8. A company has three factories at Amethi, Baghpat and Gwalior; and four distribution centers at Allahabad, Mumbai, Calcutta and Delhi. With identical cost of production at the three factories the only variable cost involved is transportation cost. The production at the three factories is 5, 000 tonnes; 6, 000 tonnes and 2, 500 tonnes respectively. The demand at four distribution centers is 6, 000 tonnes; 4, 000 tonnes; 2, 000 tonnes and 1, 500 tonnes respectively. The transportation costs per tone from different centers are given below:

Factory	Distribution centre			
	Allahabad	Mumbai	Calcutta	Delhi
Amethi	3	2	7	6
Baghpat	7	5	2	3
Gwalior	2	5	4	5

Suggest the optimum transportation schedule and find the minimum cost of transportation.

9. A wholesale company has three warehouse from which supplies are drawn for four retail customers. The company deals in a single product, the supplies of which at each warehouse are:

Warehouse no.	Supply (units)	Customer no.	Demand (units)
1	20	1	15
2	28	2	19
3	17	3	13
		4	18

Conveniently, total supply at the warehouse is equal to total demand from the customers. The following table gives the transportation costs per unit shipment from each warehouse to each customer.

Warehouse	Customer			
	1	2	3	4
1	3	6	8	5
2	6	1	2	5
3	7	8	3	9

Determine what supplies to dispatch from each of the warehouse to each customer so as to minimize overall transportation cost.

UNIT – 7

INVENTORY CONTROL

Introduction:

Inventory may be defined as the stock of goods, commodities or other economic resources that are stored or reserved for smooth and efficient running of business affairs. The Inventory may be kept in any one of the following forms:

(i) Raw material Inventory.

Raw materials which are kept in stock for using in production of goods.

(ii) Work in Process Inventory.

Semi finished goods which are stored during production process.

(iii) Finished goods Inventory.

Finished goods awaiting shipments from the factory.

Objectives:

Structure:

7.1 Types of Inventory and Economic order Quantity:

Types of Inventory

(i) **Fluctuation Inventories**

In real – life problems, there are fluctuations in the demand and lead times that affect the production of the items. Such type of safety stock are called fluctuation Inventories.

(ii) **Anticipated Inventories**

These are built up in advance for the season of large sales, promotion programme or a plant shut down period. Anticipated Inventories keep men and machine hours for future participation.

(iii) **Lot – Size Inventories**

Generally rate of consumption is different from rate of production or purchasing. Therefore the items are produced in large quantities, which result in lot – size Inventories.

Reasons for maintaining Inventory

1. Inventory helps in smooth and efficient running of business.
2. It provides service to the customers at short notice.
3. Because of long – uninterrupted runs, production cost is less.
4. It acts as a buffer stock if shop rejections are too many.
5. It takes care of economic fluctuations.

The Inventory Decisions

Inventory Control is the process of deciding what and how much of various items are to be kept in stock. It also determines the time and quantity of various items to be produced. The basic objective of inventory control is to reduce investment in inventories and ensuring that production process does not suffer at the same time. To attain various objectives in an inventory control situation, there are two basic questions to be answered.

They are:

(i) How much to order? That is what is the optimum quantity of an item that should be ordered.

(ii) When should the order be placed?

The answer to the first question determines the economic order quantity (EOQ) by minimizing the total inventory cost, which is given by

Purchase Cost + Set up Cost + Carrying Cost + Shortage Cost

The answer to the second question (when to order?) depends on the type of inventory system with which we are dealing. If the system requires periodic review (e.g., every month or year), the time for receiving a new order coincides with the start of each period. Alternatively, if the system is based on continuous review, new orders are placed when the inventory level drops to a pre – specified level, called the reorder point.

Costs Associated with Inventories:

Various costs associated with inventory control are often classified as follows:

Set – up cost:

This is the cost associated with the setting up of machinery before starting production. Set – up cost is generally assumed to be independent of the quantity ordered for or produced.

Ordering cost:

This is a cost associated with ordering of raw material for production purposes. Advertisements, consumption of stationery and postage, telephone charges telegrams, rent for space used by the purchasing department, travelling expenditures incurred, etc., constitute the ordering cost.

Purchase (or production cost)

The cost of purchasing (or producing) a unit of an item is known as purchase (or production) cost. The purchase price will become important when quantity discounts are allowed for purchases above a certain quantity or when economies of scale suggest that the per unit production cost can be reduced by a larger production run.

Carrying (or holding) cost:

The carrying cost is associated with carrying (or holding) inventory. This cost generally includes the costs such as rent for space used for storage, interest on the money locked – up, insurance of stored equipment, production, taxes depreciation of equipment and furniture used, etc.

Shortage (or stock out) cost:

The penalty cost for running out of stock (i.e., when an item cannot be supplied on the customer's demand) is known as shortage cost. This cost includes the loss of potential profit through sales of items and loss of goodwill, in terms of permanent loss of customers and its associated lost profit in future sales.

Salvage cost (or selling price):

When the demand for certain commodity is affected by the quantity stocked, decision problem is based on a profit maximization criterion that includes the revenue from selling. Salvage value may be combined with the cost of storage and hence is generally neglected.

Revenue Cost:

When it is assumed that both the price and the demand of the production are not under control of the organization, the revenue from the sales is independent of the company's inventory policy and may be neglected except for the situation when the organization cannot meet the demand and the sale is lost. Therefore, the revenue cost may or may not be included in the study of inventory policy.

Factors Affecting Inventory Control:

Besides the costs that determine the profitability, other factors which play an important role in the study of inventory control are the following:

Demand:

The number of units required per period is called demand. The demand pattern of a commodity may be either deterministic or probabilistic.

Lead time:

The time gap between placing of an order and its actual arrival in the inventory is known as lead time.

Lead time has two components, namely the administrative lead time – from initiation of procurement action until the placing of an order, and the delivery lead time – from placing of an order until the delivery of the ordered material.

Order cycle:

The time period between placement of two successive orders is referred to as an order cycle. The order may be placed on the basis of following two types of inventory review system:

(a) Continuous review: The record of the inventory level is checked continuously until a specified point (called recorded point) is reached where a new order is placed. This is often referred to as the two – bin system. This divides the inventory into two parts and places it physically, or on paper, in two bins. Items are drawn from only one bin and when it is empty, a new order is placed. Demand is then satisfied from the second bin until the order is received. Upon receipt of the order, enough items are placed in the second bin to make up the earlier total. The remaining items are placed in the first bin. This procedure is then repeated.

(b) Periodic Review: In this system the inventory levels are viewed at equal time intervals and orders are placed at such intervals. The quantity ordered each time depends on the available inventory level at the time of review.

Time Horizon:

The time period over which the inventory level will be controlled is called the time horizon. This horizon may be finite or infinite depending upon the nature of the demand for the commodity.

Re-order level:

The level between maximum and minimum stock, at which purchasing (or manufacturing) activities must start for replenishment, is known as re – order level.

Stock replenishment:

Although an inventory problem may operate with lead time the actual replacement of stock may occur instantaneously or uniformly. Instantaneous replenishment occurs in case the stock may occur instantaneously or uniformly. Instantaneous replenishment occurs in case the stock is purchased from outside sources whereas the uniform replenishment may occur when the product is manufactured by the company.

Economic Order Quantity (EOQ)

By the order quantity we mean the quantity produced or procured during one production cycle. When the size of order increases, the ordering cost (cost of purchasing, inspection, etc.) will increase. Thus in the production process there are two opposite costs, one encourages the increase in the order size and the other discourages. Economic Order Quantity (EOQ) is that size of order which minimizes total annual costs of carrying inventory and cost of ordering.

The two opposite costs can be shown graphically by plotting them against the order size as shown in Fig. 7.1 below:

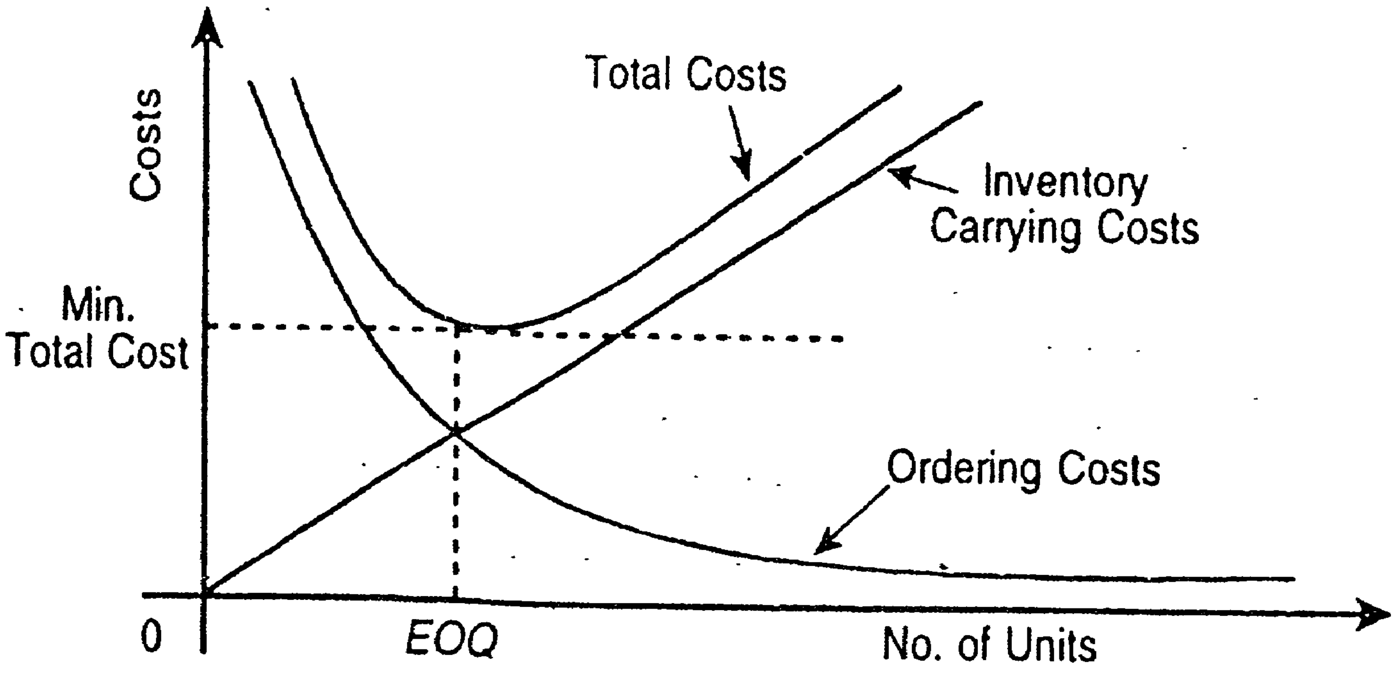


Figure 7.1 Graph of EOQ

It is evident from above that the minimum total cost occurs at the point where the ordering costs and inventory carrying costs are equal.

7.2 Deterministic Inventory Problems with no Shortages:

This section presents five variations of the EOQ problem when demand is assumed to be fixed and completely pre – determined.

Case: 1

The Fundamental EOQ Problem:

The objective of the study of this problem is to determine an optimum order quantity (EOQ) such that the total inventory cost is minimized. We illustrate the problem, under consideration, using the following assumptions:

Demand is known and uniform.

(i) Let D denote the total number of units purchased/produced or supplied per time period and Q denote the lot size in each production run.

(ii) Shortages are not permitted, i.e., as soon as the level of the inventory reaches zero, the inventory is replenished.

(iii) Production or supply of commodity is instantaneous (Abundant Availability).

(iv) Lead time is zero.

(v) Set – up cost per production run on procurement cost is C_s (or A).

(vi) Holding cost is C_1 per unit in inventory for a unit, i.e., $C_1 = IC$, where C is the unit cost, I is called inventory carrying charge expressed as a % of the value of the average inventory.

This fundamental situation can be shown on an inventory – time diagram, with Q on the vertical axis and time on the horizontal axis. The total time period (one year) is divided into n parts:

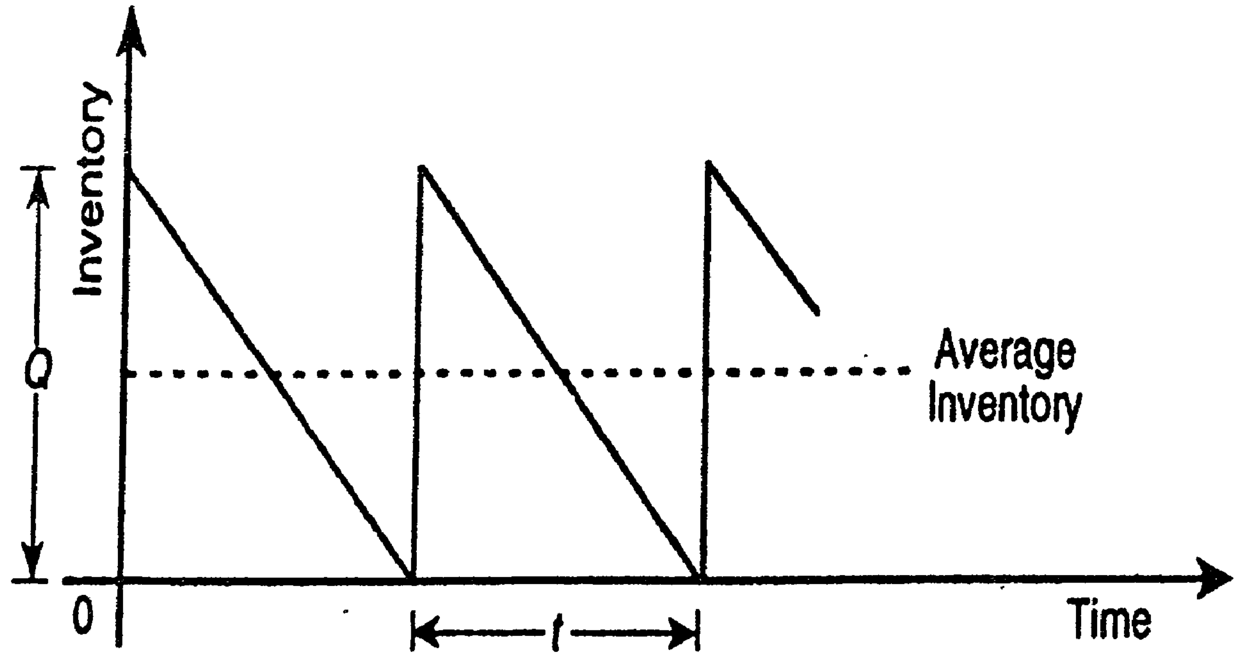


Figure 7.2 EOQ Problem with uniform demand

Here it is assumed that after each time t, the quantity Q is produced/purchased on supplied throughout the entire time period, say one year. Now, if n denotes the total number of runs of the quantity produced or purchased during the year, then clearly we have

$$1 = nt \text{ and } D = nQ.$$

It may be clear that the average amount of inventory at hand on any day is then $\frac{1}{2}Qt$ as shown by dotted line in Fig. 7.2. Total inventory over the time period t days is clearly the area of the first triangle $\left(= \frac{1}{2}Qt\right)$. Thus the average inventory at any time on any given day in the t period is $\frac{1}{2}Qt/t = \frac{1}{2}Q$.

Now, since each of the triangles in Fig. 19.2 over a year period looks the same, $\frac{1}{2}Q$ remains the average amount of inventory in each interval of length t during the entries period. Annual inventory holding cost is therefore given by

$$f(Q) = \frac{1}{2}QC_1$$

Annual costs associated with runs of size Q are given by

$$g(Q) = nC_s = \frac{D}{Q}C_s$$

Since the minimum total cost occurs at the point where ordering cost and the total inventory carrying cost are equal, we must have

$$f(Q) = g(Q).$$

This implies, that $\frac{1}{2}QC_1 = \frac{D}{Q}C_s$

Hence, the optimum value of Q is $Q^\circ = \sqrt{\frac{2DC_s}{C_1}}$

This is known as the economic (optimum) lot size formula due to R.H. Wilson.

The above EOQ formula can also be expressed in terms of the economic order value terms as follows:

$$Q^\circ = \sqrt{\frac{2AD}{IC}}$$

Characteristics of Case: 1

1. Optimum number of orders placed per year

$$n^\circ = D/Q^\circ = \sqrt{DC_1/2C_s}$$

2. Optimum length of time between orders

$$t^\circ = T/n^\circ = T\sqrt{2C_s/DC_1} \text{ or } \sqrt{2C_s/DC_1},$$

when T (total time horizon) is one year.

3. Minimum total annual inventory cost

$$TC^\circ = \frac{1}{2}Q^\circ C_1 + DC_s/Q^\circ = \sqrt{2DC_1C_s}$$

Remark:

If the carrying cost is given as a percentage of average value of inventory held, then total annual carrying cost may be expressed as $C_1 = Cx1$ or P x 1. The total annual inventory cost then becomes

$$TC = \frac{1}{2}Q \text{ x } C \text{ x } I + DC_s / Q$$

The optimum order quantity Q° will, then, be

$$Q^\circ = \sqrt{2DC_s / CI}$$

Corollary: 1

In the EOQ problem discussed above, if the set – up cost is $C_S + bQ$ instead of being fixed (where b is set – up cost per unit item produced) then there is no change in the optimum order quantity produced due to change in the set – up cost.

Proof:

In this case, the annual cost is given by

$$TC = \frac{1}{2}QC_1 + \frac{D}{Q}(C_S + bQ).$$

For the optimum value of Q, we see that

$$\frac{d}{dQ}(TC) = 0 \Rightarrow Q = \sqrt{\frac{2DC_s}{C_1}} \text{ and } \frac{d^2}{dQ^2}(TC) > 0 \text{ for } Q > 0$$

Hence $$Q^\circ = \sqrt{\frac{2DC_S}{C_1}}$$

This shows that there is no change in Q° in spite of change in the set – up cost.

Corollary: 2

In the above EOQ problem lead time has been assumed to be zero. But in the most of the business situations, there exists a positive lead time, say L, from the time the order is placed until it is actually supplied.

Proof:

If the inventory consumption rate is 'K' units per day and L is the lead time in days, the total inventory requirements during the lead time will be 'LK'. Therefore, as soon as the inventory level becomes 'LK' an order Q is placed. This is called the re – order point $p = LK$. This is equivalent to continuously observing the level of inventory, until the re – order point is obtained. Because of this reason, EOQ problem is sometimes called the continuous review problem. Fig. 7.3 shows the re – order points:

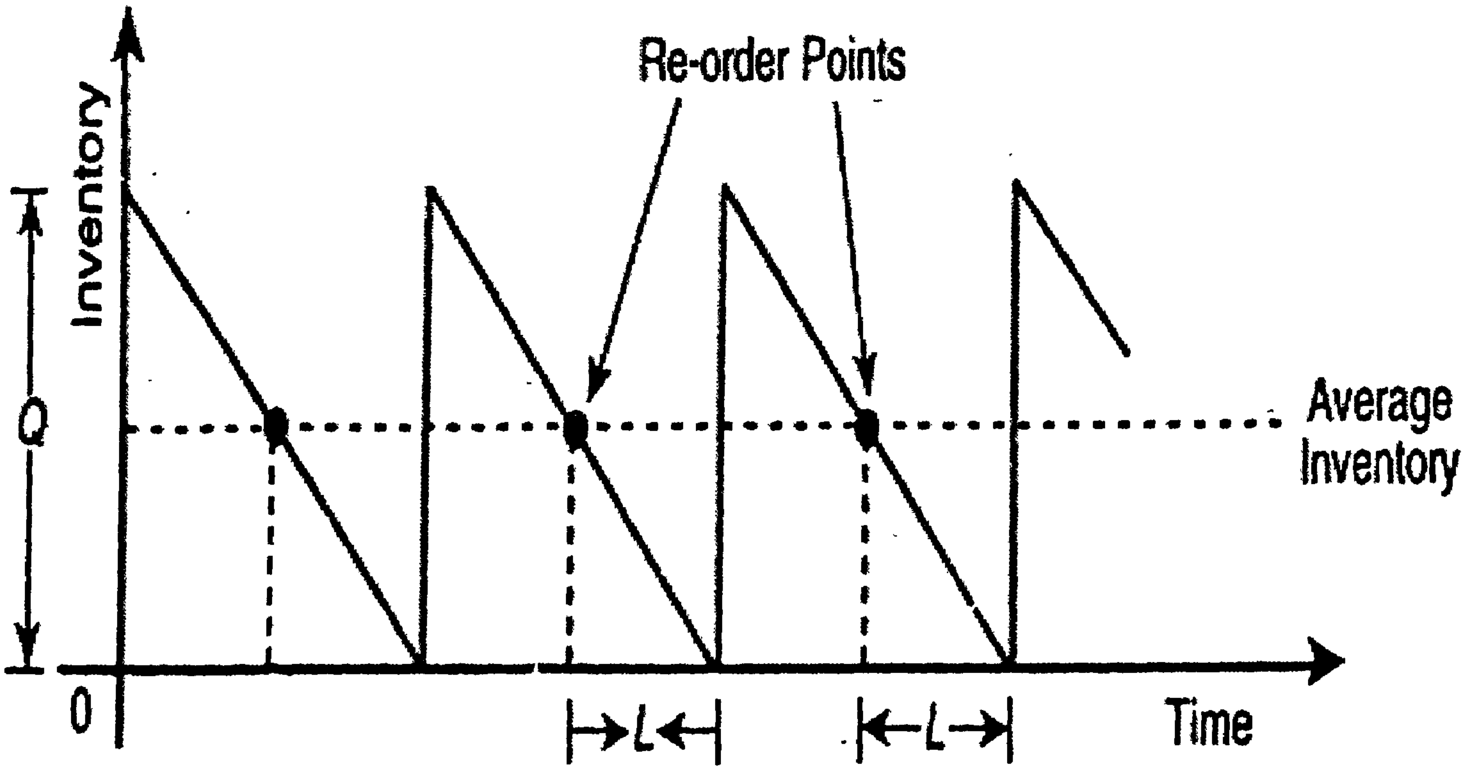


Figure 7.3

Figure given above assumes that the lead time is less than the cycle length, which is not necessarily the case in general. To account for this situation, we define the effective lead time as

$$L_e = L - mt^{\circ},$$

where m is the largest integer not exceeding L/t°. This result is justified because after m cycles of t° each, the inventory situation acts as if the interval between placing an order and receiving another is L_e.

Case: 2

EOQ Problem with Several Production Runs of Unequal Length

In this problem all the assumptions are same as in Case 1 except that the demand is uniform and the production runs differ in units.

Let $t_1, t_2, t_3, ..., t_n$ denote the times of successive production runs, such that

$$t_1 + t_2 + ... + t_n = 1\,\text{year}.$$

Thus, the fundamental situation can be represented graphically as shown in Fig 7.4.

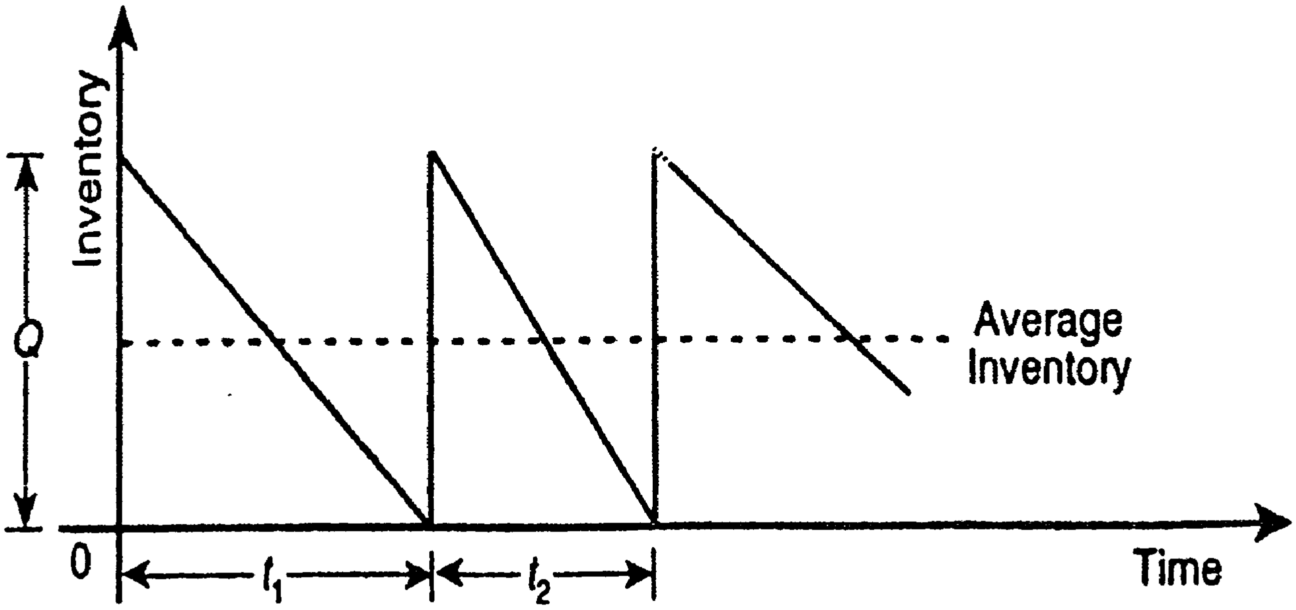


Figure 7.4

Obviously, the annual inventory holding cost is given by

$$f(Q)=\left(\frac{1}{2}Qt_1\right)C_1+\left(\frac{1}{2}Qt_2\right)C_1+......+\left(\frac{1}{2}Qt_n\right)C_1$$

$$=\frac{1}{2}Q(t_1+t_2+........+t_n)C_1=\frac{1}{2}QC_1$$

and the set – up costs associated with runs of size Q are given by

$$g(Q)=\frac{D}{Q}C_s\text{, since } nQ=D.$$

$\therefore$ Total annual cost is

$$TC=f(Q)+g(Q)=\frac{1}{2}QC_1+\frac{D}{Q}C_s.$$

This cost is the same as was obtained in Case 1 and hence the optimum quantities are

$$Q^\circ=\sqrt{2DC_s/C_1}\text{ and } TC^\circ=\sqrt{2\,DC_1C_s}$$

Remark:

If the total time period is T instead of one year, then the optimum order quantity becomes $Q^\circ=\sqrt{2C_sD/C_1T}$ and the minimum cost becomes $TC^\circ=\sqrt{2C_1C_sD/T}$. Thus, the uniform rate of demand is replaced by average rate of demand, i.e., D is replaced by D/T.

Example: 7.2.1

An oil engine manufacturer purchases lubricants at the rate of Rs. 42 per piece from a vendor. The requirement of these lubricants is 1, 800 per year. What should be the order quantity per order, if the cost per placement of an order is Rs. 16 and inventory carrying charge per rupee per year is only 20 paise.

Solution:

We are given

$D = 1{,}800$ lubricants per year, C_s = Rs.16 per order

C_1 = Rs.42 x Re.0.20 = Rs.8.40

$Q^\circ = \sqrt{2 \text{ x } 1800 \text{ x } 16/8.40} = 82.8$ or 83 lubricants.

Example: 7.2.2

A manufacturing company purchases 9,000 parts of a machine for its annual requirements ordering one month usage at a time. Each part costs Rs. 20. The ordering cost per order is Rs. 15 and the carrying charges are 15% of the average inventory per year. You have been assigned to suggest a more economical purchasing policy for the company. What advice would you offer and how much would it save the company per year?

Solution:

We are given

$D = 9{,}000$ parts per year, C_s = Rs.15 per order

C_1 = 15% of the average inventory per year

= Rs.20 x 15/100 = Rs.3 each part per year

$$Q^\circ = \sqrt{\frac{2\text{x}15\text{x}9{,}000}{3}} = 300 \text{ units}$$

and $$t^\circ = \frac{300}{9{,}000} = \frac{1}{30} \text{year} .$$

Minimum average yearly cost

$= \sqrt{2 \text{ x } 3 \text{ x } 15 \text{ x } 9{,}000} = \text{Rs.}900$. If the company follows the policy of ordering every month, then the annual ordering cost is Rs. 12 x 15 or Rs. 180, and lot size of inventory each month $= 9{,}000/12 = 750(= Q)$.

Average inventory at any time $= \frac{1}{2}Q = 750/2 = 375$.

$\therefore$ Storage cost at any time $= 375C_1 = 375 \text{ x } 3 = \text{Rs.}1{,}125$.

Total annual cost $= \text{Rs.}1{,}125 + \text{Rs.}180 = \text{Rs.}1{,}305$.

Hence, the company should purchase 300 parts at time intervals of 1/30 year instead of ordering 750 parts each month. The net saving of the company will be

$\text{Rs.}1{,}305 - \text{Rs.}900 = \text{Rs.}405$ per year.

Example: 7.2.3

A chemical company holds its inventory of raw materials in special containers, with each container occupying 10 square feet of floor space. There are only 5, 000 square feet of storage space available. Each year, this company uses 9, 000 special containers of raw materials, paying Rs. 8 per container of raw material. If ordering cost is Rs. 40 per order and annual holding costs are 20 per cent of the average inventory value, how much is it worth for this company to increase its container of raw material storage area? How many days (maximum) supply of inventory can be stored with the 5, 000 square feet storage limitation, assuming that this company works a 300 – day year?

Solution:

We are given

$$D = 9{,}000 \text{ containers}, \; C_s = \text{Rs.}40 \text{ per order}$$

$$C_1 = \text{Rs.}8.00 \text{ x Re.}0.20 = \text{Rs.}1.60$$

$$Q^{\circ} = \sqrt{\frac{2 \text{ x } 9000 \text{ x } 40}{1.60}} = 670.8 \text{ or } 671 \text{ approx.}$$

Hence, the optimum order size is 671 containers. Since each container occupies 10 square feet of floor space, the company will require 6,710 square feet of floor space to store the optimum order quantity. The total storage space available with the company currently is only 5, 000 square feet, therefore, it can order only 500 containers at a time. Hence, in order to calculate how much is worth for this company to increase the storage area, we will calculate the total variable cost that the company will have to incur for two alternatives, viz., 671 containers per order and 500 containers per order.

Alternatives	Annual ordering cost (Rs.)	Annual carrying cost (Rs.)	Total cost (Rs.)
671 per order	$\frac{9{,}000}{671} \text{x} 40 = 537$	$\frac{671}{2} \text{x} 8 \text{x} 0.2 = 537$	1,074
500 per order	$\frac{9{,}000}{500} \text{x} 40 = 720$	$\frac{500}{2} \text{x} 8 \text{x} 0.2 = 400$	1,120

Thus, the company will save $Rs.(1{,}120-1{,}074)=Rs.46$ if it follows economic order policy it is Rs. 46 worth for the company to increase the storage space.

With 5, 000 sq. ft. limitation only 500 units can be ordered and each day usage is 30 units $9{,}000/300$ and so nearly $500/30=17$ days supply of inventory will be there in each order.

Example: 7.2.4

Neon lights in an industrial part are replaced at the rate of 100 units per day. The physical plant orders the neon lights periodically. It costs Rs. 100 to initiate a purchase order. A neon light kept in storage is estimated to cost about Rs. 02 per day. The lead time between placing and receiving an order is 12 days. Determine the optimum inventory policy for ordering the neon lights.

Solution:

We are given

$$D=100 \text{ units per day, } C_s=\text{Rs.}100 \text{ per order,}$$

$$C_1=\text{Re.}02 \text{ per unit per day, } L=12 \text{ days.}$$

$$Q^{\circ}=\sqrt{\frac{2 \text{ x } 100 \text{ x } 100}{.02}}=1{,}000 \text{ neon lights}$$

$$t^{\circ}=\frac{Q^{\circ}}{D}=10 \text{ days}$$

Now, since the lead time, $L(=12\,\text{days})$ exceeds the cycle length $(=10\text{ days})$, we compute L_e.

Thus, $L_e = L - mt^\circ = 12 - 1 \times 10 = 2$ days

Where $m =$ (largest integer $\leq L/t^\circ$) $=$ (largest integer $\leq 12/10) = 1$

.

The re – order point, therefore, occurs when the inventory level drops to

$$L_e \times D = 2 \times 100 = 200 \text{ neon lights.}$$

Hence, the inventory policy for ordering neon lights is: Order 100 units whenever the inventory level drops to 200 units.

Example: 7.2.5

A manufacturing company purchases 9000 parts of machine for its annual requirements, ordering one month usage at a time. Each part costs Rs. 20. The ordering cost per order is Rs. 15 and the carrying charges are 15% of the average inventory per year. You have been asked to suggest a more economical purchasing policy for the company. What advice would you offer, and how much would it save the company per year?

Solution:

Here $D = 9000$ parts per year

$C_1 = 15\%$ unit cost

(Here 15 % of average Inventory per year means that the carrying cost per year is 15% of the unit cost)

$$= 20 \times \frac{15}{100} = \text{Rs.}3 \text{ each part per year}$$

$$C_s = \text{Rs.}15 \text{ per order}$$

Space for Hints

$$\therefore Q^{\circ} = \sqrt{\frac{2DC_s}{C_1}}$$

$$= \sqrt{\frac{2 \text{ x } 15 \text{ x } 9000}{3}} = 300 \text{ units}$$

$$t_{\circ} = \frac{Q^{\circ}}{D} = \frac{300}{9000}$$

$$= \frac{1}{30} \text{ year } = \frac{365}{30} = 12 \text{ days}$$

$$C_{min} = \sqrt{2C_1C_sD}$$

$$= \sqrt{2 \text{ x } 3 \text{ x } 15 \text{ x } 9000}$$

$$= \text{Rs.}900.$$

If the company follows the policy of ordering every month, then the annual ordering cost becomes

$$= 12 \text{ x } 15 = \text{Rs.}180.$$

and lot – size of Inventory each month $q = \frac{9000}{12} = 750$ parts.

Average Inventory at any time $= \frac{1}{2}q = 375$ parts

Storage cost at any time $= 375C_1$

$$= 375 \text{ x } 3 = \text{Rs.}1125.$$

Total annual cost $= 1125 + 180 = \text{Rs.}1305$.

$\therefore$ The company purchases 300 parts at time intervals of 12 days instead of ordering 750 parts each month. So there will be a net saving of Rs.1305 – Rs.900 = Rs.405 per year.

Example: 7.2.6

A certain item costs Rs. 235 per ton. The monthly requirements are 5 tons, and each time the stock is replenished, there is a setup cost of Rs. 1000. The cost of carrying inventory has been estimated at 10 % of the average inventory per year. What is the optimum order quantity.

Solution:

$$D = 5 \text{ tons/month}$$

$$= 60 \text{ tons/year}$$

$$C_s = \text{Rs.}1000$$

$$C_1 = 10\% \text{ of unit cost per year}$$

$$= \text{Rs.}235\text{x}\frac{10}{100}$$

$$= \text{Rs.}23.5 \text{ per item per year}$$

$$Q^\circ = \sqrt{\frac{2DC_s}{C_1}}$$

$$= \sqrt{\frac{2\text{x}1000\text{x}60}{23.5}}$$

$$= 71.458 \text{ tons}$$

Example: 7.2.7

For an item, the production is instantaneous. The storage cost of one item is Rs. One per month and the set up cost is Rs. 25 per run. If the demand is 200 units per month, find the optimum quantity to be produced per set – up and hence determine the total cost of storage and set – up per month.

