R 280

Directorate of Distance Education

MATHEMATICS

M.Sc. – First Year

26

Paper - II

REAL AND COMPLEX ANALYSIS

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SYLLABUS

M.Sc. – First Year

Paper – II : REAL AND COMPLEX ANALYSIS

Part A : Real Analysis

Unit I : The Real field - the extended Real number system - the complex field - Euclidean space, Finite - Countable, uncountable sets - convergent sequences - subsequences - Cauchy sequences - Upper and Lower limits - some special sequences

Unit II : Series - Series of non-negative terms - The number e - the root and ratio tests - Power series - summation by parts - Absolute convergence - addition and multiplication of series - Rearrangements.

Unit III : Limits of functions - continuous functions - continuity and compactness - continuity and connectedness - Discontinuities - Monotone functions - Infinite limits and limits at infinity

Unit IV : The derivative of a real function - Mean value theorems - the continuity of derivatives - L'Hospital rule - derivatives of higher order - Taylor's theorem.

Unit V : Definition and existence of the integral, properties of the integral, Integration and Differentiation.

Part B : Complex Analysis

Unit VI : Algebra of complex numbers, Geometric representation of complex numbers

Unit VII : Complex function : concept of analytic function, elementary theory of power series, exponential and Trigonometric functions.

Unit VIII : Analytic functions : Elementary point set topology, conformality, Linear transformations (symmetry cross ratio, group property of the linear transformation, elementary conformal mapping)

Unit IX : Complex integration : Fundamental theorems, Cauchy integral formula, Local properties of analytic functions, Statement of general forms of Cauchy theorem, calculus of residues

Unit X : Power series expansions, Wieierstrass theorem, Taylor's theorem, Laurent series

Text Books

For Real Analysis

1. Lecture material prepared by DDE, Madurai Kamaraj University
2. Walter Rudin : Principles of Mathematical Analysis, IIIrd Edition, MCGraw Hill International student Edition, 1976.

For Complex Analysis

1. Lecture mateial prepared by DDE, Madurai Kamaraj University

Reference Books

1. Ahlfors : Complex Analysis, Third edition, McGraw Hill International Book Company
2. V.Karunakaran : Complex Analysis, Narosa Publication House, New Delhi 2002.

ANALYSIS – I

REAL AND COMPLEX ANALYSIS

UNIT – A : REAL ANALYSIS

UNIT – I

SECTION–1 : THE REAL FIELD

Section 1.1.1 :

The rational number system is inadequate for many purposes, both as a field and as an ordered set. There is no rational number p such that $p^2 = 2$. This leads to the introduction of so-called "irrational numbers" that are often written as infinite decimal expansions.

Example 1.1.2 :

Prove that the equation $p^2 = 2$ has no solutions in Q, the set of rationals.

Proof :

Suppose $p = \frac{m}{n}$, m & n are not both even,

satisfying $p^2 = 2$

Then $2 = \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2}$

$2n^2 = m^2$

$\therefore$ m^2 is even, and so m is even

$\Rightarrow$ 4 divides m^2

$\Rightarrow$ 4 divides $2n^2$

$\Rightarrow$ 2 divides n^2

$\therefore$ n^2 is even & so n is even.

Hence we get a contradiction.

$\therefore$ $p^2 = 2$ is not satisfied by an rational number.

Definition 1.1.3 :

Let `S′ be any set. An order on S is a relation (denoted by <), with the following properties.

(i) $x \in S$, $y \in S \Rightarrow$ one and only one the statements $x < y$, $x = y$, $y < x$ is true.

(ii) If $x, y, z \in S$; $x < y$; $y < z$, then $x < z$.

Definition 1.1.4 :

An ordered set S is a set in which an order is defined.

Definition 1.1.5 :

Suppose S is an ordered set, and $E \subset S$. Then E is bounded above if there exists a $\beta \in S$ satisfying $x \leq \beta$, $\forall x \in E$. Here β is called an upper bound of E.

Lower bounds are defined in the same way with $\geq$ in place of $\leq$.

Definition 1.1.6 :

Let S be an ordered set, $E \subset S$, and E is bounded above. Then α is least upper bound (lub) of E, if (i) α is an upper bound of E; (ii) if $r < \alpha$, then r is not an upper bound of E. Infact, $\alpha = \text{Sup } E$.

Similarly, the greatest lower bound (glb) or infimum of a set E is K means that (i) K is a lower bound of E, and no β with $\beta > \alpha$ is a lower bound of E.

Example 1.1.7 :

Let A be the set of all positive rationals p such that $p^2 < 2$, and let B consist of all positive rationals with $p^2 > 2$. The set `A' is bounded above. Infact, the uper bounds of A are exactly the members of B. Since B contains no smallest member, A has no least upper bound in Q.

Similarly B is bounded below. The collection of all lower bounds of B consists of A, and of all $r \in Q$ with $r \leq 0$.

Since A has no largest member, B has no greatest lower bound in Q.

Example 1.1.8 :

If $\alpha = \sup E$ exists, then α' may (or) may not be a member of E.

Assume that $E_1 = \{r \in Q : r < 0\}$

$E_2 = \{r \in Q : r \leq 0\}$

Then $\sup E_1 = \sup E_2 = 0$; $0 \notin E_1$; $0 \in E_2$.

Example 1.1.9 :

Let $E = \{1/n : n \in N\}$. Then $\sup E = 1 \in E$; $\inf E = 0 \notin E$.

Definition 1.1.10 :

An ordered set S is called the least-upper-bound property, if the following is true.

$E \subset S$, $E \neq \{\}$ and E is bounded above

$\Rightarrow$ sup E exists in S.

By example (1.1.7), Q does not have the least-upper-bound property.

Theorem 1.1.11 :

Suppose S is an ordered set with the least-upper-bound property. Assume $B \subset S$, $B \neq \{\}$, and B is bounded below. Let L be the set of all lower bounds of S. Then show that α = sub L exists in S and α = inf B. In particular inf B exists in S.

Proof :

Given that B is bounded below.

$\therefore L \neq \{\}$. It follows that L is the collection of those $y \in S$ satisfying $y \leq x$, $\forall x \in B$. Thus every $x \in B$ is an upper bound of L, and so L is bounded above.

By assumption, L has a supremum in S, say 'α'.

If $r < \alpha$, then r is not an upper bound of L $\Rightarrow r \notin B$. Then $\alpha \leq x$, $\forall x \in B$.

Thus $\alpha \in L$.

If $\alpha < \beta$, then $\beta \notin L$ (since 2 is an upper bound of L).

$\therefore \alpha \in L$, but $\beta \notin L$ if $\beta > \alpha$.

Hence α is a lower band of B, but β is not if $\beta > \alpha$. This implies that α = inf B.

(Fields)

Definition 1.12 :

A field is a set F with two binary operations called addition and multiplication, that satisfy the following so-called field axioms. (A), (M) and (D).

(A) Axioms for addition :

(A1) $x, y \in F \Rightarrow x+y \in F$

(A2) $x+y = y+x$, $\forall x, y \in F$

(A3) $(x+y)+z = x+(y+z), \forall\ x,y,z \in F$

(A4) F has an element `0′ such that $x+0 = 0+x, \forall\ x \in F$.

(A5) Every element $x \in F$ has at least one addition inverse $(-x)$ in F itself.

(M) Axioms for multiplication :

(M1) $x,y \in F \Rightarrow xy \in F$

(M2) $xy = yx, \forall\ x, y \in F$

(M3) $(xy)z=x(yz), \forall\ x, y, z \in F$

(M4) F has an element $1\neq0$ satisfying $1.x = x = x.1, \forall\ x \in F$

(M5) Every non-zero element $x\in F$ has at least one multiplicative inverse.

(D) The distributive law

$x.(y+z) = x.y+x.z, \forall\ x, y, z \in F$

Example 1.1.13 :

The set Q, the collection of all rationals is a field under usual addition and usual multiplication.

Theorem 1.1.14 :

The axioms for addition imply the following statements.

(i) $x+y = x+z \Rightarrow y = z$ (ii) $x+y = x \Rightarrow y = 0$

(iii) $x+y = 0 \Rightarrow y = -x$ (iv) $-(-x) = x$

Proof :

(i) Let $x+y = x+z$. Then from axioms (A),

$y = 0+y = (-x+x)+y = (-x)+(x+y) = (-x)+(x+z) = (-x+x)+z = 0+z = z$

(ii) $x+y = x = x+0 \Rightarrow y = 0$ by (i)

(iii) $x+y = 0 = x+(-x) \Rightarrow y = -x$ by (i)

(iv) $x+(-x) = 0 = -(-x)+(-x) \Rightarrow x = -x$ by (i)

Corollary 1.1.15 :

The axioms for multiplication imply the following statements :

(i) If $x \neq 0$ and $xy = xz$, then $y = z$

(ii) $x \neq 0; xy = x \Rightarrow y = 1$

(iii) $x \neq 0; xy = 1 \Rightarrow y = 1/x$

(iv) $x \neq 0 \Rightarrow 1/(1/x) = x$

Proof :

They are similar to previous theorem.

Theorem 1.1.16 :

The field axioms imply the following statements for all $x, y, z \in F$.

(i) $0x = 0$ (ii) $x \neq 0; y \neq 0 \Rightarrow xy \neq 0$

(iii) $(-x)y = -(xy) = x(-y)$; (iv) $(-x)(-y) = xy$

Proof :

(i) $0 = 0+0 \Rightarrow 0x = (0+0)x = 0x+0x$ by ----- (D)

$\therefore 0x = 0$ by 1.1.14 (ii)

(ii) Let $x \neq 0 \neq y$; but $xy = 0$

Then $1 = (1)(1) = \left(\frac{1}{y}y\right)\left(\frac{1}{x}x\right)$

$= \left(\frac{1}{y}\right)\left(\frac{1}{x}\right)(xy) = \left(\frac{1}{y}\right)\left(\frac{1}{x}\right)(0) = 0$, a contradiction. Thus (ii) holds

(iii) $(-x)y+xy = (-x+x)y = 0y = 0$

$-(xy)+xy = 0 \Rightarrow (-x)y = -(xy)$

(iv) $(-x)(-y) = -x(-y)$ by (iii)

$= -(-(xy))$ by (iii)

$= xy$ by 1.1.14 (iv).

Definition 1.1.17 :

An ordered field is a field F that is an ordered set such that (i) $x+y < x+z$, if $x,y,z \in F$; $y < z$. (ii) $xy > 0$, if $x \in F$, $y \in F$; $x > 0$, $y > 0$.

Example 1.1.18 :

Q is an ordered field.

Theorem 1.1.19 :

The following statements are true in every ordered field.

(i) If $x > 0$, then $-x < 0$ and viçe versa.

(ii) $x > 0$; $y < z \Rightarrow xy < xz$.

(iii) $x < 0$, $y < z \Rightarrow xy > xz$

(iv) $x \neq 0 \Rightarrow x^2 > 0$. In particular $1 \neq 0$

(v) $0 < x < y \Rightarrow 0 < 1/y < 1/x$.

Proof :

(i) $x > 0 \Rightarrow 0 = -x+x > -x+0 = -x$

$x < 0 \Rightarrow 0 = -x+x < -x+0 = -x$

(ii) $x > 0$; $y > z \Rightarrow z-y > y-y = 0$

$\therefore x(z-y) > 0 \Rightarrow xz - xy > 0$

$\Rightarrow \quad xz > xy$

(iii) $x < 0$; $y < z$

$\Rightarrow (-x) > 0$

$\therefore -(x.(z-y) = (-x)(z-y) > 0$

$\therefore x(z-y) < 0 \Rightarrow xz-xy < 0$

$\Rightarrow xz < xy$

(iv) Let $x \neq 0$. If $x > 0$, then $xx = x^2 > 0$

If $x < 0$, then $(-x) > 0 \Rightarrow (-x)^2 > 0$

$\Rightarrow (-x)(-x) > 0$

$1 \neq 0 \Rightarrow 1^2 = 1$

(v) Let $0 < x < y$

$y > 0$, and $v \leq 0 \Rightarrow yv \leq 0$

But $y(1/y) = 1 > 0$

Thus $1/y > 0$; similarly $1/x > 0$.

$\therefore x < y; \frac{1}{x}\frac{1}{y} > 0$

$\Rightarrow x\frac{1}{x}\frac{1}{y} < y\frac{1}{x}\frac{1}{y}$

$\Rightarrow 0 < \frac{1}{y} < \frac{1}{x}$

SECTION 2 : THE REAL FIELD

Theorem 1.2.1 :

If $x \in R$, $y \in R$, and $x > 0$, there is a positive integer n such that $nx > y$.

Proof :

Let $A = \{nx : n \in N\}$

Suppose $nx \leq y$, $\forall x,y \in R$; $\forall n$.

Then y is an upper bound of A.

$\Rightarrow$ A has lub in R, say α

$x > 0 \Rightarrow \alpha - x < \alpha$, and $\alpha - x$ is not an upper bound of A.

Therefore $\alpha - x < mx$ for some positive integer `m'.

$\Rightarrow \alpha < mx + x = (m+1)x \in A$, which is a contradiction,

$\therefore nx > y$ for some `n'.

Theorem 1.2.2 :

If $x, y \in R$; $x < y$, then there exists a $p \in Q$ such that $x < p < y$.

Proof :

Let $x, y \in R$; $x < y \Rightarrow y - x > 0$

By Theorem 1.2.1, there is a positive integer n satisfying $n(y-x) > 1$.

Further there are positive integers m_1 & m_2 such that $m_1 > nx$; $m_2 > (-nx)$

Then $-m_2 < nx < m_1$

There is an integer $m(-m_2 \leq m \leq m_1)$ such that $m-1 \leq nx < m$

Hence $nx < m \le 1+nx < ny$

$n>0 \Rightarrow x < m/n < y$

$\Rightarrow x < p < y$ with $p = m/n \in Q$

Theorem 1.2.3 :

For every real $x > 0$, and every integer $n > 0$, show that there is only one real y satisfying $y^n = x$.

Proof :

Let E be the set containing of all positive reals `t' such that $t^n < x$. Then if $t = x|(1+x)$, then $0 < t < 1$.

So $t^n < t < x \Rightarrow t \in E \quad \therefore E \neq \{\}$

If $t > (1+x)$, then $t^n > t > x$.

So that $t \notin E$. Therefore $1+x$ is an upper bound of E.

Let $y = \text{Sup } E$.

Then the identity $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + ... + a^{n-1})$

gives $b^n - a^n < (b-a)n\, b^{n-1}$, where $0 < a < b$.

Assume $y^n < x$.

Choose h such that $0 < h < 1$, and $h < \dfrac{x - y^n}{n(y+1)^{n-1}}$

Let $a = y$; $b = y+h$;

$$(y+h)^n - y^n < hn(y+h)^{n-1}$$
$$\le hn(y+1)^{n-1}$$
$$< x - y^n.$$

Therefore $(y+h)^n < x$ and $y+h \in E$.

$y+h > y$ which is a contradiction.

Assume $y^n > x$.

Let $k = \dfrac{y^n - x}{n.y^{n-1}} \Rightarrow 0 < k < y$

If $t \ge (y-k)$, we conclude that $y^n - t^n \le y^n - (y-k)^n < kny^{n-1} = y^n - x$

So $t^n > x$, and $t \notin E$

If follows that $y-k$ is an upper bound of E.

$\therefore y-k < y$, which is a contradiction.

Hence $y^n = x$.

Corollary 1.2.4 :

If a & b are positive real numbers and n is a positive integer, then show that $(ab)^{1/n} = a^{1/n} b^{1/n}$.

Proof :

There exist α and β satisfying

$\alpha^n = a; \beta^n = b \Rightarrow \alpha = a^{1/n}; \beta = b^{1/n}$.

$ab = \alpha^n\beta^n = (\alpha\beta)^n$

$(ab)^{1/n} = \alpha\beta \Rightarrow (ab)^{1/n} = a^{1/n} b^{1/n}$

The Extend Real Number System

Definition 1.2.5 :

The extended real number system is the real field R containing $+\infty$ and $-\infty$.

We have $-\infty < x < +\infty, \forall x \in R$

$x+\infty = +\infty; x-\infty = -\infty; x/+\infty = x/-\infty = 0$

$x.\infty = +\infty; x.(-\infty) = -\infty$, if $x > 0$

$x.\infty = -\infty; x(-\infty) = +\infty$, if $x < 0$.

Definition 1.2.6 :

A complex number is an ordered pair (a, b) of reals (a+ib). Addition and multiplication of complex numbers are defined as

$(a+ib)+(c+id) = (a+c)+i(b+d)$

$(a+ib)(c+id) = (ac-bd)+i(ad+bc)$.

Theorem 1.2.7 :

Prove that these definitions of addition and multiplication in the set of all complex numbers is a field.

Proof :

Assume $x = a+ib$; $y = c+id$; $z = e + if$.

From definition 1.1.12, (A1) is an obvious.

(A2) $x+y = (a+c)+i(b+d) = (c+a)+i(d+b) = y+x$

(A3)
$$\begin{aligned}(x+y)+z &= [(a+c)+i(b+d)]+(e+if)\\ &= ((a+c)+e)+i((b+d)+f)\\ &= (a+(c+e))+i(b+(d+f))\\ &= x+(y+z)\end{aligned}$$

(A4) $x+0 = (a+ib)+0+i0 = a+ib = x$

(A5) Let $-x = -a-ib \Rightarrow x+(-x) = (a+ib)+(-a-ib) = 0+i0 = 0$

(M1) is obvious

(M2)
$$\begin{aligned}xy &= (ac-bd) + i(ad+bc)\\ &= (ca-db) + i(da+cb)\\ &= yx\end{aligned}$$

(M3)
$$\begin{aligned}(xy)z &= [(ac+bd)+i(ad+bc)][e+if]\\ &= (ace-bde-adf-bcf) + i(acf-bdf+ade+bce)\\ &= (a+ib)[(ce-df)+i(cf+de)]\\ &= x(yz)\end{aligned}$$

(M4) $1x = (1+0i)(a+ib) = a+ib = x$

(M5) if $x \neq 0$, then $a+ib \neq 0+i0$

$\therefore$ Either $a \neq 0$ (or) $b \neq 0$

Then $$\frac{1}{x} = \frac{a}{\left(a^2+b^2\right)} - i\frac{b}{\left(a^2+b^2\right)}$$

$$x\left(\frac{1}{x}\right) = (a+ib)\left(\frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}\right) = 1+i0 = 1.$$

(D)
$$\begin{aligned}x.(y+z) &= (a+ib)[(c+e)+i(d+f)]\\ &= (ac+ae-bd-bf)+i(ad+af+bc+be)\\ &= [(ac-bd)+i(ad-bf)]+[(ae-bf)+i(af+be)]\\ &= xy+xz\end{aligned}$$

Corollary 1.2.8 :

For reals a & b,

$(a+i0)+(b+i0) = a+ib$; $(a+i0)(b+i0) = ab+i0$

Remark 1.2.9 :

(a) $i = 0+(1)i$

$i^2 = (0+(1)i)(0+i) = -1+i(0) = -1$

(b) $z = a+ib \Rightarrow a = \mathrm{Re}(z)$; $b = \mathrm{Im}(z)$.

Conjugate of z is $\bar{z} = a-ib$.

Then $|z| = +\sqrt{z\bar{z}} \Rightarrow |z|^2 = z\bar{z}$

Theorem 1.2.10 :

Let z and w be complex numbers. Prove that (i) $|z| > 0$ unless $z = 0$; (ii) $|\bar{z}| = z$; (iii) $|zw| = |z||w|$; (iv) $|\mathrm{Re}(z)| \le |z|$; (v) $|z+w| \le |z|+|w|$.

Proof :

(i) Let $z = a+ib$

Then $|z| = +\sqrt{a^2+b^2} > 0$

$$|z| = 0 \Leftrightarrow \sqrt{a^2+b^2} = 0 \Leftrightarrow a^2+b^2 = 0$$

$$\Leftrightarrow a = 0;\ b = 0 \Leftrightarrow z = a+ib = 0$$

(ii)
$$\bar{z} = a-ib \Rightarrow |\bar{z}| = \sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2} = |z|$$

(iii) Let $w = c+id$;

Then
$$\begin{aligned} |zw|^2 &= |(a+ib)(c+id)| \\ &= |(ac-bd)+i(ad+bc)|^2 \\ &= (ac-bd)^2+(ad+bc)^2 \\ &= a^2c^2+b^2d^2 - 2abcd + a^2d^2+b^2c^2+2abcd \\ &= a^2(c^2+d^2)+b^2(c^2+d^2) \\ &= (a^2+b^2)(c^2+d^2) \end{aligned}$$

$$= \quad |z|^2\,|w|^2$$

$$\therefore \quad |zw| = |z||w|$$

(iv) $z = a+ib;\ a^2 \le (a^2+b^2)$

$$|\text{Re } z|^2 \le |z|^2 \Rightarrow |\text{Re}(z)| \le |z|$$

(v) $\bar{z}\,w$ is the conjugate of $z\bar{w}$.

$\Rightarrow \bar{z}w + z\bar{w} = 2\,\text{Re}(z\bar{w})$

$$\begin{aligned}
\text{Therefore } |z+w|^2 &= (z+w)\overline{(z+w)} = (z+w)(\bar{z}+\bar{w}) \\
&= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\
&= |z|^2 + 2\,\text{Re}(z\bar{w}) + |w|^2 \\
&\le |z|^2 + 2|z\bar{w}| + |w|^2 \\
&= |z|^2 + 2|z||w| + |w|^2 \\
&= (|z|+|w|)^2 \\
|z+w| &\le |z|+|w|
\end{aligned}$$

Notation 1.2.11 :

$$x_1+x_2+\ldots\ldots+x_n = \sum_{j=1}^{n} x_j$$

Theorem 1.2.12 :

If $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ are complex numbers, then show that

$$\left|\sum_{j=1}^{n} a_j \overline{b_j}\right|^2 \le \left(\sum_{j=1}^{n} |a_j|^2\right)\left(\sum_{j=1}^{n} |b_j|^2\right)$$ (Schwarz inequality).

Proof :

Assume $A = \sum |a_j|^2$

$B = \sum |b_j|^2$

$C = \sum a_j \overline{b_j}$

If $B = 0$, then $b_1^2+b_2^2+\ldots+b_n^2 = 0$

$\Rightarrow b_1 = 0;\ b_2 = 0;\ \ldots;\ b_n = 0$

So the conclusion is trivial.

Assume B > 0.

Then
$$\Sigma\left|Ba_j - Cb_j\right|^2$$
$$= \Sigma\left(Ba_j - Cb_j\right)\left(B\overline{a_j} - C\overline{b_j}\right)$$
$$= B^2\Sigma\left|a_j\right|^2 - B\overline{C}\Sigma a_j\overline{b_j} - BC\Sigma\overline{a_j}b_j + |C|^2\Sigma\left|b_j\right|^2$$
$$= B^2A - B|C|^2$$
$$= B(AB - |C|^2)$$

Since each term in the first sum is non-negative, we get

$$B(AB-|C|^2) \geq 0$$
$$B > 0 \geq AB-|C|^2 \geq 0$$
$$\Rightarrow \quad |C|^2 \leq AB$$

Euclidean Spaces :

Definition 1.2.13 :

For all positive integer n, R^n is the set of all ordered n-tuples $(x_1,\ldots,x_n)$, where $x_1,\ldots,x_n$ are reals, called the co-ordinates of x.

$$x = (x_1,\ldots,x_n);\ y = (y_1,\ldots,y_n)$$
$$\Rightarrow\ x+y = (x_1+y_1,\ldots,x_n+y_n), \text{ and } \alpha x = (\alpha x_1,\ldots,\alpha x_n);$$

Theorem 1.2.14 :

Let x, y, z $\in R^n$ and α is real.

Prove that (i) $|x| \geq 0$; (ii) $|x| = 0 \Leftrightarrow x = 0$; (iii) $|\alpha x| = |\alpha||x|$; (iv) $|xy| \leq |x||y|$; (v) $|x+y| \leq |x|+|y|$; (vi) $|x-z| \leq |x-y|+|y-z|$

Proof :

(i) $x = (x_1,\ldots,x_n);\ y = (y_1,\ldots,y_n)$

$$|x| = \sqrt{x_1^2+\ldots+x_n^2} > 0$$

(ii) $|x| = 0 \Leftrightarrow x_1^2 = 0 = x_2^2 = \ldots = x_n^2$

$$\Leftrightarrow x_1 = 0 = x_2 = \ldots = x_n \Leftrightarrow x = 0.$$

(iii) $|\alpha x| = |(\alpha x_1, \alpha x_2, \ldots, \alpha x_n)| = \sqrt{\alpha^2 x_1^2 + \ldots + \alpha^2 x_n^2}$

$$= |\alpha|\sqrt{x_1^2 + \ldots + x_n^2} = \alpha|x|$$

(iv) It is followed by Schwarz inequality.

