



MADURAI KAMARAJ UNIVERSITY

(University with Potential for Excellence)

DISTANCE EDUCATION



B.Sc., (Mathematics)

THIRD YEAR

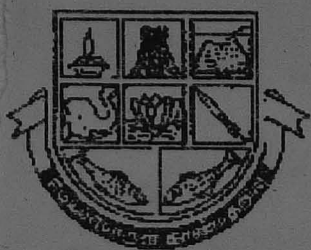
GRAPH THEORY

UNIT : 6-10 (VOLUME - 2)

PAPER - 1

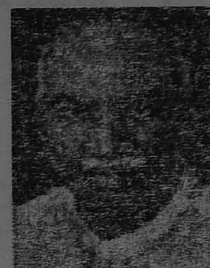
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Ancillary Paper- I

GRAPH THEORY

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B.Sc., Mathematics

Third Year

Dear Student,

We welcome you as a Student of the Third year B.Sc degree Course in Mathematics.

This Ancillary paper – I deals with Graph Theory. The learning material for this paper will be supplemented by Contact seminars.

Learning through the Distance Education mode, as you are all aware, involves self – learning and self – assessment and in this regard you are expected to put in disciplined and dedicated effort.

On our part, we assume of our guidance and support.

With best wishes.

S. Ramachandran, New Gamma, June 2001.

GRAPH THEORY

Ancillary Paper - I

SYLLABUS

B.Sc., Mathematics – Third year.

Ancillary Paper – 1

Graph theory

UNIT – 1:

Introduction – The konigsberg Bridge Problem - Four Colour Problem -- Graphs - Definition and Examples with pictorial Representation – Sub graphs – Isomorphism – Degrees.

Walks and Connected graphs – Cycles in Graphs.

UNIT – 3:

Eulerian Graphs – Fleury's Graphs.

UNIT – 4:

Hamiltonian Graphs – Weighted Graphs.

UNIT – 5:

Bipartite Graphs – Marriage Problem – Trees – Connector Problem

UNIT – 6:

Matrix representation – Vector Spaces Associated with Graphs

UNIT – 7:

Cycle space – Cut – Set space

UNIT – 8:

Planar Graphs – Euler formula – Platonic Solids – Dual of plane graph – characterization of Planar graphs.

UNIT – 9:

Colourings – Vertex Colouring – Edge Colouring – An Algorithm for colouring of a graph.

UNIT – 10:

Directed Graphs – Connectivity in digraphs – Strong – Orientation of Graphs – Eulerian digraphs – Tournaments.

Text Book:

1. “Invitation to Graph Theory” by Dr. S. Arumugam and S. Ramachandran, New Gamma, June 2001.

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Adjacency matrix

Let G be a simple graph of n vertices $v_1, v_2, \dots, v_n$.
 Then the $n \times n$ matrix $A(G) = (a_{ij})$ is called the adjacency matrix
 of G where

$$a_{ij} = 1 \text{ if } v_i \text{ is adjacent to } v_j$$

$$= 0 \text{ otherwise}$$

Example:

A graph G and its adjacency matrix are given below.

UNIT - 6

MATRIX REPRESENTATION

6.0 Introduction:

The study of representation of graphs by matrices and vector spaces is both practical and theoretical. Practical because graphs are stored as matrices in computers; theoretical because these representations provide us an opportunity to use already well founded algebraic techniques.

An elementary knowledge of linear algebra is a prerequisite for this unit. Mathematically matured reader, having the knowledge of finite dimensional vector space, will probably anticipate the theorems and proofs before they are stated.

6.1 Matrix Representation:

Essentially, there are two ways of representing a graph by a matrix: through adjacency matrix and through incidence matrix.

Definition: 6.1.1

Adjacency matrix

Let G be a simple graph of n vertices $v_1, v_2, \dots, v_n$. Then the $n \times n$ matrix $A(G) = (a_{ij})$ is called the **adjacency matrix** of G where

$$a_{ij} = 1 \text{ if } v_i \text{ is adjacent to } v_j$$

$$= 0 \text{ otherwise}$$

Example:

A graph G and its adjacency matrix are given below.

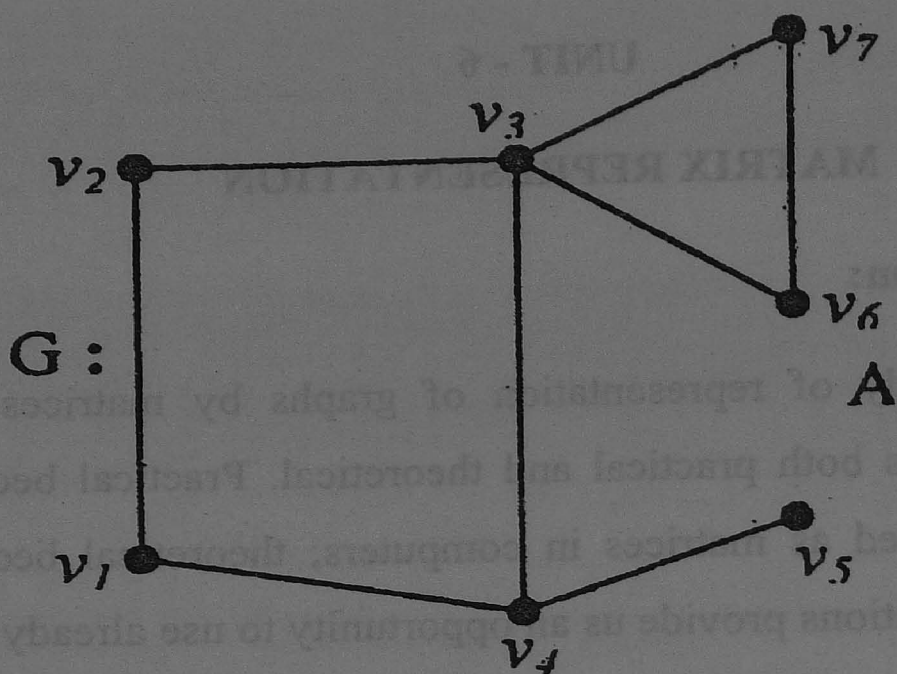


Figure 6.1

A(G):

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	0	1	0	1	0	0	0
v_2	1	0	1	0	0	0	0
v_3	0	1	0	1	0	1	1
v_4	1	0	1	0	1	0	0
v_5	0	0	0	1	0	0	0
v_6	0	0	1	0	0	0	1
v_7	0	0	1	0	0	1	0

Remarks:

1. $A(G)$ is a square matrix of order n .
2. Row headings and column headings are the vertices taken in the same order.
3. It is a binary matrix with 0 and 1 as entries.
4. The diagonal elements are all zero.
5. It is a symmetric matrix.
6. Any row total or column total is the degree of the corresponding vertex.
7. Corresponding to every simple graph there is an adjacency matrix and conversely a matrix of the form $A(G)$ determines a simple graph up to isomorphism.

8. If the graph G is disconnected with components G_1 and G_2 then the adjacency matrix $A(G)$ is partitioned as follows.

$$A(G) = \begin{pmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{pmatrix}$$

where $A(G_1)$ and $A(G_2)$ are adjacency matrices of the components G_1 and G_2 and O are null matrices.

Note:

(1) From the above we observe there is no path connecting vertices of G_1 and vertices of G_2 , which should be the case as G_1 and G_2 are components of a disconnected graph G .

(2) $A(G)$ is also called a **bit matrix** or a **Boolean matrix**.

Theorem: 6.1

Two simple graphs G_1 and G_2 are isomorphic if and only if their adjacency matrices A_1 and A_2 are related by $A_1 = P^{-1} A_2 P$ where P is a permutation matrix.

Proof:

Given that A_1 and A_2 are adjacency matrices of the graphs G_1 and G_2 such that

$$A_1 = P^{-1} A_2 P$$

This shows that A_1 and A_2 are similar matrices and so they are square matrices of the same order. So one can be obtained from the other by interchange of rows and the interchange of corresponding columns. In doing so the effect is a relabelling of the vertices and so the graphs are isomorphic.

Space for Hints

Conversely, if the graphs G_1 and G_2 are isomorphic their structure is the same except for the labelling of their vertices. They have the same number of vertices and same number of edges. So the adjacency matrices of A_1 and A_2 are square matrices of the same order and so they are similar.

$$\therefore A_1 = P^{-1} A_2 P$$

where P is a permutation matrix.

Theorem: 6.2

Let A be the adjacency matrix of a simple graph G . Then the $(i, j)^{\text{th}}$ entry in A^2 is the number of different paths of length 2 between the i^{th} and j^{th} vertices of G . Also the i^{th} diagonal elements is the degree of the i^{th} vertex.

Proof:

Let G be a simple graph of n vertices $v_1, v_2, \dots, v_n$ so that the adjacency matrix $A = (a_{ij})$ is a square matrix of order n given by

$$a_{ij} = 1 \text{ if } v_i \text{ is adjacent to } v_j$$

$$= 0 \text{ otherwise.}$$

Since A is a square matrix of order n , A^2 exists and it is also a square matrix of order n .

Now the $(i, j)^{\text{th}}$ entry of A^2

$$= \text{Dot product of } i^{\text{th}} \text{ row and } j^{\text{th}} \text{ column}$$

$$= \text{Dot product of } i^{\text{th}} \text{ row and } j^{\text{th}} \text{ row (by symmetry)}$$

- = Number of 1's in the dot product is the number of positions in which both i^{th} and j^{th} rows have 1's
- = Number of vertices which are adjacent to both i^{th} and j^{th} vertices.
- = Number of different paths of length 2 between the i^{th} and j^{th} vertices

Again the i^{th} diagonal element that is the $(i,i)^{\text{th}}$ entry in A^2 is the dot product of the i^{th} row by itself. So it is the number of 1's in the i^{th} row of the matrix A . Therefore it is the number of edges incident on the i^{th} vertex. Hence it is the degree of the i^{th} vertex.

Theorem: 6.3

Let $A = [a_{ij}]$ be the adjacency matrix of a simple graph G then the entry $(i,j)^{\text{th}}$ in $[A^m]_{ij}$ is the number of walks of length m between the i^{th} and j^{th} vertices.

Proof:

We prove the theorem by induction on m .

When $m = 1$, the result is true by definition of the adjacency matrix A .

When $m = 2$, the result is proved in the previous theorem.

Space for Hints

Let us assume that the result is true for m and we prove it for $m+1$. That is in A^m the $(i, j)^{\text{th}}$ entry $[A^m]_{ij}$ is the number of walks of length m between v_i and v_j . We have to prove that the $(i, j)^{\text{th}}$ entry $[A^{m+1}]_{ij}$ in A^{m+1} is the number of walks of length $(m+1)$ between v_i and v_j .

We have

$$A^{m+1} = A^m \cdot A$$

$$\therefore [A^{m+1}]_{ij} = \sum_{k=1}^n [A^m]_{ik} a_{kj}$$

Take

$$[A^m]_{ik} \cdot a_{kj} = \begin{cases} [A^m]_{ik} & \text{if } a_{kj} = 1, \text{ that is if } v_k \text{ and } v_j \text{ are adjacent} \\ 0 & \text{if } a_{kj} = 0, \text{ that is if } v_k \text{ and } v_j \text{ are non adjacent.} \end{cases}$$

By induction hypothesis $[A^m]_{ik}$ is the number of walks of length m between v_i and v_k . If v_k and v_j are adjacent, then the above walks can be made into walks of length $m+1$ between v_i and v_j .

$\therefore [A^{m+1}]_{ij}$ is the number of walks of length $m+1$ between v_i and v_j .

That is the $(i, j)^{\text{th}}$ entry in A^{m+1} is the number of walks of length $(m+1)$ between i^{th} and j^{th} vertices.

Hence the theorem holds.

Theorem: 6.4

The simple graph G of n vertices is connected if and only if $(A + I)^{n-1}$ has no zero entries, where A is the adjacency matrix of G and I is the identity matrix of order n .

Proof:

A being the adjacency matrix of the simple graph G of n vertices, is a square matrix of order n. Also the identity matrix I is a square matrix of order n.

$\therefore A + I$ exists and is a square matrix of order n. So integral powers of $(A + I)$ are defined.

Now

$$(A + I)^{n-1} = A^{n-1} + \binom{n-1}{1} A^{n-2} I + \binom{n-1}{2} A^{n-3} I^2 + \dots + I^{n-1}$$

i.e.,

$$(A + I)^{n-1} = A^{n-1} + \binom{n-1}{1} A^{n-2} + \binom{n-1}{2} A^{n-3} + \dots + I \quad (1)$$

Since all the diagonal elements of the last term I are 1, the diagonal elements of the sum in the RHS must be atleast 1. So the diagonal elements $(A + I)^{n-1}$ are all non zero, that is $[(A + I)^{n-1}]_{ii} \neq 0$. Therefore it is enough we discuss the nature of the off diagonal entries of $(A + I)^{n-1}$, that is $[(A + I)^{n-1}]_{ij}$, $i \neq j$

From (1)

$$[(A + I)^{n-1}]_{ij} = [A^{n-1}]_{ij} + \binom{n-1}{1} [A^{n-2}]_{ij} + \binom{n-1}{2} [A^{n-3}]_{ij} + \dots + [I]_{ij}$$

Suppose G is connected. Let $1 \leq i, j \leq n$ and $i \neq j$. By previous theorem the $(i, j)^{\text{th}}$ element of A^m gives the number of walks of length m between the i^{th} and j^{th} vertices. Since G is

connected there is a path (walk) of length $m \leq n-1$. So $[A^m]_{ij} \neq 0$ for $m \leq n-1$.

$$\therefore [(A + I)^{n-1}]_{ij} \neq 0$$

$$\text{Already } [(A + I)^{n-1}] \neq 0$$

Therefore $(A + I)^{n-1}$ has no zero entry.

Conversely, let $(A + I)^{n-1}$ have no zero entries

$$[(A + I)^{n-1}]_{ij} \neq 0 \text{ for all } i, j \leq n$$

By (1) there exists $m \leq n-1$ such that $[A^m]_{ij} \neq 0$. Hence there is a walk of length m between v_i and v_j . As v_i and v_j are arbitrary it follows that there is a walk (path) between every pair of vertices of G . Hence G is connected.

Theorem: 6.5

If G is a connected graph, then the distance between v_i and v_j is the smallest integer $n(\geq 0)$ such that $[A^n]_{ij} \neq 0$

Proof:

Let n be the smallest integer such that $[A^n]_{ij} \neq 0$. By the minimality of n , $[A^m]_{ij} = 0$, for all m , $0 \leq m \leq n-1$. So, by theorem 6.3, there is no walk of length $\leq n-1$, connecting v_i and v_j ; hence $d(v_i, v_j) \geq n$. Since $[A^n]_{ij} \neq 0$, there does exist a walk, say $W(v_i, v_j)$, from v_i to v_j . Since $[A^m]_{ij} = 0$, for all m , $0 \leq m \leq n-1$, $W(v_i, v_j)$ is a walk of minimum length and hence it is a path (by theorem). So $d(v_i, v_j) \leq n$. Combining the two inequalities we get $d(v_i, v_j) = n$.

Definition: 6.1.2

Incidence Matrix:

Let G be a (p, q) graph with vertices $v_1, v_2, \dots, v_p$ edge $e_1, e_2, \dots, e_q$ and without self loops. Then the $p \times q$ matrix $B(G) = (b_{ij})$ is called the **incidence matrix** of G , where

$b_{ij} = 1$ if the vertex v_i and the edge e_j are incident on each other
 $= 0$ otherwise.

Example:

A graph G and its incidence matrix given below

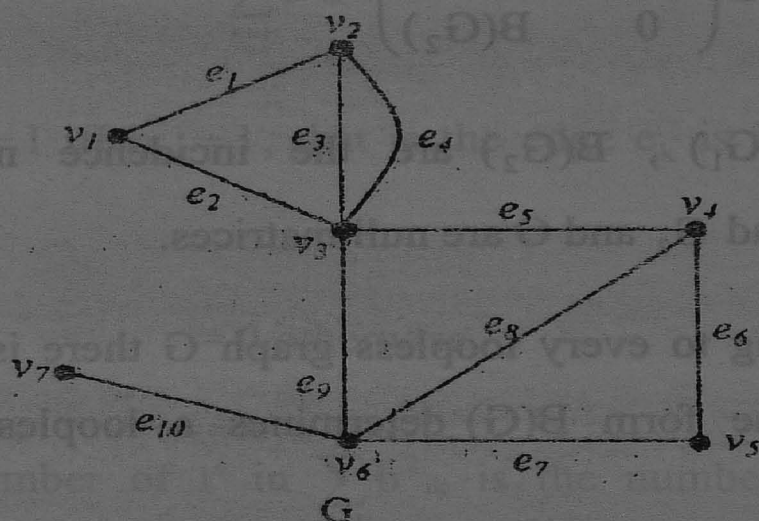


Figure 6.2

[illegible]

Remarks:

- (1) $B(G)$ is a matrix of order $p \times q$
- (2) As the entries are only 0 and 1, it is a binary matrix.
- (3) The row headings are the vertices and the column headings are the edges.
- (4) As every edge is incident exactly on two vertices there are exactly two 1's in each column and so the column total is 2.
- (5) Any row total is the degree of the corresponding vertex.
- (6) Parallel edges give identical column in the matrix.
- (7) If the graph is simple, no two columns of the matrix are identical.
- (8) If a graph G is disconnected with components G_1 and G_2 then the incidence matrix $B(G)$ is partitioned as follows.

$$B(G) = \begin{pmatrix} B(G_1) & 0 \\ 0 & B(G_2) \end{pmatrix}$$

Where $B(G_1)$, $B(G_2)$ are the incidence matrix of the components G_1 and G_2 and O are null matrices.

- (9) Corresponding to every loopless graph G there is an incidence matrix of the form $B(G)$ determines a loopless graph upto isomorphism.

Theorem: 6.6

If B is the incidence matrix of a graph G and B^T its transpose then the i^{th} diagonal entry of BB^T is the degree of the vertex v_i

Proof:

Let $B = (b_{ij})_{q \times p}$ be the incidence matrix of the graph G .

Its transpose $B^T = (b'_{ji})_{p \times q}$ where $b'_{ji} = b_{ij}$.

Therefore BB^T exists and is a square matrix of order $p \times p$.

Now $BB^T = (C_{ij})$

where $C_{ij} = \sum_{k=1}^q b_{ik} b'_{kj}$

$$= \sum_{k=1}^q b_{ik} b_{jk}$$

The i^{th} diagonal element of BB^T is

$$C_{ii} = \sum_{k=1}^q b_{ik} b_{ik}$$

$$= \sum_{k=1}^q b_{ik}^2$$

Now $b_{ik}^2 = 1$ iff $b_{ik} = 1$, that is the edge e_k is incident on the vertex v_i

$= 0$ otherwise.

So the number of 1's in $\sum_{k=1}^q b_{ik}^2$ is the number of edges incident on the vertex v_i

$\therefore e_v$ is the number of edges incident on the vertex v_i and so is the degree of the vertex v_i .

6.2 Operation on graphs:

Since a graph is basically a set of vertices and a set of edges, the usual set operations are also applicable to graphs.

Definition: 6.2.1

Union:

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs, then their union denoted by $G_1 \cup G_2$ is the graph $(V_1 \cup V_2, E_1 \cup E_2)$.

Example:

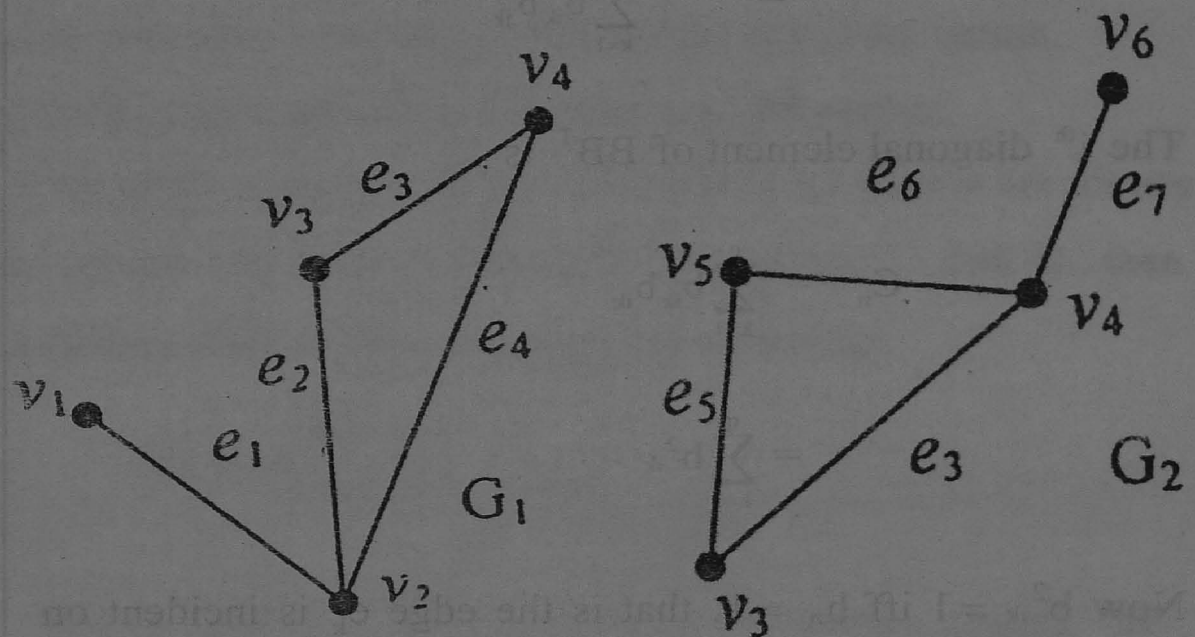


Figure 6.3

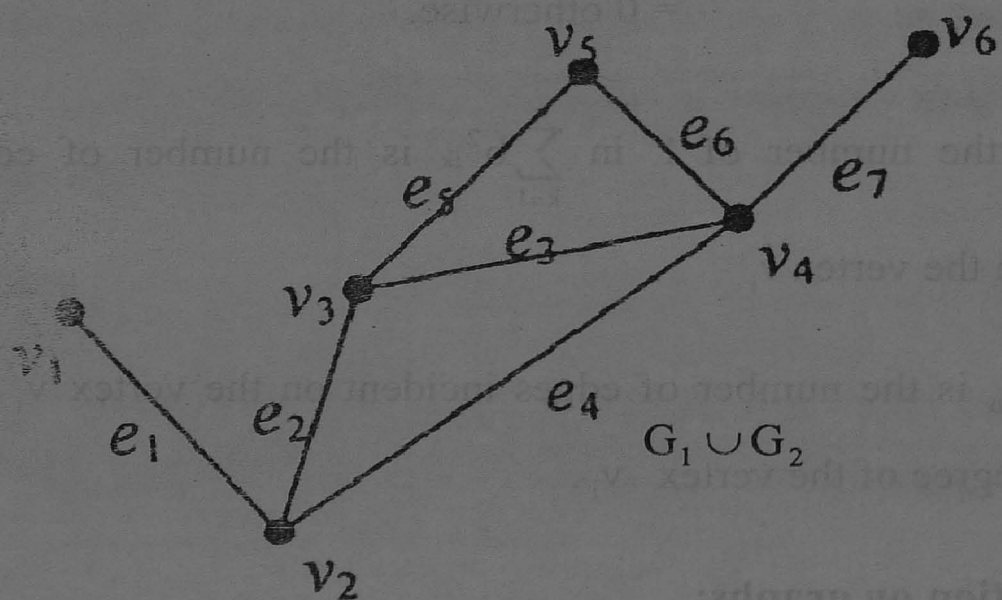


Figure 6.3(a)

$$V_1 = \{v_1, v_2, v_3, v_4\}$$

$$E_1 = \{e_1, e_2, e_3, e_4\}$$

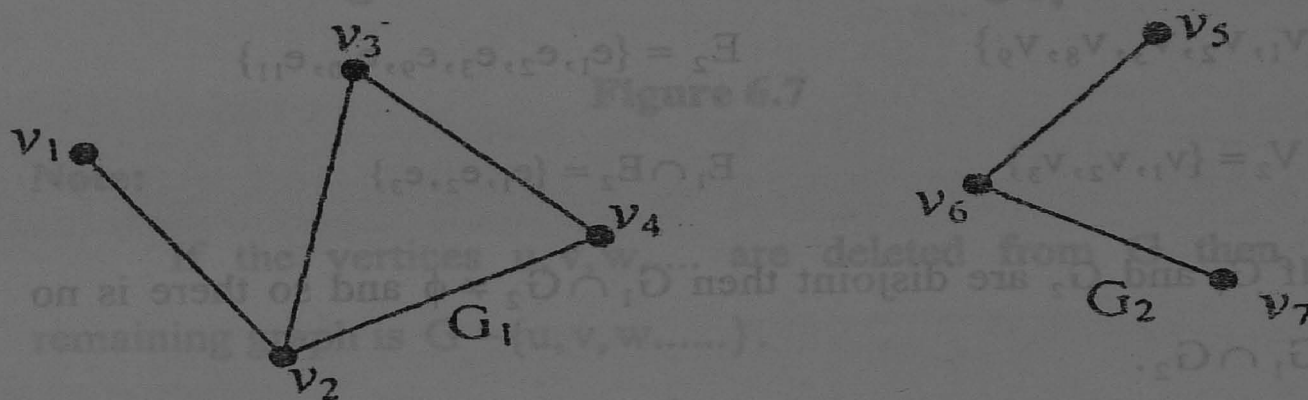
$$V_2 = \{v_3, v_4, v_5, v_6\}$$

$$E_2 = \{e_3, e_5, e_6, e_7\}$$

$$\therefore V_1 \cup V_2 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

$$E_1 \cup E_2 = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$$

If G_1 and G_2 are disjoint (vertex disjoint) then their union is given by the distinct graphs G_1 and G_2 and is denoted by $G_1 + G_2$.



$G_1 + G_2$

Figure 6.4

Definition: 6.2.2

Intersection

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs, then their **intersection** denoted by $G_1 \cap G_2$ is the graph $(V_1 \cap V_2, E_1 \cap E_2)$.

Example:

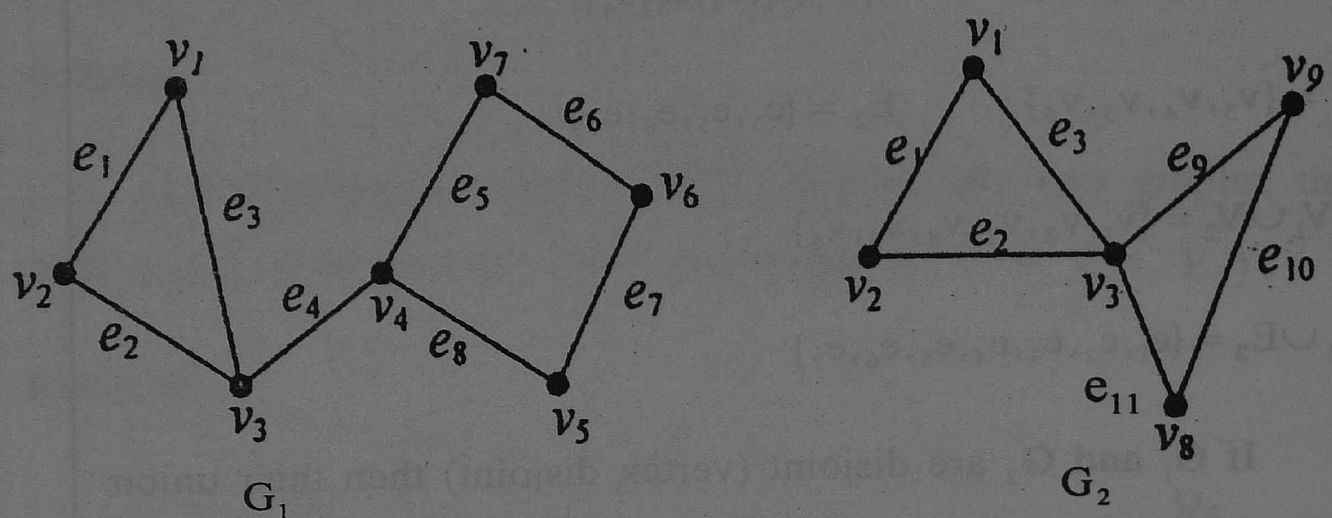


Figure 6.5

$$V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\} \quad E_1 = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$$

$$V_2 = \{v_1, v_2, v_3, v_8, v_9\} \quad E_2 = \{e_1, e_2, e_3, e_9, e_{10}, e_{11}\}$$

$$\therefore V_1 \cap V_2 = \{v_1, v_2, v_3\} \quad E_1 \cap E_2 = \{e_1, e_2, e_3\}$$

If G_1 and G_2 are disjoint then $G_1 \cap G_2 \neq \emptyset$ and so there is no graph $G_1 \cap G_2$.

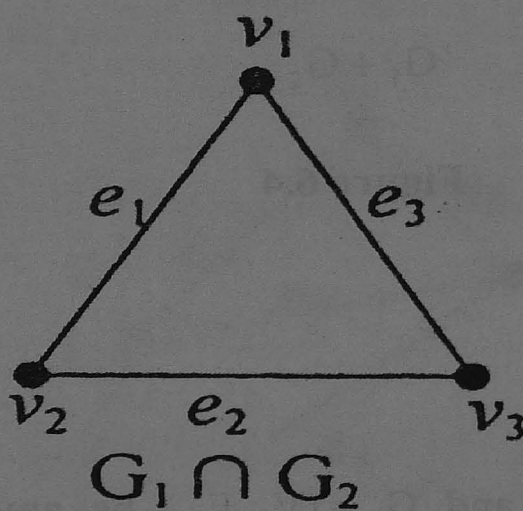


Figure 6.6

Definition: 6.2.3

Deletion of a vertex:

Let G be any graph. If a vertex v and all edges incident on it are deleted then remaining graph is denoted by $G - v$

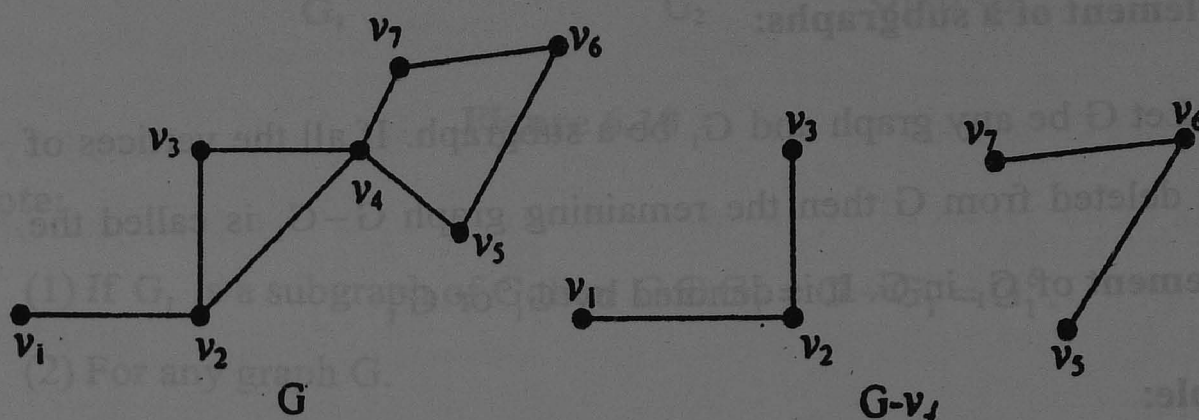


Figure 6.7

Note:

If the vertices $u, v, w, \dots$ are deleted from G then the remaining graph is $G - \{u, v, w, \dots\}$.

Definition: 6.2.4

Deletion of an edge

Let G be any graph and e an edge. If the edge e is deleted without deleting its end vertices then the remaining graph is denoted by $G - e$.

Example:

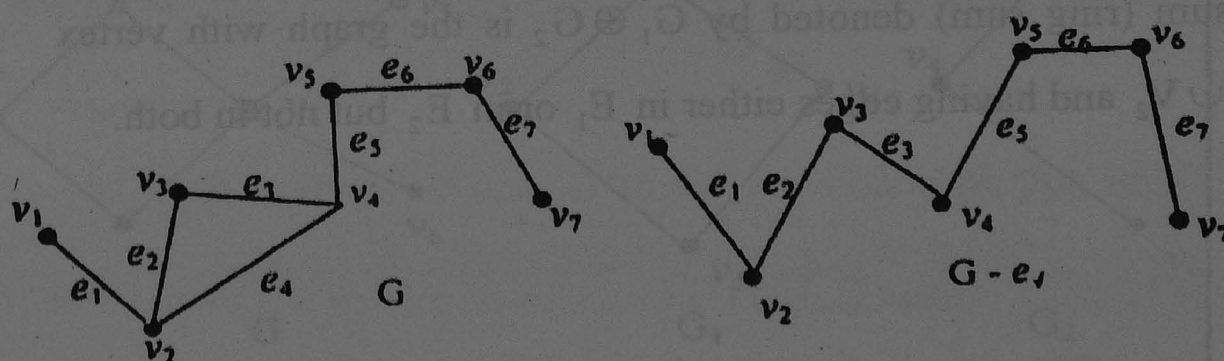


Figure 6.8

Note:

If the edges $e_1, e_2, e_3, \dots$ are deleted from G then the remaining graph is $G - \{e_1, e_2, e_3, \dots\}$.

Definition: 6.2.5

Complement of a subgraphs:

Let G be any graph and G_1 be a subgraph. If all the vertices of G_1 are deleted from G then the remaining graph $G - G_1$ is called the **complement** of G_1 in G . It is denoted by G_1^c or G'_1 .

Example:

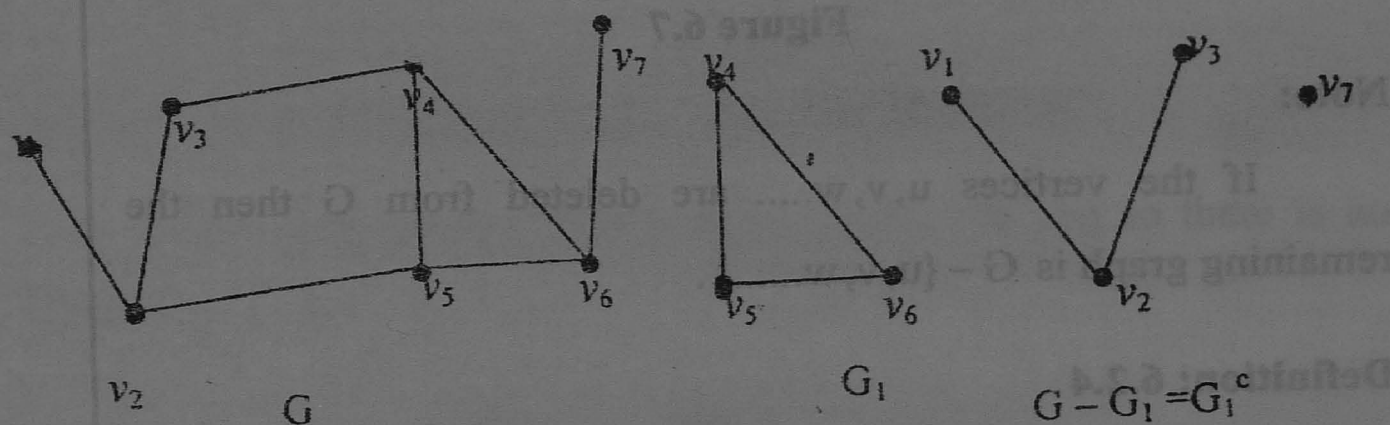
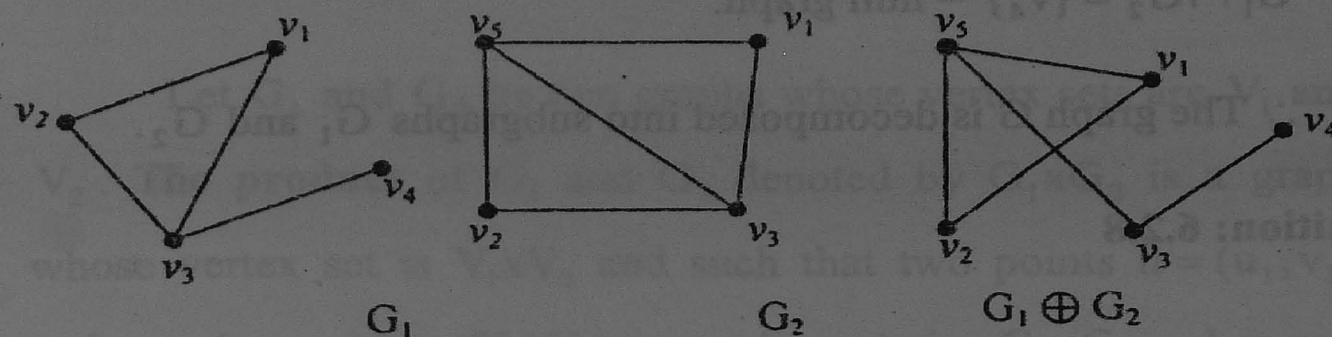


Figure 6.9

Definition: 6.2.6

Direct sum:

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs. Their direct sum (ring sum) denoted by $G_1 \oplus G_2$ is the graph with vertex set $V_1 \cup V_2$ and having edges either in E_1 or in E_2 but not in both.