Solution:

$$D = 200 \text{ units/month}$$

$$C_s = \text{Rs.}1 \text{ per unit per month}$$

$$C_1 = \text{Rs.}25 \text{ per run}$$

$$\therefore Q^\circ = \sqrt{\frac{2DC_s}{C_1}}$$

$$= \sqrt{\frac{2x25x2000}{1}} = 100 \text{ units.}$$

$$TC^\circ = \sqrt{2C_1C_sD}$$

$$= \sqrt{2x1x25x200}$$

$$= \text{Rs.}100.$$

Total costs of storage and set up

$$= 25 + (1x100)$$

$$= \text{Rs.}125.$$

Example: 7.2.8

A manufacturer has to supply his customer with 600 units of his products per year. Shortage are not allowed and storage cost amounts to 60 paise per unit per year. The set up cost is Rs. 80.00 find

(i) The economic order quantity
(ii) The minimum average yearly cost
(iii) The optimum period of supply per optimum order

Solution:

$$D = 600 \text{ units/year}$$

$$C_s = \text{Rs.}80$$

$$C_1 = 0.60 \text{ per unit/year}$$

(i) $$Q^\circ = \sqrt{\frac{2DC_s}{C_1}} = \sqrt{\frac{2x600x80}{0.60}}$$

$$= 400 \text{ units/years}$$

(ii) $$TC^\circ = \sqrt{2C_1C_sD} = \sqrt{2x0.60x80x600} = \text{Rs. } 240$$

(iii) $$t^\circ = \frac{Q^\circ}{D} = \frac{400}{600} = \frac{2}{3}$$

Check your progress: 7.1

1. A shipbuilding firm uses rivets at a constant rate of 20,000 numbers per year. Ordering costs are Rs. 30 per year. The rivets cost Rs. 1.50 per number. The holding cost of rivets is estimated to be 12.5 % of unit cost per year. Determine the EOQ.
2. A manufacturer has to supply his customer with 600 units of his product per year. Shortages are not allowed and the storage cost amounts to Rs. 0.60 per unit per year. The set – up cost per run is Rs. 80.00. Find the optimum run – size and the minimum average yearly cost.
3. A shopkeeper has a uniform demand of an item at the rate of 600 items per year. He buys from a supplier at a cost of Rs. 8 per item and the cost of ordering is Rs. 12 each time. If the stock holding costs are 20% per year of stock value, how frequently should he replenish his stocks and what is the Optimal Order Quantity?

4. A certain item costs Rs. 235 per tone. The monthly requirement is 5 tonnes and each time the stock is replenished there is a set – up cost of Rs. 1, 000. The cost of carrying inventory has been estimated at 10% of the value of the stock per year. What is the Optimal Order Quantity?
5. A manufacturer has to supply his customer with 24, 000 units of his product per year. This demand is fixed and known. Since the unit is used by the customer is an assembly line operation and the customer has no storage space for the units, the manufacturer must ship a day's supply each day. If the manufacturer fails to supply the required units, he will lose the amount and probably his business. Hence, the cost of a shortage is assumed to be infinite and, consequently, none will be tolerated. The inventory holding cost amounts to 0.10 per unit per month, and the set – up cost per production run is Rs. 350. Find the optimum lot size and the length of optimum production run.
6. An aircraft uses rivets at an approximately constant rate of 5,000 kg. per year. The rivets cost Rs. 20 per kg. and the company personnel estimate that it costs Rs. 200 to place an order, and the carrying cost of inventory is 10% per year.

   (i) How frequently should orders for rivets be placed, and what quantities should be ordered for?

   (ii) If the actual costs are Rs. 500 to place an order and 15% for carrying cost, the optimum policy would change. How much is the company loosing per year because of imperfect cost information?

Case: 3

EOQ Problem with finite Replenishment (Production)

In this problem all the assumptions are same as in case 1, except that of instantaneous replenishment. Assume that each production run of length t consists of two parts, say t_1 and t_2, such that

(i) The inventory is building up at a constant rate of (k – r) units, per unit of time during t_1, $k > r$;

(ii) There is no replenishment (or production) during time t_2 and the inventory is decreasing at the rate of r per unit of time.

The graphical representation of the situation is shown in Fig. 7.5.

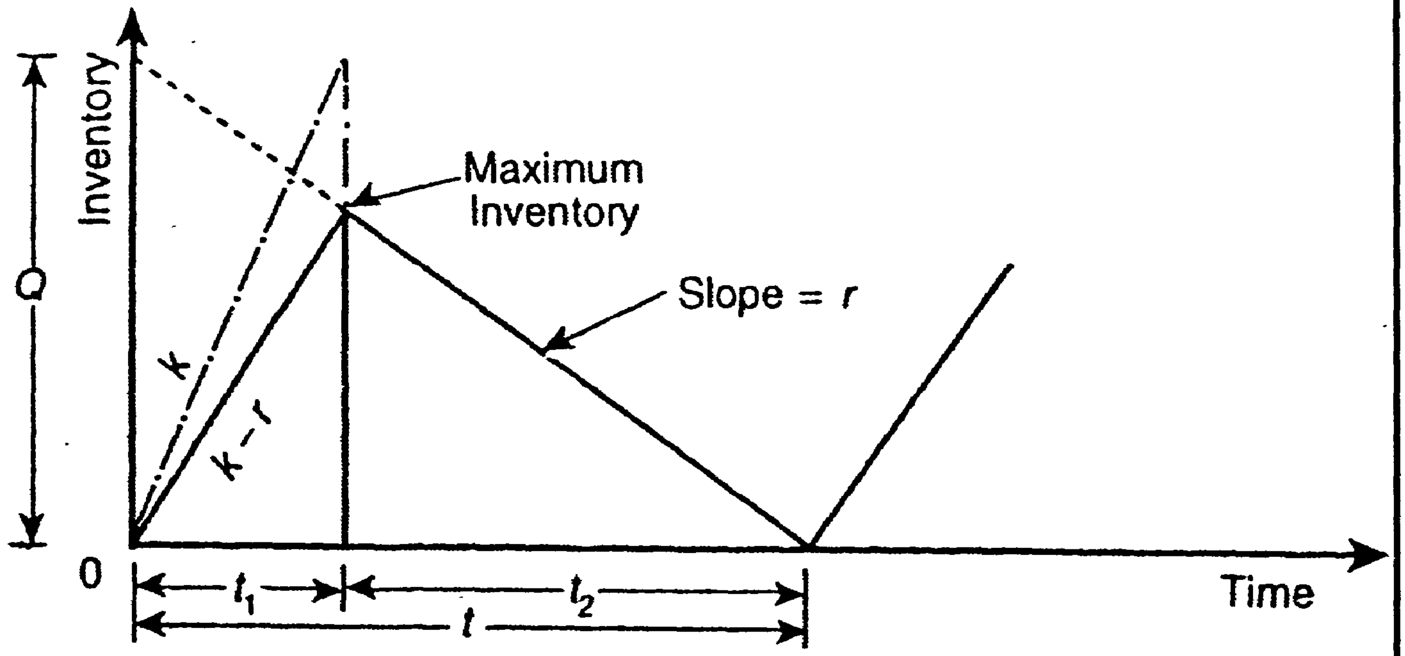


Figure 7.5

Here, the total (order) quantity Q is produced over a period, t_1, which is defined by the production rate k. Since the inventory does not pile up in one shot but rather continuously over a time period and is also consumed simultaneously, the average inventory level would be determined not only by the lot size Q, but also be affected by the production rate k and depletion (demand) rate r.

To determine the average inventory, we proceed as follows:

Since t_1 is the time required to produce Q at a rate k, we shall have

$$Q = kt_1 \text{ or } t_1 = Q/k.$$

During production period t_1, inventory is increasing at the rate of k and simultaneously decreasing at the rate of r. Thus inventory accumulates at the rate of $(k-r)$ units.

Therefore, the maximum inventory level shall be equal to $t_1(k-r)$.

$\therefore$ Average inventory $= \frac{1}{2}t_1(k-r) = \frac{1}{2}Q(1-r/k)$, since $t_1 = Q/k$

Now, with C_1 as the holding cost per unit per year, the total annual holding cost is given by

$$f(Q) = \text{Average inventory x } C_1 = \frac{1}{2}QC_1(1-r/k)$$

Annual ordering cost is given by

$$g(Q) = C_s \text{ x } D/Q,$$

Since D is the total demand in a year.

Since the minimum total cost occurs at the point where annual ordering cost and annual holding cost are equal, we must have

$$f(Q) = g(Q)$$

This implies that

$$\frac{1}{2}QC\left(1-\frac{r}{k}\right) = C_s\frac{D}{Q}$$

Hence, the optimum value of Q is

$$Q^\circ = \sqrt{\frac{2DC_s}{C_1(1-r/k)}} = \sqrt{\frac{2DC_s}{C_1}\left(\frac{k}{k-r}\right)}$$

Characteristic of Case: 3

1. Optimum number of production runs per year

$$n^\circ = D/Q^\circ = \sqrt{\frac{DC_1}{2C_s}(1-r/k)}$$

2. Optimum length of each lot size production run

$$t_1^\circ = Q^\circ/k = \sqrt{\frac{2DC_s}{C_1 k(k-r)}}$$

3. Total minimum production inventory cost

$$TC^\circ = \frac{C_s D}{Q^\circ} + \frac{1}{2}Q^\circ\left(1-\frac{r}{k}\right)C_1 = \sqrt{2DC_sC_1(1-r/k)}$$

Note:

If $k \to r$, then $C \to 0$. This shows that there will be no holding cost and no set – up cost.

If $k \to \infty$, i.e., when the production rate becomes infinite, the above problem reduces to the one considered in case 1. The inventory holding cost per unit of time is reduced from the cost discussed in case 1 in the ratio $(1-r/k):1$ for minimum cost, although the set – up cost remains the same.

Example: 7.2.9

A contractor has to supply 10,000 bearings per day to an automobile manufacturer. He finds that, when he starts a production run, he can produce 25,000 bearings per day. The cost of holding a bearing in stock for one year is Rs. 2 and the set – up cost of a production run is Rs. 1,800. How frequently should production run be made?

Solution:

We are given

$D = 10{,}000 \text{ x } 300 = 30{,}00{,}000$ bearings (Assuming 300 working days in a year)

C_1 = Rs.2 per bearing per year, C_s = Rs.1,800 per production run

$r = 10{,}000$ bearing per day, $k = 25{,}000$ bearings per day

$$\therefore \qquad Q^\circ = \sqrt{\frac{2 \text{ x } 30{,}00{,}000 \text{ x } 1{,}800}{2}}\sqrt{\frac{25{,}000}{25{,}000 - 10{,}000}} = 94{,}868$$

bearings

$$t^\circ = \frac{94{,}868}{10{,}000} = 9.49 \text{ days}$$

$$\text{Length of production cycle } = \frac{94{,}868}{25{,}000} = 4 \text{ days (approx.)}$$

Thus the production cycle starts at an interval of 9.49 days and production continues for 4 days so that in each cycle a batch of 94,868 bearings is produced.

Example: 7.2.10

An item is produced at the rate of 50 items per day. The demand occurs at the rate of 25 items per day. If the set up cost is Rs. 100 per set up and holding cost is Rs. 0.01 per unit of item per day, find the economic lot size for one run, assuming that shortages are not permitted. Also find the time of cycle and minimum total cost for one run.

Solution:

$D = 25$ items per day

$r = 25$ items per day

C_1 = Rs.0.01 per unit per day

C_s = Rs.100 per set up

$K = 50$ items per day

$$Q^\circ = \sqrt{\frac{2DC_s}{C_l}\left(\frac{k}{k-r}\right)}$$

$$= \sqrt{\frac{2x100x25}{0.01}} x \sqrt{\frac{50}{25}}$$

$$= 1000 \text{ items}$$

$$t^\circ = \frac{Q^\circ}{D} = \frac{1000}{25} = 40 \text{ days}$$

$$\text{Minimum daily cost} = \sqrt{2C_lC_sD}\sqrt{\frac{K-r}{K}}$$

$$= \text{Rs.}\sqrt{2 \text{ x } 0.01 \text{ x } 100 \text{ x } 25 \text{ x } \frac{25}{50}}$$

$$= \text{Rs. } 5$$

$$\text{Minimum total cost per run} = 5 \text{ x } 40$$

$$= \text{Rs. } 200$$

Check your progress: 7.2

1. An item is produced at the rate of 128 units per day. The annual demand is 6,400 units. The set – up cost for each production run is Rs. 24 and inventory carrying cost is Rs. 3 per unit per year. There are 250 working days for production each year. Develop an inventory policy for this item.
2. A contractor has to supply 20,000 units per day. He can produce 30,000 units per day. The cost of holding a unit in stock is Rs. 3 per year and the set – up cost per run is Rs. 50. How frequently, and of what size, the production runs be made?
3. Amit manufactures 50,000 bottles of tomato ketchup in an year. The factory cost per bottle is Rs. 5, the set – up cost per production run is estimated to be Rs. 90, and the carrying costs on finished goods inventory amount to 20%

of the cost per annum. The production rate is 600 bottles per day, and sales amount to 150 bottles per day. What is the optimum production lot size and the number of production runs?

If the factor costs increase to Rs. 7.50 per bottle, what will be the optimum production lot size?

4. The details of a part to be machined are as follows:

   Annual requirement = 2,400 pieces, Machine rate = 10 pieces/shift

   No. of working days in the year = 320 shifts

   Cost of machining a component = Rs. 100 per piece

   Inventory carrying cost per annum = 12 % of value

   Set – up cost per production run = Rs. 400

5. A manufacturing company uses an EOQ (Economic order quantity) approach in planning its production of gears. The following information is available.. Each gear costs Rs. 250 per unit, annual demand is 60,000 gears, set – up cost are Rs. 4,000 per set – up and the inventory carrying cost per month is established at 2 per cent of the average inventory value. When in production, these gears can be produced at the rate of 400 units per day and this company works only for 300 days in a year. Determine the economic lot size, the number of production runs per year and the total inventory costs.

7.3 Deterministic Inventory problem with shortages:

In a business concern, if shortages occur then these can be classified into the following two categories:

(a) As soon as the desired units of a certain commodity arrive in inventory, the back orders are satisfied.

(b) Shortages are lost sales.

In the first category demand of the customer is met in the beginning of new production run, whereas in the second category the customer moves to some other firm to fulfil his requirements. This case deals with those problems of shortages where back orders are entertained.

Case: 1

EOQ problem with Instantaneous Production and Variable Order Cycle Time:

The problem that we now discuss is same as was discussed in Section 7.2 (Case 1) with the difference that the shortages are now permitted. Let C_2 be the shortage cost per unit of time per unit quantity. This inventory situation can also be illustrated graphically as shown in Fig. 7.6:

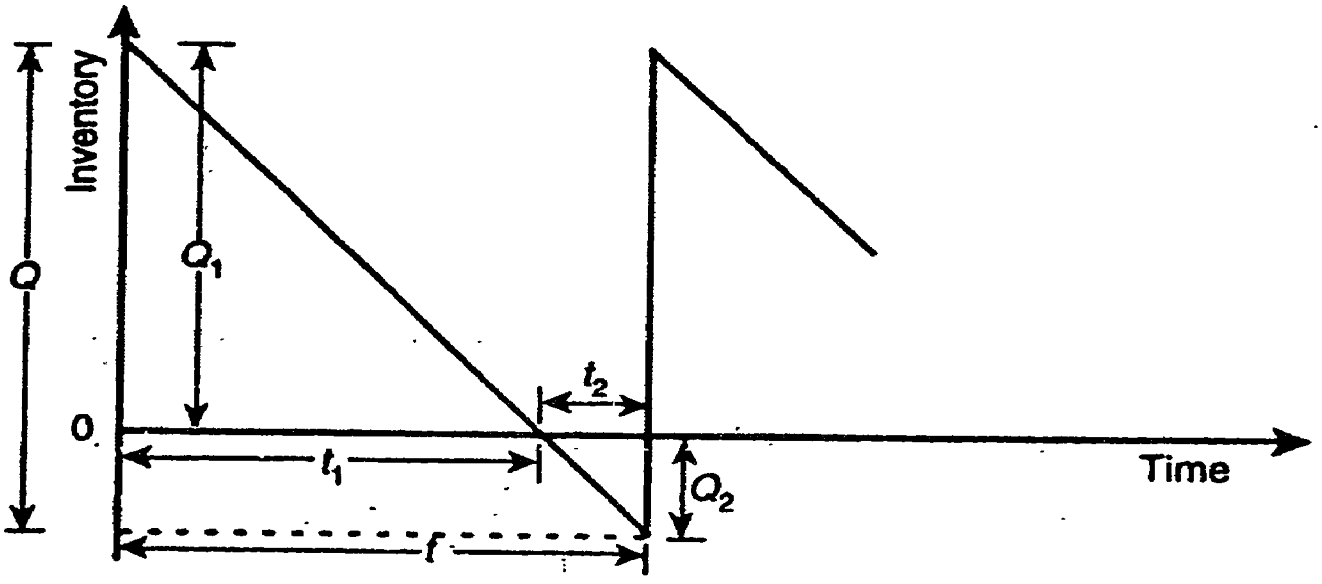


Figure 7.6

Here the total time period is one year and is divided into equal parts, say of interval t. Further this time interval t is divided into two parts t_1 and t_2, such that $t = t_1 + t_2$.

During the interval t_1, the items are drawn from the inventory as needed and during t_2, orders for the item are being accumulated but not filled. Then at the end of the interval t an

Space for Hints

amount Q is produced (or delivered). The amount Q has been divided into Q_1 and Q_2 such that $Q = Q_1 + Q_2$, where Q_1 denotes the amount which goes into inventory, and Q_2 denotes the amount which is immediately taken to satisfy past orders or unfilled demand.

The problem now is concerned with the areas of triangles above the time axis (representing items in inventory) and below the same axis (representing items in shortage).

Now, Total inventory over the time period $t = \frac{1}{2}Q_1 t_1$

Average inventory at any time $= \left(\frac{1}{2}Q_1 t_1\right)/\ t$

Annual inventory holding cost $= C_1\left(\frac{1}{2}Q_1 t_1\right)/\ t$

Similarly,

Total amount of shortage over time period t $= \frac{1}{2}Q_2 t_2$

Annual shortage costs $= C_2\left(\frac{1}{2}Q_2 t_2\right)/\ t$

Annual costs associated with runs of size $Q = nC_s = \frac{D}{Q}C_s$,

Since D/Q runs are produced in each year.

$\therefore$ Total annual cost is given by

$$TC = \left[C_1\left(\frac{1}{2}Q_1 t_1\right) + C_2\left(\frac{1}{2}Q_2 t_2\right)\right]/\ t + \frac{D}{Q}C_s.$$

Now using the relationship for similar triangles, we have

$$\frac{t_1}{t}=\frac{Q_1}{Q} \text{ and } \frac{t_2}{t}=\frac{Q_2}{Q}$$

i.e., $$t_1=\frac{Q_1}{Q}t \text{ and } t_2=\frac{Q_2}{Q}t.$$

Making use of these values, we get

$$TC=\frac{1}{2}C_1\left(\frac{Q_1^2}{Q}\right)+\frac{1}{2}C_2\left[\frac{(Q-Q_1)^2}{Q}\right]+C_s\left(\frac{D}{Q}\right).$$

Since $Q_2=Q-Q_1$.

For determining the optimum values of Q_1 and Q so as to optimize TC, we have

$$\frac{\partial(TC)}{\partial Q_1}=0 \Rightarrow Q_1=C_2Q/(C_1+C_2).$$

$$\frac{\partial(TC)}{\partial Q}=0 \Rightarrow Q=\sqrt{\frac{2C_sD+C_1Q_1^2}{C_2}+Q_1^2}$$

and $\frac{\partial^2(TC)}{\partial Q_1^2}>0,\ \frac{\partial^2(TC)}{\partial Q^2}>0$ for these values of Q_1 and Q_2.

Thus the optimum quantities are given by (on simplification)

$$Q^\circ=\sqrt{\frac{2C_sD}{C_1}}\sqrt{\frac{(C_1+C_2)}{C_2}}$$

$$Q_1^\circ=\left(\frac{C_2}{C_1+C_2}\right)Q^\circ=\sqrt{\frac{C_2}{C_1+C_2}}\sqrt{\frac{2C_sD}{C_1}} \text{ (Optimum Stock Level)}$$

Characteristic of Case: 1

1. Time between receipt of orders (when to order)

$$t^\circ = Q^\circ / D = \sqrt{\frac{2C_s}{DC_1}}\sqrt{\frac{C_1 + C_2}{C_2}}$$

2. Total optimum inventory cost

$$TC^\circ = \sqrt{2DC_sC_1}\sqrt{\frac{C_2}{C_1 + C_2}}$$

3. Maximum inventory level

$$Q^\circ - Q_2^\circ = Q^\circ\left(1 - \frac{C_1}{C_1 + C_2}\right).$$

$$= \sqrt{\frac{2C_sD}{C_1}}\sqrt{\frac{C_2}{C_1 + C_2}}$$

Remarks:

1. If $C_1 > 0$ and $C_2 = \infty$, shortages are prohibited. In this case $Q_1^\circ = Q^\circ = \sqrt{2C_sD/C_1}$ and each batch Q° is used entirely for inventory.

2. If $C_1 = \infty$ and $C_2 > 0$, inventories are prohibited. In this case $Q_1^\circ = 0,\ Q^\circ = \sqrt{2C_sD/C_2}$ and each batch is used only to fill back orders.

3. If shortage costs are negligible, then $C_1 > 0$ and $C_2 \to 0$. In this case $Q_1^\circ \to 0$ and $Q^\circ \to \infty$.

4. If inventory costs are negligible, then $C_1 \to 0$ and $C_2 > 0$. In this case $Q^\circ \to \infty$ and $Q_1^\circ \to \infty$, i.e., $Q_1^\circ \to Q^\circ$. Thus as inventory costs become very small, increasingly large batches should be produced and used entirely as inventory for future demands.

5. When inventories and shortages are equally costly, i.e., when

$$C_1 = C_2,\ \frac{C_2}{C_1 + C_2} = \frac{1}{2}.$$

Thus in this case

$$Q^\circ = \sqrt{2}\sqrt{\frac{2C_s'D}{C_1}} = (1.414)\sqrt{\frac{2C_sD}{C_1}}$$

This shows that the lot size is .414 times as large as earlier when no shortages were allowed.

Case: 2

EOQ Problem with Instantaneous Production and Fixed Order Cycle

Let t be fixed, i.e., inventory is to be replenished after every time period t. Also let us assume that items are being supplied or produced at the rate of r units per unit of time during this fixed time period.

Here, total inventory over the time period $t_1 = \frac{1}{2}Q_1t_1$

and total amount of shortages over time period $t_2 = \frac{1}{2}Q_2t_2$

∴ Total production cost is given by

$$TC = \frac{1}{2}Q_1t_1C_1 + \frac{1}{2}Q_2t_2C_2$$

Using now the relationship of similar triangles, viz.,

$$\frac{t_1}{t} = \frac{Q_1}{Q} \text{ and } \frac{t_2}{t} = \frac{Q_2}{Q}$$

The cost equation reduces to

$$TC = \frac{1}{2Q}.C_1Q_1^2t + \frac{1}{2Q}.C_2Q_2^2t + \frac{1}{2r}C_1Q_1^2 + \frac{1}{2r}C_2(rt - Q_1)^2,$$

Since $Q_2 = Q - Q_1$ and $Q = rt$.

The optimum value Q_1° is obtained as follows:

$$\frac{\partial(TC)}{\partial Q_1} > 0 \Rightarrow C_1Q_1 + C_2(Q_1 - rt) = 0 \text{ or } Q_1 = rtC_2/(C_1 + C_2).$$

Now

$$\frac{\partial^2(TC)}{\partial Q_1^2} > 0$$ for all values of Q_1, therefore,

$$Q_1^{\circ} = rt\left(\frac{C_2}{C_1 + C_2}\right). \qquad \text{(Optimum Inventory Level)}$$

Note:

In this problem, set – up cost is not considered, because of it being fixed as the time of one production run is fixed. Substituting the value of Q_1° in the total production cost equation, the optimum inventory cost is

$$TC^{\circ} = \left(\frac{C_1C_2}{C_1 + C_2}\right) rt$$

Case: 3

EOQ Problem with Finite Replenishment (Production)

In this problem all the assumptions are same in case 1 except that the rate of replenishment of inventory is finite, say k units per unit of time.

Assume that each production run of length t consists of two parts t_1 and t_2 which are further sub – divided into two parts say t_{11} and t_{12}, t_{21} and t_{22}, where:

(i) Inventory is building up at a constant rate of $(k-r)$ units per unit of time during time t_{11}.

(ii) No replenishment during time t_{12} and inventory is decreasing at the rate r per unit of time,

(iii) Shortage is building up at a constant rate of r per unit of time during time t_{21}.

(iv) Shortages are being filled immediately at the rate of $(k-r)$ units per unit of time during time t_{22}.

The graphical representation of the situation is as follows:

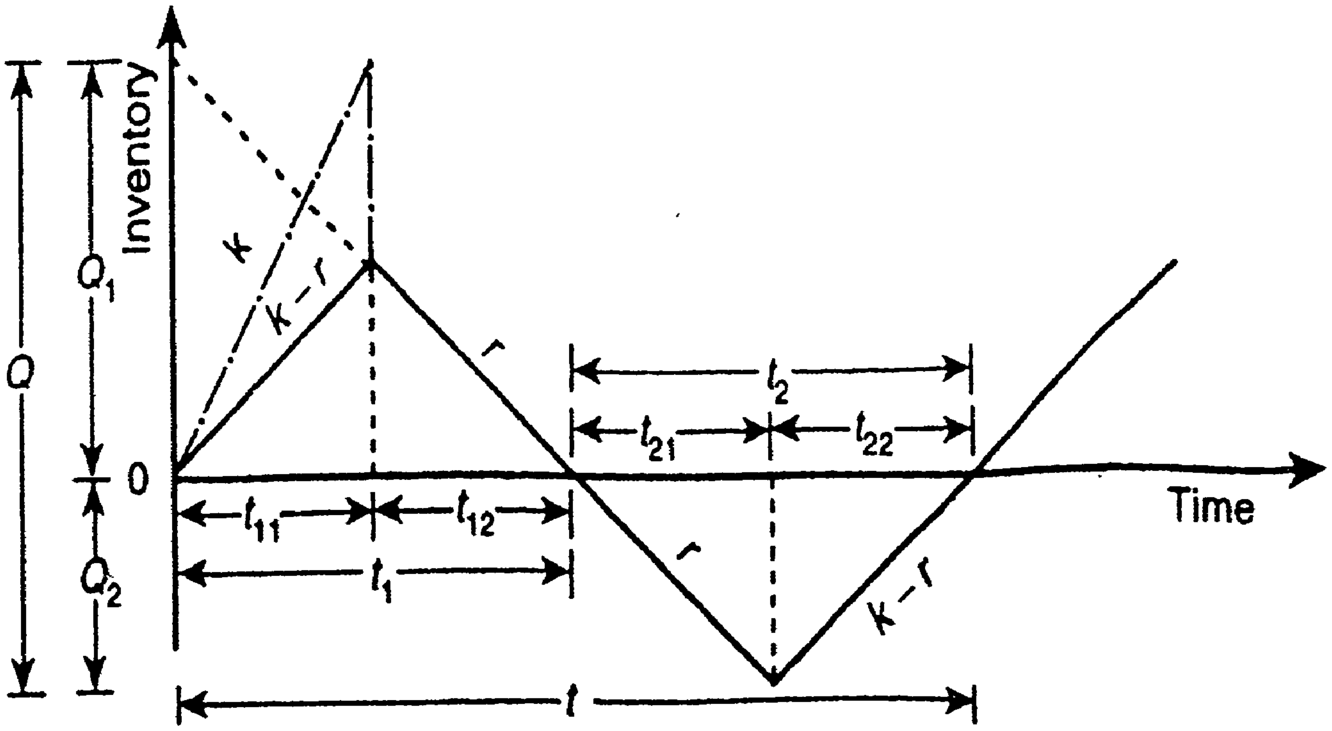


Figure 7.7

From Fig. 7.7, we see that at the end of t_{11}, the level of inventory is Q_1 and at the end of period t_{12} inventory becomes nil. Now shortages start and suppose that the shortages build up of quantity Q_3 up to time t_{21} and let then these shortages be filled up during time t_{22}.

Then obviously,

$$Q_1 = t_{11}(k-r), \qquad Q_1 = t_{12}r,$$

$$Q_2 = t_{21}r \quad \text{and} \quad Q_2 = t_{22}(k-r).$$

Now if Q is the lot size, then

$$Q_1 = Q - Q_2 - rt_{11} - rt_{22}.$$

Eliminating t_{11} and t_{22} from this, we have

$$Q_1 = Q - Q_2 - r\left(\frac{Q_1}{k-r} + \frac{Q_2}{k-r}\right) \qquad \text{or}$$

$$Q_1 + Q_2 = (k-r)Q/k$$

Production cycle is

$$t = t_{11} + t_{12} + t_{21} + t_{22} = \frac{Q_1}{k-r} + \frac{Q_1}{r} + \frac{Q_2}{r} + \frac{Q_2}{k-r}$$

$$= k(Q_1 + Q_2)/r(k-r)$$

Substituting the value of $Q_1 + Q_2$, we get $t = Q/r$.

The average inventory and amount of shortage during production cycle time t are:

Average inventory $= \frac{1}{2}Q_1(t_{11} + t_{12})/t$, and Average shortage $= \frac{1}{2}Q_2(t_{21} + t_{22})/t$

∴ The total inventory cost is

$$TC = \frac{1}{2}Q_1C_1\frac{(t_{11} + t_{12})}{t} + \frac{1}{2}Q_2C_2\frac{(t_{21} + t_{22})}{t} + rC_s/Q$$

$$= \frac{1}{2Q} \text{ x } \frac{k}{k-r}\left[C_1\left(\frac{k-r}{k}Q - Q_2\right)^2 + C_2Q_2^2\right] + \frac{r}{Q}Cs$$

Since

$$Q_1 + Q_2 = \frac{r(k-r)}{k} = t = \frac{r(k-r)}{k} x \frac{Q}{r} = \frac{k-r}{k} Q.$$

Now, $\frac{\partial}{\partial Q_2} TC = 0 \Rightarrow Q_2 = \frac{C_1 Q}{C_1 + C_2}(1 - r/k)$

and $\frac{\partial}{\partial Q} TC = 0 \Rightarrow Q = \sqrt{\frac{2C_s(C_1 + C_2)}{C_1 C_2}} \sqrt{\frac{kr}{k-r}}$

Since $\frac{\partial^2 TC}{\partial Q^2} > 0$, and $\frac{\partial^2 (TC)}{\partial Q_2^2} > 0$ for all values of Q and Q_2, the optimum values of Q and Q_2 so as to minimize TC are given by

$$Q^\circ = \sqrt{\frac{2C_s(C_1 + C_2)}{C_1 C_2}} \sqrt{\frac{kr}{k-r}} \text{ and } Q_2^\circ = \sqrt{\frac{2C_s C_1 r}{(C_1 + C_2)} \frac{k-r}{k}}$$

Characteristic of Case: 3

1. Production cycle time is

$$t^\circ = Q^\circ / r = \sqrt{\frac{2C_s(C_1 + C_2)}{C_1 C_2 r(1 - r/k)}}$$

2. Maximum inventory level is

$$Q_1^\circ = \frac{k-r}{k} Q^\circ - Q_2^\circ = \sqrt{\frac{2C_2 C_s r}{C_1(C_1 + C_2)}\left(1 - \frac{r}{k}\right)}$$

3. Total minimum production inventory cost is

$$TC^\circ = \frac{1}{2Q^\circ} x \frac{k}{k-r}(C_1 Q_1^{\circ 2} + C_2 Q_2^{\circ 2}) + \frac{r}{Q^\circ} C_s$$

$$= \sqrt{\frac{2C_1 C_2 C_s r}{C_1 + C_2}(1 - r/k)}$$

Remark:

1. If $k = \infty$, the problem is in complete agreement with the problem discussed in case 1.
2. If $C_3 = \infty$, the problem reduces to that of case 3 of Section 7.2.
3. If $k = \infty$ and $C_2 = \infty$, the problem reduces to that discussed in case 1 of Section 7.2.

Example: 7.3.1

A dealer supplies you the following information with regard to a product dealt in by him:

Annual demand: 10,000 units; ordering cost: Rs. 10 per order; price: Rs. 20 per unit.

Inventory carrying cost: 20% of the value of inventory per year.

The dealer is considering the possibility of allowing some back – order (stock – out) to occur. He has estimated that the annual cost of back – ordering will be 25% of the value of inventory.

(i) What should be the optimum number of units of the product he should buy in one lot?

(ii) What quantity of the product should be allowed to be back – ordered, if any?

(iii) What would be the maximum quantity of inventory at any time of the year?

(iv) Would you recommend to allow back – ordering? If so, what would be the annual cost saving by adopting the policy of back – ordering.

Solution:

In the usual notations, we are given:

$D = 10{,}000$ units, $C_s = \text{Rs.}10$ per order, $C_1 = 20\%$ of Rs. 20 = Rs. 4 per unit per year and $C_2 = 25\%$ of Rs. 20 = Rs. 5 per unit per year.

(i) (a) When stock – outs are not permitted:

$$Q^\circ = \sqrt{\frac{2DC_s}{C_1}} = \sqrt{\frac{2x10{,}000x10}{4}} = 223.6 \text{ units}$$

(b) When back – ordering is permitted:

$$Q^\circ = \sqrt{\frac{2DC_s}{C_1} x \frac{C_1 + C_2}{C_2}} = \sqrt{\frac{2x10{,}000x10}{4} x \frac{4+5}{5}} = 300 \text{ units}$$

(ii) Optimum quantity of the product to be back – ordered is given by:

$$Q_2^\circ = 300x\frac{4}{4+5} = 133 \text{ units (approx.)}$$

(iii) Maximum inventory level $= 300 - 133 = 167$ units.

(iv) Minimum total variable inventory cost in the cases (a) and (b) are:

$$TC(223.6) = \sqrt{2x10{,}000x10x4} = \text{Rs.}894.43$$

$$TC(300) = \sqrt{2x10{,}000x10x4x\frac{5}{4+5}} = \text{Rs.}666.67$$

Since $TC(223.6) > TC(666.67)$, the dealer should accept the proposal for back – ordering as this will result in a saving of $(894.43 - 666.67) = \text{Rs.}227.76$ per year.