(v) $$\begin{aligned} |x+y|^2 &= (x+y).(x+y) = x.x + 2x.y + y.y \\ &\leq |x|^2 + 2|x||y| + |y|^2 \\ &= (|x|+|y|)^2 \\ \therefore |x+y| &\leq |x|+|y|. \end{aligned}$$

(vi) $$\begin{aligned} |x-z| &= |(x-y)+(y-z)| \\ &\leq |x-y|+|y-z| \text{ by case (v).} \end{aligned}$$

Existence Theorem 1.2.15 :

Show that there exists an ordered field R that has the least-upper-bound property.

Prove that R contains Q as a subfield.

Proof :

Step 1 :

The members of R are certain subsets of Q, called cuts.

By definition, a cut is any set $\alpha \subset Q$ with the following three properties.

(1) α is not empty; $\alpha \neq Q$.

(2) $p \in \alpha$, $q \in Q$; $q < p \Rightarrow q \in \alpha$.

(3) $p \in \alpha \Rightarrow p < r$ for some $r \in \alpha$.

From these definitions, $p \in \alpha$, and $q \notin \alpha \Rightarrow p < q$

$r \notin \alpha$; and $r < s$ and $s \notin \alpha$.

Step 2 :

Define $\alpha < \beta$ means that α is a proper subset of β. If $\alpha < \beta$ and $\beta < r$, it is clear that $\alpha < r$. We have $\alpha < \beta$ (or) $\alpha = \beta$ (or) $\alpha > \beta$.

Assume that the first two fail. Then α is not a subset of β. Hence there is a $p \in \alpha$ with $p \notin \beta$. If $q \in \beta$, it follows that $q < p$. (since $p \notin \beta$).

$\therefore q \in \alpha$ by (2)

Thus $\beta \subset \alpha$.

Since $\beta \neq \alpha$, we conclude $\beta < \alpha$. Therefore R is an ordered set.

Step 3 :

The ordered set R has the least-upper-bound property.

Let A be a non-empty subset of R. Assume that $\beta \in R$, is an upper bound of A.

Define r to be the union of all $\alpha \in A$. Then $p \in r$ iff $p \in \alpha$ for some $\alpha \in A$.

Claim :

$r \in R$ and $r = \text{sub } A$.

Since A is not empty, there exists an $\alpha_0 \in A$.

This α_0 is not empty. $\alpha_0 \subset r \Rightarrow r \neq \{\}$.

Next $r \subset \beta$ (Since $\alpha \subset \beta$, $\forall\ \alpha \in A$).

$r \neq Q$

Therefore r satisfies property (I).

To prove (II) and (III), let $p \in r$.

Then $p \in \alpha_1$ for some $\alpha_1 \in A$

If $q < p$, then $q \in \alpha_1$ and so $q \in r$. This proves (II).

If $r \in \alpha_1$, is so chosen that $r > p_j$ then $r \in r$ (since $\alpha_1 \subset r$) and so r satisfies (III). Thus $r \in R$.

It is clear that $\alpha \leq r$, $\forall\ \alpha \in A$.

Suppose $\delta < r$. Then there is an $s \in r$ and that $s \notin \delta$.

Since $s \in r$, $s \in \alpha$ for some $\alpha \in A$. $\delta < d$ and δ is not an upper bound of A. This gives the desired result $r = \sup A$.

Step 4 :

If $\alpha \in R$ and $\beta \in R$, we define $\alpha+\beta$ is $\{r+s : r \in \alpha; s \in \beta\}$.

We denote 0^* to be the set of all negative rational numbers. It is clear that 0^* is a cut. We verify the conditions for addition hold in R with 0^* playing the role of 0.

(A1) Claim :

$\alpha+\beta$ is a cut. We have $\alpha+\beta$ is a non-empty subset of Q. Assume $r' \notin \alpha$; $s' \notin \beta$.

Then $r'+s' > r+s$, $\forall\ r \in \alpha$; $s \in \beta$.

$\Rightarrow r'+s' \notin \alpha+\beta$. Then $\alpha+\beta$ satisfies (1)

Let $p \in \alpha+\beta$; Therefore $p = r+s$ for some $r \in \alpha$, $s \in \beta$.

If $q < p$, then $q-s < r \Rightarrow q-s \in \alpha$, and

$q = (q-s)+s \in (\alpha+\beta)$. Thus (2) holds.

Choose $t \in \alpha$ sothat $f > r$.

Then $p < t+s$, and $t+s \in (\alpha+\beta)$

Therefore (3) holds.

(A2)

$\alpha+\beta$ is that set of all $(r+s)$ with $r \in \alpha$; $s \in \beta$. Then $(p+\alpha)$ is the set of all $(s+r)$.

Since $r+s = s+r$, $\forall\ r \in Q$, $\forall\ s \in Q$, We obtain $\alpha+\beta = \beta+\alpha$.

(A3)

This follows association law in Q.

(A4)

If $r \in \alpha$ and $s \in 0^*$, then $r+s < r$.

So $r+s \in \alpha \Rightarrow \alpha+0^* \subset \alpha$.

Let $p \in \alpha$; $r \in \alpha$ with $r > p$.

Then $p-r \in 0^*$, and $p = r+(p-r) \in \alpha+0^*$ $\alpha \subset \alpha+0^*$

Hence $\alpha = \alpha+0^*$

(A5)

Let $\alpha \in R$; β is $\{p$: There is an $r > 0$ satisfying $-p-r \notin \alpha\}$

Claim :

$\beta \in R$ and $\alpha+\beta = 0^*$.

If $s \notin \alpha$ and $p = -s-1$, then $-p-1 \notin \alpha$

$\Rightarrow p \in \beta$. So β is not empty.

If $q \in \alpha$, then $-q \notin \beta \Rightarrow \beta \neq Q$

$\therefore$ β satisfies (1)

Let $p \in \beta$ and $r > 0$ such that $-p-r \notin \alpha$.

If $q < p$, then $-q-r > -p-r$

$\Rightarrow -q-r \notin \alpha$. Thus $q \in \beta$ and (2) holds

If $t = p+(r/2)$, $t > p$ and $-t-(r/2) = -p-r \notin \alpha$.

$\therefore$ $t \in \beta$ and β satisfies (3)

$\Rightarrow \beta \in R$.

If $r \in \alpha$ and $s \in \beta$, then $-s \notin \alpha$

$\Rightarrow r < -s$, $rts < 0$. Therefore $\alpha+\beta \subset 0^*$.

To prove opposite inclusion, let $0 \in 0^*$; $w = -v/2$. Then $w > 0$.

There is an integer `n' such that $nw \in \alpha$, but $(n+1)w \notin \alpha$.

If $p = -(n+2)w$, then $p \in \beta$(since $-p-w \notin \alpha$) and $v = nw+p \in \alpha+\beta \Rightarrow 0^* \subseteq \alpha+\beta$.

We get $\alpha+\beta = 0^*$.

Step 5 :

We follow that $\alpha, \beta, r \in R$ and $\beta < r \Rightarrow \alpha+\beta < \alpha+r$.

It is obvious from the definition of addition in R that $\alpha+\beta \subset \alpha+r$, and $\alpha > 0^*$ iff $(-\alpha) < 0^*$.

Step 6 :

Products of negative rationals are positive $\Rightarrow R^+ = \{\alpha \in R : \alpha > 0^*\}$

If $\alpha, \beta \in R^+$, then $\alpha\beta = \{p : p \leq rs$, for some choice of $r \in \alpha$; $s \in \beta$; $r, s > 0\}$

Thus $1^* = \{q : q < 1\}$

The axioms (M) and (D) hold in R^+ in place of R, and 1^* as multiplicative identity.

Step 7 :

We complete the definition of multiplication by taking $\alpha.0^* = 0^*\alpha = 0^*$ and by setting

$$\alpha\beta = \begin{cases} (-\alpha)(-\beta);\ \alpha < 0^*, \beta < 0^* \\ -[(-\alpha)\beta];\ \alpha < 0^*, \beta > 0^* \\ -[\alpha(-\beta)];\ \alpha > 0^*, \beta < 0^* \end{cases}$$

By step 6, the axioms (M) hold in R^+. For distributive law $\alpha(\beta+r) = \alpha\beta+\alpha r$, whenever $\alpha > 0^*$, $\beta < 0^*$, $\beta+r > 0^*$.

Then $\quad r = (\beta+r)+(-\beta)$

$\alpha r = \alpha(\beta+r)+\alpha.(-\beta)$

But $\alpha.(-\beta) = -(\alpha\beta)$

Thus $\alpha(\beta+r) = \alpha\beta+\alpha r$

Hence R is an ordered field with the least - upper - bound property.

Step 8 :

We define $r^* = \{p\in Q : p < r\}$ where $r\in Q$.

Then r^* is a cut.

$\therefore$ $r^*\in R$ satisfying

(i) $r^*+s^* = (r+s)^*$; (ii) $r^*s^* = (rs)^*$; (iii) $r^* < s^*$ iff $r < s$.

To prove (i), choose $p\in r^*+s^*$

Then $p = u+v$, where $u < r$; $v < s$

Therefore $p < (r+s) \Rightarrow p\in (r+s)^*$

Conversely, let $p\in(r+s)^*$, and choose `t' such that $2t = r+s-p$.

Let $r' = r-t$; $s' = s-t$;

Then $r'\in r^*$; $s'\in s^*$; $p = r'+s'$

$\Rightarrow p\in r^*+s^*$

(i) is true.

The proof of (ii) is similar (iii)

If $r < s$, then $r\in s^*$, but $r \notin r^* \Rightarrow r^* < s^*$

If $r^* < s^*$, there is $p\in s^*$ such that $p\notin r^*$.

So $r \le p \le s \Rightarrow r < s$.

Step 9 :

By step 8, the replacement of the rational numbers r by the corresponding "rational cuts" $r^* \in R$ preserves sums, products and order. Hence the ordered field Q is isomorphic to the ordered field Q^* whose elements are the rationals cuts. Further any two ordered fields with the least-upper-bound property are isomorphic.

SECTION – 3 : FINITE, COUNTABLE AND UNCOUNTABLE SETS

Definition 1.3.1 :

A function or map $f : A \to B$ is a rule that assigns a unique element in B for every element of A. The set A is called the domain of f, and B is codomain of f

$x \in A \Rightarrow f(x) \in B$

Range of $f = \{f(x) : x \in A\}$; $f^{-1}(x) = \{x \in A : f(x) \in x\}, \forall x \subset B$.

Definition 1.3.2 :

Let $f : A \to B$ be a map. Then $f(E) = \{f(x) : x \in E\}, \forall\ E \subseteq A$.

Definition 1.3.3 :

If there is a 1–1 map from A onto N (the set of all natural numbers), then A is countable.

Here A is uncountable if it is neither finite nor countable.

A is at most countable (or simply countable) if it is finite (or) countable (enumerable or denumerable).

Theorem 1.3.4 :

Prove that Every infinite subset of a countable set is countable.

Proof :

Let $E \subset A$, and A is infnite. We have known that A is countable $\Rightarrow$ The elements of A are arranged in a sequence $\{x_n\}$ of distinct elements.

Let n_1 be the smallest integer such that $x_{n_1} \in E$. After choosing $n_1, n_2, \ldots\ldots n_{k-1}$, let n_k be the smallest integer greater than n_{k-1} and $x_{n_k} \in E$.

Let $f(k) = x_{n_k}$ $(k = 1, 2, 3, \ldots\ldots,)$

Then f is bijection from E onto N

$\therefore$ E is a countable set.

Definition 1.3.5 :

$$\bigcup_{\alpha} E_\alpha = \{x : x \in E_\alpha \text{ for at least one } \alpha \in A\}$$

$$\bigcap_{\alpha} E_\alpha = \{x : x \in E_\alpha, \text{ for each } \alpha\}$$

Theorem 1.3.6 :

Show that countable union of countable sets is countable.

Proof :

Assume $\{E_\alpha\}_{\alpha=1}^{\infty}$ is a sequence of countable sets. Then each E_α is a countable set. This implies that each set E_n is arranged in a sequence $\{x_{n_k}\}$, k=1, 2, 3,......

$\therefore$ The elements of $\left(\bigcup_n E_n\right)$ are arranged in the following manner

x_{11} x_{12} x_{13} x_{14} -----

x_{21} x_{22} x_{23} x_{24} -----

....

....

....

x_{n1} x_{n2} x_{n3} x_{n4} -----

....

....

....

Now these above elements can be arranged in a sequence

$x_{11}; x_{21}, x_{12}; x_{31}, x_{22}, x_{13}, x_{41}, x_{32}, x_{23}, x_{14}, \ldots$

If any two of the sets `E_n' have elements in common, these will appear more than once in the above sequence.

Therefore there is a subset T of the set of all positive integers such that $\left(\bigcup_{n=1}^{\infty} E_n\right)$ is bijective with T.

$\therefore \bigcup_{n=1}^{\infty} E_n$ is countable.

Theorem 1.3.7 :

Let A be a countable set, and let B_n be the set of all n-tuples $(a_1,\ldots,a_n)$, where $a_k \in A$ (k=1,2,...,n) and the elements $a_1,\ldots,a_n$ need not be distinct. Show that B_n is countable.

Proof :

Since $A = B_1$, it is countable. Suppose B_{n-1} is countable. (n=2, 3, 4,...)

The elements of B_n are of the form (b, a) where $b \in B_{n-1}$; $a \in A$. For every fixed b, the set of pairs (b, a) is equivalent to A, and so it is countable. Therefore B_n is the union of a countable set of countable sets. Thus B_n is countable.

Corollary 1.3.8 :

Show that the set of all rationals is countable.

Proof :

We apply the above theorem with n=2. Every rational number is of the form b/a, where a&b are integers. The set of pairs (a, b) and so the set of fractions b/a is countable.

Theorem 1.3.9 :

Let 'A' be the set of all sequences whose elements are the digits 0 & 1. Then the set A is uncountable.

Proof :

The elements of A are sequences like 1, 0, 0, 1, 0, 1, 1, 1,....

Let E be a countable subset of A, and E consists of the sequences s_1, s_2 ,.... We construct a sequence s_n as follows :

If the n^{th} digit in s_n is 1, we let the n^{th} digit of s_n be 0, and vice versa. Therefore the sequence s_n differs from every member of E in at least one place. So $s_n \notin E$. But we have $s_n \in A \Rightarrow E$ is a proper subset of A and it is a proper subset of itself $\Rightarrow$ A is uncountable.

SECTION – 4 : CONVERGENT SEQUENCES

Definition 1.4.1 :

A sequence $\{p_n\}$ in a metric space (X, d) is called converge sequence if there is a point $p \in X$ with the property; $\forall \epsilon > 0$, there is an integer N satisfying $d(p_n, p) < \epsilon$, $\forall\, n \geq N$.

Further $p_n \to p$ (or) $\underset{n\to\infty}{\text{Lt}}\ p_n = p$. If $\{p_n\}$ does not converge, it is said to be "diverge.

Example 1.4.2 :

(i) $\{1/n\}$ converges to `0'.

(ii) $\{n^2\}$ does not converge (diverges)

(iii) $\{1+(-1)^n/n\}$ converges to 1.

(iv) $\{i^n\}$ is divergent bounded.

Theorem 1.4.3 :

Let $\{p_n\}$ be a sequence in a metric space X.

Prove :

(i) $\{p_n\}$ converges to $p \in X$ iff every neighborhood of p contains all but finitely many of the terms of $\{p_n\}$

(ii) If $p, q \in X$, and $\{p_n\}$ converges to p and q, then $p=q$

(iii) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.

(iv) If $E \subset X$, and p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p = \underset{n\to\infty}{\text{Lim}}\ p_n$.

Proof :

(i) Suppose $p_n \to p$, and let V be a neighborhood of p. For some $\epsilon > 0$, $d(p, q) < \epsilon$, $q \in X \Rightarrow q \in V$. Corresponding to this ϵ, there is N satisfying $n \geq N$, implies that $d(p_n, p) < \epsilon$.

$\therefore p_n \in V, \forall n \geq N$.

Conversely, let every neighborhood of p contains all, but finitely many of the p_n. Fix $\epsilon > 0$, and let V be the set of all $q \in X$ such that $d(p, q) < \epsilon$. By assumption, there is N (depending V) satisfying $p_n \in V$, whenever $n \geq N$.

So $d(p_n, p) < \epsilon$, if $n \geq N$

$\Rightarrow p_n \to p$

(ii) Let $\epsilon > 0$ be given. There exist integers N, N' satisfying

$n \geq N \Rightarrow d(p_n, p) < \epsilon/2$

$n \geq N' \Rightarrow d(p_n, q) < \epsilon/2$

Whenever $n \geq \max\{N, N'\}$, we have $d(p, q) \leq d(p, p_n) + d(p_n, q)$

$< \in/2 + \in/2 = \in$.

$\in$ is arbitrary $\Rightarrow d(p, q) = 0 \Rightarrow p = q$.

(iii) Let $p_n \to p$. There is an integer N such that $n > N \Rightarrow d(p_n, p) < 1$.

Let $r = \max\{1, d(p_1, p), \ldots\ldots. d(p_N, p)\}$

Then $d(p_n, p) \leq r, \quad \forall\, n = 1, 2, \ldots\ldots$

$\Rightarrow \{p_n\}$ is bounded.

(iv) $\forall$ integer $n > 0$, there is a point $p_n \in E$ satisfying $d(p_n, p) < 1/n$. Given $\in > 0$, choose N such that $N\in > 1$.

Whenever $n > N$, $d(p_n, p) < \in \Rightarrow p_n \to p$.

Theorem 1.4.4 :

Let $\{s_n\}$ & $\{t_n\}$ be two complex sequences, and $\lim\limits_{n\to\infty} s_n = s$, $\lim\limits_{n\to\infty} t_n = t$.

Then prove that (i) $\lim\limits_{n\to\infty} (s_n + t_n) = s + t$; (ii) $\lim\limits_{n\to\infty} (cs_n) = cs$; $\lim\limits_{n\to\infty} (c + s_n) = c + s$;

(iii) $\lim\limits_{n\to\infty} s_n t_n = st$; (iv) $\mathop{\mathrm{Lt}}\limits_{n\to\infty} \dfrac{1}{s_n} = \dfrac{1}{s}$ provided $s_n \neq 0$ (n=1, 2, 3,...) and $s \neq 0$.

Proof :

(i) Given $\in > 0$, there exist integers N_1 & N_2 satisfying

$n \geq N_1 \Rightarrow |s_n - s| < t/2$

$n \geq N_2 \Rightarrow |t_n - t| < t/2$

If $N = \text{Max}\ \{N_1, N_2\}$, then $\forall\, n \geq N \Rightarrow |(s_n + t_n) - (s + t)| \geq |s_n - s| + |t_n - t| < \in/2 + \in/2 = \in$

(ii) It is obvious

(iii) $s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)$ ------ (1)

$\forall \in > 0$, there are integers N_1 & N_2 such that

$n \geq N_1 \Rightarrow |s_n - s| < \sqrt{\in}$

$n \geq N_2 \Rightarrow |t_n - t| < \sqrt{\in}$

Whenever $N = \text{Max}\{N_1, N_2\}$,

$\forall n \geq N \Rightarrow |(s_n - s)(t_n - t)| < \epsilon$.

$$\Rightarrow \lim_{n\to\infty}(s_n - s)(t_n - t) = 0$$

By (i) & (ii), we have $\lim_{n\to\infty}(s_n t_n - st) = 0$

$$\Rightarrow \lim_{n\to\infty} s_n t_n = st$$

(iv) For $\epsilon = \frac{1}{2}|s| > 0$, there is an integer m such that $|s_n - s| < \frac{1}{2}|s|, \forall n \geq m$

$\therefore |s_n| > \frac{1}{2}|s|, \forall n \geq m$.

For $\epsilon > 0$, there is an integer $N > m$ satisfying $n \geq N \Rightarrow |s_n - s| < \frac{1}{2}|s|^2\epsilon$,

For $n \geq N$, $\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s_n - s|}{|s_n s|}$

$$< \frac{2}{|s|^2}|s_n - s|$$

$$< \epsilon$$

Theorem 1.4.5 :

Let $x_n \in R^k$, and $x_n = (\alpha_{1,n}, \ldots\ldots, \alpha_{k,n})$. Prove that

(i) $\{x_n\}$ converges to $x = (\alpha_1, \ldots, \alpha_n)$ iff $\lim_{n\to\infty}\alpha_{j,n} = \alpha_j, \forall\ 1 \leq j \leq k$ ------- (2)

(ii) Let $\{x_n\}$, $\{y_n\}$ be two sequences in R^k, $\{\beta_n\}$ be a sequence of reals. $x_n \to x$, $y_n \to y$, $\beta_n \to \beta$. Then

$$\lim_{n\to\infty}(x_n + y_n) = x + y; \quad \lim_{n\to\infty} x_n y_n = xy;$$

$$\lim_{n\to\infty} \beta_n x_n = \beta x.$$

Proof :

(i) If $x_n \to x$, the inequalities $|\alpha_{j,n} - \alpha_j| \leq |x_n - x|$ that follow immediately from the definition of the norm in R^k, show that (2) holds.

Conversely, if (2) holds, then to each $\epsilon > 0$, there corresponds an integer N such that $n \geq N$ implies

$$|\alpha_{j,n} - \alpha_j| < \frac{\epsilon}{\sqrt{k}}; \ (1 \leq j \leq k)$$

$\forall\, n \geq N$ implies

$$|x_n - x| = \left[\sum_{j=1}^{k} \left|\alpha_{j,n} - \alpha_j\right|^2\right]^{1/2} < \in$$

so $\{x_n\} \to x$

(ii) It follows from case (i).

Subsequences :

Definition 1.4.6 :

Given a sequence $\{p_n\}$, we consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < n_3 < \ldots$. Then the sequence $\{p_{n_k}\}$ is called a subsequence of $\{p_n\}$. If $\{p_n\}$ converges, its limit is called a subsequential limit of $\{p_n\}$.

It is clear that $\{p_n\}$ converges to p iff every subsequence of $\{p_n\}$ converges to p.

Theorem 1.4.7 :

(i) If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X.

(ii) Every bounded sequence in R^k contains a convergent subsequence.

Proof :

(i) Let E be the range of $\{p_n\}$. If E is finite, there is a $p \in E$, and a sequence $\{n_i\}$ with $n_1 < n_2 < n_3 < \ldots$ satisfying $p_{n_1} = p_{n_2} = \ldots\ldots = p$.

Then the subsequence $\{p_{n_i}\}$ so obtained coverges obviously to p.

If E is infinite, and a known result, E has a limit point $p \in X$. We select n_1, so that $d(p_n, p) < 1$. After choosing $n_1, n_2, \ldots n_{i-1}$, we see that there is an integer $n_i > n_{i-1}$ such that $d(p, p_{n_i}) < 1/i$. Then $\{p_{n_i}\}$ converges to p.

(ii) It follows from case (i), that every bounded subset of R^k lies in a compact subset of R^k.

Theorem 1.4.8 :

Prove that the subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.

Proof :

Let E* be the set of all subsequential limits of $\{p_n\}$, and q be a limit point of E*.

Claim :

$q \in E^*$.

We select n_1 so that $p_{n_1} \neq p$.

Let $\delta = d(q, p_{n_1})$. After choosing $n_1, \ldots\ldots n_{i-1}$ and q is a limit point of E*, there is an $x \in E^*$ with $d(x, q) < 2^{-i}\delta$.

$x \in E^* \Rightarrow$ There is an $n_i > n_{i-1}$

satisfying $d(x, p_{n_i}) \leq 2^{1-i}\delta$, $\forall$ i = 1, 2, 3,.......

This implies that $\{p_{n_i}\}$ converges to q.

Hence $q \in E^*$.

SECTION 5 : CAUCHY SEQUENCES

Definition 1.5.1 :

A sequence $\{p_n\}$ in a metric space X is said to be a Cauchy sequence, if $\forall \in > 0$, there is an integer N such that $d(p_n, p_m) < \in$, whenever $n, m \geq N$.