Example:**Figure 6.10****Note:**

(1) If G_1 is a subgraph of G then $G \oplus G_1 = G - G_1 = G_1^c$.

(2) For any graph G .

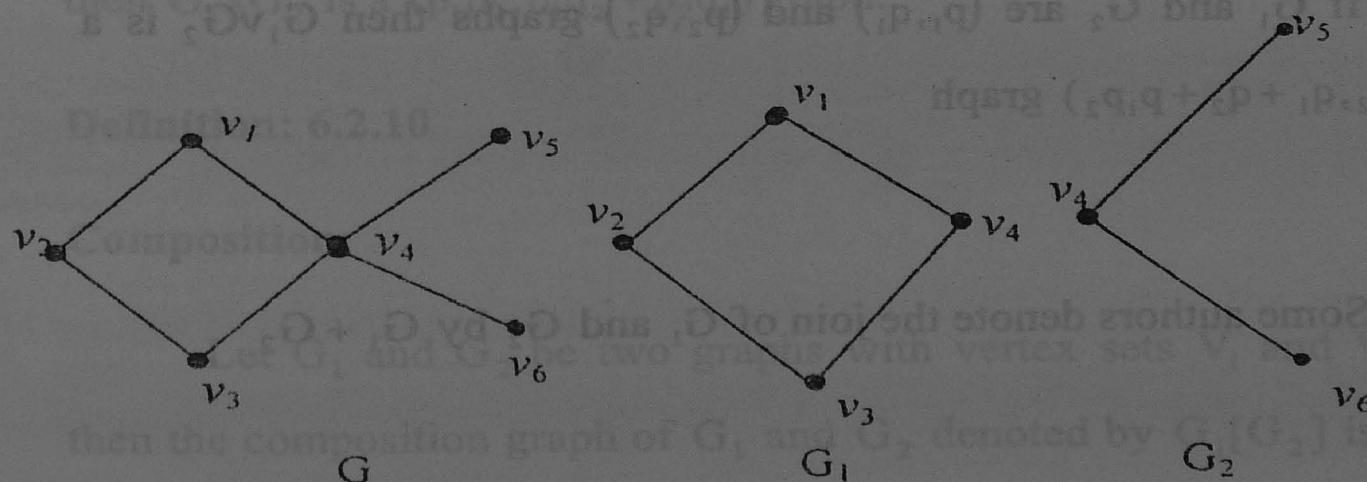
$$G \cup G = G \cap G = G$$

(3) For any graph G

$G \oplus G$ is a null graph consisting of all the vertices of G .

Definition: 6.2.7**Decomposition:**

Let G be any graph. It is said to be **decomposed** into subgraphs G_1 and G_2 if $G_1 \cup G_2 = G$ and $G_1 \cap G_2$ is a null graph

Examples:**Figure 6.11**

Clearly $G_1 \cup G_2 = G$

$G_1 \cap G_2 = \{v_4\} = \text{null graph.}$

$\therefore$ The graph G is decomposed into subgraphs G_1 and G_2 .

Definition: 6.2.8

Join:

Let G_1 and G_2 be two distinct graphs. The graph obtained by joining every vertex of G_1 to every vertex of G_2 is called the **join** of the graphs G_1 and G_2 and it is denoted by $G_1 \vee G_2$.

Example:

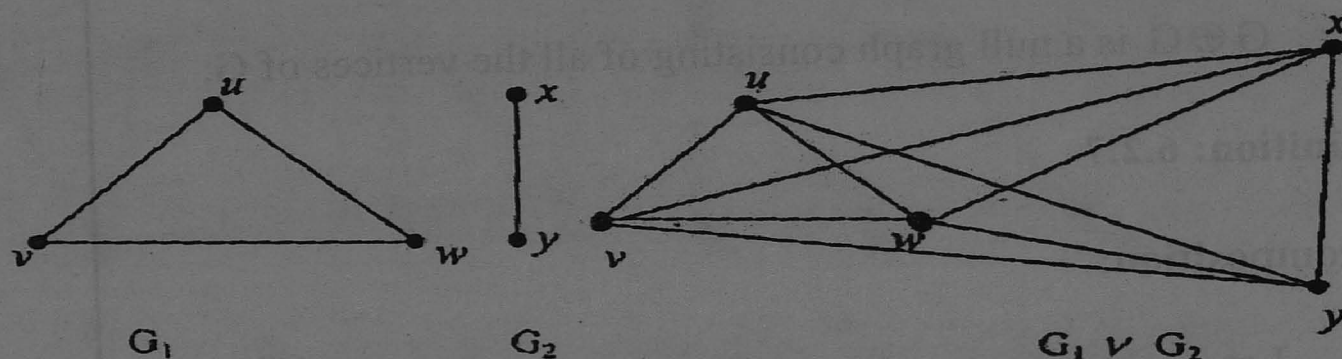


Figure 6.12

Remarks:

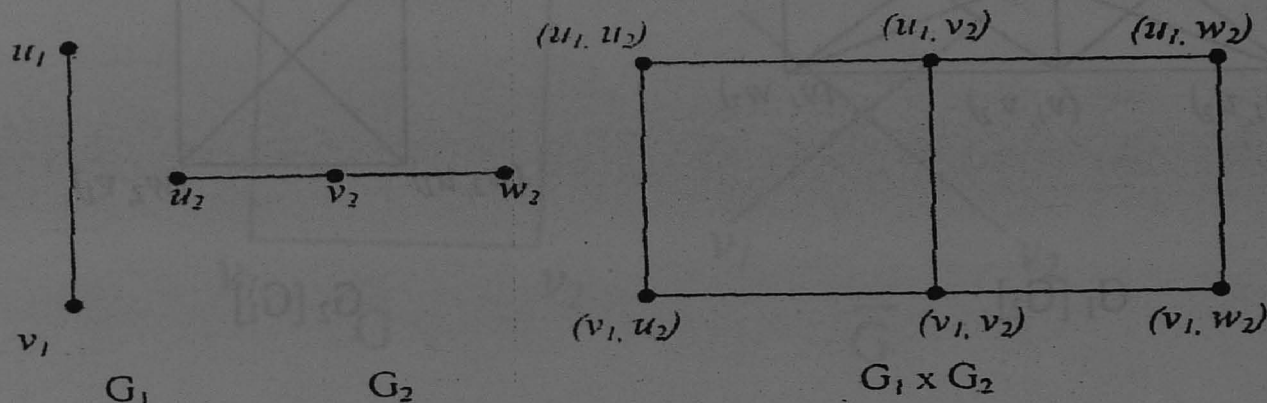
If G_1 and G_2 are (p_1, q_1) and (p_2, q_2) graphs then $G_1 \vee G_2$ is a $(p_1 + p_2, q_1 + q_2 + p_1 p_2)$ graph

Note:

Some authors denote the join of G_1 and G_2 by $G_1 + G_2$

Definition: 6.2.9**Product:**

Let G_1 and G_2 be two graphs whose vertex sets are V_1 and V_2 . The **product** of G_1 and G_2 denoted by $G_1 \times G_2$ is a graph whose vertex set is $V_1 \times V_2$ and such that two points $u = (u_1, v_1)$ and $v = (u_2, v_2)$ in $V_1 \times V_2$ are adjacent in $G_1 \times G_2$ whenever $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1 .

Example:**Figure 6.13****Remarks:**

If G_1 and G_2 are respectively (p_1, q_1) and (p_2, q_2) graphs then $G_1 \times G_2$ is a $(p_1 p_2, p_1 q_2 + p_2 q_1)$ graph.

Definition: 6.2.10**Composition:**

Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2 then the composition graph of G_1 and G_2 denoted by $G_1[G_2]$ is a graph whose vertex set is $V_1 \times V_2$ and such that the vertices

$u = (u_1, u_2)$ and $v = (v_1, v_2)$ of $V_1 \times V_2$ are adjacent whenever u_1 is adjacent to v_1 or $[u_1 = v_1 \text{ and } u_2 \text{ is adjacent to } v_2]$. Similarly we get the composition graph $G_2[G_1]$.

Example:

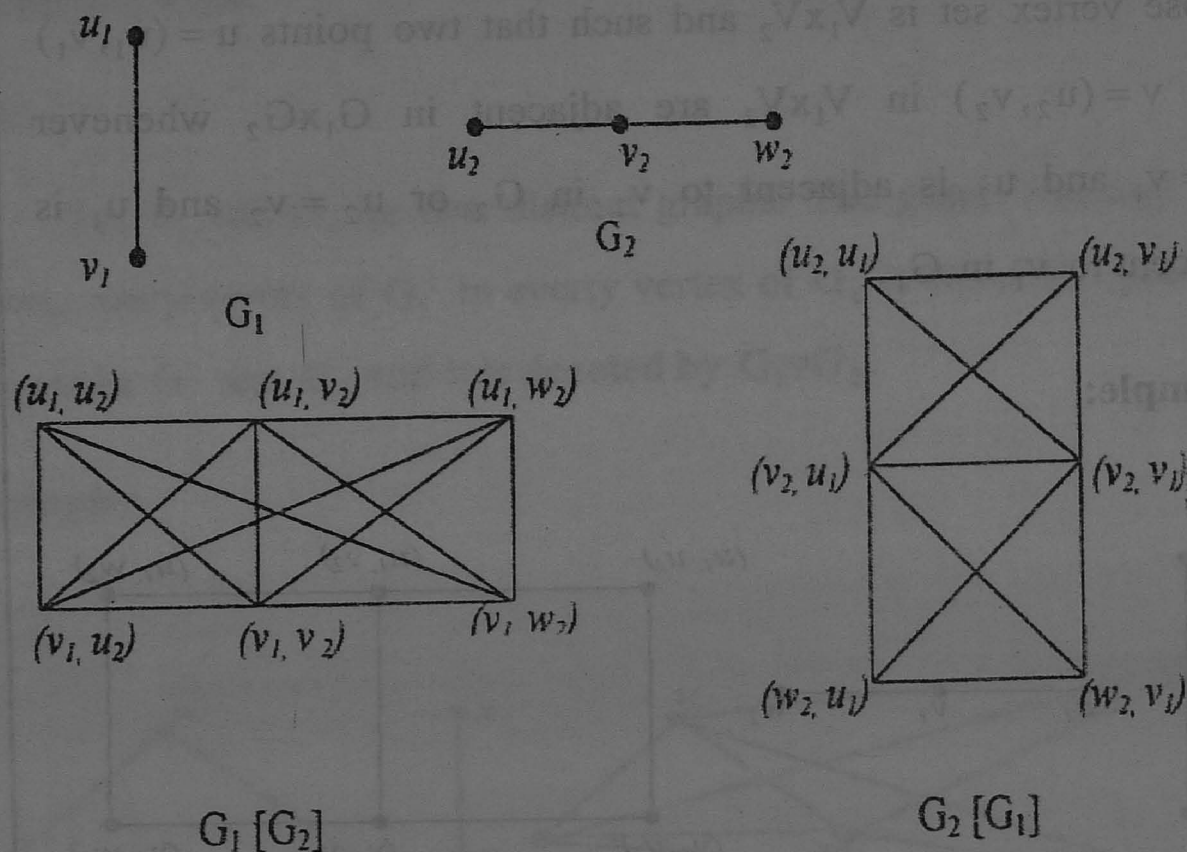


Figure 6.14

Remark:

If G_1 and G_2 are (p_1, q_1) and (p_2, q_2) graphs then,

$G_1[G_2]$ is a $(p_1 p_2, p_1 q_2 + p_2^2 q_1)$ graph

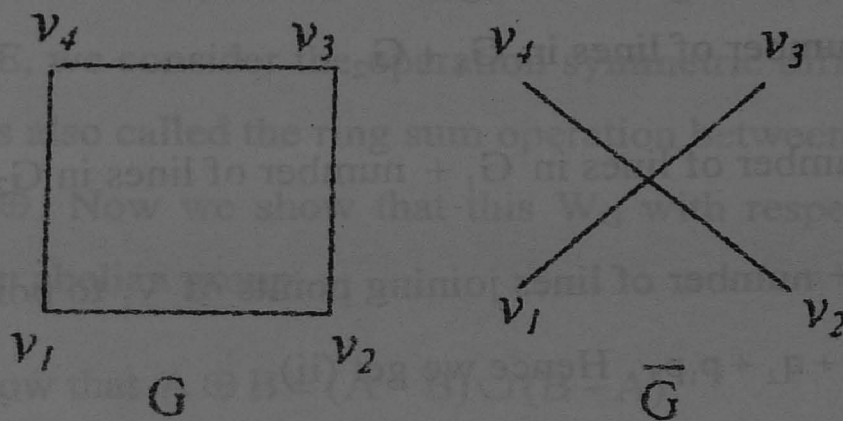
and $G_2[G_1]$ is a $(p_1 p_2, p_2 q_1 + p_1^2 q_2)$ graph.

Note:

There are some special operations on graphs and they will be discussed in the appropriate places.

Definition: 6.2.11**Complement graphs:**

Let G be any simple graph, then the graph $\bar{G}$ is called the **complement graph** of G if the vertex set of $\bar{G}$ is the same as that of G and such that two vertices of $\bar{G}$ are adjacent if and only if those vertices are non adjacent in G .

Example:**Figure 6.15****Remarks:**

(1) $\bar{\bar{G}} = G$.

(2) In a complete graph K_n every pair of vertices are adjacent.

Therefore in its complement $\bar{K}_n$ no pair of vertices are adjacent. That is it has no edges. So $\bar{K}_n$ is a null graph of n vertices (i.e.,) $\bar{K}_n = (n, 0)$ graph.

(3) Let G be any graph of n vertices. The vertices which are not adjacent in G are adjacent in $\bar{G}$. So in $G \cup \bar{G}$ every pair of vertices are adjacent and so is the complete graph K_n .

(i.e.,) $G \cup \bar{G} = K_n$.

Theorem: 6.7

Let G_1 be a (p_1, q_1) graph and G_2 a (p_2, q_2) graph. Prove that

- (i) $G_1 \cup G_2$ is a $(p_1 + p_2, q_1 + q_2)$ graph.
- (ii) $G_1 + G_2$ is a $(p_1 + p_2, q_1 + q_2 + p_1 p_2)$ graph.
- (iii) $G_1 \times G_2$ is a $(p_1 p_2, q_1 p_2 + q_2 p_1)$ graph.
- (iv) $G_1[G_2]$ is a $(p_1 p_2, p_1 p_2 + q_2^2 q_1)$ graph.

Proof:

(i) Is obvious.

(ii) Number of lines in $G_1 + G_2$

= number of lines in G_1 + number of lines in G_2

+ number of lines joining points of V_1 to points of V_2 .

= $q_1 + q_2 + p_1 p_2$. Hence we get (ii)

(iii) Clearly number of points in $G_1 \times G_2$ is $p_1 p_2$.

Now, let $(u_1, u_2) \in V_1 \times V_2$

The points adjacent to (u_1, u_2) are (u_1, v_2) where u_2 is adjacent to v_2 and (v_1, u_2) where v_1 is adjacent to u_1 .

$$\therefore \deg(u_1, u_2) = \deg u_1 + \deg u_2$$

The total no of lines in $G_1 \times G_2$

$$= \frac{1}{2} \left[\sum_{i,j} \deg(u_i) + \deg(v_j) \right] \text{ (by theorem 1.1)}$$

$$= \frac{1}{2} \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} (\deg u_i + \deg v_j) \text{ where } u_i \in V_1, v_j \in V_2$$

$$= \frac{1}{2} \left[\sum_{i=1}^{p_1} (p_2 \deg u_i) + \sum_{j=1}^{p_2} \deg v_j \right]$$

$$\begin{aligned}
&= \frac{1}{2} \left[\sum_{i=1}^{p_1} (p_2 \deg u_i + 2q_2) \right] \\
&= \frac{1}{2} (p_2 2q_1 + p_1 2q_2) \\
&= p_2 q_1 + p_1 q_2
\end{aligned}$$

The proof of (iv) is left to the reader.

6.3 Vector spaces associated with graphs

Definition: 6.3.1

Consider a graph $G = (V, E, \chi_G)$. Let $W_G = P(E)$. On this power set of E , we consider the operation symmetric difference of sets; which is also called the ring sum operation between sets. We denote it by $\oplus$. Now we show that this W_G with respect to this operation is an abelian group.

We know that $A \oplus B = (A - B) \cup (B - A)$.

- (i) W_G is closed under $\oplus$, since W_G contains all subsets of E , including ϕ .
- (ii) For any $E_i \in W_G$, $E_i \oplus \phi \oplus E_i = E_i$ and hence ϕ is the identity element of W_G .
- (iii) Since $E_i \oplus E_i = \phi$ each E_i in W_G has self inverse.
- (iv) $\oplus$ is associative is clear.
- (v) By definition of $\oplus$, $E_i \oplus E_j = E_j \oplus E_i$ $E_i, E_j \in W_G$. Hence $(W_G, \oplus)$ is an abelian group.

Now we consider the field $(Z_2, \oplus, \odot)$. We show that W_G is a vector space over this field Z_2 , by defining a multiplication operation between W_G and Z_2 by,

$1 * E_i = E_i$ and $0 * E_i \neq \phi$ where $E_i \in W_G$. It can be easily seen that $*$ satisfies other conditions of a vector space.

Hence $(W_G, \oplus, *)$ is a vector space over this field Z_2 .

Theorem 6.8:

The dimension of the vector space W_G over Z_2 is the number of edges in G .

Proof:

We know that W_G is a vector space over Z_2 . Consider the edge set $E(G) = \{e_1, e_2, \dots, e_q\}$ (say). Then $\{\{e_1\}, \{e_2\}, \dots, \{e_q\}\}$ is a basis for W_G . Since, let $\{e_{\alpha_1}, e_{\alpha_2}, \dots, e_{\alpha_k}\}$ be a member of W_G , where

$(\alpha_1, \alpha_2, \dots, \alpha_k) \subseteq \{1, 2, \dots, q\}$. Now,

$\{e_{\alpha_1}, e_{\alpha_2}, \dots, e_{\alpha_k}\} = \{e_{\alpha_1}\} \oplus \dots \oplus \{e_{\alpha_k}\}$. Also no member of $\{\{e_1\}, \{e_2\}, \dots, \{e_q\}\}$ can be written as a linear combination of preceding members. Hence the dimension W_G is q , which is the number of edges in G .

Remark:

Consider a graph $G = (V, E, \chi_G)$. Since each edge-induced subgraph of G corresponds to unique subset of E and the ring sum of any two edge-induced subgraphs corresponds to the ring sum of their corresponding edge sets, the set of all edge-induced subgraphs of G is also a vector space over Z_2 under the multiplication operation,

$1 * G_i = G_i$ and $1 * G_i = \phi$, G_i is an edge - induced subgraph of G . This vector space also will be denoted by W_G .

Remark:

When we consider vector spaces of graphs, isolated vertices are of no consequence.

Remark:

Now we recall a known theorem from linear algebra, which states that if V is a finite dimensional vector space of dimension n over the field F then V is isomorphic to $V_n(F)$, the vector space of all n -vectors over the same field F .

Since the dimension of the vector space W_G over Z_2 is q , the number of edges in G , W_G is isomorphic to the vector space of all q -vectors over Z_2 . And so, we use the same symbol W_G to denote the vector space of all q -vectors associated with the subgraphs of the graph G .

CYCLE SPACE

7.0 Introduction:

In 1847, while establishing the well-known Kirchhoff laws governing electrical networks, G. Kirchhoff initiated Algebraic graph theory.

Cutsets play a vital role in communication and transportation problems. Suppose a few cities are connected by telephone lines, to find and strengthen the weak spots of the network, we concentrate on cutsets with smallest number of edges.

Cutsets are very useful in the development of graph related vector spaces. Why graph related vector spaces?

Modern Algebra is more useful to analyse the problems of other fields. Analysing the characteristics of problems of other areas through algebraic structures yields better results. Moreover, if we wish to use computers in graph theory, it would be better to represent graphs algebraically and analyse them algebraically.

In this unit we introduce the concept of cycle space and cutset space. Now we consider two important subspaces of W_G and discuss the relationship between them.

7.1 Cutsets and cuts

Definition: 7.1.1

A cutset of a connected graph G is a set of edges whose removal disconnects G , but no proper subset of these edges disconnects G .

Some times it is also called as a **cocycle**.

Example: Consider the graph of figure 7.1

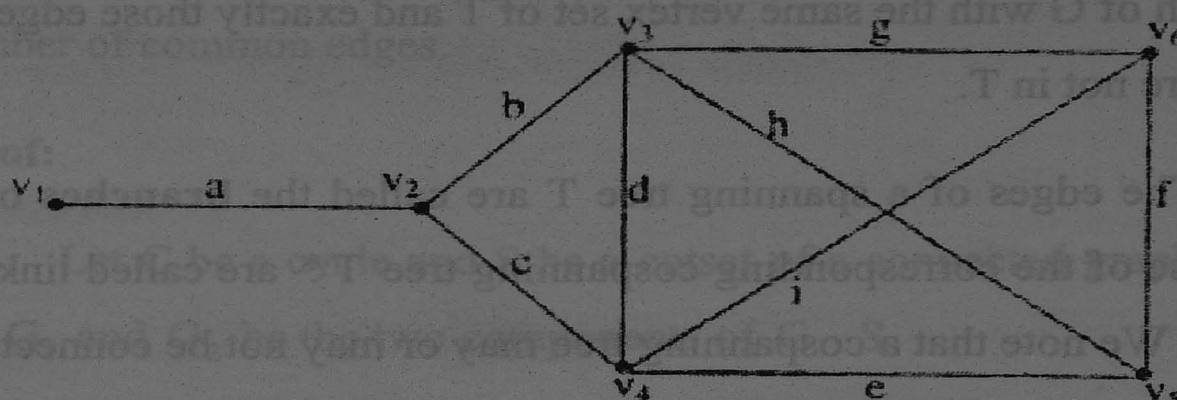


Figure 7.1

Here, $\{a\}$ is a cutset with one edge. $\{b, c\}$ is a cutset with two edges. $\{g, i, f\}$ is a cutset with three edges. $\{e, g, h, i\}$ is a cutset with four edges. But $\{b, c, d\}$ is not a cutset since its proper subset $\{b, c\}$ disconnects this graph.

Theorem 7.1:

A cutset of a connected graph G disconnects G into exactly two components.

Proof:

Let $A = \{e_1, e_2, \dots, e_k\}$ be a cutset of G . Now remove all the edges of A except one, say e_1 , from G . Now, $G - \{A - \{e_1\}\}$ is still connected, since A is cutset. Now remove e_1 . Since removing an edge can always give at most two components and here, since A is a cutset, we get exactly two components. Hence the theorem.

Definitions: 7.1.2

Consider a connected graph G with p vertices and q edges. Let T be a spanning tree of G . The **cospanning tree** T^* of T is the subgraph of G with the same vertex set of T and exactly those edges of G that are not in T .

The edges of a spanning tree T are called the **branches** of T , and those of the corresponding cospanning tree T^* are called **links** or **chords**. We note that a cospanning tree may or may not be connected.

Theorem: 7.2

Every cutset in a connected graph G must contain at least one branch of every spanning tree of G .

Proof:

Consider a spanning tree T of G and an arbitrary cutset S of G . Suppose S has no edge of T , then S cannot disconnect G , since $G-S$ contains the spanning tree T and hence $G-S$ is connected. This is a contradiction to S being a cutset. Hence the theorem.

Now we consider the converse of the above theorem.

Theorem: 7.3

In a connected graph G , any minimal set of edges containing at least one branch of every spanning tree of G is a cutset.

Proof:

Let A be a minimal set of edges containing at least one branch of every spanning tree of the connected graph G . So, $G - A$ does not have any spanning tree of G and hence $G - A$ is disconnected. Since A is a minimal set of edges with this property, $G - (A - \{e\})$ contains

every branch of some spanning tree and therefore $G - (A - \{e\})$ is connected. Hence A is a minimal set of edges, whose removal disconnects G . Therefore A is a cutset.

Theorem: 7.4

A cycle and a cutset of a connected graph have an even number of common edges.

Proof:

Let C be a cycle and S be a cutset of a connected graph G . Let G_1 and G_2 be the two components of $G - S$.

If C is a subgraph of G_1 or G_2 , then the number of edges common to C and S is equal to zero, which is an even number.

Otherwise, C and S have common edges. Take a vertex v_1 of C from G_1 and traverse C . Since the traversing should end at v_1 , whenever we meet an edge of S leading us from a vertex in G_1 to a vertex in G_2 , there should be another edge S leading us from a vertex in G_2 to a vertex in G_1 .

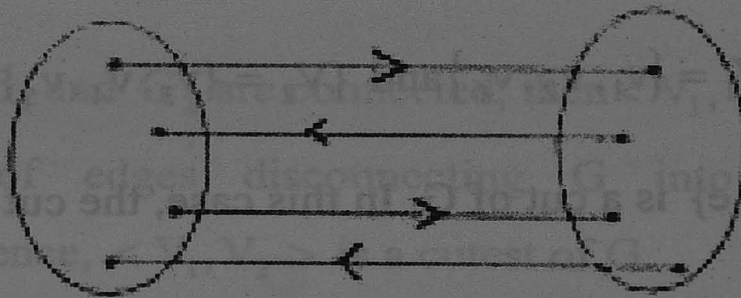


Figure 7.2

This implies that C and S have even number of common edges.

Definition: 7.1.3

Let G be a connected graph with vertex set V and let (V_1, V_2) be a partition of V ; that is $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$. Then the set of all edges of G having one end in V_1 and the other end in V_2 is called a **cut** of G .

We denote this by $\langle V_1, V_2 \rangle$

Note:

A cut is different from a cutset.

Example:

Consider the graph G of figure 7.3.

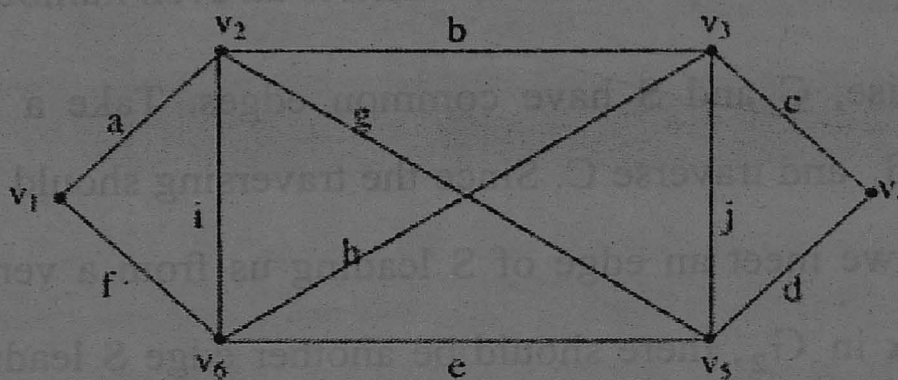


Figure 7.3

If we take $V_1 = \{v_1, v_2, v_6\}$ and $V_2 = \{v_3, v_4, v_5\}$ then

$\langle V_1, V_2 \rangle = \{b, g, h, e\}$ is a cut of G . In this case, the cut $\langle V_1, V_2 \rangle$ is a cutset of G .

If we take $V_1 = \{v_2, v_6, v_4\}$ and $V_2 = \{v_1, v_3, v_5\}$ then $\langle V_1, V_2 \rangle = \{a, f, c, d, b, g, h, e\} = \{a, f\} \cup \{c, d\} \cup \{b, g, h, e\}$. In this case, the cut $\langle V_1, V_2 \rangle$ is a union of edge-disjoint cutsets of G .

Theorem: 7.5

If S is a cutset of a connected graph G , and V_1 and V_2 are the vertex sets of the two components of $G - S$, then S is a cut of G .

Proof:

This proof is immediate.

Theorem: 7.6

A cut $\langle V_1, V_2 \rangle$ of a connected graph G is a cutset of G if the induced subgraphs of G on vertex sets V_1 and V_2 are connected.

Proof:

Consider a cut $\langle V_1, V_2 \rangle$ of a connected graph G , which is the minimal set of edges of G whose removal disconnects G into two graphs G_1 and G_2 , where G_1 and G_2 are induced subgraphs of G on the vertex sets V_1 and V_2 respectively. G_1 and G_2 may or may not be connected.

If both G_1 and G_2 are connected, then $\langle V_1, V_2 \rangle$ is also the minimal set of edges disconnecting G into exactly two components. Hence, $\langle V_1, V_2 \rangle$ is a cutset of G .

Theorem 7.7

A cut in a connected graph G is a cutset or union of edge-disjoint cutsets of G .

Proof:

Consider a cut $[V_1, V_2]$. Let $A_1, A_2, \dots, A_n$ be the components of $\langle V_1 \rangle$. Then,

$$[V_1, V_2] = [A_1, V_2] = [A_1, V_2] \cup [A_2, V_2] \cup \dots \cup [A_n, V_2] \quad (1)$$

We note that, $[A_i, A_j] = \emptyset$ for every $i \neq j$ and $1 \leq i, j \leq n$. Consider $[A_1, V_2]$. If $\langle V - A_1 \rangle$ is connected, then $[A_1, V_2] = [A_1, V - A_1]$ is a cutset by theorem 7.6.

Suppose $\langle V - A_1 \rangle$ is not connected, then let $B_1, B_2, \dots, B_m$ be its components. Then $[A_1, V_2] = [A_1, V - A_1] = [A_1, B_1] \cup \dots \cup [A_1, B_m]$.

Clearly each $[A_1, B_i]$, $1 \leq i \leq m$ is a cutset of G . So $[A_1, V_2]$ is a disjoint union of cutsets or a cutset. Hence, we get $[V_1, V_2]$ is either a cut set or a disjoint union of cutsets, by (1).

7.2 Cycle space:

Definition: 7.2.1

The set of all cycles and unions of edge-disjoint cycles of a graph G , say W_C is a subspace of the vector space W_G of G .

W_C is called as the **cycle subspace** of the graph G .

Theorem: 7.8

The set of all cycles and unions of edge-disjoint cycles of a graph G , say W_C is a subspace of the vector space W_G of G .

Proof:

Without loss of generality we assume that W_C consists of all edge-induced subgraphs of G , in which all vertices are of even degree.

It is enough if we prove that W_C is closed under the ring sum operation. Let $C_1, C_2 \in W_C$. Now we prove that $C_3 = C_1 \oplus C_2 \in W_C$. Let $v \in C_3$. This vertex should belong to at least one of the subgraphs C_1 and C_2 . Let X_i denote the set of all edges incident on v in $C_i, i=1,2,3$. Thus $|X_i|$ is the degree of the vertex v in C_i . We note that $|X_1|$ and $|X_2|$ are even, one of them may be zero and $|X_3|$ is non-zero. Since $C_3 = C_1 \oplus C_2$, we get $X_3 = X_1 \oplus X_2$. So $|X_3| = |X_1| + |X_2| - 2|X_1 \cap X_2|$.

That is, degree of v in C_3 is even. Since v is arbitrary, $C_3 \in W_C$. Hence the theorem.

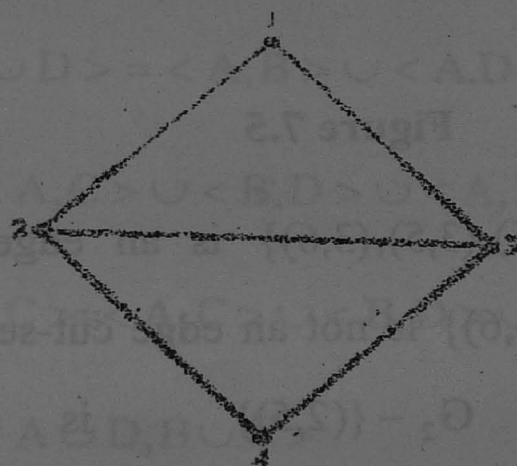
Illustration:

Figure 7.4

The graph of Figure 7.4 contains 3 cycles namely, $C_1 : 1,2,3, 1, C_2 : 2,3,4,2$ and $C_3 : 1,2,4,3, 1$. We find that

$$C_1 \oplus C_2 = C_3, C_1 \oplus C_2 = C_2, C_2 \oplus C_3 = C_1.$$

7.3 Cut-set space:

Definition: 7.3.1:

Let G be a graph, A subset $F \subseteq E(G)$ is called a edge cut-set (or simply cut-set) of G , if

- (i) $\omega(G - F) > \omega(G)$; and
- (ii) $\omega(G - F') = \omega(G)$ for every subset F' of F .

Thus if G is a connected graph, then $F \subseteq E(G)$ is a cut-set of G , if

- (i) $G - F$ is disconnected; and
- (ii) $G - F'$ is connected for every proper subset F' of F .

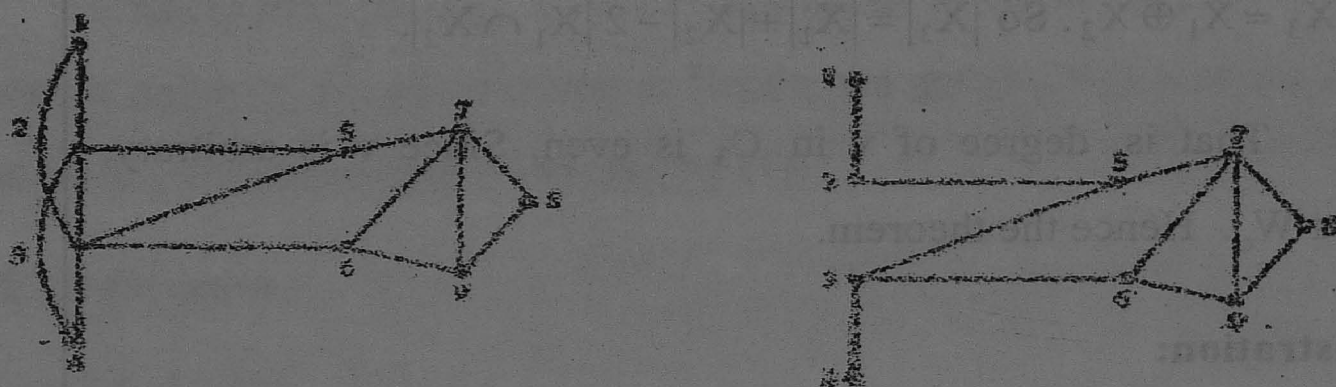


Figure 7.5

In figure 7.5, $\{(2,5), (3,5), (3,6)\}$ is an edge cut-set of G_1 ; whereas $F = \{(2,5), (3,5), (3,6)\}$ is not an edge cut-set of G_2 , because $\{(2,5)\} \subseteq F$ and $G_2 - \{(2,5)\}$ is disconnected; $\{(1,2)\}, \{(3,5), (3,6)\}, \{(5,7), (6,7), (6,9)\}$ are some of the cut-sets of G_2 .