Example: 7.3.2

A contractor undertakes to supply Diesel engines to a truck manufacturer at the rate of 25 per day. There is a clause in the contract penalizing him Rs. 10 per engine per day late for missing the scheduled delivery data. He finds that the cost of holding a complete engine in stock is Rs. 16 per month. His production process is such that each month he starts a batch of engines through the shops, and all these engines are available for delivery any time after the end of the month. What should his inventory level be at the beginning of each month?

Solution:

We are given

$C_1 = \text{Rs.}16.00$ per engine per month,

$C_2 = \text{Rs.}10.00$ per engine per day

$r = 25$ engines per day and $t =$ one month (= 30 days)

The optimal value of Q_1 is,

$$Q_1^\circ = rt\frac{C_2}{C_1 + C_2} = 25\text{x}30\frac{10}{10+16/30} = \frac{10\text{x}25\text{x}30\text{x}30}{316} = 712 \text{ engines.}$$

Example: 7.3.3

The demand for an item in a company is 18,000 units per year, and the company can produce the items at a rate of 3,000 per month. The cost of one set – up is Rs. 500.00 and the holding cost of 1 unit per month is 15 paise. The shortage cost of one unit is Rs. 20.00 per month. Determine (i) Optimum production batch quantity and the number of strategies, (ii) Optimum cycle time and production time, (iii) Maximum inventory level in the cycle, and (iv) Total associated cost per year if the cost of the item is Rs. 20 per unit.

Solution:

Here,

$C_1 = \text{Rs.}0.15$ per month, $C_2 = \text{Rs.}20.00$, $C_s = \text{Rs.}500.00$, $k = 3{,}000$ units per month, $r = 18{,}000$ units per year or 1,500 units per month.

(i) Optimum production batch quantity is given by

$$\therefore \quad Q^\circ = \sqrt{\frac{2C_s(C_1 + C_2)}{C_1C_2}}\sqrt{\frac{kr}{k-r}}$$

$$= \sqrt{\frac{2\text{x}500(0.15+20)}{0.15\text{x}20}}\sqrt{\frac{1{,}500\text{x}3{,}000}{3{,}000-1{,}500}}$$

$= 4{,}489$ units approx.

Number of shortages is given by

$$Q_2^\circ = \frac{C_1}{C_1 + C_2}Q^\circ\left(1-\frac{r}{k}\right) = \frac{0.15}{0.15+20}\text{x}4{,}889\left[1-\frac{1{,}500}{3{,}000}\right]$$

$= 18$ units approx.

(ii) Optimum production time $= \frac{Q^\circ}{k} = \frac{4{,}489}{3{,}000} = 1.5$ months.

Optimum cycle time between set – ups

$= \frac{Q^\circ}{r} = \frac{4{,}489}{1{,}500} = 3$ months.

(iii) Maximum inventory level is

$$Q_1^\circ = \frac{k-r}{k}Q^\circ - Q_2^\circ = \frac{1{,}500}{3{,}000}\text{x}4{,}489 - 17 = 2{,}227 \text{ units}$$

approx.

(iv) Total associated cost is given by

$$TC^{\circ} = \sqrt{\frac{2C_1C_2C_s^r}{C_1 + C_2}\frac{k-r}{k}} = \sqrt{\frac{0.15x20x50x1{,}500}{0.15+20}}$$

$$= \text{Rs.}106\,.$$

Example: 7.3.4

The demand for an Item is 18,000 units per year. The holding cost per unit time is Rs. 1.20 and the cost of shortage is Rs. 5.00, the production cost is Rs. 400. Assuming that replenishment rate is Instantaneous, determine the optimal order quantity.

Solution:

$$D = 18{,}000,\ C_1 = \text{Rs.}1.20,\ C_2 = \text{Rs.}5.00,\ C_s = \text{Rs.}400$$

$$\therefore Q^{\circ} = \sqrt{\frac{C_1 + C_2}{C_2}} \quad \sqrt{\frac{2C_sD}{C_1}}$$

$$= \sqrt{\frac{1.20+5}{5}} \quad \sqrt{\frac{2x400x18{,}000}{1.20}}$$

$$= 1.113 \text{ x } 3{,}464.10$$

$$= 3856 \text{ units (app)}$$

$$t^{\circ} = \frac{Q^{\circ}}{D} = \frac{3856}{18{,}000}$$

$$= 0.214\,\text{year}$$

$$N^{\circ} = \frac{D}{Q^{\circ}}$$

$$= 4.67 \text{ orders per year}$$

Example: 7.3.5

A certain product has a demand of 25 units per month and the items are withdrawn uniformly. Each time a production run is made the set up cost is Rs. 15. The production cost is Rs. 1 per item and inventory holding cost is Rs. 0.30 per Item per month. If shortage cost is Rs. 1.50 per item per month, determine how often to make a production run and what size it should be?

Solution:

Through the production cost is given, the cost equation remains the same.

$$\Rightarrow Q^\circ = \sqrt{\frac{C_1 + C_2}{C_2}} \quad \sqrt{\frac{2DC_s}{C_1}}$$

Given $C_1 = 0.30$ per item per month

$C_s = \text{Rs.}15$

$C_2 = \text{Rs.}1.50$ per item per month

$D = 25$ units/month

$$\Rightarrow Q^\circ = \sqrt{\frac{(1.50 + 0.30)2x15x25}{0.30x1.50}}$$

$$= \sqrt{\frac{1350}{0.45}}$$

$= 54.77$ units

$$t^\circ = \frac{Q^\circ}{D}$$

$$= \frac{54.77}{25}$$

$$= 2.19 \text{ month}$$

Check your progress: 7.3

1. The demand for a purchased item is 1,000 units/month, and shortages are allowed. If the unit cost is Rs. 1.50 per unit, the cost of making one purchase is Rs. 600, the holding cost for one unit is Rs. 2 per year, and the cost of one shortage is Rs. 10 per year, determine:
   (i) The optimum purchase quantity.
   (ii) The number of orders per year.
   (iii) The optimum total yearly cost.
2. A commodity is to be supplied at a constant rate of 200 units per day. Supplies of any amounts can be had at any required time, but each ordering costs Rs. 50,00, cost of holding the commodity in inventory is Rs. 2.00 per unit per day while the delay in the supply of the item induces a penalty of Rs. 10.00 per unit per delay of 1 day. (i) Find the optimal policy (q, t), where t is the re – order cycle period and q is the inventory level after re – order. (ii) What would be the best policy, if the penalty cost becomes ∞ ?
3. The demand for product is 25 units per month and the items are withdrawn uniformly. The set – up cost each time a production is run is Rs. 15. The inventory holding cost is Rs. 0.30 per item per month:

(i) Determine how often to make production run, if shortages are not allowed.

(ii) Determine how often to make production run, if shortages cost Rs. 1.50 per item per month.

7.4 Key words:

Raw material inventory, work in progress inventory, ordering cost, production cost, shortage cost, revenue cost, economic order quantity.

7.5 Answers to check your progress questions:

Check your progress: 7.1

1. $Q^\circ = 1{,}005$ number
2. $Q^\circ = 400$ units, and $TC = Rs.240$.
3. $Q^\circ = 95$ items and $t^\circ = 57$ months,
4. $Q^\circ = 71.5$ tons
5. $Q^\circ = 3{,}740$ units, $t^\circ = 8$ months, $TC = Rs.240$
6. (i) $Q^\circ = 1{,}000$ units, $t^\circ = 73$ days and $n^\circ = 5$ orders

   (ii) Loss per year will be of Rs. 1,873.

Check your progress: 7.2

1. $Q^\circ = 358$ units, $t^\circ = 14$ working days, $t_1^\circ = 2.8$ days

   $TC = Rs.858.65$ per year.
2. $Q^\circ = 1{,}414$ units, $t^\circ = 1.68$ hours.
3. $Q^\circ = 3{,}464$ units.
4. $Q^\circ = 800$ pieces
5. $Q^\circ = 4{,}000$ gears, $n^\circ = 15$ production runs per year,

   $TC(Q^\circ) = Rs.1{,}20{,}000$.

Check your progress: 7.3

1. (i) $Q^\circ = 3{,}600$,

   (ii) 50 Orders per year

   (iii) $TC = Rs.6{,}000$.

2. (i) $Q^{\circ} = 109.5$ units and $t^{\circ} = 12$ hours

   (ii) $Q^{\circ} = 100$ units and $t^{\circ} = 12$ hours

3. (i) $Q^{\circ} = 50$ units

   (ii) $Q^{\circ} = 54.7$ units.

7.6 Model Questions:

1. A purchase manager places order each time for a lot of 500 units of a particular item. From the available data the following results are obtained:

   Inventory carrying cost = 40%
   Ordering cost per order = Rs.600
   Cost per unit = Rs.40
   Annual demand = 6 orders each of 200 units.

   Find out the loss to the organization due to his ordering policy.

2. Find the EOQ for the following data:

   Annual usage = 1,000 pieces
   Cost per piece = Rs.250
   Ordering cost = Rs.6 per order
   Expediting cost = Rs.4 per order
   Inventory holding cost = 20% of average inventory
   Material holding cost = Rs.1 per piece.

3. A factory follows an economic order quantity system for maintaining stocks of one of the component requirements. The annual demand is for 24,000 units, the cost of placing an order is Rs. 300, and the component cost is Rs. 60 per unit. The factory has imputed 24 per cent as the inventory carrying rate.

(i) Find the optimal interval for placing orders, assuming a year is equivalent to 360 days.

(ii) If it is decided to place only one order per month, how much extra cost does the factor incur per year as a consequence of this decision?

4. The annual demand of a particular item by a company is 10,000 units. This item may be obtained from either an outside supplier or subsidiary company. The relevant data for the procurement of the item are given below:

Costs	From outside supplier Rs.	From subsidiary company Rs.
Cost per unit	12	13
Cost of placing an order	10	10
Cost of receiving an order	20	15
Storage and all carrying costs, including capital cost per unit per annum	2	2

(a) What purchase quantity and from which source would you recommend to procure?

(b) What would be the minimum total cost in that case?

[Hint: Ordering cost = Cost of (Placing + Receiving) an order]

5. A product is to be manufactured on a machine. The cost, production and demand etc. are follows:

Fixed costs per lot = Rs.30

Variable costs per unit = Rs.0.10

Percentage charges for interest, taxes

Insurance and storage = 50%

Production rate = 1,00,000 units per year

Demand rate = 10,000 units per year

Determine the economic manufacturing quantity.

6. Company A wants to know what production cost its major competitor, company B, has assigned to product item p_7. After a bit of investigation, company A has collected the following data about company B's production of item p_7:

Production lot size = 2,600 units

Set – up cost = Rs.135

Annual demand = 30,000 units

Daily demand = 100 units

Production rate = 200 units per day

Inventory holding costs = 28% of the average value per year.

Company A has further learnt that company B produces according to 'economic lot size' model. What is the company B's cost of producing product item p_7?

UNIT – 8

REPLACEMENT AND SEQUENCING PROBLEMS

Introduction:

The replacement problems are concerned with the situations that arise when some items such as men, machines, electric light bulbs etc need replacement due to their decreased efficiency, failure or breakdown. Such decreased efficiency or complete breakdown may either be gradual or all of a sudden.

The replacement problem arises because of the following factors:

i). The old item has becomes worse or requires expensive maintenance.
ii) The old item has failed due to accident.
iii) A more efficient design of equipment has become available in market.

Thus the problem of replacement is to decide the best policy to determine the age at which the replacement is most economical instead of continuing at increased cost due to factor like maintenance. The objective is to find the optimum period of replacement. We shall discuss the following main type of replacement situations:

(i) Replacement of items that deteriorate with time.
(ii) Replacement of items which do not deteriorate but fail after certain amount of use.

The replacement situations may be placed into four categories:

(i) Replacement of capital equipment that become worse with time

Example: machine tools, Planes etc.

(ii) Group replacement of items that fail completely.

Example: Radio tubes, Light bulbs etc.

(iii) Problems of mortality and staffing.

(iv) Miscellaneous problems.

Structure:

8.1 Replacement of Equipment/Asset that deteriorates Gradually.

8.2 Replacement of Equipment that fails suddenly.

8.3 Sequencing Problems

8.4 Key words

8.5 Answers to check your progress Questions

8.6 Model Questions

8.1 Replacement of Equipment/Asset that Deteriorates Gradually:

Generally, the cost of maintenance and repair of certain items (equipments) increases with time and a stage may come when these costs become so high that it is more economical to replace the item by a new one.

Replacement Policy when Value of Money does not change with time

The aim here is to determine the optimum replacement age of an equipment/item whose running/maintenance cost increases with time and the value of money remains static during that period. Let

C: capital cost of equipment.

S: scarp value of equipment.

n: number of years that equipment would be in use,

f(t) : maintenance cost functions, and

A(n) : Average total annual cost.

Case: 1

When t is a continuous variable. If the equipment is used for 'n' years, then the total cost incurred during this period is given by

TC = Capital cost – Scrap value + Maintenance cost

$$= C - S + \int_n^0 f(t)\,dt.$$

Average annual total cost, therefore is

$$A(n) = \frac{1}{n} TC = \frac{C-S}{n} + \frac{1}{n}\int_0^n f(t)dt.$$

For minimum cost, we must have $\frac{d}{dn}[A(n)] = 0$

or $$\frac{-(C-S)}{n^2} - \frac{1}{n^2}\int_0^n f(t)dt + \frac{1}{n}f(n) = 0$$

or $$f(n) = \frac{C-S}{n^2} - \frac{1}{n}\int_0^n f(t)dt \equiv A(n).$$

Clearly,

$$\frac{d^2}{dn^2}[A(n)] > 0 \text{ at } f(n) = A(n).$$

This suggests that the equipment should be replaced when maintenance cost equals the average annual total cost.

Case: 2

When t is a discrete variable. Here, the period of time is considered as fixed and n, t take the values 1, 2, 3,... Then

$$A(n) = \frac{C-S}{n} + \frac{1}{n}\sum_{t=1}^{n} f(t)$$

Now, A(n) will be a minimum for that value of n, for which

$$A(n+1) \geq A(n) \text{ and } A(n-1) \geq A(n).$$

or $$A(n+1) - A(n) \geq 0 \text{ and } A(n) - A(n-1) \leq 0$$

For this, we write

$$A(n+1) = \frac{C-S}{n+1} + \frac{1}{n+1}\sum_{t=1}^{n+1} f(t)$$

$$= \frac{1}{n+1}\left[C - S + \sum_{t=1}^{n} f(t)\right] + \frac{1}{n+1} f(n+1)$$

$$= \frac{1}{n+1}[nA(n) + f(n+1)]$$

$\therefore$ $$A(n+1) - A(n) = \frac{1}{n+1}[f(n+1) - A(n)]$$

Thus $$A(n+1) - A(n) \geq 0 \Rightarrow f(n+1) \geq A(n).$$

Similarly, it can be shown that

$$A(n) - A(n-1) \leq 0 \Rightarrow f(n) \leq A(n-1).$$

This suggests the optimal replacement policy:

Replace the equipment at the end of n years, if the maintenance cost in the $(n+1)^{th}$ year is more than the average total cost in the nth year and the nth year's maintenance cost is less than the previous year's average total cost.

Example: 8.1.1

A firm is considering replacement of a machine, whose cost price is Rs. 12,200 and till scrap value. Rs. 200. The running (maintenance and operating) costs in rupees are found from experience to be as follows:

Year:	1	2	3	4	5	6	7	8
Running cost:	200	500	800	1,200	1,800	2,500	3,200	4,000

When should the machine be replaced?

Solution:

We are given the running cost, $f(n)$, the scrap value $S = \text{Rs.}200$ and the cost of the machine, $C = \text{Rs.}12{,}200$. In order to determine the optimal time n when the machine should be replaced we calculate an average total cost per year during the life of the machine as shown in table given below:

Year of service (n) (1)	Running cost (Rs.) (2)	Cumulative running cost (Rs.) $\sum f(n)$ (3)	Depreciation cost (Rs.) C – S (4)	Total cost (Rs.) TC (3) + (4) (5)	Average cost (Rs.) A(n) (5)/t (6)
1	200	200	12,000	12,200	12,200
2	500	70	12,000	12,700	6,350
3	800	1,500	12,000	13,500	4,500
4	1,200	2,700	12,000	14,700	3,675
5	1,800	4,500	12,000	16,500	3,300
6	2,500	7,000	12,000	19,000	3,167
7	3,200	10,200	12,000	22,200	3,171
8	4,000	14,200	12,000	26,200	3,275

From the table it is noted that the average total cost per year, A(n) is minimum in the 6th year (Rs. 3,167). Also the average cost in 7th year (Rs. 3,171) is more than the cost in the 6th year. Hence the machine should be replaced after every 6 years.

Example: 8.1.2

The data collected in running a machine, the cost of which is Rs. 60,000 are given below:

Year	**1**	**2**	**3**	**4**	**5**
Resale value (Rs.)	42,000	30,000	20,400	14,400	9,650
Cost of spares (Rs.)	4,000	4,270	4,880	5,700	6,800
Cost of labour (Rs.)	14,000	16,000	18,000	21,000	25,000

Determine the optimum period for replacement of the machine.

Solution:

The operating or maintenance cost of machine in successive years is as follows:

Year	**1**	**2**	**3**	**4**	**5**
Operating cost (Rs.)	18,000	20,270	22,880	26,700	31,800

(The cost of spares and labour together determine operating or running or maintenance cost.)

The average total annual cost is computed below:

Year of service	Operating cost (Rs.)	Cum. Operating cost (Rs.)	Resale value (Rs.)	Depreciation cost (Rs.)	Total cost (Rs.) TC	Average cost (Rs.)
n	f(n)	$\sum f(n)$	S	C – S		A(n)
1	18,000	18,000	42,000	18,000	36,000	36,000.00
2	20,270	38,270	30,000	30,000	68,270	34,135.00
3	22,880	61,150	20,400	39,600	1,00,750	33,583,30
4	26,700	87,850	14,400	45,600	1,33,450	**33,362.50**
5	31,800	1,19,650	9,650	50,350	1,70,000	35,000.00

The calculations in the above table show that the average cost is lowest during the fourth year. Hence the machine should be replaced after every fourth year.

Example: 8.1.3

(a) Machine A costs Rs. 9,000. Annual operating costs are Rs. 200 for the first year, and then increase by Rs. 2,000 every year. Determine the best age at which to replace the machine. If the optimum replacement policy is followed, what will be the average yearly cost of owning and operating the machine?

(b) Machine B costs Rs. 10,000. Annual operating costs are Rs. 400 for the first year, and then increase by Rs. 800 every year. You now have a machine of type A which is one – year old. Should you replace it with B, if so when?

Solution:

(a) Let the machine have no resale value when replaced. Then, for machine A, the average total annual cost ATC (n) is computed as follows:

Year (n)	f(n)	$\sum f(n)$	C – S	TC	A(n)
1	200	200	9,000	9,200	9,200
2	2,200	2,400	9,000	11,400	5,700
3	4,200	6,600	9,000	15,600	**5,200**
4	6,200	12,800	9,000	21,800	5,450
5	8,200	21,000	9,000	30,000	6,000

This table shows that the best age for the replacement of machine A is 3rd year. The average yearly cost of owning and operating for this period is Rs. 5,200.

(b) For machine B, the average cost per year can similarly be computed as given in the following table:

Year (n)	f(n)	$\sum f(n)$	C – S	T	A (n)
1	400	400	10,000	10,400	10,400
2	1,200	1,600	10,000	11,600	5,800
3	2,000	3,600	10,000	13,600	4,533
4	2,800	6,400	10,000	16,400	4,100
5	3,600	10,000	10,000	20,000	4,000
6	4,400	14,400	10,000	24,400	4,066

Since the minimum average cost for machine B is lower than that for machine A, machine should be replaced by machine A.

To decide the time of replacement, we should compare the minimum average cost for B (Rs. 4,000) with yearly cost of maintaining and using the machine A. Since there is no salvage vale of the machine, we shall consider only the maintenance cost. We would keep the machine A so long as the yearly maintenance cost is lower than Rs. 4,000 and replace when it exceeds Rs. 4,000.

On the one – year old machine A, Rs. 2,200 would be required to be spent in the next year; while Rs. 4,200 would be needed in year following. Thus, we should keep machine A for one year and replace it thereafter.

Example: 8.1.4

A machine owner finds from his past records that the costs per year of maintaining a machine whose purchase price is Rs. 6000 are as given below:

Year	**1**	**2**	**3**	**4**	**5**	**6**
Maintenance Cost (Rs)	1000	1200	1400	1800	2300	2800
Resale Value (Rs)	3000	1500	750	375	200	200

Determine at what age is replacement due?

Solution:

See Table below. Here C = Rs. 6000, S = scrap value.

Year (n) (1)	Main. Cost f(n) (Rs) (2)	Total Main. Cost $\sum f(n)$ (Rs) (3)	C – S (Rs) (4)	Total Cost = (3) + (4) = TC (5)	Ave. cost $= \frac{(5)}{(1)} = A(n)$ (6)
1	1000	1000	3000	4000	4000
2	1200	2200	4500	6700	3350
3	1400	3600	5250	8850	2950
4	1800	5400	5625	11,025	2756
5	2300	7700	5800	13,500	2700*
6	2800	10,500	5800	16,300	2717

Minimum cost is in 5th year ⇒ optimum replacement plan: replace the machine at the end of 5th year

Check your progress: 8.1

1. A fleet owner finds from his past records that the costs per year of running a vehicle whose purchase price is Rs. 50,000 are as under:

Year	1	2	3	4	5	6	7
Running cost (Rs.)	5,000	6,000	7,000	9,000	11,500	16,000	18,000
Resale value (Rs.)	30,000	15,000	7,500	9,750	2,000	2,000	2,000

2. Fleet cars have increased their costs as they continue in service due to increased direct operating cost (gas and oil) and increased maintenance (repairs, tyres, batteries, etc.). The initial cost is Rs. 3,500, and the trade – in value drops as time passes until it reaches a constant value of Rs. 500.

Given the cost of operating, maintaining and the trade – in value, determine the proper length of service before cars should be replaced:

Year of service	1	2	3	4	5
Year end trade–in value (Rs.)	1,900	1,050	600	500	500
Annual operating cost (Rs.)	1,500	1,800	2,100	2400	2,700
Annual maintaining cost (Rs.)	300	400	600	800	1,000

3. The data on the operating costs per year and resale price of equipment A whose purchase price is Rs. 10,000 are given below:

Year	1	2	3	4	5	6	7
Running cost (Rs.)	1,500	1,900	2,300	2,900	3,600	4,500	5,500
Resale value (Rs.)	5,000	2,500	1,250	600	400	400	400

(i) What is the optimum period for replacement?

(ii) When equipment A is 2 years old, equipment B, which is a new model for the same usage is available. The optimum period for replacement is 4 years with an average cost of Rs. 3,600. Should we change Equipment A with that of B? If so when?

4. A new tempo costs Rs. 80,000 and may be sold at the end of any year at the following prices:

Year (end)	1	2	3	4	5	6
Selling price (in Rs.) (at present value)	50,000	33,000	20,000	11,000	6,000	1,000

The corresponding annual operating costs are:

Year (end)	1	2	3	4	5	6
Cost/year (in Rs.) (at present value)	10,000	12,000	15,000	20,000	30,000	50,000

It is not only possible to sell the tempo after use but also to buy a second hand tempo.

It may be cheaper to do so that to replace by a new tempo.

Age of tempo	0	1	2	3	4	5
Purchase price (in Rs.) (at present value)	80,000	58,000	40,000	26,000	16,000	10,000

What is the age to buy and to sell tempo so as to minimize average annual cost?

5. A transport manager finds from his past records that the cost per year of running a truck whose purchase price is Rs. 6,000 are as given below:

Year	1	2	3	4	5	6	7	8
Running cost (Rs.)	1,000	1,200	1,400	1,800	2,300	2,800	3,400	4,000
Resale value (Rs.)	3,000	1,500	750	375	200	200	200	200

Determine at what age is replacement due?

Replacement Policy when Value of Money changes with time

When the time value of money is taken into consideration, we shall assume that (i) the equipment in question has no salvage value, and (ii) the maintenance costs are incurred in the beginning of the different time periods.

Since it is assumed that the maintenance cost increases with time and each cost is to be paid just in the start of the period, let the money carry a rate of interest r per year. Thus a rupee invested now will be worth (1 + r) after a year, $(1+r)^2$ after two years, and so on. In this way a rupee invested today will be worth $(1+r)^n$, n years hence, or, in other words, if we have to make a payment of one rupee in n years time, it is equivalent to making a payment of $(1+r)^{-n}$ rupees today. The quantity $(1+r)^{-n}$ is called the present worth factor (Pwf) of one rupee spent in n years time from now onwards. The expression $(1+r)^n$ is known as the payment compound amount factor (Caf). of one rupee spent in n years time.

Let the initial cost of the equipment be C and let R_n be the operating cost in year n. Let v be the rate of interest in such a way that $v = (1+r)^{-1}$ is the discount rate (present worth factor). Then the present value of all future discounted costs V_n associated with a policy of replacing the equipment at the end of each n years is given

$$V_n = \{(C+R_o)+vR_1+v^2R_2+....v^{n-1}R_{n-1}\}+$$

$$\{(C+R_o)v_n+v^{n+1}R_1+v^{n+2}R_2+.....+v^{2n-1}R_{n-1}\}+.....$$

$$=\left[C+\sum_{k=0}^{n-1}v^kR_k\right]x\sum_{k=0}^{\infty}(v^n)^k=\left[C+\sum_{k=0}^{n-1}v^kR_k\right]/\ (1-v^n)^{-1}$$

Now, V_n will be a minimum for that value of n, for which

$$V_{n+1}-V_n>0 \text{ and } V_{n-1}-V_n>0.$$

For this, we write

$$v_{n+1} - V_n = \left[C + \sum_{k=0}^{n} v^k R_k\right](1 - v^{n+1})^{-1} - V_n$$

$$= v^n [R_n - (1 - v)V_n]/(1 - v^{n+1})$$

and similarly $V_n - V_{n-1} = V^{n-1}[R^{n-1} - (1 - v)V_n]/(1 - v^{m-1})$

Since v is the depreciation value of money, it will always be less than 1 and therefore v will always be positive. This implies that $v^n/(1 - v^{n+1})$ will always be positive.

Hence, $V_{n+1} - V_n > 0 \Rightarrow R_n > (1 - v)V_n$ and

$V_n - V_{n-1} < 0 \Rightarrow R_{n-1} < (1 - v)V_n$

Thus, $R_{n-1} < (1 - v)V_n < R_n$.

$$R_{n-1} < \frac{C + R_\circ + vR_1 + v^2R_2 + \ldots.. + v^{n-1}R_{n-1}}{1 + v + v^2 + \ldots... + v^{n-1}} < R_n,$$

Since $(1 - v^n)(1 - v)^{-1} = \sum_{n=0}^{n-1} v^k$.

The expression which lies between R_{n-1} and R_n is called the "weighted average cost" of all the previous n years with weights $1, v, v^2, \ldots.., v^{n-1}$ respectively.

Hence, the optimal replacement policy of the equipment after n period is:

(a) Do not replace the equipments if the next period's operating cost is less than the weighted average of previous costs.

(b) Replace the equipments if the next period's operating cost is greater than the weighted average of previous costs.

Remark:

Procedure for determining the weighted average of costs (annualized cost) may be summarized in the following steps:

Step: 1

Find the present value of the maintenance cost for each of the years, i.e., $\sum R_1^{n-1}(n = 1,2.......,)$; where $v = (1+r)^{-1}$.

Step: 2

Calculate cost plus the accumulated present values obtained in step 1. i.e.; $C + \sum R_1^{n-1}$

Step: 3

Find the cumulative present value factor up to each of the year 1, 2, 3.... i.e., $\sum v^{n-1}$.

Step: 4

Determine the annualized cost W(n), by dividing the entries obtained in step 2 by the corresponding entries obtained in step 3, i.e., $[C + \sum v^{n-1}]/\sum v^{n-1}$.

Corollary:

When the time value of money is not taken into consideration, the rate of interest becomes zero and hence v approaches unity. Therefore, as $v \to 1$, we get

$$R_{n-1} < \frac{C + R_o + R_1 + ... + R_{n-1}}{1 + 1 + ... + n \text{ times}} < R_n$$

or

$$R_{n-1} < W(n) < R_n$$

Selection of the Best Equipment Amongst Two

Following is the procedure for determining a policy for the selection of an economically best item amongst the available equipments:

Step: 1

Considering the case of two equipments, say A and B, we first find the best replacement age for both the equipments by making use of

$$R_{n-1} < (1-v)V_n < R_n .$$

Let the optimum replacement age for A and B comes out to be n_1 and n_2 respectively.

Step: 2

Next, compute the fixed annual payment (or weighted average cost) for each equipment by using the formula

$$W(n) = \frac{C + R_\circ + vR_1 + v^2R_2 + \ldots\ldots. + v^{n-1}R_{n-1}}{1 + v + \ldots\ldots. + v_{n-1}}$$

and substitute $n = n_1$ for equipment A and $n = n_2$ for equipment B in it.

Step: 3

(i) If $W(n_1) < W(n_2)$, choose equipment A.

(ii) If $W(n_1) > W(n_2)$, choose equipment B.

(iii) If $W(n_1) = W(n_2)$, both equipments and equally good.

Example: 8.1.5

Let the value of money be assumed to be 10% per year and suppose that machine A is replaced after every 3 years whereas machine B is replaced after every six years. The yearly costs of both the machines are given below:

Year	1	2	3	4	5	6
Machine A	1,000	200	400	1,000	200	400
Machine B	1,700	100	200	300	400	500

Determine which machine should be purchased.

Solution:

Since the money carries the rate of interest, the present worth of the money to be spent over in a period of one year is

$$v = \frac{100}{100+10} = \frac{10}{11} = 0.9091$$

∴ The total discounted cost (present worth) of A for 3 years is

$$1000 + 200x(0.9091) + 400x(0.9091)^2 = \text{Rs.}1512 \text{ approx.}$$

Again, the total discounted cost of B for six years is

$$1{,}700 + 100x(0.9091) + 200x(0.9091)^2 + 300x(0.9091)^3 + 400 + (0.9091)^4 + 500x(0.9091)^5 = \text{Rs.}2{,}765$$

Average yearly cost of machine $A = \text{Rs.}1{,}512/3 = \text{Rs.}504$.

Average yearly cost of machine $B = \text{Rs.}2{,}765/6 = \text{Rs.}461$.

This shows that the apparent advantage is with machine B. But, the comparison is unfair since the periods for which the costs are considered are different. So, if we consider 6 years period for machine A also, then the total discounted cost of A will be

$$1{,}000 + 200x(0.9091) + 400x(0.9091)^2 + 1{,}000x(0.9091)^3 + 200x(0.9091)^4 + 400x(0.9091)^5$$

After simplification this comes out to be Rs. 2,647 which is Rs. 118 less costlier than machine B over the same period.

Hence machine A should be purchased.

Example: 8.1.6

A pipeline is due for repairs. It will cost Rs. 10,000 and last for 3 years. Alternatively, a new pipeline can be laid at a cost of Rs. 30,000 and lasts for 10 years. Assuming cost of capital to be 10% and ignoring salvage value, which alternative should be chosen?

Solution:

Consider the two types of pipeline for infinite replacement cycles of 10 years for the new pipeline, and 3 years for the existing pipeline.

Since, the discount rate of money per year is 10%, therefore the present worth of money to be spent over in a period of one year is

$$v = \frac{100}{100+10} = 0.9091$$

Let k_n denote the discounted value of all future costs associated with a policy of replacing the equipment after n years. Then, if we designate the initial outlay by C,

$$k_n = C + Cv^n + Cv^{2n} + \ldots\ldots. + \infty = C(1 + v^n + v^{2n} + \ldots.. + \infty) = C/(1 - v^n)$$

Making use of values of C, v and n for two types of pipelines, the discounted value, therefore, yields

$$k_3 = \frac{10,000}{1-(09091)^3} = \text{Rs.}4,021$$

for the existing pipeline, and

$$k_{10} = \frac{30,000}{1-(0.9091)^{10}} = \frac{30,000}{1-0.3855} = \text{Rs.}48,820$$

for the new pipeline.

Since $k_3 < k_{10}$, the existing pipeline should be continued. Alternatively, the comparison may be made over $3x10 = 30$ years.

Example: 8.1.7

The cost of a new machine is Rs. 5,000. The maintenance cost of nth year is given by $C_n = 500(n-1); n = 1,2,........$ Suppose that the discount rate per year is 0.5. After how may years it will be economical to replace the machine by a new one?

Solution:

Since the discount rate of money per year is 0.05, the present worth of the money to be spent over a period of one year is

$$v = (1+0.05) = 0.9523.$$

The optimum replacement time is determined in the following table:

Year (n)	R_{n-1} (2)	v^{n-1} (3)	$R_{n-1}\ v^{n-1}$ (4)	$C+\sum_k R_{k-1}v^{k-1}$ (5)	$\sum_k v^{k-1}$ (6)	W(n) (7)
1	0	1.0000	0	5,000	1.0000	5,000
2	500	0.9523	476	6,476	1.9523	2,805
3	1,000	0.9070	907	6,383	2.8593	2,232
4	1,500	0.8638	1,296	7,679	3.7231	2,063
5	2,000	0.8227	1,645	9,324	4.5458	**2,051***
6	2,500	0.7835	1,959	11,283	5.3293	2,117

Since, $W(n)$ is minimum for $n = 5$ and $R_4 = 1{,}500 < W(5)$ as well as $W(5) > R_6 = 2{,}500$; it is economical to replace the machine by a new one at the end of five years.

Example: 8.1.8

A manufacturer is offered two machines A and B. A is priced at Rs. 5,000, and running costs are estimated at Rs. 800 for each of the first five years, increasing by Rs. 200 per year in the sixth and subsequent years. Machine B, which has the same capacity as A, costs Rs. 2,500 but will have running costs of Rs. 1,200 per year for six years, increasing by Rs. 200 per year thereafter.

If money is worth 10% per year, which machine should be purchased? (Assume that the machine will eventually be sold for scrap at a negligible price.)

Solution:

Since the money is worth 10% per year, the discount rate for both the machines is given by

$$v = \frac{1}{1 + 0.10} = 0.9091$$

For the solution of this problem, we compute the following tables for machines A and B separately, by using Pwf table given at the end of the book.