Definition 1.5.2 :

Let E be a subset of a metric space X, and S be the set of all reals of the form d(p, q) with $p, q \in E$.

The Sup of S is called the diameter of E.

$\{p_n\}$ is a Cauchy sequence iff $\lim\limits_{n\to\alpha}$ diam $E_N = 0$.

Theorem 1.5.3 :

Prove that

(i) If $\overline{E}$ is the closure of a set E in a metric space X, then diam $\overline{E}$ = diam E.

(ii) If K_n is a sequence of compact sets in X satisfying $K_n \supset K_n$, $\forall$n, and if $\lim\limits_{n\to\alpha}$ diam K_n=0, then $\bigcap\limits_{n=1}^{\propto} K_n$ consists of exactly one point.

Proof (i)

$E \subset \overline{E} \Rightarrow$ It is obvious that diam $E \leq$ diam $\overline{E}$.

Fix $\epsilon > 0$, and let $p, q \in \overline{E}$. By definition of $\overline{E}$, there are $p_1, q_1 \in E$ satisfying $d(p, p_1) < \epsilon$; $d(q, q_1) < \epsilon$.

Hence $\quad d(p, q) \leq d(p, p_1)+d(p_1, q_1)+d(q_1, q)$

$$< \epsilon + d(p_1, q_1) + \epsilon$$

$$\leq 2\epsilon + \text{diam } E, \forall \epsilon > 0.$$

$\therefore$ case (i) is verified.

(ii) Let $K = \left(\bigcap\limits_{K=1}^{\infty} K_n \right)$. By known result, $K \neq \{\}$. If it contains more than one point, then diam $K > 0$. For each n, $K_n \supset K \Rightarrow$ diam $K_n \geq$ diam K,

which is a contradiction to the assumption that diam $K_n \to 0$.

Theorem 1.5.4 :

Prove that (i) In any metric space X, every convergent sequence is a Cauchy sequence. (ii) If X is a compact metric space, and $\{p_n\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges to some point of X (iii) In R^k, every Cauchy sequence converges.

Proof :

(i) If $p_n \to p$, and $\forall \epsilon > 0$, there is an integer N satisfying $d(p, p_n) < \epsilon$, $\forall$ $n \geq N$.

Therefore $d(p_n, p_m) \leq d(p_n, p)+d(p, p_m)$

$$< \epsilon + \epsilon = 2\epsilon, \quad \forall n, m \geq N.$$

Thus $\{p_n\}$ is a Cauchy sequence.

(ii) Let $\{p_n\}$ be a Cauchy sequence in the compact space X. For N = 1, 2, 3,..., let E_N be the set consisting of $p_n, p_{n+1}, p_{N+2},\ldots$ Then $\lim\limits_{N\to\infty}$ diam $\overline{E_N} = 0 \ldots (*)$.

Being a closed subset of the compact space X, each $\overline{E_N}$ is compact.

Further $E_N \supset E_{N+1} \Rightarrow \overline{E_N} \supset \overline{E_{N+1}}$.

A known result gives now that there is a unique $p \in X$ that lies in every $\overline{E_N}$.

Let $\epsilon > 0$ be given. By (*), there is an integer N_0 such that diam $\overline{E_N} < \epsilon$, whenever $N \geq N_0$.

Since $p \in \overline{E_N}$, it follows that $d(p, q) < \in$, $\forall\ q \in \overline{E_N}$ (hence $\forall q \in E_N$).

$\therefore\ d(p, p_n) < \in, \forall n \geq N_0 \Rightarrow \{p_n\} \rightarrow p.$

(iii) Let $\{x_n\}$ be a Cauchy sequence in R^k. Define E_N in (ii) with x_i in place of p_i. For some N, diam $E_N < 1$. Then the range of $\{x_n\}$ is the union of E_N, and the finite set $\{x_1, x_2, \ldots x_{N-1}\}$.

Therefore $\{x_n\}$ is bounded.

Since every bounded subset of R^K has compact closure in R^K, (iii) is followed from (ii).

Definition 1.5.5 :

A metric space in which every cauchy sequence converges, is said to be complete.

Definition 1.5.6 :

A sequence $\{s_n\}$ of reals is said to be (i) monotonically increasing if $s_n \leq s_{n+1}$, $\forall n$ (ii) monotonically decresing if $s_n \geq s_{n+1}$, $\forall n$.

Theorem 1.5.7 :

Let $\{s_n\}$ is monotonic. Prove that $\{s_n\}$ converges iff it is bounded.

Proof :

Let $s_n \leq s_{n+1}$. Assume E is the range of $\{s_n\}$. If $\{s_n\}$ is bounded, let `s' be the least upper bound of E.

Then $s_n \leq s, \forall n$

$\forall\ \in > 0$, there is an integer N such that $s-\in\ < s_N \leq s$

(Otherwise $s-\in$ is an upper bound of E)

Since $\{s_n\}$ increases, $n \geq N$ thus implies $s-\in\ < s_0 < S$

which shows that $\{s_n\}$ converges to s. Similarly, the converse follows.

Upper and Lower Limits :

Definition 1.5.8 :

Let $\{s_n\}$ be a sequence of reals.

(i) $\forall$ real M, there is an integer N such that $n \geq N$ implies $s_n \geq M$.

This $s_n \rightarrow \propto$.

(ii) $\forall$ real M, there is an integer N such that $n \geq N$ implies, $s_n \leq M$

Then $s_n \to (-\infty)$

Definition 1.5.9 :

Let $\{s_n\}$ a sequence of reals and E be the set of numbers `x' such that $s_{n_k} \in x$ for some subsequence $\{s_{n_k}\}$.

$s^* = \sup E$; $s_* = \inf E$.

Theorem 1.5.10 :

Let $\{s_n\}$ be a sequence of reals; E and s^* have the same meaning in definition 1.5.9. Prove that (i) $s^* \in E$; (ii) If $x > s^*$, there is an integer N such that $s_n < x$, $\forall\, n \geq N$. s^* is only number satisfying (i) & (ii).

Proof :

If $s^* = +\infty$, then E is not bounded above. So $\{s_n\}$ is not bounded above.

$\therefore$ There is a subsequence $\{s_{n_k}\}$ such that $\{s_{n_k}\} \to \infty$.

If s^* is real, then E is bounded above and at least one subsequential limit exists so that (i) follows from known results.

If $s^* = -\infty$, then E contains only one element, namely $-\infty$, and there is no subsequential limit. $\forall$ real M, $s_n > M$ for at most a finite number of values of n, so that $s_n \to (-\infty)$. This establishes (i) in all cases.

(ii) Suppose there is a number $x > s^*$ such that $s_n \geq x$, for infinitely many values of n. Then there is a number $y \in E$ satisfying $y \geq x > s^*$ contradicting the definition of s^*.

Thus s^* satisfies (i) & (ii).

To prove uniqueness, suppose there are two numbers p & q which satisfy (i) and (ii) and suppose $p < q$.

We select x such that $p < x < q$.

Since p satisfies (ii), we obtain $s_n < x$, $\forall\, n \geq N$.

Then q can not satisfy (i).

Example 1.5.11 :

(i) Let $\{s_n\}$ be a sequence containing all rationals. Then every real number is a subsequential limit, and $\lim_{n\to\infty} \sup s_n = +\infty$; $\lim_{n\to\infty} \inf s_n = -\infty$.

(ii) Let $s_n = (-1)^n/[1+(1/n)]$

Then $\lim_{n\to\infty} \sup s_n = 1$; $\lim_{n\to\infty} \inf s_n = -1$.

(iii) For a real–valued sequence $\{s_n\}$,

$$\lim_{n\to\infty} s_n = s \quad \text{iff} \quad \lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} \inf s_n = s$$

Theorem 1.5.12 :

If $s_n \leq t_n$; $\forall n \geq N$, then $\lim_{n\to\infty} \inf s_n \leq \lim_{n\to\infty} \inf t_n$, and

$\lim_{n\to\infty} \sup s_n \leq \lim_{n\to\infty} \sup t_n$.

Proof :

It is trivial.

Some Special Sequences

Theorem 5.12 :

Prove that

(i) $p > 0 \Rightarrow \lim_{n\to\infty} n^{-p} = 0$

(ii) $p > 0 \Rightarrow \lim_{n\to\infty} p^{1/n} = 1$

(iii) $\lim_{n\to\infty} n^{1/n} = 1$

(iv) If $p > 0$, and α is real, $\lim_{n\to\infty} \dfrac{n^n}{(1+p)^n} = 0$

(v) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$

Proof :

(i) We choose $N > (1/\epsilon)^{1/p}$

Thus $|n^{-p}-0| < \epsilon, \forall n \geq N$.

(ii) Let $p > 1$. Choose $x_n = p^{1/n}-1$

Then $x_n > 0$.

By binomial theorem, $1+nx_n \leq (1+x_n)^n = p$

$$\Rightarrow 0 < x_n \leq \left(\frac{p-1}{n}\right)$$

Thus $x_n \to 0$. If $p=1$, (ii) is trivial. Whenever $0 < p < 1$, the result is obtained.

(iii) Let $x_n = n^{1/n}-1$. Then $x^n \geq 0$.

By binomial theorem,

$$n = (1+x_n)^n \geq \frac{n(n-1)}{2}\left(x_n^{\,2}\right)$$

$$\Rightarrow 0 \leq x_n \leq \sqrt{\frac{2}{n-1}} \quad (\forall n \geq 2)$$

(iv) Let k be an integer such that $k > \alpha$, $k > 0$.

$\forall\, n \geq 2k$, we have

$$(1+p)^n > (n_{c_k})pk = \frac{n(n-1)\ldots\ldots\ldots(n-k+1)}{k!}p^k$$

$$> n^k p^k / 2^k k!$$

Therefore $0 < \dfrac{n^\alpha}{(1+p)^n} < \dfrac{2^k k!}{p^k} n^{\alpha-k} \quad (\forall n \geq 2k)$

Since $\alpha-k < 0$, $n^{\alpha-k} \to 0$ by (i)

(v) Choose $\alpha=0$ in (iv).

UNIT – II

SECTION–1 : SERIES

Definition 2.1.1 :

Given a sequence $\{a_n\}$, we denote $\sum_{n=p}^{q} a_n$ $(p \leq q)$ as $a_p + a_{p+1} + \ldots a_q$.

Now a series $\sum_{n=1}^{\infty} a_n$ converges if its partial sequence $\{s_n\}$ converges where $s_n = \sum_{k=1}^{n} a_k$.

If $\{s_n\}$ diverges, then the series is said to be divergent.

Theorem 2.1.2 :

Prove that Σa_n converges if and only if, for every $\in > 0$, there is an integer N satisfying

$$\left| \sum_{k=n}^{m} a_k \right| \leq \in, \forall m \geq n \geq N.$$

It is followed by cauchy criterion $m=n \Rightarrow |a_n| \geq \in, \forall n \geq N$.

Corollary 2.1.3 :

If Σa_n converges, then $\lim\limits_{n \to \infty} a_n = 0$.

Proof :

Let $\Sigma a_n = l$. Then $\{s_n\} \to l$, where $s_n = \sum_{k=1}^{n} a_k$.

Then $a_n = s_n - s_{n-1}$

$$\lim_{n \to \infty} = \lim_{n \to \infty} s_n - \lim_{n \to \infty} s_{n-1} = l - l = 0.$$

Theorem 2.1.4 :

A series of non-negative terms converges if and only if its partial sums form a bounded sequence.

Proof :

It is followed from the fact that $\{s_n\}$ converges iff it is bounded.

Theorem 2.1.5 :

(i) If $|a_n| \leq C_n$, for all $n \geq N_0$ (where N_0 is some fixed integer), and if ΣC_n converges, then Σa_n converges.

(ii) If $a_n \geq d_n \geq 0$, for all $n \geq N_0$; and if Σd_n diverges, then Σa_n diverges.

Proof :

(i) Given $\epsilon > 0$, and ΣC_n converges, there is an integer $N \geq N_0$ such that $\sum\limits_{k=n}^{m} C_k$, $\forall m \geq n \geq N$, by the cauchy criterion.

Thus $\left|\sum\limits_{k=n}^{m} a_k\right| \leq \sum\limits_{k=n}^{m} |a_k|$

$\leq \sum\limits_{k=n}^{m} C_k \leq \epsilon.$

$\therefore \Sigma a_n$ converges.

(ii) If Σa_n converges, then Σd_n converges by case (i), and a known result (2.1.4).

SECTION – 2 : SERIES OF NON-NEGATIVE TERMS

Theorem 2.2.1 :

If $0 \leq x < 1$, then $\sum\limits_{n=0}^{\infty} x^n = \frac{1}{1-x}$. If $x \leq 1$, this series diverges.

Proof :

If $x \neq 1$, $s_n = \sum\limits_{k=0}^{n} x^k = \frac{1-x^{n+1}}{(1-x)}$.

As $n \to \infty$, and $0 \leq x \leq 1$, $x^{n+1} \to 0$.

$\therefore s_n \to \frac{1}{1-x}$; $\sum\limits_{n=0}^{\infty} x^n = \frac{1}{1-x}$

For $x=1$, $\sum\limits_{n=0}^{\infty} = 1+1+1+\ldots\ldots$ which is divergent series.

Similarly $x > 1 \Rightarrow$ the series diverges.

Theorem 2.2.2 :

Assume $a_1 \geq a_2 \geq ... \geq 0$.

Prove that Σa_n converges iff the series $\sum_{k=0}^{\infty} 2^k a_{(2^k)} = a_1+2a_2+4a_4+8a_8+...$ converges.

Proof :

By (2.1.4), it remains to the boundedness of the partial sums.

Assume $s_n = a_1+a_2+......+a_n$

$t_k = a_1+2a_2+4a_4+..........+2^k(a_{2^k})$

$\forall\, n < 2^k$, we have

$$s_n \leq a_1+(a_2+a_3)+........+(a_{2^k}+.......+a_{2^{k+1}-1})$$
$$\leq a_1+2a_2+..........+2^k a_{(2^k)}$$
$$= t_k \qquad \text{--------- (*1)}$$

so that $s_n \leq t_k$

But $n > 2^k$, we get

$$s_n \geq a_1+a_2+(a_3+a_4)+......+(a_{2^{k-1}+1}+.......+a_{2^k})$$
$$\geq (1/2)a_1+a_2+2a_4+.........+2^{k-1}a_{2^k}$$
$$= (1/2)t_k$$

so that $2s_n \geq t_k$ --------- (*2)

By (*1) & (*2), the sequences $\{s_n\}$ & $\{t_n\}$ are either bounded or both unbounded.

Theorem 2.2.3 :

Show that $\Sigma \frac{1}{n^p}$ converges if $p > 1$, and diverges if $p \leq 1$.

Proof :

If $p \leq 0$, then Σn^{-p} (with $-p \geq 0$) is divergent. Otherwise $p > 0$.

To show that Σa_k converges, It suffices to verify that $\Sigma 2^k a_{2^k}$ converges by (2.2.2.)

(i.e) $\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} 2^{(1-p)k}$

Now $2^{1-p} < 1$ iff $1-p < 0$, and the result follows by comparison with the geometric series.

Corollary 2.2.4 :

If $p > 1$, $\sum\limits_{n=2}^{\infty} \frac{1}{n(\log_n)^p}$ converges; if $p \leq 1$, the series diverges.

Proof :

Logarithmic function is an increaing sequence. Then $1/(n \log_n)$ decreases.

By the previous theorem,

$$\text{Given series} = \sum_{k=1}^{\infty} 2^k \frac{1}{2^k (\log 2^k)^p}$$

$$= \sum_{k=1}^{\infty} \frac{1}{2^k (\log 2^k)^p}$$

$$= \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}$$

It is convergent if $p > 1$

It is divergent if $p \leq 1$

Remark 2.2.5 :

By an application,

$\sum\limits_{n=3}^{\infty} \frac{1}{n \log n \log \log n}$ is divergent, but $\sum\limits_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^2}$ is convergent.

SECTION – 3 : "THE NUMBER e"

Definition 2.3.1 :

$$e = 1 + \frac{1}{\angle 1} + \frac{1}{\angle 2} + \ldots\ldots\ldots\ldots$$

where $\angle n = 1.2.3\ldots\ldots\ldots n$ if $n \geq 1$, and $\angle 0 = 1$.

$$s_n = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \ldots\ldots\ldots$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots\ldots\ldots + \frac{1}{2^{n-1}} < 3$$

$\therefore \quad e < 3.$

Theorem 2.3.2 :

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Proof :

Let $s_n = \sum_{k=0}^{n}\frac{1}{\angle k}$; $\quad t_n = \left(1+\frac{1}{n}\right)^n$

By the Binomial theorem,

$$t_n = 1+1+\frac{1}{\angle 2}\left(1-\frac{1}{n}\right)+\frac{1}{\angle 3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots\ldots+\frac{1}{\angle n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\frac{n-1}{n}\right)$$

Thus $t_n \le s_n \Rightarrow \lim_{n\to\infty}\sup t_n \le e$ ------ (1)

$\forall n \ge m$, we have

$$t_n \ge 1+1+\frac{1}{\angle 2}\left(1-\frac{1}{n}\right)+\ldots\ldots+\frac{1}{\angle m}\left(1-\frac{1}{n}\right)\ldots\ldots\left(1-\frac{m-1}{n}\right)$$

when m is fixed, and $n \to \infty$,

we get $\lim_{n\to\infty}\inf t_n \ge 1+1+\frac{1}{\angle 2}+\ldots\ldots+\frac{1}{\angle m}$

$\Rightarrow s_n \le \lim_{n\to\infty}\inf t_n$.

As $m\to\infty$, we get $e \le \lim_{n\to\infty}\inf t_n$ ---------- (2)

By (1) & (2), $\quad \lim_{n\to\infty}\sup t_n \le \lim_{n\to\infty}\inf t_n$

Always $\quad \lim_{n\to\infty}\inf t_n \le \lim_{n\to\infty}\sup t_n$

Hence $\quad \lim_{n\to\infty}\inf t_n = \lim_{n\to\infty}\sup t_n$.

$\Rightarrow \quad \lim_{n\to\infty}(1+(1/n))^n = \lim_{n\to\infty} t_n = e.$

Theorem 2.3.3 :

Prove that e is irrational.

Proof :

Suppose e is rational.

e = p/q, where p & q are positive integers.

s_n = partial sum of $\Sigma \frac{1}{\angle n}$.

Then $$e - s_n = \frac{1}{\angle n+1} + \frac{1}{\angle n+2} + \frac{1}{\angle n+3} + \ldots\ldots\ldots$$

$$< \frac{1}{\angle n+1}\left[1 + \frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \ldots\ldots\right]$$

$$= \frac{1}{\angle n+1}\left(1 - \frac{1}{n+1}\right)^{-1}$$

$$= \frac{1}{\angle n+1}\frac{(n+1)}{n} = \frac{1}{n\angle n}$$

$$\Rightarrow \quad 0 \le (e - s_n) \le \frac{1}{n\angle n} \quad \text{-------- } (*)$$

Further $\quad 0 \le \angle q(e - s_q) < 1/n.$

By assumption, $e\angle q$ is an integer.

$\angle q(s_n) = \angle q \left[1 + 1 + \frac{1}{\angle 2} + \ldots\ldots + \frac{1}{\angle q}\right]$ is an integer implies that $\angle q(e - s_q)$ is an integer.

Since $q \ge 1$, (*) implies that there is an integer between 0 and 1, which is a contradiction.

Hence e is an irrational number.

SECTION – 4 : THE ROOT AND RATIO TESTS

Theorem 2.4.1 : Root Test

Given Σa_n, put $\alpha = \lim_{n\to\infty} \sup |a_n|^{1/n}$. Then (i) if $\alpha < 1$, Σa_n converges (ii) $\alpha > 1$, Σa_n diverges (iii) if $\alpha = 1$, this test has no conclusion.

Proof :

(i) $\alpha < 1$. Let β such that $\alpha < \beta < 1$. There is an integer N satisfying $|a_n|^{1/n} < \beta$, $\forall\ n \ge N$, by (1.5.10).

(i.e.) $n \geq N \Rightarrow |a_n| < \beta^n$.

Since $0 < \beta < 1$, $\Sigma\beta^n$ converges $\Rightarrow \Sigma a^n$ converges by comparison test.

(ii) $\alpha > 1$. There is a sequence $\{n_k\}$ satisfying $(a_{n_k})^{1/n_k} \to \alpha$. Thus $|a_n| > 1$ for infinitely many values of n, so that the condition $a_n \to 0$, necessary for convergence of Σa_n does not hold

(iii) $\Sigma \frac{1}{n}$ is divergent

$\Sigma \frac{1}{n^2}$ is convergent.

This follows the case.

Theorem 2.4.2 : Ratio Test

The series Σa_n (i) converges if $\lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right| < 1$ (ii) diverges, if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$, $\forall\ n \geq n_0$, where n_0 is some fixed integer.

Proof :

(i) $\lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right| < 1$

We choose $\beta < 1$, and an integer N satisfying $\left|\frac{a_{n+1}}{a_n}\right| < \beta$, $\forall\ n \geq N$.

In particular, $|a_{N+1}| < \beta|a_N|$

$|a_{N+2}| < \beta|a_{N+1}| < \beta^2|a_N|$

$|a_{N+p}| < \beta^p|a_N|$

(i.e) $|a_n| < |a_N|\, \beta^{-N}\beta^n$, $\forall\ n \geq N$.

(i) follows from the comparison test, and $\Sigma\beta^n$ converges.

(ii) $|a_{n+1}| \geq |a_n|$, $\forall\ n \geq n_0$, it follows that $a_n \to 0$ does not hold. Thus Σa_n diverges.

Example 2.4.3 :

(i) Let $\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\ldots\ldots.$ be a series. Then

$$\lim_{n\to\infty} \inf \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} (2/3)^n = 0$$

$$\lim_{n\to\infty} \inf (a_n)^{1/n} = \lim_{n\to\infty} (1/3^n)^{1/2n} = \frac{1}{\sqrt{3}}$$

$$\lim_{n\to\infty} \sup (a_n)^{1/n} = \lim_{n\to\infty} (1/2^n)^{1/2n} = \frac{1}{\sqrt{2}}$$

$$\lim_{n\to\infty} \sup \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} (2/2)n = +\infty$$

The root test indicates convergence. The ratio test does not apply.

(ii) Consider $\frac{1}{2}+1+\frac{1}{8}+\frac{1}{4}+\frac{1}{32}+\frac{1}{16}+\frac{1}{128}+\frac{1}{64}+\ldots\ldots$

Here $\lim\limits_{n\to\infty} \inf \frac{a_{n+1}}{a_n} = \frac{1}{8}$

$\lim\limits_{n\to\infty} \sup \frac{a_{n+1}}{a_n} = 2$

but $\lim a_n^{1/n} = \frac{1}{2}$.

Theorem 2.4.4 :

For any sequence $\{C_n\}$ of positive numbers, show that

(i) $\lim\limits_{n\to\infty} \inf \frac{C_{n+1}}{C_n} \leq \lim\limits_{n\to\infty} \inf (C_n)^{1/n}$

(ii) $\lim\limits_{n\to\infty} \sup (C_n)^{1/n} \leq \lim\limits_{n\to\infty} \sup \frac{C_{n+1}}{C_n}$

Proof :

To prove (ii), let $\alpha = \lim\limits_{n\to\infty} \sup \frac{C_{n+1}}{C_n}$

If $\alpha = +\infty$, there is nothing to show.

Otherwise α is finite. Let $\beta > \alpha$.

There is an integer N satisfying

$\frac{C_{n+1}}{C_n} \leq \beta, \forall n \geq N.$

In particular, $C_{N+k+1} \leq \beta C_{N+k}$, $\forall p > 0$

where $k = 0, 1, 2, \ldots\ldots, (p-1)$

Finally, we get $C_{N+p} \leq \beta^p C_N$ (or)

$C_n \leq C_N \beta^{-N} \beta^n$; $\forall n \geq N$

Therefore $C_n^{1/n} \leq \left(\sqrt{C_N \beta^{-N}}\right)^n . \beta$

$$\Rightarrow \lim_{n\to\alpha} \sup (C_n)^{1/n} \leq \beta$$

Since the above equation holds for all $\beta > \alpha$, we obtain $\lim\limits_{n\to\infty} \sup (C_n)^{1/n} \leq \alpha$.