(OR)

Definition: 7.3.1

The set of all cutsets and unions of edge-disjoint cutsets in a graph G , say W_S is a subspace of the vector space W_G of G .

W_S is called the **cutset subspace** of the graph G .

Theorem: 7.9

The ring sum of any two cuts in a graph G is also a cut in G .

Proof:

Consider any two cuts $S_1 = \langle V_1, V_2 \rangle$ and $S_2 = \langle V_3, V_4 \rangle$ in a graph G .

Now, $V_1 \cup V_2 = V = V_3 \cup V_4$ and $V_1 \cap V_2 = \phi = V_3 \cap V_4$.

Let $A = V_1 \cap V_3$, $B = V_1 \cap V_4$, $C = V_2 \cap V_3$ and $D = V_2 \cap V_4$

Clearly these sets are mutually disjoint.

Consider

$S_1 = \langle A \cup B, C \cup D \rangle = \langle A, C \rangle \cup \langle A, D \rangle \cup \langle B, C \rangle \cup \langle B, D \rangle$ and
 $S_2 = \langle A \cup C, B \cup D \rangle = \langle A, B \rangle \cup \langle A, D \rangle \cup \langle C, B \rangle \cup \langle C, D \rangle$

Hence, $S_1 \oplus S_2 = \langle A, C \rangle \cup \langle B, D \rangle \cup \langle A, B \rangle \cup \langle C, D \rangle$

But, $\langle A \cup D, B \cup C \rangle = \langle A, C \rangle \cup \langle B, D \rangle \cup \langle A, B \rangle \cup \langle C, D \rangle$

Hence, $S_1 \oplus S_2 = \langle A \cup D, B \cup C \rangle$

Since

$(A \cup D) \cap (B \cup C) = \phi$ and $(A \cup D) \cup (B \cup C) = V$, $S_1 \oplus S_2$ is a cut in G .

Hence the theorem

Corollary: 7.9.1

The union of any two edge-disjoint cuts in a graph G is also a cut in G .

Proof:

Since the ring sum of two disjoint sets is the same as their union, the result follows immediately.

Theorem: 7.10

W_s , the set of all cutsets and unions of edge-disjoint cutsets in a graph G , is nothing but the set of all cuts in G .

Proof:

Since each cutset is also a cut and by the previous corollary union of any two edge-disjoint cuts in a graph G is also a cut in G , W_s is contained in the set of all the cuts in G .

Also, since a cut in a connected graph G is a cutset or union of edge-disjoint cutsets of G , the set of all cuts in G is contained in W_s . Hence, W_s is nothing but the set of all cuts in G .

Theorem: 7.11

W_s , the set of all cutsets and unions of edge-disjoint cutsets in a graph G , is a subspace of the vector space W_G of G .

Proof:

Since W_s is the set of all cuts in G and the ring sum of two cuts is again a cut, the theorem holds.

Rank and Nullity:

Consider a graph with p vertices, q edges and k components. The **rank of G** , denoted by $\rho(G)$, is defined to be equal to $p - k$.

The **nullity of G** , denoted by $\mu(G)$, is defined to be equal to $q - p + k$. It is obvious to note that, $\rho(G) + \mu(G) = q$, the number of edges of G .

1. If G is a connected graph with p vertices and q edges then the rank of G is $p - 1$ and the nullity of G is $q - p + 1$.
2. We know that any graph without cycles is a forest. If G is a forest with p vertices, q edges and k components then $q = p - k$. So the rank of a forest is the number of edges in the forest; and hence its nullity is zero.
3. The nullity of a graph is also called as its cyclomatic number or first Betti number.

Fundamental cycles:

Consider a connected graph G with p vertices and q edges. Let T be a spanning tree of G . Let the branches of T be denoted by $b_1, b_2, \dots, b_{p-1}$ and the chords of T be denoted by $c_1, c_2, \dots, c_{q-p+1}$.

Each $T \cup C_i$ contains exactly one cycle C_i . This cycle consists of the chord c_i and the unique path between the end vertices of c_i in T . This cycle C_i is called the **fundamental cycle** of G with respect to the chord c_i of the spanning tree T .

Observation:

1. Each fundamental cycle C_i contains exactly one chord, namely c_i .
2. Each chord c_i is not present in any other fundamental cycle with respect to T .
3. Hence the edge set of no fundamental cycle can be expressed as the ring sum of the edge sets of some or all of the remaining fundamental cycles.

Also we will be proving that every cycle of a graph G can be expressed as the ring sum of some fundamental cycles of G with respect to spanning tree of G . It is for these reasons, we call them as fundamental cycles.

Fundamental cutsets:

Consider a connected graph G and T to be a spanning tree T of G . Let b a branch of T . When we remove b from T , we get exactly two components T_1 and T_2 . Let V_1 and V_2 denote the vertex sets of T_1 and T_2 . Let G_1 and G_2 be the induced subgraphs of G on V_1 and V_2 . Since T_1 and T_2 are spanning trees of G_1 and G_2 are connected. By theorem 7.6 the cut $\langle V_1, V_2 \rangle$ becomes a cutset of G . This cutset is called as the **fundamental cutset** of G with respect to the branch b of the spanning tree T of G .

Observation:

1. Each fundamental cutset contains exactly one branch of T .
2. All other edges of this fundamental cutsets are chords of T .
3. The branch b is not present in any other fundamental cutset with respect to T .

4. Hence, the edge set of no fundamental cutset can be expressed as the ring sum of the edge sets of some or all the remaining fundamental cutsets.

Also we will prove that every cutset of a graph G can be expressed as the ring sum of the fundamental cutsets of G with respect to a spanning tree T of G . That is why, we call them as fundamental cutsets.

Definition: 7.3.2

Let $G = (V, E)$ be a connected graph and T be a spanning tree of G . (Such a T exists by Theorem). The edges of T are called the branches of T and the edges of $E(G) - E(T)$ are called the chords of T .

If e is a chord of T , then $T + e$ contains a unique cycle. Such a cycle is called a fundamental cycle of G , relative to T .

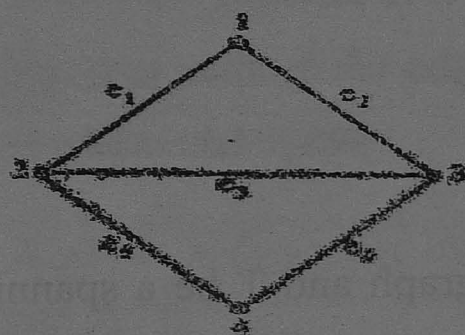
Thus, if C is a fundamental cycle (relative to spanning tree T) in a graph G , then exactly one of its edges is a chord of T and all the other edges of C are branches of T .

Example:

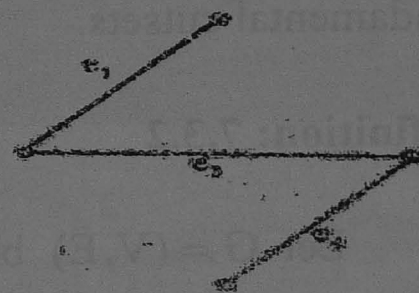
In the following graph e_1, e_3, e_4 are branches of T ; and e_2, e_5 are chords of T .

Remark:

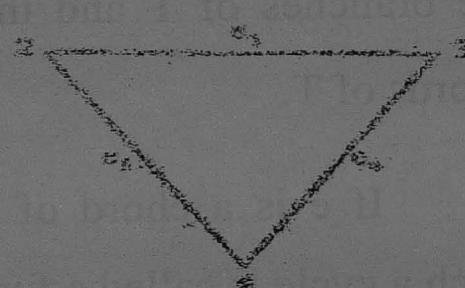
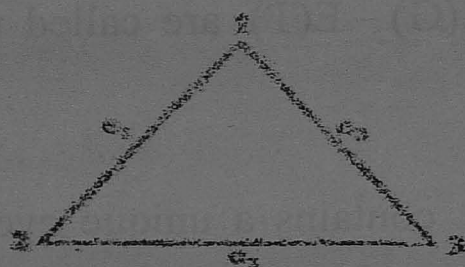
Let G be a connected (p, q) – graph and let T be an arbitrary spanning tree of G . The number of branches of T is $p - 1$; Hence, the number of chords of T is $q - p + 1$; so, the number of fundamental cycles in G relative to any spanning tree T of G is $q - p + 1$.



A graph G



A spanning tree T of G



Two fundamental cycles relative to T

Figure 7.6

Dimensions:

Here we find the dimension of cycle and cutset subspace of a graph G .

Theorem: 7.12

Let G be a connected graph with p vertices and q edges. Then the dimension of the cycle subspace of G is equal to the nullity of G .

Proof:

Let T be a spanning tree of G and let $b_1, b_2, \dots, b_{p-1}$ be the branches of T and $c_1, c_2, \dots, c_{q-p+1}$ be the chords of T . Let C_i be the fundamental cycle corresponding to c_i .

Now we prove that the fundamental cycles $C_1, C_2, \dots, C_{q-p+1}$ form a basis for cycle subspace W_C .

Since each fundamental cycle contains exactly one chord, and this chord is not present in any other fundamental cycle, no fundamental cycle can be expressed as the ring sum of other fundamental cycles. Hence the fundamental cycles $C_1, C_2, \dots, C_{q-p+1}$ are linearly independent.

Now we prove that every subgraph in W_C can be expressed as a ring sum of these fundamental cycles. Let C be a subgraph in W_C and let C contains the chords $c_{i_1}, c_{i_2}, \dots, c_{i_r}$. Let C' denotes the ring sum of the fundamental cycles $c_{i_1}, c_{i_2}, \dots, c_{i_r}$. Clearly the chords $c_{i_1}, c_{i_2}, \dots, c_{i_r}$ are all in C' and C' contains no other chords of T . Since C also contains these chords and no other chords. $C' \oplus C$ contain no chords.

Now we prove that $C' \oplus C$ is empty. Suppose not, then $C' \oplus C$ contains only branches and hence no cycles. Since $C' \oplus C$ is a ring sum of cycles and W_C is a subspace of W_G , $C' \oplus C$ is a cycle or union of some edge-disjoint cycles. Thus we get a contradiction. Hence $C' \oplus C$ is empty.

$$\text{So, } C = C' = c_{i_1} \oplus c_{i_2} \oplus \dots \oplus c_{i_r}.$$

Therefore, the fundamental cycles $C_1, C_2, \dots, C_{q-p+1}$ form a basis for W_C . Thus, the dimension of W_C is $p - q + 1$, which is the nullity of G .

Remark:

In the previous theorem, apart from proving the dimension, we have proved the set of fundamental cycles form a basis for the cycle subspace of G .

Theorem: 7.13

Let G be a connected graph with p vertices and q edges. Then the dimension of the cutset subspace of G is equal to the rank of G .

Proof:

The proof is very similar to the previous theorem; the set of fundamental cutsets with respect to a spanning tree of G form a basis for the cutset subspace of G .

If G is a disconnected graph with p vertices, q edges and k components, then each component being connected has a spanning tree. Let $T_1, T_2, \dots, T_k$ be the spanning trees in G . Then the set of fundamental cycles and fundamental cutsets with respect to these spanning trees form a basis for the cycle and cutset subspace of G , respectively. Hence the following corollary.

Corollary: 7.13.1

Let G be a graph with p vertices, q edges and k components. Then

1. The dimension of the cycle subspace of G is equal to the nullity of G .

2. The dimension of the cutset subspace of G is equal to the rank of G .

Theorem: 7.14

Every subgraph in the cycle subspace of a graph G has an even number of common edges with every subgraph in the cutset subspace of G .

Proof:

Let C be a subgraph of cycle subspace of a graph G . Then C is a cycle or union of edge-disjoint cycles. Also we know that every subgraph of cutset subspace of G is a cutset or union of edge-disjoint cutsets. By theorem 7.4, a cycle and a cutset have an even number of common edges. Hence C has an even number of common edges with every subgraph in the cutset subspace of G .

Theorem: 7.15

A subgraph of a graph G belongs to W_C , the cycle subspace of G if it has an even number of common edges with every subgraph in the cutset subspace of G .

Proof:

With out loss of generality, we assume that G is connected.

Let C be a subgraph of G which has an even number of common edges with every subgraph in the cutset subspace of G . Let T be a spanning tree of G and let $b_1, b_2, \dots, b_\alpha$ be its branches and $c_1, c_2, \dots, c_\beta$ be its chords. Let C contain the chords $c_1, c_2, \dots, c_r$ only. Let the fundamental cycles corresponding to these chords be $C_1, C_2, \dots, C_r$ respectively.

Let $C' = C_1 \oplus C_2 \oplus \dots \oplus C_r$.

Clearly C' contains the chords $c_1, c_2, \dots, c_r$ and no other chords. Hence $C' \oplus C$ consists of no chords. By previous theorem, C' has an even number of common edges with every subgraph in the cutset subspace of G . C also has this property and hence $C' \oplus C$ has this property.

Now we prove that $C' \oplus C$ is empty. Suppose not, then $C' \oplus C$ has only branches. Let b_i be a branch in $C' \oplus C$. Then b_i is the only edge common between $C' \oplus C$ and the fundamental cutset with respect to b_i . This is a contradiction, since $C' \oplus C$ must have an even number of common edges with every cutset.

Therefore, $C' \oplus C = \phi$.

Hence $C = C' = C_1 \oplus C_2 \oplus \dots \oplus C_r \in W_c$.

Theorem: 7.16

A subgraph of a graph G belongs to the cutset space of G if it has an even number of common edges with every subgraph in the cycle subspace of G .

Proof:

The proof is very similar to the previous theorem.

Orthogonality:

Consider a graph G with q edges. Let $E(G) = \{e_1, e_2, \dots, e_q\}$.

We associate with each edge-induced subgraph G_i of G an q -vector $v = \{v_1, v_2, \dots, v_q\}$ such that

$v_j = 1$ if and only if the edge e_j is in G_i .

$= 0$ otherwise.

That is, define $\phi : W_G \rightarrow V_q(Z_2)$ such that

$\phi(G_i) = v$ where $v = \{v_1, v_2, \dots, v_q\}$ and

$v_j = 1$ if and only if the edge e_j is in G_i .

$= 0$ otherwise.

Now for two subgraphs G_i and G_j , the image of $G_i \oplus G_j$ under ϕ is $v + w$. The modulo 2 sum of v and w , where $\phi(G_i) = v$ and $\phi(G_j) = w$.

Clearly ϕ is an isomorphism between W_G and $V_p(Z_2)$.

We denote the vector space of all q -vectors associated with the subgraphs of the graph G as W_G . Also, here, W_C and W_S will denote the cycle and cutset subspace respectively.

Note:

The tree vector spaces W_G, W_C and W_S can be made into inner product spaces by using the following dot product; if $u = \{u_1, u_2, \dots, u_q\}$ and $v = \{v_1, v_2, \dots, v_q\}$ then $\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \dots + u_q v_q \pmod{2}$.

Theorem: 7.17

The cycle and cutset subspaces of a graph are orthogonal to each other.

Proof:

Let u and v be two vectors such that $u \in W_C$ and $v \in W_S$.

Consider the dot product $\langle u, v \rangle$. Since every subgraph of W_C had an even number of common edges with those in W_S .

$$\begin{aligned}\langle u, v \rangle &= \text{Sum of even number of 1's (mod 2)} \\ &= 0\end{aligned}$$

Hence W_C and W_S are orthogonal to each other.

Theorem: 7.18

W_C and W_S are orthogonal compliments W_G if and only if $W_C \cap W_S$ is the zero vector.

Proof:

We know that W_C and W_S are orthogonal subspaces of W_G .

Therefore $W_C \cap W_S = \{0\}$

We know that,

$$\dim(W_C + W_S) = \dim(W_C) + \dim(W_S) - \dim(W_C \cap W_S)$$

But $\dim(W_C) + \dim(W_S) = q - p + k + p - k - 0 = q$, where q is the dimension of W_G and hence $W_C + W_S = W_G$.

W_C and W_S will be orthogonal compliments

If and only if $W_C + W_S = W_G$ and $W_C \cap W_S = \{0\}$

If and only if $W_C + W_S = W_G$ and $W_C \cap W_S = \{0\}$

If and only if $\dim(W_C + W_S) = q$ and $\dim(W_C \cap W_S) = 0$

Hence the theorem

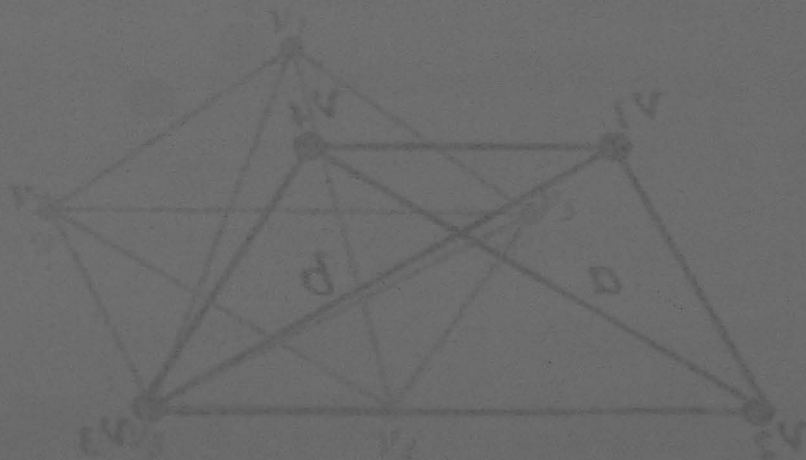


Figure 8.1
G 3.8 graph

PLANAR GRAPHS

8.0 Introduction:

In this chapter we consider the embedding (drawing without crossings) of graphs on surfaces, especially the plane. We study the properties of planar graphs and state Kuratowski's famous theorem, on the characterization of planar graphs. The materials in this chapter form part of what is known as topological graph theory.

8.1 Definition and properties:

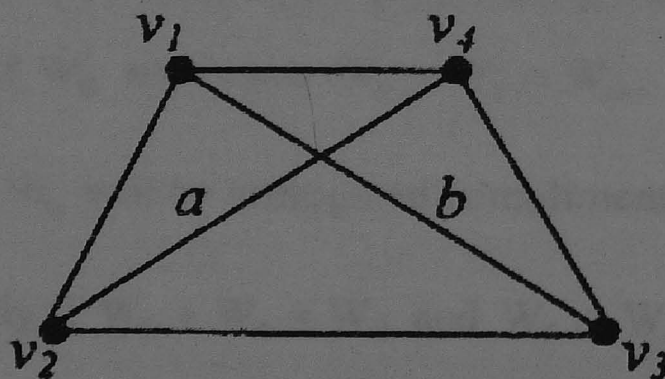
Definition: 8.1.1

A graph is said to be **planar** if there exists a geometric representation of the graph which can be drawn on a plane such that no two edges intersect (cross over) at a point other than a vertex.

A graph that cannot be drawn on a plane without cross over of its edges is a **non planar** graph.

A geometric representation of a graph without cross over of its edges is called a **plane embedding** of the graph.

Thus a planar graph is one which has a plane embedding. Consider the graph G . Here the edges a and b appear to cross over each other. This can be avoided by pulling out a or b so that there is no cross over.



G
Figure 8.1

Thus, a graph has a plane embedding and so it is a planar graph.

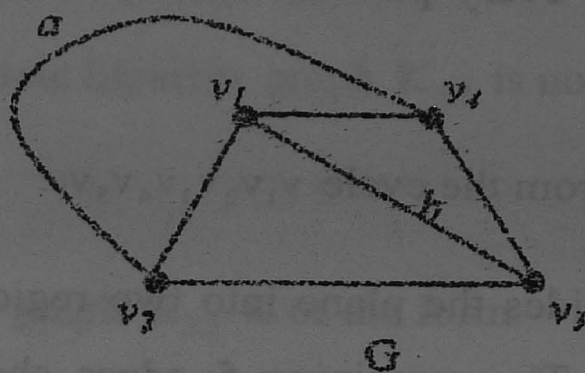


Figure 8.2

Plane graph and planar graph:

A graph is said to be planar if it is embeddable in the plane, that is if it can be drawn on the plane so that its edges intersect only at its vertices (there is no cross over). Such a drawing of a planar graph G is a plane embedding of G .

A plane embedding of a planar graph is called a **plane graph**.

If $\tilde{G}$ is a plane embedding of a planar graph G then $\tilde{G} \cong G$.

Remarks:

Any two plane embeddings of a planar are isomorphic.

Theorem: 8.1

A complete graph of 5 vertices is non planar (i.e.,) K_5 is non planar.

Proof:

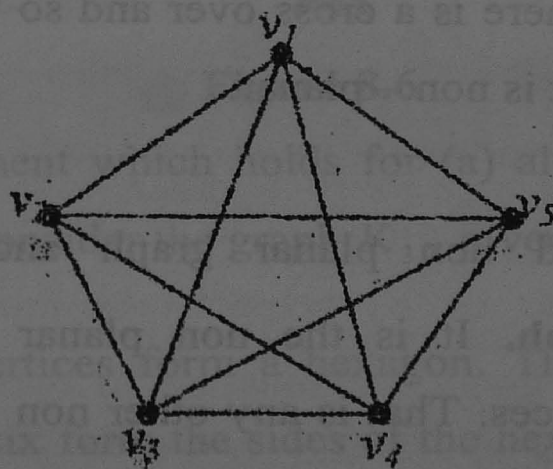


Figure 8.3

Let v_1, v_2, v_3, v_4, v_5 be a vertices of K_5 . Since it is complete there is an edge between every pair of vertices. So the number of edges $C_2 = 10$.

Of these 10 edges from the cycle $v_1v_2v_3v_4v_5v_1$.

Now this cycle divides the plane into two regions namely the interior and the exterior. The remaining 5 edges should be drawn either in the interior or in the exterior.

The edges v_2v_5 and v_3v_5 can be drawn in the interior without cross over.

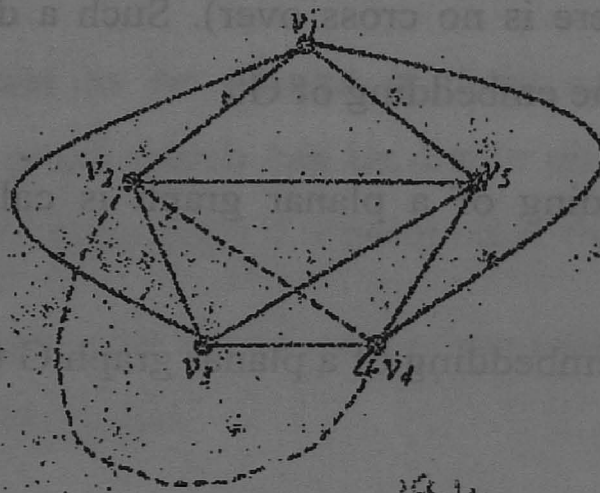


Figure 8.4

The edges v_1v_3 and v_1v_4 can be drawn in the exterior without cross over. Now the edges v_2v_4 remains. If it is drawn inside, it crosses over the edge v_3v_5 . If it is drawn out side it crosses over the edge v_1v_3 .

Thus in any case there is a cross over and so the graph has no plane embedding. Hence it is non – planar.

Remarks:

K_5 is a standard non planar graph and is known as **Kuratowski's first graph**. It is the non planar graph with the minimum number of vertices. That is any other non planar graph has more than 5 vertices.

Theorem: 8.2

The complete bipartite graph $K_{3,3}$ is non planar.

Proof:

The given graph $K_{3,3}$ is of the form.

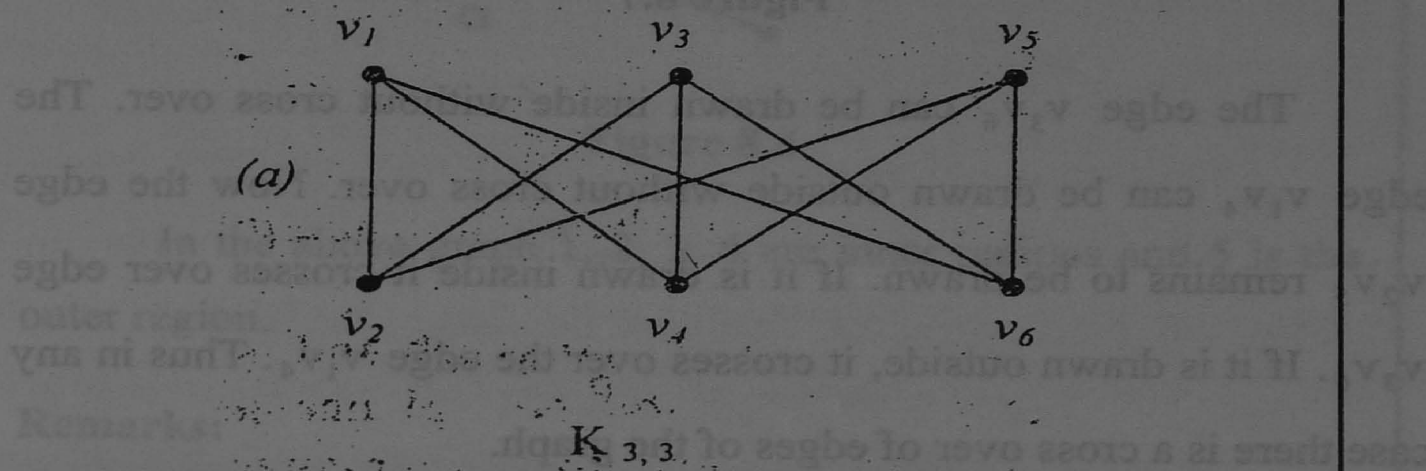


Figure 8.5

It has an equivalent (isomorphic) representation.

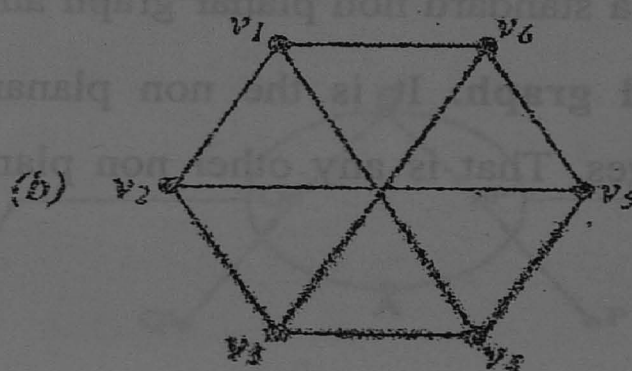


Figure 8.6

Any argument which holds for (a) also holds for (b). For convenience we consider the graph $K_{3,3}$ given in (b).

The six vertices form a hexagon. There are 9 edges. Of these nine edges six form the sides of the hexagon. The remaining three edges should be drawn inside or outside the hexagon.

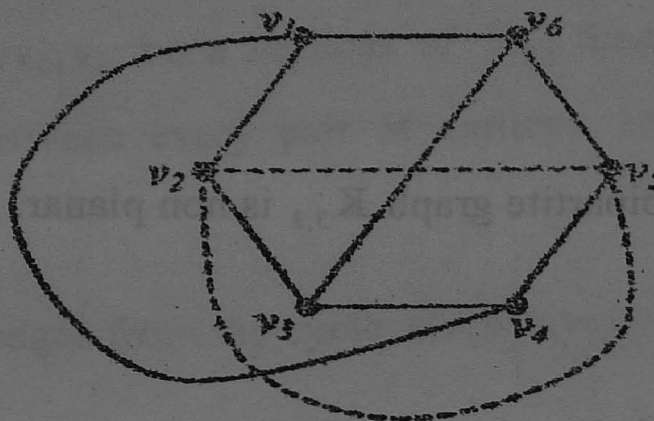


Figure 8.7

The edge v_3v_6 can be drawn inside without cross over. The edge v_1v_4 can be drawn outside without cross over. Now the edge v_2v_5 remains to be drawn. If it is drawn inside it crosses over edge v_3v_6 . If it is drawn outside, it crosses over the edge v_1v_4 . Thus in any case there is a cross over of edges of the graph.

So there is no plane embedding and hence it is non-planar.

Remarks:

The graph $K_{3,3}$ is a standard non planar graph and it is known as **Kuratowski's second graph**. It is the non planar graph with minimum number of edges. That is any other non planar graph has more than 9 edges.

Regions of a graph:

A plane representation of a graph divides the plane into a number of regions. The regions bounded by cycles or by cycles and cut edges of the graph are called the **inner** or **interior regions**. The unbounded region is called the **outer** or **exterior region**. The inner and outer regions are also called respectively the **bounded** and **unbounded** regions of the graph. An inner or an outer region is also known as a **face** of the graph.

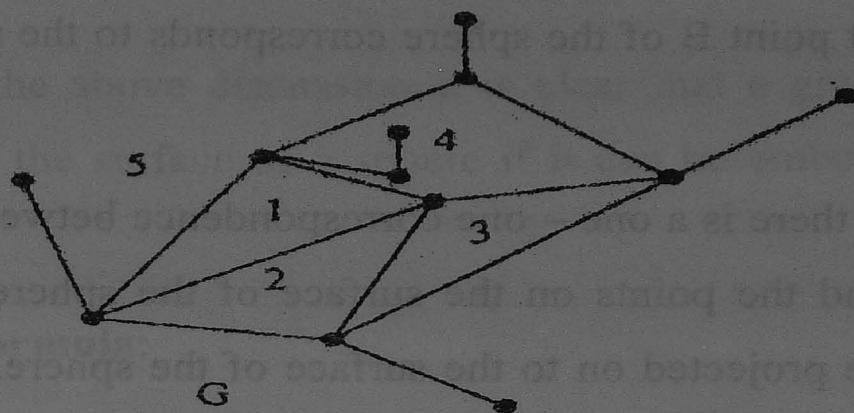


Figure 8.8

In the above graph 1, 2, 3, 4 are inner regions and 5 is the outer region.

Remarks:

(1) There is only one outer region.

(2) A tree has no inner region.

Stereographic projection:

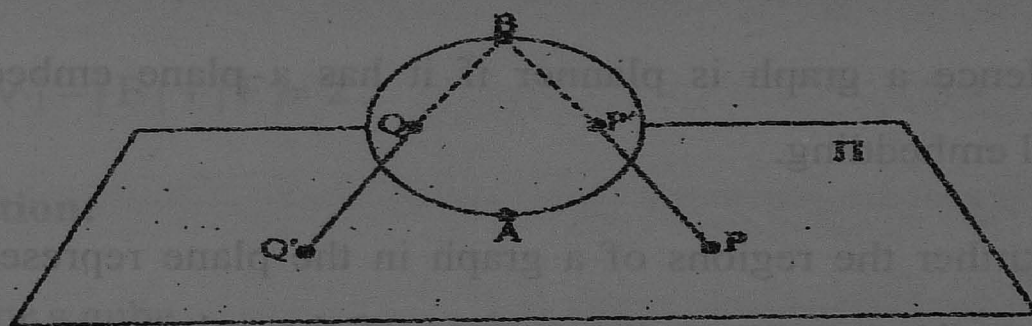


Figure 8.9

Let a sphere of any radius touch a plane Π at the point A and let B be the highest point on the sphere. Let P be any point on the plane. Join BP. Clearly it meets the surface of the sphere at a unique point P' . This is true for every point on the plane.

Again let Q be a point on the sphere. Join BQ. It will meet the plane Π at a unique point Q' . This is true for every point on the sphere.

Further the point A is common to both the plane and the sphere. Also highest point B of the sphere corresponds to the point at infinity on the plane.

Thus there is a one – one correspondence between the points of the plane and the points on the surface of the sphere. So the entire plane can be projected on to the surface of the sphere. This is known as **stereographic projection**.

Since the plane is projected on to the surface of the sphere and every point on the plane has a unique corresponding point on the surface of the sphere, we find that incidence and adjacency of the vertices and edges of a graph on the plane are preserved in the projected graph on the surface of the sphere. So any graph G on the plane has an isomorphic projection G' on the surface of the sphere.

So if a graph has a plane embedding then it has an embedding on the surface of the sphere. That is corresponding to a plane embedding, there is a spherical embedding of the graph.

Hence a graph is planar if it has a plane embedding or a spherical embedding.

Further the regions of a graph in the plane representation are mapped on to regions on the surface of the sphere in the spherical embedding of the graph. Since the spherical surface is finite any region on it is also finite. Thus we find that any region finite or infinite on the plane can be mapped onto a finite region on the surface of the sphere by stereographic projection. The region on the surface of the sphere representing the infinite region (on the plane) will contain the highest point B as an interior point.

Remarks:

From the above discussion it is clear that a graph can be embedded on the surface of a sphere if it can be embedded on a plane.

8.2 Euler's formula:**Polyhedron:**

A solid bounded by plane surfaces is called a **polyhedron**. The plane surfaces are called the **faces**, the bounding lines are called the **edges** and the corners are called the **vertices** of the polyhedron. Cube and tetrahedron are two examples of a polyhedron.

Euler formula for a polyhedron:

The number of vertices $|V|$, the number of edges $|E|$ and the number of faces $|F|$ of a polyhedron are related by the following formula known as Euler's formula.

$$|V| - |E| + |F| = 2.$$

Verification:

(1) For a cube

$$|V| = 8, |E| = 12, |F| = 6,$$

$$|V| - |E| + |F| = 8 - 12 + 6 = 2.$$

(2) For a tetrahedron

$$|V| = 4, |E| = 6, |F| = 4$$

$$|V| - |E| + |F| = 4 - 6 + 4 = 2.$$

Remark:

We have already seen that a graph partitions the plane into a number of regions and they are called faces. There are vertices and edges in the graph. The number of vertices p , the number of edges q and the number of regions (faces) r of a planar graph are connected by a formula analogous to the above one and is also known after Euler.

Euler formula for planar graphs:**Theorem: 8.3****Statement:**

If p be the number of vertices, q the number of edges and r the number of faces (regions) of a planar graph then

$$p - q + r = 2$$

Proof:**Case (i)**

Suppose the graph is a tree.

A tree has only one face namely the infinite region and so $r = 1$.

Also the number of edges $q = p - 1$.

$$\therefore p - q + r = p - (p - 1) + 1 = 2.$$

Thus the result is true for a tree.

Case (ii)

Suppose the graph G is not a tree.

We prove the result by induction on the number of edges q .

When $q = 0$, the graph is only an isolated vertex and there is only one face namely the infinite region.

$$\therefore p - q + r = 1 - 0 + 1 = 2.$$

Thus the result is true when $q = 0$.

Let us assume that the result is true for all graphs of less than q edges.

As G is not a tree it has cycles.

Let e be an edge of a cycle.

Consider $G' = G - e$.

Clearly the number of vertices of G' is $p' = p$, the number of edges is $q' = q - 1$ and the number of faces $r' = r - 1$.