For Machine A

Year (n) (1)	R_{n-1} (2)	v^{n-1} (3)	$v^{n-1}R_{n-1}$ (4)	$C+\sum_k v^{k-1}R_{k-1}$ (5)	$\sum_k v^{k-1}$ (6)	W(n) (7) = (5)/(6)
1	800	1.0000	800	5,800	1.0000	5,800.00
2	800	0.9091	727	6,527	1.9091	3,418.88
3	800	0.8264	661	7,188	2.7355	2,627.67
4	800	0.7513	601	7,789	3.4868	2,233.85
5	800	0.6830	546	8,335	4.1698	1,998.89
6	1,000	0.6209	621	8,956	4.7907	1,869.45
7	1,200	0.5645	677	9,633	5.3552	1,798.81
8	1,400	0.5132	718	10,351	5.8684	1,763.85
9	1,600	0.4665	746	11,097	6.3349	**1,751.72**
10	1,800	0.4241	63	11,860	6.7590	1,754.70

For Machine B

Year (n) (1)	R_{n-1} (2)	v^{n-1} (3)	$v^{n-1}R_{n-1}$ (4)	$C+\sum_{k} v^{k-1}R_{k-1}$ (5)	$\sum_{k} v^{k-1}$ (6)	W(n) (7) = (5)/(6)
1	1,200	1.0000	1,200.00	3,700.00	1.0000	3,700.00
2	1,200	0.9091	1,090.91	4,790.91	1.9091	2,509.51
3	1,200	0.8264	991.98	5,782.59	2.7353	2,113.91
4	1,200	0.7513	901.56	6,684.15	3.4868	1,916.99
5	1,200	0.6830	819.60	7,503.75	4.1698	1,799.55
6	1,200	0.6209	745.08	8,248.83	4.7907	1,721.84
7	1,400	0.5645	790.30	9,039.13	5.3552	1,687.92
8	1,600	0.5132	821.12	9,860.25	**5.8684**	**1,680.23**
9	1,800	0.4665	839.70	10,699.95	6.3349	1,689.05
10	2,000	0.4241	848.20	11,548.15	6.7590	1,708.56

From the above tables we observe that for machine A, $1{,}600 < 1{,}751.72 < 1{,}800$.

Now since the running cost of 9th year is Rs. 1,600 and that of 10th year is Rs. 1,800 and since $1{,}800 > 1{,}751.72$, it is better to replace the machine A after 9th year.

Similarly, for machine B since $1{,}800 > 1{,}680.23$, it is better to replace the machine B after 8th year.

Further since the weighted average cost in 9 years of machine A is Rs. 1751.72 and the weighted average cost in 8 years of machine B is Rs. 1,680.23, it is advisable to purchase machine B.

Example: 8.1.9

Assume that the present value of one rupee to be spent in a year's time is Rs. 0.9 and C = Rs. 3000, capital cost of equipment and the running costs are given in the table below. When should the machine be replaced?

Year	1	2	3	4	5	6	7
Ung. Cost (Rs)	500	600	800	1000	1300	1600	2000

Solution:

Consider the following table:

Year (n) (1)	R_{n-1} (2)	v^{n-1} (3)	$R_{n-1}V^{n-1}$ (4)	$ER_{n-1}V^{n-1}$ (5)	$C+\sum R_{n-1}V^{n-1}$ (6)	$\sum V^{n-1}$ (7)	$W(n)=\frac{6}{7}$
1	500	1	500	500	3500	1	3500
2	600	0.90	540	1040	4040	1.90	2126.32
3	800	0.81	648	1688	4688	2.71	1729.88
4	1000	0.73	730	2418	5418	3.44	1575
5	1300	0.66	858	3276	6276	4.10	1530.73
6	1600	0.59	944.784	4220.78	7220.78	4.69	1539.61

Since $W(n)$ is minimum at 6th year optimum replacement plan is end of sixth year.

Example: 8.1.10

The cost of a new machine is Rs. 5000. The maintenance cost of nth year is given by $C_n = 500(n-1); n = 1,2,.......$suppose that money is worth 5% per year, after how many years will it be economical to replace the machine by a New one?

Solution:

The present worth of the money to be spent a year from now is

$$V = (1+0.05)^{-1} = 0.9523$$

The optimum replacement time is determined in the following table.

Year n	R_{n-1}	v^{n-1}	$R_{n-1}V_{n-1}$	$C+\sum R_{n-1}V^{n-1}$	$\sum V^{n-1}$	$W(n)=\frac{5}{6}$
1	2	3	4	5	6	7
1	0	1.0000	0	5000	1.000	5000
2	500	0.9523	476	6476	1.9523	2805
3	1000	0.9073	907	6383	2.593	2232
4	1500	0.8638	1296	7679	3.7231	2063
5	20000	0.8227	1645	9324	4.5458	2061
6	2500	0.7835	1959	11,283	5.3293	2117

Since $W(n)$ is minimum for $n = 5$ and $R_4 = 1500 < w(5)$ as well as $w(5) > R_6 = 2500$, it is economical to replace the machine by a new one at the end of 5 years.

Check your progress: 8.2

1. Let $v = 0.9$ and initial price is Rs. 5,000. Running cost varies as follows:

Year	1	2	3	4	5	6	7
Running cost (in Rs.)	400	500	700	1,000	1,300	1,700	2,100

What would be the optimum replacement interval?

2. The initial cost of an item is Rs. 15,000 and maintenance or running costs for different years are given below:

Year	1	2	3	4	5	6	7
Running cost (in Rs.)	2,500	3,000	4,000	5,000	6,500	8,000	10,000

What is the replacement policy to be adopted if the capital is worth 10% and there is no salvage value?

3. The yearly cost of 2 machines A and B when the money value is neglected is as follows:

Year	1	2	3	4	5
Machine A	1,800	1,200	1,400	1,600	1000
Machine B	2,800	200	1,400	1,100	600

Find their cost patents if money value is 10% per year and hence find which machine is most economical.

4. A manual stamper currently valued at Rs. 1000 is expected to last 2 years and costs Rs. 4,000 per year to operate. An automatic stamper which can be purchase for Rs. 3,000 will last 4 years and can be operated at an annual cost of Rs. 3,000. If money carries the rate of interest 10% per annum, determine which stamper should be purchased.

8.2 Replacement of Equipment that fails suddenly:

It is difficult to predict that a particular equipment will fail at a particular time. This difficulty can be overcome by determining the probability distribution of failures. Here it is assumed that the failures occurs only at the end of the period, say t. Thus the objective becomes to find the value of t which minimizes the total cost involved for the replacement.

We shall consider the following two types of replacement policies:

Individual Replacement Policy

Under this policy, an item is replaced immediately after its failure.

Group Replacement Policy

Under this policy, we take decisions as to when all the items must be replaced, irrespective of the fact that items have failed or have not failed, with a provision that if any item fails before the optimal time, it may be individually replaced.

Mortality Tables:

These are used to derive the probability distribution of the life span of an equipment. Let

$M(t)$ = number of survivors at any time t,

$M(t-1)$ = number of survivors at any time t – 1, and

N = initial number of equipments

Then the probability of failure during time period t is given by

$$p(t) = [M(t-1) - M(t)]/N$$

The probability that an equipment survived till age (t – 1), will fail during the interval (t – 1) to t can be defined as the conditional probability of failure. It is given by

$$p_c(t) = [M(t-1) - M(t)]/M(t-1)$$

The probability of survival till age t is given by

$$p_s(t) = M(t)/N.$$

Theorem: 1 (Mortality)

A large population is subject to a given mortality law for a very long period of time. All deaths are immediately replaced by births and there are no other entries or exists. Then the age distribution ultimately becomes stable and that the number of deaths per unit time becomes constant (which is equal to the size of the total population divided by the mean age of death).

Proof:

Let k be a constant such that no item of the system can survive upto and beyond time $k+1$, i.e., the life span of any item lies between $t=0$ and $t=k$. We define

$f(t)$: the number of births at time t, and

$p(x)$: the probability that an equipment will die (fail) just before achieving the age $x+1$, i.e., at age x.

It is easy to note that $\sum_{x=0}^{k} p(x) = 1$.

Now $f(t-x)$ births take place at time $t-x$, $t=k,k+1,\dots\dots$ Such newly born items attain the age x at time t. Therefore, the expected number of deaths of such alive numbers at time t is $p(x)f(t-x)$. Thus the mathematical expectation of the number of deaths before time $t+1$ is $\sum f(t-x)p(x), t=k,k+1,\dots\dots$ Moreover, since deaths are immediately replaced by births, we must have

$$f(t+1) = \sum_{x=0}^{k} f(t-x)p(x),\ t=k,k+1,\dots\dots$$

The solution to this difference equation in t can be obtained by making use of

$f(t) = A\alpha^{1}$, where A is some constant and $1\alpha 1 < 1$.

$$\therefore \quad A\alpha^{t+1} = \sum_{x=0}^{k} A\alpha^{t-x}p(x) = A[\alpha^{t}p(0)+\alpha^{t-1}p(1)+\dots\dots+\alpha^{t-k}p(k)] \text{ Or}$$

$$\alpha^{k+1} = \alpha^{k}\left[\sum_{x=0}^{k} \alpha^{-x}p(x)\right] = \alpha^{k}\left[p(0)+\alpha^{-1}p(1)+\dots\dots+\alpha^{-k}p(k)\right]$$

Thus

$$\alpha^{k+1}[\alpha^{k}p(0)+\alpha^{k-1}p(1)+.......+p(k)]=0.$$

This is a linear homogeneous difference equation of degree k + 1 and thus has exactly k + 1 roots. Let the roots be $\alpha_0,\alpha_1,......,\alpha_k$.

For $\alpha=1$, the equation yields

$$\text{L.H.S} = 1-\sum_{x=0}^{k} p(x)=1-1=\text{R.H.S.}$$

Thus $\alpha=1$ is a root of the above equation. Let us denote it by $\alpha_0=1$. The most general solution of the difference equation will be the form

$$f(t)=A_{\circ}\alpha_{\circ}^{!}+A_1\alpha_1^{t}+....+A_k\alpha_k^{t}$$

$$=A_{\circ}+A_1\alpha_1^{t}+......+A_k\alpha_k^{t}$$

where $A_{\circ},A_1,.....,A_k$ are constants whose values are to be determined. We observe that since $|\alpha_1|<1$ as $t\to\infty$, $\lim_{t\to\infty}f(t)=A_{\circ}$ Thus under our assumption for a long period t, the number of deaths per unit time is equal to $A_{\circ}$.

Now the problem is to determine the value of the constant $A_{\circ}$.

Let $g(x)=$ Probability of surviving for more than x years.

or $g(x)=1-P$ (survivor will die before attaining the age x)

$=1-\{p(0)+p(1)+......+p(x-1)\}$

Obviously, it can be assumed that $g(0)=1$.

Since the number of births as well as deaths have become constant, each equal to $A_\circ$, therefore expected number of survivors of age x is given by $A_\circ.g(x)$.

As deaths are immediately replaced by births and therefore size N of the population remains constant. Thus, we must have

$$N = A_\circ \sum_{x=0}^{k} g(x) \text{ or } A_\circ = N / \sum_{x=0}^{k} g(x).$$

From finite differences, we know that

$$\Delta(x) = (x+1) - x = 1 \text{ and}$$

$$\sum_{x=0}^{b} f(x)\Delta h(x) = f(b+1)h(b+1) - f(a)h(a) - \sum_{x=a}^{b} h(x+1)\Delta f(x)$$

Therefore, we can write,

$$\sum_{x=0}^{k} g(x) = \sum_{x=0}^{k} g(x)\Delta(x) = (k+1)g(k+1) - 0 \text{ x } g(0) - \sum_{x=0}^{k} (x+1)\Delta g(x)$$

$$= (k+1)g(k+1) - \sum_{x=0}^{k} (x+1)\Delta g(x).$$

But $\quad g(k+1) = 1 - \{p(0) + p(1) + \ldots\ldots\ldots + p(k)\} = 0,$

and $\quad \Delta g(x) = g(x+1) - g(x)$

$$= [1 - p(0) - p(1) - \ldots\ldots. - p(x)] - [1 - p(0) - \ldots.. - p(x-1)]$$

$$= -p(x).$$

$$\therefore \quad \sum_{x=0}^{k} g(x) = (k+1)g(k+1) - \sum_{x=0}^{k} (x+1)[-p(x)]$$

$$= \sum_{x=0}^{k} (x+1)p(x);$$

Which happens to be the mean (expected age at death).

Hence, A_o = N/Mean age at death.

Theorem: 2 (Group Replacement)

Let all the items in a system be replaced after a time interval 't' with provisions that individual replacements can be made if and when any item fails during this time period. Then

(a) Group replacement must be made at the end of t^{th} period if the cost of individual replacement for the period is greater than the average cost per unit time period through the end of t periods.

(b) Group replacement is not advisable at the end of period t if the cost of individual replacement at the end of period t − 1 is less than the average cost per unit period through the end of period t.

Proof:

Let,

N = total number of items in the system,

C_2 = cost of replacing an individual item,

C_1 = cost of replacing an item in group,

C(t) = total cost of group replacement after time period t,

f(t) = number of failures during time period t.

Then, clearly

$$C(t) = NC_1 + C_2 \sum_{x=0}^{t-1} f(x)$$

The average cost of group replacement per unit period of time during a period t, is thus given by

$$A(t) = \frac{C(t)}{t} = \left[NC_1 + C_2 \sum_{x=0}^{t-1} f(x)\right] / t.$$

We shall determine the optimum t so as a minimize $C(t)/t$.

Note that whenever $\frac{C(t-1)}{t-1} > \frac{C(t)}{t}$ and $\frac{C(t+1)}{t+1} > \frac{C(t)}{t}$, it is better to replace all items after time period t.

Now,

$$\frac{C(t+1)}{t+1} - \frac{C(t)}{t} > 0 \Rightarrow C_2 f(t) > C(t)/t;$$

and $$\frac{C(t-1)}{t-1} - \frac{C(t)}{t} > 0 \Rightarrow C_2 f(t-1) < C(t)/t$$

∴ $$tC_2 f(t-1) < C(t) < t.C_2 f(t).$$

or $$t\, f(t-1) - \sum_{x=0}^{t-1} f(x) < \frac{NC_1}{C_2} < t\, f(t) - \sum_{x=0}^{t-1} f(x).$$

Example: 8.2.1

The following failure rates have been observed for a certain type of transistors in a digital computer:

End of the week	1	2	3	4	5	6	7	8
Probability of failure to date	.05	.13	.25	.43	.68	.88	.96	1.00

The cost of replacing an individual failed transistor is Rs. 1.25. The decision is made to replace all these transistors simultaneously at fixed intervals, and to replace the individual transistors as they fail in service. If the cost of group replacement is 30 paise per transistor, what is the best interval between group replacements? At what group replacement price per transistor would a policy of strictly individual replacement become preferable to the adopted policy?

Solution:

Suppose there are 1,000 transistors in use. Let p_i be the probability that a transistor, which was new when placed in position for use, fails during the ith week of its life. Thus, we have

$$p_1 \equiv 0.05, \qquad p_2 \equiv 013 - 0.5 = 0.08,$$

$$p_3 \equiv 0.25 - 0.13 = 0.12, \qquad p_4 \equiv 0.43 - 0.25 = 0.18,$$

$$p_5 \equiv 0.68 - 0.43 = 0.25, \qquad p_6 \equiv 0.88 - 0.68 = 0.20,$$

$$p_7 \equiv 0.96 - 0.88 = 0.08, \qquad p_8 \equiv 1.00 - 0.96 = 0.04.$$

Let N_i denote the number of replacements made at the end of the ith week. Then, we have

N_o = number of transistors in the beginning

$= 1{,}000$

$N_1 = N_o p_1 = 1.000 \text{ x } 0.05$

$= 50$

$N_2 = N_o p_2 + N_1 p_1 = 1.000\text{x}0.08 + 50\text{x}0.05$

$= 82$

$N_3 = N_o p_3 + N_1 p_2 + N_2 p_1 = 1.000\text{x}0.12 + 50\text{x}0.08 + 82\text{x}0.05$

$= 128$

$N_4 = N_o p_4 + N_1 p_3 + N_2 p_3 + N_3 p_2 + N_4 p_1$

$= 199$

$$N_5 = N_o p_5$$

$$= 289$$

$$N_6 = N_o p_6 + N_1 p_5 + N_2 p_4 + N_3 p_3 + N_4 p_2 + N_5 p_1$$

$$= 272$$

$$N_7 = N_o p_6 + N_1 p_6 + N_2 p_5 + N_3 p_4 + N_4 p_3 + N_5 p_2 + N_6 p_1$$

$$= 194$$

$$N_8 = N_o p_8 + N_1 p_7 + N_2 p_6 + N_3 p_5 + N_4 p_4 + N_5 p_3 + N_6 p_2 + N_7 p_1$$

$$= 195$$

From the above calculations, we observe that the expected number of transistors failing each week increases till 5th week and then starts decreasing and later again increasing from 8th week.

Thus, N_i will oscillate till the system acquires a steady state; The expected life of each transistor is

1 x 0.5+2 x .08+3 x 12+4 x .18+5 x .25+6 x .2+7 x .08+8 x .04
= 4.62 weeks.

Average number of failures per week

$$= 1{,}000/4.62 = 216 \text{ approximately.}$$

Therefore, the cost of individual replacement

$$= 216 \times 1.25 = \text{Rs.}270.00 \text{ per week.}$$

Now, since the replacement of all the 1,000 transistors simultaneously cost 30 per transistors and the replacement of an individual transistor on failure cost Rs. 1.25, the average cost for different group replacement policies is given as under:

End of week	Individual replacement	Total cost (Rs.) Individual + Group	Average cost (Rs.)
1	50	50 x 1.25+1,000 x 0.30=363	363
2	132	132 x 1.25+1,000 x 0.30=465	232.50
3	260	260 x 1.25+1,000 x 0.30=625	208.30
4	459	459 x 1.25+1,000 x 0.30=874	218.50

Since the average cost is lowest against week 3, the optimum interval between group replacement is 3 weeks. Further, since the average cost is less than Rs. 270 (for individual replacement), the policy of group replacement is better.

Example: 8.2.2

At time zero all items in a system are new. Each item has a probability p of failing immediately before the end of the first month of life, and a probability $q = 1 - p$ of failing immediately before the end of the second month (i.e., all items fail by the end of the second month). If all items are replaced as they fail, show that the expected number of failures $f(x)$ at the end of month x is given by

$$f(x) = \frac{N}{1+q}[1-(-q)^{x+1}],$$

where N is the number of items in the system.

If the cost per item of individual replacement is C_1, and the cost per item of group replacement is C_2, find the condition under which

(a) A group replacement policy at the end of each month is the most profitable.

(b) No group replacement policy is better than a policy of pure individual replacement.

Solution:

Let

N = number of items in the system in the beginning

N_1 = number of items expected to fail at the end of 1st month

$= N_{\circ}p = N(1-q)$, since $p = 1 - q$.

N_2 = number of items expected to fail at the end of 2nd month

$= N_{\circ}q + N_1 p = Nq + N(1-q)^2 = N(1-q+q^2)$,

N_3 = number of items expected to fail at the end of 3rd month

$= N_1 q + N_2 p = N(1-q)q + N(1-q+q^2)(1-q) = N(1-q+q^2+q^3)$,

and so on. In general,

$$N_k = N[1 - q + q^2 - q^3 + \dots\dots + (-q)^k].$$

$$\therefore \quad N_{k+1} = N_{k-1}q + N_k p$$

$$= N[1 - q + q^2 + \dots\dots + (-q)^{k-1}]q + N[1 - q + q^2 + \dots\dots\dots + (-q)^k](1-q)$$

$$= N[1 - q + q^2 + \dots\dots + (-q)^{k+1}]$$

Hence by mathematical induction, the expected number of failures at the end of month x will be given by

$$f(x) = N[1 - q + q^2 + \dots\dots + (-q)^x] = N[1-(-q)^{x+1}]/(1+q).$$

The value of $f(x)$ at the end of month x will vary for different values of $(-q)^{x+1}$ and it will reach the steady – state as $x \to \infty$.

Hence, in the stead state case, the expected number of failures will be

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} N[1-(-q)^{x+1}]/(1+q)$$

$$= N/(1+q)\text{; since } q<1 \text{ and } (-q)^{x+1} \to 0 \text{ as } x \to \infty.$$

= Total number of items in the system/Mean age,

Now, since C_1 is the cost of replacement per item individually and C_2 is the cost of an item in group, therefore

(i) If we have a group replacement at the end of each month, then the cost of replacement is NC_2.

(ii) If we have a group replacement policy at the end of every other month, then the cost is $NC_2 + NpC_1$.

The average cost per month, therefore, is $(NC_2 + NpC_1)/2$, and

(iii) Average life of an item

$$= 1 \text{ x } p + 2 \text{ x } q = 1 \text{ x } (1-q) + 2q = 1+q.$$

Therefore, the average number of failures is $N/(1+q)$ and hence the cost of individual replacement is $NC_1/(1+q)$.

(a) A group replacement at the end of first month will be better than individual replacement, if total cost of group replacement is less than the average monthly cost of individual replacement.

Thus, $N(1-q)C_1 + NC_2 < NC_1/(1+q)$, i.e., $C_2 < c_1q^2/(1+q)$.

For a group replacement at the end of every second month, the total cost of replacement will be

$$(N_1 + N_2)C_1 + NC_2 = N(2 - 2q + q^2)C_1 + NC_2 .$$

Average monthly cost of group replacement at the end of second month is

$$[N(2 - wq + q^2)C_1 + NC_2]/2$$

In this case, the group replacement policy will be better than the individual replacement policy, if

$$\begin{matrix}\text{Average monthly cost of} \\ \text{group replacement}\end{matrix} < \begin{matrix}\text{Average monthly cost of} \\ \text{individual replacement}\end{matrix}$$

or $$[N(1 - q + q^2 / 2\ C_1 + NC_2 / 2] < NC_1 /(1 + q),$$

or $$C_2 < q^2(1 - q)C_1 /(1 + q) .$$

(b) For the individual replacement policy to be better than any of the group replacement policies discussed above, we must have

$$C_2 > C_1 q^2 /(q + 1) \text{ and } C_2 > C_1 q^2 (1 - q)/(q + 1)$$

or $$C_1 < C_2(1 + q)/q^2 \text{ and } C_1 < C_2(1 + q)/[q^2(1 - q)]$$

But $q < 1$, therefore $(1 + q)/q^2 < (1 + q)/q^2(1 - q)$

Hence, $C_1 < (1 + q)C_2 / q^2$.

8.3 Sequencing Problems:

Introduction:

The selection of an appropriate order for a series of jobs to be done on a finite number of service facilities, in some pre – assigned order, is called sequencing. A practical situation may correspond to an industry producing a number of products, each of which is to be processed through different machines, of course, finite in number.

Problem of Sequencing:

Consider a problem of machine operator who has to perform three operations, namely (i) turning, (ii) threading, and (iii) knurling on a finite number of different jobs. Let there be six jobs and the time required to perform these operations/(in minutes) for each job is known. Also it is given that each job first goes for turning, then for threading, and lastly for knurling. The problem of the machine operator is to decide which job should be for knurling. The problem of the machine operator is to decide which job should be processed first, which to process next, and so on, i.e., the order (sequence) of the jobs for the above – mentioned operations in order to minimize the total time required to turn out all the jobs. This is an example of six – job and three – machine sequencing problem. We now consider the general case.

The general sequencing problem may be defined as: Let there be n jobs to be performed one at a time on each of m machines. The sequence (order) of the machines in which each job should be performed is given. The actual or expected time required by the jobs on each of the machine is also given. The general sequencing problem, therefore, is to find the sequence out of $(n!)^m$ possible sequences which minimize the total elapsed time between the start of the job in the first machine and the completion of the last job on the last machine.

Given below are the assumptions underlying a sequencing problem:

1. Each job, once started on a machine, is to be performed up to completion on that machine.
2. The processing time on each machine is known. Such a time is independent of the order of the jobs in which they are to be processed.
3. The time taken by each job in changing over from one machine to another is negligible.
4. A job starts on the machines as soon as the job and the machine both are idle and job is next to the machine and the machine is also next to the job.
5. No machine may process more than one job simultaneously.
6. The order of completion of job has no significance, i.e., no job is to be given priority. The order of completion of jobs is independent of sequence of jobs.

Basic Terms used in Sequencing:

Some of the basic terms in sequencing are:

1. **Number of machines:** It refers to the number of service facilities through which a job must pass before it is assumed to be completed.
2. **Processing order:** It refers to the order (sequence) in which given machines are required for completing the job.
3. **Processing time:** It indicates the time required by a job on each machine.
4. **Total elapsed time:** It is the time interval between starting the first job and completing the last job including the idle time (if any) in a particular order by the given set of machines.
5. **Idle time on a machine:** It is the time for which a machine does not have a job to process, i.e., idle time from the end of job $(i-1)$ to the start of job i.

6. **No passing rule:** It refers to the rule of maintaining the order in which jobs are to be processed on given machines. For example, if n jobs are to be processed through three machines M_1, M_2 and M_3 in the order M_1, M_2 and M_3, then this rule will mean that each job will go to machine M_1 first, then to M_2 and lastly to M_3.

Step – wise procedure for determining the optimal sequence for n jobs on 2 machines [Johnson's Method]

Let $A_{11}, A_{21}, A_{31}, \ldots\ldots, A_{n1}$ be processing times of n jobs on Machine 1 and let $A_{12}, A_{22}, A_{32}, \ldots\ldots, A_{n2}$ be the processing times of n jobs on Machine 2.

Step: 1

Find Min (A_{i1}, A_{i2}) $i = 1,2,\ldots\ldots,n$.

Step: 2

(a) If this minimum be A_{l1} for $i = \ell$ process the ℓ^{th} job first

(b) If this minimum be A_{m2} for some $i = m$ process the m^{th} job last of all.

Step: 3

(a) If there is a tie, i.e., $A_{l1} = A_{m2}$ process the ℓ^{th} job first and m^{th} job in the last.

(b) If the tie for the minimum occurs among the $A_{i1} - s$, choose the job corresponding to the minimum of $A_{i2} - s$ and process it first of all

(c) If the tie for minimum occurs among the $A_{i2} - s$ choose the job corresponding to the minimum of $A_{i1} - s$ and process it in the last.

Step: 4

Cancel the jobs already assigned and repeat steps 1 to 3 until all the jobs have been assigned.

Example: 8.3.1

There are five jobs, each of which is to be processed through two machines M_1, M_2 in the order M_1M_2. Processing hours are as follows:

Job	1	2	3	4	5
M_1	3	8	5	7	4
M_2	4	10	6	5	8

Determine the optimum sequence for the five jobs, and minimum total elapsed time. Find also the idle time of machines M_1 and M_2.

Solution:

Machine/Job No.	M_1	M_2	Order of cancellation
1	3	4	(1)
2	8	10	(5)
3	5	6	(3)
4	7	5	(4)
5	4	8	(2)

Optimal sequence

1	5	3	2	4

Explanation:

The smallest entry is 3 and it is against job 1 occurring under machine 1. $\therefore$ Process job 1 first

Next smallest entry is 4 against the job 5 under machine M_1, so next do the job 5. Now the smallest entry is 5. But there is a tie. 5 occurs against job 3 and 4. But job 3 is chosen since the entry corresponding to 5 under machine 2 is 6 smaller than 7 found for job 4 under Machine 1.

Obviously job 4 is chosen next and lastly the job 2 is chosen.

To find the minimum total lapsed time

Job	M_1		M_2	
No	Time in	Time out	Time in	Time out
1	0	3	3	7
5	3	7	7	15
3	7	12	15	21
2	12	20	21	31
4	20	27	31	36

$\therefore$ Minimum total elapsed time $= 36$ hrs.

Idle time for machine 1 $= 36 - 27 = 9$ hrs.

Idle time for machine 2 $= 3$ hrs.

Example: 8.3.2

Find the sequence that minimizes the total elapsed time required to complete the following tasks on machine M_1 and M_2 in the order M_2 in the order M_1, M_2. Also, find the minimum total elapsed time.

Task	A	B	C	D	E	F	G	H	I
M_1	2	5	4	9	6	8	7	5	4
M_2	6	8	7	4	3	9	3	8	11

Solution:

Machine/Task	M_1	M_2	Order of cancellation
A	2	6	(1)
B	5	8	(7)
C	4	7	(4)
D	9	4	(6)
E	6	3	(2)
F	8	9	(9)
G	7	3	(3)
H	5	8	(8)
I	4	11	(5)

Optimal sequence

A	C	I	B	H	F	D	G	E

To find the minimum total lapsed time.

Job	M_1		M_2	
No	Time in	Time out	Time in	Time out
A	0	2	2	8
C	2	6	8	15
I	6	10	15	26
B	10	15	26	34
H	15	20	34	42
F	20	28	42	51
D	28	37	51	55
G	37	44	55	58
E	44	50	58	61

$\therefore$ Minimum total elapsed time = 61 time units.

Note: Idle time machine M_1 = 11 time units.

Idle time of machine M_2 = 2 time units.

Is the sequence ACIHBFDGE Optimal?

Processing n jobs on three machines

Let A_1, A_2, A_3 be the three machines. Let the order of operations be $A_1\ A_2\ A_3$. This problem can be converted into a two machine problem if any one of the following conditions is satisfied.

Condition (1) $\underset{i}{\text{Min}}\ A_{i1} \geq \text{Max}\ A_{i2}$ or (2) $\underset{i}{\text{Min}}\ A_{i3} \geq \underset{i}{\text{Max}}\ A_{i2}$

The method fails if none of these conditions is satisfied.

If one of the conditions is satisfied, we introduce two fictitious machines H and K in such a way that the processing times on H and K are given by

$$H_i = A_{i1} + A_{i2}, \qquad i = 1,2,3,\ldots\ldots\ldots,n$$

$$K_i = A_{i2} + A_{i3}, \qquad i = 1,2,3,\ldots\ldots\ldots,n$$

Now proceed to determine the optimal sequence using Johnson's Method.

Example: 8.3.3

Find the sequence that minimizes the total elapsed time required to complete the following tasks on the machines in the order 1 – 2 – 3. Find also the minimum total elapsed time (hours) and the idle times on the machines.

Task	A	B	C	D	E	F	G
Time on							
Machine 1	3	8	7	4	9	8	7
Machine 2	4	3	2	5	1	4	3
Machine 3	6	7	5	11	5	6	12

Solution:

Min. time on Machine 3 is 5. Max time on Machine 2 is 5.

Min. time on machine 3 ≥ max. time on machine 2.

∴ The required condition to convert this into a two machine problem is satisfied.

Let H and K be the two fictitious machines. Then

Machine/Task	H	K	Order of cancellation
A	7	10	(2)
B	11	10	(6)
C	9	7	(3)
D	9	16	(4)
E	10	6	(1)
F	12	10	(7)
G	10	15	(5)

Optimal sequence

A	D	G	F	B	C	E

To find the total elapsed time

	Machine 1		Machine 2		Machine 3	
Job	Time in	Time out	Time in	Time out	Time in	Time out
A	0	3	3	7	7	13
D	3	7	7	12	13	24
G	7	14	14	17	24	36
F	14	22	22	26	36	42
B	22	30	30	33	42	49
C	30	37	37	39	49	54
E	37	46	46	47	54	$\boxed{59}$

The minimum total Elapsed Time $= 59$ hrs.

Idle time on Machine 1 $= 59 - 46 = 13$ hrs.

Idle time on Machine 2 $= 3 + 2 + 5 + 4 + 4 + 7 + 12 = 37$ hrs.

Idle time on Machine 3 $= 7$ hrs.

Example: 8.3.4

Find the sequence that minimizes the total elapsed time required to complete the following jobs on machines M_1, M_2 and M_3 in the order M_1, M_2, M_3.

Task	A	B	C	D	E	F
M_1	8	3	7	2	5	1
M_2	3	4	5	2	1	6
M_3	8	7	6	9	10	9

Solution:

Min Processing time on $M_1 = 1$

Max Processing time on $M_2 = 6$

Min Processing time on $M_3 = 6$

Min Processing time on $M_3 \geq$ Max Processing time on M_2

∴ The required condition to reduce this three machine problem to a two machine problem is satisfied.

Let A and K be two fictitious machines such that

$$H_i = M_{i1} + M_{i2}$$

$$K_i = M_{i2} + M_{i3}$$

Where $i = A,B,C,D,E$ and F and $H_i, M_{1i}, M_i, K_i, M_{i2}, M_{i3}$ are all processing times on machines H, M_1, M_2, K_3 and M_3 :

	Machines	
Job	$H_i = M_{i1} + M_{i2}$	$K_i = M_{i2} + M_{i3}$
A	11	11
B	7	11
C	12	11
D	4	11
E	6	11
F	7	15

Optimal Sequence

D	E	B	F	C	A

Processing n jobs on m machines

Let there be m machines $A_1, A_2, \ldots\ldots, A_m$. This problem can be converted to a two machine problem if one of the following conditions is satisfied.

Let $A_{i1}, A_{i2}, \ldots\ldots\ldots, A_{im}$ be the processing times on machines

$A_1, A_2, \ldots\ldots, A_m$ respectively.

Then if $\underset{i}{\text{Min}}\ A_{i1} \geq \underset{i}{\text{Max}}\ A_{ij},\ j = 2,3,\ldots\ldots, m-1$

or $\underset{i}{\text{Min}}\ A_{im} \geq \underset{i}{\text{Max}}\ A_{ij}$, $j = 2,3,\ldots\ldots, m-1$

Then this problem can be converted to a two machine problem.

Introduce two fictitious machines H and K such that

$$H_i = A_{i1} + A_{i3} + \ldots\ldots.. + A_{im-1}$$

$$K_i = A_{i2} + A_{i3} + \ldots\ldots.. + A_{im} \qquad i = 1,2,3,\ldots.,n$$

Where H_i and K_i are the processing times for job i on machines H and K respectively.

Example: 8.3.5

Solve the following sequencing problem giving an optimal solution if passing is not allowed.

		Machines			
		M_1	M_2	M_3	M_4
	A	13	8	7	14
Jobs	B	12	6	8	19
	C	9	7	8	15
	D	8	5	6	15

Solution:

$\text{Min } M_{i1} \geq \text{Max } M_{ij}$, $j = 2,3$

Even $\text{Min } M_{i4} \geq \text{Max } M_{ij}$, $j = 2,3$.

Hence the required condition is satisfied.

$$H_i = M_{i1} + M_{i2} + M_{i3}$$

$$K_i = M_{i2} + M_{i3} + M_{i4}$$

Machine/Task	H	K	Order of cancellation
A	28	29	(4)
B	26	33	(3)
C	24	30	(2)
D	19	26	(1)

Optimal sequence

D	C	B	A

	M_1		M_2		M_3		M_4	
Job	Time in	Time out	Time in	Time out	Time in	Time out	Time in	Time out
D	0	8	8	13	**13**	19	**19**	34
C	8	17	**17**	24	**24**	32	34	49
B	17	29	**29**	35	**35**	43	49	68
A	29	**42**	**42**	**50**	**50**	**57**	68	$\boxed{82}$

Minimum total elapsed Time $= 82$ time units

Idle time on $M_1 = 82 - 42 = 40$ time units

Idle time on $M_2 = 8 + 4 + 5 + 7 + 32 = 56$ time units

Idle time on $M_3 = 13 + 5 + 3 + 7 + 25 = 53$ time units

Idle time on $M_4 = 19$ time units

Example: 8.3.6

Solve the following sequencing problem of 4 jobs on 6 machines (Processing time in hrs) Machines:

	Machines					
Job	M_1	M_2	M_3	M_4	M_5	M_6
A	19	8	8	3	11	24
B	18	6	9	6	9	18
C	12	5	8	5	7	15
D	20	5	3	4	8	11

Solution:

Solving a sequencing problem means determining (i) the optimal sequence (ii) Minimum total time and (iii) idle times of all the machines.

$$\text{Min} \quad M_{i1} = 12, \qquad \text{Min } M_{i6} = 11$$

$$\text{Max of } M_{i2} = 8$$

$$\text{Max} \quad M_{i3} = 9$$

$$\text{Max} \quad M_{i4} = 6$$

$$M_{i5} = 11$$

The required condition to reduce this problem to a two machine problem is satisfied.