SECTION – 5 : POWER SERIES

Definition 2.5.1 :

Given a sequence $\{C_n\}$ of complex numbers, the series $\sum\limits_{n=0}^{\infty} C_n . z^n$ is called a poewr series. The numbers C_n are called the coefficients of the series. Here z is a complex number.

Theorem 2.5.2 :

Given the power series $\Sigma C_n z^n$, let $\alpha = \lim\limits_{n\to\infty} \sup |C_n|^{1/2}$; $R = (1/\alpha)$ Prove that $\Sigma C_n z^n$ converges if $|z| < R$, and diverges if $|z| > R$.

Proof :

Let $a_n = C_n z^n$. Applying the root-test, $\lim\limits_{n\to\infty} \sup |a_n|^{1/n} = |z| \lim\limits_{n\to\infty} \sup |C_n|^{1/n} = |z|/R$.

Here R is said to be the radius of convergence of $\Sigma C_n . z^n$.

Example 2.5.3:

(i) The series $\Sigma n^n z^n$ has $R = 0$

(ii) The series $\Sigma z^n / \angle n$ has $R = \infty$

(iii) The series Σz^n has $R=1$. If $z = 1$, the series diverges ($\because$ $\{z^n\}$ does not tends to 0 as $n \to \infty$)

(iv) The series $\Sigma z^n/n$ has $R = 1$, if $z = 1$, it diverges. It converges for all z with $|z| = 1$.

(v) The series $\Sigma z^n/n^2$ has $R = 1$. It converges for all z with $|z| = 1$.

SECTION – 6 : SUMMATION BY PARTS

Theorem 2.6.1 :

Given two sequences $\{a_n\}$, $\{b_n\}$, let $A_n = \sum_{k=0}^{n} a_k$, $\forall n \geq 0$.

Assume $A_{-1}=0$. Then show that if $0 \leq p \leq q$,

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

Proof :

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n \text{ (by assumption)}$$

$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}$$

The above expression is equal to the right side of the requirement.

Theore 2.6.2 :

Assume

(i) the partial sums A_n of of Σa_n form a bounded sequence.

(ii) $b_0 \geq b_1 \geq b_2 \geq \ldots$

(iii) $\lim_{n\to\infty} b_n = 0$

Prove that $\Sigma a_n b_n$ converges.

Proof :

Since $\{A_n\}$ is bounded, $|A_n| \leq M$, $\forall_n$ (for some M). $\forall \in > 0$, there is an integer N satisfying $b_N \leq \in/2M$.

For $N \leq p \leq q$, we have

$$\left|\sum_{n=p}^{q} a_n b_n\right| = \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right|$$

$$\leq M\left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + b_q + b_p\right|$$

$$= 2M(b_q) \leq 2M(b_N) \leq \in.$$

$\therefore$ Convergence follows from the cauchy criterian. We note that the first inequality in the above chain depends on the fact that $b_n - b_{n+1} \geq 0$.

Theorem 2.6.3 :

Let

(i) $|C_1| \geq |C_2| \geq \ldots\ldots.$

(ii) $C_{2m-1} \geq 0; C_{2m} \geq 0; m = 1, 2, 3, \ldots\ldots.,$

(iii) $\lim\limits_{n\to\infty} C_n = 0$. Show that ΣC_n converges.

Proof :

It follows from 2.6.2 with $a_n = (-1)^{n+1}$, and $b_n = |C_n|$.

Theorem 2.6.4 :

Let the radius of convergence of $\Sigma C_n z^n$ be 1, and $C_0 \geq C_1 \geq C_2 \ldots$, $\lim\limits_{n\to\infty} C_n = 0$. Prove that $\Sigma C_n z^n$ converges at every point on the circle $|z|=1$, except possibly $z = 1$.

Proof :

Let $a_n = z^n$; $b_n = C_n$.

$$|A_n| = \left|\sum_{m=0}^{n} z^m\right| = \left|\frac{1-z^{m+1}}{1-z}\right| \leq \frac{2}{|1-z|} \text{ whenever } |z| \neq 1.$$

$\therefore$ The conditions of (2.6.2) are satisfied.

Hence $\Sigma a_n z^n$ converges.

Absolute Convergence :

Definition 2.6.5 :

A series Σa_n is said to converges absolutely, if the series $\Sigma |a_n|$ converges.

Theorem 2.6.6 :

If Σa_n converges absolutely, then Σa_n converges.

Proof :

We have the inequality $\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{n=k}^{m} |a_k|$

By the cauchy criterion, the convergence of Σa_n is followed.

Addition and Multiplication of Series :

Theorem 2.6.7 :

Let $\Sigma a_n = A$, $\Sigma b_n = B$; Then show that $\Sigma(a_n+b_n) = A+B$, and $\Sigma C.a_n = CA$, for any fixed C.

Proof :

(i) Let $A_n = \sum_{k=0}^{n} a_k$; $B_n = \sum_{k=0}^{n} b_k$.

$$\Rightarrow \quad A_n+B_n = \sum_{k=0}^{n}(a_k + b_k)$$

Now $\lim_{n\to\infty} A_n = A$; $\lim_{n\to\infty} B_n = B$

$$\Rightarrow \quad \lim_{n\to\infty} (A_n+B_n) = \lim_{n\to\infty} \sum_{K=0}^{n}(a_k + b_k)$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} a_k + \lim_{n\to\infty} \sum_{k=0}^{n} b_k$$

$$= A+B$$

(ii) $$\lim_{n\to\infty} CA_n = \lim_{n\to\infty} C\left(\sum_{k=0}^{n} a_k\right)$$

$$= C. \lim_{n\to\infty} C\left(\sum_{k=0}^{n} a_k\right)$$

$$= C.A.$$

Definition 2.6.8 :

For two series Σa_n & Σb_n, let $C_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0b_n+a_1b_{n-1}+.......+a_nb_0$

(n=0,1,2,......,).

Then ΣC_n is the product of the given two series.

If $\Sigma a_n z^n$ & $\Sigma b_n z^n$ are two power series, then $(\Sigma a_n z^n)(\Sigma b_n z^n)$ is defined as

$$(a_0+a_1z+a_2z^2+\ldots\ldots.)\ (b_0+b_1z+b_2z^2+\ldots\ldots.)$$

$$= a_0b_0+(a_0b_1+a_1b_0)z + (a_0b_2+a_1b_1+a_2b_0)z^2+\ldots\ldots\ldots.$$

$$= C_0+C_1z+C_2z^2+\ldots\ldots..$$

Example 2.6.9 :

Let $A_n = \sum_{k=0}^{n} a_k$; $B_n = \sum_{k=0}^{n} b_k$; $C_n = \sum_{k=0}^{n} C_k$; $\{A_n\}\to A$; $\{B_n\}\to B$.

It is not clear that $\{C_n\}$ converges to AB. $\{\because$ we do not have $C_n = A_n B_n\}$

The dependent relation of $\{C_n\}$ on $\{A_n\}$ & $\{B_n\}$ is very complicated one.

Also we shall give here that the product of two convergent series may be divergent.

The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = 1-\left(\frac{1}{\sqrt{2}}\right)+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots\ldots$ converges. We form the product of this series with itself, and we get,

$$\sum_{n=0}^{\infty} C_n = 1-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{3}}+\frac{1}{2\sqrt{2}}+\frac{1}{\sqrt{3}}\right)-\left(\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{3}\sqrt{2}}+\frac{1}{\sqrt{2}\sqrt{3}}+\frac{1}{\sqrt{4}}\right)+\ldots..$$

so that $C_n = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)(k+1)}}$

Since $(n-k+1)(n+k) = \left(\frac{n}{2}+1\right)^2-\left(\frac{n}{2}-k\right)^2, \ = \left(\frac{n}{2}+1\right)^2$

we obtain $|C_n| \geq \sum_{k=0}^{n} \frac{2}{(n+2)} = \frac{2(n+1)}{(n+2)}$

$\Rightarrow C_n$ does not tend to zero, as $n\to\infty$.

$\Rightarrow \Sigma(C_n)$ is not convergent.

Theorem 2.6.10 :

Suppose (i) $\sum_{n=0}^{\infty} a_n$ converges absolutely (ii) $\sum_{n=0}^{\infty} a_n = A$; (iii) $\sum_{n=0}^{\infty} b_n = B$;

(iv) $C_n = \sum_{k=0}^{n} a_k b_{n-k}$, $n = 0, 1, 2, \ldots.$

Then show that $\sum_{n=0}^{\infty} C_n = AB$.

(i.e.) The product of two convergent series is convergent if at least one of the two series converges absolutely.

Proof :

Let $A_n = \sum_{k=0}^{n} a_k$; $\quad B_n = \sum_{k=0}^{n} b_k$;

$C_n = \sum_{k=0}^{n} C_k$; $\quad \beta_n = B_n - B$

$$\begin{aligned}\text{Then } C_n &= a_0b_1+(a_0b_1+a_1b_1)+.....+(a_1b_n+a_2b_{n-2}+......+a_nb_0)\\ &= a_0B_0+a_1B_{n-1}+........+a_nB_0\\ &= a_0(B+\beta_n)+a_1(B+\beta_{n-1})+......+a_n(B+\beta_0)\\ &= A_nB+a_0\beta_n+a_1\beta_{n-1}+.......+a_n\beta_0\end{aligned}$$

Let $r_n = a_0\beta_n+a_1\beta_{n-1}+......+a_n\beta_0$.

Claim :

$C_n \to AB$.

We know that $A_n.B \to AB$

It remains to verify that $\lim_{n\to\infty} r_n = 0$

Let $\alpha = \sum_{n=0}^{\infty} |a_n|$

Let $\epsilon > 0$ be given. By (iii), $\beta_n \to 0$.

There is a positive integer N such that $|\beta_n| \le \epsilon$, $\forall n \ge N$, in which case

$$\begin{aligned}|r_n| &\le |\beta_0a_n+.....+\beta_Na_{n-N}| + |\beta_{N+1}a_{n-N-1}+.......+\beta_na_0|\\ &\le |\beta_0a_n+.....+\beta_Na_{n-N}| + \epsilon\alpha.\end{aligned}$$

Whenever N is fixed, and letting $n\to\infty$, we get $\limsup_{n\to\infty}|r_n| \le \epsilon\alpha$.

($\because a_k \to 0$ as $k \to \infty$).

Since ϵ is arbitrary, we followed $\lim_{n\to\infty} r_n = 0$.

Corollary 2.6.11 :

If the series Σa_n, Σb_n, Σa_n converage to A, B, C respectively, and

$C_n = a_0b_n + a_1b_{n-1} + \ldots\ldots + a_nb_0$.

Prove that C = AB.

Rearrangements

Definition 2.6.12 :

Let $\{k_n\}$ be a sequence in which every positive integer appears once and only once. Let $a_n' = a_{k_n}$ (n = 1,2,3,.......).

We say that $\Sigma a_n'$ is a rearrangement of Σa_n.

Example 2.6.13 :

Consider the convergent series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots\ldots\ldots$ ------ (1)

One of its rearrangement is $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\ldots\ldots.$ ------- (2)

in which two positive terms are followed by one negative.

If s is the sum of (1), then $s < 1-\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$.

Since $\frac{1}{4k-3}+\frac{1}{4k-1}-\frac{1}{2k} > 0, \forall\, k \geq 1.$

We have $s_3' < s_6' < s_9' < \ldots\ldots$, when s_n' is the n^{th} partial sum of (2).

Therefore $\lim_{n\to\infty} \sup s_n' > s_3' = 5/6$

$\Rightarrow$ Equation (2) does not converge to s

However it is convergent.

Theorem 2.6.14 :

Let Σa_n be a series of reals which converges, but not absolutely.

Assume $-\infty \leq \alpha \leq \beta \leq \infty$. Then show that a rearrangement $(\Sigma a_n')$ with partial sums s_n' such that $\lim_{n\to\infty} \inf s_n' = \alpha$; $\lim_{n\to\infty} \sup s_n' = \beta$. -------- (*)

Proof :

Let $p_n = \frac{|a_n| + a_n}{2}$; $q_n = \frac{|a_n| - a_n}{2}$;

Then $p_n - q_n = a_n$; $p_n + q_n = |a_n|$; $p_n \geq 0$; $q_n \geq 0$.

Therefore the series Σp_n, Σq_n must be both divergent.

For this, both are convergent.

$\Rightarrow \Sigma(p_n + q_n) = \Sigma p_n + \Sigma q_n = \Sigma |a_n|$

which converges, contrary to assumption.

Since $\sum\limits_{n=1}^{N} a_n = \sum\limits_{n=1}^{N} (p_n - q_n) = \sum\limits_{n=1}^{N} p_n - \sum\limits_{n=1}^{N} q_n$

Then the divergence of Σp_n, and the convergence of Σq_n (or conversely) implies that Σa_n is divergent, again contrary to hypothesis.

Finally, let $P_1, P_2, P_3, \ldots\ldots$ denote the non-negative terms of Σa_n, in the order in which they occur, and let $Q_1, Q_2, Q_3, \ldots\ldots$ be the absolute values of the negative terms of Σa_n, and also in their original order.

Now, the series ΣP_n, ΣQ_n differ from Σp_n, Σq_n only by zero terms, and they are divergent.

We shall construct sequences $\{m_n\}, \{k_n\}$ such that the series

$$P_1 + \ldots + P_{m_1} - Q_1 - \ldots - Q_{k_1} + P_{m_1+1} + \ldots + P_{m_2} - Q_{k_1+1} - \ldots - Q_{k_2} + \ldots \qquad \text{-------}(**)$$

which clearly is a rearrangement of Σa_n, satisfying (*).

We choose two real-valued sequence $\{\alpha_n\}$, $\{\beta_n\}$ such that

$\alpha_n \rightarrow \alpha$, $\beta_n \rightarrow \beta$; $\beta_n < \beta_n$; $\beta_1 > 0$.

Let m_1, k_1 be the smallest integers such that

$P_1 + P_2 + \ldots + P_{m_1} > \beta_1$; $P_1 + \ldots + P_{m_1} - Q_1 - \ldots - Q_{k_1} < \alpha_1$;

Let m_2, k_2 be the smallest integers such that

$P_1 + \ldots + P_{m_1} - Q_1 - \ldots - Q_{k_1} + P_{m_1+1} + \ldots + P_{m_2} - Q_{k_1+1} - \ldots - Q_{k_2} < \alpha_2$;

and continue in this similar manner. This is possible since ΣP_n & ΣQ_n are divergent.

If x_n, y_n denote the partial sums of (**),

Whose last terms are $P_{m_n} - Q_{k_n}$, then $|x_n - \beta_n| \leq P_{m_n}$; $|y_n - \alpha_n| \leq Q_{k_n}$.

Since $P_n \to 0$ and $Q_n \to 0$ as $n \to \infty$,

We see that $x_n \to \beta$; $y_n \to \alpha$.

Finally it is clear that no number less than α or greater than β can be a subsequential limit of the partial sums of (**).

Theorem 2.6.15 :

If Σa_n is a series of complex numbers which converges asbolutely then every rearrangement of Σa_n converges and all converge to the same sum.

Proof :

Let $\Sigma a_n'$ be a rearrangement with partial sums s_n'. $\forall \in > 0$, there is an integer N such that $m \geq n \geq N$ implies $\sum_{i=n}^{m} |a_i| \leq \in (*)$. Now, we choose p such that the integers 1, 2,....N are all contained in the set $k_1, k_2, \ldots, k_p$.

Then if $n > p$, then the numbers $a_1, \ldots a_n$ will cancel in the difference $s_n - s_n' \Rightarrow |s_n - s_n'| \leq \in$ by (*). Therefore $\{s_n'\}$ converges to the same sum as $\{s_n\}$.

SECTION - 1 : LIMITS OF FUNCTIONS

Definition 3.1.1 :

Let X and Y be metric spaces. Assume $E \subset X$. f maps E into Y, and p is a limit point of E. Then $f(x) \to q$ as $x \to p$ (or) $\lim_{x\to p} f(x)=q$, if there is a point $q \in y$ with the following property :

$\forall \in > 0$, there is a $\delta > 0$ such that $d_y(f(x), q) < \in$, for all points $x \in E$ for which $0 < d_x(x,p) < \delta$.

Theorem 3.1.2 :

Let X, Y, E, f and p be in the above definition. Then show that $\lim_{x\to p} f(x)=q$ ----(1) iff $\lim_{n\to\infty} f(p_n)=q$ -----(2) for every sequence $\{p_n\}$ in E such that $p_n \neq p$; $\lim_{n\to\infty} p_n=p$-----(3).

Proof :

Assume (1) holds.

We choose $\{p_n\}$ in E satisfying (3). Let $\in > 0$ be given. Then there exists $\delta > 0$ satisfying $d_y(f(x), q) < \in$, whenever $x \in E$ and $0 < d_x(x,p) < \delta$.

Further there is an `N' such that $n > N$ implies $0 < d_x(p_n, p) < \delta$. Therefore $\forall n > N \Rightarrow d_y(f(p_n),q) < \in$ from which $f(p_n) \to q$.

Conversely, suppose (2) holds, and equation (1) is false. Then there is some $\in > 0$ satisfying $\forall \delta > 0 \Rightarrow$ there is a point $x \in E$ (depending on δ), for which $d_y(f(x), q) \geq \in$, but $0 < d_x(x,p) < \delta$. If $\delta_n=1/n$ (n=1,2,......), we thus find a sequence in E satisfying (3) for which the equation (2) is false.

Corollary 3.1.3 :

If f has a limit at p, this limit is unique.

Solution :

It is obvious.

Theorem 3.1.4 :

Suppose $E \subset X$, a metric space; p is a limit point of E; f and g are complex functions on E, and

$$\lim_{x \to p} f(x) = A; \lim_{x \to p} g(x) = B$$

Show that (i) $\lim_{x \to p} (f+g)(x) = A+B$

(ii) $\lim_{x \to p} (fg)(x) = AB$

(iii) $\lim_{x \to p} (f/g)(x) = A/B$, if $B \neq 0$

Proof :

It is obvious.

Continuous Functions

Definition 3.1.5 :

Let X and Y be metric spaces, $E \subset X$, $p \in E$, and f maps E into Y. Then f is said to be continuous at p, whenever for every $\in > 0$, there is a $\delta > 0$ such that $d_y(f(x), f(y)) < \in$ for all points $x \in E$ for which $d_x(x,p) < \delta$. If f is continuous at every point of E, then f is continuous on E.

Theorem 3.1.6 :

Assume that p is a limit point of E. Show that f is continuous at p iff $\lim_{x \to p} f(x)=f(p)$.

Proof :

It is clear.

Theorem 3.1.7 :

Let X, Y, Z be metric spaces $E \subset X$; f maps E into Y; g maps the range of f, f(E) into Z, and h is the mapping of E into Z defined by

$h(x) = g(f(x)), \forall x \in E$.

If f is continuous at $p \in E$, and g is continuous at f(p), then h is continuous at p.

Proof :

Let $\in > 0$ be given.

Since g is continuous at f(p), there is an $\eta > 0$ satisfying $d_Z(g(y), g(f(p))) < \in$, if $d_y(y, f(p)) < \eta$, and $y \in f(E)$.

Since f is continuous at p, there is $\delta > 0$ such that

$d_y(f(x), f(p)) < \eta$, if $d_x(x, p) < \delta$, and $x \in E$.

It follows that

$d_z(h(x), h(p)) = d_z(g(f(x)), g(f(p))) < \in$ whenever $d_x(x,p) < \delta$, and $x \in E$.

Therefore h is continuous at p.

Theorem 3.1.8 :

Show that a mapping f of a metric space X into a metric space Y is continuous on X iff $f^{-1}(V)$ is open in X for every open set V in Y.

Proof :

Assume f is continuous on X, and V is an open set in Y.

Claim :

$f^{-1}(V)$ is open in X.

Let $p \in f^{-1}(V) \Rightarrow f(p) \in V$.

Since V is open, there is $\in > 0$ satisfying $y \in V$ if $d_y(f(p), y) < \in$.

Since f is continuous at p, there is $\delta > 0$ satisfying $d_y(f(x), f(p)) < \in$,

if $d_x(x, p) < \delta$, where $y = f(x)$

Then $x \in f^{-1}(V)$ as soon as $d_x(x,p) < \delta$.

Conversely, assume $f^{-1}(V)$ is open in X for every open set V in Y. Let $p \in X$ and $\in > 0$. Let V be the set of all $y \in Y$ such that $d_y(y, f(p)) < \in$. Then V is open $\Rightarrow f^{-1}(V)$ is open.

Therefore there is $\delta > 0$ satisfying $x \in f^{-1}(V)$ as soon as $d_x(p, x) < \delta$.

If $x \in f^{-1}(V)$, then $f(x) \in V$

$d_y(f(x), f(p)) < \in$.

This completes the proof.

Theorem 3.1.9 :

Let f and g be complex continuous functions on a metric space X. Then f+g, fg, and f/g are all continuous on X.

Proof :

It is clear.

Corollary 3.1.10 :

A mapping f of a metric space X into a metric space Y is continuous iff $f^{-1}(C)$ is closed in X for every closed set C in Y.

Proof :

It is obvious.

Theorem 3.1.11 :

(i) Let $f_1, f_2, \ldots f_k$ be real functions on a metric space X, and f be the mapping of X into R^K defined by $f(x) = (f_1(x), f_2(x), \ldots.. f_k(x)), \forall x \in X$.

Then show that f is continuous iff each of the functions $f_1, f_2, \ldots f_k$ is continuous.

(ii) If f & g are continuous mappings of X into R^k, then f+g, fg are both continuous on X.

Proof :

(1) is followed from

$|f_j(x)-f_j(y)| \leq |f(x)-f(y)|$

$$= \left(\sum_{j=1}^{k} |f_i(x) - f_i(y)|^2 \right)^{1/2} \quad \forall j = 1, 2, \ldots\ldots k$$

case (ii) is followed by (i).

SECTION – 2 : CONTINUITY AND COMPACTNESS

Definition 3.2.1 :

A mapping f of a set E into R^K is called bounded set if there is a real number M satisfying $|f(x)| \leq M$ for all $x \in E$.

Theorem 3.2.2 :

Let f be a continuous mapping of a compact metric space X into a metric space Y. Prove that f(X) is compact.

Proof :

Given that $f : X \to Y$ is continuous; X is compact; Y is metic space.

Claim :

f(X) is compact.

Let $\{V_\alpha\}$ be an open cover of f(X). Since f is continuous, then $f^{-1}(V_\alpha)$ is open set in "Y" for each α.

Since X is compact, and $\{f^{-1}(V_\alpha)\}$ is an open cover for X, then this open cover has a finite subcover, and so there are finitely many indices, say $\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_n$ satisfying

$X \in f^{-1}(V_{\alpha_1}) \cup \ldots \cup f^{-1}(V_{\alpha_n})$ ------- (*)

Now $f(f^{-1}(E)) \subset E$, for every $E \subset Y$. Then (*) implies that

$f(X) \in V_{\alpha_1} \cup \ldots \cup V_{\alpha_n}$

This completes the proof.

Corollary 3.2.3 :

If f is a continuous mapping of a compact metric space X into R^k, then f(x) is closed and bounded (hence f is bounded).

Proof :

It follows from Heine-Borel theorem.

Theorem 3.2.4 :

Let f be a continuous real function on a compact metric space X, and $M = \sup\limits_{p \in X} f(p)$; $m = \inf\limits_{p \in X} f(p)$.

Then show that there exist points $p, q \in X$ satisfying $f(p)=M$, and $f(q)=m$.

Proof :

We note that M is the least upper bound of the set of all numbers f(p), where p ranges over X, and m is the greatest lower bound of this set of numbers.

There exist points p & $q \in X$ satisfying $f(q) \leq f(x) \leq f(p)$, $\forall x \in X$.

$\therefore$ f attains its maxmum at p, and also its minimum at q.

By previous result, f(X) is closed and bounded set of real numbers.

Hence f(x) contains M & m.

Theorem 3.2.5 :

Let f be a continuous 1–1 mapping of a compact metric space X onto a metric space Y. Then show that the inverse mapping f^{-1} defined on Y by $f^{-1}(f(x))=x$, $\forall x \in X$ is a continuous mapping of Y onto X.

Proof :

It suffices to verify that f(V) is an open set in Y, for every open set V in X.

Let V be an open set in X.

$\Rightarrow V'$ is closed in X. But X is compact.

$\Rightarrow$ closed subset of compact is compact.

$\Rightarrow f(V')$ is a compact subset of Y.