Since the number of edges of G' is less than q , the result is true for G' .

$$p' - q' + r' = 2$$

$$\text{i.e., } p - (q - 1) + (r - 1) = 2$$

$$\therefore p - q + r = 2$$

Theorem: 8.4

In any simple connected planar graph with p vertices q edges ($q > 2$) and r faces the following inequalities hold

$$(i) \quad q \geq \frac{3r}{2}$$

$$(ii) \quad q \leq 3p - 6.$$

Proof:

In a simple connected planar graph each face must have atleast 3 edges. So if r is the number of faces there must be atleast $3r$ edges. But in counting the edges each edge is counted twice, once in each of the adjacent faces. Also the number of edges is q .

$\therefore 2q$ must be atleast $3r$.

i.e., $2q \geq 3r$

$$q \geq \frac{3r}{2}$$

Again by Euler's formula

$$p - q + r = 2$$

$$r = 2 - p + q$$

From (i) $r \leq \frac{2q}{3}$

$$\therefore 2 - p + q \leq \frac{2q}{3}$$

$$6 - 3p + 3q \leq 2q$$

$$q \leq 3p - 6$$

Remarks:

The results (i) and (ii) are necessary conditions for a graph to be planar. If the conditions are not satisfied the graph is non planar.

Corollary:

- (i) If all the faces of the planar graph are triangles

$$2q = 3r$$

$$2q = 3(2 - p + q)$$

$$q = 3q - 6$$

- (ii) If all the faces of the planar graph are 4 - sided.

$$2q = 4r$$

$$q = 2r$$

$$q = 2(2 - p + q)$$

$$q = 2p - 4$$

- (iii) If all the faces of the planar graph are n - sided.

$$2q = nr$$

$$2q = n(2 - p + q)$$

$$n(p - 2) = (n - 2)q$$

$$\therefore q = \frac{n(p - 2)}{n - 2}$$

- (iv) If the graph is free from triangles each face must be atleast 4 - sided.

$$\therefore 2q \geq 4r$$

$$q \geq 2r$$

$$q \geq 2(2 - p + q)$$

$$\therefore q \leq 2p - 4$$

Theorem: 8.5

Kuratowski's graphs K_5 and $K_{3,3}$ are non planar.

Proof:

- (i) For K_5 , the number of vertices $p = 5$ and the number of edges $q = 10$.

If the graph is to be planar

$$q \leq 3p - 6$$

Clearly this inequality is not satisfied when $p = 5$, $q = 10$.

$\therefore K_5$ is non planar.

- (ii) For $K_{3,3}$, $p = 6$, $q = 9$.

Now $K_{3,3}$ is free from triangles. So if it is to be planar

$$q \leq 2p - 4$$

Clearly this inequality is not satisfied when $p = 6$, $q = 9$.

$\therefore K_{3,3}$ is non planar.

Theorem: 8.6

A graph G is embeddable in a plane iff it is embeddable on a sphere.

Proof:

Let G' be a plane embedding of the graph G . Consider a sphere, which touches the plane. Projective G' stereographically on the sphere we get a spherical embedding of the graph G .

Conversely, let G' be a spherical embedding of the graph G . Choose any point B on the sphere but not in G' . Then the image of G' under stereographic projection from B onto a plane is a plane embedding of G .

Theorem: 8.7

A planar graph G can be embedded in the plane in such a way that any specified vertex of G is on the exterior face of G .

Proof:

Let v be any specified vertex of the graph G . Let G' be a spherical embedding of G . Clearly G' is divided into interior and exterior faces. Let z be a point in the exterior face adjoining some face of G' which contains v . Then the image of G' under stereographic projection from z onto the plane contains v on the exterior face of G .

Boundary of a face:

Definition: 8.2.1

The **boundary** of a face f of a plane connected graph G is denoted by $b(f)$ is traversed twice.

A face is said to be incident with each one of its edges and vertices of its boundary. The number of edges incident with a face f is called the **degree** of the face f in G and is denoted by $d_G(f)$.

Every cut edge contributes 2 degrees to the face.

Theorem: 8.8

In a plane graph G the sum of degrees of all the faces is twice the number of edges.

Proof:

The boundaries of the faces of a plane graph are cycles and cut edges. In taking the sum of degrees of the faces each cyclic edge is counted twice, once for each adjacent faces. Also each cut edge is counted twice. Hence the sum of the degrees of the faces is twice the number of edges.

$$d_G(f_1) = 3$$

$$d_G(f_2) = 3$$

$$d_G(f_3) = 10$$

$$d_G(f_4) = 8$$

$$\sum_{i=1}^4 d_G(f_i) = 3 + 3 + 10 + 8$$

$$= 24$$

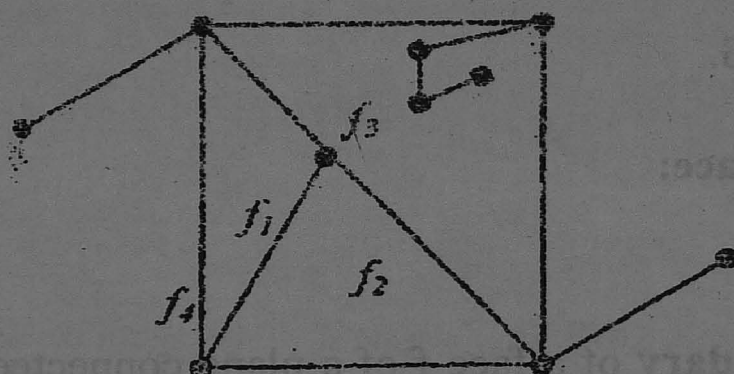


Figure 8.10

Now there are 12 edges.

Hence the sum of degrees of the faces is twice the number of edges.

Theorem: 8.9

Every planar graph G with $p \geq 3$ vertices has atleast three vertices of degree less than 6.

Proof:

Let q be the number of edges of the planar graph G of $p \geq 3$ vertices.

We know $q \leq 3p - 6$

$$\therefore 2p \leq 6p - 12$$

But $2q = \sum d(v_i)$

$$\therefore \sum d(v_i) \leq 6p - 12 \quad (1)$$

Since G is connected $d(v_i) \geq 1 \forall i$

Suppose there are atmost 2 vertices of degree less than 6 then,

$$\sum d(v_i) \geq 1 + 1 + 6 + 6 + \dots (p - 2) \text{ terms}$$

$$= 2 + 6(p - 2)$$

$$= 6p - 10$$

$$\sum d(v_i) \geq 6p - 10 \quad (2)$$

which is contradiction to (1)

Hence there must be atleast three vertices of degree less than 6.

Theorem: 8.10

If G is a (p, q) plane graph with $\delta(G) \geq 3$ then there is a face in G of degree ≥ 5 .

Proof:

Let r be the number of faces (region) of the given graph. If there is no face of degree ≤ 5 then the degree of each face is ≥ 6 .

$$\sum d_G(f) \geq 6r$$

$$\text{i.e., } 2q \geq 6r$$

$$\therefore r \leq \frac{q}{3}$$

Again since $\delta(G) \geq 3$

Sum of degrees of vertices $\geq 3p$

$$\therefore 2q \geq 3p$$

$$\therefore p \leq \frac{2q}{3}$$

By Euler's formula

$$p - q + r = 2$$

Here

$$p - q + r \leq \frac{2q}{3} - q + \frac{q}{3}$$

i.e.,

$$p - q + r \leq 0$$

This is a contradiction to Euler's formula.

Therefore our assumption that there is no face of degree ≤ 5 is wrong.

Hence there exists a face of degree ≤ 5 .

8.3 Platonic solids:

Let P be any polyhedron. Paint its vertices (corners) and edges (sides) by some colours and roll it on a sheet of white paper. Now we get a plane graph having the same number of vertices, edges and faces as the polyhedron. This plane graph is called the graph of the polyhedron. Thus a **Polyhedral graph** is a plane embedding of a polyhedron and so is a planar graph.

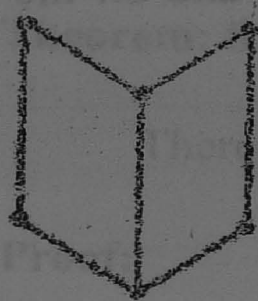
Definition: 8.3.1

Regular polyhedron:

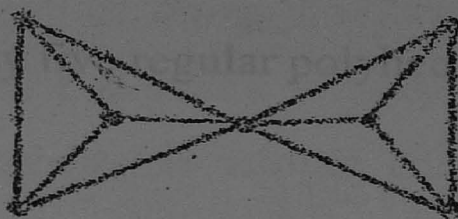
A polyhedron is said to be **regular** if all its faces are congruent polygons and there are equal number of faces meeting at each vertex.

Remarks:

A polyhedron is said to be convex, if the line segment joining any two points belonging to two distinct polygons, has only its end points common with the polyhedron.



Violates (a)



Violates (b)



These polygons are not connected

Figure 8.11: Arrangements of polygons which do not yield polyhedrons

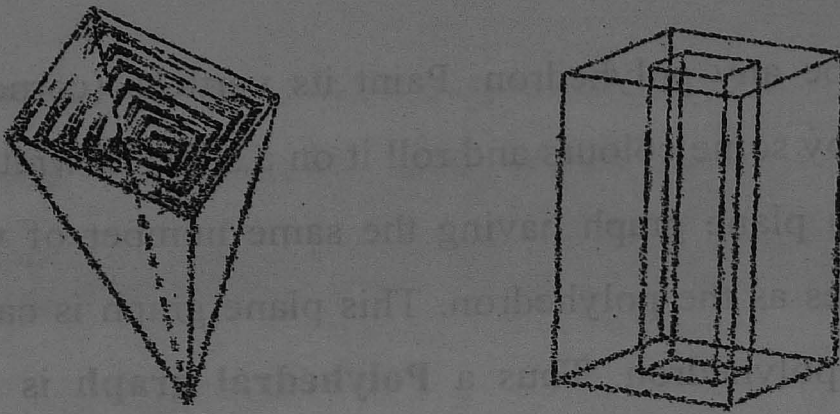


Figure 8.12 Two polyhedrons which are not convex

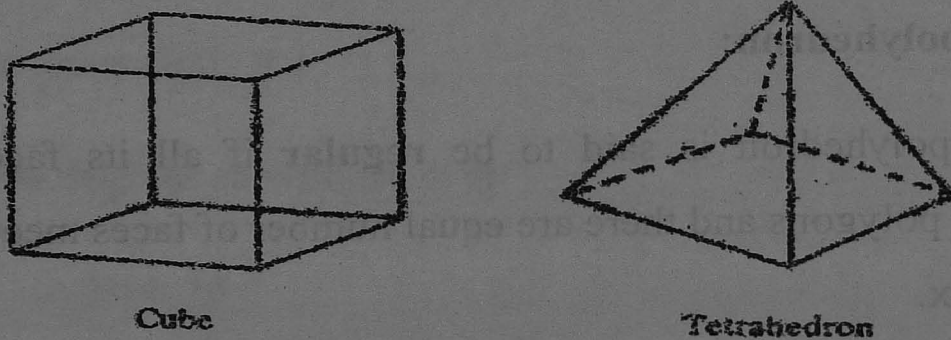


Figure 8.13: Two convex polyhedrons

Remarks:

In Figure 8.14 on the left we have 5 regular polyhedra (known also Platonic solids, because of Plato's interest in them), and on the right their associated plane graphs.



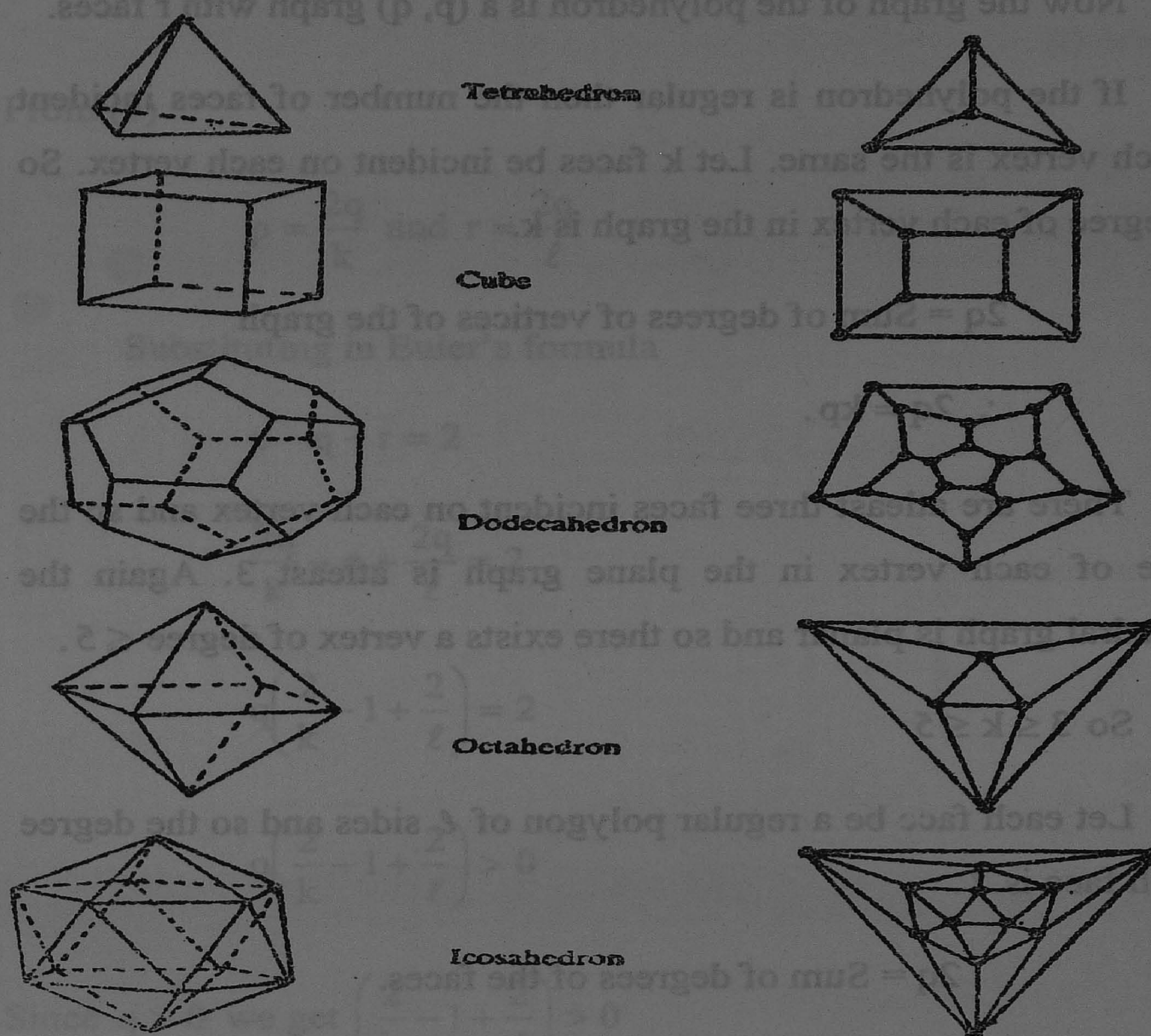


Figure 8.14

Theorem: 8.11

There are only five regular polyhedra.

Proof:

Let p , q , r be respectively the number of vertices, the number of sides and the number of faces of any polyhedron.

By Euler's formula

$$p - q + r = 2$$

Now the graph of the polyhedron is a (p, q) graph with r faces.

If the polyhedron is regular then the number of faces incident on each vertex is the same. Let k faces be incident on each vertex. So the degree of each vertex in the graph is k .

$$2q = \text{Sum of degrees of vertices of the graph}$$

$$\therefore 2q = kp.$$

There are atleast three faces incident on each vertex and so the degree of each vertex in the plane graph is atleast 3. Again the polyhedral graph is planar and so there exists a vertex of degree ≤ 5 .

$$\text{So } 3 \leq k \leq 5$$

Let each face be a regular polygon of ℓ sides and so the degree of each face is ℓ .

$$2q = \text{Sum of degrees of the faces.}$$

$$2q = \ell r$$

$$\text{Thus } 2q = kp = \ell r$$

Each face of the polyhedron graph is bounded by atleast 3 edges and so the minimum degree of a face is 3. By theorem there is face of degree ≤ 5 . So for the regular polyhedron each face is of degree ≤ 5 .

$$\therefore 3 \leq \ell \leq 5$$

Thus we have

$$3 \leq k, \ell \leq 5$$

From (1)

$$p = \frac{2q}{k} \text{ and } r = \frac{2q}{\ell}$$

Substituting in Euler's formula

$$p - q + r = 2$$

$$\frac{2q}{k} - q + \frac{2q}{\ell} = 2$$

$$q\left(\frac{2}{k} - 1 + \frac{2}{\ell}\right) = 2$$

$$q\left(\frac{2}{k} - 1 + \frac{2}{\ell}\right) > 0$$

Since $q > 0$ we get $\left(\frac{2}{k} - 1 + \frac{2}{\ell}\right) > 0$

$$2\ell - k\ell + 2k \geq 0$$

$$k\ell - 2\ell - 2k < 0$$

$$k\ell - 2\ell - 2k + 4 < 4$$

$$(k-2)(\ell-2) < 4$$

The only solution of the inequalities (2) and (3) are

$$(k, \ell) = (3, 3), (3, 4), (4, 3), (3, 5), (5, 3)$$

Hence there are only five regular polyhedrons.

Let us examine the nature of these regular polyhedrons.

Case (i)

$$k = 3, \ell = 3$$

$$\text{i.e., } 2q = kp = \ell r$$

$$2q = 3p = 3r$$

$$\therefore p = \frac{2q}{3}, r = \frac{2q}{3}$$

Substituting in

$$p - q + r = 2$$

$$\frac{2q}{3} - q + \frac{2q}{3} = 2$$

$$\therefore q = 6$$

$$\text{So } p = 4 \text{ and } r = 4$$

The regular polyhedron is the tetrahedron.

Case (ii)

$$k = 3, \ell = 4$$

$$\therefore 2q = 3p = 4r$$

$$p = \frac{2q}{3}, r = \frac{q}{2}$$

Substituting in

$$p - q + r = 2$$

$$\frac{2q}{3} - q + \frac{q}{2} = 2$$

$$q = 12$$

$$\therefore p = 8 \text{ and } r = 6$$

The regular polyhedron is the **cube**.

Case (iii)

$$k = 4, \ell = 3$$

$$2q = 4p = 3r$$

Proceeding as in case (ii), we get

$$q = 12, p = 6, r = 8$$

The regular polyhedron is the **Octahedron**.

Case (iv)

$$k = 3, \ell = 5$$

$$2q = 3p = 5r$$

$$p = \frac{2q}{3}, r = \frac{2q}{5}$$

Substituting in

$$p - q + r = 2$$

$$\frac{2q}{3} - q + \frac{2q}{5} = 2$$

$$q = 30$$

$$\therefore p = 20 \text{ and } r = 12$$

The regular polyhedron is **Dodecahedron**.

Case (v)

$$k = 5, \ell = 3$$

$$2q = 5p = 3r$$

Proceeding as in case (iv), we get

$$q = 30, p = 12, r = 20$$

The regular polyhedron is **Icosahedron**.

Thus the five regular polyhedrons are

S. No.	Name	p	q	r
1.	Tetrahedron	4	6	4
2.	Cube	8	12	6
3.	Octahedron	6	12	8
4.	Dodecahedron	20	30	12
5.	Icosahedron	12	30	20

The above regular polyhedrons are also known as **Platonic solids**. Thus there are only five Platonic solids viz tetrahedron, Cube, Octahedron, Dodecahedron and Icosahedron. These are the only five regular solids.

8.4 Dual of a plane graph:

A plane graph G partition the rest of the plane into a number of connected regions; the closures of these regions are called the faces of G . Fig. 8.15 shows a plane graph with six faces f_1, f_2, f_3, f_4, f_5 and f_6 . The notion of a face applies also to embeddings of other surfaces. We denote $F(G)$ and $\phi(G)$ respectively as the set of faces and the number

of faces of a plane graph G . Each plane graph has exactly one unbounded face, called the exterior face. In the plane graph of Fig. 8.15, f_1 is the exterior face.

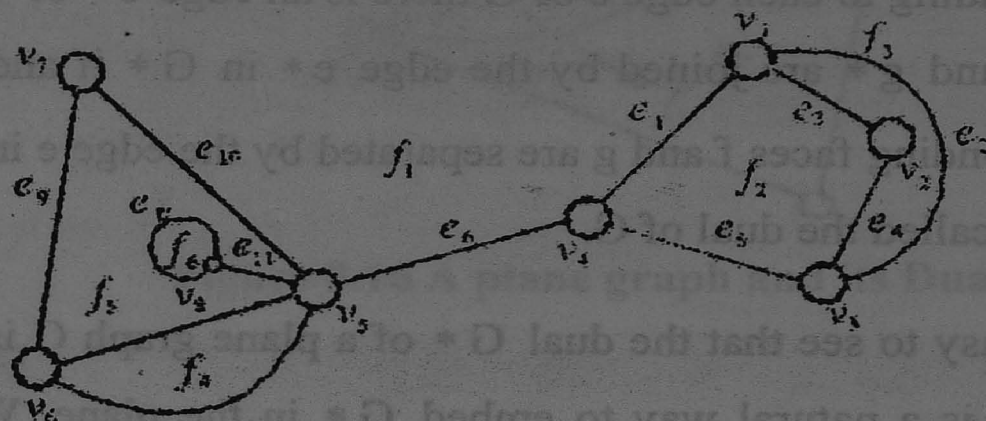


Figure 8.15: A plane Graph with six faces

We denote the boundary of a face f of a plane graph G by $b(f)$. If G is connected, then $b(f)$ can be regarded as a closed walk in which each cut edge of G in $b(f)$ is traversed twice; when $b(f)$ contains no cut edges, it is a cycle of G . For example, in the plane graph of Fig. 8.15

$$b(f_2) = v_1 e_3 v_2 e_4 v_3 e_5 v_4 e_1 v_1$$

$$b(f_5) = v_7 e_{10} v_5 e_{11} v_8 e_{12} v_8 e_{11} v_5 e_8 v_6 e_9 v_7$$

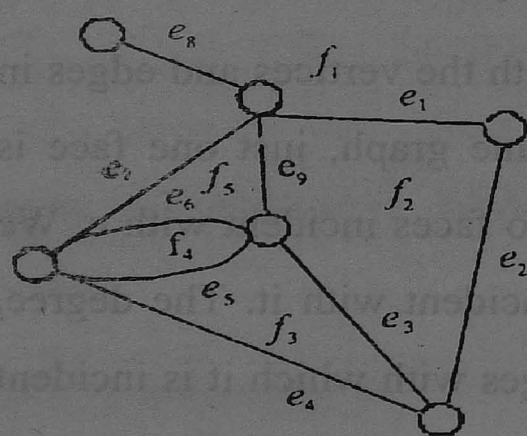
A face f is said to be incident with the vertices and edges in its boundary. If e is cut edge in a plane graph, just one face is incident with e , otherwise, there are two faces incident with e . We say that an edge separates the faces incident with it. The degree, $d_G(f)$, of a face f is the number of edges with which it is incident (that is, the number of edges in $b(f)$), cut edges being counted twice. In Fig. 8.15 f_1 is incident with the vertices $v_1, v_3, v_4, v_5, v_6, v_7$ and the edges $e_1, e_2, e_5, e_6, e_7, e_9, e_{10}, e_{11}$ separates f_1 and f_2 and e_{11} separates f_5 from f_5 ; $d(f_2) = 4$ and $d(f_5) = 6$.

Definition: 8.4.1

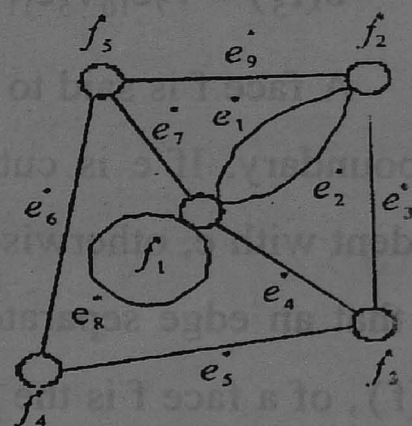
Given a plane graph G , one can define another graph G^* as follows: corresponding to each face f of G there is a vertex f^* of G^* , and corresponding to each edge e of G there is an edge e^* of G^* , two vertices f^* and g^* are joined by the edge e^* in G^* if and only if their corresponding faces f and g are separated by the edge e in G . The graph G^* is called the dual of G .

It is easy to see that the dual G^* of a plane graph G is planar; in fact there is a natural way to embed G^* in the plane. We place each vertex f^* in the corresponding face f of G , and then draw each edge e^* in such a way that it crosses the edge e and G exactly once (and crosses no other edge of G). It is intuitively clear that we can always draw the dual as a plane graph in this way, but we shall not prove this fact. Note that if e is a loop of G , then e^* is a cut edge of G^* , and vice versa.

Example:



G
(a)



G^*
(b)

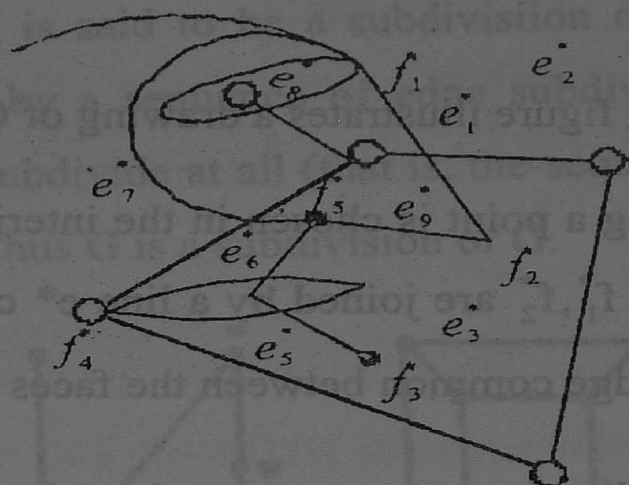


Figure 8.16 A plane graph and its Dual

Note:

1. G^* of a plane graph G as a plane graph we can consider the dual G^{**} if G^* and G^* and $G^{**} \equiv G$.

2. Isomorphic plane graphs may have nonisomorphic duals.

3. The following relations are direct consequences of the definition of G^* .

$$(i) \quad v(G^*) = \phi(G)$$

$$(ii) \quad \varepsilon(G^*) = \varepsilon(G)$$

$$(iii) \quad d_G^*(f^*) = d_G(f) \forall f \in F(G)$$

(OR) Dual of a plane graph:

Definition: 8.4.2

Let G be a plane graph with $V(G) = \{v_1, v_2, \dots, v_p\}$, $E(G) = \{e_1, e_2, \dots, e_q\}$ and $F(G) = \{f_1, f_2, \dots, f_r\}$. The dual (general) graph G^* of G has vertices $f_1^*, f_2^*, \dots, f_r^* (f_i \leftrightarrow f_i^*)$ and edges $e_1^*, e_2^*, \dots, e_q^* (e_j \leftrightarrow e_j^*)$ where the edge e_j^* joins the vertices f_m^* and f_n^* iff the edge e_j is common to the faces f_m and f_n in G .

The following figure illustrates a drawing of G^* .

In this drawing a point is chosen in the interior of every region and two such points f_1^*, f_2^* are joined by a line e^* crossing the edge e only, where e is an edge common between the faces f_1, f_2 .

Though the following statements are intuitively obvious from the definition of

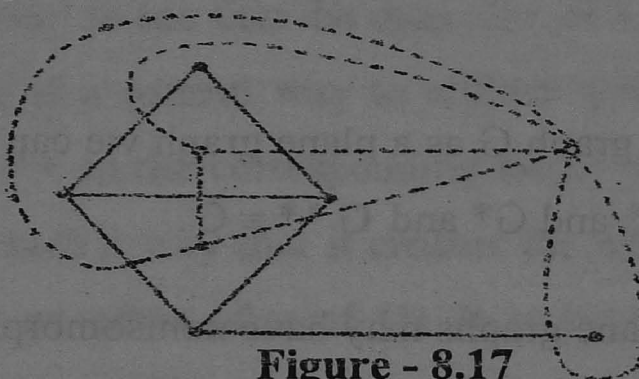


Figure - 8.17

The dual and its above drawing, the proofs are tedious and hence they are omitted.

- (i) G^* is plane; infact the above drawing gives a plane embedding of G .
- (ii) $(G^*)^* = G$.
- (iii) e is a cut-edge in G iff e^* is a loop in G^* .

Remark:

We have defined the dual of a plane graph and not of a planar graph.

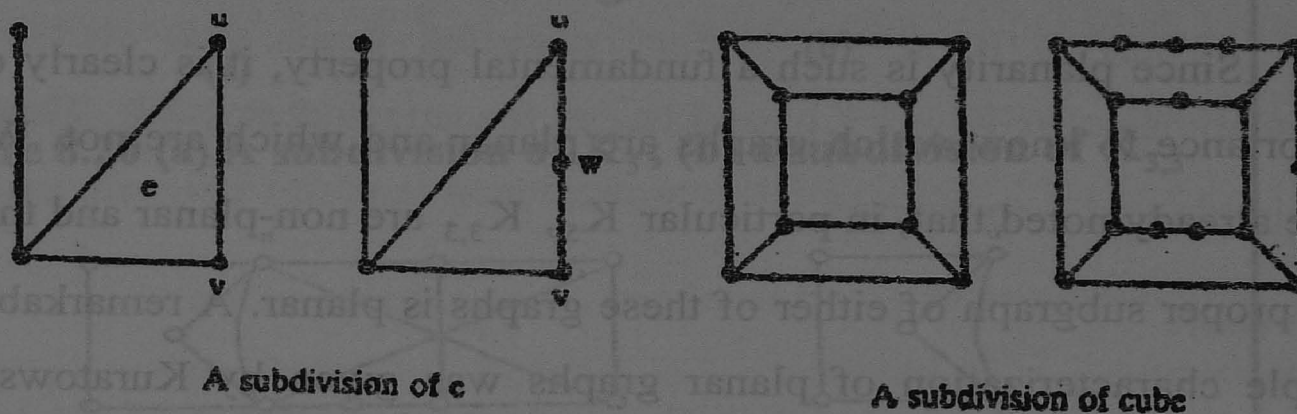
8.5 Characterization of planar graphs:

Definition: 8.5.1

The operation of replacing an edge $e = (u, v)$ in a graph G , by a path u, w, v ($w \notin V(G)$) of length 2, is called a subdivision of e .

Definition: 8.5.2

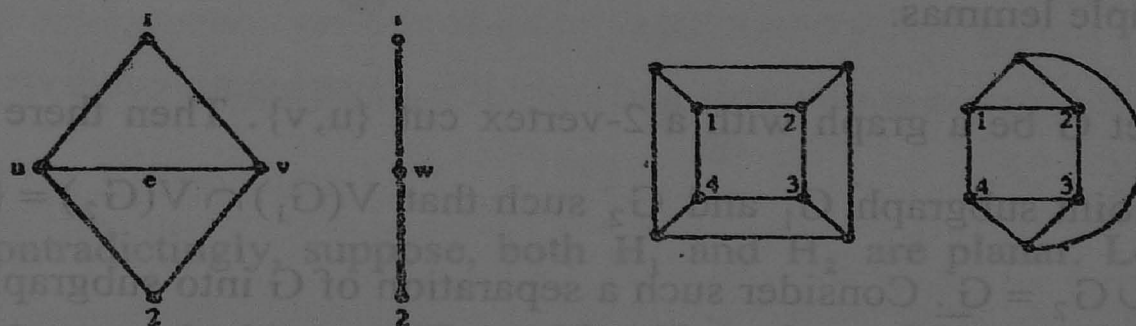
A graph H is said to be a subdivision of G if H can be obtained from G by a sequence of edge subdivisions. The case where we do not subdivide at all (that is, the sequence is empty) is also considered. Thus G is a subdivision of G .

**Figure 8.18****Definition: 8.5.3**

Let $e = (u, v)$ be an edge in graph G . Delete the vertices u, v from G . Add a new vertex w to $G - (u, v)$ and join w to all those vertices in $V - (u, v)$ which are adjacent to u or v in G . This operation is called an edge contraction of G .

Definition: 8.5.4

A graph G' is said to be a contraction of a graph G if G' can be obtained from G by a sequence of edge contractions. The case where we do not contract at all is also considered. Thus G is a contraction of G .

**Figure 8.19**

Kuratowski's Graphs:

The graphs K_5 and $K_{3,3}$ are called Kuratowski's graphs. A graph is planar if and only if it has no subgraph, homcomorphic to K_5 or $K_{3,3}$.

Since planarity is such a fundamental property, it is clearly of importance to know which graphs are planar and which are not. We have already noted that, in particular K_5 , $K_{3,3}$ are non-planar and that any proper subgraph of either of these graphs is planar. A remarkable simple characterization of planar graphs was given by Kuratowski (1930). This can be proved by two lemmas.

Lemma-8.1: If G is non-planar, then every subdivision of G is non-planar.

Lemma-8.2: If G is planar, then every subgraph of G is planar.

Proof:

Since K_5 , and $K_{3,3}$ are nonplanar, we can see that if G is planar, then G cannot contain a subdivision of K_5 or of $K_{3,3}$ (Fig 8.20). Kuratowski showed that this necessary condition is also sufficient.

Before proving Kuratowski's theorem, we need to establish two more simple lemmas.

Let G be a graph with a 2-vertex cut $\{u, v\}$. Then there exist edge-disjoint subgraph G_1 and G_2 such that $V(G_1) \cap V(G_2) = \{u, v\}$ and $G_1 \cup G_2 = G$. Consider such a separation of G into subgraphs. In both G_1 and G_2 join u

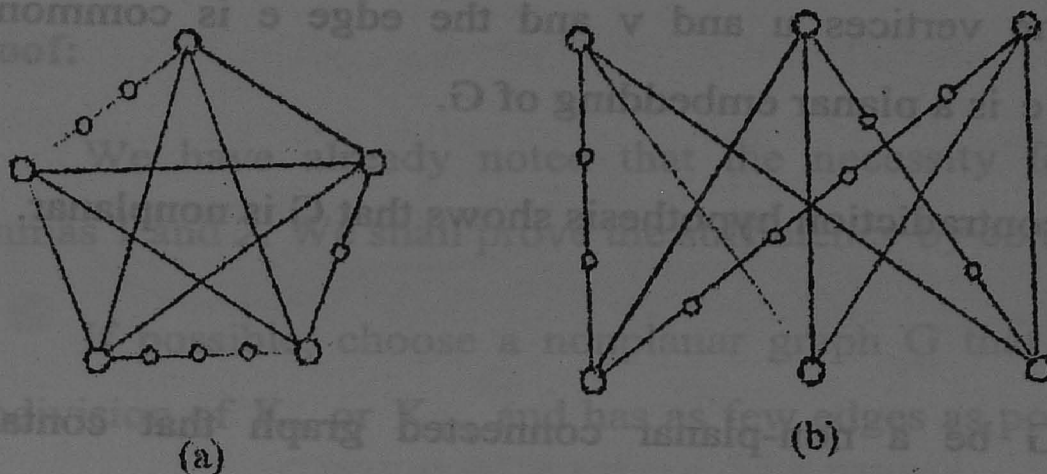


Figure 8.20 (a) A subdivision of K_5 ; (b) a subdivision of $K_{3,3}$

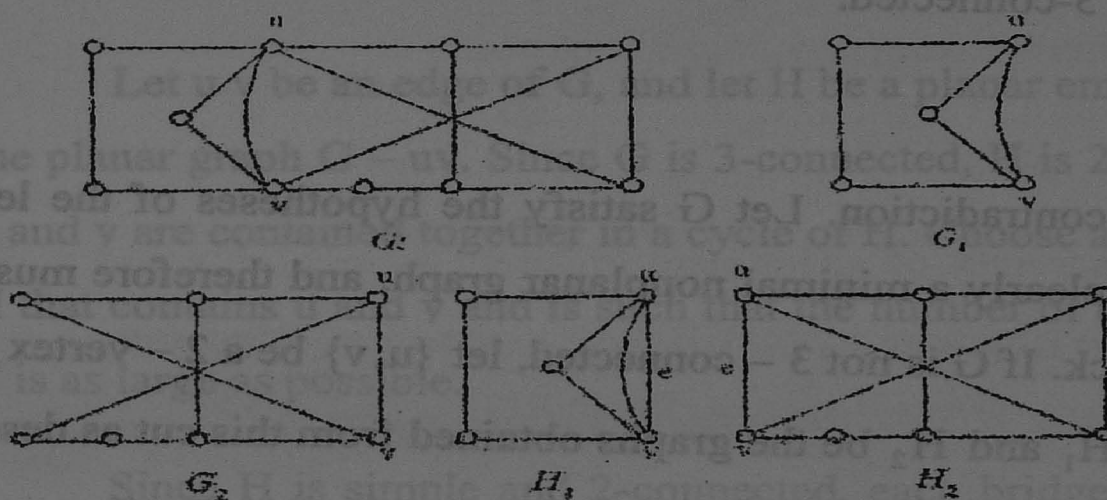


Figure 8.2.1

and v by a new edge e to obtain graphs H_1 and H_2 , as in Fig. 8.21.