Let H and K be two fictitious machines.

$$H_i = M_{i1} + M_{i2} + M_{i3} + M_{i4} + M_{i5}$$

$$K_i = M_{i2} + M_{i3} + M_{i4} + M_{i5} + M_{i6}$$

Job	H	K	Order of cancellation
A	49	54	(4)
B	48	48	(3)
C	37	40	(2)
D	40	31	(1)

Optimal sequence

C	B	A	D

To find Total elapsed time

	M_1		M_2		M_3		M_4		M_5		M_6	
Job	T1	To	T1	To	T1	To	T1	To	T1	To	T1	To
C	0	12	12	17	17	25	25	30	30	37	37	52
B	12	30	30	36	36	45	45	57	57	60	60	78
A	30	49	49	57	57	65	65	68	68	79	79	103
D	49	69	69	74	74	77	77	81	81	89	103	114

Minimum Total elapsed time $= 114$hrs.

T_1 =Time in

To = Time out

Idle time:

$M_1 : 45$hrs; $M_2 : 90$hrs; $M_3 : 86$hrs; $M_4 : 96$hrs; $M_5 : 79$hrs; $M_6 : 46$hrs;

Is CABD an optimal sequence.

Check your progress: 8.3

1. We have five jobs, each of which must go through the two machines A and B in the order AB. Processing times in hours are given in the table below:

Job (i)	1	2	3	4	5
Machine A (A)	5	1	9	3	10
Machine B (B_1)	2	6	7	8	4

Determine a sequence for the five jobs that will minimize the elapsed time.

2. Six jobs go first over machine I and then over machine II. The order of the completion of jobs has no significance. The following table gives the machine time in hours for six jobs and the two machines:

Job No.	1	2	3	4	5	6
Time on Machine 1 (A_i)	5	9	4	7	8	6
Time on Machine II (B_i)	7	4	8	3	9	5

Find the sequence of jobs that minimizes the total elapsed time to complete the jobs. Find the minimum time by using Gnatt's chart or by any other method. Also compute the idle time for both the machines.

3. Determine the optimal sequence of jobs that minimizes the total elapsed time based on the following information processing time on machines given in hours and passing is not allowed:

Job	1	2	3	4	5
Machine A	3	8	7	5	2
Machine B	3	4	2	1	5
Machine C	5	8	10	7	6

4. Find the sequence that minimizes the total time required in performing the following jobs on three machines in the order ABC:

Processing time (in hours) on	Jobs					
	1	2	3	4	5	6
Machine A	8	3	7	2	5	1
Machine B	3	4	5	2	1	6
Machine C	8	7	6	9	10	9

8.4 Keywords:

Replacement policy, Group replacement policy. Individual replacement policy, problem of sequencing, Total elapsed time, processing time.

8.5 Answers to check your progress questions:

Check your progress: 8.1

1. After 6 years
2. End of 5th year
3. After 2 years
4. Four years, net saving will be Rs. 1,250
5. End of 5th year.

Check your progress: 8.2

1. Six years
2. End of 5th year
3. Machine B must be purchased.
4. Automatic stamper should be purchased

Check your progress: 8.3

1. $2 \to 4 \to 3 \to 5 \to 1$, minimum total time = 30 hours
2. $3 \to 1 \to 5 \to 6 \to 2 \to 4$, minimum total time = 42 hours
3. $1 \to 4 \to 5 \to 3 \to 2$, minimum time = 42 hours

4. $4 \rightarrow 5 \rightarrow 6 \rightarrow 2 \rightarrow 1 \rightarrow 3$, minimum time is 53 hours

8.6 Model Questions:

1. The cost of a machine is Rs. 6,100 and its scrap value is Rs. 100. The maintenance costs found from experience are as follows:

Year	1	2	3	4	5	6	7	8
Maintenance cost (Rs.)	100	250	400	600	900	1,200	1,600	2,000

When should the machine be replaced?

2. A firm is considering replacement of a machine whose cost price is Rs. 17,500 and the scrap value is Rs. 500. The maintenance costs (in Rs.) are found from experience to be as follows:

Year	1	2	3	4	5	6	7	8
Maintenance cost (Rs.)	200	300	3,500	1,200	1,800	2,400	3,300	4,500

3. A firm is using a machine whose purchase price is Rs. 13,000. The installation charges amount to Rs. 3,600 and the machine has a scrap value of only Rs. 1,600, because the firm has a monopoly of this type of work.

The maintenance cost in various years is given in the following table:

Year	1	2	3	4	5	6	7
Cost (Rs.)	250	750	1,000	1,500	2,100	2,900	4,000

4. Following table gives the running costs per year and resale price of a certain equipment whose purchase price is Rs. 5,000.

Year	1	2	3	4	5	6	7	8
Running cost (Rs.)	1,500	1,600	1,800	2,100	2,500	2,900	3,400	4,000
Resale value (Rs.)	3,500	2,500	1,700	1,200	800	500	500	500

At what year is the replacement due?

5. A manufacturer is offered two machines A and B. A is priced at Rs. 5,000 and running cost are estimated at Rs. 800 for each of the first five years, increasing by Rs. 200 per year in the sixth subsequent years. Machine B, which has the same capacity as A, costs Rs. 2,500 but will have running costs of Rs. 1,200 per year for six years, increasing by Rs. 200 per year thereafter.

   If money is worth 10% per year, which machine should be purchased? (Assume that the machines will eventually be sold for scrap at a negligible price.)

6. An engineering company is offered two types of material handling equipment A and B. A is priced at Rs. 60,000 including cost of installation, and the costs for operation and maintenance are estimated to be Rs. 10,000 for each of the first five years, increasing by Rs. 3,000 per year in the sixth Rs. 30,000 but in terms of operation and maintenance costs more than A. these costs more than A. these costs for B are estimated to be Rs. 13,000 per year for the first six years, increasing by Rs. 4,000 per year for each year from the 7th year onwards. The company expects a return of 10 per cent on all its investments. Neglecting the scrap value of the equipment at the end of its economic life, determine which equipment the company should buy.

7. An individual is planning to purchase a car. A new car will cost Rs. 1,20,000. The resale value of the car at the end of the year is 85% of the previous year value. Maintenance and operation value of car ~~can be~~ Rs. 40,000.

Space for Hints

(i) When should the car be replaced to minimize average annual cost (ignore interest)?

(ii) If interest of 12% is assumed, when should the car be replaced?

8. A company has six jobs on hand coded 'A' to 'F'. All the jobs have to go through two machines 'M I' and 'M II'. The time required for each job on each machine, in hours, is given below:

	A	B	C	D	E	F
M I	3	12	18	9	15	6
M II	9	18	24	24	3	15

Draw a sequence table scheduling the six jobs on the two machines.

9. (a) A book binder has one printing press, one binding machine and the manuscripts of a number of different books. The times required to perform the printing and binding operation for each the total time required to process all the books. Find also the total time required. (Clearly state any algorithm you might use.)

Processing time (in minutes)

Book	1	2	3	4	5
Printing time	40	90	80	60	50
Binding time	50	60	20	30	40

(b) Suppose that an additional operation is added to the process described in (a), viz., finishing. The times required for operations are given below:

Finishing time (in minutes)

Book	1	2	3	4	5
Finishing time	80	100	60	70	100

What is the order in which the books should be processed? Find also the minimal total elapsed time.

10. A readymade garments manufacturer has to process 7 items through two stages of production, viz., cutting and sewing. The time taken for each of these at the different stages are given below in appropriate units:

	Item	1	2	3	4	5	6	7
Processing time	Cutting	5	7	3	4	6	7	12
	Sewing	2	6	7	5	9	5	8

(a) Find an order in which these items are to be processed through these stages so as to minimize the total processing time.

(b) Suppose a third stage of production is added, viz., pressing and packing, with processing time for these items as follows:

Item	1	2	3	4	5	6	7
Processing time	10	12	11	13	12	10	11

(Pressing and packing)

Find an order in which these seven items are to be processed so as to minimize the time taken to process all the items through all the three stages.

UNIT – 9

GAME THEORY

Introduction:

In many practical problems, it is required to take decision in a situation where there are two (or more) opposite parties with conflicting interests and the action of one depends upon the action which the opponent takes. The outcome of the situation is controlled by the decisions of all the parties involved. Such a situation is controlled by the decisions of all the parties involved. Such a situation is termed as a **competitive situation.** Such problem occurs frequently in economic military, social, political, advertising and marketing by competing business forms. The mathematical treatment is due to **Von Neumann.**

A competitive situation is called a **game** if it has the following properties:

1. There are finite number of competitors called **players**.
2. A list of finite or infinite number of possible courses of action is available to each player.
3. A play is played when each player chooses one of his courses of action. The courses are assumed to be made simultaneously so that, no player knows his opponent's choice until he has decided his course of action.
4. Every play i.e., combination of courses of action is associated with an outcome, known as the pay off (generally money) which determine a set of gains, one to each player. Here a loss is considered as a negative gain (for e.g., If there are two players A and B the pay off, generally referred in Matrix form indicate gains to A for each possible outcome of the game. Then negative entries in the table

denote the payments from A to B. Player A is called maximizing player and B is called minimizing player) other obvious assumptions are (a) All players act rationally (b) Each player attempts to maximize his gain or minimize his loss (c) complete relevant information is known to each player and (d) each player makes individual decisions without direct communications.

5. A game involving n players is called n – person game. In practice two – person games are of more importance.

Structure:

9.1 Two Person Zero Sum Games and Some Basic Terms:

Two – Person Zero – Sum Games

When there are two competitors playing a game, it is called a 'two – person game'. In case the number of competitors exceeds two, say n, then the games is termed as a 'n – person game'.

Games having the 'zero – sum' character that the algebraic sum of gains and losses of all the players is zero are called zero – sum games. The play does not add a single paisa to the total wealth of all the players; it merely results in a new distribution of initial money among them. Zero – sum games with two players are called two – person zero – sum games. In this case the loss (gain) of one player is exactly equal to the gain (loss) of the other. If the sum of gains or losses is not equal to zero, then the game is of non – zero sum character or simply a non – zero sum game.

Basic Terms

1. **Player:** The competitors in the game are known as players. A player may be individual or group of individuals, or an organization.
2. **Strategy:** A strategy for a player is defined as a set of rules or alternative courses of action available to him in advance, by which player decides the course of action that he should adopt. Strategy may be of two types:

(a) **Pure strategy:** If the players select the same strategy each time, then it is referred to as pure – strategy. In this case each player knows exactly what the other player is going to do, the objective of the players is to maximize gains or to minimize losses.

(b) **Mixed strategy:** When the players use a combination of strategies and each player always kept guessing as to which course of action is to be selected by the other player at a particular occasion then this is known as mixed strategy. Thus, there is a probabilistic situation and objective of the player is to maximize expected gains or to minimize losses.

3. **Optimum strategy**: A course of action or play which puts the player in the most preferred position, irrespective of the strategy of his competitors, is called an optimum strategy.

4. **Value of the game**: It is the expected payoff of play when all the players of the game follow their optimum strategies. The game is called fair if the value of the game is zero and unfair if it is non – zero.
5. **Pay off matrix**: When the players select their particular strategies, the payoffs (gains or losses) can be represented in the form of a matrix called the payoff matrix. Since the game is zero – sum, therefore gain of one player is equal to the loss of other and vice – versa. In other words, one player's payoff table would contain the same amounts in payoff table of other player with the sign changed. Thus, it is sufficient to construct payoff only for one of the players.

   Let player A have m strategies $A_1, A_2, \ldots\ldots, A_m$ and player B have n strategies $B_1, B_2, \ldots\ldots\ldots, B_n$. Here, it is assumed that player A is always the gainer and player B is always the loser. That is, all payoffs are assumed in terms of player A. Let a_{ij} be the payoff which player A gains from player B if player A chooses strategy A_i and player B chooses strategy B_j. Then the payoff matrix to play A is:

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \text{Player A} & \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array} & \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right] \end{array}$$

   The payoff matrix to player B is $(-a_{ij})$.

The Maxi Min – Mini Max Principle

We shall now explain the so – called Maxi Min – Mini Max Principle for the selection of the optimal strategies by the two players.

For player A, minimum value in each row represents the least gain (payoff) to him if he choose his particular strategy. These are written in the matrix by row minima. He will then select the strategy that maximizes his minimum gains. This choice of player A is called the Maxi Min principle, and the corresponding gain is called the Maxi Mini value of the game.

For player B, on the other hand, likes to minimize his losses. The maximum value in each column represents the maximum loss to him if he chooses his particular strategy. These are written in the matrix by column maxima. He will then select the strategy that minimizes his maximum losses. This choice of player B is called the Mini Max principle, and the corresponding loss is the Mini Max value of the game.

If the Maxi Min value equals the Mini Max value, then the game is said to have a saddle (equilibrium) point and the corresponding strategies are called optimum strategies. The amount of payoff at an equilibrium point is known as the value of the game.

Theorem: 1

Let (a_{ij}) be the $m \times n$ payoff matrix for a two –'person zero – sum game. If $\underline{v}$ denotes the Maxi Min value and $\bar{v}$ the Mini Max value of the game, then $\bar{v} \geq \underline{v}$. That is

$$\min_{1\le j\le n}\left[\max_{1\le i\le m} \{a_{ij}\}\right] \geq \max_{1\le i\le m}\left[\min_{1\le j\le n} \{a_{ij}\}\right].$$

Proof:

We have

$$\max_{1\le i\le m}\{a_{ij}\} \ge a_{ij} \qquad \text{for all } j = 1,2,\ldots\ldots,n$$

$$\min_{1\le j\le n}\{a_{ij}\} \le a_{ij} \qquad \text{for all } i = 1,2,\ldots\ldots.,m$$

and

Let the above maximum be attained at $i = i'$ and the minimum be attained at $j = j'$, i.e.,

$$\max_{1\le j\le n}.\{a_{ij}\} = a_{i'j} \text{ and } \min_{1\le j\le n}.\{a_{ij}\} = a_{ij'}$$

Then, we must have

$$a_{i'j} \ge a_{ij} \ge a_{ij'} \qquad \text{for all } i = 1,2,\ldots..,m;\ j = 1,2,\ldots\ldots,n$$

From this, we get

$$\min_{1\le j\le n}.\{a_{i'j}\} \ge a_{ij} \ge \max_{1\le i\le m}.\{a_{ij'}\} \text{ for all}$$

$i = 1,2,\ldots..,m; j = 1,2,\ldots..m$

$$\min_{1\le j\le n}\left[\max_{1\le j\le m}.\{a_{ij}\}\right] \ge \max_{1\le j\le m}\left[\min_{1\le j\le m}.\{a_{ij}\}\right]$$

or $\qquad \bar{v} \ge \underline{v}$.

9.2 Games with Saddle Point:

Saddle Point and Value of the Game

Definition: A **saddle point** of a payoff matrix is that position in the payoff matrix where maximum of row minima coincides with the minimum of the column maxima. The payoff at the saddle point is called the **value of the game denoted by v**. The saddle point need not be unique. We shall denote the Maxi Min value of the game by $\underline{v}$ and the Mini Max value of the game by $\bar{v}$. A game is said to be **fair** if $\underline{v} = 0 = \bar{v}$. A game is said to **strictly determinable** if $\underline{v} = v = \bar{v}$.

Mixed Strategy:

When Maxi Min $\neq$Mini Max, then pure strategy fails. Therefore each player with certain probabilistic fixation. This type of strategy is called **mixed strategy.**

Example: 9.2.1

Solve the game whose pay – off matrix is given by

$$\text{Player A}\;\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\overset{\text{Player B}}{\begin{bmatrix} B_1 & B_2 & B_3 \\ 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix}}$$

Solution:

$$\begin{array}{ccc} & \text{Player B} & \text{Row Minima} \\ \begin{array}{r} \\ A_1 \\ \text{Player A}\;A_2 \\ A_3 \end{array} & \begin{bmatrix} B_1 & B_2 & B_3 \\ 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} & \begin{array}{r} \\ 1 \\ -4 \\ -1 \end{array} \\ \text{Column maxima} & \begin{array}{ccc} 1 & 5 & 1 \end{array} & \end{array}$$

Mini Max = 1, Maxi Min = 1 $\Rightarrow$Saddle Point (1, 1)

$S_0 = (A_1, B_1)$ or (A_1, B_3)

Value of the game $V = 1$.

Example: 9.2.2

Solve the following game whose pay – off matrix is given below:

$$\begin{bmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 3 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 3 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{bmatrix} \qquad \begin{array}{c} \text{Row minima} \\ 0 \\ 4 \\ 2 \\ 1 \end{array}$$

Column maxima $= 9 \quad 6 \quad 4 \quad 8 \quad 8$

Max of Row minima $= 4$

Minimum of column maxima $= 4$

S_0 = Row 2 and Column 3 .

Value of the game V $= 4$

Example: 9.2.3

For what value of λ, the game with the following matrix is strictly determinable.

Player B

$$\text{Player A} \quad \left[\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ A_1 & \lambda & 6 & 2 \\ A_2 & -1 & \lambda & -7 \\ A_3 & -2 & 4 & \lambda \end{array}\right]$$

Ignoring the value of λ,

Solution:

Player A

Player B

$$\left[\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ A_1 & \lambda & 6 & 2 \\ A_2 & -1 & \lambda & -7 \\ A_3 & -2 & 4 & \lambda \end{array}\right] \qquad \begin{array}{c} \text{Row Minima} \\ \\ 2 \\ -7 \\ -2 \end{array}$$

Column maxima $\quad -1 \quad 6 \quad 2$

We know if $\underline{v} = v = \overline{v}$, then the game is strictly determinable. Here $\underline{v} = 2$ and $\overline{v} = -1 \Rightarrow -1 \leq \lambda \leq 2$.

Example: 9.2.4

For the game with payoff matrix

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix} \end{array}$$

Determine the best strategies for players A and B and also the value of the game. Is this game (i) fair (ii) strictly determinable?

Solution:

$$\begin{array}{ccc} & \text{Player B} & \text{Row min ima} \\ \text{player A} & \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix} & \begin{matrix} -2 \\ -6 \end{matrix} \\ \text{Column maxima} & \begin{matrix} 6 & 4 & -2 \end{matrix} & \end{array}$$

$$\therefore \text{Max Row Minima} = -2,$$

$$\text{Min Column Maxima} = -2$$

$\Rightarrow S_0 = (\text{I}, \text{III})$

$V = -2$, the game is not fair but strictly determinable.

Example: 9.2.5

Determine the range of value of p and q that will make the payoff element a_{22} a saddle point for the game whose payoff matrix (a_{ij}) is given below:

Player B

$$\text{Player A}\begin{bmatrix} 2 & 4 & 5 \\ 10 & 7 & q \\ 4 & p & 8 \end{bmatrix}$$

Solution:

Ignoring p, q we will determine the Maxi Min and Mini Max values of the pay of matrix.

For this, we have

Player A

Player B Row Minima

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{bmatrix} B_1 & B_2 & B_3 \\ 2 & 4 & 5 \\ 10 & 7 & q \\ 4 & p & 8 \end{bmatrix} \qquad \begin{array}{c} \\ 2 \\ 7 \\ 4 \end{array}$$

Column Maxima $\quad 10 \quad 7 \quad 8$

Therefore, Maxi min value $= 7(= \underline{v})$

Mini max value $= 7(= \overline{v})$

Thus there exists a saddle point at position (2, 2). This implies the condition on p as $p \leq 7$ and on q as $q \geq 7$

$\therefore$ Range required is $p \leq 7,\ q \geq 7$.

Check your progress: 9.1

1. Determine which of the following two – person zero – sum games are strictly determinable and fair. Give the optimum strategies for each player in the case of strictly determinable games:

(a)

Player B

$$\text{Player A}\quad \begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} -5 & 2 \\ -7 & -4 \end{bmatrix} \end{array}$$

(b)

Player B

$$\text{Player A}\quad \begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 10 & 6 \\ 8 & 2 \end{bmatrix} \end{array}$$

(a) Show that G is strictly determinable whatever μ may be.

(b) Determine the value of G.

2. Consider the game G with the following payoff matrix:

Player B

$$\text{Player A}\quad \begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & 6 \\ -2 & \mu \end{bmatrix} \end{array}$$

3. For the game with payoff matrix:

Player A

$$\text{Player B}\quad \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix}$$

Determine the best strategies for player A and B and also the values of the game for them. Is this game (i) fair? (ii) strictly determinable?

4. For what value of λ, the game with following payoff matrix is strictly determinable?

Player B

$$\text{Player A}\quad \begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & \lambda & 6 & 2 \\ A_2 & -1 & \lambda & -7 \\ A_3 & -2 & 4 & \lambda \end{array}$$

5. Solve the game whose payoff matrix is given by

(a)
$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & -3 & -2 & 6 \\ A_2 & 2 & 0 & 2 \\ A_3 & 5 & -2 & -4 \end{array}$$

(b)
$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & -2 & 15 & -2 \\ A_2 & -5 & -6 & -4 \\ A_3 & -5 & 20 & -8 \end{array}$$

6. Solve the following 2 – person zero – sum game:

Player B

$$\text{Player A}\begin{bmatrix} 10 & 5 & -2 \\ 6 & 7 & 3 \\ 4 & 8 & 4 \end{bmatrix}$$

9.3 Games without saddle points – Mixed strategies:

As determining the minimum of column maxima and the maximum of row minima are two different operations, there is no reason to expect that they should always lead to unique payoff position – the saddle point.

In all such cases to solve games, both the players must determine an optimal mixture of strategies to find a saddle (equilibrium) point. The optimal strategy mixture for each player may be determined by assigning to each strategy its probability of being chosen. The strategies so determined are called mixed strategies because they are probabilistic combination of available choices of strategy.

The value of game obtained by the use of mixed strategies represents which least player A can expect to win and the least which player B can lose. The expected payoff of a player in a game with arbitrary payoff matrix (a_{ij}) of order mxn is defined as:

$$E(p,q) = \sum_{i=1}^{m} \sum_{j=1}^{n} P_i a_{ij} q_j = p^T Aq$$

where p and q denote the mixed strategies for player A and B respectively.

Maxi min – Mini max Criterion:

Consider an $m \ x \ n$ game (a_{ij}) without any saddle point. i.e., strategies are mixed. Let $p_1, p_2, \ldots\ldots, p_m$ be the probabilities with which player A will play his moves $A_1, A_2, \ldots\ldots, A_m$ respectively; and let $q_1, q_2, \ldots\ldots, q_n$ be the probabilities with which player B will play his moves $B_1, B_2, \ldots\ldots, B_n$ respectively. Obviously, $p_i \geq 0$ $(i = 1,2,\ldots..,m)$, $q_j \geq 0 (j = 1,2,\ldots..,n)$, and

$p_1 + p_2 + \ldots + p_m = 1$; $q_1 + q_2 + \ldots\ldots\ldots + q_n = 1$.

The expected payoff function for player A, therefore, will be given by

$$E(p,q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j$$

Making use of maxi min – mini max criterion, we have

For Player A.

$$\underline{v} = \max_{p} . \min_{q} . E(p,q) = \max_{p} . \left[\min_{j} \left\{ \sum_{i=1}^{m} p_i a_{ij} \right\} \right]$$

$$= \max_{p} . \left[\min_{j} . \left\{ \sum_{i=1}^{m} p_i a_{i1}, \sum_{i=1}^{m} p_i a_{i2}, \ldots\ldots, \sum_{i=1}^{m} p_i a_{in} \right\} \right]$$

Here $\min_{j} . \left\{ \sum_{i=1}^{m} p_i a_{ij} \right\}$ denotes the expected gain to player A when player B uses his jth pure strategy.

For player B.

$$\overline{v} = \min_{q} . \left[\max_{i} . \left\{ \sum_{j=1}^{n} q_j a_{1j}, \sum_{j=1}^{n} a_j a_{2j}, \ldots\ldots\ldots, \sum_{j=1}^{n} q_j a_{mj} \right\} \right] .$$

Here $\max . \left\{ \sum_{j=1}^{n} q_j a_{ij} \right\}$ denotes the expected loss to player B when player A uses his ith strategy.

The relationship $\underline{v} \leq \overline{v}$ holds good in general and when p_i and q_j correspond to the optimal strategies the relation holds in 'equality' sense and the expected value for both the players becomes equal to the optimum expected value of the game.

Definition:

A pair of strategies (p,q) for which $\underline{v} = \overline{v} = v$ is called a **saddle point** of $E(p,q)$.

Theorem: 2

For any 2 x 2 two – person zero – sum game without any saddle point having the payoff matrix for player A.

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{array}$$

The optimum mixed strategies

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \text{ and } S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}.$$

are determined by

$$\frac{p_1}{p_2} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}}, \frac{q_1}{q_2} = \frac{a_{22} - a_{12}}{a_{11} - a_{21}}$$

Where $p_1 + p_2 = 1$ and $q_1 + q_2 = 1$. The value v of the game to A is given by

$$v = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}.$$

Proof:

Let a mixed strategy for player A be given by $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ where $p_1 + p_2 = 1$. Thus, if player B moves B_1 the net expected gain of A will be

$$E_1(p) = a_{11}p_1 + a_{21}p_2$$

And if B moves B_2, the net expected gain of A will be

$$E_2(p) = a_{12}p_1 + a_{22}p_2.$$

Similarly, if B plays his mixed strategy $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ where $q_1 + q_2 = 1$, then B's net expected loss will be

$$E_1(q) = a_{11}q_1 + a_{12}q_2$$

If A plays A_1, and

$$E_2(q) = a_{21}q_1 + a_{22}q_2$$

If A plays A_2.

The expected gain of player A, when B mixes his moves with probabilities q_1 and q_2 is therefore given by

$$E(p,q) = q_1[a_{11}p_1 + a_{21}p_2] + q_2[a_{12}p_1 + a_{22}p_2].$$

Player A would always try to mix his moves with such probabilities so as to maximize his expected gain.

Now, $E(p,q) = q_1[a_{11}p_1 + a_{21}(1-p_1)] + (1-q_1)[a_{12}p_1 + a_{22}(1-p_1)]$

$$= [a_{11} + a_{22} - (a_{12} + a_{21})]p_1q_1 + (a_{12} - a_{22})p_1 + (a_{21} - a_{22})q_1 + a_{22}$$

$$= \lambda\left(p_1 - \frac{a_{22} - a_{21}}{\lambda}\right)\left(a_1 - \frac{a_{22} - a_{12}}{\lambda}\right) + \frac{a_{11}a_{22} - a_{12}a_{21}}{\lambda},$$

where $\lambda = a_{11} + a_{22} - (a_{12} + a_{21})$.

We see that if A chooses $p_1 = \frac{a_{22} - a_{21}}{\lambda}$, he ensures an expected gain of at least $(a_{11}a_{22} - a_{12}a_{21})/\lambda$. Similarly if B chooses $q_1 = \frac{a_{22} - a_{12}}{\lambda}$, then B will limit his expected loss to at most $(a_{11}a_{22} - a_{12}a_{21})/\lambda$. These choices of p_1 and q_1 will thus be optimal to the two players.

Thus we get

$$p_1 = \frac{a_{22} - a_{21}}{\lambda} = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} \text{ and}$$

$$p_2 = 1 - p_1 = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})};$$

$$q_1 = \frac{a_{22} - a_{12}}{\lambda} = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})} \text{ and}$$

$$q_2 = 1 - q_1 = \frac{a_{11} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})};$$

$$\text{and } v = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}.$$

Hence, we have

$$\frac{p_1}{p_2} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}}, \ \frac{q_1}{q_2} = \frac{a_{22} - a_{12}}{a_{11} - a_{21}}; \text{ and } v = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

Note:

The above formulae for p_1, p_2, q_1, q_2 and v are valid only for $2x2$ games without saddle points.

Example: 9.3.1

For the game with the following payoff matrix, determine the optimum strategies and the value of the game:

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

Solution:

Clearly, the given matrix is without a saddle point. So the mixed strategies of P_1 and P_2 are:

$$S_p = \begin{bmatrix} 1 & 2 \\ p_1 & p_2 \end{bmatrix}, S_{p_2} = \begin{bmatrix} 1 & 2 \\ q_1 & q_2 \end{bmatrix}; \ p_1 + p_2 = 1 \text{ and } q_1 + q_2 = 1$$

If $E(p,q)$ denotes the expected payoff function, then

$$E(p,q) = 5p_1q_1 + 3(1-p_1)q_1 + p_1(1-q_1) + 4(1-p_1)(1-q_1)$$

$$= 5p_1q_1 - 3p_1 - q_1 + 4 = 5(p_1 - 1/5)(q_1 - 3/) + 17/5.$$

If P_1 chooses $p_1 = 1/5$, he ensures that his expectation is at least 17/5. He cannot be sure of more than 17/5, because by choosing $q_1 = 3/5$, p_2 can keep $E(p_1, q_1)$ down to 17/5. So P_1 might as well settle for 17/5and P_2 reconcile to 17/5. Hence the optimum strategies for P_1 and P_2 are

$$S_{p_1} = \begin{bmatrix} 1 & 2 \\ 1/5 & 4/5 \end{bmatrix}, S_{p_2}\begin{bmatrix} 1 & 2 \\ 3/5 & 2/5 \end{bmatrix}$$

and the value of the game is $v = 17/5$.

Example: 9.3.2

Consider a "modified" form of "matching biased coins" game problem. The matching player is paid Rs. 8.00 if the two coins turn both heads and Rs. 1.00 if the coins turn both tails. The non – matching player is paid Rs. 3.00 when the two coins do not match. Given the choice of being the matching or non – matching player, which one would you choose and what would be your strategy?

Solution:

The payoff matrix for the matching player is given by

Non – matching Player

Space for Hints

$$\text{Matching Player}\;\begin{array}{c} \\ H \\ T \end{array}\!\!\begin{array}{cc} H & T \\ \left[\begin{array}{cc} 8 & -3 \\ -3 & 1 \end{array}\right] & \end{array}$$

Clearly, the payoff matrix does not possess any saddle point. The players will use mixed strategies. The optimum mixed strategy for matching player is determined by

$$p_1 = \frac{1-(-3)}{8+1-(-3-3)} = \frac{4}{15},\; p_2 = \frac{11}{15}$$

And for the non – matching player by

$$q_1 = \frac{1-(-3)}{8+1-(-3-3)} = \frac{4}{15},\; q_2 = \frac{11}{15}.$$

The expected value of the game (corresponding to the above strategies) is given by

$$v = \frac{8-3(-3)(-3)}{8+1-1(-3-3)} = -\frac{1}{15}.$$

Thus the optimum mixed strategies for matching player and non – matching player are given by

$$S_{\text{match}} = \begin{bmatrix} H & T \\ 4/15 & 11/15 \end{bmatrix} \text{ and } S_{\text{non-match}} = \begin{bmatrix} H & T \\ 4/15 & 11/15 \end{bmatrix}$$

Clearly, we would like to be the non – matching player.

Example: 9.3.3

In a game of matching coins with two players suppose A wins one unit value when there are two heads, wins nothing when there are two tails, and looses $\frac{1}{2}$ unit value when there are one head and one tail. Determine the payoff matrix, the best strategy for each player, and the value of the game.

Solution:

Formulation

$$\begin{array}{cc|cc|} & & \multicolumn{2}{c}{B} \\ & & H & T \\ A & H & 1 & -\frac{1}{2} \\ & T & -\frac{1}{2} & 0 \end{array}$$

$$\text{Let this be } \begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{c} \begin{matrix} B_1 & B_2 \end{matrix} \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

The optimum mixed strategies

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \text{and} \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

$$\text{Now } \lambda = a_{11} + a_{22} - (a_{12} + a_{21})$$

$$= 1 + 0 - \left(-\frac{1}{2} - \frac{1}{2}\right)$$

$$= 2$$

$$p_1 = \frac{a_{22} - a_{21}}{\lambda}$$

$$= \frac{0 - \left(-\frac{1}{2}\right)}{2}$$

$$= \frac{1}{4}$$

$$\Rightarrow p_2 = 1 - p_1. = \frac{3}{4}$$

$$q_1 = \frac{a_{22} - a_{12}}{\lambda}$$

$$= \frac{0 - \left(-\frac{1}{2}\right)}{2}$$

$$= \frac{1}{4}$$

$$\Rightarrow q_2 = 1 - q_1$$

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

Value of the game V $= \dfrac{a_{11}a_{22} - a_{12}a_{21}}{\lambda}$

$$= \frac{(1)(0) - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}$$

$$= -\frac{1}{8}$$

Example: 9.3.4

Solve the following game without Saddle point.

$$\begin{array}{c} \\ A \end{array}\begin{array}{c} B \\ \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$$

Solution:

Let the given payoff – matrix be

$$\begin{array}{ccc} & B_1 & B_2 \\ A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{array}$$

The optimum mixed strategies

where Strategy for $A : S_A : \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$

and Strategy for $B : S_B : \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$

Now $\lambda = a_{11} + a_{22} - (a_{12} + a_{21}) = 2 + 1 - (5 + 4)$

$= -6$

$$p_1 = \frac{a_{22} - a_{21}}{\lambda} = \frac{1-4}{-6} = \frac{1}{2}.$$

$\Rightarrow \quad p_2 = \frac{1}{2}$

$$q_1 = \frac{a_{22} - a_{12}}{\lambda} = \frac{1-5}{-6} = \frac{2}{3} = \frac{1}{3} = 1 - \frac{1}{3} = \frac{2}{3}$$

$$S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Value of the game

$$= \frac{a_{11}a_{22} - a_{12}a_{21}}{\lambda} = \frac{2-20}{-6} = 3$$

Check your progress: 9.2

1. Solve the following game and determine the value of the game:

(a)
$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 6 & -3 \\ -3 & 0 \end{bmatrix} \end{array}$$

(b)
$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \end{array}$$

2. In a game of matching coins with two players, suppose A wins one unit of value, when there are two heads, wins nothing when there are two tails and loses $\frac{1}{2}$ unit of value when there are one head and one tail. Determine the payoff matrix, the best strategies for each player and the value of the game to A.

3. Two players A and B match coins. If the coins match, then A wins two units of value, if the coins do not match, then B wins 2 units of value. Determine the optimum strategies for the players and the value of the game.

4. A and B each take out one or two matches and guess how many matches opponent has taken. If one of the players guesses correctly then the loser has to pay him as many rupees as the sum of the number held by both players. Otherwise, the payout is zero. Write down the payoff matrix and obtain the optimal strategies of both players.