$\therefore f(V')$ is closed in Y.

Since f is one-to-one and onto, then f(V) is the complement of $f(V')$.

Therefore f(V) is open set in Y.

Definition 3.2.6 :

Let f be a mapping of a metric space X into a metric space Y. Then prove that f is uniformly continuous on X.

Proof :

Let $\in > 0$ be given. Since f is continuous function, we can associate to each point $p \in X$, a positive number $\phi(p)$ satisfying

$q \in X;\ d_x(p,q) < \phi(p)$

$\Rightarrow d_y(f(p), f(q)) < \in/2$ -------- (1)

Let J(p) be the set of all $q \in X$ for which $d_x(p, q) < \phi(p)/2$ -------- (2)

Since $p \in J(p)$, the collection of all sets J(p) is an open cover of X.

Since X is compact, there is a finite set of points $p_1, p_2, \ldots, p_n \in X$ satisfying

$X \subset J(p_1) \cup \ldots \cup J(p_n)$ -------- (3)

Take $\delta = \frac{1}{2}\min\{\phi(p_1), \ldots, \phi(p_n)\}$ -------- (4)

Here $\delta > 0$.

Let p & q be points of X, such that $d_x(p, q) < \delta$. By (3), there is an integer m, $1 \le m \le n$, satisfying $p \in J(p_m) \Rightarrow d_x(p, p_m) < \frac{1}{2}\phi(p_m)$

$\therefore d_x(q, p_m) \le d_x(p, q) + d_x(p, p_m)$

$$< \delta + \frac{1}{2}\phi(p_m) \le \phi(p_m)$$

$\therefore$ The equation (1) shows that $d_y(f(p), f(q)) \le d_y(f(p), f(p_m)) + d_y(f(q), f(p_m)) < \in$

This completes the proof.

Theorem 3.2.8 :

Let f be a non-empty set in R. Then prove

(i) There is continuous function on E which is not bounded.

(ii) There is a continuous and bounded function on E which has no maximum

(iii) E is bounded $\Rightarrow$ There is a continuous function on E, which is not uniformly continuous.

Proof :

Assume that E is bounded, so that there is a limit point x of E, which is not a point of E (i.e) $x_0 \in \overline{E} - E$.

Let $f(x) = 1/(x - x_0)$, $\forall x \in E$ ------ (*)

This function is continuous on E, but clearly unbounded.

To show that f(x) is not uniformly continuous, let $\in > 0$, and $\delta > 0$ be arbitrary. We choose a point $x \in E$ such that $|x - x_0| < \delta$.

Taking $t \to x_0$, we have $|f(t) - f(x)| > \in$, whenever $|t - x| < \delta$.

Since this holds for all $\delta > 0$, f is not uniformly continuous on E.

This shows (i).

If boundness is omitted from assumption, let E be the set of all integers.

Every function defined on E is uniformly continuous on itself (taking $\delta < 1$).

$\therefore$ (iii) is false.

To prove (ii), we consider the function $g(x) = \frac{1}{1+(x-x_0)^2}$, $\forall\ x \in E$ is continuous function on E, and it is bounded (because $0 < g(x) < 1$).

Also $\sup\limits_{x \in E} g(x) = 1$, whereas $g(x) < 1$; $\forall x \in E$.

Therefore g has no maximum on E.

To show (i), let E is unbounded. then $f(x) = x$ gives the requirement, where as $h(x) = \frac{x^2}{1+x^2}$; $\forall\ x \in E$ gives (ii) (because $\sup\limits_{x \in E} h(x) = 1$; $h(x) < 1$, $\forall\ x \in E$).

SECTION – 3 : CONTINUITY AND CONNECTEDNESS

Theorem 3.3.1 :

If f is a continuous mapping of a metric space X into a metric space Y, and E is connected in X, then show that f(E) is connected.

Proof :

Given that $f: X \to Y$ is continuous.

E is a connected subset in X.

Claim :

f(E) is connected in Y.

Assume the contrary.

Then $f(E) = A \cup B$, where A & B are non-empty (separated) disjoint subsets of Y. Let $G = E \cap f^{-1}(A)$; $H = E \cap f^{-1}(B)$.

Then $E = G \cup H$, and both ets G & H are non-empty.

$A \subset \overline{A}$ (the closure of A)

$\Rightarrow \overline{G} \subset f^{-1}(\overline{A})$

$\Rightarrow$ Here $f^{-1}(\overline{A})$ is closed ($\because$ f is continuous)

It follows that $f(\overline{G}) \subset \overline{A}$.

Since f(H)=B, and $\overline{A} \cap B$ is empty, we conclude that $\overline{G} \cap H$ is empty.

Similarly, $G \cap \overline{H}$ is empty.

Thus G & H are separated.

$\Rightarrow$ E is disconnected which is a contradiction.

Hence f(E) is connected.

Theorem 3.3.2 :

Let f be a continuous real function on the interval [a b]. If f(a) < f(b), and c is a number satisfying f(a) < c < f(b), show that there exists a point $x \in (a\ b)$ such that f(x)=c.

Proof :

By known result, [a b] is connected.

Since f is continuous real function on the interval [a b], we follow that f([a b]) is a connected subset of R.

We proved that a subset E of the real line R is connected iff it has the property that $x, y \in E;\ x < z < y \Rightarrow z \in E$. (i.e., it is an interval).

We have $f(a) \in f([a\ b])$

$f(b) \in f([a\ b])$

$f(a) < c < f(b)$

Thus $[f(a), f(a)] \subseteq f([a\ b])$

$\therefore$ There is $x \in (a\ b)$ satisfying f(x) = c.

Discontinuities

Definition 3.3.3 :

Let f be defined on (a b). Take a point x such that $a \le x < b$. Then f(x+) = q, whenever $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (x b) satisfying $t_n \to x$.

To define the definition of f(x–), for $a < x \le b$, we consider only the sequence $\{t_n\}$ in (a x).

It follows that $\lim_{t\to x} f(t)$ exists iff $f(x+) = f(x-) = \lim_{t\to x} f(t)$; $\forall\ x \in (a\ b)$.

Definition 3.3.4 :

Let f be defined on (a b): If f is discontinuous at x, and f(x+), and f(x–) exist, then f is said to have a discountinuity of the first kind or a simple discontinuity at x.

Otherwise the discontinuity is called the second kind.

Example 3.3.5 :

$$f(x) = \begin{cases} 1\ (x \text{ rational}) \\ 0\ (x \text{ irrational}) \end{cases}$$

Then f has a discontinuity of the second kind at every point x

(since neither f(x+) nor f(x–) exists).

Example 3.3.6 :

$$f(x) = \begin{cases} x\ \ (x \text{ rational}) \\ 0\ (x \text{ irrational}) \end{cases}$$

Then f is continuous at x = 0, and has a discontinuity of the second kind at every other point.

Example 3.3.7 :

$$f(x) = \begin{cases} x+2 \quad (-3 < x < -2) \\ -x-2\ (-2 \le x < 0) \\ \ \ x+2\ \ (0 \le x < 1) \end{cases}$$

Then f has a simple discontinuity at x=0, and is continuous at every other point of (–3 1).

Example 3.3.8 :

$$f(x) = \begin{cases} \sin(1/x) & (x \ne 0) \\ 0 & (x = 0) \end{cases}$$

Since neither f(0+) nor f(0–) exists, f has a discontinuity of the second kind at x = 0. Also f is continuous at every point $\ne 0$.

Monotonic Functions :

Definition 3.3.9 :

Let f be a real function on (a b). Then f is said to be monotonically increasing on (a b) if $a < x < y < b$ implies $f(x) \le f(y)$.

If the last inequality is reversed, we obtain the definition of a monotonically decreasing function.

Theorem 3.3.10 :

Let f be monotonically increasing on (a b). Then show that f(x+) and f(x–) exist at every point of x of (a b). (i.e.)

$$\sup_{a < t < x} f(t) = f(x-) \le f(x) \le f(x+) = \inf_{x < t < b} f(t) \qquad \text{------ (1)}$$

Also if $a < x < y < b$, then $f(x+) \le f(y-)$ ------ (2)

Proof :

By hypothesis, the set of numbers f(t), where $a < t < x$, is bounded above by the number f(x). It has a least upper bound, say A.

Clearly $A \le f(x)$.

Claim :

$A = f(x)$ be given. By definition, A is lub that there exists $\delta > 0$ satisfying $a < x - \delta < x$ and $a - \in < f(x-\delta) \le A$ ------- (3)

Since f is monotonic, we have

$$f(x-) \le f(t) \le A \qquad (x-\delta < t < x) \qquad \text{------- (4)}$$

By (3) & (4), we get

$$|f(t)-A| < \in, \ (x-\delta < t < x)$$

Therefore $f(x-) = A$.

The remaining proof of (1) is showed in the same way.

If $a < x < y < b$, we follow from (1) that $f(x+) = \inf_{x < t < b} f(t) = \inf_{x < t < y} f(y)$.

Now the last inequilty is obtained by applying (1) to (a y) in stead of (a b) ----- (5)

Similarly $f(y-) = \sup_{a < t < y} f(t) = \sup_{x < t < y} f(t)$ ----- (6)

By (5) & (6), we get (2).

Corollary 3.3.11 :

Monotonic functions have no discontinuities of the second kind.

Theorem 3.3.12 :

Let f be monotonic on (a b). Show that the set of prints of (a b) at which f is discontinuous is at most countable.

Proof :

Without loss of generality, let f be an increasing function, and E be the set of points at which f is discontinuous.

By assumption and $x \in E$, we get a rational number r(x) satisfying $f(x-) < r(x) < f(x+)$.

Since $x_1 < x_2$ implies $f(x_1+) \leq f(x_2-)$, we see that $r(x_1) \neq r(x_2)$, if $x_1 \neq x_2$.

Hence we have obtained a one-to-one correspondence (bijection) between the set E and a subset of the set of rational numbers, which is countable.

$\therefore$ E is at most countable.

Infinite Limits and Limits at Infinity

Definition 3.3.13 :

For every real c, the set of real numbers x satisfying $x > c$ is called a neighborhood of ∞ and is denoted as $(c\ \infty)$.

The set $(-\infty\ c)$ is a neighborhood of $-\infty$.

Definition 3.3.14 :

Let f be a real function defined on E. Then $f(t) \to A$ as $t \to x$, where A & x are in the extended real number system, if every neighborhood u of A, there is a neighborhood V of x satisfying $V \cap E$ is non-empty, and $f(t) \in U$; $\forall t \in V \cap E$; $t \neq x$.

Theorem 3.3.15 :

Let f & g be defined on E. Assume $f(t) \to A$, $g(t) \to B$, as $t \to x$.

Then (i) $f'(t) \to A'$

(ii) $(f+g)(t) \to (A+B)$

(iii) $(fg)(t) \to AB$

(iv) $(f/g)(t) \to A/B$ provided the right members of (ii), (iii) & (iv) are defined.

Proof :

It is obvious by earlier results.

SECTION–1 : DERIVATIVE OF A REAL FUNCTION

Definition 4.1.1 :

Let f be defined (real-valued) on [a b]; $\forall$ x $\in$ [a b].

let $\phi(t) = \dfrac{f(t)-f(x)}{t-x}$ $(a < t < b;\ t \neq x)$

Define $f'(x) = \lim\limits_{t\to x} \phi(t) = \lim\limits_{t\to x} \dfrac{f(t)-f(x)}{t-x}$ provided that this limit exists and finite.

f' is defined at a point x $\Rightarrow$ we call that f is differentiable at x.

If f is differentiable at every point of a set $E \subset [a\ b]$, then f is differentiable on E.

Theorem 4.1.2 :

Every differentiable function on an interval [a b] is continuous on itself.

Proof :

Let f be defined on [a b] and differentiable at a point x $\in$ [a b].

Claim :

f is continuous at x.

As $t\to x$,

$$f(t)-f(x) = \frac{f(t)-f(x)}{(t-x)}.(t-x)$$

$$\therefore \lim_{t\to x}(f(t)-f(x)) = \lim_{t\to x}\frac{f(t)-f(x)}{(t-x)}.\left[\operatorname*{Lt}_{t\to x}(t-x)\right]$$

$$= f'(x).0$$

$$= 0$$

$$\therefore \lim_{t\to x} f(t) = f(x)$$

Therefore f is continuous at x.

Thereom 4.1.3 :

Let f & g be defined on [a b] and are differentiable at a point x$\in$[a b]. Then f+g, fg, and f/g are all differentiable at x, and

(i) $(f+g)'(x) = f'(x)+g'(x)$

(ii) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

(iii) $(f/g)'(x) = \dfrac{g(x)f'(x) = g'(x)f(x)}{g^2(x)}$

Proof :

Case (i)

Let

$$
\begin{aligned}
h &= (f+g) \\
h(t)-h(x) &= (f+g)(t) - (f+g)(x) \\
&= [f(t)+g(t)]-[f(x)+g(x)] \\
&= [f(t)-f(x)]+[g(t)-g(x)] \\
\lim_{t\to x}\frac{h(t)-h(x)}{(t-x)} &= \lim_{t\to x}\frac{f(t)-f(x)}{(t-x)} + \lim_{t\to x}\frac{g(t)-g(x)}{(t-x)} \\
\therefore \quad h' &= (f+g)'(x) = f'(x)+g'(x)
\end{aligned}
$$

Case (ii)

Let

$$
\begin{aligned}
h &= (fg) \\
h(t)-h(x) &= (fg)(t) - (fg)x \\
&= f(t)g(t) - f(x)g(x) \\
&= f(t)g(t)-f(t)g(x)+f(t)g(x)-f(x)g(x) \\
&= f(t)[g(t)-g(x)]+g(x)[f(t)-f(x)] \\
\lim_{t\to x}\frac{h(t)-h(x)}{t-x} &= \lim_{t\to x} f(t)\frac{[g(t)-g(x)]}{(t-x)} + \lim_{t\to x} g(x)[f(t)-f(x)] \\
h'(x) &= f(x)g'(x)+g(x)f'(x)
\end{aligned}
$$

Case(iii)

Let

$$
\begin{aligned}
h &= f/g \\
\frac{h(t)-h(x)}{t-x} &= \frac{f(t)/g(t)-f(x)/g(x)}{(t-x)} \\
&= \frac{g(x)f(t)-g(t)f(x)}{g(t)g(x)(t-x)}
\end{aligned}
$$

$$= \frac{1}{g(t)g(x)} \frac{[g(x)f(t) - g(x)f(x) + f(x)g(x) - g(x)f(x)]}{(t-x)}$$

$$\lim_{t \to x} \frac{h(t) - h(x)}{(t-x)} = \left(\lim_{t \to x} \frac{1}{g(t)g(x)} \right) \left[\lim_{t \to x} g(x) \frac{f(t) - f(x)}{(t-x)} - \lim_{t \to x} f(x) \frac{g(t) - g(x)}{(t-x)} \right]$$

$$= \frac{1}{g(x)g(x)} [g(x)f'(x) - f(x)g'(x)]$$

Theorem 4.1.4 : Chair Rule

Let f be continuous on [a b], $f'(x)$ exists at some point $x \in (a\ b)$, g defined on an interval I that contains the range of f, and g is differentiable at the point f(x).

If $h(t) = g(f(t))$; $a \leq t \leq b$, then show that h is differentiable at x, and $h'(x) = g'(f(x))\ f'(x)$ ------ (1)

Proof :

Let $y = f(x)$

By the definition of the derivative, we get

$$f(t)-f(x) = (t-x)[f'(x)+u(t)] \quad \text{------ (2)}$$

$$g(s)-g(y) = (s-y)[g'(y)+t(s)] \quad \text{------ (3)}$$

where $t \in [a\ b]$, $s \in I$, and $u(t) \to 0$ as $t \to x$; $t(s) \to 0$ as $s \to y$. Assume $s=f(t)$.

Using (3), and then (2), we obtain

$$h(t)-h(x) = g(f(t))-g(f(x))$$

$$= f[(t)-f(x)]\ [g'(y)+v(s)]$$

$$= (t-x)[f'(x)+u(t)]\ [g'(y)+v(s)]$$

If $t \neq x$, then

$$\frac{h(t)-h(x)}{(t-x)} = [f'(x)+u(t)]\ [g'(y)+v(s)]$$

As $t \to x$, we get $s \to y$ by the continuity of f so that

$$h'(x) = g'(y)\ f'(x) = g'(f(x))\ f'(x)$$

Example 4.1.5 :

Consider the function

$$f(x) = \begin{cases} x.\sin\frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

$$f'(x) = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x} \quad (x \neq 0)$$

At x=0, $\sin\frac{1}{x}$ is not defined.

For $t \neq 0$, $\dfrac{f(t)-f(0)}{t-0} = \sin\dfrac{1}{t}$

As $t \to 0$, this does not have any limit.

$\Rightarrow f'(0)$ does not exist.

Example 4.1.6 :

Let
$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

$$f'(x) = 2x.\sin\frac{1}{x} - \cos\frac{1}{x}; \ (x \neq \theta) \text{ ------ } (1)$$

$$\left|\frac{f(t)-f(0)}{t-0}\right| = \left|t\sin\frac{1}{t}\right| \leq |t| \quad (t \leq 0)$$

As $t \to 0$, we see that $f'(0) = 0$

Thus f is differentiable at all points `x', but f' is not a continuous function (since cos(1/x) in (1) does not have a limit as $x \to 0$)

SECTION – 2 : MEAN VALUE THEOREMS

Definition 4.2.1. :

Let f be a real function defined on a metric space X. Then f has a local maximum at a point $p \in X$, if there exists $\delta > 0$ satisfying $f(q) \leq f(p)$ for all $q \in X$ with $d(p,q) < \delta$.

Local **minimum is defined** likewise.

Theorem 4.2.2 :

Let f be defined on [a b]. If f has a local maximum at a point $x \in (a\ b)$, and $f'(x)$ exists, then $f'(x)=0$.

Proof :

We choose δ so that $a < x-\delta < x < x+\delta < b$.

If $x-\delta < t < x$, then $\dfrac{f(t)-f(x)}{t-x} \geq 0$

As $t \to x$, we get $f'(x) \geq 0$.

$$x < t < x+\delta \Rightarrow \frac{f(t)-f(x)}{(t-x)} \leq 0 \Rightarrow f'(x) \leq 0$$

Therefore $f'(x) = 0$

Theorem 4.2.3 :

If f & g are continuous real functions on [a b], that are differentiable in (a b), show that there is a point $x \in (a\ \ b)$ at which $[f(b)-f(a)]g'(x) = [g(b)-g(a)]f'(x)$.

Proof :

Let $h(t) = [f(b)-f(a)]g(t)-[g(b)-g(a)]f(t); \quad a \leq t \leq b$

Clearly f & g are continuous on $[a\ b] \Rightarrow$ h is continuous on [a b].

f & g are both differentiable on (a b) $\Rightarrow$ h is differentiable in (a b).

$h(a) = f(b)g(a) - f(a)g(b) = h(b)$ ------- (1)

If h is constant function, $h'(x) = 0$ for every $x \in (a\ \ b)$. If $h(t) > h(a)$ for some $t \in (a\ \ b)$, let x be a point on [a b] at which h attains its maximum. By (1), $x \in (a\ b)$ and known result implies that $h'(x) = 0$.

Similiarly, let $h(t) < h(a)$ for some $t \in (a\ b)$. There is a point $x \in (a\ \ b)$ at which h attains its minimum $\Rightarrow h'(x) = 0$.

In all cases, $h'(x)=0$ for sense $x \in (a\ \ b)$

(i.e.) $[f(b)-f(a)]g'(x) - [g(b)-g(a)]f'(x) = 0$

(i.e.) $[f(b)-f(a)]g'(x) = [g(b)-g(b)]f'(x)$

This theorem is called generalized mean value theorem.

Corollary 4.2.4 :

If f is a real continuous function on [a b] whichis differentiable in (a b), show that there is a point $x \in (a\ \ b)$ at which $f(b)-f(a) = (b-a)f'(x)$.

Proof :

Take $g(x) = x$ in (4.2.3)

Theorem 4.2.5 :

Suppose f is a real differentiable in (a b). (i) If $f'(x) \geq 0$, for all $x \in$ (a b), then f is monotonically increasing (ii) if $f'(x)=0$ for all $x\in$(a b), f is constant (iii) If $f'(x) \leq 0$ for all $x\in$(a b), f is monotonically decreasing.

Proof :

Since f is differentiable in (a b), then $f(x)-f(y) = (x-y)f'(x_0)$, $\forall x, y \in$ (a b) and for some $x_0\in$(a b).

All the conclusions are followed by the above equation.

The Continuity of Derivatives :

Theorem 4.2.6 :

Suppose f is a real differential function on [a b], and $f'(a) < \lambda < f'(b)$. Show that there is a point $x \in$ (a b) satisfying $f'(x) = \lambda$.

Proof :

Let $g(t) = f(t)-\lambda t$. Then $g'(a) < 0 \Rightarrow g(t_1) < g(a)$ for some $t_1\in$(a b), and $g'(b) > 0 \Rightarrow g(t_2) < g(b)$ for some $t_2\in$(a b). Thus g attains its minimum on [a b] at some point x satisfying $a < x < b$. By (4.2.2), $g'(x) = 0 \Rightarrow f'(x) = \lambda$.

Corollary 4.2.7 :

If f is differentiable on [a b], then show that f′ can not have any simple discontinuities on [a b].

Proof :

It is clear.

L' Hospital's Rule :

Theorem 4.2.8 :

Let f & g be both real valued differentiable in (a b), and $g'(x) \neq 0$ for all $x \in$ (a b), where $-\infty \leq a < b \leq +\infty$.

Assume $f'(x)/g'(x) \to A$ as $x \to a$ ------ (1)

If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ ----- (2)

or $g(x) \to \infty$ as $x \to a$ ------ (3)

then prove that $f(x)/g(x) \to A$ as $x \to a$ ------ (4)

Proof :

We consider the case in which $-\infty \le A < \infty$. Choose a real number q such that $A < q$, and then select r such that $A < r < q$.

By (1), there is point $c \in (a\ b)$ satisfying $a < x < c$ implies $f'(x)/g'(x) < r$ ------ (5)

If $a < x < y < c$, and (4.2.3), there is a point $t \in (x\ y)$ satisfying

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(t)}{g'(t)} < r \qquad \text{------ (6)}$$

Assume (2) holds.

As $x \to a$ in (6), we get

$$\frac{f(y)}{g(y)} \le r < q \ (a < y < c).$$

Suppose (3) holds.

Keeping y fixed in (6), we can select a point $c_1 \in (a\ \ y)$ satisfying $g(x) > g(y)$; $g(y) > 0$ if $a < x < c$.

Multiplying (6) by $[g(x)-g(y)]/g(x)$,

we obtain $\dfrac{f(x)}{g(x)} < r < r\dfrac{g(y)}{g(x)} + \dfrac{f(y)}{f(x)}$ $(a < x < c_1)$ ------ (7)

If we let $x \to a$ in (7) then the equation (3) gives that this is a point

$c_2 \in (a\ c_1)$ satisfying $f(x)/g(x) < q$; $a < x < c_2$. -------- (8)

By (6) & (8), for any q with $A < q$, there is a point c_2 such that $f(x)/g(x) < q$ whenever $a < x < c_2$.

If $-\infty < A \le +\infty$, and p is selected so that $p < A$, we can find a point c_2 such that $p < f(x)/g(x)$ $(a < x < c_2)$

$\therefore$ we get (4).

Derivatives of Higher Order :

Definition 4.2.9 :

If f has a derivative f' on an interval, and f' is itself differentiable, we denote the derivative of f' by f''.

Continuing in this manner, we get $f, f', f'', \ldots\ldots f^{(n)}$, each of which is the derivative of the preceding one.

$f^{(n)}$ is called the n^{th} derivative (or) the derivative of order n of f.

Taylor's Theorem 4.2.10 :

Suppose f is a real function on [a b]; n is a positive integer; $f^{(n-1)}$ is continuous on [a b], $f^{(n)}(t)$ exists for every $t \in (a\ b)$.