Clearly $G = (H_1 \cup H_2) - e$. It is also easily seen that $\varepsilon(H_i) < \varepsilon(G)$ for $i = 1, 2$.

Lemma 8.3

If G is non-planar, then at least one of H_1 and H_2 is also non-planar.

Proof:

Contradictingly, suppose, both H_1 and H_2 are planar. Let H_1 be a planar embedding of H_1 , and let f be a face of $\bar{H}_1$ incident with e . If $\bar{H}_2$ is an embedding of H_2 in f such that H_1 and $\bar{H}_2$

have only the vertices u and v and the edge e is common, then $(\bar{H}_1 \cup H_2) - e$ is a planar embedding of G .

This contradiction hypothesis shows that G is nonplanar.

Lemma 8.4

Let G be a non-planar connected graph that contains no subdivision of K_5 or $K_{3,3}$ and has as few edges as possible. Then G is simple and 3-connected.

Proof:

By contradiction. Let G satisfy the hypotheses of the lemma. Then G is clearly a minimal nonplanar graph, and therefore must be a simple block. If G is not 3 – connected, let $\{u, v\}$ be a 2 – vertex cut of G and let H_1 and H_2 be the graphs obtained from this cut as described above. By lemma at least one of H_1 and H_2 , say H_1 , is nonplanar. Since $\varepsilon(H_1) < \varepsilon(G)$, H_1 must contain a subgraph K which is a subdivision of K_5 or $K_{3,3}$; moreover $K \supseteq G$, and so edge e is in K . Let P be a (u, v) – path in $H_2 - e$. Then G contains the subgraph $(K \cup P) - e$, which is a subdivision of K and hence a subdivision of K_5 or $K_{3,3}$. This contradiction establishes the lemma.

Theorem 5.12 Kuratowski's theorem

Statement:

A graph is planar if and only if it contains no subdivision of K_5 and $K_{3,3}$.

Proof:

We have already noted that the necessity follows from lemmas 1 and 2. We shall prove the sufficiency by contradiction.

If possible, choose a nonplanar graph G that contains no subdivision of K_5 or $K_{3,3}$ and has as few edges as possible. From lemma 8.4 it follows that G is simple and 3-connected. Clearly G must also be a minimal nonplanar graph.

Let uv be an edge of G , and let H be a planar embedding of the planar graph $G - uv$. Since G is 3-connected, H is 2-connected and u and v are contained together in a cycle of H . Choose a cycle C of H that contains u and v and is such that the number of edges in $\text{Int } C$ is as large as possible.

Since H is simple and 2-connected, each bridge of C in H must have at least two vertices of attachment. Now all outer bridges of C must be 2-bridges that overlap uv because, if some outer bridge were a k -bridge for $k \geq 3$ or a 2-bridge that avoided uv , then there would be an cycle C' containing u and v with more edges in its interior than C , contradicting the choice of C . These two cases are illustrated in Fig 8.22 (with C' indicated by heavy lines)

In fact, all outer bridges of C in H must be single edges. For if a 2-bridge with vertices of attachment x and y had a third vertex, the set $\{x, y\}$ would be a 2-vertex cut of G , contradicting the fact that G is 3-connected.

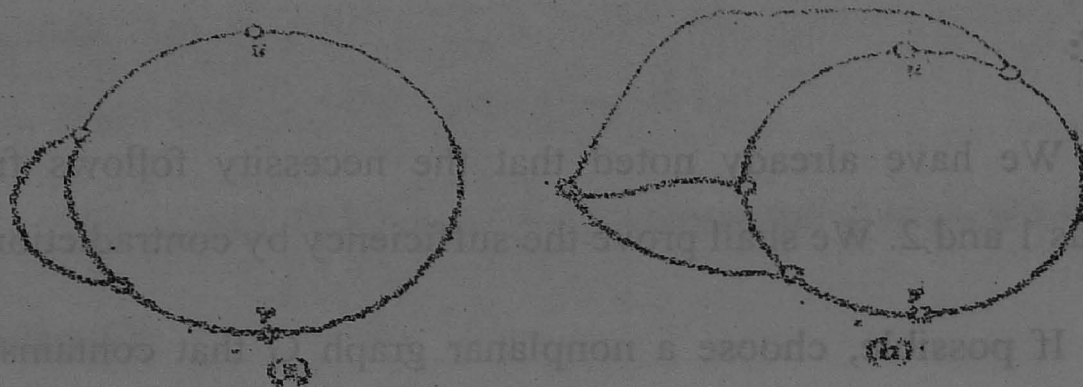


Figure 8.22 Vertex attachment

Two cases now arise, depending on whether B has a vertex of attachment different from u, v, x and y or not.

Case 1

B has a vertex of attachment different from u, v, x and y . We can choose the notation so that B has a vertex of attachment v_1 in $C(x, u)$ (see Fig 8.23). We consider two subcases, depending on whether B has a vertex of attachment in $C(y, v)$ or not.

Case 1(a)

B has a vertex of attachment v_2 in $C(y, v)$. In this case there is a (v_1, v_2) -path P in B that is internally-disjoint from C . But then $(C \cup P) + \{uv, xy\}$ is a subdivision of $K_{3,3}$ in G , a contradiction (see Fig 8.23)

Case 2(b)

B has no vertex of attachment in $C(y, v)$. Since B is skew to uv and to xy , B must have vertices of attachment v_2 in $C\{u, y\}$ and v_3 in $C\{v, x\}$. Thus B has three vertices of attachment v_1, v_2 and v_3 . There exists a vertex v_0 in $V(B) \setminus V(C)$ and three paths P_1, P_2 and P_3 in B joining v_0 to v_1, v_2 and v_3 , respectively, such that, for $i \neq j$, P_i and P_j have only the vertex v_0 in common. But now

$(C \cup P_1 \cup P_2 \cup P_3) + \{uv, xy\}$ contains a subdivision of $K_{3,3}$ a contradiction. This case is illustrated in Fig 8.24. The subdivision of $K_{3,3}$ is indicated by heavy lines.

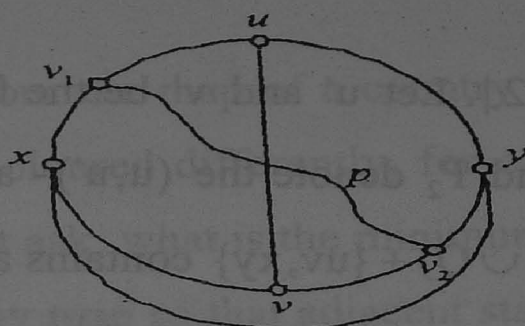


Figure 8.23

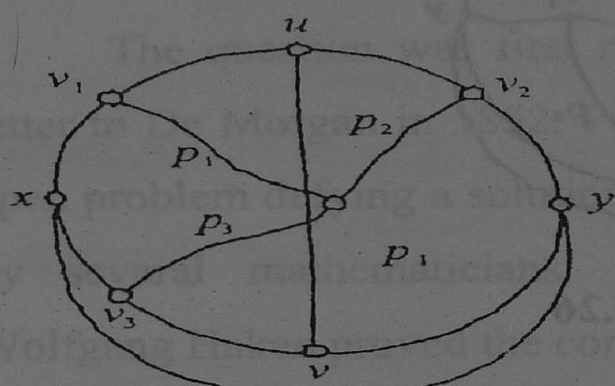


Figure 8.24

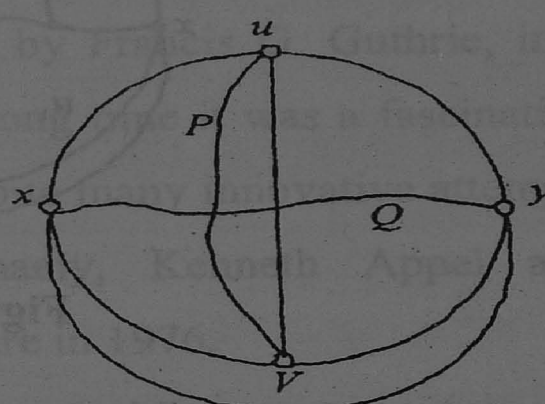


Figure 8.25

Case 2

B has no vertex of attachment other u, v, x and y . Since B is skew to both uv and xy , it follows that u, v, x and y must all be vertices of attachment of B. Therefore there exists a (u,v) -path P and an (x,y) -path Q in B such that (i) P and Q are internally-disjoint from "C, and (ii) $|V(P) \cap V(Q)| \geq 1$. We consider two subcase, depending on whether P and Q have one or more vertices in common.

Case 2a

$|V(P) \cap V(Q)| = 1$. In this case $(C \cup P \cup Q) + \{uv, xy\}$ is a subdivision of K_5 in G , again a contradiction (see Fig. 8.25)

Case 2b

$|V(P) \cap V(Q)| \geq 2$. Let u' and v' be the first and last vertices of P on Q , and let P_1 and P_2 denote the (u, u') - and (v', v) -sections of P . Then $(C \cup P_1 \cup P_2 \cup Q) + \{uv, xy\}$ contains a subdivision of $K_{3,3}$ in G , once more a contradiction (see Fig. 8.26)

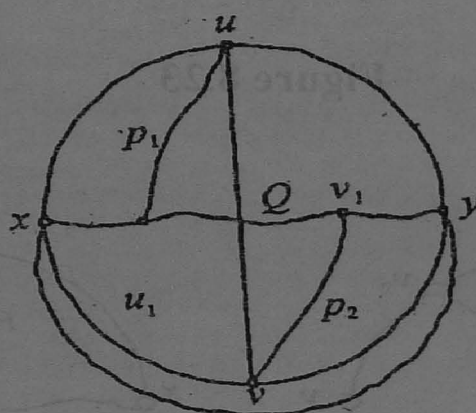


Figure 8.26

Thus all the possible cases lead to contradictions, and the proof is complete.

UNIT- 9

COLOURINGS

9.0 Introduction:

When we colour the map of a country we see to it that its adjacent states are coloured differently, for easy identification of the states. One might ask: what is the minimum number of colours required to colour any map so that adjacent states receive different colours?

The famous four colour conjecture states that four colours are sufficient.

The question was first raised by Francis G. Guthrie, in a letter to De Morgan in 1852. For a long time it was a fascinating open problem defying a solution despite many innovative attempts by several mathematicians. Ultimately, Kenneth Appel and Wolfgang Haken proved the conjecture in 1976.

Clearly, a map is a plane graph with states corresponding to faces. As per 4 C. C. we should be able to colour the faces of a plane graph with 4 colours in such a way that adjacent faces (that is faces sharing a common line) receive different colours.

Let G be a plane graph and G^* be its dual. The face colouring of G is equivalent to the 'vertex colouring' of G^* . Though in the beginning, 4 C. C. was the main motivation for the study of colourings, subsequently however, the concept of colourings has found applications in such diverse fields as the theory of automata and storage problems in operations research.

Generally we can use any finite set as the colour set. The nature of the colouring problem depends on the number of colours but not on what they are.

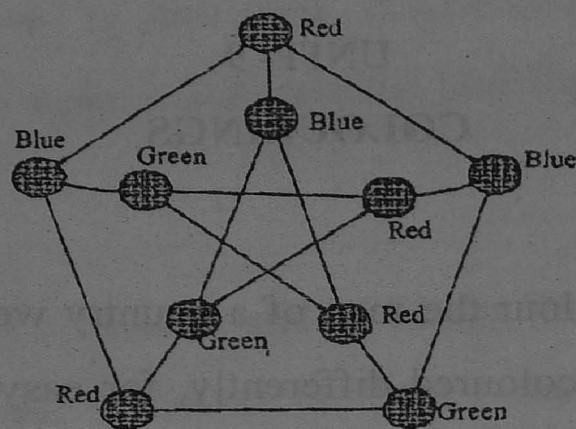


Figure 9.0: A proper vertex colouring of the Petersen Graph with 3 colours, the minimum Number possible

Colouring of graph enjoys many practical applications as well as theoretical challenges. Beside the classical types of problems, different limitations can be set on the graph, or on the way a colour is assigned, or even on the colour itself. It has reached popularity with the general public in the form of popular number puzzle Sudoku. Graph colouring is very active in the field of research.

In this unit we introduce the concept of vertex colouring and Edge colouring. We give some simple theorems on colouring and its equivalent views. Finally we give an algorithm for colouring of a graph.

9.1 Vertex colouring:

Definition: 9.1.1

By **vertex colouring** or **colouring** of a graph we mean a assigning different colours to the vertices of the graph such that no two adjacent vertices get the same colour.

Suppose G is a graph of n vertices. It is always possible to colour the vertices with n different colour. But our purpose is to find the minimum number of colours that will be sufficient to colour the graph.

We use the numbers 1, 2, 3, for denoting the different colours.

Consider the different colourings of the graph.

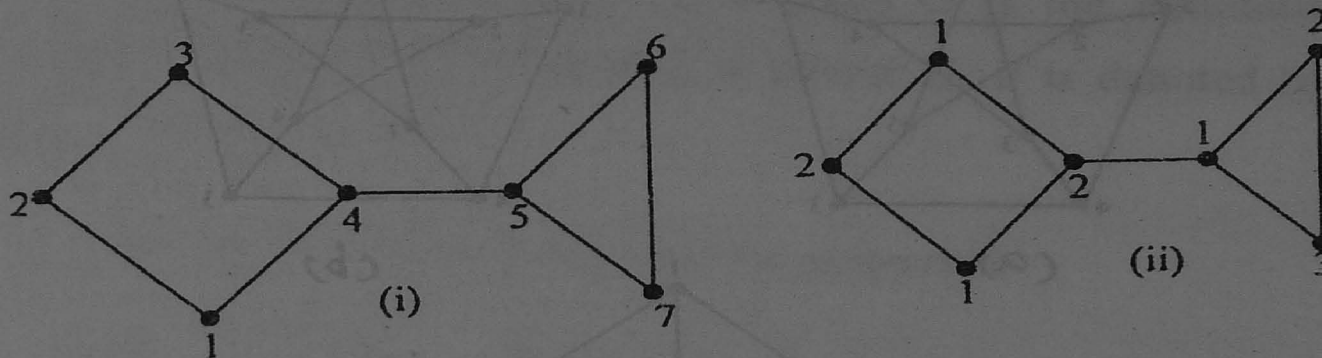


Figure 9.1

In (i) we have used the maximum number of colours namely the number of vertices of the graph.

In (ii) we have used the minimum number of colours namely three only.

In (iii) the graph is said to be properly coloured.

Thus if a graph is coloured with the minimum possible number of colours it is said to be **properly coloured**. However we may use “coloured” to mean “properly colored”.

Note: The above definition is also stated as,

A k -vertex colouring (or briefly, a k -colouring) of a graph G is an assignment of k colours to the vertices of G such that no two adjacent vertices receive the same colour.

Colours are usually, though not always, represented by positive integers. Thus a k -colouring of G is a function $f : V(G) \rightarrow (1, 2, \dots, k)$ such that $f(u) \neq f(v)$, whenever $(u, v) \in E(G)$.

Example:

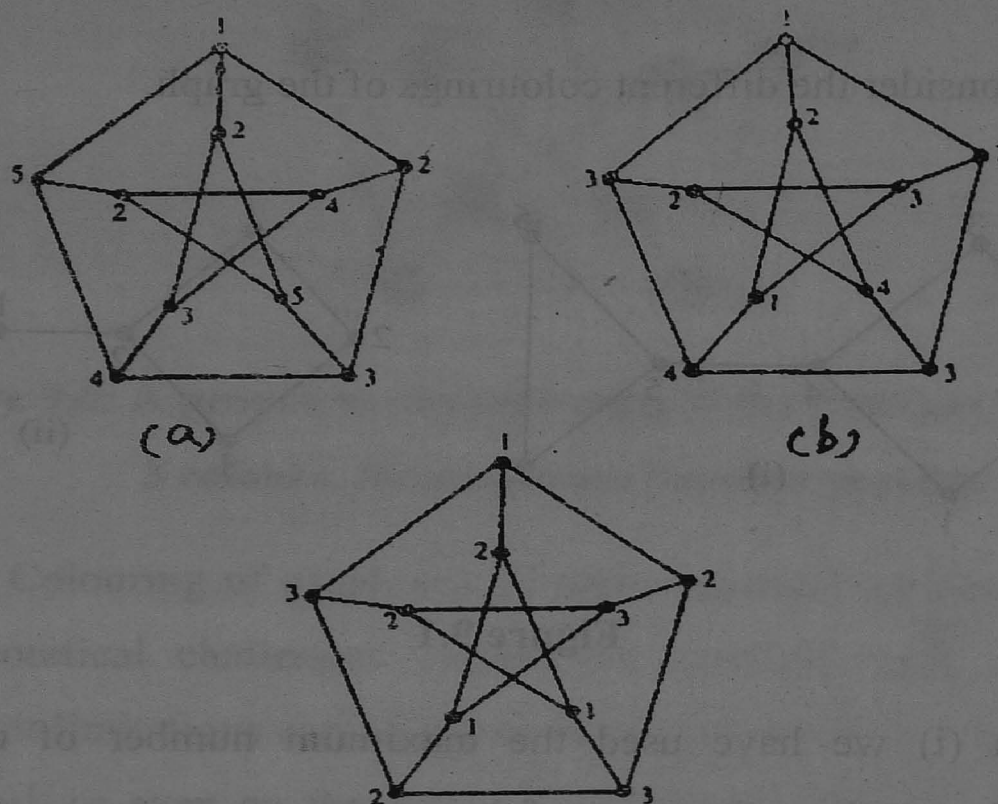


Figure 9.2

Here Figure 9.2 (a) $\rightarrow$ 5 colours

Figure 9.2 (b) $\rightarrow$ 4 colours

Figure 9.2 (c) $\rightarrow$ 3 colours

Remarks:

- In a disconnected graph the colouring of different components are independent. Therefore it is enough we consider only connected graphs for the purposes of colouring.
- (2) If G consists of only isolated vertices then it can be coloured with only one colour.
- (3) Since every pair of vertices of a complete graph K_n are adjacent we require n different colours for colouring K_n .

Definition: 9.1.2**Chromatic number:**

The **chromatic number** of a graph G is the minimum number of colours that is required to colour G . It is denoted by $\chi(G)$ or $\Psi(G)$

If $\chi(G) \leq k$ then G is said to be **k-colourable**

Thus a graph is k -colourable if the number of colours required to colour it is less than or equal to k .

If $\chi(G) = k$, then G is said to be **k=chromatic**.

Note:

If H is any sub graph of G then $\chi(H) \leq \chi(G)$.

A graph G is said to be **critical** if $\chi(H) < \chi(G)$ for every proper subgraph H of G . A graph G is said to be **k-critical** if G is critical and $\chi(G) = k$.

Thus a k -critical graph is a critical graph with chromatic number k .

Note:

Suppose G is a disconnected graph with connected components $G_1, G_2, \dots, G_n$ whose chromatic numbers are $\chi_1, \chi_2, \dots, \chi_n$ then the chromatic number of G is $\chi = \max \{\chi_i\}, i = 1, 2, 3, \dots, n$.

Theorem: 9.1

A critical graph must be connected.

Proof:

If possible let G be a critical graph which is disconnected with components $G_1, G_2, \dots, G_n$. Let $\chi_1, \chi_2, \dots, \chi_n$ be the chromatic numbers of these components. Then the chromatic number of G is $\chi = \max \{\chi_i\}, i = 1, 2, \dots, n$

$$\chi = \chi_p \text{ (say)}$$

Since G is critical we have

$\chi(H) < \chi(G)$ for every subgraph H of G .

$$\therefore \chi(G_p) < \chi(G)$$

$$\text{(i.e.) } \chi_p < \chi.$$

This is a contradiction to $\chi = \chi_p$

Therefore G cannot be disconnected.

Hence the critical graph G is connected.

Remarks:

- (1) Let G be a graph of n vertices. Clearly it is n -colourable. Let $\chi(G)$ be the chromatic number then $\chi(G)$ is the minimum number of colours required for (proper) colouring of G . So if G is k -colourable then we must have $\chi(G) \leq k \leq n$.
- (2) Let K_n be a complete graph of n vertices. In this graph every pair of vertices are adjacent and so no two vertices can be coloured with the same colour. Therefore we require as many colours as the number of vertices. That is we require n colours so that the chromatic number of K_n is n , i.e. $\chi(K_n) = n$.

- (3) If any one vertex v is removed from the complete graph K_n then $K_n - v$ is also a complete graph of $(n - 1)$ vertices K_{n-1} .
 $\therefore \chi(K_n - v) = \chi(K_{n-1}) = n - 1$
- (4) If G is a totally disconnected graph, the graph contains only isolated vertices and so no two vertices are adjacent. Therefore we can colour the vertices with just one colour.
 $\therefore \chi(G) = 1$
- (5) If the graph G has atleast one edge we require atleast two colours to colour the graph
 $\therefore \chi(G) \geq 2$
- (6) If the connected graph G has a complete subgraph of r vertices then $\chi(G) \geq r$.
- (7) If G is connected graph $\chi(G) \geq 2$.
- (8) A triangle is a complete graph of 3 vertices and so it is 3-chromatic
 $\therefore \chi(G) = 3$.
- (9) If a graph G contains a triangle as a subgraph then
 $\chi(G) \geq 3$.
- (10) An even cycle is 2-chromatic and an odd cycle is 3-chromatic. Therefore if a graph G contains an even cycle then $\chi(G) \geq 2$.
 If G contains an odd cycle then $\chi(G) \geq 3$.
- (11) A tree is 2 - chromatic. That is $\chi(T) = 2 \forall T$.

Theorem: 9.2

A graph is 2-chromatic if and only if it has no odd cycles.

Proof:

Suppose the graph G has an odd cycle C then C is 3-chromatic and so $\chi(G) = 3$. As C is a subgraph of G , $\chi(G) \geq \chi(C) = 3$.

(i.e) $\chi(G) \geq 3$.

So G is not 2-chromatic.

Suppose G has no odd cycles, we have to prove G is 2-chromatic.

Consider a spanning tree T of the graph G . We know T is 2-chromatic.

To the tree T we add the chords so as to get the given graph G .

Since G has no cycle of odd length, the end vertices of every chord are differently coloured in T and so in G . Thus G is coloured with 2 colours. Therefore G is 2-chromatic.

Corollary: 9.2.1

A bipartite graph is 2-chromatic.

Theorem: 9.3

If G is k -critical then $\delta \geq k - 1$.

Proof:

Given that the graph G is k -critical we have to prove that the minimum degree $\delta \geq k - 1$.

Suppose $\delta < k - 1$. Therefore there is a vertex v of degree $\delta < (k - 1)$. Clearly $G - v$ is a proper subgraph of G which is k -critical, $\chi(G) = k$.

$$\therefore \chi(G - v) < \chi(G)$$

$$\chi(G - v) < k.$$

So $G - v$ is $(k - 1)$ colourable.

Now the set of vertices of $G - v$ can be partitioned into $k - 1$ sets $V_1, V_2, \dots, V_{k-1}$ such that the vertices in the set V_i are coloured with the colour i ($1 \leq i \leq k - 1$).

Again since $d(v) = \delta < k - 1$ the vertex v is adjacent to $\delta < k - 1$ vertices in G . We find that there is a colour j in the above set of $k - 1$ colours which is not assigned to the vertices adjacent to v . Assign the colour j to v . Thus the vertices of G are coloured with $(k - 1)$ colours. This is a contradiction because G is k -critical.

So our assumption that $\delta < (k - 1)$ is wrong. Hence $\delta \geq k - 1$.

Corollary: 9.3.1

In a k -chromatic graph the number of vertices of degree at least $k - 1$ is at least k .

Proof:

Given that G is a k -chromatic graph. Let H be a k -critical subgraph of G . We know the minimum degree of any vertex in H is $\delta_H \geq k - 1$

$\therefore$ The degree of any vertex v in H is at least $k - 1$

$$(i.e.,) d_H(v) \geq k - 1$$

$$\text{Also we have } d_G(v) \geq d_H(v)$$

$$\therefore d_G(v) \geq v - 1$$

Again since H is k – critical atleast k colours are used to colour H so that each vertex receives different colour.

So there must be atleast k -vertices in H . So the number of vertices of degree atleast $(k - 1)$ in G is atleast k .

Theorem: 9.4

For any graph G , $\chi \leq \Delta + 1$

Proof:

Suppose $\chi > \Delta + 1$

$$\therefore \chi \geq \Delta + 2$$

$$\chi - 1 \geq \Delta + 1$$

By previous corollary there are atleast χ vertices each of whose degree is atleast $\chi - 1$. Therefore atleast there are χ vertices each of degree atleast $\Delta + 1$.

This is a contradiction for no vertex can be of degree $> \Delta$, the maximum degree of the graph. So our assumption $\chi > \Delta + 1$ is wrong.

$$\therefore \chi \leq \Delta + 1$$

Remarks:

(i) From $\chi \leq \Delta + 1$, we find that the maximum value of the chromatic number of any graph is $\Delta + 1$. Therefore any graph whose maximum degree is Δ is $(\Delta + 1)$ – colourable.

(ii) We know $\chi(K_n) = n$ and $\Delta(K_n) = n - 1$

$$\therefore \chi = \Delta + 1$$

Again for an odd cycle C , $\chi(C) = 3$, $\Delta(C) = 2$

$$\therefore \chi = \Delta + 1$$

Definition: 9.1.3

Chromatic Partition:

Consider any colouring of vertices of a graph G . The set of vertices having the same colour is said to form a **colour class**. If n colours are used then there are n colour classes. Let the colour classes be $C_1, C_2, \dots, C_n$. Clearly they are non empty and mutually disjoint. Further no two vertices in the same class are adjacent and the union of the colour classes is the vertex set of the graph.

$$C_1 \cup C_2 \cup \dots \cup C_n = V(G)$$

Therefore $\{C_1, C_2, \dots, C_n\}$ is a partition of the vertex set of G . This partition of the vertex set of G is called a **chromatic partition** of the graph. The number of members in the partition is the number of colours used. So the minimum number of members in any chromatic partition is equal to the chromatic number of the graph.

Consider the colouring of the following graph with five colours 1,2,3,4,5.

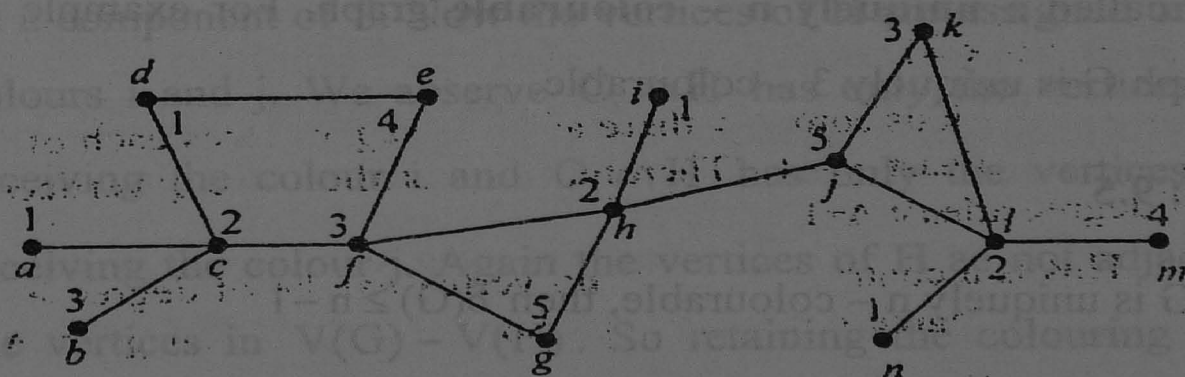


Figure 9.3

The corresponding Chromatic partition is

$$\{\{a, b, i, n\}, \{c, h, \ell\}, \{b, f, k\}, \{e, m\}, \{g, j\}\}$$

Note:

Generally the chromatic partition of a graph is not unique

Definition: 9.1.4

Uniquely colourable graphs:

A graph is said to be **uniquely colourable** if it has a unique chromatic partition that is there is only one way of partitioning the vertices into colour classes.

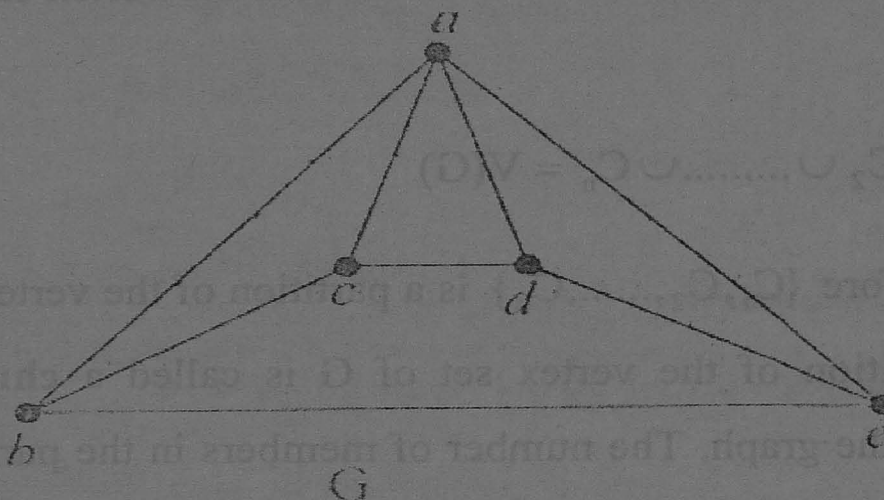


Figure 9.3

Here $\{\{a\}, \{b, d\}, \{c, e\}\}$ is the only chromatic partition of the graph and so the graph G is uniquely colourable.

If the chromatic number of a uniquely colourable graph is n then it is called a **uniquely n – colourable** graph. For example the above graph G is uniquely 3 – colourable.

Theorem: 9.5

If G is uniquely n – colourable, then $\delta(G) \geq n - 1$

Proof:

Given that the graph G is uniquely n – colourable. Consider a colouring of G with n colours. Let v be a vertex. After assigning a colour to v , $(n - 1)$ colours remain. All these $(n - 1)$ colours must be assigned to the vertices adjacent to v , if not any one of the remaining colours may be assigned to v and a different n – colouring of the graph may be obtained. This is a contradiction as the graph is uniquely colourable. Therefore there must be $(n - 1)$ vertices adjacent to v to receive the $(n - 1)$ colours. So the number of edges incident on v must be atleast $(n - 1)$. That is $\deg(v)$ must be atleast $(n - 1)$. Since v is arbitrary we find that the minimum degree of the graph must be atleast $(n - 1)$. Hence $\delta(G) \geq n - 1$.

Theorem: 9.6

Let G be a uniquely n – colourable graph. In any colouring of G , the subgraph induced by the union of any two colour classes is connected.

Proof:

Consider a colouring of the uniquely n – colourable graph G . Let the n colours be denoted by $1, 2, 3, \dots, n$ and the colour classes by $C_1, C_2, C_3, \dots, C_n$. Let S be the subgraph induced by the union of the colour classes C_i and C_j . If S is disconnected let H be a component of S . Now the vertices of H are assigned only the colours i and j . We observe $C_i \cap H$ has only the vertices of H receiving the colour i and $C_j \cap H$ has only the vertices of H receiving the colour j . Again the vertices of H are not adjacent to the vertices in $V(G) - V(H)$. So retaining the colouring of the vertices of $V(G) - V(H)$ and interchanging the colours i and j in

$C_i \cap H$ and $C_j \cap H$ we get a new n – colouring of G . This is a contradiction as G is uniquely colourable. So G is not disconnected. That is G is connected.

Note:

The converse of the above theorem is not true. For example consider the 3 – colourable graph G .

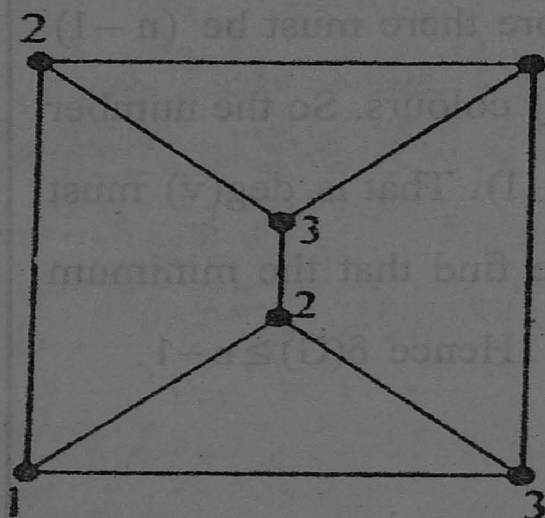


Figure 9.4

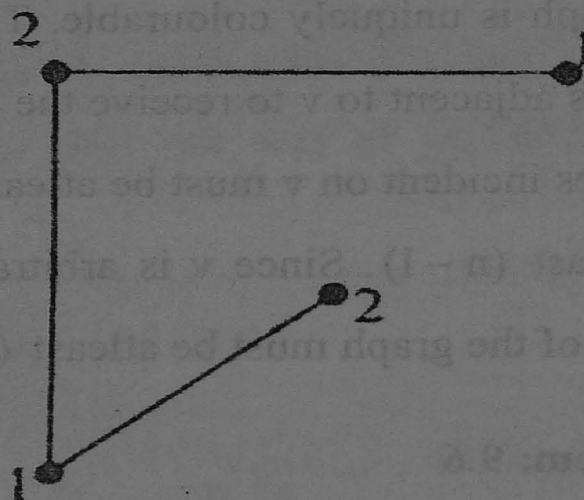


Figure 9.5

The subgraph induced by the colour classes C_1 and C_2 is clearly connected. (Figure 9.5)

But G is not uniquely 3 – colourable.

Theorem: 9.7

Every uniquely n – colourable graph is $(n - 1)$ connected.

Proof:

Let G be a uniquely n – colourable graph. We have to prove G is $(n - 1)$ – connected. Suppose G is not $(n - 1)$ connected then there exists a set S of atmost $(n - 2)$ – vertices such that $G - S$ is either trivial or disconnected.