9.4 Graphical solution of $2xn$ and $mx2$ Games:

The procedure described in the last section will generally be applicable for any game with 2×2 payoff matrix unless it possesses a saddle point. moreover, the procedure can be extended to any square payoff matrix of any order. But it will not work for the game whose payoff matrix happens to be a rectangular one, say mxn. In such cases a very simple graphical method is available if either m or n is two. The graphic short – cut enables us to reduce the original $2xn$ or $mx2$ game to a much simpler $2x2$ game. Consider the following $2xn$ game:

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{array}{c|cccc} & B_1 & B_2 & \cdots & B_n \\ A_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ A_2 & a_{21} & a_{22} & \cdots & a_{2n} \end{array} \end{array}$$

It is assumed that the game does not have a saddle point. Let the optimum mixed strategy for A be given by $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ where $p_1 + p_2 = 1$. The average (expected) payoff for A when he plays S_A against B's pure moves $B_1, B_2, \ldots\ldots, B_n$ is given by

B's pure move A's expected payoff E(p)

B_1 $E_1(p_1) = a_{11}p_1 + a_{21}p_2 = a_{11}p_1 + a_{21}(1 - p_1)$

B_2

$E_2(p_1) = a_{12}p_1 + a_{22}p_1 + a_{22}p_2 = a_{12}p_1 + a_{22}(1 - p_1)$

B_n $E_n(p_1) = a_{1n}p_1 + a_{2n}p_2 = a_{1n}p_1 + a_{2n}(1 - p)$.

According to the Maxi Min criterion for mixed strategy games, player A should select the values of p_1 and p_2 so as to maximize his minimum expected payoffs. This may be done by plotting the expected payoff lines:

$$E_j(p_1) = (a_{1j} - a_{2j})p_1 + a_{2j}(j = 1,2,.......,n).$$

The highest point on the lower envelope of these lines will give maximum of the minimum (i.e., Maxi Min) expected payoffs to player A as also the maximum value of p_1.

The two lines passing through the Maxi Min point identify the two critical moves of B which, combined with two of A, yield the 2×2 matrix that can be used to determine the optimum strategies of the two players, for the original game, using the results of the previous section.

The $(m\text{x}2)$ games are also treated in the same way where the upper envelope of the straight liens corresponding to B's expected payoffs will give the maximum expected payoff to player B and the lowest point on this then gives the minimum expected payoff (Mini Max value) and the optimum value of q_1.

Example: 9.4.1

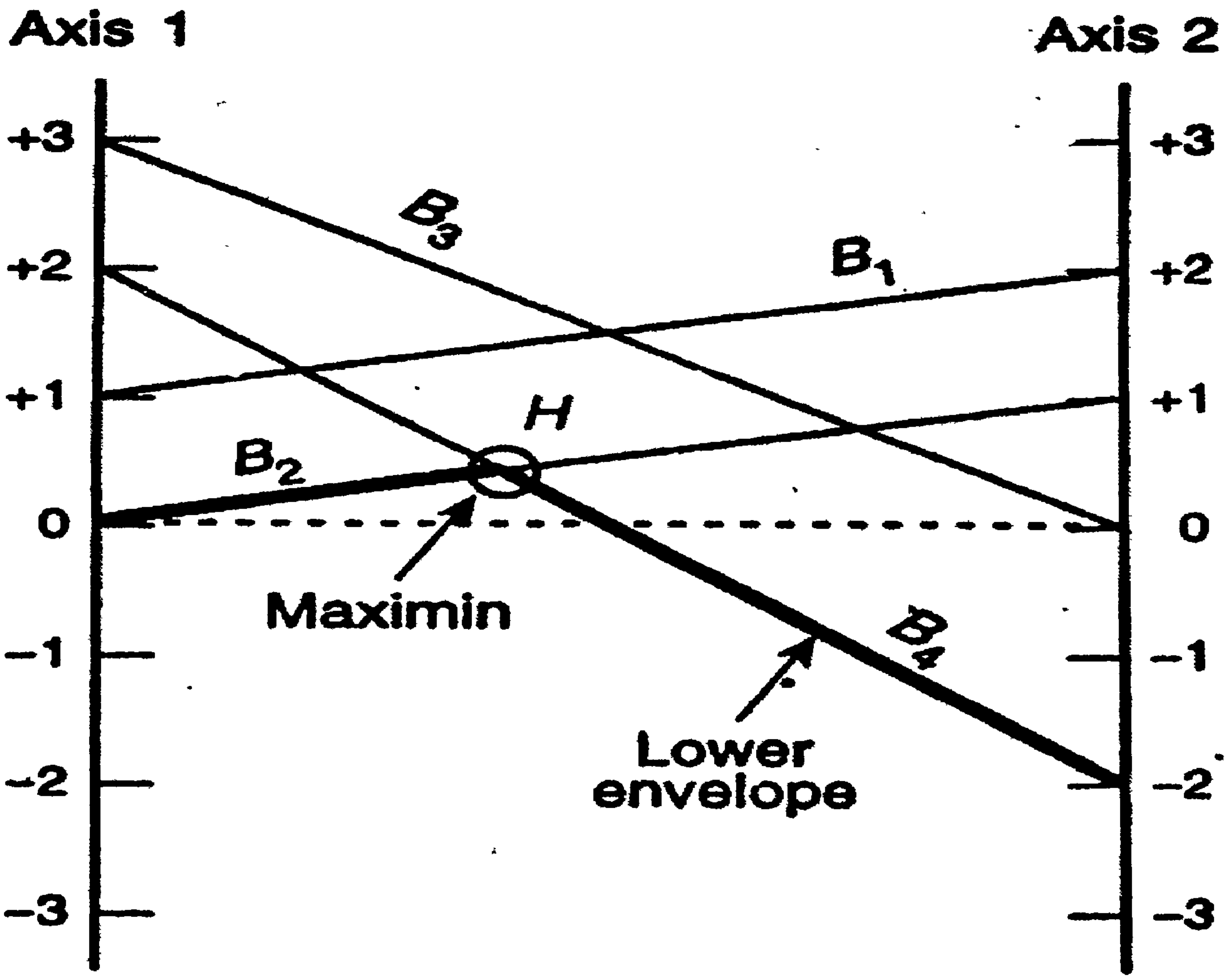
Solve the following 2x2 game graphically:

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \text{Player A} & \begin{array}{c} A_1 \\ A_2 \end{array}\begin{bmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & 3 & 2 \end{bmatrix} \end{array}$$

Solution:

Clearly, the problem does not possess a saddle point. Let the player A play the mixed strategy $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ where $p_2 = 1 - p_1$, against B. Then A's expected payoffs against B's pure moves are given by

B's pure move	A's expected payoff $E(p_1)$
B_1	$E_1(p_1) = p_1 + 1$
B_2	$E_2(p_1) = p_1$
B_3	$E_3(p_1) = -3p_1 + 3$
B_4	$E_4(p_1) = -4p_1 + 2$



The maxi min value

These expected payoff equations are then plotted as functions of p_1 as shown in Fig. which shows the payoffs of each column represented as points on two vertical axis 1 and 2, unit distance apart. Thus line B_1 joins the first payoff element 2 in the first column represented by +2 on axis 2, and the second payoff element 1 in the first column represented by +1 on axis 1. Similarly, lines B_2, B_3 and B_4 joins the corresponding representation of payoff elements in the second, third and fourth columns. Since the player A wishes to maximize his minimum expected payoff we consider the highest point of intersection H on the lower envelope of the A's expected payoff equations. This point H represents the Maxi Min expected value of the game for A. the lines B_2 and B_4, passing through H, define the two relevant moves B_2 and B_4 that alone B needs to play. The solution to the original 2 x 4 game, therefore, boils down that the simpler game with the 2 x 2 payoff matrix:

$$\begin{array}{cc} & \begin{array}{cc} B_2 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix} \end{array}$$

Now if $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ and $S_B = \begin{bmatrix} B_2 & B_4 \\ q_2 & q_4 \end{bmatrix}$.

be the optimum strategies for A and B, then we have

$$p_1 = \frac{2-0}{1+2(-2)} = 2/5, \; p_2 = 1 - p_1 = 3/5.$$

$$q_2 = \frac{2-(-2)}{1+2-(-2)} = 4/5, \; q_4 = 1 - q_1 = 1/5.$$

Hence the solution to the game is

(i) The optimum strategy for A is $S_A = \begin{bmatrix} A_1 & A_2 \\ 2/5 & 3/5 \end{bmatrix}$,

(ii) The optimum strategy for B is $S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 4/5 & 0 & 1/5 \end{bmatrix}$

(iii) The expected value of the game is $v = \dfrac{2x1 - 0x(-2)}{1+2-(0-2)} = \dfrac{2}{5}$.

Example: 9.4.2

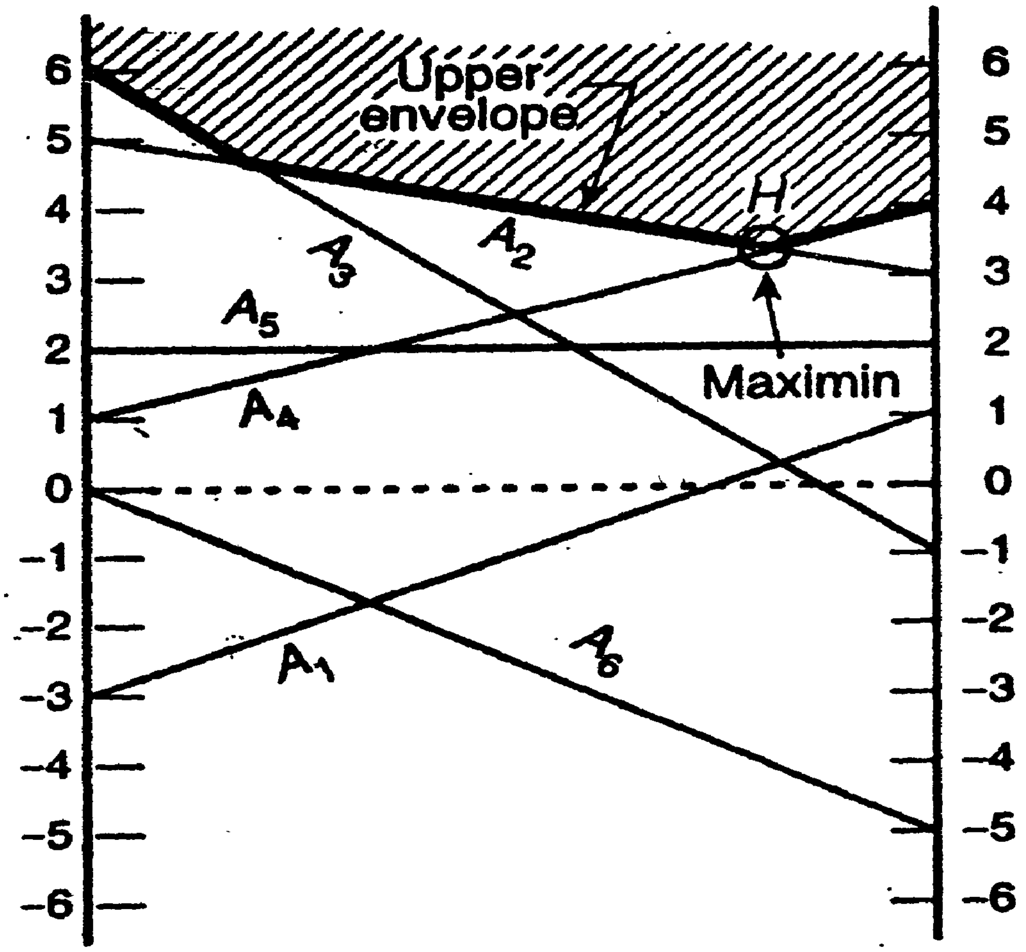
Obtain the optimal strategies for both – persons and the value of the game for zero – sum two – person game whose payoff matrix is as follows:

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix}$$

Solution:

Clearly, the given problem does not posses any saddle point. So, let the player B play the mixed strategy $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ with $q_2 = 1 - q_1$ against player A. Then B's expected payoffs against A's pure moves are given by

A's pure move	B's expected payoff $E(q_1)$
A_1	$E_1(q_1) = 4q_1 - 3$
A_2	$E_2(q_1) = -2q_1 + 5$
A_3	$E_3(q_1) = -7q_1 + 6$
A_4	$E_4(q_1) = 3q_1 + 1$
A_5	$E_5(q_1) = 2$
A_6	$E_6(q_1) = -5q_1$



The Mini max value

The expected payoff equations are then plotted as functions of q_1 as shown in Fig.

Since the player B wishes to minimize his maximum expected payoff, we consider the lowest point of intersection H on the upper envelope of B's expected payoff equations. This point H represents the Mini Max expected value of the game for player B. The lines A_2 and

A_4 passing through H, define the two relevant moves A_2 and A_4 that alone the player A needs to play. The solution to the original 6x2 game therefore reduces to that of the simpler game with 2 x 2 payoff matrix:

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$$

If we now let

$$S_A = \begin{bmatrix} A_2 & A_4 \\ p_1 & p_2 \end{bmatrix}, \ p_1 + p_2 = 1; \ S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}, \ q_1 + q_2 = 1$$

Then using the usual method of solution for 2 x 2 games, the optimum strategies can easily be obtained as

$$S_B = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ 0 & 3/5 & 0 & 2/5 & 0 & 0 \end{bmatrix}, S_B = \begin{bmatrix} B_1 & B_2 \\ 4/5 & 1/5 \end{bmatrix}$$

And the value of the game as $v = 17/5$.

Check your progress: 9.3

1. $$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 1 & 3 & -3 & 7 \\ 2 & 5 & 4 & -6 \end{bmatrix} \end{array}$$

2. $$\begin{array}{ccc} & & \text{B's strategy} \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ & \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 3 & -4 \\ 2 & 5 \\ -2 & 8 \end{bmatrix} \end{array}$$

A's strategy

9.5 Dominance Property:

Sometimes, it is observed that one of the pure strategies of either player is inferior to atleast one of the remaining ones. The superior strategies are said to dominate the inferior ones. In such cases of dominance, we can reduce the size of the payoff matrix by deleting those strategies which are dominated by others.

General rule:

i) If all the elements of a row, say k^{th}, are less than or equal to the corresponding elements of any other row, say r^{th}, then k^{th} row is dominated by r^{th} row.

ii) If all the elements of a column, say k^{th}, are greater than or equal to the corresponding elements of any other column, say r^{th}, then k^{th} column is dominated by r^{th} column.

iii) Omit dominated rows or columns.

iv) If some linear combination of some rows dominates i^{th} row, then i^{th} row will be deleted. Similar arguments follow for columns.

Example: 9.5.1

Two firms are competing for business under the condition so that one firm's gain is another firm's loss. Firm A's payoff matrix is given below:

$$\begin{array}{llccc}
 & & & \text{Firm B} & \\
 & & \text{No ad} & \text{Medium ad} & \text{Heavy ad} \\
 & \text{No advertising} & 10 & 5 & -2 \\
\text{Frim A} & \text{Medium advertising} & 13 & 12 & 15 \\
 & \text{Heavy avertising} & 16 & 14 & 10
\end{array}$$

Suggest optimum strategies for the two firms and the net outcome thereof.

Solution:

Clearly, the first column is dominated by the second column as the elements of the first column are greater than elements of second column. Thus eliminating first column, we get

$$\begin{array}{ccc} & & \text{Firm B} \\ & & \begin{array}{cc}\text{Medium ad} & \text{Heavy ad}\\ B_2 & B_3\end{array} \\ \text{Frim A} & \begin{array}{r}\text{No advertising}\\ \text{Medium advertising}\\ \text{Heavy avertising}\end{array} & \begin{bmatrix}5 & -2\\ 12 & 15\\ 14 & 10\end{bmatrix}\end{array}$$

Again, first row is dominated by second and third row as all the elements of first row are less than the respective elements of second, and third row. Hence eliminating first row, we obtain the following 2 x 2 payoff matrix.

$$\begin{array}{ccc} & & \text{Firm B} \\ & & \begin{array}{cc}\text{Medium ad} & \text{Heavy ad}\\ B_2 & B_3\end{array} \\ \text{Frim A} & \begin{array}{r}\text{Medium advertising} \quad A_2\\ \text{Heavy advertising} \quad A_3\end{array} & \begin{bmatrix}12 & 15\\ 14 & 10\end{bmatrix}\end{array}$$

Since the reduced payoff matrixes do not have any saddle point, the strategies are mixed. So, let

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{bmatrix}, S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 5/7 & 2/7 \end{bmatrix} \text{ and } v = 90/7 .$$

Hence, firm A should adopt strategy A_2 and A_3 with 57% of time and 43% of time respectively (or with 57% and 43% probability on any one play of the game respectively). Similarly, firm B should adopt strategy B_2 and B_3 with 71% of time and 29% of time respectively (or with 71% and 29% probability on any one play of the game respectively).

Example: 9.5.2

Solve the following game:

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{array}{c|cccc} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \hline \text{I} & 3 & 2 & 4 & 0 \\ \text{II} & 3 & 4 & 2 & 4 \\ \text{III} & 4 & 2 & 4 & 0 \\ \text{IV} & 0 & 4 & 0 & 8 \end{array} \end{array}$$

Solution:

From the above payoff matrix, we observe that first row is dominated by third row and first column is dominated by third column. The reduced payoff matrix is

$$\begin{array}{c|ccc} & \text{II} & \text{III} & \text{IV} \\ \hline \text{II} & 4 & 2 & 4 \\ \text{III} & 2 & 4 & 0 \\ \text{IV} & 4 & 0 & 8 \end{array}$$

Now, none of the pure strategies of player B is inferior to any of his other strategies. However, a convex linear combination (average) of strategies III and IV dominates strategy II of player B, yielding the reduced payoff matrix.

$$\begin{array}{c|cc} & \text{III} & \text{IV} \\ \hline \text{II} & 2 & 4 \\ \text{III} & 4 & 0 \\ \text{IV} & 0 & 8 \end{array}$$

Again, we observe that none of the pure strategies of player A is inferior to any of his other strategies. However, a convex linear combination of strategies III and IV dominates strategy II of player A, yielding the reduced payoff matrix.

$$\begin{array}{ll} & \text{Player B} \\ & \begin{array}{cc} \text{III} & \text{IV} \end{array} \\ \text{Player A} \begin{array}{c} \text{III} \\ \text{IV} \end{array} & \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix} \end{array}$$

Now, letting

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & p_1 & p_2 \end{bmatrix}, S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & q_1 & q_2 \end{bmatrix}.$$

where $p_1 + p_2 = 1, q_1 + q_2 = 1$, and then using the method of solving 2 x 2 games, we can easily obtain the optimum strategies as

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & 2/3 & 1/3 \end{bmatrix}, S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & 2/3 & 1/3 \end{bmatrix}$$

and the value of the game $v = 8/3$

Example: 9.5.3

Using Dominance property solve

$$\begin{array}{cc} & \text{B} \\ \text{A} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \text{I} & \text{II} & \text{III} & \text{IV} \\ -5 & 3 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ -4 & -2 & 0 & -5 \end{bmatrix} \end{array}$$

Solution:

Row III is dominated by Row II and column II is dominated by column I

$\Rightarrow$ We omit Row III, Column II

$$\begin{array}{c|ccc} & & B & \\ & I & III & IV \\ A\ 1 & -5 & 1 & 20 \\ 2 & 5 & 4 & 6 \end{array}$$

Column IV is dominated by Column I

$\therefore$ We omit column IV

$$\begin{array}{c|cc|c} & B & & \text{Row minima} \\ & I & III & \\ A\ 1 & -5 & 1 & -5 \\ 2 & 5 & 4 & 4 \\ \text{Column maxima} & 5 & 4 & \end{array}$$

Minimum of column maxima = 4

Max of row minima = 4

Saddle point = (2, III)

Value of the game = 4

Example: 9.5.4

A and B play a game in which each has three coins, 5p, a 10p and a 20p. Each selects a coin without the knowledge of other's choice. If the sum of the coins is an odd amount, A wins B's coin; If the sum is even B wins A's coin. Find the best strategy for each player and the value of the game.

Solution:

Formulation of the Game

$$\begin{array}{cc|ccc} & & & B & \\ & & 5p & 10p & 20p \\ \hline & 5p & -5 & 10 & 20 \\ A & 10p & 5 & -10 & -10 \\ & 20p & 5 & -20 & -20 \end{array}$$

Column III is dominated by Column II,

Row III is dominated by Row II

We omit column III, Row III

$$\begin{array}{cc|cc} & & B & \\ & & 5p & 10p \\ \hline A & 5p & -5 & 10 \\ & 10p & 5 & -10 \end{array}$$

Let $S_B : \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$, $S_A : \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$

Where matrix is $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\lambda = a_{11} + a_{22} - (a_{12} + a_{21})$$

$$= -5 - 10 - (5 + 10)$$

$$= -30.$$

$$p_1 = \frac{a_{22} - a_{21}}{\lambda}$$

$$= \frac{-10-5}{-30} = \frac{1}{2}$$

$$\Rightarrow p_2 = 1 - \frac{1}{2} = \frac{1}{2}$$

Space for Hints

$$S_B : \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$q_1 = \frac{a_{22} - a_{12}}{\lambda}$$

$$= \frac{-10 - 10}{-30} = \frac{2}{3}$$

$$= \frac{2}{3}$$

$$\Rightarrow q_2 = \frac{1}{3}$$

$$S_A : \left(\frac{2}{3}, \frac{1}{3}, 0\right)$$

$$\text{Value of the game} = \frac{a_{22}a_{11} - a_{12}a_{21}}{\lambda}$$

$$= \frac{50 - 50}{\lambda}$$

$$v \quad = 0$$

Check your progress: 9.4

1. Two leading firms A and B are planning to make fund allocation for advertising their product. The matrix given below show the percentage of matrix shares of firm A and B for their various advertising policies:

Firm A	Firm B		
	No advertising	Medium advertising	Heavy advertising
No advertising	60	50	40
Medium advertising	70	70	50
Heavy advertising	80	60	75

Find the optimum strategies for the two firms and the expected outcome when both the firms follow their optimum strategies.

2. Even though there are several manufacturers of scooters, two firms with branch names Janta and Praja, control their market in Western India. If both manufacturers make model changes of the same type of their market segment in the same year, their respective market shares remain constant. Likewise, if neither makes model changes, then also their market shares remain constant. The payoff matrix in terms of increased/decreased percentage market share under different possible conditions is given below:

Janta	Praja		
	No change	Minor change	Major change
No change	0	-4	-10
Minor change	3	0	5
Major change	8	1	0

(i) Find the value of the game.

(ii) What change should Janata consider if this information is available only to itself?

9.6 Linear Programming Method:

A Two person zero – sum game can also be solved by linear programming approach. The major advantage of using linear programming technique is that it solves mixed strategy of any size.

To illustrate the connection between a game problem and linear programming, let us consider $(m x n)$ payoff matrix (a_{ij}) for player A.

$$S_m = \begin{bmatrix} A_1 & \cdots\cdots & A_m \\ p_1 & \cdots\cdots & p_m \end{bmatrix}$$

$$S_n = \begin{bmatrix} A_1 & \cdots\cdots & A_n \\ a_1 & \cdots\cdots & a_n \end{bmatrix}$$

where $\sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j = 1$

the mixed strategies for 2 players respectively. Then the expected gains $(j = 1,\dots\dots,n)$ of player A against B's pure strategies will be

$$g_1 = a_{11}p_1 + a_{21}p_2 + \dots\dots + a_{m1}p_m$$

$$g_2 = a_{12}p_1 + a_{22}p_2 + \dots\dots + a_{m2}p_m$$

$$g_n = a_{1n}p_1 + a_{2n}p_2 + \dots\dots + a_{mn}p_m$$

and the expected losses $\ell_i (i = 1,2,\dots\dots,m)$ of player B against A's pure strategies will be

$$\ell_1 = a_{11}q_1 + a_{12}q_2 + \dots\dots + a_{1n}q_n$$

$$\ell_2 = a_{12}q_1 + a_{22}q_2 + \ldots\ldots.. + a_{2n}q_n$$

....

$$\ell_m = a_{m1}q_1 + am_2q_2 + \ldots\ldots. + a_{mn}q_n$$

The objective of player A is to select $p_i(i = 1,2,\ldots\ldots.,m)$ such that he can maximize his minimum expected gains and the player B desires to select $q_j(j = 1,2,\ldots\ldots..,n)$ that will minimize his expected losses.

Thus if we let $U = \min_{j} \sum_{i=1}^{m} a_{ij}p_j \qquad (j = 1,2,\ldots..,n)$

and $V = \max_{i=1} \sum_{i=1}^{n} a_{ij}q_j \qquad (i = 1,2,\ldots\ldots.,m)$

the problem of two players could be written as

Player A Maximize $U =$ minimize $\frac{1}{U} = \sum_{i=1}^{m} \frac{p_i}{U}$

Subject to constraints $\sum_{i=1}^{m} a_{ij}p_iu \geq V$ and $\sum p_i = 1$,

$$p_i \geq 0(i = 1,2,\ldots..,m).$$

Player B Minimize $V =$ maximize $\frac{1}{V} = \sum_{j=1}^{n} . \frac{q_i}{V}$

Subject to constraints $\sum_{i=1}^{m} a_{ij}a_j \leq V$ and $\sum q_J = 1$,.

$$q_J \geq 0 \qquad (J = 1,2,\ldots\ldots,n)$$

Assuming $u > 0, v > 0$, introduce a new variable defined by

$$p_1' = \frac{p_i}{U} \text{ and } q_J' = \frac{q_J}{V} \qquad (i = 1,2,.....,m,\ j = 1,2,.....,n)$$

Then this pair of Linear Programming problems can be re – written as

Player A Minimize $p_0 = p_1 + p_2' + ... + p_m'$

Subject to the constraints

$$a_{1j}p_1' + a_{2j}p_2 + +a_{mj}p_m' \geq 1$$

$$p_i' \geq 0 \qquad (i = 1 \text{ to } m \text{ and } j = 1 \text{ to } n)$$

Player B Maximize $q_0 = q_1' + q_2' + ... + q_n$

Subject to the constraints

$$a_{i1}q_1' + a_{j2}q_2' + +a_{in}q_n' \leq 1$$

$$q_j' \geq 0. \qquad (i = 1 \text{ to } m \quad j = 1 \text{ to } n)$$

It is easy to note that the L.P.P's of the 2 players represent a primal dual pair.

Note:

In case there are negative elements in the payoff matrix add a suitable constant. $\therefore$ Value of the original game = Value of the game minus constant.

Example: 9.6.1

Solve the following game by using simplex method.

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{bmatrix} \end{array}$$

Solution:

Since some of the entries in the payoff matrix are negative, we add a suitable constant say $c = 4$ to each element

$$\begin{array}{cc} & B \\ \Rightarrow A & \begin{bmatrix} 5 & 3 & 7 \\ 7 & 9 & 1 \\ 10 & 6 & 2 \end{bmatrix} \end{array}$$

Let the strategies for 2 players be

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{bmatrix},$$

$$S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{bmatrix}$$

where $p_1 + p_2 + p_3 = 1,\ q_1 + q_2 + q_3 = 1$.

The linear programming for B is

$$\text{Minimize } V = \text{Maximize } \frac{1}{V} = y_1 + y_2 + y_3$$

Subject to $5y_1 + 3y_2 + 7y_3 \le 1,$

$7y_1 + 9y_2 + y_3 \le 1,$

$10y_1 + 6y_2 + 2y_3 \le 1,$

$y_j \ge 0,\ j = 1,2,3$ where $y_j = q_j / v,\ j = 1,2,3$

Introduce slack variables $S_1 \geq 0,\ S_2 \geq 0,\ S_3 \geq 0$.

Simplex Table is

Initial Iteration

Introduce y_3 and drop y_4

C_B	Y_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	1	5	3	7*	1	0	0
0	y_5	1	7	9	1	0	1	0
0	y_6	1	10	6	2	0	0	1
	$\frac{1}{V}$	0	-1	-1	-1	0	0	0

First Iteration

Introduce y_2 and drop y_5

C_B	Y_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6
1	y_3	$\frac{1}{7}$	$\frac{5}{7}$	$\frac{3}{7}$	1	$\frac{1}{7}$	0	0
0	y_5	$\frac{6}{7}$	$\frac{44}{7}$	$\frac{60}{7}$ *	0	$\frac{-1}{7}$	1	0
0	y_6	$\frac{5}{7}$	$\frac{60}{7}$	$\frac{36}{7}$	0	$\frac{-2}{7}$	0	1
	$\frac{1}{V}$	$\frac{1}{7}$	$\frac{-2}{7}$	$\frac{-4}{7}$	0	$\frac{1}{7}$	0	0

Final Iteration

Optimal Solution

C_B	Y_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6
1	y_3	$\frac{1}{10}$	$\frac{2}{5}$	0	1	$\frac{3}{20}$	$\frac{-1}{20}$	0
0	y_2	$\frac{1}{10}$	$\frac{11}{15}$	1	0	$\frac{-1}{60}$	$\frac{7}{60}$	0
0	y_6	$\frac{1}{5}$	$\frac{24}{5}$	0	0	$\frac{-1}{5}$	$\frac{-3}{5}$	1
	$\frac{1}{v}$	$\frac{1}{5}$	$\frac{2}{5}$	0	0	$\frac{2}{15}$	$\frac{1}{15}$	0

Value of the game V = V – 4

$$= 5 - 4$$

$$= 1$$

Optimum strategies for B

$$q_1^\circ = 0, q_2^\circ = \frac{1}{10}x5 = \frac{1}{2}, q_3^\circ = \frac{1}{10}x5 = \frac{1}{2}$$

Making use of duality, the optimum strategies for player A are obtained as

$$p_1^\circ = \frac{2}{15}x5 = \frac{2}{3},\ p_2^\circ = \frac{1}{10}x5 = \frac{1}{2}\ p_3^\circ = 0$$

Example: 9.6.2

Two companies A and B competing for the same product. Their different strategies are given in the following payoff matrix

Company A

$$\text{Company } \mathbf{B} = \begin{bmatrix} 2 & -2 & 3 \\ -3 & 5 & -1 \end{bmatrix}$$

Solution:

Add a constant $c = 4$.

$$\Rightarrow B = \begin{bmatrix} 6 & 2 & 7 \\ 1 & 9 & 3 \end{bmatrix}$$

From A's point of view

$$\text{Minimize } V = \text{maximize } \frac{1}{V} = y_1 + y_2 + y_3$$

$$\text{Subject to} \quad 6y_1 + 2y_2 + 7y_3 \le 1,$$

$$y_1 + 9y_2 + 3y_3 \le 1,$$

$$y_1, y_2, y_3 \ge 0,$$

Where V is the game value, $y_j = \frac{q_j}{V}$, $j = 1,2,3$

Introduce slack variables $S_1 \ge 0$, $S_2 \ge 0$.

Iterative Simplex table is

Initial Iteration

Introduce y_1 and drop y_4

C_B	Y_B	X_B	y_1	y_2	y_3	y_4	y_5
0	y_4	1	6*	2	7	1	0
0	y_5	1	1	9	3	0	1
	$\frac{1}{V}$	0	−1	−1	−1	0	0

First Iteration

Introduce y_2 and drop y_5

C_B	Y_B	X_B	y_1	y_2	y_3	y_4	y_5
1	y_1	$\frac{1}{6}$	1	$\frac{1}{3}$	$\frac{7}{6}$	$\frac{1}{6}$	0
0	y_5	$\frac{5}{6}$	0	$\frac{26}{3}*$	$\frac{11}{6}$	$\frac{-1}{6}$	1
	$\frac{1}{V}$	$\frac{1}{6}$	0	$\frac{-2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	0

Final Iteration

Optimal Solution

C_B	Y_B	X_B	y_1	y_2	y_3	y_4	y_5
1	y_1	$\frac{7}{52}$	1	0	$\frac{57}{52}$	$\frac{9}{52}$	$\frac{-1}{26}$
1	y_2	$\frac{5}{52}$	0	1	$\frac{11}{52}$	$\frac{-1}{52}$	$\frac{3}{26}$
	$\frac{1}{V}$	$\frac{3}{13}$	0	0	$\frac{4}{13}$	$\frac{1}{13}$	$\frac{1}{13}$

Optimum strategies for A

$$q_1^\circ = y_1 x V = \frac{7}{52} x \frac{13}{3} = \frac{7}{12}$$

$$q_2^\circ = y_2 x V = \frac{5}{52} x \frac{13}{3} = \frac{5}{12} \text{ and } q_3^\circ = 0$$

Expected value of the game is $V^\circ = v - 4 = \frac{13}{3} - 4 = \frac{1}{3}$

For company B; We read the value of the dual variables x_1 and x_2 from the net evaluation row of the final iteration. These are respectively $\frac{1}{13}$ and $\frac{2}{13}$. From these we get

$$p_1^{\circ} = x_1 \text{ x } u = \frac{2}{13} \text{x} \frac{13}{3} = \frac{2}{3}.$$

$$p_2^{\circ} = x_2 \text{ x } u = \frac{1}{13} \text{x} \frac{13}{3} = \frac{1}{3}.$$

$$A\left(\frac{7}{12},\frac{5}{12},0\right), B\left(\frac{2}{3},\frac{1}{3}\right)$$

$$\text{Value of the Game } = \frac{1}{3}.$$

Check Your Progress: 9.5

1. For the following payoff matrix, find the value of the game and the strategies of players A and B by using linear programming:

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \text{Player A} & \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 3 & -1 & 4 \\ 6 & 7 & --2 \end{bmatrix} \end{array}$$

2. For the following payoff table, transform the zero – sum game into an equivalent linear programming problem and solve it by simplex method:

$$\begin{array}{ccc} & & \text{Player Q} \\ & & \begin{array}{ccc} Q_1 & Q_2 & Q_3 \end{array} \\ \text{Player P} & \begin{array}{c} P_1 \\ P_2 \\ P_3 \end{array} & \begin{bmatrix} 9 & 1 & 4 \\ 0 & 6 & 3 \\ 5 & 2 & 8 \end{bmatrix} \end{array}$$

9.7 Key words

Two person zero – sum game, Maxi min – Mini max principle, saddle point, Graphical solution, Linear programming Method, Dominance Property.

9.8 Answers to check your progress questions:

Check your progress: 9.1

1. a) Not fair, $S_{\circ} = (A_1, B_1); v = -5$

   b) Not fair, $v = 6$.

2. $S_{\circ} = (A_1, B_1); v = 2$.

3. $S_{\circ} = (B_1, A_3); v = -2$, Game is strictly determinable and not fair.

4. $-1 \leq \lambda \leq 2$

5. (a) $S_{\circ} = (A_2, B_2)$ (or) $(A_1, B_3); r = 0$

   (b) $S_{\circ} = (A_1, B_1); v = -2$

6. $S_{\circ} =$ Row 3 and column 3, $v = 4$.