Let α, β be distinct points of [a b] and define

$$p(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{\angle k}(t-\alpha)^k \qquad \text{------ (1)}$$

Prove that there exists a point x between α & β such that

$$f(\beta) = p(\beta) + \frac{f^{(n)}(x)}{\angle n}(\beta-\alpha)^n \qquad \text{------ (2)}$$

Proof :

When n=1, this is mean value theorem. This theorem shows that f can approximated by a polynomial of degree (n–1), and that (2) allows us to estimate the error. Let M be the number defined by

$$f(\beta) = p(\beta)+M(\alpha-\beta)^n \qquad \text{------ (3) and}$$

Consider $g(t) = f(t)-p(t)-M(t-\alpha)^n$; $(a \le t \le b)$ ------ (4)

Claim :

$(\angle n)M = f^{(n)}(x)$ for some x between α and β.

By (1) & (4), $g^{(n)}(t) = f^{(n)}(t)-(\angle n)M$; $(a < t < b)$

Therefore the proof is completed if we can show that $g^{(n)}(x)=0$ for some x between α and β.

Since $p^{(k)}(\alpha) = f^{(k)}(\alpha), \forall k = 0,...n-1$.

We have $g(\alpha) = g'(\alpha) = ... = g^{(n-1)}(\alpha) = 0$

By choice of M, $g(\beta) = 0$

$\Rightarrow g'(x_1) = 0$ for some $x_1 \in (\alpha\ \beta)$ by mean value theorem.

Since $g'(\alpha) = 0$, we conclude similarly that $g'(x_2) = 0$ for some $x_2 \in (\alpha\ x_1)$.

After 'n' steps, we arrive at the conclusion that $g^{(n)}(x_n) = 0$ for some x_n between α and x_{n-1}.

((i.e.) between α & β).

This completes the proof.

SECTION – 1 : DEFINITION AND EXISTENCE OF INTEGRAL

Definition 5.1.1 :

Let [a b] be a given integral. By a partition p of [a b], we define a finite set of points $x_0, \ldots, x_n$, where $a = x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n = b$

Assume $\Delta x_i = x_i - x_{i-1}$ (i = 1, 2, ..., n)

Definition 5.1.2 :

Let f be a bounded real function defined on [a b]. Corresponding to each partition p of [a b], let

$$M_i = \sup f(x) \qquad (x_{i-1} \leq x \leq x_i)$$

$$M_i = \inf f(x) \qquad (x_{i-1} \leq x \leq x_i)$$

$$U(p, f) = \sum_{i=1}^{n} M_i \Delta x_i;$$

$$L(p, f) = \sum_{i=1}^{n} m_i \Delta x_i;$$

$$\overline{\int_a^b} f dx = \inf (p, f) \qquad \text{-------} (1)$$

$$\underline{\int_a^b} f dx = \sup L(p, f) \qquad \text{-------} (2)$$

where sup & inf are taken over all partitions p of [a b]. The left members of (1) & (2) are called the upper and lower Riemann integrals of f over [a b].

If the upper & lower integrals are equal, f is Riemann integrable on [a b]. We write $f \in R$, the set of all Riemann integrable functions.

Definition 5.1.3 :

If f is bounded, there exist two numbers m & M satisfying $m \leq f(x) \leq M$ $(a \leq x \leq b)$

For each partition p,

$$m(b-a) \leq L(p, f) \leq U(p, f) \leq M(b-a)$$

so that the numbers L(p, f) & U(p, f) form a bounded set. This implies that the upper & lower integrals are defined for every bounded function.

Definition 5.1.4 :

Let α be a monotonically increasing function and bounded on [a b]. Corresponding to each partition p of [a b],

let $\Delta x_i = \alpha(x_i) - \alpha(x_{i-1})$.

Then $\Delta\alpha_i \geq 0$.

For any real valued function `f' which is bounded on [a b], let

$$U(p, f, \alpha) = \sum_{i=1}^{n}(M_i \Delta\alpha_i)$$

$$L(p, f, \alpha) = \sum_{i=1}^{n}(m_i \Delta\alpha_i)$$

where M_i & m_i are defined in above definition.

$$\overline{\int_a^b} f(x)dx = \inf U(p, f, \alpha) \quad \text{------ (3)}$$

$$\underline{\int_a^b} f(x)dx = \sup L(p, f, \alpha) \quad \text{------ (4)}$$

where inf & sup are taken over all partitious.

If (5) & (6) are equal, then their common value is denoted as $\int_a^b fdx$ and this is the Riemann –Stielties integral of f with respect to α. We denote this as $f \in R(\alpha)$.

Definition 5.1.5 :

A partition p* is a refinement of p if $p^* \supset p$ ((i.e.) every point of p is a point of p*).

Given two partitions p_1 & p_2, we say that p* is their common refinement if $p^* = p_1 \cup p_2$.

Theorem 5.1.6 :

If p* is a refinement of a partition p, then prove that

(i) $L(p, f, \alpha) \leq L(p^*, f, \alpha)$ ------ (1)

(ii) $U(p^*, f, \alpha) \leq U(p, f, \alpha)$ ------ (2)

Proof :

Case (i) To prove (i), assume that p* contains just one point more than p.

Let one extra point be x^*. Suppose $x_{i-1} \le x^* \le x_i$, where x_{i-1} & x_i are two consecutive points of p.

Let $w_1 = \inf f(x) \qquad (x_{i-1} \le x \in x^*)$

$w_2 = \inf f(x) \qquad (x^* \le x \le x_i)$

Clearly $w_1 \ge m_i$; $w_2 \ge m_i$, where $m_i = \inf f(x)\ (x_{i-1} \le x \le x_i)$

Therefore $L(p^*, f, \alpha) - L(p, f, \alpha)$

$\le w_1[\alpha(x^*)-\alpha(x_{i-1})]+w_2[\alpha(x_i)-\alpha(x^*)] - m_i[\alpha(x_i)-\alpha(x_{i-1})]$

$= (w_1-m_i)[\alpha(x^*)-\alpha(x_{i-1})]+(w_2-m_1)[\alpha(x_i)-\alpha(x^*)]$

≥ 0

If p* contains `k' - points more than p, we repeat this reasoning k times, and arrive at equation (1).

Similarly the equation (2) is followed.

Theorem 5.1.7 :

Prove that $\underline{\int_a^b} f d\alpha \le \overline{\int_a^b} f d\alpha$.

Proof :

Let p* be the common refinement of two partitions p_1 & p_2.

By (5.1.6),

$$\begin{aligned} L(p_1, f, \alpha) &\le L(p^*, f, \alpha) \\ &\le U(p^*, f, \alpha) \\ &\le U(p_2, f, \alpha) \end{aligned}$$

Thus $L(p, f, \alpha) \le U(p_2, f, \alpha)$ ------ (*)

If p_2 is fixed, and the sup is taken over all p_1, the equation (*) gives

$$\int_{-}^{+} f d\alpha \le U(p_2, f, \alpha)$$

Then we take inf over all p_2 in the above equation, we get the theorem.

Theorem 5.1.8 :

Show that $f \in R(\alpha)$ on [a b] iff $\forall \in > 0$, there exists a partition P satisfying

$U(P, f, \alpha) - L(P, f, \alpha) < \in$ ------ (1)

Proof :

Case (i)

Assume the condition (1).

For each partition P, we have $L(P, f, \alpha) \leq \underline{\int} f d\alpha \leq \overline{\int} f d\alpha \leq U(P, f, \alpha)$

By assumption (1), $0 \leq \overline{\int} f d\alpha - \underline{\int} f d\alpha < \in; \ \forall \in > 0$

$\therefore \ \overline{\int} f d\alpha = \underline{\int} f d\alpha$

$\Rightarrow f \in R(\alpha)$

Case (ii)

Let us assume that $f \in R(\alpha)$.

Let $\in > 0$ be given.

Then there exist partitions P_1 and P_2 satisfying

$$U(P_2, f, \alpha) - \int f d\alpha < \frac{\in}{2}; \quad \text{------ (2)}$$

$$\int f d\alpha - L(P_1, f, \alpha) < \frac{\in}{2}; \quad \text{------ (3)}$$

Assume $P = P_1 \cup P_2$.

By (5.2.6), and the equation (2) & (3), we get

$$\begin{aligned} U(P, f, \alpha) &\leq U(P_2, f, \alpha) \\ &\leq \int f dx + \in/2 \\ &< L(P_1, f, \alpha) + \in/2 + \in/2 \\ &\leq L(P, f, \alpha) + \in; \ \forall \in > 0 \end{aligned}$$

$U(P, f, \alpha) - L(P, f, \alpha) < \in$ for partition `P'.

Theorem 5.1.9 :

(i) If (1) holds for some P and some $\in$ in (5.1.8), then show that (i) holds (with the same $\in$) for every refinement of P.

(ii) If (1) hoids for $P = \{x_0, \ldots\ldots x_n\}$, and s_i, t_i are arbitrary points in $[x_{i-1}\ \ x_i]$, then prove that $\sum_{i=1}^{n} |f(s_i) - f(t_i)|(\Delta\alpha_i) < \in$

(iii) If $f \in R(\alpha)$, and the hypothesis of (ii) hold, then show that $\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha \right| < \in$.

Proof :

Theorem (5.1.6) implies case (i).

From the assumption of (ii), both $f(s_i)$ & $f(t_i)$ lie in $[m_i, M_i]$, so that $|f(s_i)-f(t_i)| \leq (M_i-m_i)$.

Therefore $\sum_{i=1}^{n} |f(s_i) - f(t_i)| (\Delta\alpha_i)$

$\leq U(P, f, \alpha)-L(P, f, \alpha)$ which shows (ii).

We know that $L(P, f, \alpha) \leq \Sigma f(t_i)\Delta\alpha_i$

$\leq U(P, f, \alpha)$ and

and $L(P, f, \alpha) \leq \int f d\alpha \leq u(P, f, \alpha)$

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha \right| < U(P,f,\alpha) - L(P,f,\alpha) < \in$$

This gives (iii).

Theorem 5.1.10 :

If f is continuous on [a b], show that $f \in R(\alpha)$ on [a b].

Proof :

Let $\in > 0$ be given.

Choose $\delta > 0$ so that $[\alpha(b)-\alpha(a)]\delta < \in$.

Given that f is continuous on [a b]

$\Rightarrow$ f is uniformly continuous on [a b]

$\therefore$ There exists a $\delta > 0$ satisfying $|f(x)-f(t)| < \delta$ -----(1) whenever $x, t \in [a\ b]$ & $|x-t| < \delta$.

Let P be a partition of [a b] such that $\Delta x_i < \delta$ for all i.

Then (1) becomes

$M_i-m_i < \delta$ $(i = 1, 2, \ldots\ldots, n)$

Thus $U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n}(M_i - m_i)(\Delta\alpha_i)$

$$\leq \delta \sum_{i=1}^{n} \Delta\alpha_i$$

$$= \delta[\alpha(b)-\alpha(a)]$$

$$< \in.$$

$\therefore$ By (5.1.6), $f \in R(\alpha)$.

Theorem 5.1.11 :

If f is monotonic on [a b], and α is continuous on [a b], then $f \in R(\alpha)$.

Proof :

Let $\alpha > 0$ be given. For any positive integer n, we choose a partition such that

$$\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n} \quad (i = 1, 2,...,n) \text{ (since } \alpha \text{ is continuous)}$$

Assume f is monotonically increasing (the proof is similar in the other case). Then $M_i=f(x_i)$; $m_i=f(x_{i-1})$; i=1,2,...n. So that,

$$U(P, f, \alpha)-L(P, f, \alpha) = \frac{\alpha(b)-\alpha(a)}{n}\left[\sum_{i=1}^{n} f(x_i) - f(x_{i-1})\right]$$

$$= \frac{\alpha(b)-\alpha(a)}{n}[f(b)-f(a)]$$

$$< \in, \text{ if n is taken large enough.}$$

By (5.1.6), $f \in R(\alpha)$.

Theorem 5.1.12 :

Assume f is bounded on [a b]; f has only finitely many points of discountinuity on [a b], and α is continuous at every point at which f is discontinuous. Show that $f \in R(\alpha)$.

Proof :

Let $\in > 0$ be given, and $M=\sup|f(x)|$.

Assume E is the set of points at which f is discountinuous.

Since E is finite and α is continuous at every point of E, we can cover E by finitely many disjoint intervals $[u_j, v_j] \subset [a\ b]$ such that the sum of the corresponding differences $\alpha(v_j)-\alpha(u_j)$ is less than $\in$. Also we can choose these intervals in such a way that every point of $E \cap (a\ b)$ lies in the interior of some $[u_j, v_j]$.

Remove the segments $(u_j\ v_j)$ from $[a\ b]$. The remaining set K is compact. Therefore f is uniformly continuous function on K, and $\exists\ \delta > 0$ such that $|f(s)-f(t)| < \in$, whenever $|s-t| < \delta$; $\forall s, t \in K$.

Now we form a partition $P=\{x_0, x_1, \ldots\ldots, x_n\}$ of $[a\ b]$ as follows :

Each u_j occurs in P, and each v_j occurs in P. No point of any segment (u_j, v_j) occurs in P. If x_{i-1} is not one of the u_j, then $\Delta x_i < \delta$.

Note that $(M_i-m_i) \leq 2M$ for each i, and that $M_i-m_i \leq \in$ unless x_{i-1} is one of the u_j.

By (5.1.9), $U(P, f, \alpha)-L(P, f, \alpha) \leq [\alpha(b)-\alpha(a)]\in + 2M\in$

Since $\in$ is arbitrary, $f \in R(\alpha)$.

Theorem 5.1.13 :

Suppose $f \in R(\alpha)$ on $[a\ b]$, $m \leq f \leq M$, ϕ is continuous on $[m, M]$ and $h(x) = \phi(f(x))$ on $[a\ b]$. Show that $h \in R(\alpha)$ on $[a\ b]$.

Proof :

Let $\in > 0$ be given.

Since ϕ is uniformly continuous on $[m\ M]$, there exists $\delta > 0$ such that $\delta < \in$, and $|Q(s)-Q(t)| < \in$, if $|s-t| \leq \delta$, and $s, t \in [m, M]$; $f \in R(\alpha) \Rightarrow$ There is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a\ b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \delta^2 \quad \text{------ (1)}$$

Let M_i, m_i have already defined in (5.1.1). Assume M_i^*, m_i^* be the analogous numbers for h.

Divides the numbers 1, 2, 3,...n into two classes :

$i \in A$ if $M_i - m_i < \delta$; $i \in B$ if $M_i - m_i \geq \delta$.

For $i \in A$, choice of δ shows that $M_i^* - m_i^* \leq \epsilon$

For $i \in B$, $M_i^* - m_i^* \leq (2k)$ where $k = \sup|\phi(t)|$; $m \leq t \leq M$.

By (1), $\delta \sum_{i \in B} \Delta\alpha_i \leq \sum_{i \in B} (M_i - m_i)\Delta\alpha_i \leq \delta^2$ so that $\sum_{i \in B} (\Delta\alpha_i) < \delta$

It follows that

$$U(P, h, \alpha) - L(P, h, \alpha)$$

$$= \sum_{i \in A} (M_i^* - m_i^*)\Delta\alpha_i + \sum_{i} (M_i^* - m_i^*)\Delta\alpha_i$$

$$\leq \epsilon[\alpha(b) - \alpha(a)] + 2k\delta$$

$$< \epsilon[(\alpha(b) - \alpha(a) + 2k]$$

Since ϵ is arbitrary, $h \in R(\alpha)$.

SECTION - 2 : PROPERTIES OF THE INTEGRAL

Theorem 5.2.1 :

Prove the following

(i) $f_1 \in R(\alpha)$; $f_2 \in R(\alpha)$ on $[a \ \ b]$

$\Rightarrow f_1 + f_2 \in R(\alpha)$ & $cf \in R(\alpha)$ for every constant `c'

$$\int_a^b (f_1 + f_2)d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$$

(ii) If $f_1(x) \leq f_2(x)$ on $[a \ \ b]$, then

$$\int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha$$

(iii) If $f \in R(\alpha)$ on $[a \ \ b]$, and $a < c < b$, then $f \in R(\alpha)$ on $[a \ \ c]$ & $[c \ \ b]$, and

$$\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha$$

(iv) If $f \in R(\alpha)$ on $[a\ \ b]$, and $|f(x)| \le M$ on $[a\ \ b]$, then

$$\left|\int_a^b f d\alpha\right| \le M[\alpha(b)-\alpha(a)]$$

(v) If $f \in R(\alpha_1)$ & $g \in R(\alpha_2)$, then prove that $f \in R(\alpha_1+\alpha_2)$ and

$$\int_a^b f\, d(\alpha_1+\alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\,\alpha_2$$

If $f \in R(\alpha)$ and c is positive constant, then $f \in R(c\alpha)$, and

$$\int_a^b f\, d(c\alpha) = c\int_a^b (f\, d\,\alpha)$$

Proof :

If $f = f_1+f_2$, and P is any partition of $[a\ \ b]$, we have

$$\begin{aligned} & L(P, f_1, \alpha) + L(P, f_2, \alpha) \\ \le\ & L(P, f, \alpha) \\ \le\ & U(P, f, \alpha) \\ \le\ & U(P, f_1, \alpha)+u(P, f_2, \alpha) \qquad \text{------ (1)} \end{aligned}$$

If $f_1 \in R(\alpha)$ & $f_2 \in R(\alpha)$, let $\in > 0$ be given. There exist partitions P_1 & P_2 such that

$$U(P_j, f_j, \alpha)-L(P_j, f_j, \alpha) < \in \quad \forall i=1, 2. \qquad \text{------ (2)}$$

These inequalities persist whenever P_1 & P_2 are replaced by their common refinement P.

By (1), $U(P, f, \alpha)-L(P, f, \alpha) < 2\in$ which gives $f \in R(\alpha)$.

With the same partition P, we get $U(P, f_j, \alpha) < \int f_j d\alpha + \in$ (j=1, 2)

$\therefore$ Equation (2) gives $\int f d\alpha \le U(P,f,\alpha) < \int f_1 d\alpha + \int f_2 d\alpha + 2\in$

Since $\in$ is arbitrary, we conclude that $\int f d\alpha \le \int f_1 d\alpha + \int f_2 d\alpha$

If we replace f_1 & f_2 in above equation by $(-f_1)$ and $(-f_2)$, then the inequality is reversed, and the inequality is proved.

The proofs of the other assertions are so similar.

Theorem 5.2.2 :

If $f \in R(\alpha)$, $g \in R(\alpha)$ on $[a\ \ b]$, then prove that

(i) $fg \in R(\alpha)$

(ii) $|f| \in R(\alpha)$ and $\left|\int_a^b f d\alpha\right| \le \int_a^b |f| d\alpha$

Proof :

Let $\phi(t) = t^2$

By (5.1.13), $f^2 \in R(\alpha)$

Consider the identity

$$4fg = (f+g)^2 - (f-g)^2$$

$$\therefore \quad fgR \in (\alpha)$$

If $\phi(t) = |t|$, (5.1.13) shows similarly that $|f| \in R(\alpha)$. We choose $c = \pm 1$

show that $\quad c \int f d\alpha \ge 0$

$$\therefore \quad \left|\int f d\alpha\right| = c \int f d\alpha = \int cf d\alpha$$

$$\le \int |f| d\alpha \ (\because cf \le |f|)$$

This completes the proof.

Definition 5.2.3 :

The unit step function I is defined by

$$I(x) = \begin{cases} 0 & (x \le 0); \\ 1 & (x > 0) \end{cases}$$

Theorem 5.2.4 :

If $a < s < b$; f is bounded on [a b], f is continuous at s, and

$\alpha(x) = I(x-s)$, then prove that $\int_a^b f \, d\alpha = f(s)$.

Proof :

Consider a partition $P = \{x_0, x_1, x_2, x_3\}$ where $x_0 = a$; $x_1 = s < x_2 < x_3 = b$.

Then $M_2 = U(P, f, \alpha)$; $m_2 = L(P, f, \alpha)$

Since f is continuous at s, M_2 & m_2 converge to f(s) as $x_2 \to s$.

Theorem 5.2.5 :

Suppose $c_n \ge 0$, for 1, 2, 3.... Σa_n converges; $\{s_n\}$ is a sequence of distinct points in (a b), and

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n) \qquad \text{------ (1)}$$

Let `f' be continuous on [a b]. Prove that

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n) \qquad \text{------ (2)}$$

Proof :

The comparison test shows that the series (1) converges for every x. Its sum $\alpha(x)$ is evidently monotonic, and $\alpha(a) = 0$;

$\alpha(b) = \Sigma c_n$.

Let $\in > 0$ be given.

We choose N so that $\sum_{N+1}^{\infty} c_n < \in$

Assume $\alpha_1(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$;

$\alpha_2(x) = \sum_{N+1}^{\infty} c_n I(x - s_n)$;

By (5.2.3) & (5.2.4), we have

$$\int_a^b f(d\alpha_1) = \sum_{i=1}^{N} c_n f(s_n) \qquad \text{------ (3)}$$

Since $\alpha_2(b) - \alpha_2(a) < \in$, we have

$$\left|\int_a^b f d\alpha_2\right| \leq M \in \text{ where } M = \sup |f(x)| \qquad \text{------ (4)}$$

$\alpha = \alpha_1 + \alpha_2$ implies that from (3) & (4) that

$$\left|\int_a^b f d\alpha - \sum_{i=1}^{N} c_n f(s_n)\right| \leq M \in$$

As $N \to \infty$, we obtain (2).

Theorem 5.2.6 :

Assume α increases monotonically, and $\alpha' \in R$ on [a b]. Assume that f be a bounded real function on [a b]. Show that

$f \in R(\alpha)$ iff $f\alpha' \in R(\alpha)$, and

$$\int_a^b f d\alpha = \int_a^b f(x)\, \alpha'(x).dx$$

Proof :

Let $\in > 0$ be given.

Applying (5.1.8) to α', there is a partition $P = \{x_0, \ldots\ldots, x_n\}$ of $[a\ \ b]$ such that

$$U(P, \alpha') - L(P, \alpha') < \in \qquad \text{------ (3)}$$

The mean value theorem furnishes points $t_i \in [x_{i-1}\ \ x_i]$ such that

$$\Delta\alpha_i = \alpha'(t_i)\Delta x_i, \quad \forall i = 1, 2, \ldots\ldots, n.$$

If $s_i \in [x_{i-1}, \ \ x_i]$ then

$$\sum_{i=1}^{n} \left|\alpha'(s_i) - \alpha'(t_i)\right| \Delta x_i < \in \text{ by (3) and 5.1.9 (ii).}$$

Let $M = \sup |f(x)|$

$$\sum_{i=1}^{n} f(s_i)\Delta x_i = \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i$$

$$\Rightarrow \left|\sum_{i=1}^{n} f(s_i)\Delta x_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i\right| \leq M \in \qquad \text{------- (4) by above equation.}$$

In particular, $\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \leq U(P, f\alpha^1) + M \in$ for all choices $s_i \in [x_{i-1}, \ \ x_i]$ so that

$$U(P, f, \alpha) \leq U(P, f\alpha') + M\in$$

(4) gives

$$U(P, f\alpha') \leq U(P, f, \alpha) + M\in$$

$$\Rightarrow |U(P, f, \alpha) - U(P, f\alpha')| \leq M\in \qquad \text{------- (5)}$$

Note that (1) holds, whenever P is replaced by any refinement.

Hence the equation (5) also remains true. We conclude that

$$\left|\overline{\int_a^b} f d\alpha - \overline{\int_a^b} f(x)\alpha'(x) dx\right| \leq M \in$$

Since $\in$ is arbitrary, we have $\overline{\int_a^b} f d\alpha = \overline{\int_a^b} f(x)\alpha'(x).dx$ for any bounded `f'.

The equality of the lower integrals follows from (4), in exactly the same way.

This completes the proof.

Theorem 5.2.7 :

Suppose ϕ is a strictly increasing continuous function that maps an interval [A, B] onto [a b]. Let α be monotonically increasing on [a b] and $f \in R(\alpha)$ on [a b].

Define β & g on [A B] by

$$\beta(y) = \alpha(\phi(y)); \; g(y) = f(\phi(y)) \qquad \text{------ (1)}$$

Then show that $g \in R(\beta)$, and

$$\int_A^B g\,d\beta = \int_a^b f(d\alpha) \qquad \text{------ (2)}$$

Proof :

To each partition $P = \{x_0,...,x_n\}$ of [a b] corresponds a partition $Q=\{y_0,y_1,...,y_n\}$ of [A B], so that $x_i = \phi(y_i)$. All partitions of [A B] are obtained in this way. Since the values taken by f on $[x_{i-1}, x_i]$ are exactly the same as those taken by g on $[y_{i-1}, y_i]$, it follows that

$$\left.\begin{array}{l} U(Q,g,\beta) = U(P,f,\alpha); \\ L(Q,g,\beta) = L(P,f,\alpha); \end{array}\right\} \qquad \text{------ (3)}$$

Since $f \in R(\alpha)$, P can be chosen so that both $U(P, f, \alpha)$ & $L(P, f, \alpha)$ are close to $\int f d\alpha$.