If $G - S$ is trivial then G is of at most $(n - 1)$ vertices only and so G is not uniquely n - colourable.

If $G - S$ is disconnected, then it has at least two components. Consider a n - colouring of G . As S has almost $(n - 2)$ vertices there are at least two colours say i and j which are not assigned to any of the vertices of S . If a component of $G - S$ has colours different from i and j , then assigning the colours i or j to some vertices of this component we get a different n - colouring of G . If any component $G - S$ contain the colours i and j , simply by interchanging these colours we get a different n colouring of G . Thus in any case it is possible to get a n - colouring different from the given n -colouring of G . This is contradiction as G is uniquely n - colourable. Hence G must be $(n - 1)$ connected.

Theorem: 9.8

The following statements are equivalent for any graph G .

- (i) G is 2 - colourable
- (ii) G is bipartite
- (iii) G has no odd cycle.

Proof

(i) $\Rightarrow$ (ii)

Given that the graph G is 2 - colourable. So the vertex set of G can be partitioned into two colour classes X and Y such that no two vertices of the same class are adjacent. Therefore (X, Y) is a bipartition of G and hence G is a bipartite graph.

(ii) $\Rightarrow$ (i)

Space for Hints

This is a contradiction to the assumption that the graph is without odd cycles.

$\therefore$ No two vertices of X are adjacent. Again we prove that no two vertices of Y are adjacent.

Suppose two vertices $u, v \in Y$ be adjacent so that uv is an edge.

Consider any vertex $y \in Y$.

Now the $(y - u)$ path the $(y - v)$ path and the edge uv together form a cycle.

$$\text{Length of this cycle} = d(y, u) + d(y, v) + 1$$

$$= \text{odd} + \text{odd} + 1$$

$$= \text{odd}$$



Figure (ii)

(i.e.,) This cycle is an odd cycle. It is a contradiction to the assumption that the graph is without odd cycles.

Therefore no two vertices of Y are adjacent.

Thus the vertex set of the graph G is divided into the non empty disjoint subsets X , Y such that no two vertices of X are adjacent and no two vertices of Y are adjacent.

$\therefore (X, Y)$ is bipartition of G and so the graph G is a bipartite graph.

Hence the theorem.

Theorem: 9.9

For any simple graph G

$$\chi(G) \leq 1 + \max \delta(H)$$

Where the maximum is taken over all the induced subgraph H of G .

Proof:

The result is obvious for a totally disconnected graph. So we assume that the graph G is connected or has a connected component so that $\chi(G) \geq 2$. Let $\chi(G) = n$. Let G' be the smallest induced subgraph of G such that $\chi(G') = n$. Let v be any vertex of G' so that $\chi(G' - v) = n - 1$. That is $G' - v$ can be coloured with $(n - 1)$ colours.

We claim: $d_{G'}(v) \geq n - 1$.

Suppose not, then $d_{G'}(v) < n - 1$. In this case we can assign the vertex v a colour (out of the $n - 1$ colours) which is not assigned to any of its adjacent vertices in G' . So G' is colourable with $(n - 1)$ colours. This is a contradiction as $\chi(G') = n$.

$$\therefore d_{G'}(v) \geq n - 1$$

Since the vertex v is arbitrary the above is true for every vertex of G' .

Space for Hints

$\therefore$ minimum degree of $G' \geq n - 1$.

$$\text{(i.e.,)} \quad \delta(G') \geq n - 1$$

$$n - 1 \leq \delta(G')$$

$$n \leq 1 + \delta(G')$$

for any induced subgraph G' .

Taking the maximum of δ among all the induced subgraph H we get

$$\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H)$$

Theorem: 9.10

If G is a simple graph of n vertices and $\bar{G}$ is its complement then

$$\chi(G) + \chi(\bar{G}) \leq n + 1$$

Proof:

G is a simple graph of n vertices and $\chi(G)$ is its chromatic number.

We shall prove the result by induction on the number of vertices n .

When $n = 2$.

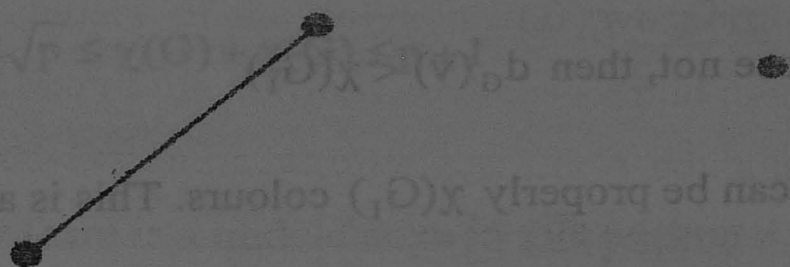


Figure 9.6

$$\chi(G) = 2 \quad \chi(\bar{G}) = 1$$

$$\chi(G) + \chi(\bar{G}) = 2 + 1$$

$$\therefore \chi(G) + \chi(\bar{G}) \leq n + 1 \text{ holds in this case}$$

Let us assume that the result be true for all graphs of less than n vertices.

Let v be any vertex and $G_1 = G - v$. Clearly G_1 has $(n - 1)$ vertices and so the result holds for G_1 .

$$\therefore \chi(G_1) + \chi(\bar{G}_1) \leq (n - 1) + 1$$

$$\chi(G_1) + \chi(\bar{G}_1) \leq n \quad (1)$$

Since $G_1 = G - v$

$$\chi(G_1) = \chi(G - v)$$

$$\chi(G) = \chi(G_1) + 1$$

$$\therefore \chi(G) = \chi(G_1) + 1 \quad (2)$$

$$\text{Similarly} \quad \chi(\bar{G}) = \chi(\bar{G}_1) + 1 \quad (2)$$

Space for Hints

We claim $d_G(v) \geq \chi(G_1)$

Suppose not, then $d_G(v) < \chi(G_1)$

So G can be properly $\chi(G_1)$ colours. This is a contradiction as

$$\chi(G) = \chi(G_1) + 1$$

$$\therefore d_G(v) \geq \chi(G_1)$$

Similarly $d_{\bar{G}}(v) \geq \chi(\bar{G}_1)$

$$\therefore d_G(v) + d_{\bar{G}}(v) \geq \chi(G_1) + \chi(\bar{G}_1) \quad (3)$$

Now

$$G \cup \bar{G} = K_n$$

$$\text{and } d_{K_n}(v) = n - 1$$

$$\therefore d_{G \cup \bar{G}}(v) = n - 1$$

$$\text{i.e., } d_G(v) + d_{\bar{G}}(v) = n - 1 \quad (4)$$

From (3) and (4)

$$\chi(G_1) + \chi(\bar{G}_1) + 1 \leq n - 1$$

$$\text{i.e., } \chi(G) + \chi(\bar{G}) \leq n + 1$$

Theorem: 9.11

If G is a graph on p vertices then

$$(i) \quad p \leq \chi(G) + \chi(\bar{G}) < \left(\frac{p+1}{2} \right)^2$$

$$(ii) \quad 2\sqrt{p} \leq \chi(G) + \chi(\overline{G}) \leq p + 1$$

Proof:

Let $\chi(G) = k$. So there exists a partition $\{V_1, V_2, \dots, V_k\}$ of the vertex set V of G such that no two vertices of V_i are adjacent for $1 \leq i \leq k$. Let $t = \max_{1 \leq i \leq k} |V_i|$. Since no two vertices of V_i are adjacent, the induced graph $G(V_i) = K_t$. Therefore $\overline{G}$ contains K_t as a subgraph

$$\therefore \chi(\overline{G}) \geq t.$$

Since p vertices are distributed in k classes the average number of vertices in each class is $\frac{p}{k}$. As t is the maximum number of vertices in any class we find,

$$t \geq \frac{p}{k}$$

$$\therefore \chi(\overline{G}) \geq \frac{p}{k}$$

$$\chi(G) \cdot \chi(\overline{G}) \geq k \cdot \frac{p}{k}$$

$$p \leq \chi(G) \cdot \chi(\overline{G}) \quad (1)$$

Again by previous theorem

$$\chi(G) + \chi(\overline{G}) \leq (p + 1) \quad (2)$$

We know

$$AM \geq GM$$

Space for Hints

$$\frac{\chi(G) + \chi(\bar{G})}{2} \geq \sqrt{\chi(G) \cdot \chi(\bar{G})} \quad (ii)$$

$$\therefore \chi(G) \cdot \chi(\bar{G}) \leq \left(\frac{\chi(G) + \chi(\bar{G})}{2} \right)^2$$

$$\chi(G) \cdot \chi(\bar{G}) \leq \left(\frac{p+1}{2} \right)^2 \quad (3)$$

From (1) and (3)

$$p \leq \chi(G) \cdot \chi(\bar{G}) \leq \left(\frac{p+1}{2} \right)^2$$

Again

$$\chi(G) + \chi(\bar{G}) \geq 2\sqrt{\chi(G) \cdot \chi(\bar{G})}$$

$$\geq 2\sqrt{p}$$

$$\therefore 2\sqrt{p} \leq \chi(G) + \chi(\bar{G}) \quad (4)$$

From (2) and (4)

$$2\sqrt{p} \leq \chi(G) + \chi(\bar{G}) \leq p+1$$

Theorem: 9.12

There exists a k -colouring of a graph G iff $V(G)$ can be partitioned into k subsets $V_1, V_2, \dots, V_k$ such that no two vertices in $V_i, i = 1, 2, \dots, k$ are adjacent.

Proof Necessity:

Let $f : V(G) \rightarrow (1, 2, \dots, k)$ be a k -colouring of G .

$V_i = \{u \in V(G) : f(u) = i\}$, ($1 \leq i \leq k$), that is V_i is the set of vertices coloured i . Then $V = V_1 \cup \dots \cup V_k$ is a partition and moreover no two vertices in V_i ($1 \leq i \leq k$) are adjacent.

Sufficiency:

Let $V = V_1 \cup V_2 \cup \dots \cup V_k$ be a partition of $V(G)$ such that no two vertices in V_i ($1 \leq i \leq k$) are adjacent. Then the function $f : V(G) \rightarrow (1, 2, \dots, k)$, defined by $f(u) = i$, if $u \in V_i$, is a k -colouring of G .

Theorem: 9.13

If H is a subgraph of a graph G , then

$$\chi(G) \geq \chi(H)$$

Proof:

Let $\chi(G) = k$ and let f be a k -colouring of G . Then f restricted to $V(H)$ is a k -colouring of H . Hence, $\chi(H) \leq k = \chi(G)$.

An important consequence is the following.

Corollary 9.13.1:

If G contains a complete subgraph K_n , then $\chi(G) \geq n$.

Problem: 1

Prove that $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$

$G_1 \vee G_2$ is the join of the graphs G_1 and G_2 . It is obtained by joining the vertices of G_1 with vertices of G_2 . So the vertices of G_1 are adjacent to the vertices of G_2 . Therefore the colours used for colouring G_1 cannot be used for colouring G_2 . The number of colours required for colouring $G_1 \vee G_2$ is $\chi(G_1) + \chi(G_2)$.

$$\text{Hence } \chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$$

Problem: 2

Prove that a non-trivial tree is 2-chromatic.

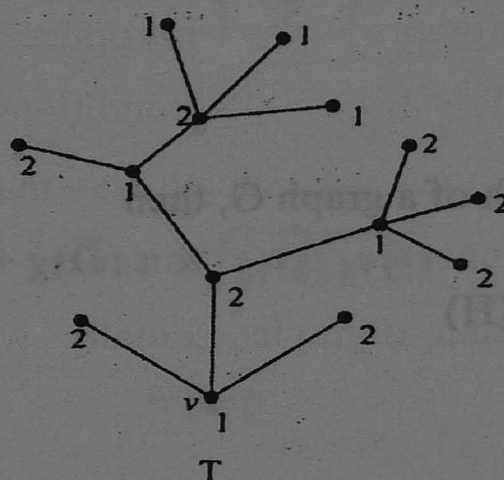


Figure 9.7

T is a non trivial tree and so it has atleast one edge and so $\chi(T) \geq 2$. So T cannot be coloured with one colour. Consider two colours 1 and 2. Take any vertex v and assign colour 1 to it. Assign colour 2 to the vertices adjacent to v . Now assign colour 1 to those vertices adjacent to the vertices which have received colour 2. Proceed like this, assigning colours 1 and 2 alternately till all the vertices are assigned colours. We observe that vertices at odd distances from v are assigned colour 2 and at even distances are assigned colour 1. As the tree is free from cycles no two adjacent vertices receives the same

colour. So the tree is properly coloured with 2 colours. Hence $\chi(T) = 2$. (i.e.,) T is 2-chromatic.

Problem: 3

If G is a (p, q) -graph, prove that $\chi(G) \geq \frac{p^2}{p^2 - 2q}$

Let $\chi(G) = k$. So the vertex set V can be partitioned into the non empty, mutually disjoint subsets (colour classes) $V_1, V_2, \dots, V_k$ such that $V_1 \cup V_2 \cup \dots \cup V_k = V$.

$$\therefore |V| = \sum_{i=1}^k |V_i|$$

$$P = \sum_{i=1}^k p_i \text{ where } p_i = |V_i|$$

In each of the colour class V_i the vertices are non adjacent.

Therefore the number of edges of G is.

$$q = \sum_{i=1}^k \frac{p_i(p_i - 1)}{2}$$

$$= \frac{p(p-1)}{2} - \sum_{i=1}^k \frac{p_i(p_i - 1)}{2}$$

$$2q = p^2 - p - \sum_{i=1}^k p_i^2 + \sum_{i=1}^k p_i$$

$$= p^2 - p - \sum_{i=1}^k p_i^2 + p$$

$$2q = p^2 - \sum_{i=1}^k p_i^2$$

$$\sum_{i=1}^k p_i^2 = p^2 - 2q \quad (1)$$

By Cauchy Schwartz inequality

$$\sum_{i=1}^k p_i^2 \geq \frac{\left(\sum_{i=1}^k p_i\right)^2}{k}$$

$$\sum_{i=1}^k p_i^2 \geq \frac{p^2}{k} \quad (2)$$

From (1) and (2)

$$p^2 - 2q \geq \frac{p^2}{k}$$

$$\therefore k \geq \frac{p^2}{p^2 - 2q}$$

9.2 Edge colouring:

Definition 9.2.1:

By edge colouring of a graph we mean assigning colours so edges of the graph. The edge colouring is said to be proper if no two adjacent edges get the same colour.

A graph G is said to be **k -edge colourable** if its edges can be properly coloured with k or less than k colours. Clearly a graph of n edges is n – edge colourable.

If k is the minimum number of colours required to properly colour the edges of a graph G then k is said to be the **edge chromatic number** of G and is denoted by $\chi'(G)$ or χ' .

If $\chi'(G) = k$, then G is said to be **k – edge chromatic**.

Theorem: 9.14

For any simple graph G , $\chi' \geq \Delta$

Proof:

Δ is the maximum degree of the graph G . Therefore there is a vertex v such that $\deg v = \Delta$. So there are Δ edges incident on this vertex v . All these Δ edges are adjacent. So we require atleast Δ colours to properly colour the edges of the graph G .

Hence, $\chi' \geq \Delta$.

Corollary: 9.14.1

For any simple graph G

$$\chi' \geq \Delta \geq \chi - 1$$

Proof:

If χ' is the edge chromatic number then

$$\chi' \geq \Delta$$

If χ is the (vertex) chromatic number, then

$$\chi \leq \Delta + 1$$

$$\therefore \Delta \geq \chi - 1$$

$$\chi' \geq \Delta \geq \chi - 1$$

Remarks:

From the above $\chi' \geq \chi - 1$

$$\therefore \chi \leq \chi' + 1$$

Thus the (vertex) chromatic number of a simple graph is at most 1 greater than the edge chromatic number.

Edge chromatic partition:

The edge of a graph receiving the same colour form an edge colour class. Therefore a proper edge colouring of a graph partitions the edge set into a number of edge colour classes. This partition of the edge set is called the **edge chromataic partition** of the graph.

Improvement of a colouring:

In a proper edge colouring of a graph, if an edge of colour i is incident on a vertex v then we say that the colour i is **represented** by the vertex v .

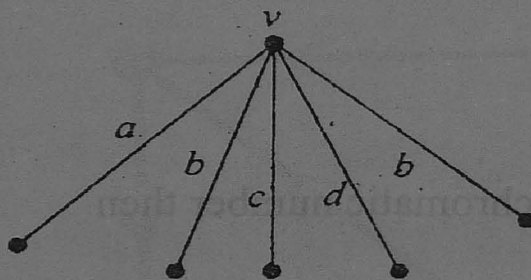


Figure 9.8

If C denote an edge colouring of a simple graph G then for each vertex v of G the number of distinct colours represented by v is denoted by $C(v)$. If a, b, c, d are the distinct colours represented by v then $C(v) = 4$.

If the edge colouring is proper then all the edges incident on v get different colours and so the number of colours represented by v is the same as the number of edges incident on v . Therefore $C(v) = d(v)$.

Let C be an edge colouring of a graph G so that $C(v)$ is the number of colours represented by the vertex v . Find the sum $\sum C(v)$ of the colours represented by the all vertices of G . Let C' be another colouring and the find sum $\sum C'(v)$ of the colours represented by the vertices of G . If $\sum C'(v) > \sum C(v)$ then the colouring C' is said to be an **improvement** of the colouring of C .

An edge colouring C is said to be an optimal edge colouring if there is no improvement of C .

Clearly a proper edge colouring is an optimal edge colouring.

Theorem: 9.15

If G be a connected graph which is not an odd cycle, then G has a 2-edge colouring in which both colours are represented at each vertex of degree atleast 2.

Proof:

The given connected graph G is either Eulerian or non Eulerian.

Case (i) G is Eulerian

Suppose G is a cycle, Since G is given to be not an odd cycle, clearly G must be an even cycle. Therefore there are an even number of edges which can be alternately assigned the colours 1 and 2. So G is 2 – edge colourable. Further it is obvious that every vertex represents both colours. So the theorem is true in this case.

Suppose G is not a (even) cycle. Since G is Eulerian every vertex is of even degree. Again since G is not a cycle there is

atleast vertex v (say) of degree atleast 4. Let $ve_1v_1e_2v_2\ldots\ldots\ldots e_qv$ be an Euler tour of G .

$$\text{Let } E_1 = \{E_i \mid i \text{ is odd}\}$$

$$E_2 = \{e_i \mid i \text{ is even}\}$$

Now edges of the graph can be coloured with two colours 1 and 2 such that the edges in E_1 receive colour 1 and the edges in E_2 receive colour 2. Thus (E_1, E_2) is a 2-edge colouring of G . Every vertex has edges of the form e_{2k+1} and e_{2k} . So the colours 1 and 2 are represented at each vertex of the graph.

Case (ii)

G is non – Eulerian graph.

Since G is non Eulerian there are some odd vertices and the number of odd vertices is even. Let v be a new vertex. Join v to the odd vertices G and get a new graph G' . Clearly G' is Eulerian.

Therefore as before the edge set of G' can be partitioned into E'_1 and E'_2 such that (E'_1, E'_2) is a 2 – edge colouring of G' in which both colours are represented at each vertex. Now $E \cap E'_1$, and $E \cap E'_2$ form a partition of the edge set E of the given graph G . Therefore $(E \cap E'_1, E \cap E'_2)$ is a 2 – edge colouring of G having the required property.

Theorem: 9.16

Let $C = \{E_1, E_2, \ldots, E_k\}$ be an optimal k – edge colouring of a graph G . If there exists a vertex u and colour i and j such that i is not represented at u and j is represented atleast twice at u , then the

component H of $G[E_i \cup E_j]$ that contains u is an odd cycle.

Proof:

Given $C = \{E_1, E_2, \dots, E_k\}$ is an optimal k – edge colouring of the graph G . u is a vertex of G such that the colour i is not represented at u and the colour j is represented atleast twice at u .

E_i is the set of edges of G getting the colour i and E_j is the set of edges of G getting the colour j . H is given to be the component $E_i \cup E_j$ of G . We have to prove that H is an odd cycle containing u .

Suppose H is not an odd cycle, then by previous theorem, H has a 2 – edge colouring in which both colours are represented at each vertex of degree atleast 2 in H . We can recolour the edges of H with the colours i and j with the above property. So we obtain a new k – edge colouring $C' = \{E'_1, E'_2, E'_3, \dots, E'_k\}$ of G .

If $C'(u)$ denotes the number of distinct colours at u in the colouring C' then we have $C'(u) = C(u) + 1$.

Since now both the colours i and j are represented at u in C'

$$C'(u) > C(u)$$

Again for any vertex v in H of degree greater than or equal to 2, two colours i and j are represented at v in C' while atmost two colours are represented at v in C .

$$\therefore C'(v) > C(v) \forall v \in H$$

$$\therefore \sum C'(v) > \sum C(v)$$

This shows that the colouring C' is an improvement of the colouring C of the graph G . This is a contradiction as the colouring C is an optimal edge colouring of G .

So our assumption that H is not an odd cycle is wrong. Hence H must be an odd cycle.

Theorem: 9.17

If G is a bipartite graph then $\chi' = \Delta$

Proof:

Suppose $\chi' > \Delta$

Let $C = \{E_1, E_2, \dots, E_\Delta\}$ be an optimal Δ -edge colouring of G . Then there must exist two adjacent edges of G which receive the same colour in C for otherwise C will be a proper edge colouring of G .

So there exists a vertex u such that the number of colours represented at u is less than the number of edges incident on it.

$$\therefore C(u) < d(u)$$

So there is a colour not represented at u and a colour represented at least twice at u . Therefore by previous theorem, G contains an odd cycle.

This is a contradiction as G is a bipartite graph.

So our assumption that $\chi' > \Delta$ is wrong

$$\therefore \chi' \leq \Delta$$

Again for any graph G

$$\chi' \geq \Delta$$

Hence $\chi' = \Delta$

Note:

From above we observe that a bipartite graph G is Δ - edge chromatic where Δ is the maximum degree of G .

Theorem: 9.18

Vizing's Theorem:

Statement:

For a simple graph G , either $\chi' = \Delta$ or $\chi' = \Delta + 1$.

Proof:

Let G be a simple graph whose edge chromatic number is χ' . We have to prove $\chi' = \Delta$ or $\chi' = \Delta + 1$.

For any simple graph

$$\chi' \geq \Delta \quad [\because \text{by theorem 9.14}]$$

Therefore in order to prove the result it is enough we prove that $\chi' \leq \Delta + 1$. Suppose $\chi' > \Delta + 1$, then G is $(\Delta + 1)$ - edge colourable. Let $C = \{E_1, E_2, \dots, E_{\Delta+1}\}$ be an optimal edge colouring of G . Let u be a vertex of G such that $C(u) < d(u)$ then there exists colours i_0 and i_1 and i_0 is not represented at u and i_1 is represented twice at u .

Space for Hints

Let uv_1 be an edge with colour i_1 since $d(v_1) \leq \Delta$ for any vertex of G we get,

$$d(v_1) < \Delta + 1$$

$\therefore$ Number of edges incident at v_1 is less than the number of given colours.

So some colour i_2 is not represented at v_1 , for otherwise a recolouring of uv_1 , with the colour i_2 gives an improvement of C which is impossible as C is an optimal edge colouring.

This colour i_2 must be represented at u . Let uv_2 be an edge with colour i_2 . Again since $d(v_2) < \Delta + 1$ some colour i_3 is not represented at v_2 . But this colour i_3 must be represented at u , for otherwise a recolouring of uv_1 , with i_2 and uv_2 with i_3 gives an improvement on the colouring C which is impossible.

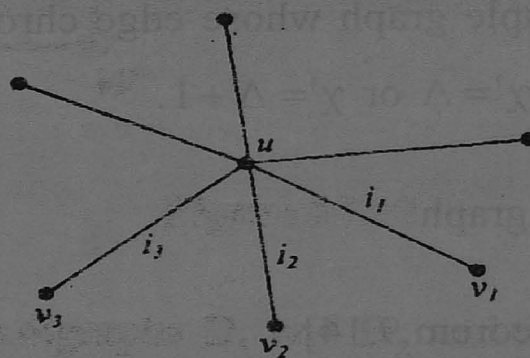


Figure 9.9

Let uv_3 be an edge with colour i_3 proceeding like this we get a sequence of vertices $v_1, v_2, v_3, \dots$ and a sequence of colours $i_1, i_2, i_3, \dots$ such that the edge uv_j gets the colour i_j and i_{j+1} is not represented at v_j .

Since the degree of u is finite there exists a smallest integers such that for some positive integer $r < s, i_r = i_s + 1$. We obtain a recolouring of G as follows.

For $i \leq j \leq r-1$, recolour uv_j with the colour i_{j+1} . The recolouring is a new $(\Delta + 1)$ - edge colouring $C' = \{E'_1, E'_2, \dots, E'_{\Delta+1}\}$.

Clearly $C'(v) \geq C(v)$ for all v in G . So C' is an optimal $(\Delta + 1)$ - edge colouring of G . By theorem the component $H' = G(E'_{i_0}, \cup E'_{i_r})$ that contains u in an odd cycle.

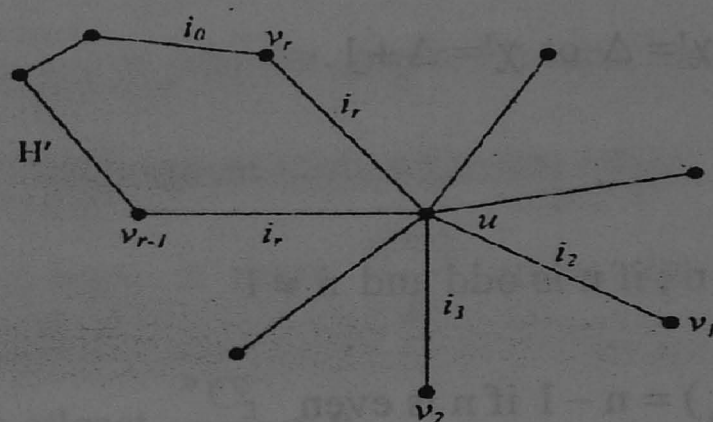


Figure 9.10

As uv_r is coloured with i_r, v_r is a vertex of H' . H' being a cycle the degree of V_r in H' is 2. So either i_r is represented twice at v_r or each of i_r and i_0 is represented once at v_r .

Now further recolour the edges of G as follows. Assign colour i_{j+1} to the edge uv_j for $i \leq j \leq s-1$ and uv_s . This gives another $(\Delta + 1)$ - edge colouring.

$$C'' = \{E''_1, E''_2, \dots, E''_{\Delta+1}\}$$

As before C'' is also an optimal $(\Delta + 1)$ -edge colouring of G . As i_o is not represented at u and i_r is represented by uv_{r-1} and uv_s we find the component H'' of $G - (E_{i_o}'' \cup E_{i_r}'')$ that contains u is an odd cycle.

In the above colouring only i_r and i_o is represented only once at v_r . In any case v_r is a vertex of degree 1 only in H'' . This is a contradiction as H'' is a cycle. The contradiction proves that our assumption that $\chi' > (\Delta + 1)$ is wrong.

$$\therefore \chi' \leq \Delta + 1$$

$$\text{Thus } \chi' \geq \Delta \text{ and } \chi' \leq \Delta + 1$$

$$\therefore \chi' = \Delta \text{ or } \chi' = \Delta + 1.$$

Theorem: 9.19

$$\chi'(K_n) = n, \text{ if } n \text{ is odd and } n \neq 1$$

$$\text{and } \chi'(K_n) = n - 1 \text{ if } n \text{ is even}$$

Proof:

Case (i)

$$n \text{ is odd and } n \neq 1$$

When $n = 3, K_n = K_3$ is a triangle and $\chi'(K_3) = 3$. Thus the result is true.

Assume $n > 3, K_n$ is a complete graph of n vertices. Since every pair of vertices are adjacent we get a n -sided polygon whose vertices are internally connected. All the n sides of the polygon can be coloured with n different colours. Take any internal edge e . This

the boundary of the polygon into two portions A and B. Since n is odd one of the portions contains an odd number of edges and the other an even number of edges. Let A contain odd number of edges. Now there is a middle edge in this portion A. Clearly this edge and e are non adjacent. So we can assign the edge e the same colour as that of the middle edge of A. This method of colouring can be extended to all the internal edges. Thus the internal edges receive some of the different colours which are already assigned to the boundary edges. Thus the edges of the graph K_n can be coloured with n colours. So the edge chromatic number $\chi'(K_n) \leq n$.

Again since n is odd the graph K_n can have at most $\frac{n-1}{2}$ sets of mutually independent (non adjacent) edges. Therefore each colour class can have at most $\frac{n-1}{2}$ edges. So the number of colour classes is atleast $\frac{n C_2}{n-1} = n$.

i.e., the number of colours required is atleast n

$$\therefore \chi'(K_n) \geq n.$$

$$\text{Hence } \chi'(K_n) = n$$

Case (ii)

n is even

When $n = 2$, the result is obvious

Assume $n \geq 4$.

Space for Hints

Let $v_1, v_2, \dots, v_{n-1}, v_n$ be the vertices of K_n . Consider the subgraph induced by the vertices $v_1, v_2, \dots, v_{n-1}$. Clearly it is K_{n-1} . As $n-1$ is odd, by case (i) $\chi'(K_{n-1}) = n-1$. Consider the colouring of K_{n-1} with $(n-1)$ colours. Since $(n-1)$ is odd the boundary polygon of $(n-1)$ sides is such that there is a middle edge starting from the vertex v_1 . This edge is not incident on v_1 and so the colour of this edge is not represented at v_1 of K_{n-1} . Assign this colour to the edge $v_1 v_n$ of K_n . Thus the $(n-1)$ colours are represented at v_1 . The same procedure can be applied to the other vertices $v_2, v_3, \dots, v_{n-1}$. Clearly the colours assigned to the edges $v_1 v_n, v_2 v_n, \dots, v_{n-1} v_n$ by this method are all distinct any they are the same as the given $(n-1)$ colours. Thus K_n is coloured with $(n-1)$ colours.

$$\therefore \chi'(K_n) \leq n-1$$

$$\text{Also } \chi'(K_n) \geq \Delta(K_n) \text{ and } \Delta(K_n) = n-1$$

$$\therefore \chi'(K_n) \geq n-1$$

$$\text{Hence } \chi'(K_n) = n-1.$$

9.3 An algorithm for vertex colouring of a graph:

In a sense there are no 'good' algorithms for vertex colouring of a graph. However, attempts have been made to obtain algorithms and we describe one below. It is called sequential (or greedy) algorithm.

Step 1:

Order the vertices of G in some order; say, $v_1, v_2, \dots, v_p$.

Step 2:

Colour v_1 with colour 1.

Step 3:

Having coloured the vertices $v_1, v_2, \dots, v_i$ with k (say) colours $1, 2, \dots, k$, let $S \subseteq \{1, 2, \dots, k\}$ be the set of colours that have been used to colour the vertices adjacent with v_{i+1} ; then colour v_{i+1}

- (i) With $m = \min[\{1, 2, \dots, k\} - S]$, if $S \neq \{1, 2, \dots, k\}$;
- (ii) With $k + 1$, if $S = \{1, 2, \dots, k\}$.

Step 4:

Stop when v_p is coloured.

Remarks:

(1) Step 3 ensures that no two adjacent vertices receive the same colour; so the algorithm is a proper colouring of G .

(2) At each stage i , we have used at most $1 + \deg_G(v_1)$ colours.

Hence, the algorithm describes a $(1 + \Delta(G))$ – colouring of G ; and thus provides an algorithmic proof of Theorem 9.4.

(3) The algorithm may not colour with the minimum number of colours. The number depends on the order we choose in Step 1.

However, we can order the vertices of G in such a way that the sequential algorithm uses only $X(G)$ colours. This can be done as follows.

Let $V(G) \cup V_1 \cup \dots \cup V_t$, (where $\chi(G) = t$)

Be a partition such that no two vertices in $v_j (1 \leq j \leq t)$ are adjacent. Then order sequentially the vertices of V_1 first and then the vertices of V_2 and so on and lastly the vertices of V_t .

Note that this ordering assumes the knowledge of $\chi(G)$ a priori!

Example:

Consider the following graph.

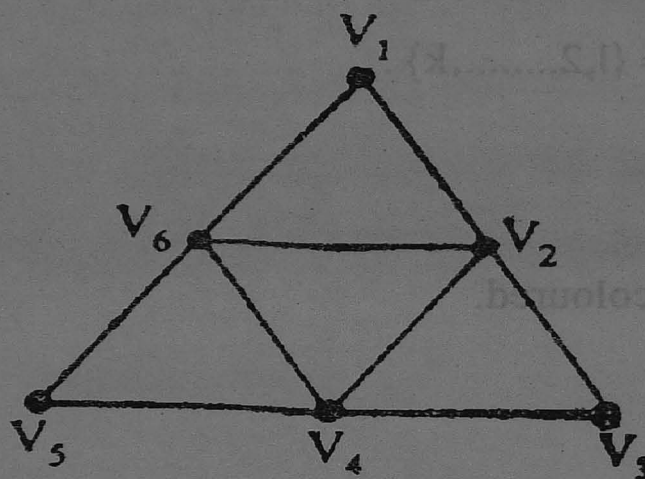


Figure 9.11

If we choose the ordering: $V_1, V_2, V_3, V_4, V_5, V_6$, then the algorithm colours the graph with 4 colours; if we choose the ordering: $V_6, V_2, V_4, V_5, V_1, V_3$, then it colours with 3 colours.

UNIT – 10

DIRECTED GRAPHS

10.0 Introduction:

The necessity for the generalization of graphs to directed graphs is obvious. We have physical situations, like one-way road maps, tournaments, where we have to have the notion of directedness. Moreover, a graph can be thought of as a special kind of directed graph: the meaning of this statements will be clear soon.

In this unit, we give the basic theory of digraphs. First we give some types of digraphs. We introduce concepts of connectivity of digraphs. We discuss the conditions under which one can direct the edges of a graph in such a way that the resulting digraph is strongly connected. We discuss some simple characterization for Eulerian digraphs. Finally we give some properties of tournaments.

10.1 Directed graphs:

Definition 10.1.1:

Let V be a non empty set and A be a subset of $V \times V$, then the ordered pair $D = (V, A)$ is called a **directed graph** or a **digraph**. The elements of V are called **vertices** or **points** and those of A are called **directed edges** or arcs. If $a \in A$ is such that $a = (u, v)$ then the arc a is said to be directed from u to v . The initial and terminal vertices u and v are respectively called the **tail** and **head** of the arc. Also the arc $a = (u, v)$ is said to join u to v .

In representing a digraph the arcs are marked by arrows in the direction from tail to head. An arc for which the tail and head coincide is called a **loop** or a **self loop**. Two directed edges are said to be **parallel** if they have the same tail and the same head. Two arcs are said to be **symmetric arcs** if the tail of one is the head of the other and vice versa.

A loop is of the form (u, u)

Parallel edges are of the form $(u, v), (u, v)$

Symmetric edges are of the form $(u, v), (v, u)$

Definition: 10.1.2

In degrees and out degrees:

A arc $a = (u, v)$ is said to be **incident out** of the tail u and **incident into** the head v .

Consequently the number of arcs incident into a vertex v is called its **in-degree** and denoted by $d^-(v)$, the number of arcs incident out of the vertex v is called its **out-degree** and is denoted by $d^+(v)$. Thus the indegree $d^-(v)$ is the number of arcs for which v is the head and the out degree $d^+(v)$ is the number of arcs for which v is the tail.

The ordered pair $(d^-(v)), (d^+(v))$ is called the **degree pair** of the vertex v , and the sum $d^-(v) + d^+(v)$ is called the **total degree** of the vertex v .