Check your progress: 9.2

1. (a) $S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \qquad S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}; \qquad v = \frac{-3}{4}$

   (b) $S_B = \begin{bmatrix} X_1 & X_2 \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \qquad S_B = \begin{bmatrix} y_1 & y_2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}; \qquad v = \frac{5}{2}$

2. $S_A = S_B = \begin{bmatrix} H & T \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}; \qquad v = -\frac{1}{8}$

3. $S_A = S_B = \begin{bmatrix} H & T \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix};$ Value of game $= 0$

4. $S_A = \begin{bmatrix} 1 & 2 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}; S_B = \begin{bmatrix} 1 & 2 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix};$ Value of the game $= \frac{4}{3}$

Space for Hints

Check your progress: 9.3

1. $S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & \frac{13}{20} & \frac{7}{20} \end{bmatrix}; v = \frac{1}{2}$

2. $S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ \frac{3}{10} & \frac{7}{10} & 0 \end{bmatrix}, S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{9}{10} & \frac{1}{10} \end{bmatrix}; v = \frac{23}{10}$

Check your progress: 9.4

1. $S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & \frac{3}{7} & \frac{4}{7} \end{bmatrix}; S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & \frac{5}{7} & \frac{2}{7} \end{bmatrix}; v = \frac{450}{7}$

2. (i) $S_{\text{Janta}} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & \frac{1}{6} & \frac{5}{6} \end{bmatrix}, S_{\text{praja}} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & \frac{5}{6} & \frac{1}{6} \end{bmatrix}; v = \frac{5}{6}.$

(ii) Janta may consider to have minor change with probability $\frac{5}{6}$ and of major change with probability $\frac{1}{6}$.

Check your progress: 9.5

1. $S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{9}{14} & \frac{5}{14} \end{bmatrix}, S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & \frac{3}{7} & \frac{4}{7} \end{bmatrix}; v = \frac{13}{7}$

2. $S_P = \begin{bmatrix} p_1 & p_2 & p_3 \\ \frac{3}{8} & \frac{13}{24} & \frac{1}{12} \end{bmatrix}, S_Q = \begin{bmatrix} Q_1 & Q_2 & Q_3 \\ \frac{7}{24} & \frac{5}{9} & \frac{11}{72} \end{bmatrix}; v = \frac{91}{24}$

9.9 Model Questions:

1. Solve the game whose payoff matrix is given below:

$$\begin{bmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 3 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{bmatrix}$$

2. Assume that two firms are competing for market share for a particular product. Each firm is considering what promotional strategy to employ for the coming period. Assume that the following payoff matrix describes the increase in market share for Firm A and the decrease in market share for Firm B. Determine the optimum strategies for each firm.

$$\begin{array}{llccc} & & & \text{Firm B} & \\ & & \text{No promoion} & \text{Moderate Promotion} & \text{Much promotion} \\ & \text{No promotion} & 5 & 0 & -10 \\ \text{Firm A} & \text{Moderate Promotion} & 10 & 6 & 2 \\ & \text{Much promotion} & 20 & 15 & 10 \end{array}$$

(i) Which firm would be the winner, in terms of market share?

(ii) Would the solution strategies necessarily maximize profits for either of the firms?

3. Solve the game whose payoff matrix is given below:

$$\begin{bmatrix} -2 & 0 & 0 & 5 & 3 \\ 3 & 2 & 1 & 2 & 2 \\ -4 & -3 & 0 & -2 & 6 \\ 5 & 3 & -4 & 2 & -6 \end{bmatrix}$$

4. Consider a modified form of "matching biased coins" The matching player is paid Rs. 8.00 if the 2 coins turn both heads and Rs. 1 if the coins turn both tails. The non – matching player is paid Rs. 3.00 when the coins do not match. Given the choice of being the matching (or) non – matching player, which one would you choose and what would be your strategy?

5. Solve the following problem graphically:

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix} \end{array}$$

6. Use graphically method in solving the following game:

$$\begin{array}{cc} & \text{Player A} \\ \text{Player B} & \begin{bmatrix} 2 & 2 & 3 & -2 \\ 4 & 3 & 2 & 6 \end{bmatrix} \end{array}$$

7. Solve the following games graphically:

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 6 & -3 & 7 \\ -3 & 0 & -6 \end{bmatrix} \end{array}$$

8. Solve the following '2 – person, zero – sum game':

Player A	Player B		
	B_1	B_2	B_3
A_1	15	10	4
A_2	10	-3	10
A_3	12	23	0

9. Two separate firms A and B have for years been selling a competitive product which forms a part of both firms' total sales. The marketing executive of firm A raised the question. "What should be the firm's strategies in terms of advertising for the product in question?" The market research team of firm A developed the following data for varying degrees of advertising:

(i) No advertising, medium advertising, and large advertising for both firms will result in equal market shares.

(ii) Firm A with no advertising: 40% of the market with medium advertising by firm B and 28% of the market with large advertising by firm B.

(iii) Firm A using medium advertising: 70% of market with no advertising by firm B and 45% of the market with large advertising by firm B.

(iv) Firm A using large advertising: 75% of the market with no advertising by firm B and 47.5% of the market with medium advertising by firm B.

(v) Based upon the foregoing information, answer the marketing executive's question.

10. The following matrix represents the payoff to P_1 in a rectangular game between two persons P_1 and P_2:

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 8 & 15 & -4 & -2 \\ 19 & 15 & 17 & 16 \\ 0 & 20 & 15 & 5 \end{bmatrix} \end{array}$$

By the notion of dominance, reduce the game to a 2 x 4 game and solve it graphically.

UNIT 10

QUEUEING THEORY

Introduction:

In every day life it is seen that a number of people arrive a cinema ticket window. If the people arrive too frequently they will have to wait for getting tickets or sometimes do without it. Such problems arise in Railways, Airline etc., Under such circumstances the only alternative form a Queue called the Waiting Line in order to get the service more effectively. If we have too many counters for service then expenditure may be high. On the other hand if we have only few counters then Queue may become longer resulting in the dissatisfaction or loss of customers Queuing models are aids to determine the optimal number of counters so as to satisfy the customers keeping the total cost minimum. Here the arriving people are called customers and the person issuing the tickets is called a Server.

Servers may be in parallel or in series. When in parallel, the arriving customers may form a single Queue or several Queues as is common in big post offices. Service time may be constant or variable and customers may be served singly or in batches.

OBJECTIVE

STRUCTURE

10.1 Queueing System.

10.2 $\{(M/M/1):(\infty/FIFO)\}$ Model

10.3 $\{(M/M/1):(N/FIFO)\}$Model

10.4 Key words

10.5 Answers to check your progress Questions

10.6 Model Questions

10.1 Queueing System

The mechanism of a queueing process is very simple. Customers arrive at a service counter and are attended to by one or more of the servers. As soon as a customer is served, it departs from the system. Thus a queueing system can be described as consisting of customers arriving for service, waiting for service if it is not immediate, and leaving the system after being served.

The general framework of a queueing system is shown in Fig.:

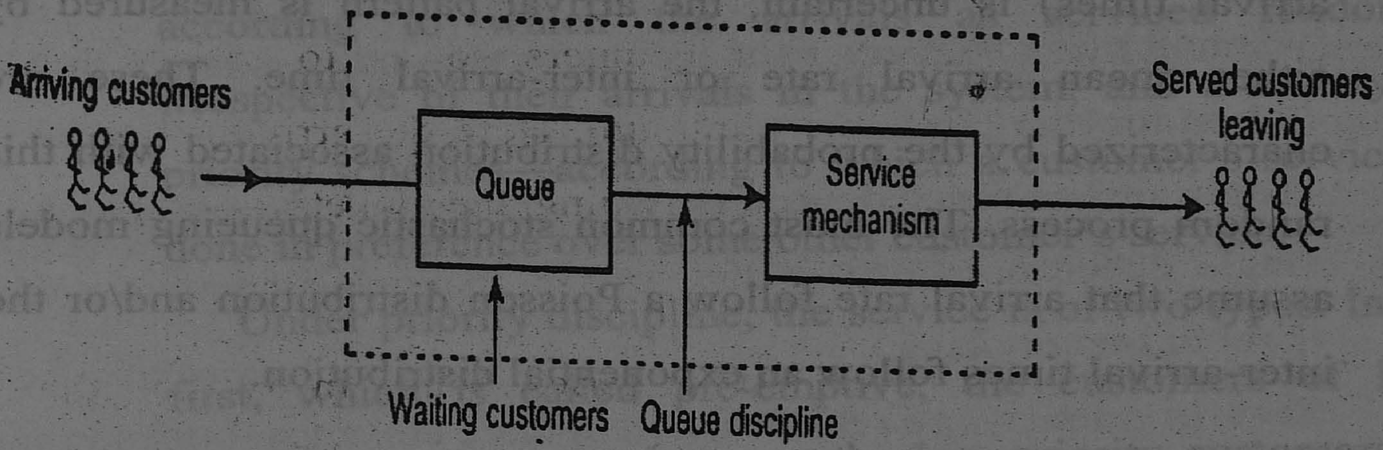


Figure: Queueing System

ELEMENTS OF A QUEUEING SYSTEM

The basic elements of a queueing system are as follows:

1. Input Process

This element of queueing system is concerned with the pattern in which the customers arrive for service. Input source can be described by following three factors:

(a) **Size of the queue:** If the total number of potential customers requiring service are only few, then size of the input source is said to be finite. On the other hand, if potential customers requiring service are sufficiently large in number, then the input source is considered to be infinite. Also, the customers may arrive at the service facility in batches of fixed size or of variable size or one by one. In the case when more than one arrival is allowed to enter the system simultaneously (entering the system does not necessarily mean entering into service), the input is said to occur in bulk or batches.

(b) **Arrival distribution:** If the pattern at which customers arrive at the service system or time between successive arrivals (inter-arrival times) is uncertain, the arrival pattern is measured by either mean arrival rate or inter-arrival time. These are characterized by the probability distribution associated with this random process. The most common stochastic queueing models assume that arrival rate follow a Poisson distribution and/or the inter-arrival times follow an exponential distribution.

(c) **Customer's behavior:** It is also necessary to know the reaction of a customer upon entering the system. A customer may decide to wait no matter how long the queue becomes, or if the queue is too long to suit him, may decide not to enter it. If a customer decides not to enter the queue because of its huge length, he is said to have balked. On the other hand, a customer may enter the queue, but after some time loses patience and decides to leave. In this case he is said to have reneged. In the case when there are two or more queues, customer may move from one queue to another for his personal economic gains, that is jockey for position.

The final factor to be considered regarding the input process is the manner in which the arrival pattern changes with time. The input process which does not change with time is called a stationary input process. If it is time dependent then the process is termed as transient.

2. **Queue Discipline:** It is a rule according to which customers are selected for service when a queue has been formed. The most common queue discipline is the "first come, first served" (FCFS), or the "first in, first out" (FIFO) rule under which the customers are serviced in the strict order of their arrivals. Other queue discipline include: "last in, first out" (LIFO) rule according to which the last arrival in the system is serviced first, "selection for service in random order" (SIRO) rule according to which the arrivals as serviced randomly irrespective of their arrivals in the system; and a variety of priority schemes—according to which a customer's service is done in preference over some other customer's service.

   Under priority discipline, the service is of two types. In the first, which is called pre-emptive, the customers of high priority are given service over the low priority customers. In the second type, called the non-pre-emptive, a customer of low priority is serviced before a customer of high priority.

   In the case of parallel channels "fastest server rule" (FSR) is adopted. For its discussion we suppose that the customers arrive before parallel service channels. If only one service channel is free, then incoming customer is assigned to free service channel. But it will be more efficient to assume that an incoming customer is to be assigned a server of largest service rate among the free ones.

3. **Service Mechanism:** The service mechanism is concerned with service time and service facilities. Service time is the time interval from the commencement of services to the completion of service. If there are infinite number of servers then all the customers are served instantaneously on arrival and there will be no queue.

If the number of servers is finite, then the customers are served according to a specific order. Further, the customers may be served in batches of fixed size or of variable size rather than individually by the same server, such as a computer with parallel processing or people boarding a bus. The service system in this case is termed as bulk service system.

Service facilities can be of the following types:

a) **Single queue-one server, i.e.,** one queue-one service channel, wherein the customers waits till the service point is ready to take him in for servicing.

b) **Single queue-several servers** Wherein the customers wait in a single queue until one of the service channels is ready to take them in for servicing.

c) **Several queues-one server** Wherein there are several queues and the customer may join any one of these but there is only one service channel.

d) **Several servers.** When there are several service channels available to provide service, much depends upon their arrangements. They may be arranged in parallel or in series or a more complex combination of both, depending on the design of the system's service mechanism.

By parallel channels, we mean a number of channels providing identical service facilities. Further, customers may wait in a single queue until one of the service channels is ready to serve, as in a barber shop where many chairs are considered as different service channels;

or customers may form separate queues in front of each service channel as in the case of super markets.

For series channels, a customer must pass through all the service channels is sequence before service is completed. The situations may be seen in public officers where parts of the service are done at different service counters.

4. **Capacity of the System:** The source from which customers are generated may be finite or infinite. A finite source limits the customers arriving for service, i.e., there is a finite limit to the maximum queue size. The queue can also be viewed as one with forced balking where a customer is forced to balk if he arrives at a time when queue size is at its limit. Alternatively, an infinite source is forever "abundant" as in the case of telephone calls arriving at a telephone exchange.

OPERATING CHARACTERISTICS QUEUEING SYSTEM

Some of the operational characteristics of queueing system those of a general interest for the evaluation of the performance of an existing queueing system and to design a new system are as follows:

1. Expected number of customers in the system denoted by E (n) or L is the average number of customers in the system, both waiting and in service. Here, n stands for the number of customers in the queueing system.

2. Expected number of customers in the queue denoted by E (m) or L_q is the average number of customers waiting in the queue. Here $m = n - 1$, i.e., excluding the customer being served.

3. Expected waiting time in the system denoted by E (v) or W is the average total time spent by a customer in the system. It is generally taken to be the waiting time plus servicing time.

4. Expected waiting time in queue denoted by E (w) or W_q is the average time spent by a customer in the queue before the commencement of his service.

5. The server utilization factor (or busy period) denoted by $P(=\lambda/\mu)$ is the proportion of time that a server actually spends with the customers. Here, λ stands for the average number of customers arriving per unit of time and μ stands for the average number of customers completing service per unit of time.

The server utilization factor is also known as traffic intensity.

PROBABILITY DISTRIBUTIONS IN QUEUEING SYSTEMS

It is assumed that customers joining the queueing system arrive in a random manner and follow a Poisson distribution or equivalently the inter-arrival times obey exponential distribution. In most of the cases, service times are also assumed to be exponentially distributed. It implies that the probability of service completion in any short time period is constant and independent of the length of time that the service has been in progress.

In this section, the arrival and service distributions for Poisson queues are derived. The basic assumption (axioms) governing this type of queues are stated below:

Axiom: 1

The number of arrivals in non-overlapping intervals are statistically independent, that is, the process has independent increments.

Axiom: 2

The probability of more than one arrival between time t and time $t + \Delta t$ is $0(\Delta t)$; that is, the probability of two or more arrivals during the small time interval Δt is negligible.

Thus $$P_0(\Delta t) + P_1(\Delta t) + o(\Delta t) = 1.$$

Axiom: 3

The probability that an arrival occurs between time t and time $t + \Delta t$ is equal to $\lambda \Delta t + o(\Delta t)$.

Thus $$P_1(\Delta t) = \lambda \Delta t + o(\Delta t).$$

Where λ is a constant independent of the total number of arrivals upto time t, Δt is an incremental element, and $o(\Delta t)$ represents the terms such that $\lim_{\Delta t \to 0} \frac{\sigma(\Delta t)}{\Delta t} = 0$.

Distribution of Arrivals "The Poisson Process" Arrival Distribution Theorem. (Pure Birth Process)

If the arrivals are completely random, then the probability distribution of a number of arrivals in a fixed – time interval follows a Poisson distribution.

Proof:

We make the following assumptions:

1) Assume that there are n units in the system at time t, and the probability that exactly one arrival (birth) will occur during the small time interval Δt be given by $\lambda \Delta t$.

2) Δt is considered to be so small that the probability of more than one arrival in time Δt is almost zero.

3) The process has independent increments. We wish to determine the probability of n arrivals in a time interval of length t, denoted by $P_n(t)$.

Case: 1

When $n > 0$, there may be two mutually exclusive ways of having n units at time $t + \Delta t$.

(a) There are n units in the system at time t and no arrival takes place during time interval Δt.

(b) There are $(n-1)$ units in the system at time t, and one arrival takes place during Δt.

Adding the 2 probabilities,

$$P_n(t+\Delta t) = P_n(t)(1-\lambda\Delta t) + P_{n-1}(t)\lambda\Delta t$$

Case: 2

When $n = 0$

$$P_0(t+\Delta t) = P_0(t)(1-\lambda\Delta t)$$

Rewriting (1), (2) and dividing thro' out by Δt we get

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -\lambda P_n(t) + \lambda P_{n-1}(t)$$

and $$\frac{P_o(t+\Delta t) - P_o(t)}{\Delta t} = -\lambda P_o(t)$$

as $\Delta t \to 0, \Rightarrow p_o'(t) = -\lambda P_o(t)$ for $n = 0$

(3)

and $\Rightarrow P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t),\ n > 0$

(4)

(3) can be written as

$$\frac{P_o'(t)}{P_o(t)} = -\lambda \text{ or } \frac{d}{dt}[\log P_o(t)] = -\lambda$$

Integrating w.r.t't'

$$\log P_o(t) = -\lambda t + A \qquad (5)$$

The constant 'A' can be found out using boundary conditions

$$P_n(0) = \begin{cases} 1, & n = 0 \\ 0, & n > 0 \end{cases}$$

Substituting $n = 0, P_o(O) = 1$, Using this in (5) A =0

$$\therefore \log P_o(t) = -\lambda t (\text{or}) P_o(t) = e^{-\lambda t}$$

Putting n =1 in (4)

$$P_1'(t) = -\lambda P_1(t) + \lambda P_o(t)$$

$$\text{Or } P_1'(t) = -\lambda P_1(t) + \lambda e^{-\lambda t}$$

$$\text{Or } P_1'(t) + \lambda P_1(t) = \lambda e^{-\lambda t}$$

This is differential equation of order one.

This can be solved by multiplying thro' out by integrating factor

$$\text{I.F} = e^{\int \lambda dt} = e^{\lambda t}$$

Now we have $e^{\lambda t}[P_1(t) + \lambda P_1(t)] = \lambda$

or $\frac{d}{dt}[e^{\lambda t} . P_1(t)] = \lambda$

Integrating both sides w.r.t 't'

$$e^{\lambda t} P_1(t) = \lambda t + B$$

Putting $t = 0 \Rightarrow B = 0$

$$\therefore P_1(t) = \frac{\lambda t e^{-\lambda t}}{1!}$$

Similarly $P_2(t) = \frac{(\lambda t)^2 e^{-\lambda t}}{2!}$ for n = 2 etc.,

In general $P_m(t) = \frac{(\lambda t)^m e^{-\lambda t}}{m!}$ for n = m

We can prove this for n = m + 1 $\therefore$ By induction hypothesis, the result is true for general value of n.

$\Rightarrow P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$ Which is a Poisson distribution formula.

Kendal's Notation for representing Queueing models

Generally Queueing Mode may be completely specified in the following symbol form: (a /b /c): (d / e):

Where a = probability law for the arrival.

b = probability law according to which customers are served.

c = Number of channels (or Service stations)

d = capacity of the system.

e = Queue discipline.

DEFINITION OF TRANSIENT AND STEDY STATES

A queueing system is said to be a transient state when its operating characteristic (like input, output, mean queue length, etc.) are dependent upon time.

If the characteristic of the queueing system becomes independent of time, then an at steady-state condition is said to prevail.

If P_n (t) denotes the probability that there are n customers in the system at time t, then in the steady-state case, we have

$$\lim_{t \to \infty} P_n(t) = P_n \text{ (independent of t).}$$

Due to practical viewpoint of the steady-state behaviour of the systems, the present chapter is amply focused on studying queueing systems under the existence of steady-state conditions. However, the differential-difference equations which can be used for deriving transient solutions will be presented.

POISSON QUEUEING SYSTEMS

Queues that follow the Poisson arrivals (exponential inter-arrival time) and Poisson services (exponential service time) are called Poisson queues. In this section, we shall study a number of Poisson queues with different characteristics.

10.2 {(M/M/1): (∞ / FIFO)} Model:

{(M/M/1): (∞/FIFO)}: This model deals with a queueing system having single service channel. Poisson input, Exponential service and there is no limit on the system capacity while the customers are served on a "first in, first out" basis.

Step: 1

Construction of Differential-Difference Equations. Let $P_n(t)$be the probability that there are n customers in the system at time t. The probability that the system has n customers at time $(t + \Delta t)$ can be expressed as the sum of the joint probabilities of the four mutually exclusive and collectively exhaustive events as follows:

$$P_n(t+\Delta t) = P_n(t).P\ [\text{no arrival in } \Delta t].\ P[\text{no service completion in } \Delta t]$$

$$+ P_n(t).P\ [\text{one arrival in } \Delta t].\ P\ [\text{one service completed in } \Delta t]$$

$$+ P_{n+1}(t).P\ [\text{no arrival in } \Delta t].\ P\ [\text{one service completed in } \Delta t]$$

$$+ P_{n-1}(t).P\ [\text{one arrival in } \Delta t].\ P\ [\text{no service completion in } \Delta t]$$

This is re – written as:

$$P_n(t+\Delta t) = P_n(t)[1+\lambda t+o(\Delta t)]+o(\Delta t)]\ [1-\mu\Delta t+o(\Delta t)]+P_n(t)[\lambda\Delta t]\ [\mu\Delta t]$$

$$+P_{n+1}(t)[1-\lambda\Delta t+o(\Delta t)]\ [\mu\Delta t+o(\Delta t)]+P_{n-1}(t)[\lambda\Delta t+o(\Delta t)]\ [1-\mu\Delta t+o(\Delta t)]$$

or

$$P_n(t+\Delta t)-P_n(t) = -(\lambda+\mu)\Delta t P_n(t)+\mu\Delta t P_{n+1}(t)+\lambda\Delta t P_{n-1}(t)+o(\Delta t)$$

Since Δt is very small, terms involving $(\Delta t)^2$ can be neglected. Dividing the above equation by Δt on both sides and then taking limit as $\Delta t \to 0$, we get

$$\frac{d}{dt}P_n(t) = -(\lambda+\mu)P_n(t)+\mu P_{n+1}(t)+\lambda P_{n-1}(t); \qquad n \geq 1.$$

Similarly, if there is no customer in the system at time $(t+\Delta t)$, there will be no service completion during Δt. Thus for n = 0 and $t \geq 0$, we have only two probabilities instead of four. The resulting equation is

$$P_o(t+\Delta t) = P_o(t)\{1-\lambda\Delta t+o(\Delta t)\}+P_1(t)\{\mu\Delta t+o(\Delta t)\}\ \{1-\lambda\Delta t+o(\Delta t)\}$$

or $\quad P_o\ (t+\Delta t)-P_o(t) = -\lambda\Delta t P_o(t)+\mu\Delta t P_1(t)+o(\Delta t)$

Dividing both sides of this equation by Δt and then taking limit as $\Delta t \to 0$, we get

$$\frac{d}{dt}P_o(t) = -\lambda P_o(t) + \mu P_1(t);\ n = 0.$$

Step: 2

Deriving the Steady – State Difference Equations. In the steady – state, $P_n(t)$ is independent of time t and $\lambda < \mu$ when $t \to \infty$. Thus $P_n(t) \to P_n$ and

$$\frac{d}{dt}P_n(t) \to 0 \qquad \text{as } t \to \infty.$$

Consequently the differential – difference equations obtained in Step 1 reduce to

$$0 = -(\lambda + \mu)P_n + \mu P_{n+1} + \lambda P_{n-1};\ n \geq 1$$

and $$0 = -\lambda P_n + \mu P_1;\ n = 0.$$

These constitute the steady – state difference equations.

Step: 3

Solution of the Steady – State Difference Equations. For the solution of the above difference equations there exist three methods, namely, the iterative method, use of generating functions and the use of linear operators. Out of these three the first one is the most straight forward and therefore the solution of the above equations will be obtained here by using the iterative method.

Using iteratively, the difference – equations yield

$$P_1 = \frac{\lambda}{\mu}P_o, \qquad P_2 = \frac{\lambda + u}{\mu}\ P_1 - \frac{\lambda}{\mu}P_o = \left(\frac{\lambda}{\mu}\right)^2 P_o$$

$$P_3 = \frac{\lambda + \mu}{\mu} \quad P_2 - \frac{\lambda}{\mu}P_1 = \left(\frac{\lambda}{\mu}\right)^3 P_o, \text{ and in general } P_n = \left(\frac{\lambda}{\mu}\right)^n P_o.$$

Now, $$P_{n+1} = \frac{\lambda+\mu}{\mu}P_{\text{1}} - \frac{\lambda}{\mu}P_{n-1}, n \geq 1$$

Substituting the values of P_n and P_{n-1}, the equation yields

$$P_{n+1} = \frac{\lambda+\mu}{\mu}\left(\frac{\lambda}{\mu}\right)^n P_o - \frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}\right)^{n-1} P_o = \left(\frac{\lambda}{\mu}\right)^{n+1} P_o.$$

To obtain the value of P_o, we make use of the boundary condition $\sum\limits_{n=0} P_n = 1$.

$$\therefore \qquad 1 = \sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n P_o = P_o\sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n; \qquad \text{since } P_o = \left(\frac{\lambda}{\mu}\right)^n P_o$$

$$= P_o\frac{1}{1-\lambda/\mu}, \text{ Since } \lambda/\mu < 1.$$

This gives $P_o = 1 - \lambda/\mu$.

Hence, the steady – state solution is

$$P_n = (\lambda/\mu)^n(1-\lambda/\mu) = \rho^n(1+\rho); \rho = \lambda/\mu < 1, \text{ and } n \geq 0.$$

This expression gives us the probability distribution of queue length.

Characteristics of Model I

(i) Probability of queue size being greater than or equal to n, the number of customers is given by

$$P(\geq n) = \sum_{k=n}^{\infty} P_k = \sum_{k=n}^{\infty}(1-\rho)\rho^k = (1-\rho)\rho^n\sum_{k=n}^{\infty}\rho^{k-n} = (1-\rho)\rho^n\sum_{k-n=0}^{\infty}\rho^{k-n}$$

$$= \frac{(1-\rho)\rho^n}{1-\rho} = \rho^n$$

(ii) Average number of customers in the system is given by

$$E(n) = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = (1-\rho)\sum_{n=0}^{\infty} n\rho^n = \rho(1-\rho)\sum_{n=1}^{\infty} n\rho^{n-1}$$

$$= \rho(1-\rho)\sum_{n=0}^{\infty} \frac{d}{d\rho}\rho^n = \rho(1-\rho)\frac{d}{d\rho}\sum_{n=0}^{\infty} \rho^n, \text{ since } \rho < 1$$

$$= \rho(1-\rho)\frac{1}{(1-\rho)^2} = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}.$$

(iii) Average queue length is given by

$$E(m) = \sum_{m=0}^{\infty} mP_n .$$

Where m = n − 1being the number of customers in the queue, excluding the customer which is in service.

∴

$$E(m) = \sum_{n=1}^{\infty} (n-1)P_n = \sum_{n=1}^{\infty} nP_n - \sum_{n=0}^{\infty} nP_n - [\sum_{n=0}^{\infty} P_n - P_o]$$

$$= \frac{\rho}{1-\rho} - [1-(1-\rho)] = \frac{\rho}{1-\rho} - \rho$$

$$= \rho^2/(1-\rho) = \lambda^2/\mu(\mu-\lambda).$$

(iv) Average length of non – empty queue is given by

$$E(m \mid m > 0) = \frac{E(m)}{P(m > 0)} = \frac{\lambda^2}{\mu(\mu-\lambda)} \cdot \frac{1}{(\lambda/\mu)^2} = \frac{\mu}{\mu-\lambda},$$

Since $P(m > 0) = P(n > 1) = \sum_{n=0}^{\infty} P_n - P_o - P_1 = \left(\frac{\lambda}{\mu}\right)^2$

(v) The fluctuation (variance) of queue length is given by

$$V(n) = \sum_{n=0}^{\infty} [n - E(n)]^2 P_n = \sum_{n=0}^{\infty} n^2 P_n - [E(n)]^2 .$$

Using some algebraic transformations and the value of P_n, the result reduces to

$$V(n) = (1-\rho)\frac{\rho+\rho^2}{(1-\rho)^3} - \left[\frac{\rho}{1-\rho}\right]^2 = \frac{\rho}{(1-\rho)^2} = \frac{\lambda\mu}{(\mu-\lambda)^2}$$

Waiting Time Distribution for Model: $\{(M/M/1):(\infty/FIFO)\}$

The waiting time of a customer in the system is, for the most part, a continuous random variable except that there is a non-zero probability that the delay will be zero, that is a customer entering service immediately upon arrival. Therefore, if we denote the time spent in the queue by w and if $\psi_w(t)$ denotes its cumulative probability distribution then from the complete randomness of the Poisson distribution, we have

$$\psi_w(0) = P(w=0)P\,\text{(No customer in the system upon arrival)}$$

$$= P_o = (1-\rho).$$

It is now required to find $\psi_w(t)$ for $t > 0$.

Let there be n customers in the system upon arrival, then in order for a customer to go into service at a time between 0 and t, all the n customers must have been served by time t. Let $s_1, s_2, \ldots, s_n$ denote service times of n customers respectively. Then

$$w = \sum_{i=1}^{n} s_i, (n \geq 1) \text{ and } w = 0 (n = 0).$$

The distribution function of waiting time, w, for a customer who has to wait is given by

$$P(w \le t) = P\left[\sum_{i=1}^{n} s_i \ge t\right];\ n \ge 1 \text{ and } t > 0.$$

Since the service time for each customer is independent and identically distributed, therefore its probability density function is given by $\mu e^{-\mu t}(t > 0)$, where μ is the mean service rate. Thus

$$\psi_n(t) = \sum_{n=1}^{\infty} P_n \times P\ (n-1 \text{ customer are served at time } t) \times P\ (1 \text{ customer is served in time } \Delta t)$$

$$= \sum_{n=1}^{\infty} \left[1 - \frac{\lambda}{\mu}\right]\left(\frac{\lambda}{\mu}\right)^n \frac{(\mu t)^{n-1}\ e^{-\mu t}}{(n-1)!}.\mu \Delta t.$$

The expression for $\psi_w(t)$, therefore, can be written as

$$\psi_w(t) = P(w \le 1) = \sum_{n=1}^{\infty} P_n \int_0^t \psi_n(t)dt$$

$$= \sum_{n=1}^{\infty} (1-\rho)P \setminus \rho^n \int_0^t \frac{(\mu t)^{n-1}}{(n-1)!} e^{-\mu t}\mu dt = (1-\rho)\rho \int_0^t \mu e^{-\mu t} \sum_{n=1}^{\infty} \frac{(\mu t \rho)^{n-1}}{(n-1)!}.dt$$

$$= (1-\rho)\rho \int_0^{\infty} \mu e^{-\mu t(1-p)} dt.$$

Hence, the waiting time of a customer who has to wait is given by

$$\psi(w) = \frac{d}{dt}[\psi_w(t)] = \rho(1-\rho).\mu e^{-\mu t(1-p)} = \lambda\left(1 - \frac{\lambda}{\mu}\right) e^{-(\mu-\lambda)t}.$$

Characteristic of Waiting Time Distribution

(i) Average waiting time of a customer (in the queue) is given by

$$E(w \mid w > 0) = \int_0^\infty t\psi(w)dt = \int_0^\infty t.\rho\mu(1-\rho)e^{-\mu t(1-\rho)}dt$$

$$= \rho\int_0^\infty \frac{xe^{-x}}{\mu(1-\rho)}dx, \text{ for } \mu(1-\rho)t = x$$

$$= \frac{\rho}{\mu(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)}.$$

(ii) Average waiting time of an arrival who has to wait is given by

$$E(w \mid w > 0) = \frac{E(w)}{P(w > 0)} = \left\{\frac{\lambda}{\mu(\mu-\lambda)}\right\} / \left(\frac{\lambda}{\mu}\right) = \frac{1}{\mu-\lambda}$$

$[\text{Here } P(w > 0) = 1 - P(w = 0) = 1 - (1-\rho) = \rho]$

(iii) For the busy period distribution, let the random variable v denote the total time that a customer has to spend in the system including service. Then the probability density of its cumulative density function is given by

$$\psi(w \mid w > 0) = \frac{\psi(w)}{P(w > 0)} = \left[\lambda\left(1 - \frac{\lambda}{\mu}\right) = e^{-(\mu-\lambda)t}\right] / \left(\frac{\lambda}{\mu}\right)$$

$$= (\mu - \lambda)e^{-(\mu-\lambda)t}, \ t > 0$$

(iv) Average waiting time that a customer spends in the system including service is given by

$$E(v) = \int_0^\infty t.\psi(w \mid w > 0)dt = \int_0^\infty t.(\mu - \lambda)e^{-(\mu-\lambda)t}dt$$

$$= \frac{1}{\mu-\lambda}\int_0^\infty xe^{-x}dx, \text{ for } (\mu-\lambda)t = x$$

$$= \frac{1}{\mu-\lambda}.$$

Relations between Average Queue Length and Average Waiting Time – Little's Formula.

We have derived above the following important characteristics of M/M/1 queueing system:

$$E(n)=\frac{\lambda}{\mu-\lambda},\ E(m)=\frac{\lambda^2}{\mu(\mu-\lambda)},\ E(w)=\frac{\lambda}{\mu(\mu-\lambda)}\ \text{and}$$

$$E(v)=\frac{1}{\mu-\lambda}.$$

Using these expression, we observe that

$$E(n)=\lambda E(v),\ E(m)=\lambda E(w)\ \text{and}\ E(v)=E(w)+1/\mu.$$

Example: 10.2.1

A T.V. repairman finds that the time spent on his jobs has an exponential distribution with mean 30 minutes. If he repairs sets in the order in which they came in, and if the arrival of sets is approximately Poisson with an average rate of 10 per 8 – hour day, what is repairman's expected idle time each day? How many jobs are ahead of the average set just brought in?

Solution:

We are given,

$$\lambda = 10\text{sets per day, and } \mu = 16\text{sets per day.}$$

$$\therefore \qquad \rho=\lambda/\mu=10/16=0.625$$

The probability for the repairman to be idle is

$$P_\circ = 1-\rho=1-0.625=0.375$$

(i) Expected idle time per day $= 8 \text{ x } 0.375 = 3$ hours.

(ii) Expected (or average) number of T.V. sets in the system

$$E(n) = \frac{\rho}{1-\rho} = \frac{0.625}{1-0.625} = \frac{5}{3} = 2 \text{(approx.) T.V. sets.}$$

Example: 10.2.2

In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter – arrival time follows an exponential distribution and the service time distribution is also exponential with an average 36 minutes. Calculate the following:

(i) The mean queue size (line length), and

(ii) The probability that the queue size exceed 10.