The equation (3) & (5.1.8) give that $g \in R(\beta)$, and the equation (2) holds.

This completes the proof.

Integration and Differentiation :

Theorem 5.2.7 :

Let $f \in R$ on [a b]. For $a \le x \le b$, let $F(x) = \int_a^x f(t)dt$.

Prove that F is continuous at a point x_0 of [a b], then F is differentiable at x_0, and $F'(x_0) = f(x_0)$.

Proof :

Case (i)

$f \in R \Rightarrow f$ is bounded.

Suppose $|f(t)| \leq M$, for $a \leq t \leq b$.

If $a \leq x < y \leq b$, then

$$|f(y)-f(x)| = \left|\int_a^x f(t)dt - \int_a^y f(t)dt\right|$$

$$= \left|\int_a^b f(t)dt\right| \leq M(y-x)$$

Given $\in > 0$, we observe that

$|f(y)-f(x)| < \in$, provided that $|y-x| < \in/M$.

$\therefore$ f is continuous on F (in fact uniform continuous function).

Case (ii)

Suppose f is continuous at x_0.

Given $\in > 0$, we choose $\delta > 0$ satisfying $|f(t)-f(x_0)| < \in$, if $|t-x_0| < \delta$; $a \leq t \leq b$.

If $x_0-\delta < s \leq x_0 \leq t < x_0+\delta$, and $a \leq s < t \leq b$, we have

$$\left|\frac{f(t)-f(s)}{(t-s)} - f(x_0)\right| = \left|\frac{1}{(t-s)}\int_s^t [f(u)-f(x_0)]du\right| < \in$$

If follows that $f'(x_0) = f(x_0)$.

The Fundamental Theorem of Calculus 5.2.8 :

If $f \in R$ on [a b], and there is a differentiable function F on [a b] such that $F' = f$, then show that

$$\int_a^b f(x)dx = F(b) - F(a)$$

Proof :

Let $\in > 0$ be given. We choose a partition $P = \{x_0, x_1, \ldots, x_n\}$ of [a b] so that

$U(p, f) - L(p, f) < \in$.

The mean value theorem furnishes points $t_i \in [x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i, \ \forall i = 1, 2, \dots n$$

Therefore $\sum_{i=1}^{n} f(t_i)\Delta x_i = F(b) - F(a)$

$$\therefore \left| F(b) - F(a) - \int_a^b f(x)dx \right| < \in, \ \forall \in > 0$$

This completes the proof.

Theorem (Integration by Parts) 5.2.9 :

Suppose F & G are differentiable functions on [a b]; $F' = f \in R$, and $G' = g \in R$. Show that

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)Gg(x)dx$$

Proof :

Let $H(x) = F(x)G(x)$.

Applying (5.2.8) to H, and its derivatives, we note that $H' \in R$.

The remaining conclusions are obvious.

REAL AND COMPLEX ANALYSIS

PART – B : COMPLEX ANALYSIS

UNIT – VI

SECTION – I : THE ALGEBRA OF COMPLEX NUMBERS

It is the fact that the real and complex numbers satisfy the basic laws of arithmetic.

From elementary algebra, the imaginary unit is i with $i^2 = -1$. Any complex number is of the form a+ib, where a & b are reals. If a=0, the number is called purely imaginary. If b=0, it is called real.

$$a+ib = c+id \text{ iff } a = c;\ b = d.$$

The addition and multiplication of complex numbers are defined as

$$(a+ib) + (c+id) = (a+c) + i(b+d)$$

$$(a+ib).(c+id) = (ac-bd) + i(ad+bc)$$

The division of two complex number is defined as

$$\frac{a+ib}{c+id} = \frac{ac+bd}{(c^2+d^2)} + i\left(\frac{bc-ad}{c^2+d^2}\right) \text{ provided that } (c+id) \neq 0.$$

The reciprocal of non-zero complex number is

$$\frac{1}{a+ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{(a^2+b^2)} - i\frac{b}{(a^2+b^2)}$$

i^n takes any one of the values 1, –1, i, –i, for every positive integer n.

Example 6.1.1 :

Find the values of (i) $(1+2i)^3$ (ii) $\frac{2}{-3+4i}$, (iii) $\left[\frac{(2-i)}{(3+2i)}\right]^2$, (iv) $(1+i)^n+(1-i)^n$,

(v) $\left(\frac{1+i}{1-i}\right)^n$.

Solution :

(i) $$(1+2i)^3 = (1+2i)^2(1+2i)$$

$$= (1+4i^2+4i)(1+2i)$$

$$= (-3+4i)(1+2i)$$

$$= (-3-8)+i(-6i+4i)$$

$$= -11-2i$$

(ii) $$\frac{2}{(-3+4i)} = \frac{2}{-3+4i}\left(\frac{-3-4i}{-3-4i}\right)$$

$$= \frac{-6-8i}{(9+16)} = \frac{(-2)(3+4i)}{25}$$

(iii) $$\frac{(2-i)}{3+2i} = \frac{(2-i)}{3+2i}\,\frac{(3-2i)}{(3+2i)}$$

$$= \frac{6+2i^2-4i-3i}{9+4}$$

$$= \frac{(4-7i)}{13}$$

$$\left[\frac{(2-i)}{3+2i}\right]^2 = \frac{(4-7i)^2}{169}$$

$$= \left[\frac{1}{169}\right]\left[16+49i^2-56i\right]$$

$$= \left[\frac{1}{169}\right]\left[-33-56i\right]$$

(iv) $$(1+i)^n+(1-i)^n = ?$$

Suppose $$1+i = r(\cos\theta + i\sin\theta)$$

$$r = \sqrt{1+1} = \sqrt{2}$$

$$\left.\begin{array}{l} r\cos\theta = 1 \Rightarrow \cos\theta = 1/\sqrt{2} \\ r\sin\theta = 1 \Rightarrow \sin\theta = 1/\sqrt{2} \end{array}\right\} \theta = \frac{\pi}{4}$$

$\therefore$ $$(1+i)^n = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n$$

$$= 2^{n/2}\left[\cos\frac{n\pi}{4} + i\,\sin\frac{n\pi}{4}\right]$$

$$(1-i)^n = 2^{n/2}\left[\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right]$$

$$\therefore \quad (1+i)^n+(1-i)^n = 2.2^{n/2}\cos\left(\frac{n\pi}{4}\right)$$

$$= 2^{\left(\frac{n+2}{2}\right)}\cos\left(\frac{n\pi}{4}\right)$$

(v)
$$\frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)}$$

$$= \left(\frac{1+i^2+2i}{1+1}\right)$$

$$= \frac{(1-1+2i)}{2}$$

$$= i$$

$$\therefore \quad \left[\frac{(1+i)}{(1-i)}\right]^n = i^n$$

$$= 1, -1, i, -i \quad \text{if } n = 0, 1, 2, 3 \pmod 4).$$

Example 6.1.2 :

If $z = x+iy$ (x, y are reals), find real & maginary parts of z^4, $\frac{1}{z}$, $\frac{z-1}{z+1}$, $\frac{1}{z^2}$.

Solution :

Given that $z = x+iy$.

(i)
$$z^2 = (x+iy)(x+iy)$$

$$= x^2+y^2i^2+2iyx$$

$$= (x^2-y^2)+i(2xy)$$

$$z^4 = [(x^2-y^2)+i\,2xy]^2$$

$$= (x^2-y^2)^2+4i^2x^2y^2+4ixy(x^2-y^2)$$

$$= [(x^2-y^2)^2-4x^2y^2]+i[4xy(x^2-y^2)]$$

(ii) $$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)}$$

$$= \frac{x-iy}{x^2+y2} = \frac{x}{(x^2+y^2)} - i\frac{y}{(x^2+y^2)}$$

(iii) $$\frac{1}{z^2} = \left(\frac{1}{z}\right)^2 = \left(\frac{x-iy}{x^2+y^2}\right)^2 = \frac{(x-iy)^2}{(x^2+y^2)^2}$$

$$= \frac{x^2+i^2y^2-2ixy}{(x^2+y^2)^2}$$

$$= \frac{(x^2-y^2)}{(x^2+y^2)} - i\frac{2xy}{(x^2+y^2)^2}$$

(iv) $$\frac{z-1}{z+1} = \frac{(x+iy)-1}{(x+iy)+1} = \frac{(x-1)+iy}{(x+1)+iy}$$

$$= \frac{[(x-1)+iy]}{[(x+1)+iy]}\frac{[(x+1)-iy]}{[(x+1)-iy]}$$

$$= \frac{[(x-1)(x+1)+y^2]+i[y(x+1)-y(x-1)]}{(x+1)^2+y^2}$$

$$= \frac{(x^2+y^2-1)+i(2y)}{(x+1)^2+y^2}$$

$$= \frac{(x^2+y^2-1)}{(x+1)^2+y^2} + i\cdot\frac{2y}{(x+1)^2+y^2} \quad \text{(Provided } (x+1)+iy\neq 0)$$

Example 6.1.3 :

Prove that $\left(\dfrac{-1\pm i\sqrt{3}}{2}\right)^3 = 1$; $\left(\dfrac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1$.

Proof :

(i) $$\left(\frac{-1\pm i\sqrt{3}}{2}\right)^3 = \frac{1}{4}[-1\pm i\sqrt{3}]^2$$

$$= \frac{1}{4}[1+i^2(3)\pm 2\sqrt{3}\,i]$$

$$= \frac{1}{4}[-2\pm 2\sqrt{3}\,i] = \frac{1}{2}[-1\pm\sqrt{3}\,i]$$

$$\left(\frac{-1\pm i\sqrt{3}}{2}\right)^3 = \frac{1}{2}[-1\mp\sqrt{3}\,i]\ \frac{1}{2}[-1\pm i\sqrt{3}]$$

$$= \frac{1}{4}[1+3]$$

$$= \frac{1}{4}(4) = 1$$

We follow that $\left(\frac{-1+i\sqrt{3}}{2}\right)^6 = 1^2 = 1.$

Otherwise $\left(\frac{1\pm\sqrt{3}\,i}{2}\right)^2 = \frac{(1+3i^2\pm 2\sqrt{3}\,i)}{4}$

$$= \frac{1}{4}(-2\pm 2\sqrt{3}\,i)$$

$$= \frac{1}{2}(-1\pm\sqrt{3}\,i)$$

$$\left(\frac{1\pm\sqrt{3}\,i}{2}\right)^3 = \frac{1}{2}(-1\pm\sqrt{3}\,i)\frac{(1\pm\sqrt{3}\,i)}{2}$$

$$= \frac{1}{4}[-1\mp\sqrt{3}\,i\sqrt{3}\,i\mp 3i^2]$$

$$= \frac{1}{4}[-1-3] = (-1)$$

$\therefore$ $\left(\frac{1\pm i\sqrt{3}}{2}\right)^6 = (-1)^2 = 1$

SECTION–2 : SQUARE ROOTS

We shall try to find the square root of a complex number.

To get $\sqrt{\alpha+i\beta}$, we have to solve the equation $\sqrt{\alpha+i\beta} = x+iy$

(i.e.,) $(x+iy)^2 = \alpha+i\beta;$

$$x^2+i^2y^2+2ixy = \alpha+i\beta$$

$$(x^2-y^2)+i(2xy) = \alpha+i\beta$$

Comparing real & imaginary parts, we get

$$x^2-y^2 = \alpha^2$$

$$2xy = \beta^2$$

Now, $\quad (x^2+y^2)^2 = (x^2-y^2)2+4x^2y^2$

$$= \alpha^2+\beta^2$$

$\Rightarrow \quad x^2+y^2 = \sqrt{\alpha^2+\beta^2}$

$$x^2-y^2 = \alpha$$

$\therefore \quad 2x^2 = \alpha+\sqrt{\alpha^2+\beta^2}$

$$x^2 = \frac{1}{2}\left(\alpha+\sqrt{\alpha^2+\beta^2}\right)$$

$$y^2 = \frac{1}{2}\left(-\alpha+\sqrt{\alpha^2+\beta^2}\right)$$

$$x^2 \ \& \ y^2 \geq 0 \Rightarrow x \ \& \ y \text{ are reals}$$

$\therefore \quad \sqrt{\alpha+i\beta} = \pm\left[\sqrt{\frac{\alpha+\sqrt{\alpha^2+\beta^2}}{2}}+i\frac{\beta}{|\beta|}\sqrt{\frac{-\alpha+\sqrt{\alpha^2+\beta^2}}{2}}\right]$

(provided $\beta \neq 0$).

For $\beta = 0$, the values are $\pm\sqrt{\alpha}$ if $\alpha \geq 0$; $\pm i\sqrt{-\alpha}$ if $\alpha < 0$;

Here all square roots of positive numbers are taken with the positive sign.

Example 6.2.1 :

Find $\sqrt{i}$, $\sqrt{-i}$, $\sqrt{1-i}$, $\sqrt{\frac{1-i\sqrt{3}}{2}}$, $\sqrt{\frac{1}{1-i}}$.

Solution :

(i) $\quad \sqrt{i} = \sqrt{0+(1)i}$

$$= \pm\left(\sqrt{\frac{0+\sqrt{0+1}}{2}}+i\left(\frac{1}{1}\right)\sqrt{\frac{-0+\sqrt{0+1}}{2}}\right)$$

$$= \pm\left(\frac{1}{2}+\frac{i}{2}\right)$$

$$= \pm\frac{1}{2}(1+i)$$

(ii)

$$\sqrt{-i} = \sqrt{0+(-1)i}$$

$$= \pm\left(\sqrt{\frac{0+\sqrt{0+1}}{2}} + i\left(\frac{-1}{1}\right)\sqrt{\frac{-0+\sqrt{0+1}}{2}}\right)$$

$$= \pm\left(\frac{1}{2}-\frac{i}{2}\right) = \pm\frac{1}{2}(1-i)$$

(iii)

$$\sqrt{1-i} = \sqrt{1+(-1)i}$$

$$= \pm\left(\sqrt{\frac{1+\sqrt{1+1}}{2}} + i\left(\frac{-1}{1}\right)\sqrt{\frac{-1+\sqrt{1+1}}{2}}\right)$$

$$= \pm\frac{1}{\sqrt{2}}\left[\sqrt{1+\sqrt{2}} - i\sqrt{-1+\sqrt{2}}\right]$$

(iv)

$$\sqrt{\frac{1-i\sqrt{3}}{2}} = \sqrt{\frac{1}{2}+\left(\frac{-\sqrt{3}}{2}\right)i}$$

$$= \pm\left(\sqrt{\frac{\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{3}{4}}}{2}} + i(-1)\sqrt{\frac{-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{3}{4}}}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{\frac{1}{2}+1}{2}} - i\sqrt{\frac{1-\frac{1}{2}}{2}}\right)$$

$$= \pm\left(\frac{3}{4}-i\frac{1}{4}\right) = \pm\frac{1}{4}(1-i)$$

(v)

$$\sqrt{\frac{1}{1-i}} = \sqrt{\frac{1}{1-i}\left(\frac{1+i}{1+i}\right)} = \sqrt{\frac{1+i}{2}}$$

$$= \frac{1}{\sqrt{2}}\sqrt{1+i}$$

$$= \frac{1}{\sqrt{2}}\sqrt{1+(1)i}$$

$$= \frac{\pm}{\sqrt{2}}\left[\sqrt{\frac{1+\sqrt{1+1}}{2}} + i(1)\sqrt{\frac{-1+\sqrt{1+1}}{2}}\right]$$

$$= \frac{(\pm)}{2}\left[\sqrt{1+\sqrt{2}} + i\sqrt{-1+\sqrt{2}}\right]$$

Example 6.2.2 :

Find the four values of $4\sqrt{(-1)} = (-1)^{1/4}$.

Solution :

$$\begin{aligned}
(-1)^{1/2} &= \sqrt{-1+(0)i}\,;\quad \alpha=-1;\ \beta=0\\
&= \pm\sqrt{\alpha} = \pm\sqrt{-1} = \pm i\\
\sqrt{i} &= i^{1/2} = \sqrt{0+(1)i}\,;\quad \alpha=0;\ \beta=1\\
&= \pm\left(\sqrt{\frac{0+\sqrt{0+1}}{2}} + i(1)\sqrt{\frac{-0+\sqrt{0+1}}{2}}\right)\\
&= \pm\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = \pm\frac{(1+i)}{\sqrt{2}}\\
\sqrt{-i} &= \sqrt{0+(-1)i}\\
&= \pm\left(\sqrt{\frac{0+\sqrt{0+1}}{2}} + i(-1)\sqrt{\frac{-0+\sqrt{0+1}}{2}}\right)\\
&= \pm\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right) = \pm\frac{(1-i)}{\sqrt{2}}
\end{aligned}$$

The four values of $4\sqrt{(-1)}$ are $\pm\frac{1}{\sqrt{2}}(1+i),\ \pm\frac{1}{\sqrt{2}}(1-i)$.

Example 6.2.3 :

$$\begin{aligned}
\text{Compute } \sqrt[4]{(-i)} &= (-1)^{1/4}\\
\sqrt{-i} &= \sqrt{0+(-1)i}\\
&= \left(\sqrt{\frac{0+\sqrt{0+1}}{2}} + i(-1)\sqrt{\frac{-0+\sqrt{0+1}}{2}}\right)\\
&= \pm\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right) = \pm\frac{(1-i)}{\sqrt{2}}\\
\sqrt{\frac{1-i}{\sqrt{2}}} &= \frac{1}{2^{1/4}}\sqrt{1+(-1)i}\\
&= \frac{1}{2^{1/4}}\left(\sqrt{\frac{1+\sqrt{1+1}}{2}} + i(-1)\sqrt{\frac{-1+\sqrt{1+1}}{2}}\right)\\
&= 2^{-1/4}\left(\sqrt{\frac{1+\sqrt{2}}{2}} - i\sqrt{\frac{-1+\sqrt{2}}{2}}\right)
\end{aligned}$$

$$\sqrt{\frac{-(1-i)}{\sqrt{2}}} = 2^{-1/4}\sqrt{(-1)+(1)i}$$

$$= 2^{-1/4}\left(\sqrt{\frac{-1+\sqrt{1+1}}{2}}+i\sqrt{\frac{1+\sqrt{1+1}}{2}}\right)$$

$$= 2^{-1/4}\left(\sqrt{\frac{-1+\sqrt{2}}{2}}+i\sqrt{\frac{1+\sqrt{2}}{2}}\right)$$

The four values of $4\sqrt{(-i)}$ are $2^{-3/4}\left(\sqrt{1+\sqrt{2}}-i\sqrt{-1+\sqrt{2}}\right)$, and $2^{-3/4}\left(\sqrt{-1+\sqrt{2}}-i\sqrt{1+\sqrt{2}}\right)$.

6.2.4. Introduction to Complex Numbers :

We recall the characteristic properties of reals as follows :

Addition and multiplication in R (reals) are defined satisfying the closure law, associative law, commutative, and distributive laws. The numbers 0 & 1 are identity elements under addition & multiplication respectively. The equations $x+\beta=\alpha$ and $x\beta=\alpha$ with $\beta\neq 0$ have solutions..Thus R is a field.

It is also an integral domain because $\alpha\beta = 0 \Rightarrow \alpha = 0$ (or) $\beta = 0$. R is an ordered field (since either $\alpha < \beta$ (or) $\alpha > \beta$, for all α, β (reals)).

In R^+ (positive reals), $\beta-\alpha \in R^+$ iff $\beta > \alpha$. The product of two positive reals is positive.

By the order relation, the sums 1, 1+1, 1+1+1,..... are all different. Thus R contains the natural numbers, and the subfield of all rational numbers.

R satisfies the completeness condition: every increasing & bounded sequence of reals has a limit.

The equation $x^2+1=0$ has no solution in R (because $x^2+1 > 0$ always $\forall\ x \in R$).

Suppose there is a field F containing R, and in which the equation x^2+1 can be solved. Assume one of solutions is i.

Then $x^2+1 = (x+i)(x-i)$. Let 'C' be the subset of F consisting of all elements which can be expressed $\alpha+i\beta$ with reals α and β.

This representation is unique, for $\alpha+i\beta = \alpha'+i\beta'$ iff $\alpha=\alpha'$; $\beta=\beta'$.

The subset C is a subfield of F.

If C' is the set of all elements $\alpha+i'\beta$ where i' is another root of $x^2+1 = 0$, then C & C' are isomorphic.

6.2.4. Conjugation, Absolute Value :

If $(x+iy) = z$ is a complex number, its conjugate is $(x-iy) = \bar{z}$

Absolute value is $\sqrt{(x^2+y^2)} = |z| = \sqrt{z\,.\,\bar{z}}$

$$\text{Re}(z) = \text{Real part of } z = \frac{z+\bar{z}}{2}$$

$$\text{Im}(z) = \text{Imaginar part of } z = \frac{(z-\bar{z})}{2};$$

$$\overline{z_1+z_2} = \overline{z_1}+\overline{z_2};\quad \overline{z_1 z_2} = \overline{z_1}-\overline{z_2}$$

$$z\bar{z} = |z|^2;\quad |z_1 z_2| = |z_1|.|z_2|$$

we have $$|a+b|^2 = |a|^2+|b|^2+2\text{Re}(a\bar{b})$$

$$|a-b|^2 = |a|2+|b|^2-2\text{Re}(a\bar{b})$$

$$\therefore\quad |a+b|^2+|a-b|^2 = 2(|a|^2+|b|^2)$$

Example 6.2.5 :

Find the absolute values of 1) $(-2i)(3-i)(2+4i)(1-i)$, and 2) $\dfrac{(3-4i)(-1+2i)}{(-1-i)(3-i)}$.

Solution :

(1) $$|(-2i)(3-i)(2+4i)(1-i)| = |(-2i)|\,|3-i|\,|(2+4i)|\,|1-i|$$

$$= 2\sqrt{9+1}\sqrt{4+16}\sqrt{1+1}$$

$$= 2\sqrt{10}\,2\sqrt{5}\sqrt{2}$$

$$= 40$$

(2) $$\left|\frac{(3-4i)(-1+2i)}{(-1-i)(3-i)}\right| = \frac{|3-4i|\,|-1+2i|}{|-1-i|\,|3-i|}$$

$$= \frac{\sqrt{9+16}\sqrt{1+4}}{\sqrt{1+1}\sqrt{9+1}}$$

$$= \frac{5\sqrt{5}}{\sqrt{2}\sqrt{10}}$$

$$= \frac{5}{\sqrt{2}\sqrt{2}} = \frac{5}{2}$$

Example 6.2.6 :

Prove that $\left|\frac{a-b}{1-\bar{a}b}\right| = 1$ if and only if either $|a| = 1$ (or) $|b| = 1$. What is made if $|a| = |b| = 1$.

Proof :

(i) Suppose $|a| = 1$

$$\left|\frac{a-b}{1-\bar{a}b}\right|^2 = \frac{(a-b)\overline{(a-b)}}{(1-\bar{a}b)\overline{(1-\bar{a}b)}} = \frac{(a-b)(\bar{a}-\bar{b})}{(1-\bar{a}b)(1-a\bar{b})}$$

$$= \frac{a\bar{a}-a\bar{b}-\bar{a}b+b\bar{b}}{1-a\bar{b}-\bar{a}b+a\bar{a}b\bar{b}}$$

$$= \frac{1-a\bar{b}-\bar{a}b+b\bar{b}}{1-a\bar{b}-\bar{a}b+(a)b\bar{b}} = 1$$

$$\therefore \quad \left|\frac{a-b}{1-\bar{a}b}\right| = 1;$$

Similarly, $|b| = 1 \Rightarrow \left|\frac{a-b}{1-\bar{a}b}\right| = 1.$

Remark 6.2.7 :

Inequalities

1) By definition, $-|a| \le \mathrm{Re}(a) \le |a|$;

$$-|a| \le \mathrm{Im}(a) \le |a| ;$$

2) $\mathrm{Re}(a) = |a|$ holds iff a is real ≥ 0.