The minimum and maximum of the indegrees of all the vertices of a digraph D are called the minimum indegree and maximum indegree of D and are denoted by $\delta^-(D)$ and $\Delta^-(D)$. Similarly we

have minimum out degree $\delta^+(D)$ and the maximum out degree $\Delta^+(D)$ of the digraph D .

The vertex of a digraph is called a **pendent vertex** if there is only one arc either incident on it or incident out of it. So if v is a pendant vertex then $d^-(v) + d^+(v) = 1$, that is it is of degree 1. The degree pair of a pendant vertex is either $(1,0)$ or $(0,1)$.

A vertex having no arc incident either into it or out of it is called an **isolated vertex**. The degree pair of an isolated vertex is $(0,0)$.

Note:

A self loop at a vertex v gives one indegree and one out degree to v .

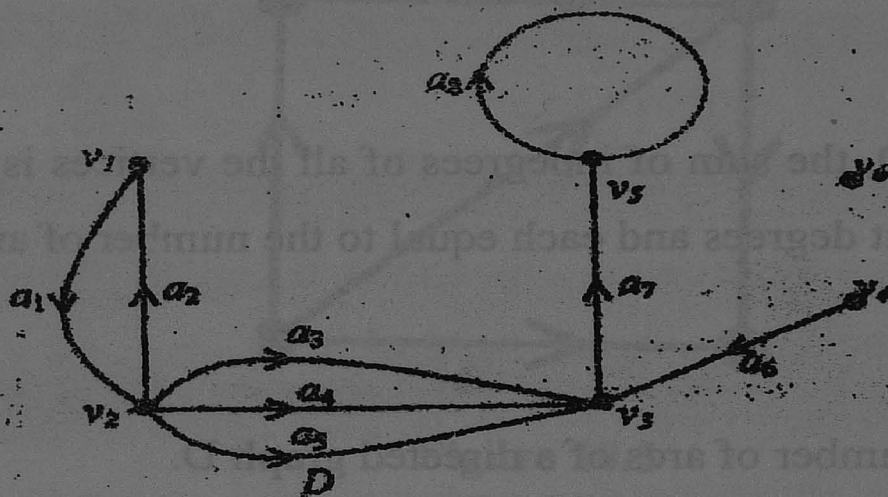


Figure 10.1

a_1, a_2 - Symmetric arcs

a_3, a_4, a_5 - Parallel arcs.

a_8 - Self loop

v_4 - Pendant vertex.

v_6 - Isolated vertex.

$$d^-(v_1) = 1$$

$$d^+(v_1) = 1$$

$$d^-(v_2) = 1$$

$$d^+(v_2) = 4$$

$$d^-(v_3) = 4$$

$$d^+(v_3) = 1$$

$$d^-(v_4) = 0$$

$$d^+(v_4) = 1$$

$$d^-(v_5) = 2$$

$$d^+(v_5) = 1$$

$$d^-(v_6) = 0$$

$$d^+(v_6) = 0$$

$$\delta^-(D) = 0$$

$$\delta^+(D) = 0$$

$$\Delta^-(D) = 4$$

$$\Delta^+(D) = 4$$

Theorem: 10.1

In a digraph D , the sum of indegrees of all the vertices is equal to the sum of their out degrees and each equal to the number of arcs.

Proof:

Let q be the number of arcs of a directed graph D .

Consider the arc $a = (u, v)$. It is incident into v and incident out of u . So it gives one indegrees to v and one out degree to u . Similarly every arc contributes 1 to the sum of indegrees and 1 to the sum of out degrees of the vertices. Hence the sum of indegrees is equal to the sum of out degrees and each equal to the number of arcs.

$$\sum_{v \in D} d^-(v) = \sum_{v \in D} d^+(v) = q$$

Types of digraphs:

- (1) A digraph without parallel arcs is called an **oriented graph**.

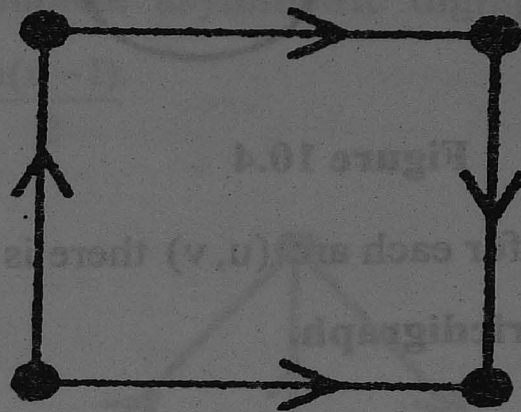


Figure 10.2

- (2) A digraph without self loops and parallel arcs is called a **simple digraph**. We also call it a **strict digraph**.

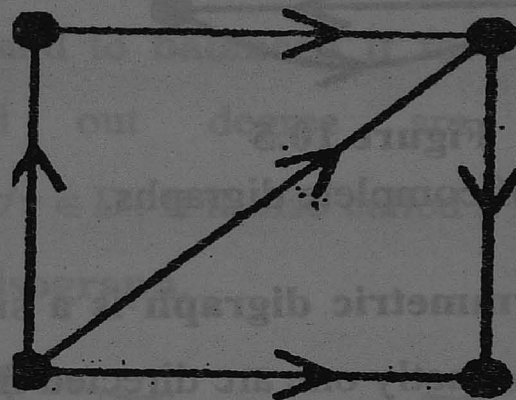


Figure 10.3

- (3) A digraph having at most one arc between a pair of vertices and with or without self-loops is called an **asymmetric or antisymmetric digraph**.

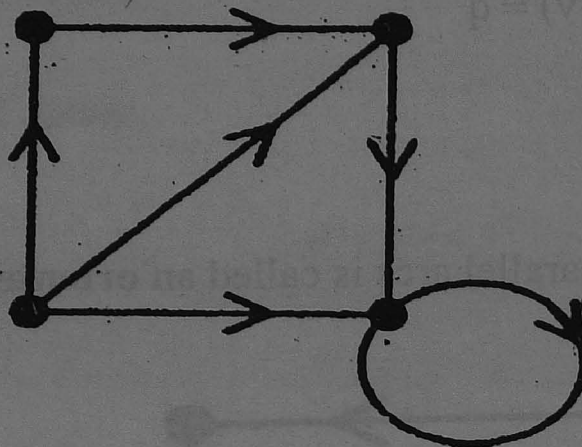


Figure 10.4

- (4) A digraph in which for each arc (u, v) there is also an arc (v, u) is called as **symmetric digraph**.

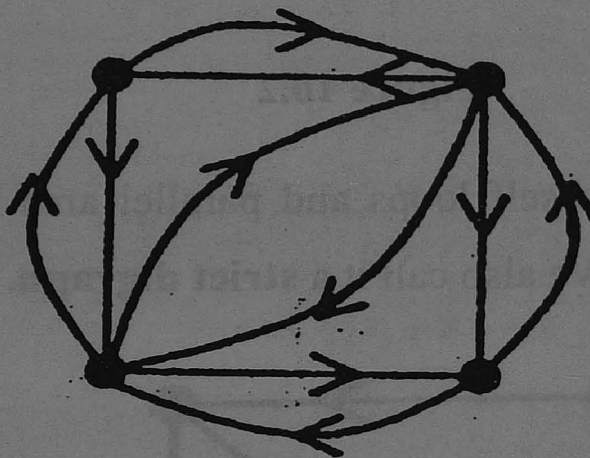


Figure 10.5

- (5) There are two kinds of complete digraphs.

- (i) A **complete symmetric digraph** is a simple digraph in which there is exactly one arc directed from every vertex to every other vertex. If D is a complete symmetric digraph on n vertices, the number of arcs is $n(n - 1)$

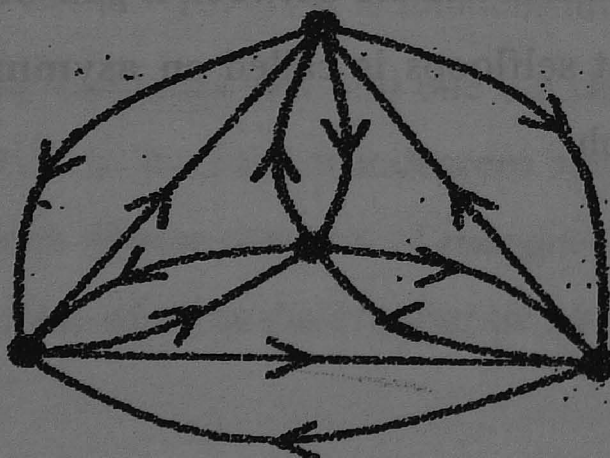


Figure 10.6

- (ii) A **complete asymmetric digraph** is an asymmetric digraph in which there is exactly one arc between every pair of vertices. If n is the number of vertices of a complete asymmetric digraph the number of arcs is $\frac{n(n-1)}{2}$

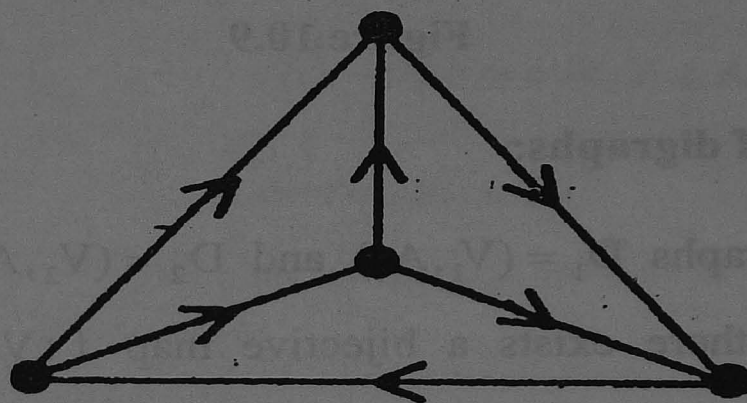


Figure 10.7

- (6) A digraph is said to be **balanced** if for every vertex v , the indegree and out degree are equal, that is $d^-(v) = d^+(v) \forall v \in D$. It is also called a **Pseudo symmetric digraph** or an **isograph**.

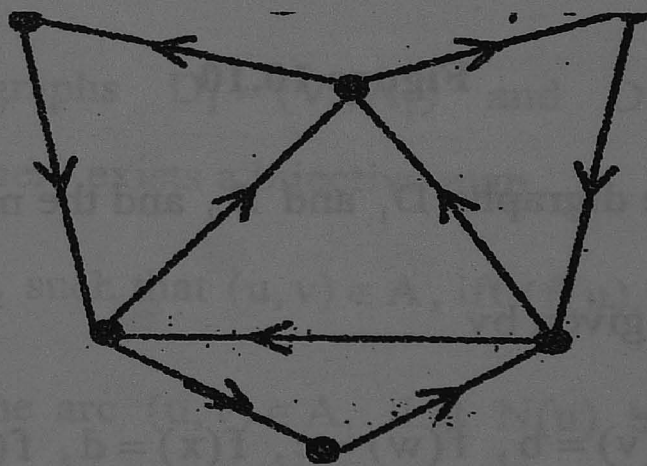


Figure 10.8

- (7) A balanced digraph in which all vertices have the same equal in-degrees and outdegrees is called a **regular digraph**.

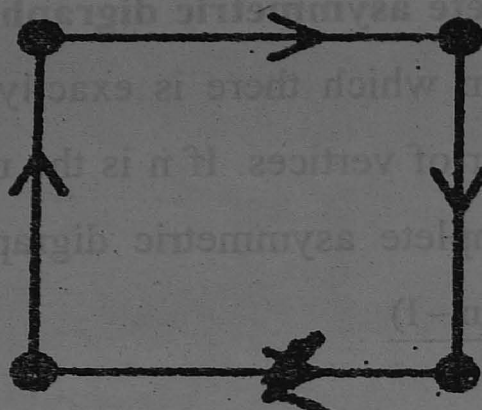
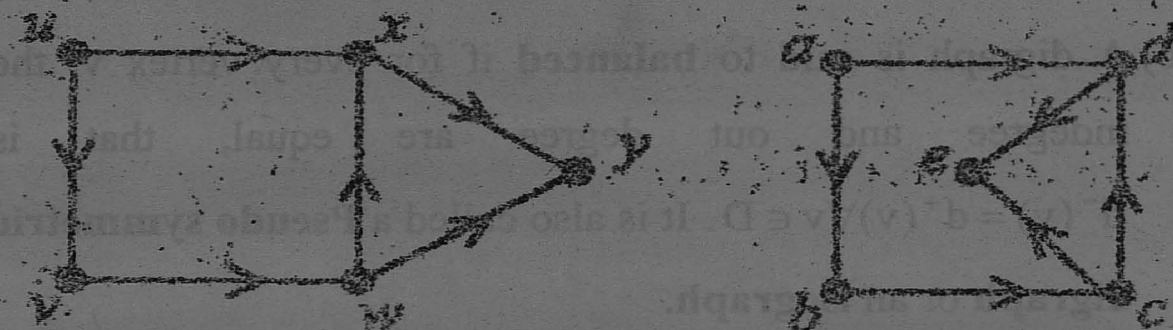


Figure 10.9

Isomorphism of digraphs:

Two digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ are said to be **isomorphic** if there exists a bijective map $f: V_1 \rightarrow V_2$ such that $(u, v) \in A_1$ iff $(f(u), f(v)) \in A_2$. We write $D_1 \cong D_2$.



D_1

Figure 10.10

D_2

Consider the digraphs D_1 and D_2 and the map

$f: V_1 \rightarrow V_2$ given by

$f(u) = a$, $f(v) = b$, $f(w) = c$, $f(x) = d$, $f(y) = e$. Clearly f is

bijective

Take

$(u, v) \in A_1$

$(f(u), f(v)) = (a, b) \in A_2$

$$(v, w) \in A_1 \quad (f(v), f(w)) = (b, c) \in A_2$$

$$(w, x) \in A_1 \quad (f(w), f(x)) = (c, d) \in A_2$$

$$(x, u) \in A_1 \quad (f(x), f(u)) = (d, a) \in A_2$$

$$(x, y) \in A_1 \quad (f(x), f(y)) = (d, e) \in A_2$$

$$(w, y) \in A_1 \quad (f(w), f(y)) = (c, e) \in A_2$$

$$\text{Hence } D_1 \cong D_2$$

Note:

Two isomorphic digraphs have the same number of vertices and the same number of arcs.

Theorem: 10.2

If two digraphs are isomorphic then the corresponding vertices have the same degree pairs.

Proof:

If the digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ are isomorphic then there exists a bijective map

$$f: V_1 \rightarrow V_2 \text{ such that } (u, v) \in A_1 \text{ iff } (f(u), f(v)) \in A_2.$$

Consider the arc $(u, v) \in A_1$. Let $N(u)$ be the number of arcs of D_1 for which u is the tail so that $|N(u)|$ is the out degree of u . So $d^+(u) = |N(u)|$. Similarly $|N(f(u))|$ is the out degree of $f(u)$. So $d^+(f(u)) = |N(f(u))|$.

Now

$$N(u) = \{v \mid v \in V_1 \text{ and } (u, v) \in A_1\}$$

$$N(f(u)) = \{f(v) \mid f(v) \in V_2 \text{ and } (f(u), f(v)) \in A_2\}$$

$$\therefore |N(u)| \Leftrightarrow (u, v) \in A_1$$

$$\Leftrightarrow (f(u), f(v)) \in A_2$$

$$\Leftrightarrow f(v) \in N(f(u))$$

Thus there is a one – one correspondence between $N(u)$ and $N(f(u))$

$$\therefore |N(u)| = |N(f(u))|$$

$$\text{i.e., } d^+(u) = d^+(f(u))$$

Similarly $d^-(u) = d^-(f(u))$

$$\therefore (d^-(u), d^+(u)) = (d^-(f(u)), d^+(f(u)))$$

Hence the corresponding vertices u and $f(u)$ have the same degree pair.

Disubgraph:

Definition: 10.1.3:

Let $D = (V, A)$ be a given digraph. Then the digraph $D' = (V', A')$ is called a disubgraph if $V' \subseteq V$ and $A' \subseteq A$.

Clearly every digraph is a disubgraph of itself.

Let V_1 be a non empty subset of V . The subgraph whose vertex set is V_1 and whose arc set A_1 is the set of those arcs of D which have

both tails and heads V_1 is called the **disubgraph** induced by V_1 and is denoted by $D(V_1)$.

As above any other disubgraph such as vertex disjoint, arc disjoint, spanning etc., disubgraphs can be defined on the same line as for the graphs.

10.2 Connectivity in digraphs:

Definitions: 10.2.1

Directed walk, trail, path and cycles:

Except for the orientation of the arcs the terms walk, trail and path in a digraph are defined as in the case of undirected graphs.

An alternating sequence of vertices and arcs starting with a vertex and ending with a vertex in a digraph is called a walk. Thus

$$W = u_1 a_1 u_2 a_2 \dots u_{n-1} a_{n-1} u_n$$

Such that each arc a_i has tail u_{i-1} and head u_i , is a **directed walk**.

A directed walk in which the end vertices coincide is called a **directed closed walk**.

A directed walk in which all arcs are distinct is called a **directed trail**.

A directed walk in which all vertices (and so also all arcs) are distinct is called a **directed path**.

Each arc together with its tail and head is itself a directed path.

A closed directed walk in which all its vertices and arcs are distinct is called a **directed cycle**. Each vertex of a directed cycle has indegree 1 and outdegree 1.

i.e., $d^-(v) = d^+(v) = 1$ for every vertex v

So the degree pair of each vertex of a directed cycle is (1,1)

The **length** of directed walks, trails paths and cycles are the number of arcs in them.

A **directed self loop** is a closed walk of unit length and an arc is a path of unit length.

Note:

The orientation of all the edges must be the same in any walk, trail path and cycle of a directed graph.

Definition: 10.2.2:

Linear subgraph:

A **linear subgraph** of a digraph D is a spanning subgraph in which each vertex has indegree 1 and out degree 1. As such it consists of a disjoint spanning collection of directed cycles.

Definition: 10.2.3

Underlying graphs:

Let D be a digraph. The graph G having the same vertex set V in which corresponding to each arc in D there is an edge is called the **underlying graph** of the digraph.

If the digraph D is simple then the underlying graph G is also simple.

Definition: 10.2.4**Orientation of a graph:**

Let G be any given graph, by giving **orientation** to the edges (attaching directions to the edges) we get a digraph. This is called an **orientation** of G .

Thus if digraph D is an orientation of a graph G then G is the underlying graph of the digraph D .

Note:

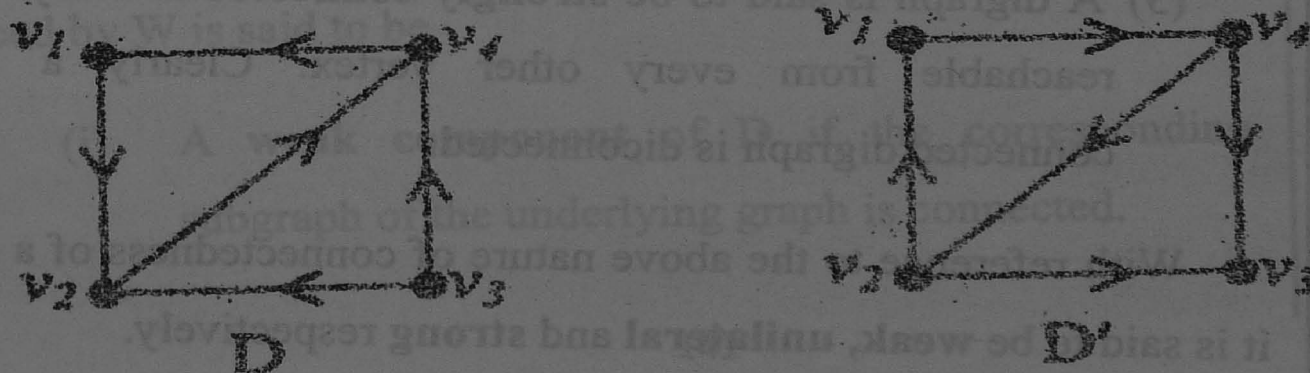
- (1) Some authors define the corresponding simple graph as the underlying graph.
- (2) In a digraph we do not usually consider multiple directed arcs and self loops, that is more than one arc (u, v) and arcs of the form (u, u)

Definition: 10.2.5**Converse Digraph:**

Let D be a given digraph. The digraph D' obtained by reversing the directions of all the arcs is called the **converse digraph** of D .

For any vertex v ,

$$d^-_D(v) = d^+_{D'}(v) \text{ and } d^+_D(v) = d^-_{D'}(v)$$

**Figure 10.11**

Note:

The two digraph D and D' have the same vertices and undirected edges that is the underlying graph is the same.

Definitions: 10.2.6

Reachability:

Let D be a digraph. A vertex v in D is said to be **reachable** from a vertex u if there exists a directed u - v path in D .

The vertices u and v are said to be **disconnected** if each is reachable from the other.

A digraph D is said to be **disconnected** if every pair of vertices are mutually reachable. That is there is a directed path connecting every pair of vertices.

Definitions: 10.2.7

Connectedness of a digraph:

There are three kinds of connected digraphs.

- (1) A digraph is said to be **weakly connected** if its underlying simple graph is connected.
- (2) A digraph is said to be **unilaterally connected** is given any two vertices u and v atleast one is reachable from the other.
- (3) A digraph is said to be **strongly connected** if every vertex is reachable from every other vertex. Clearly a strongly connected digraph is disconnected.

With reference to the above nature of connectedness of a digraph it is said to be **weak**, **unilateral** and **strong** respectively.

Vacuously the following implications hold.

$$\text{Strong} \Rightarrow \text{Unilateral} \Rightarrow \text{Weak}$$

But the reverse implications need not hold.

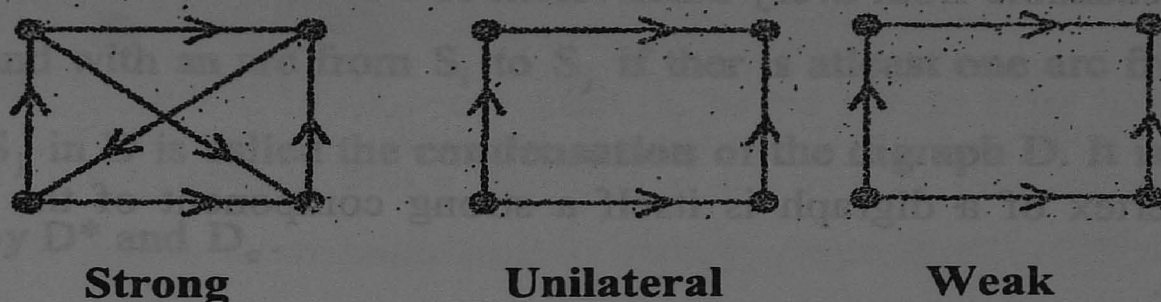


Figure 10.12

Remarks:

If the underlying graph is disconnected then the corresponding digraph is disconnected. So a disconnected digraph is not even weakly connected.

Note:

By definition the trivial digraph is strongly connected.

10.3 Strong orientation of graphs:

Definitions: 10.3.1

Components of a digraph:

There are three types of components of a digraph $D = (V, A)$.

Let W be a maximal subset of V . Then the subgraph $D(W)$ induced by W is said to be

- (i) A **weak component** of D if the corresponding subgraph of the underlying graph is connected.

- (ii) A **unilateral component** of D if, given any two vertices $u, v \in W$ at least one is reachable from the other.
- (iii) A **strong component** of D if every vertex of W is reachable from every other vertex of W .

Note:

- (1) Each vertex of a digraph is itself a strong component of the digraph.
- (2) A digraph to have a strongly connected component need not be a strongly connected digraph.
- (3) The components of a digraph need not be the components of a disconnected digraph.

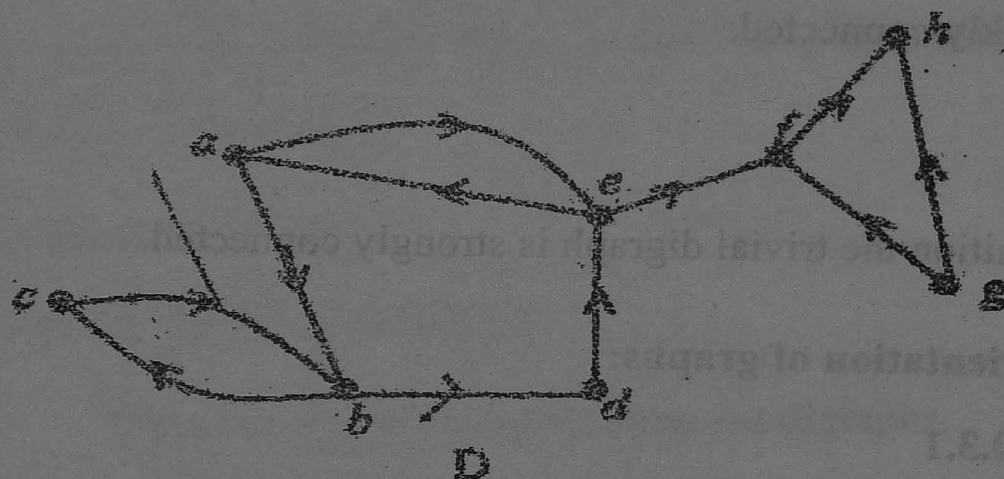


Figure 10.13

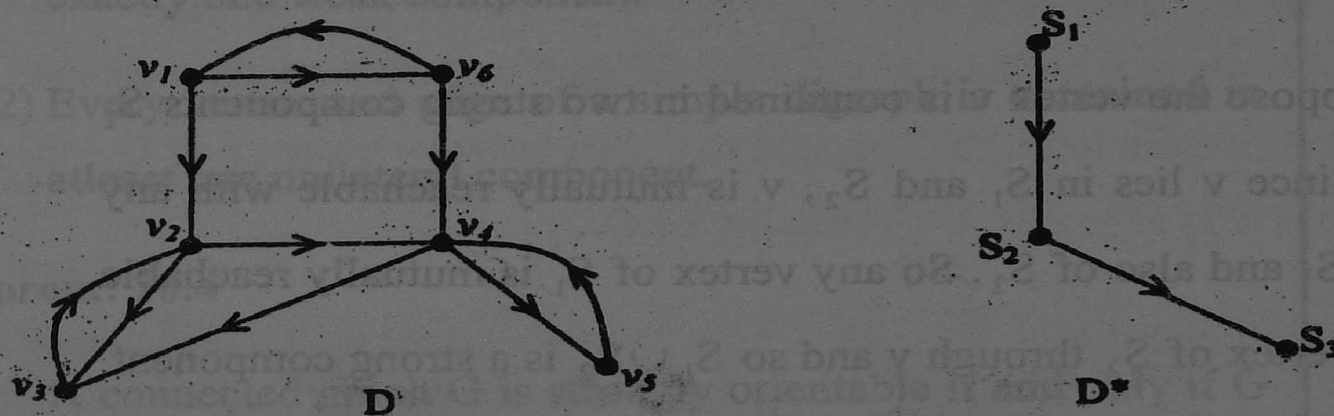
The subdigraph induced by $W_1 = \{a, b, c, d, e\}$ is a strong component

The subdigraph induced by $W_2 = \{f, g, h\}$ is a unilateral component.

The subdigraph induced by $W_3 = \{d, e, f, g, h\}$ is a weak component.

Definition: 10.3.2**Condensation:**

Let $S_1, S_2, \dots, S_K$ be the strongly connected components of a digraph D . Then the digraph with $S_1, S_2, \dots, S_K$ as vertices and with an arc from S_i to S_j if there is at least one arc from S_i to S_j in D is called the **condensation** of the digraph D . It is denoted by D^* and D_c .

**Figure 10.14**

The strong components of D are the subdigraphs induced by

$$S_1 = \{v_1, v_6\}, S_2 = \{v_2, v_3, v_4\}, S_3 = \{v_5\}$$

The condensation graph D^* is given above.

Remarks:

- (1) The condensation of a strongly connected digraph is simply a vertex D .
- (2) The condensation of a digraph has no directed cycle.

Theorem:

In a simple digraph $D = (V, A)$ every vertex lies in exactly one strong component.

Proof:

Let $v \in V$ and let S be the set of all those vertices of D which are mutually reachable with v . The set S contains v and is clearly a strong component of D . Thus every vertex is contained in a strong component.

Suppose the vertex v is contained in two strong components S_1 and S_2 . Since v lies in S_1 and S_2 , v is mutually reachable with any vertex of S_1 and also of S_2 . So any vertex of S_1 is mutually reachable with any vertex of S_2 through v and so $S_1 \cup S_2$ is a strong component. This contradicts the maximality of the strong component S_1 (or S_2).

Hence the vertex v can be in exactly one component of D .

Corollary: 10.3.1

The strong components of a digraph partitions the vertex set.

Corollary: 10.3.2

An arc of a digraph lies in exactly one strong component if it lies on a cycle.

Proof:

Let S be a strong component of a digraph D . And $a = (u, v)$ be an arc in S . Since S is strong and $u, v \in S$, the vertices u and v are mutually reachable. Therefore the arc $a = (u, v)$ lies on a cycle.

Again since a vertex lies in exactly one strong component, the vertices u and v cannot be in a strong component other than S . So the arc a lies in exactly one strong component.

Remarks:

As above we can characterize the vertices and edges in weak and unilateral components of a simple digraph as follows.

- (1) Every vertex and edge of a simple digraph is contained in exactly one weak component.
- (2) Every vertex and edge of a simple digraph is contained in at least one unilateral component.

Theorem: 10.4

A connected graph G is strongly orientable if and only if G has no cut edges.

Proof:

Let G be a strongly orientable graph and the directed graph D be a strong orientation of G . Let (u, v) be an edge of G . Then (u, v) or (v, u) will be an arc of digraph D . Let us take (u, v) as an arc of D . Since D is strongly oriented it is

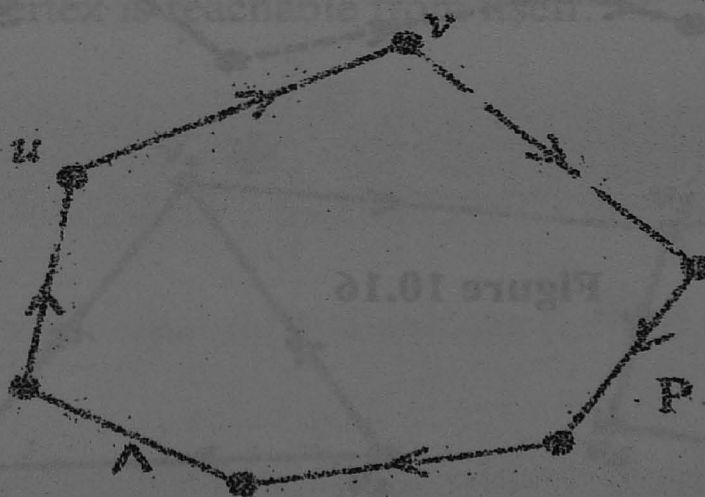


Figure 10.15

Strongly connected and so there is a directed (v, u) path P . So the arc uv and the directed $(v - u)$ path P together form a directed cycle of D . So the edge uv is contained in a cycle of the given graph G . Hence uv is not a cut edge of G .

Conversely suppose G has no cut edge then every edge of G lies on a cycle.

Let $C: v_1 v_2 v_3 \dots v_n v_1$ be a cycle in G .

Orient the edges of C so that it becomes a directed cycle and hence becomes a connected disubgraph. If C contains all the vertices of G then we are through. Suppose not, then let w be a vertex of $G - C$ such that w is adjacent to a vertex v_i of C .

By hypothesis the edges u, w lies on a cycle S (say). Now choose a direction for this edge and give the orientation determines by this direction to the edges of the cycle S which are not already oriented. The resulting digraph $C \cup S$ is also strongly connected.

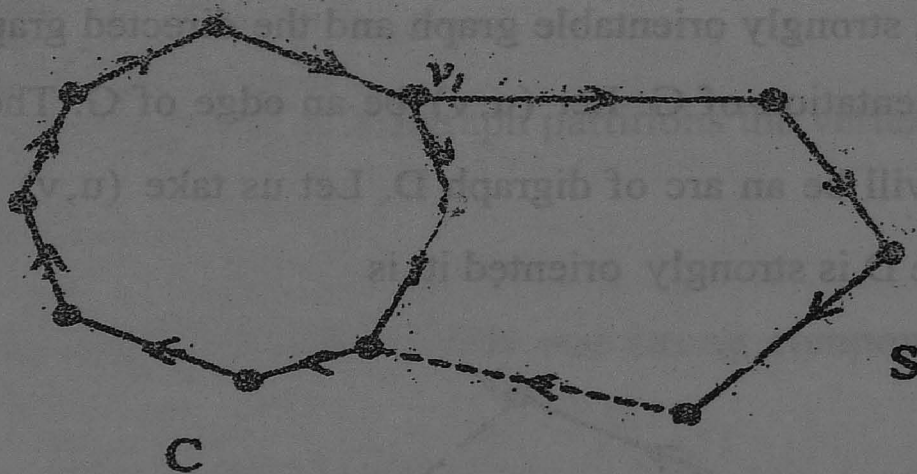


Figure 10.16

This process of adding edges and orienting cycles can be continued till we get a spanning subgraph of G which is strongly oriented. The remaining edges can be assigned any directions. Clearly the resulting is strongly oriented.

Remarks:

The above theorem may also be stated as follows.

“The edges of a connected graph can be oriented so that the resulting digraph is strongly connected if and only if every edge of G is contained in atleast one cycle”.

Definition: 10.3.3

Reachable sets:

Let D be a digraph. The set of vertices of D which are reachable from a given vertex v is said to be the **reachable set** of v and is denoted by $R(v)$.

For any subset $S \subseteq V$ the **reachable set of S** is set of vertices which are reachable from any vertex of S and it is denoted by $R(S)$.

Note:

Every vertex is reachable from itself.

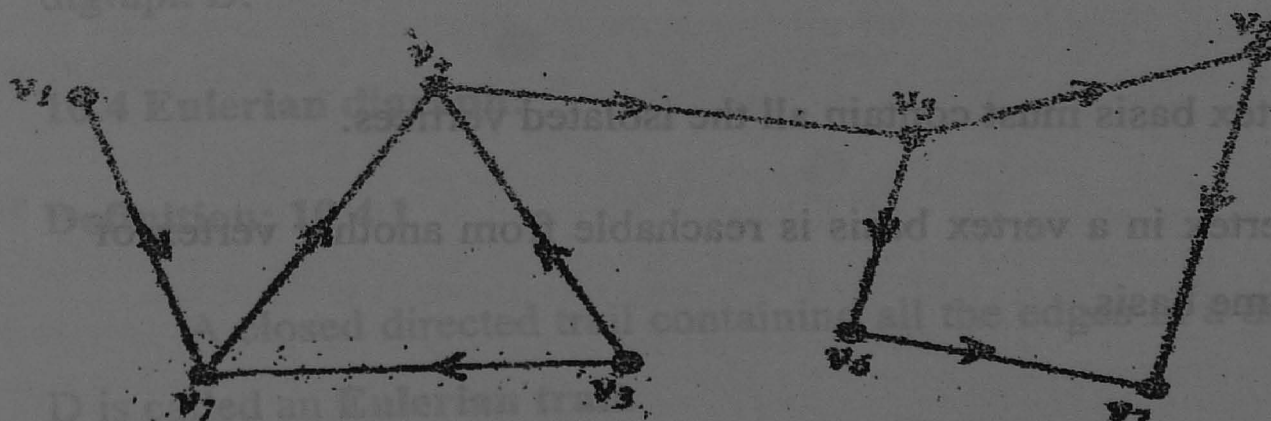


Figure 10.17

$$R(v_1) = \{v_1\}$$

$$R(v_2) = \{v_1, v_2, v_3\}$$

$$R(v_3) = \{v_3\}$$

$$R(v_4) = \{v_2, v_3, v_4\}$$

$$R(v_5) = \{v_4, v_5\}$$

$$R(v_6) = \{v_5, v_6\}$$

$$R(v_7) = \{v_6, v_7, v_8\}$$

$$R(v_8) = \{v_5, v_8\}$$

Take $S = \{v_3, v_4, v_6\}$, then $R(S) = \{v_2, v_3, v_4, v_5, v_6\}$.