If the input of trains increases to an average 33 per day, what will be the change in (i) and (ii)?

Customers at time $(t + \Delta t)$ can be expressed as the sum of the joint probabilities of the four mutually exclusive and collectively exhaustive events as follows:

$$P_n(t+\Delta t) = P_n(t).\ P[\text{no arrival in } \Delta t]\ P[\text{no service completion in } \Delta t]$$

$$+ P_n(t).\ P[\text{one arrival in } \Delta t]\ P[\text{one service completed in } \Delta t]$$

$$+ P_{n+1}(t).\ P[\text{no arrival in } \Delta t]\ P[\text{one service completed in } \Delta t]$$

$$+ P_{n-1}(t).\ P[\text{one arrival in } \Delta t]\ P[\text{on service completion in } \Delta t]$$

This is re-written as:

$$P_n(t+\Delta t) = P_n(t)[1 + \lambda t + o(\Delta t)]\,[1 - \mu\Delta t + o(\Delta t)] + P_n(t)[\lambda\Delta t]\,[\mu\Delta t]$$

$$+ P_n + 1(t)\,[1 - \lambda\Delta t + o(\Delta t)][\mu\Delta t + o(\Delta t)] + P_{n-1}(t)[\lambda\Delta t + 0(\Delta t)][1 - \mu\Delta t + 0(\Delta t)]$$

or

$$P_n(t+\Delta t)-P_n(t)=-(\lambda+\mu)\Delta t\ P_n(t)+\mu\Delta t\ P_{n+1}(t)+\lambda\Delta t\,P_{n-1}(t)+o(\Delta t)$$

Since Δt is very small, terms involving $(\Delta t)^2$ can be neglected. Dividing the above equation by Δt on both sides and then taking limit as $\Delta t \to 0$, we get

$$\frac{d}{dt}P_n(t)=-(\lambda+\mu)P_n(t)+\mu P_{n+1}(t)+\lambda P_{n-1}(t);\ n\geq 1.$$

Similarly, if there is no customer in the system at time $(t+\Delta t)$, there will be no service completion during Δt. Thus for $n=0$ and $t\geq 0$, we have only two probabilities instead of four. The resulting equation is

$$P_o(t+\Delta t)=P_o(t)\{1-\lambda\Delta t+0(\Delta t)\}+P_1(t)\{\mu\Delta t+o(\Delta t)\}\{1-\lambda\Delta t+o(\Delta t)\}$$

or $\quad P_o(t+\Delta t)-P_o(t)=-\lambda\Delta t\,P_o(t)+\mu\Delta t\,P_1(t)+o(\Delta t).$

Dividing both sides of this equation by Δt and then taking limit as $\Delta t \to 0$, we get

$$\frac{d}{dt}P_o(t)=-\lambda P_o(t)+\mu P_1(t);\ n=0.$$

Step2:

Deriving the Steady-State Difference Equations. In the steady-state, $P_n(t)$ is independent of time t and $\lambda<\mu$ when $t\to\infty$. Thus $P_n(t)\to P_n$ and

$$\frac{d}{dt}P_n(t)\to 0 \text{ as } t\to\infty.$$

Consequently the differential –difference equations obtained in Step 1 reduce to

$$0 = -(\lambda + \mu)P_n + \mu P_{n+1} + \lambda P_{n-1};\ n \geq 1$$

and $$0 = -\lambda P_n + \mu P_1;\ n = 0.$$

These constitute the steady-state difference equations.

Step3:

Solution of the Steady-State Difference Equations. For the solution of the above difference equations there exist three methods, namely, the iterative method, use of generating functions and the use of linear operators. Out of these three the first one is the most straightforward and therefore the solution of the above equations will be obtained here by using the iterative method.

Using iteratively, the difference-equations yield

$$P_1 = \frac{\lambda}{\mu}P_o,\ P_2 = \frac{\lambda+\mu}{\mu}P_1 - \frac{\lambda}{\mu}P_o = \left(\frac{\lambda}{\mu}\right)^2 P_o$$

$$P_3 = \frac{\lambda+\mu}{\mu}P_2 - \frac{\lambda}{\mu}P_1 = \left(\frac{\lambda}{\mu}\right)^3 P_o \text{ and in general } P_n = \left[\frac{\lambda}{\mu}\right]^n P_o.$$

Now, $$P_{n+1} = \frac{\lambda+\mu}{\mu}P_n - \frac{\lambda}{\mu}P_{n-1},\ n \geq 1.$$

Substituting the values of P_n and P_{n-1}, the equation yields

$$P_{n+1} = \frac{\lambda+\mu}{\mu}\left(\frac{\lambda}{\mu}\right)^n P_o = \left(\frac{\lambda}{\mu}\right)^{n-1} P_o\left(\frac{\lambda}{\mu}\right)^{n-1} P_o.$$

Thus by the principle of mathematical induction, the general formulae for P_n is valid for $n \geq 0$.

(i) To obtain the value of P_o we make use of the boundary condition $\sum_{n=0}^{\infty} P_n = 1$.

$$1 = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n P_o = P_o \sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n : \text{since, } P_n = \left(\frac{\lambda}{\mu}\right)^n P_o$$

$$= P_o \frac{1}{1-\lambda/\mu}, \text{ since } \lambda/\mu < 1.$$

This gives $P_o = 1 - \lambda/\mu$.

Hence, the steady-state solution is

$$P_n = (\lambda/\mu)^n (1 - \lambda/\mu) = \rho^n (1 + \rho); \ \rho = \lambda/\mu < 1, \text{and } n \geq 0.$$

This expression gives us probability distribution of queue length.

Characteristics:

(i) Probability of queue size being greater than or equal to n, the number of customers is given by

$$P(\geq n) = \sum_{k=n}^{\infty} P_k = \sum_{k=n}^{\infty} (1-\rho)\rho^n \sum_{k=n}^{\infty} \rho^{k-n} = (1-\rho) \sum_{k-n=0}^{\infty} \rho^{k-n}$$

$$= \frac{(1-\rho)\rho^n}{1-\rho} = \rho^n.$$

(ii) Average number of customers in the system is given by

$$E(n) = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = (1-\rho)\sum_{n=0}^{\infty} n\rho^n = \rho(1-\rho)\sum_{n=1}^{\infty} n\rho^{n-1}$$

$$= \rho(1-\rho)\sum_{n=0}^{\infty} \frac{d}{d\rho}\rho^n = \rho(1-\rho)\frac{d}{d\rho}\sum_{n=0}^{\infty} \rho^n, \text{ since } \rho < 1$$

$$= \rho(1-\rho)\frac{1}{(1-\rho)^2} = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}.$$

(iii) Average queue length is given by

$$E(m) = \sum_{m=0}^{\infty} mP_n,$$

Where $m = n - 1$ being the number of customers in the queue, excluding the customer which is in service.

$$E(m) = \sum_{n=1}^{\infty} (n-1)P_n = \sum_{n=1}^{\infty} nP_n - \sum_{n=1}^{\infty} P_n = \sum_{n=0}^{\infty} nP_n - [\sum_{n=0}^{\infty} P_n - P_\circ]$$

$$= \frac{\rho}{1-\rho} - [1-(1-\rho)] = \frac{\rho}{1-\rho} - \rho$$

$$= \rho^2/(1-\rho) = \lambda^2/\mu(\mu-\lambda).$$

(iv) Average length of non – empty queue is given by

$$E(m \mid m > 0) = \frac{E(m)}{P(m>0)} = \frac{\lambda^2}{\mu(\mu-\lambda)} \cdot \frac{1}{(\lambda/\mu)^2} = \frac{\mu}{\mu-\lambda},$$

Since

$$P(m > 0) = P(n > 1) = \sum_{n=0}^{\infty} P_n - P_\circ - P_1 = \left(\frac{\lambda}{\mu}\right)^2$$

(v) The fluctuation (variance) of queue length is given by

$$V(n) = \sum_{n=0}^{\infty} [n - E(n)]^2 P_n = \sum_{n=0}^{\infty} n^2 P_n - [E(n)]^2.$$

Using some algebraic transformations and the value of P_n, the result reduces to

$$V(n) - (1-\rho)\frac{\rho+\rho^2}{(1-\rho)^3} - \left[\frac{\rho}{1-\rho}\right]^2 = \frac{\rho}{(1-\rho)^2} = \frac{\lambda\mu}{(\mu-\lambda)^2}.$$

Solution:

Here, we have

$$\lambda = \frac{30}{60x24} = \frac{1}{48} \text{ and } \mu = \frac{1}{36} \text{ trains per minute.}$$

$$\therefore \qquad \rho = \lambda/\mu = 36/48 = 0.75$$

(i) $E(m) = \frac{\rho}{1-\rho} = \frac{0.75}{1-0.75} = 3\,\text{trains}$

(ii) $P(\geq 10) = \rho^{10} = (0.75)^{10} = 0.06$.

When the input increases to 33 trains per day, we have

$$\lambda = \frac{33}{60x24} = \frac{11}{480} \text{ and } \mu = \frac{1}{36} \text{ trains per minute.}$$

$$\therefore \qquad \rho = \frac{\lambda}{\mu} = \frac{11}{480}x36 = 0.83$$

Then, we get

(i) $E(n) = \frac{\rho}{1-\rho} = \frac{0.83}{1-.083} = 4.9$ or 5 trains (approx.)

(ii) $P(\geq 10) = \rho^{10} = (0.83)^{10} = 0.2$ (approx.)

Example: 10.2.3

The rate of arrival of customers at a public telephone booth follows Poisson distribution, with an average time of 10 minutes between one customer and the next. The duration of a phone call is assumed to follow exponential distribution, with mean time of 3 minutes.

Space for Hints

(i) What is the probability that a person arriving at the booth will have to wait?

(ii) What is the average length of the non – empty queues that form from time to time?

(iii) The Mahanagar Telephone Nigam Ltd. will install a second booth when it is convinced that the customers would expect waiting for at least 3 minutes for their turn to make a call. By how much time should the flow of customers increase in order to justify a second booth?

(iv) Estimate the fraction of a day that the phone will be in use.

(v) What is the probability that it will take him more than 10 minutes altogether to wait for phone and complete his call?

Solution:

Here we are given:

$\lambda = \frac{1}{10} x 60$ or 6 per hour and $\mu = \frac{1}{3} x 60$ or 20 per hour.

(i) Probability that a person arriving at the booth will have to wait

$$P(w > 0) = 1 - P_\circ = 1 - \left(1 - \frac{\lambda}{\mu}\right) = \frac{6}{20} \text{ or } 0.3.$$

(ii) Average length of non – empty queues

$$E(m \mid m > 0) = \frac{\mu}{\mu - \lambda} = \frac{20}{20 - 6} = 1.43.$$

(iii) The installation of a second booth will be justified if the arrival rate is greater than the waiting time. Now, if λ' denotes the increased arrival rate, expected waiting time is:

$$E(w)=\frac{\lambda'}{\mu(\mu-\lambda')}\Rightarrow\frac{3}{60}=\frac{\lambda'}{20(20-\lambda')}\text{ or }\lambda'=10.$$

Hence, the arrival rate should become 10 customers per hour to justify the second booth.

(iv) The fraction of a day that the phone will be busy = traffic intensity $\rho=\lambda/\mu=0.3$.

(v) $P(w\geq 0)=\int_{10}^{\infty}\lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)t}dt=\int_{10}^{\infty}(0.30)(0.23)e^{-0\ 23t}dt$,

where $\lambda=0.10$per minute, and $\mu=0.33$per minute.

$$\therefore\qquad P(w\geq 10)=(0.069)=\left.\frac{e^{-0\ 23t}}{(-0.23)}\right|_{10}^{\infty}=0.03.$$

This shows that 3 per cent of the arrivals on an average will have to wait for 10 minutes or more before they can use the phone.

Example: 10.2.4

On an average 96 patients per 24 – hours day require the service of an emergency clinic. Also on an average, a patient requires 10 minutes of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it costs the clinic Rs. 100 per patient treated to obtain an average servicing time of 10 minutes, and that each minute of decrease in this average time would cost Rs. 10 per patient treated. How much would have to be budgeted by the clinic to decrease the average size of the queue from $1\frac{1}{3}$ patients to $\frac{1}{2}$ a patient.

Solution:

Here,

$$\lambda = \frac{96}{24x60} = \frac{1}{15} \text{ and } \mu = \frac{1}{10} \text{ patients per minute}$$

$\therefore$ $$\rho = \frac{\lambda}{\mu} = \frac{2}{3}.$$

Average number of patients in the queue are given by,

$$E(m) = \frac{\rho^2}{1-\rho} = \frac{(2/3)^2}{1-2/3.} = \frac{4}{3}$$

Fraction of the time for which there are no patients is given by,

$$P_o = 1-\rho = 1-2/3 = 1/3.$$

Now, when the average queue size is decreased from 4/3 patients to ½ patient, we are to determine the value of μ. So, we have

$$E(m) = \frac{\lambda^2}{\mu(\mu-\lambda)} \Rightarrow \frac{1}{2} = \frac{(1/15)^2}{\mu(\mu-1/15)^2}$$

i.e., $\mu = 2/15$ patients per minute.

$\therefore$ Average rate of treatment required $= 1/\mu = 15/2 = 7.5$ minutes.

i.e., a decrease in the average rate of treatment is (10 – 0.75) minutes or 2.5 minutes.

$$\text{Budget per patient Rs.} = (100 + 2.5x10) = \text{Rs.}125.$$

Hence, in order to get the required size to the queue, the budget should be increased from Rs. 100 per patient to Rs. 125 per patient.

Example: 10.2.5

A road transport company has one reservation clerk on duty at a time. He handles information of bus schedules and makes reservations. Customers arrive at a rate of 8 per house and the clerk can service 12 customers on an average per hour. After stating your assumptions, answer the following:

(i) What is the average number of customers waiting for the service of the clerk?

(ii) What is the average time a customer has to wait before getting service?

(iii) The management is contemplating to install a computer system to handle the information and reservations. This is expected to reduce the service time from 5 to 3 minutes. The additional cost of having the new system works out to Rs. 50 per day. If the cost of goodwill of having to wait is estimated to be 12 paise per minute spent waiting before being served. Should the company install the computer system? Assume 8 hours working day.

Solution:

We are given

$\lambda = 8$ customers per hour and $\mu = 12$ customers per hour.

Average number of customers waiting for the service of the clerk (in the system):

$$E(n) = \frac{\lambda}{\mu - \lambda} = \frac{8}{12 - 8} = 2 \text{ customers}$$

Space for Hints

The average number of customers waiting for the service of the clerk (in the queue):

$$E(m) = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{8x8}{12(12-8)} \text{ or } 1.33 \text{ customers.}$$

(ii) The average waiting time of a customer (in the system) before getting service:

$$E(w) = \frac{1}{\mu - \lambda} = \frac{1}{12-8} \text{ hour or 15 minutes.}$$

The average waiting time of a customer (in the queue) before getting service:

$$E(v) = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{8}{12(12-8)} = \frac{1}{6} \text{ hour or 10 minutes.}$$

(iii) We now calculate the difference between the goodwill cost of customers with one system and the goodwill cost of customers with an additional computer system. This difference will be compared with the additional cost (of Rs. 50 per day) of installing another computer system.

An arrival waits for $E(v)$ hours before being served and there are λ arrivals per hour. Thus expected waiting time for all customers in an 8 – hour day with one system $= 8\lambda.\ E(v) = 8 \text{ x} 8 \text{ x } \frac{1}{6}$ hrs. or $\frac{64}{6}$ x 60 minutes. i.e., 640 minutes.

The goodwill cost per day with one system

$$= 640 \text{ x Rs.}0.12 = \text{Rs.}76.80$$

The expected waiting time of a customer before getting service when there is an additional computer system is:

$$E(v^*) = \frac{8}{20(20-8)} = \frac{8}{20x12} \text{ or } \frac{1}{30} \text{hrs.}$$

Thus expected waiting time of customers in an 8 – hour day with an additional computer system is 8λ x $E(v^*)$

$$= 8 \text{ x } 8 \text{ x } \frac{1}{30} \text{hr.} = 128 \text{ minutes}$$

The total goodwill cost with an additional computer system $= 128$ x Rs.0.12 = Rs.15.36.

Hence reduction in goodwill cost with the installation of a computer system

$$= \text{Rs.}76.80 - \text{Rs.}15.36 = \text{Rs.}61.44.$$

Whereas the additional cost of a computer system is Rs. 50 per day, Rs. 61.44 is the reduction in goodwill cost when additional computer system is installed, hence there will be net saving of Rs. 11.44 per day. It is, therefore, worthwhile to install a computer.

Example: 10.2.6

In the production shop of a company the breakdown of the machines is found to be Poisson with an average rate of 3 machines per hour. Breakdown time at one machine costs Rs. 40 per hour to the company. There are two choices before the company for hiring the repairmen. One of the repairmen is slow but cheap, the other fast but expensive. The slow – cheap repairman demands, Rs. 20 per hour and will repair the broken down machines exponentially at the rate of 4 per hour. The fast –

expensive repairman demands Rs. 30 per hour and will repair machines exponentially at an average rate of 6 per hour. Which repairman should be hired?

Solution:

In this problem, we compare the total expected daily cost for both the repairmen. This would equal the total wages paid plus the downtime cost.

Case: 1

Slow – cheap repairman

$\lambda = 3$ machines per hour and $\mu = 4$ machines per hour.

$\therefore$ Average downtime of a machine $= \dfrac{1}{\mu - \lambda} = \dfrac{1}{4-3} = 1$ hour.

$\therefore$ The downtime of 3 machines that arrive in an hour $1x3 = 3$ hours.

Downtime cost $= Rs.40x3 = Rs.120$, charges paid to the repairman $= Rs.20x3 = Rs.60$

Total cost $= Rs.120 + Rs.60 = Rs.180$.

Case: 2

Fast – expensive repairman

$\lambda = 3$ machines per hour and $\mu = 6$ machines per hours.

$\therefore$ Average downtime of machine $= \dfrac{1}{\mu - \lambda} = \dfrac{1}{3}$ hours

$\therefore$ The downtime of 3 machines that arrive in an hour $= \dfrac{1}{3}x3 = 1$ hours.

Downtime cost $= \text{Rs.}40\text{x}1 = \text{Rs.}40$, charges paid to the repairman $= \text{Rs.}30\text{x}1 = \text{Rs.}30$

Total cost $= \text{Rs.}40 + \text{Rs.}30 = \text{Rs.}70$.

From the above two cases, the decision of the company should be to engage the fast – expensive repairman.

Check your progress: 10.1

1) At a one – man barber shop, customers arrive according to Poisson distribution with a mean arrival rate of 5 per hour and his hair – cutting time was exponentially distributed with an average hair – cut taking 10 minutes. It is assumed that because of his excellent reputation, customers were always willing to wait. Calculate the following:

(i) Average number of customers in the shop and the average number of customers waiting for a hair – cut.

(ii) The percentage of time an arrival can walk right in without having to wait.

(iii) The percentage of customers who have to wait prior to getting into the barber's chair.

2) Customers arrive at a sales counter manned by a single person according to a Poisson process with a mean rate of 20 per hour. The time required to serve a customer has an exponential distribution with a mean of 100 seconds. Find the average waiting time of a customer.

3) At a certain filling station, customers arrive in a Poisson process with an average time of 12 per hour. The time intervals between services follow exponential distribution and as such the mean time taken to service a unit is 2 minutes. Evaluate:

i) The probability that there is no customer at the counter,

ii) The probability that there are more than two customers at the counter,

iii) The probability that there is no customer to be served,

iv) The probability that a customer is being served, but no body is waiting,

v) The expected number of customers in the waiting line, and

vi) The expected time a customer spends in the system.

10. 3: $\{(M \mid M \mid 1):(N/FIFO)\}$ Model:

$\{(M \mid M \mid 1):(N/FIFO)\}$. This model differs from that of $\{(M \mid M \mid 1):(N/FIFO)\}$ in the sense that the maximum number of customers in the system is limited to N. therefore, the difference equations of $\{(M \mid M \mid 1):(N/FIFO)\}$ I are valid for this model as long as $n < N$.

The additional difference equation for $n = N$, is

$$P_N(t+\Delta t) = P_N(t)[1-\mu\Delta t] + P_{N+1}(t).[\lambda\Delta t][1-\mu\Delta t] + \circ(\Delta t).$$

This gives, after simplification, the differential – difference equation

$$\frac{d}{dt}P_N(t) = -\mu P_N(t) + \lambda P_{N-1}(t)$$

from which the resultant steady – state difference equation is

$$0 = -\mu P_N + \lambda P_{N-1}.$$

The complete set of steady – state difference equations for this model, therefore, can be written as

$$\mu P_1 = \lambda P_\circ$$

$$\mu P_{n+1} - (\lambda + \mu)P_n - \lambda\, P_{n-1}, \qquad 1 \le n \le N-1,$$

and $$\mu P_N = \lambda P_{N-1}.$$

Using the iterative procedure (as in Model I), the first two difference equations give

$$P_n = (\lambda / \mu)^n P_\circ, \qquad n \le N-1.$$

Also, we see that for this value of P_n, the third (last) difference equation holds for $n = N$.

Therefore, we have

$$P_n = (\lambda / \mu)^n P_\circ = \rho^n P_\circ, \qquad n \le N.$$

For obtaining the value of $P_\circ$, we make use of the boundary conditions, $\sum_{n=0}^{N} P_n = 1$.

Therefore

$$1 = P_\circ \sum_{n=0}^{N} \rho^n = \begin{cases} P_\circ \dfrac{1-\rho^{N+1}}{1-\rho}, & (\rho \ne 1) \\ P_\circ (N+1), & (\rho = 1) \end{cases}$$

$$P_\circ = \begin{cases} \dfrac{1-\rho}{1-\rho^{N+1}}, & (\rho \ne 1) \\ \dfrac{1}{N+1}, & (\rho = 1) \end{cases}$$

Hence,

$$P_n = \begin{cases} \dfrac{(1-\rho)}{1-\rho^{N+1}}, & (\rho \ne 1) \qquad 0 \le n \le N \\ \dfrac{1}{N+1}, & (\rho = 1) \end{cases}$$

Space for Hints

Remark:

The steady – state solution exists even for $\rho \geq 1$. Intuitively this makes sense since the maximum limit prevents the process form "blowing up". If $N \to \infty$, then the steady – state solution is

$$P_n = (1-\rho)\rho^n ; \qquad n < \infty,$$

This result is in complete agreement with that of Model $\{(M/M/1):(\infty/FIFO)\}$.

Characteristics:

(i) Average number of customers in the system is given by

$$E(n) = \sum_{n=0}^{N} nP_n = P_o \sum_{n=0}^{N} nP^n = P_o\rho \sum_{n=0}^{N} \frac{d}{d\rho}\rho^n$$

or $$E(n) = P_o\rho\frac{d}{d\rho}\sum_{n=0}^{N} \rho^n P_o\rho\frac{d}{d\rho}\left[\frac{1-\rho^{N+1}}{1-\rho}\right]$$

$$= P_o \frac{\rho[1-(N+1)\rho^N + N\rho^{N+1}}{(1-\rho)^2} = \frac{\rho[1-(N+1)\rho^N + N\rho^{N+1}}{(1-\rho)(1-\rho^{N+1})}$$

$$\text{Since } P_o = \frac{1-\rho}{1-\rho^{N+1}}; \rho \neq 1$$

(ii) Average queue length is given by

$$E(m) = \sum_{n=1}^{N} (n-1)P_n = E(n) - \sum_{n=1}^{N} P_n = E(n) - (1-P_o)$$

$$= E(n) - \frac{\rho(1-\rho)^N}{1-\rho^{N+1}}, \qquad \text{since } P_o = \frac{1-\rho}{1-\rho^{N+1}}, (\rho \neq 1)$$

$$= \frac{\rho^2[1-N\rho^{N-1} + (N-1)\rho^N}{(1-\rho)(1-\rho^{N+1})}$$

(iii) The average waiting time in the system can be obtained by using Little's formula, that is, $E(v) = \{E(n)\}/\lambda'$, where λ' is the mean rate of customers entering the system and is equal to $\lambda(1 - P_N)$. The average waiting time in the queue can be obtained by using the relations

$$E(w) = E(v) - 1/\mu \text{ or } E(w) = \{E(m)\}/\lambda'.$$

Example: 10.3.1

At a railway station, only one train is handled at a time. The railway yard is sufficient only for two trains to wait while other is given to leave the station. Trains arrive at the station at an average rate of 6 per hour and the railway station can handle them on an average of 12 per hour. Assuming Poisson arrivals and exponential service distribution, find the steady – state probabilities for the various number of trains in the system. Also find the average waiting time of a new train coming into the yard.

Solution:

Here, $\lambda = 6$ and $\mu = 12$ so that $\rho = 6/12 = 1/2 = 0.5$.

The maximum queue length is 2, i.e., the maximum number of trains in the system is $3(= N)$.

The probability that there is no train in the system (both waiting and in service) is given by

$$P_\circ = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.5}{1-(0.5)^{3+1}} = 0.53.$$

Now, since

$$P_n = P_\circ \rho^n \text{, therefore}$$

$P_1 = (0.53)(0.5) = 0.27,\ P_2 = (0.53)(0.5)^2 = 0.13$, and

$P_3 = (0.53)(0.5)^3 = 0.07$

Hence, we get

$$E(n) = 1(0.27) + 2(0.13) + 3(0.07) = 0.74.$$

Thus the average number of trains in the system is 0.74 and each train takes on an average $\frac{1}{12} = (.08)$ hours for getting service. As the arrival of new train expects to find an average of 0.74 trains in the system before it.

$$E(w) = (0.74)(0.08)\,\text{hours} = 0.0592 \text{ hours or } 3.5 \text{ minutes.}$$

Example: 10.3.2

Assume that the goods trains are coming in a yard at the rate of 30 trains per day and suppose that the inter – arrival times follow an exponential distribution. The service time for each train is assumed to be exponential with an average of 36 minutes. If the yard can admit 9 trains at a time (there being 10 lines, one of which is reserved for shunting purposes), calculate the probability that the yard is empty and find the average queue length.

Solution:

We have

$$\lambda = \frac{30}{60x24} = \frac{1}{48} \text{ and } \mu = \frac{1}{16} \text{ trains per minute.}$$

$$\therefore \qquad \rho = \lambda/\mu = 36/48 = 0.75.$$

The probability that the yard is empty is given by

$$P_\circ = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.75}{1-(0.75)^{10}}, \text{ since } N=9$$

$$= \frac{0.25}{0.90} = 0.28.$$

Average queue length is given by

$$E(m) = \frac{\rho^2[1-N\rho^{N-1}+(N-1)\rho^N]}{(1-\rho)(1-\rho^{N+1})} = \frac{(0.75)^2[1-9(0.75)^8+8(0.75)^9}{0.25[(0.75)^{10}]}$$

$$= (2.22)\frac{(1-0.303)}{(1-0.005)} = (2.22)(0.70) = 1.55.$$

Example: 10.3.3

A barbershop has space to accommodate only 10 customers. He can service only one person at a time. If a customer comes to his shop and finds it full, he goes to the next shop. Customers randomly, arrive at an average rate $\lambda = 10$per hours and the barbers service time is negative exponential with an average of $\frac{1}{\mu} = 5$ minutes per customer. Find P_0, P_n.

Solution:

Here $N = 10,\ \lambda = \frac{10}{60},\ \mu = \frac{1}{5}$

$$\rho = \frac{\lambda}{\mu} = \frac{5}{6}$$

$$P_0 = \frac{1-\rho}{1-\rho^{11}} = \frac{1-5/6}{1-\left(\frac{5}{6}\right)^{11}}$$

$$= \frac{0.1667}{0.8655} = 0.1926$$

$$P_n = \left(\frac{1-\rho}{1-\rho^{N+1}} \right) \rho^n$$

$$= (0.1926) x \left(\frac{5}{6} \right)^n , n = 0,1,2 \ldots\ldots\ldots ,10$$

Example: 10.3.4

A car park contains 5 cars. The arrival of cars is Poisson at a mean rate of 10 per hour. The length of time each car spends in a car park is negative exponential distribution with mean of 2 hours. How many cars are in the care park on average?

Solution:

$$N = 5,\ \lambda = \frac{10}{60},\ \mu = \frac{1}{2x60},\ \rho = \frac{\lambda}{\mu} = 20$$

$$P_0 = \left(\frac{1-\rho}{1-\rho^{N+1}} \right)$$

$$= \frac{1-20}{1-20^6} = \frac{-19}{-6399} = 2.962x10^{-7}$$

$$E(n) = P_0 \sum_{n=0}^{N} n\rho^n$$

$$= (2.9692x10^{-3}) x \sum_{n=0}^{5} n(2.9692x10^{-3})^n$$

$$= (2.9692x10^{-3}) x [0 + (2.9692x10^{-3})$$

$$+ 2x(2.9692x10^{-3})^2$$

$$+ 3x(2.9692x10^{-3})^3$$

$$+4x(2.9692x10^{-3})^{4}$$

$$+5x(2.9692x10^{-3})^{5}$$

$$=(2.9384x10^{-3})x[0+(2.9692x10^{-3})$$

$$+2x(2.9692x10^{-3})$$

$$+3x(2.9692x10^{-3})^{2}$$

$$+4x(2.9692x10^{-3})^{3}$$

$$+5x(2.9692x10^{-3})^{4}$$

$$=5(\text{app}).$$

Check your progress: 10.2

1. A stenographer has 5 persons for whom she performs stenographer work. Arrival rate is Poisson and service time is exponential. Average arrival rate is 4 per hour with an average service time of 10 minutes. Find

(i) The average waiting time of an arrival

(ii) The average length of waiting line

(iii) The average time an arrival spends in the system.

2. Consider a single server queueing system with Poisson input, exponential service times. Suppose the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 hours and the maximum permissible calling units in the system is two. Calculate the expected number in the system.

3. At a railway station, only one train is handled at a time. The railway yard is sufficient only for two trains to wait while the other is given signal to leave the station. Trains arrive at

the station at an average rate of 6 per hour and railway station can handle them on an average 12 per hour. Assuming Poisson arrivals and exponential service distribution, find the steady – state probabilities of various number of trains in the system. Also find the average waiting time of a new train coming into the yard.

10.4 Key words:

Queuing System, Size of the queue, Queue Discipline, Single queue – one server, Several queues – one server.

10.6 Answers to check your progress questions:

Check your progress: 10.1

1. (i) 5 customers in the shop and approximately 4 customers waiting for hair cut. (ii) 16.7 % (iii) 83.3 %
2. Average waiting time in queue = 125 seconds
   Average waiting time in system = 225 seconds
3. (i) 40% (ii) 5 customers (iii) 15 minutes (iv) 4.167 (or) 4 customers (approx.), (v) 12.5 minutes.

Check your progress: 10.2

1. (i) 12.4 minutes (ii) 0.79 $\approx$ one (iii) 22.4 minutes.
2. 0.81
3. $P = 0.27$, $P_o = 0.53$, $P_2 = 0.13$, $P_3 = 0.07$, $E(W) = 3.8$

10.6 Model Questions:

1. A foreign bank is considering opening a drive – in window for customer service. Management estimates that customers will arrive for service at the rate of 12 per hour. The teller whom it is considering to staff the window can serve customers at the rate of one every three minutes. Assuming Poisson arrivals and Exponential service, find:

   (i) Utilization of teller,
   (ii) Average number in the system,
   (iii) Average waiting time in the line, and
   (iv) Average waiting time in the system.

2. A super market has a single cashier. During the peak hours, customers arrive at a rate of 20 customers per hour. The average number of customers that can be processed by the cashier is 24 per hour. Calculate:
   (i) The probability that the cashier is idle,
   (ii) The average number of customers in the queueing system,
   (iii) The average time a customer spends in the system,
   (iv) The average time a customer in the queue, and
   (v) The average time a customer spends in the queue waiting for service.
3. Cars arrive at a petrol pump with exponential inter – arrival times having mean ½ minute. The attendant takes an average of 1/5 minute per car to supply petrol, the service times being exponentially distributed. Determine (i) the average number of cars waiting to be served, (ii) the average number of cars in the queue, and (iii) the proportion of time for which the pump attendant is idle.
4. An airlines organization has one reservation clerk on duty in its local branch at any given time. The clerk handles information regarding passenger reservation and flight timings. Assume that the number of customers arriving during any given period is Poisson distribution with an arrival rate of eight per hour and that the reservation clerk can serve a customer in six minutes on an average, with an exponentially distributed service time.

Space for Hints

(i) What is the probability that the system is busy?

(ii) What is the average time a customer spends in the system?

(iii) What is the average length of the queue and what is the number of customers in the system?

5. At what average rate must a clerk at a super marked work in order to ensure a probability of 0.90 that the customer will not have to wait longer than 12 minutes? It is assumed that there is only one counter, to which customers arrive in a Poisson fashion at an average rate of 15 per hour. The length of service by the clerk has an exponential distribution.

6. Customers arrive at a one – window drive according to a Poisson distribution with mean of 10 minutes and service time per customer is exponential with mean of 6 minutes. The space in front of the window can accommodate only three vehicles including the serviced one. Other vehicles have to wait outside this space. Determine the;

(i) Probability that an arriving customer can drive directly to the space in front of the window,

(ii) Probability that an arriving customer will have to wait outside the direct space.

(iii) How long an arriving customer is expected to wait before getting the service?

7. Consider a single server queueing system with Poisson input, exponential service times. Suppose the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 hours and the maximum permissible number calling units in the system is two. Drive the steady – state probability distribution of the number of calling units in the system, and then calculate the expected number in the system.

Space for Hints

8. If for a period of 2 hours in the day (8 to 10 a.m.) trains arrive at the yard every 20 minutes but the service time continues to remain 36 minutes, then calculate for this period: (a) the probability that the yard is empty; (b) average number of trains in the system; on the assumption that the line capacity of the yard is limited to 4 trains only.

9. (a) Patients arrive at a clinic according to a Poisson distribution at a rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate 20 per hour.

   (i) Find the effective arrival rate at the clinic.

   (ii) What is the probability that an arriving patient will not wait?

   (iii) What is the expected waiting time until a patient is discharged form the clinic?

(b) In a car – wash service facility, cars arrive for service according to a Poisson distribution with mean 5 per hour. The time for washing and cleaning each car varies but is found to follow an exponential distribution with mean 10 minutes per car. The facility cannot handle more than one care at a time and has a total of 5 parking spaces.

   i. Find the effective arrival rate.

   ii. What is the probability that an arriving car will get service immediately upon arrival?

   iii. Find the expected number of parking spaces occupied.

10. A petrol station has a single pump and space for not more than 3 cars (2 waiting, 1 being served). A car arriving when the space is filled to capacity goes elsewhere for petrol. Cars arrive according to a Poisson distribution at a mean rate of one every 8 minutes. Their service time has an exponential distribution with a mean of 4 minutes.