Definition: 10.3.4

Geodesic:

Let D be a digraph. If a vertex v is reachable from a vertex u then the minimum distance from u to v is called a **geodesic**. The length of the geodesic from u to v is the distance of v from u and is denoted by $d(u, v)$. If v is not reachable u we write $d(u, v) = \infty$.

Note:

If u and v are mutually reachable then the geodesic distances $d(u, v)$ and $d(v, u)$ need not be equal. The relation reachability among the vertices of a digraph is neither symmetric nor antisymmetric.

Definition: 10.3.5

Point Basis:

A **point basis** or **vertex basis** of a digraph D is a minimal collection of vertices of D from which all vertices are reachable.

Remarks:

- (1) A vertex basis must contain all the isolated vertices.
- (2) No vertex in a vertex basis is reachable from another vertex of the same basis.

- (3) If there is a directed cycle in the digraph any one vertex of the cycle can be included in a vertex basis.
- (4) In an acyclic graph a vertex base consists of only those vertices whose indegree is 0.
- (5) If $X \subseteq V$ is a vertex basis then the reachable set of X is V , that is $R(X) = V$.
- (6) The vertex basis of a digraph is not unique.
- (7) The number of vertices in any vertex basis of a digraph is the same.

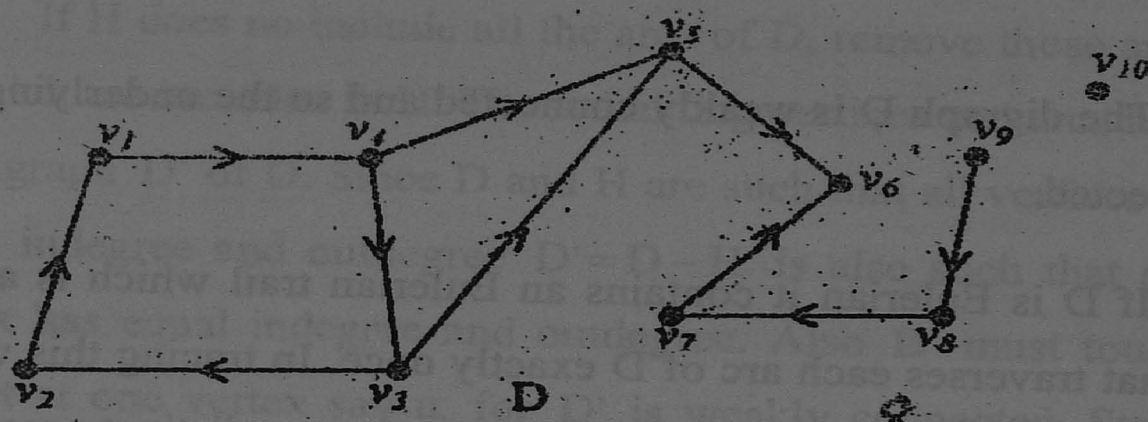


Figure 10.18

$\{v_1, v_9, v_{10}\}$, $\{v_3, v_9, v_{10}\}$ are two vertex bases, of the digraph D .

10.4 Eulerian digraph:

Definition: 10.4.1

A closed directed trail containing all the edges in a digraph D is called an **Eulerian trail**.

A digraph containing an Eulerian trail is called an **Eulerian digraph**.

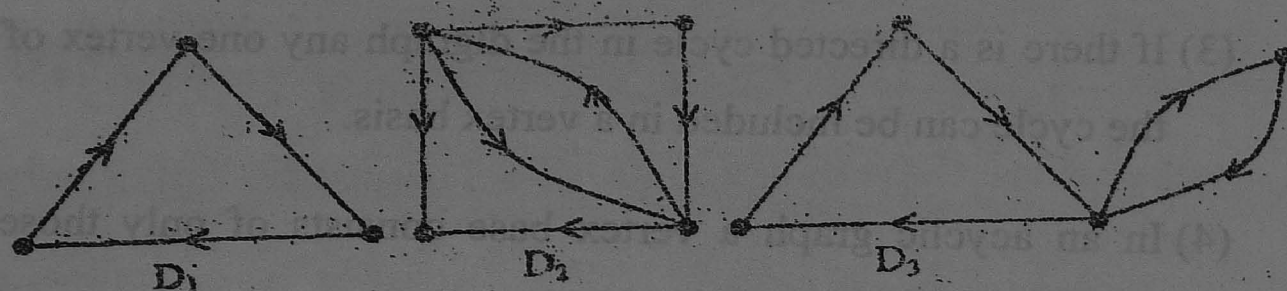


Figure 10.19

Here D_1, D_2, D_3 are Eulerian digraphs.

Theorem: 10.5

A weak digraph D is Eulerian if and if every vertex of D has equal indegree and outdegree.

Proof:

The digraph D is weakly connected and so the underlying graph is connected.

If D is Eulerian it contains an Eulerian trail which is a closed walk that traverses each arc of D exactly once. In tracing this we find, every time the walk meets a vertex there are two distinct arcs, one to enter the vertex and the other to leave the vertex. This contributes one indegree and one outdegree to the vertex. So the vertex has as many indegree and one out degrees as the number of times the walk meets the vertex. This is true for all the vertices. Hence it follows that every vertex has equal indegree and outdegree.

Conversely let the digraph D be such that every vertex has equal indegree and outdegree. We have to prove D is Eulerian.

Let us construct a directed walk starting from any arbitrary vertex v and going through the arcs of D such that no arc is traced more than once. We continue this as far as possible. Since each vertex has equal indegree and outdegree, corresponding to an arc to enter vertex there is an arc to leave the vertex. This tracing clearly stops only at the vertex v because corresponding to the arc we left v there is an arc to enter it.

Thus we get a closed walk H (say). If H includes all the arcs of D then H itself is an Eulerian trail and so the digraph D is Eulerian. Also H is such that every vertex of H has equal indegree and outdegree.

If H does not include all the arcs of D , remove those arcs of D which are included in H . The remaining vertices and arcs form a subdigraph D' of D . Since D and H are such that all vertices have equal indegree and outdegree $D' = D - H$ is also such that every vertex has equal indegree and outdegree. Also D' must touch H at least at one vertex say u , for D' is weakly connected. Starting from this vertex u we can construct a new closed trail H' in D' . Since all the vertices of D' are such that every vertex has equal indegree and outdegree, this walk H' must end in u . This walk H' is also Eulerian. Now $H \cup H'$ is a new Eulerian trail and has more arcs than H and H' .

This process can be repeated till we obtain a closed walk containing all the arcs of the digraph D . This closed walk is an Eulerian trail and so the given digraph is Eulerian.

Definitions: 10.4.2

Semi walk, semi path, semi cycle:

Let D be a digraph and G be the underlying graph. A walk W in G is said to be a **semi walk** in D if W is not a directed walk. That is the orientations of all the edges of W are not the same. Similarly we define **semi path** and **semi cycle**.

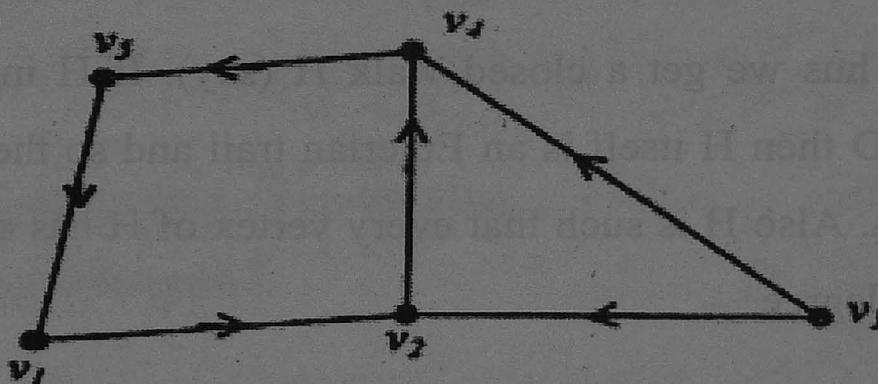


Figure 10.20

$v_1v_2v_4$ - directed walk

$v_1v_2v_4v_3$ - semi walk

$v_1v_2v_4v_5$ - directed path

$v_1v_2v_4v_3$ - semi path

$v_1v_2v_4v_5v_1$ - directed cycle

$v_1v_2v_3v_4v_5v_1$ - semi cycle.

Definition: 10.4.3

Tree with directed edges:

A digraph D is said to be **acyclic** if it contains no cycle (directed or semi)

A connected acyclic digraph is a tree with directed edges. If the tree has n vertices then it has $(n - 1)$ directed edges (arcs)

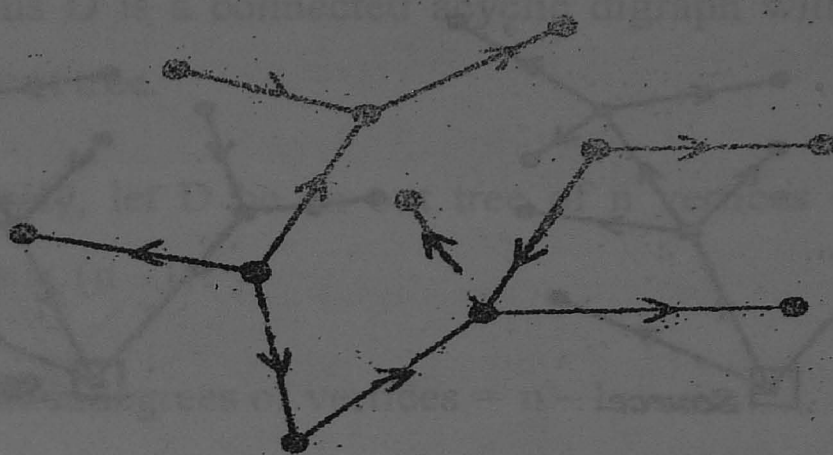


Figure 10.21

Definitions: 10.4.4**Source and Sink:**

A **source** in a digraph is a vertex from which all other edges can be reached. It has indegree zero.

A **sink** in a digraph is a vertex which is reachable from all other vertices. It has out degree zero.

Definitions: 10.4.5**Out tree and In tree:**

An acyclic graph with a source is called an **out tree**. The source has indegree zero and all other vertices have indegree one each.

An acyclic graph with a sink is called **intree**. The sink has out degree zero and all other vertices have out degree one each.

In an out tree arcs are oriented from the source and in an intree arcs are oriented towards the sink.

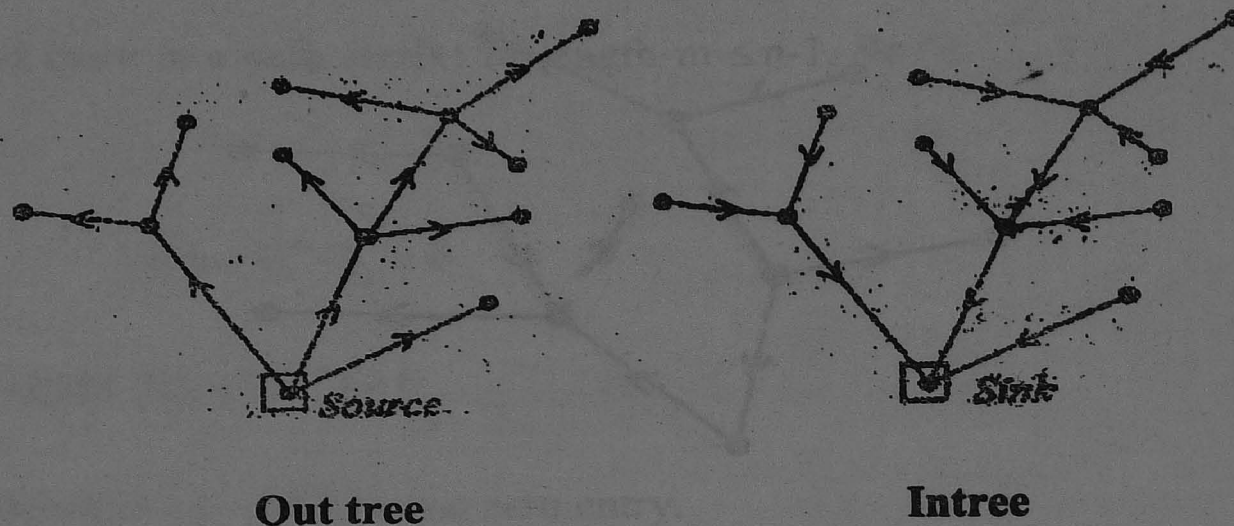


Figure 10.22

Definition: 10.4.6

Arborescence:

An out tree has one vertex of indegree zero and all other vertices have indegree exactly one. The source (vertex of indegree 0) is distinguished from the others and can be taken as the root of a rooted tree in which all edges are oriented from the root. This rooted tree is called an **arborescence**. We observe that an arborescence is nothing but an out tree with the source as the root.

Theorem: 10.6

A weak digraph is an out tree if and only if exactly one vertex has indegree 0 and all others have indegree 1.

Proof:

Let D be a weak digraph with exactly one vertex of indegree 0 and other vertices of indegree 1.

Let v be the vertex of degree 0. So it is not reachable from any other vertex. Therefore v is a source. As no vertex has indegree more than 1, the graph is acyclic. Again since D is a weak digraph it is

connected. Thus D is a connected acyclic digraph with a source. Hence it is an out tree.

Conversely, let D be an out tree of n vertices so that the number of arcs is $(n-1)$.

$\therefore$ Sum of indegrees of vertices = $n-1$

$$d^-(v_1) + d^-(v_2) + d^-(v_3) + \dots + d^-(v_n) = n-1$$

Since D is an out tree it has a source which is of indegree 0.

Let v_1 be the source so that $d^-(v_1) = 0$. Using this in the above we get.

$$d^-(v_2) + d^-(v_3) + \dots + d^-(v_n) = n-1$$

As there is no other vertex of zero indegree, the sum of indegrees of $(n-1)$ vertices must be $(n-1)$. This is possible only when each of the $(n-1)$ vertices has indegree 1.

Hence there is exactly one vertex of indegree 0 and all others have indegree 1.

Note:

On similar lines we can prove the theorem.

'A weak digraph is an intree if and only if exactly one vertex has out degree 0 and all others have out degree 1'.

Definition: 10.4.7

Principle of directional duality:

We know the converse digraph D' is obtained from D by simply reversing the directions of its edges. The indegree of a vertex in D becomes the outdegree of the same vertex in D' . Thus results related to direction in a digraph give rise to dual results in the converse digraph. This concept of dual results in a digraph is known as the **Principle of directional duality** and is stated as follows.

‘For each theorem about digraph there is a corresponding theorem by replacing every concept by its converse’

For example, the theorem

“An acyclic digraph has atleast one vertex of out degree zero” has a dual theorem.

“An acyclic digraph has atleast one vertex of indegree zero”

Similarly the theorem

“A weak digraph is an out tree if and only if exactly one vertex has indegree 0 and all others have indegree 1” has the dual theorem.

“A weak digraph is an intree if and only if exactly one vertex has out degree 0 and all others have out degree 1”

Further, sink is the dual of source and intree is the dual of out tree.

Definition: 10.4.8**Functional digraph:**

A digraph D in which every vertex has out degree 1 i.e., $d(v) = 1 \forall v \in D$ is called a **functional digraph**.

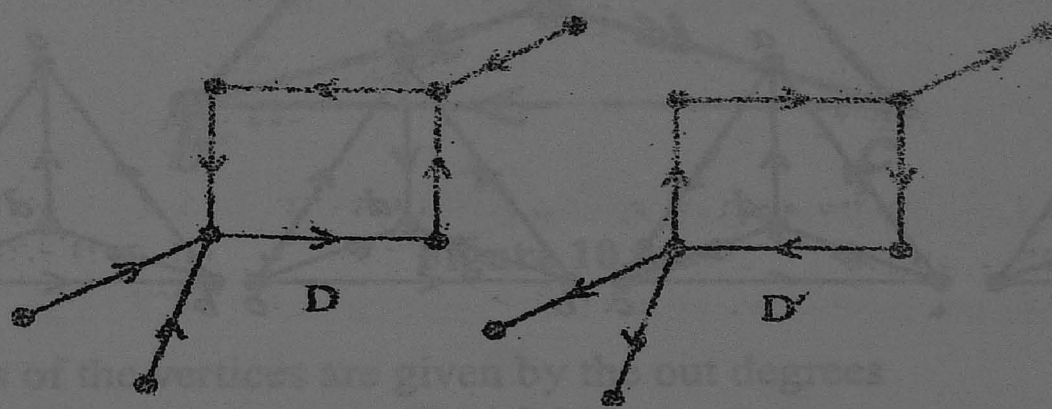
Remarks:

If a digraph D is functional then every vertex has out degree 1. So if D has n vertices then the sum of out degrees is n and so the number of arcs is n . Thus for a functional digraph of n vertices have n arcs.

Note:

A directed cyclic graph is a functional digraph. But a functional digraph need not be cyclic.

The dual of a functional digraph is called a **contra functional digraph**. It is a digraph in which every vertex has indegree 1.

**Figure 10.23**

D is a functional digraph, D' is contra functional.

Clearly a directed cyclic graph is contra functional digraph but a contra functional digraph need not be cyclic.

Note:

A cyclic digraph is both functional and contra functional.

10.5 Tournaments:

Definition: 10.5.1

A **tournament** is an oriented complete graph. So a tournament T is a digraph in which there is exactly one arc between every pair of vertices.

If (u, v) is an arc of a tournament then u is said to **dominate** v .

All the tournaments of 2, 3, 4 vertices are given below.

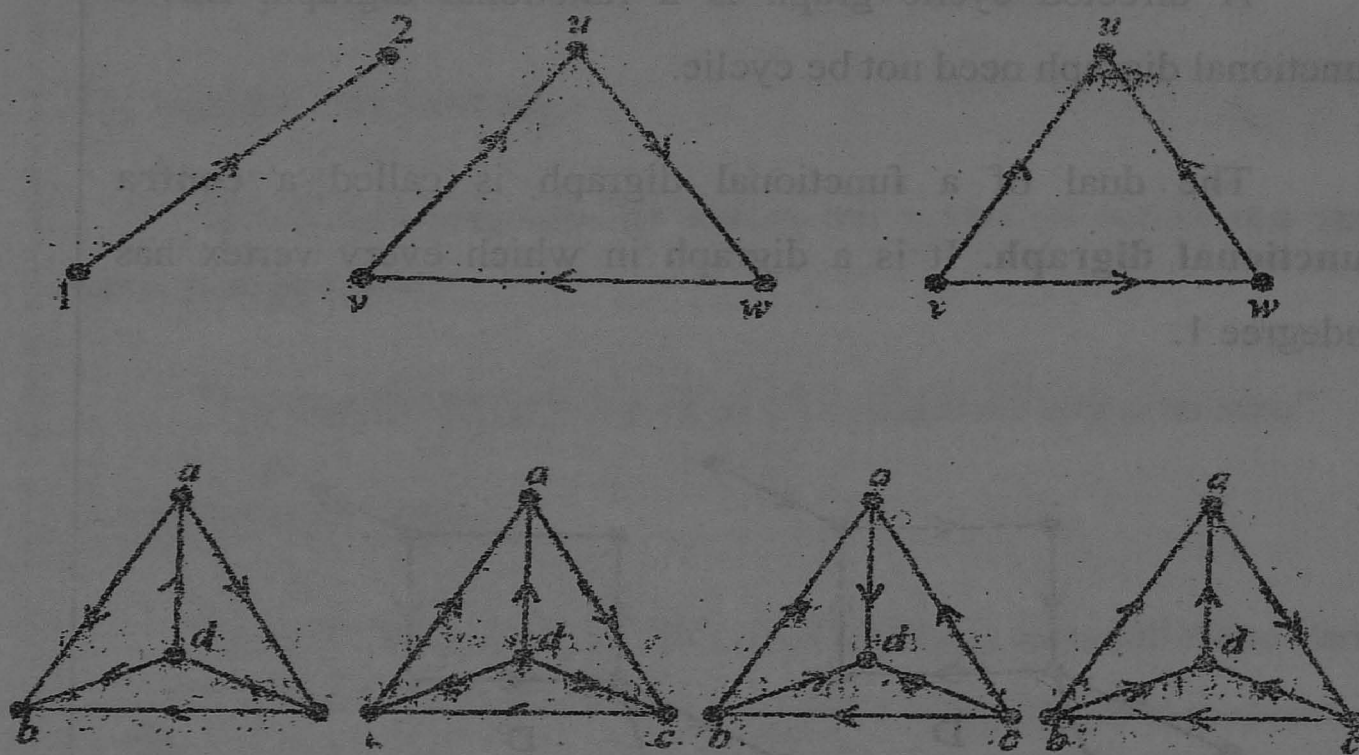
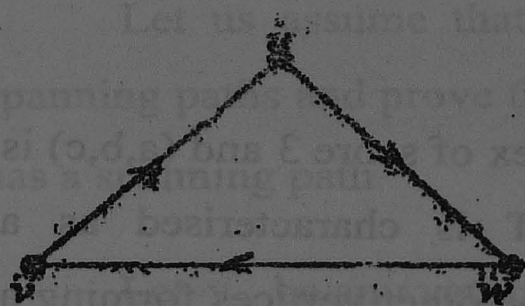


Figure 10.24

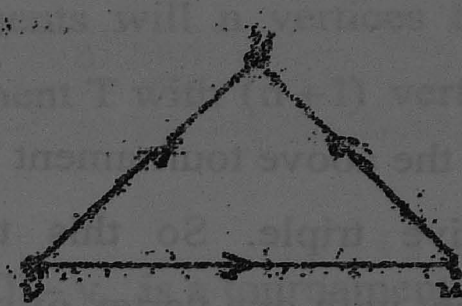
These are the only tournaments with 2, 3, 4 vertices.

Consider the tournaments of three vertices.



(i)

Figure 10.25



(ii)

Here (i) is called a **cyclic triple** and (ii) is called a **transitive triple**.

The outdegree of a vertex in a tournament is called its **score**. The scores of the vertices of a tournament taken in non increasing order is called the **score sequence** of the tournament.

Consider the tournament T of four vertices.

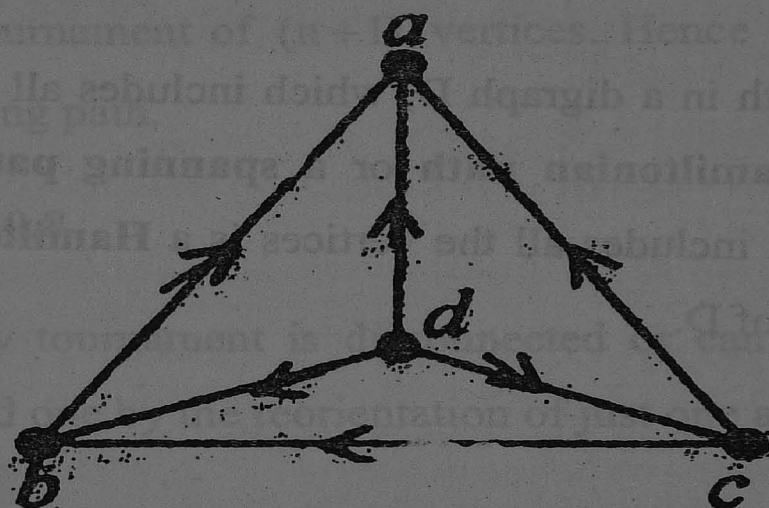


Figure 10.26

Scores of the vertices are given by the out degrees

$$d^+(a) = 0 \quad d^+(b) = 1 \quad d^+(c) = 2 \quad d^+(d) = 3$$

$\therefore$ Scores of a, b, c, d are respectively 0, 1, 2, 3.

The score sequence of T is (3, 2, 1, 0)

Note:

In the above tournament T , d is a vertex of score 3 and (a,b,c) is a transitive triple. So this tournament T is characterised as a tournament with one vertex of score 3 and the other vertices forming a transitive triple. Similarly we can characterise the other tournaments of four vertices.

Remarks:

The score of each vertex of a cyclic triple is 1. There is exactly one vertex of score 2, one vertex of score 1 and one vertex of score 0 in a transitive triple.

Definition: 10.5.2

Hamiltonian path:

A directed path in a digraph D , which includes all the vertices of D is called a **Hamiltonian path** or a **spanning path** of D . A directed cycle which includes all the vertices is a **Hamiltonian cycle** or a **spanning cycle** of D .

Theorem: 10.7

Every tournament has a spanning path.

Proof:

We prove the result by induction on the number of vertices of the tournament.

Obviously every tournament of 2, 3 or 4 vertices has a spanning path.

Let us assume that all tournaments with n vertices have spanning paths and prove that a tournament T with $(n+1)$ vertices has a spanning path.

Let v_0 be any vertex of T . Now $T - v_0$ is a tournament of n vertices and so it has a spanning path P (say). Let $P = v_1 v_2 v_3 \dots v_n$. The arc $v_0 v_1$ or $v_1 v_0$ belongs to T . If $v_0 v_1$ is in T then $v_0 v_1 \cup P$ (i.e.,) $v_0 v_1 v_2 v_3 \dots v_n$ is spanning path of T .

If $v_1 v_0$ is in T , let v_i be the first vertex of P for which the arc $v_0 v_i$ is in T . Therefore $v_0 v_{i-1}$ is not in T . So $v_{i-1} v_0$ is an arc in T and $v_1 v_2 \dots v_{i-1} v_0 v_1 \dots v_n$ is a spanning path of T . If no vertex such as v_i exists then $v_n v_0$ is an arc of T . Therefore $v_1 v_2 \dots v_n v_0$ is a spanning path of T . Thus there is a spanning path of a tournament of $(n+1)$ vertices. Hence each tournament has a spanning path.

Theorem: 10.8

Every tournament is disconnected or can be made into a disconnected one by the reorientation of just one arc.

Proof:

Let T be a tournament. If it is disconnected we are through.

Suppose T is not disconnected. Let T be a tournament on n vertices $v_1, v_2, \dots, v_n$. Since T is a tournament there is a spanning path $P: v_1 v_2 \dots v_n$. The arc between v_1 and v_n is either $v_n v_1$ or $v_1 v_n$. If it is $v_n v_1$ then $P \cup v_n v_1$, that is $v_1 v_2 \dots v_n v_1$ is a spanning cycle of T and so T is disconnected. If the arc is $v_1 v_n$

reorient it as $v_n v_1$ so that we get a new tournament T' for which $v_1 v_2 \dots v_n v_1$ is a spanning cycle and so T' is disconnected.

Theorem: 10.9

Each vertex of a disconnected tournament of p vertices $p \geq 3$ is contained in a directed cycle of length k for each k , $3 \leq k \leq p$.

Proof:

Let T be a disconnected tournament on p vertices and u be any vertex of T .

Consider,

$$N^+(u) = \{w \mid (u, w) \text{ is an arc}\}$$

$$N^-(u) = \{w \mid (w, u) \text{ is an arc}\}$$

Clearly $N^+(u) \cap N^-(u) = \Phi$ and $N^+(u) \cup N^-(u) = V - \{u\}$ where V is the vertex set of T . Again since T is disconnected neither $N^+(u)$ or $N^-(u)$ is empty. Also $N^-(u)$ is reachable from $N^+(u)$. That is there is an arc from any vertex in $N^+(u)$ to a vertex w in $N^-(u)$. So u lies on the directed cycle $u v w u$ of length 3.

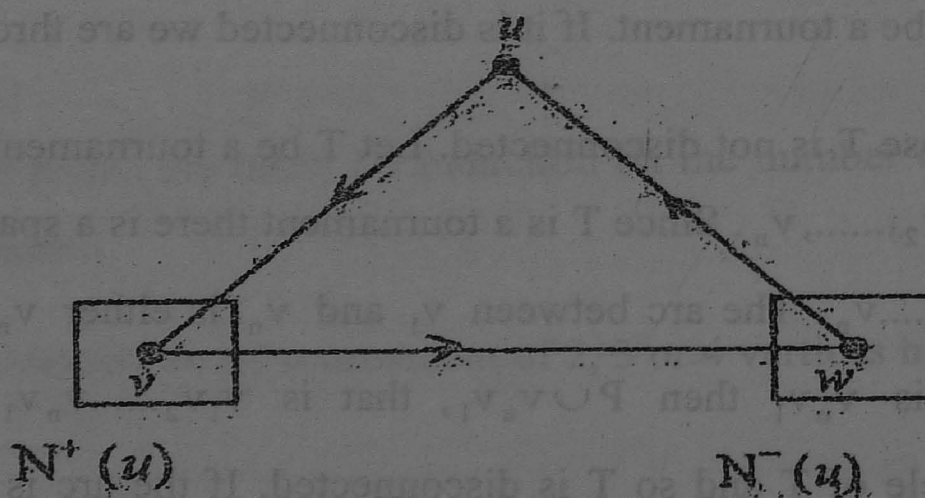


Figure 10.27

Let us assume that the vertex u is on the directed cycle of all lengths n where $3 \leq n \leq p$. We prove that u lies on a directed cycle of length $(n+1)$.

Let $C: v_1, v_2, v_3, \dots, v_n$ be a directed n -cycle on which u lies. Let $v_1 = v_n = u$. We have two cases.

Case (i)

There exists a vertex $v \in V(T) - V(C)$ and vertices v_i, v_{i+1} in C such that $v_i v$ and $v v_{i+1}$ are arcs of T .

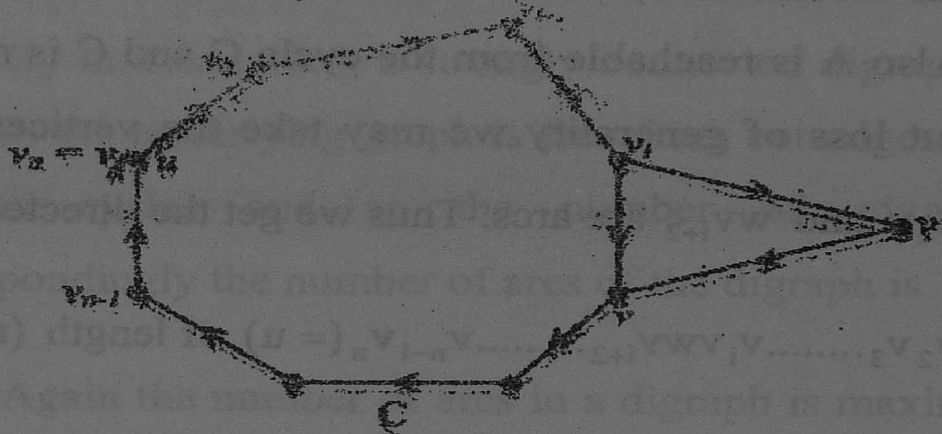


Figure 10.28

In this case $(u =) v_1 v_2 v_3 \dots v_i v v_{i+1} \dots v_{n-1} v_n (= u)$ is a directed cycle of length $(n+1)$. So u lies on a directed cycle of length $(n+1)$.

Case (ii)

There does not exist a vertex v as in case (i)

Let $A = \{w \in V(T) - V(C) \mid v_i w \text{ is an arc } \forall_i\}$

$B = \{w \in V(T) - V(C) \mid w v_{i+1} \text{ is an arc } \forall_i\}$

Clearly A and B are non empty, disjoint and $A \cup B = V(T) - V(C)$.



Figure 10.29

Since T is disconnected, there exist $v \in A$ and $w \in B$ such that vw is an arc. Also A is reachable from the cycle C and C is reachable from B . Without loss of generality we may take the vertices v_i and v_{i+2} such that $v_i v$ and $w v_{i+2}$ are arcs. Thus we get the directed cycle.

$(u =) v_1 v_2 v_3 \dots v_i v w v_{i+2} \dots v_{n-1} v_n (= u)$ of length $(n + 1)$.

So u lies on a directed cycle of lengths $(n + 1)$.

Hence the theorem is proved by induction.

Theorem: 10.10

A tournament is strong if and only if it has a spanning cycle.

Proof:

If the tournament T on p vertices is strong, it is disconnected. So, by the previous theorem it has a cycle of length p . Hence T has a spanning cycle.

Conversely if a digraph has a spanning cycle, every pair of vertices are mutually reachable. So T is disconnected and hence it is strong.

Problem: 1

If a (p, q) digraph is strongly connected prove that

$$p \leq q \leq p(p-1)$$

Solution:

By theorem 10.4, a strongly connected digraph has no cut edge. So the underlying graph is free from cut edges. Therefore it contains cycles and so the number of edges $q \geq p$. Correspondingly the number of arcs of the digraph is $q \geq p$.

Again the number of arcs in a digraph is maximum when it is a complete symmetric digraph for which $q = p(p-1)$. So for any strongly connected digraph $q \leq p(p-1)$

Hence $p \leq q \leq p(p-1)$

Problem: 2

If T is a tournament on p vertices with score sequence $(s_1, s_2, \dots, s_p)$ then show that

$$(a) \sum_{i=1}^p s_i = \frac{p(p-1)}{2}$$

$$(b) \sum_{i=1}^p s_i^2 = \sum_{i=1}^p (p-1-s_i)^2$$

Proof:

- (a) In a tournament, there is one directed edge between every pair of vertices. So the underlying graph is a complete graph of p vertices. Therefore the number of edges is $\frac{p(p-1)}{2}$ and so the number of arcs in the tournament is $q = \frac{p(p-1)}{2}$.

Now the score of a vertex is its outdegree.

$$\therefore \sum_{i=1}^p s_i = \text{sum of out degrees.}$$

$$= q.$$

$$= \frac{p(p-1)}{2}$$

$$\sum_{i=1}^p s_i = \frac{p(p-1)}{2}$$

$$(b) \sum_{i=1}^p (p-1-s_i)^2 = \sum_{i=1}^p (p-1)^2 - 2 \sum_{i=1}^p (p-1)s_i + \sum_{i=1}^p s_i^2$$

$$= p(p-1)^2 - 2(p-1) \sum_{i=1}^p s_i + \sum_{i=1}^p s_i^2$$

$$= p(p-1)^2 - 2(p-1) \frac{p(p-1)}{2} + \sum_{i=1}^p s_i^2$$

$$= \sum_{i=1}^p s_i^2$$

$$\therefore \sum_{i=1}^p s_i = \therefore \sum_{i=1}^p (p-1-s_i)^2$$

Problem: 3

Prove the number of transitive triples in a tournament T with score sequences $(s_1, s_2, \dots, s_p)$ is $\frac{1}{2} \sum_{i=1}^p s_i(s_i - 1)$

Proof:

Let $v_1, v_2, \dots, v_p$ be the p vertices of the tournament T . Let s_i be the score of the vertex v_i . In a transitive triple exactly one vertex has score 2.

So the number of transitive triples having v_i as its vertex of degree 2 is $s_i C_2 = \frac{1}{2} s_i(s_i - 1)$.

Hence the number of transitive triples in the tournament T is $\frac{1}{2} \sum_{i=1}^p s_i(s_i - 1)$.