3)
$$\begin{aligned} |a+b|^2 &= |a|^2+|b|^2+2\mathrm{Re}(a\bar{b}) \\ &\le |a|^2+|b|^2+2|a\bar{b}| \\ &= |a|^2+|b|^2+2|a||b| \\ &= (|a|+|b|)^2 \end{aligned}$$

$$\therefore \quad |a+b|^2 \le |a|+|b|^2$$

which is triangle inequality.

By induction hypothesis,

$$|a_1+a_2+.....+a_n| \leq |a_1|+|a_2|+......+|a_n|.$$

4) The Cauchy inequality states

$$|a_1b_1+....+a_nb_n|^2 \leq (|a_1|^2+.....+|a_n|^2)(|b_1|^2+....+|b_n|^2)$$

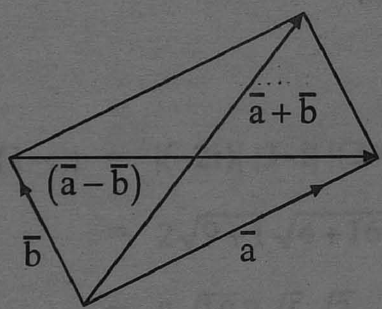
(ie) $$|\Sigma a_ib_i|^2 \leq \left(\sum_{i=1}^{n}|a_i|^2\right)\left(\sum_{i=1}^{n}|b_i|^2\right)$$

SECTION–3 : THE GEOMETRIC REPRESENTATION OF COMPLEX NUMBERS

With respect to a given rectangular Co-ordinate system in a plane, the complex number $\alpha+i\beta$ can be represented by the point with co-ordinates (α, β). The first Co-ordinate lies on x-axis (real axis), and the second coordinate lies on y-axis (imaginary axis). The plane itself is referred as the complex plane.

The addition of two complex numbers a & b can be visualized as vector addition in the Geometric representation as follows :

We consider a second vector b so that its initial point coincides with the end point of a. Therefore a+b is represented by the vector from of b To construct the difference (b–a), we draw both vectors a & b from the end point of b. Then (b–a) points from the end point of a to the end point of b. It follows that a+b & a–b are the diagonals in a parallelogram with the sides a & b.



• Vector addition
&
Vector subration

If $a = \alpha+i\beta = r_1e^{i\theta_1}$; $b = r_23^{i\theta_2}$, then $ab = r_1r_2\, e^{i(\theta_1+\theta_2)}$.

$\therefore$ $|ab| = |a|\,|b|$; arg (ab) = arg(a)+arg(b).

Similarly, $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

$\arg\left(\frac{a}{b}\right) = \arg(a) - \arg(b)$

Section 4 : The Binomial equation.

If $a = r(\cos\phi + i\sin\phi)$, then

$$a^n = r^n[\cos(n\phi)+i\sin(n\phi)];\ \forall\ n>0 \qquad \text{------(1)}$$

This formula holds if n=0;

$$a^{-1} = r^{-1}(\cos\phi - i\sin\phi) = r-1[\cos(-\phi)+i\sin(-\phi)]$$

$$a^{-n} = r^{-n}[\cos(n\phi)-i\sin(n\phi)];\ \forall\ n<0 \qquad \text{-------(2)}$$

r=1 in (1) $\Rightarrow$ we get De-Moivre's formula.

Example 6.4.1 :

To find the n^{th} root of a complex number a, we have to solve the equation $z^n=a$.

Suppose $0 \neq a = r(\cos\phi+i\sin\phi)$

and $z = p(\cos\theta+i\sin\theta)$

$z^n = a$ becomes $p^n[\cos(n\theta)+i\sin(n\theta)] = r[\cos\phi+i\sin\phi]$

$\Rightarrow p^n = r$, and $n\theta=\phi$

$\Rightarrow\ p = r^{1/n}\ \&\ \theta = -\phi/n$

$\therefore\ z = r^{1/n}(\cos(\theta/n)+i\sin(\theta/n))$

where $r^{1/n}$ denotes the positive n^{th} root of a positive number `r'.

We know that (1) is also filfilled if $(n\theta)$ differ from ϕ by a multiple of the full angle. If angles are expressed in radians the full angle is 2π, and we find that (1) is satisfied if and only if $n\theta = \theta+2k\pi$ (=) $\dfrac{\phi}{n}+k\dfrac{2\pi}{n}$ where k is any integer. In fact, the values k=0,1,2,....,(n–1) give different values of z.

Thus the complete solution of the equation $z^n = a$ is given by

$$z = r^{1/n}\left[\cos\left(\frac{\phi}{n}+k\frac{2\pi}{n}\right)+i\sin\left(\frac{\phi}{n}+k\frac{2\pi}{n}\right)\right],\quad \text{where } k = 0,1,2,\ldots\ldots,(n-1).$$

There are n^{th} roots of any complex number $\neq 0$. They have the same modules, and the their arguments are equally spaced.

Hence the roots of the equation $z^n = 1$ are called n^{th} roots of unity, and they are of the form $w^k = \cos\ k\dfrac{2\pi}{n}+i\sin\ k\dfrac{2\pi}{n}$ where k=1,2,...,n–1, and denoted by 1, w, w^2,...w^{n-1}.

Analytic Geometry :

In classical analytic geometry, the equation of a locus is expressed as a relation between x & y. The equation of a circle is $|z-a| = r$ (ie) $(z-a)\overline{(z-a)} = r^2$ $\Rightarrow (z-a)(\overline{z}-\overline{a}) = r^2$. The fact that this equation is invariant under complex conjugation is an indication that it represents a single real equation.

A straight line in the complex plane can be given by a parametric equation $z = a+bt$, where a & b are complex numbers, and $b \neq 0$. Two equation $a+bt$ & $a'+b't$ represent the same line if and only if $(a'-a)$ & b' are real multiples of b_0. The lines are parallel whenever b' is a real multiple of b, and they are equally directed if b' is a positive multiple of b. The direction of a directed line can be identified with arg(b). Note that $|z-a| < r$ describes the inside of a circle.

The spherical representation :

In the set C of complex numbers, we define $a+\infty = \infty+a = \infty$ for all finite `a' and $b-\infty = \infty-b = \infty$, $\forall\, b \neq 0$.

$\infty.\infty = \infty$; $a-\infty = -\infty$;

$a/0 = \infty$ for $a \neq 0$; $b/\infty = 0$, if $b \neq \infty$

In the plane, there is no room for a point corresponding to ∞, but we introduce an ideal point, called the point of infinity. The points together with the point at infinity forms the extended complex plane. We agree that each straight line passes through a point at infinity.

To represent a unit sphere in `three-dimensional space, its equation is $x^2+y^2+z^2+1$.

Except (0, 0, 1), every point on S is given by $z_0 = \dfrac{x+iy}{(1-z)}$ -----(*) and this function is one-to-one.

$$\therefore \quad |z_0| = \frac{x^2+y^2}{(1-z)^2} = \frac{1+z}{1-z}$$

$$\Rightarrow \quad |z| = \frac{|z_0|^2-1}{|z_0|^2+1}$$

$$x_1 = \frac{z_0+\overline{z_0}}{1+|z_0|^2} \; ; \; x_2 = \frac{z_0-\overline{z_0}}{i[i+|z_0|^2]}$$

The relation can be completed by taking the point at infinity correspond to (0, 0, 1). Hence sphere can be representation of the extend plane or the extended number system.

Note that the hemisphere $z < 0$ corresponds to the disk $|z| < 1$, and the hemisphere $z > 0$ to its outside $|z| > 1$.

The complex plane is identified with XY-plane with X-axis & Y-axis corresponding to the real & imaginary axis respectively. Thus the equation (or) transformation (*) takes on a simple geometric meaning. To write $z = x_0+iy_0$; we can check that

$x_0 : y_0 : -1 = x : y : z : -1$, and

this means that the points $(x_0, y_0, 0)$, (x, y, z) and $(0, 0, 1)$ are collinear. This method is called a central projection from the centre (0, 0, 1), and the correspondence is said to be a stereographic projection which is also regarded as a mapping from S to the extended complex plane or vice versa. In the spherical representation, there is no simple interpretation of addition and multiplication.

It is true that the stereographic projection transforms every straight line in the z-plane into a circle on S that passes through the point (0, 0, 1) and the converse also holds. (ie) any circle on the sphere corresponds to a circle (or) a straight line in the z-plane.

To verify this, we follow that a circle on the sphere, lies in a plane $ax+by+cz=d$, where $a^2+b^2+c^2 = 1$ and $0 \leq d \leq 1$.

Interms of z and $\bar{z}$, this equation takes the form

$a(z+\bar{z})-bi(z-\bar{z})+c(|z|^2-1) = d(|z|^2+1)$ (or)

$(d-c)(x^2+y^2)-2ax-2by+(d+c) = 0$

If $d \neq c$, this equation is a circle and if $d = c$, it represents a straight line.

Conversely, the equation of any circle (or) straight line can be written in this form. The correspondence is one-to-one.

The distance $d(z, z')$ is between the stereographic projections of z & z'.

If $z_1 = (x_1, x_2, x_3)$ & $z_2 = (y_1, y_2, y_3)$ are two points on the sphere, we have

$(x_1-y_1)^2+(x_2-y_2)^2+(y_3-y_3)^2 = 2-2(x_1y_1+x_2y_2+x_3y_3)$

Thus we obtain $x_1y_1+x_2y_2+x_3y_3$

$$= \frac{(z_1+\overline{z_1})(z_2+\overline{z_2})-(z_1-\overline{z_1})(z_2-\overline{z_2})+(|z_1|^2-1)(|z_2|^2-1)}{(1+|z_1|^2)(1+|z_2|^2)}$$

$$= \frac{(1+|z_1|^2)(1+|z_2|^2)-2|z_1-z_2|^2}{(1+|z_1|^2)(1+|z_2|^2)}$$

$$d(z_1, z_2) = \frac{2|z_1-z_2|}{\sqrt{(1+|z_1|^2\,(1+|z_2|^2)}}$$

SECTION–1 : LIMIT AND CONTINUITY

The theory of function of a complex variable claims at extending calculus to the complex domain. Both differentiation and integration acquire new depth and signifiance in the same direction of applications. Only analytic functions can be differentiated and integrated. Further a function is defined for all values of the independent variables. (ie) a function is defined on an open set, and f(a) is defined means that f(x) is also defined for all x sufficiently close to a.

Definition 7.1.1 : Limit and Continuity

A map f(x) is said to have the limit A as x tends to a $\left(\lim\limits_{x\to a} f(x) = A\right)$ if for every $\in > 0$, there exists a number $\delta > 0$ with the property that $|f(x)-A| < \in$, for all values of x satisfying $|x-a| < \delta$ and $x \neq a$.

(Continuous) Definition 7.1.2 :

A map f(x) is said to be continuous at x = a if $\lim\limits_{x\to a} f(x) = f(a)$.

Thus a continuous function is one that is continuous at each point where the function is defined.

It is the fact that the sum of two continuous functions is continuous; the product of two continuous functions is continuous; the quotient f(x)/g(x) is defined and continuous at a iff $g(a) \neq 0$; If f(x) is continuous, then Re f(x), Im f(x) and |f(x)| are all continuous functions.

The derivative of a function f(x) at a point x = a is defined as

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{(x-a)}$$

Then the usual aim for forming the derivative of a sum, a product or a Quotient are all valid. The derivative of a composition function is identified by the chain rule

$$\text{(ie)} \quad \frac{d}{dx} f(g(x)) = [f'(g(x))]\ [g'(x)]$$

If $z(t) = x(t)+iy(t)$, then we write $z'(t)=x'(t)+iy'(t)$ provided that the existence of $z'(t)$ is equivalent to the simultaneous existence of $x'(t)$ and $y'(t)$.

Analytic functions :

It is the class of functions such that they possess a derivative where the functions are defined. The derivative of a mapping can be rewritten in the form

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

Every analytic function is differenciable, and every differenciable function is continuous. For the latter case, f has a derivative at $z = \alpha$

$$\Rightarrow \lim_{h\to 0} \frac{f(\alpha+h)-f(\alpha)}{h} \text{ exists and finite.}$$

$$\therefore \lim_{h\to 0} [f(\alpha+h)-f(\alpha)] = \lim_{h\to 0} \frac{f(\alpha+h)-f(\alpha)}{h} . \underset{h\to 0}{\text{Lt}} (h)$$

$$= f'(\alpha).0$$

$$= 0$$

$$\therefore \lim_{h\to 0} f(\alpha+h) = f(\alpha)$$

$\therefore$ f is continuous function at $z = \alpha$.

Theorem 7.1.3 :

Derive C–R–equation for an analytic function.

Solution :

Let f(z) be an analytic function.

$\therefore$ It is differentiable at each point where the function is defined. Since z has two variables x & y (z=x+iy), the derivative can be evaluated in two different ways.

Let f(z) = u+iv;

If we choose real values for h, and the imaginary part y is kept as constant, we have

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{------(1)}$$

Similarly if we consider purely imaginary values ik for h, and the real part x is kept as constant, we obtain

$$f'(z) = (-i)\frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(z+ik)-f(z)}{(ik)}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{------(2)}$$

From (1) & (2), we follow that

$$\frac{\partial f}{\partial x} = (-i)\frac{\partial f}{\partial y}; \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{------(3)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{-------(4)}$$

(3) & (4) are called Cauchy-Riemann differential equations, that must be satisfied by the real & imaginary parts of an analytic function u+iv.

Theorem 7.1.4 :

Prove that real and imaginary parts of an analytic function satisfy Laplace's equation.

Proof :

By (7.1.3), f(z) = u+iv is analytic $\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ which are C-R-equations.

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial y \partial x}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

u satisfies the Laplace equation $\nabla^2 u = 0$

Similarly v satisfies $\nabla^2 v = 0$.

Theorem 7.1.5 :

If u & v have first-order partial derivatives that satisfy the Cauchy-Riemann differential equations, then show that f(z) = u+iv is analytic with continuous derivative f′(z), and conversely.

Proof :

By (7.1.3), it remains to verify that the function f(z) = u+iv is determined by a pair of conjugate harmonic functions (ie.) u & v have continuous first - order partial derivatives. By continuous properties of calculus, we have

$$u(x+h, y+k)-u(x, y) = \frac{\partial u}{\partial x}h+\frac{\partial u}{\partial y}k+\in_1$$

$$v(x+h, y+k)-v(x, y) = \frac{\partial v}{\partial x}h+\frac{\partial v}{\partial y}k+\in_2$$

with the assumptions that $\in_1 \to 0$; $\in_2 \to 0$ as $(h+ik) \to 0$.

Thus we obtain

$$f(z+h+ik)-f(z) = \left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(h+ik)+\in_1+i\in_2$$

$$\therefore \lim_{(h+ik)\to 0}\frac{f(z+h+ik)-f(z)}{(h+ik)} = \frac{\partial u}{\partial x}+i.\frac{\partial v}{\partial x}$$

$\therefore$ f(z) is analytic function.

Example 7.1.6 :

Consider the function $f(z) = z^2+iC$. Then $f(z) = (x+iy)2+(C_1+iC_2)$

(ie) $f(z) = u+iv = [(x^2-y^2)+C_1]+i[2xy+C_2]$

Real part $= u = x^2-y^2+C_1$

Imaginary part $= r = 2xy+C_2$

$$\frac{\partial u}{\partial x} = 2x; \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial x} = 2y; \frac{\partial v}{\partial y} = 2x;$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

u & v satisfy Cauchy-Riemann equation. And all partial derivatives are continuous.

$f(z) = z^2+ic$ is an analytic function.

Remark 7.1.7 :

Consider a complex function f(x, y) of two real variables. Introducing the complex variable $z = x+iy$, and its conjugate $\bar{z} = x-iy$, then we have

$x = (z+\bar{z})/2;\ y = (z-\bar{z})/2i$

If we assume the function f(x, y) (by changing of x & y) as a function of z and $\bar{z}$ which are independent variables, they are in fact conjugate to each other.

Finally we obtain

$$\frac{\partial f}{\partial z} = \left(\frac{1}{2}\right)\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right);$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

By comparison of C–R–equations, analytic functions are characterized by the condition $\frac{\partial f}{\partial \bar{z}} = 0$. Thus an analytic function is independent of $\bar{z}$, and a function of z only.

If f(z) is an analytic function whose real part is harmonic function u(x, y), then the conjugate function $\overline{f(z)}$ has the derivative zero with respect to z, and it is considered as a function of $\bar{z}$ denoted by $\tilde{f}(\bar{z})$. Therefore

$$u(x, y) = \left(\frac{1}{2}\right)\left[f(x+iy) + \tilde{f}(x-iy)\right]$$

This holds for complex numbers x & y.

If $x = z/2;\ y = z/2i$, then we obtain $u(z/2, z/2i) = \left(\frac{1}{2}\right)\left[f(z) + \tilde{f}(0)\right]$

If f(z) is only determined upto a purely imaginary constant, then f(0) is real, $\tilde{f}(0) = u(0, r)$, and $f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0)$.

Polynomials :

Any constant function is an analytic function with the derivative 0.

The (trivial) non-constant analytic function, whose derivative is 1, is z.

We know that the sum and product of two analytic functions are both again analytic functions.

It gives the fact that every polynomial $p(z) = a_0+a_1z+a_2z^2+....+a_nz^n$ ------(1)

is an analytic function. Its derivations is obtained as $P'(z) = a_1+2a_2z+....+na_nz^{n-1}$.

If $a_n \neq 0$ in (1), degree of P(z) is n.

By fundamental theorem of algebra, P(z)=0 has atleast one root, for n > 0.

$\therefore$ If $P(z_1) = 0$ for some z_1, then $P(z) = (z-z_1)\,P_1(z)$, where $P_1(z)$ is a polynomial of degree (n–1). Continuing this process, finally we obtain

$P(z) = a_n(z-z_1)(z-z_2).....(z-z_n)$ where $z_1, z_2,......,z_n$ are complex numbers.

We know that $P(z) \neq 0$ for all $z \neq z_1, z_2,...,z_n$.

Also the factors are unique in product. If a root z_i is a root for h times, then z_i is a zero of P(z) of the order `h'. In fact, $0 = P(z_i) = P'(z_i) = P^{(2)}(z_i) = ... = P^{(h-1)}(z_i)$ with $0 \neq P^{(h)}(z_i)$. (ie) the order of the first non-vanishing derivative equals the order of a zero.

Theorem (Lucas) 7.1.8 :

If all zeros of a polynomial P(z) lie in a half plane, then all zeros of the derivative P'(z) lie in the same half plane.

Proof :

Consider a polynomial P(z) with zeros, namely $\alpha_1, \alpha_2,.....\alpha_n$.

Then $$P(z) = a_n(z-\alpha_1)(z-\alpha_2).....(z-\alpha_n).$$

Taking `log' on both sides, we obtain

$$\log P(z) = \log a_n+\log (z-\alpha_1)+\log(z-\alpha_2)+....+\log(z-\alpha_n)$$

Differenting with respect to z,

$$\frac{P'(z)}{P(z)} = \frac{1}{(z-\alpha_1)}+\frac{1}{(z-\alpha_2)}+.....+\frac{1}{(z-\alpha_n)} \quad \text{------}(*)$$

Let H be the half-plane defined as the part of the plane such that Imaginary part of $(z-a)/b < 0$.

Then if $\alpha_k \in H$, and $z \notin H$, we have

$$\text{Im}\ ((z-\alpha_k)/b) \;=\; \text{Im}\ ((z-\alpha)/b) - \text{Im}\ ((\alpha_k-a)/b) > 0$$

Imaginary parts of reciprocal numbers have opposite sign. Hence Im $[b(z-\alpha_k)^{-1}] < 0$.

This holds for all k. Finally we get from (*) that

$$\text{Im}\ \frac{bP'(z)}{P(z)} = \sum_{k=1}^{n} \text{Im}(b/(z-\alpha_k)) < 0$$

and $P'(z) \neq 0$. It follows that zeros of P(z) contians the zeros of P'(z).

Rational Function :

It is of the form R(z) = P(z)/Q(z) as the quotient of two polynomials. We assume that P(z) and Q(z) have no common factors and no common zeros. $R(z) = \infty$ at the zeros of Q(z).

$\therefore$ The rational function is considered as a function with valued in the extended plane, and it is continuous. The zeros of Q(z) are called poles of R(z), and the order of a pole equals the order of the corresponding zero of Q(z).

Its derivative is $R'(z) = \dfrac{P'(z)Q(z) - P(z)Q'(z)}{[Q(z)]^2}$ -----(1)

exists only when $Q(z) \neq 0$.

Note that R'(z) has the same poles of R(z) and the order of each pole in R'(z) is increased by one in the order of each pole of R(z). If $R_1(z) = R(1/z)$, then $R_1(z)$ is a rational function satisfying $R_1(0) = R(\infty)$. If $R_1(0) = 0$ (or) ∞, then the order of zero of zero (or) pole at ∞ is defined as the order of the zero (or) pole of $R_1(z)$ at the origin.

If $R(z) = \dfrac{P(z)}{Q(z)} = \dfrac{a_0 + a_1 z + \ldots.. + a_n z^n}{b_0 + b_1 z + \ldots.. + b_m z^m}$

then $R_1(z) = z^{m-n} \dfrac{(a_0 z^n + a_1 z^{n-1} + \ldots. + a_n)}{(b_0 z^m + b_1 z^{m-1} + \ldots. + b_m)}$

where the power z^{m-n} belongs either to the numerator (or) to the denominator.

If m > n, R(z) has a zero of order (m–n) at ∞. If m < n, the point at ∞ is a pole of order (n–m). If m = n, $R(\infty) = a_n/b_m \neq 0, \infty$. The common number of zeros and poles is called the order of the rational function.

If a is any constant, then R(z) has the same poles as (R(z)–a) and the same order. The zeros of R(z)–a are the roots of R(z) = a. We conclude that a rational function R(z) of order p has p zeros, and so p poles, and every equation R(z)=a has exactly p-roots.

A rational function of order one is a fraction $S(z) = \frac{(az+b)}{(cz+d)}$ (with $ad-bc \neq 0$) which is called linear transformation. Here

$$z = S^{-1}(w) = \left(\frac{dw-b}{-cw+a}\right), \text{ and}$$

S and S^{-1} are inverses to each other. The linear transformation (z+a) is called a parallel translation, and y_z is an inversion.

Every rational function has a representation by partial fractions. For this, let R(2) be a pole at ∞. We find out the division of P(z) by Q(z), until the degree of the remainder is at most equal to that of the denominator.

Then R(z) = G(z)+H(z) ------(1)

where G(z) is a polynomial without constant term and H(z) is finite at ∞. The degree of G(z) is the order of the pole at ∞, and the polynomial is said to be the singular part of R(z) at ∞.

Let the distinct finite poles of R(z) be denoted by $\beta_1, \beta_2, \ldots\ldots, \beta_q$. The function $R(\beta_j+1/\epsilon)$ is a rational function of ϵ with a pole $\epsilon = \infty$.

By (1), $R(\beta_j+1/\epsilon) = G_j(\epsilon)+H_j(\epsilon)$

$\Rightarrow R(z) = G_j(1/(z-\beta_j))+H_j(1/(z-\beta_j))$

where $G_j(1/(z-\beta_j))$ is a polynomial in the variable $1/(z-\beta_j)$ without constant term and it is called the singular part of R(z) at β_j. The function $H_j(1/(z+\beta_j))$ is finite for $z=\beta_j$.

Now $R(z)-G(z)-\sum_{j=1}^{q} G_j(1/(z-\beta_j))$ is also a rational function, that has poles $\beta_1, \beta_2, \ldots\beta_q$ and ∞ only. At $z = \beta_j$, we get that the two terms which become infinite, have a difference $H_j(1/(z-\beta_j))$ with a finite limit and the same holds at ∞.

A rational function without poles must reduce to a constant, and we get

$$R(z) = \text{constant} + \sum_{j=1}^{q} G_j(1/(z-\beta_j)).$$

SECTION–2 : ELEMENTARY THEORY OF POWER SERIES

Definition 7.2.1 :

A sequence $\{a_n\}$ has a limit l if $\forall \in > 0$, there exists a positive integer N such that $|a_n - l| < \in$, for all $n \geq N$. A convergent sequence is a sequence with a finite limit. If $\lim\limits_{n\to\infty} a_n = \infty$ (or) $-\infty$, then the sequence $\{a_n\}$ is divergent.

Definition 7.2.2 :

A Cauchy sequence is a sequence $\{a_n\}$ such that $\forall \in > 0$, there exists a positive integer N satisfying $|a_n - a_m| < \in$; $\forall n, m \geq N$.

Definition 7.2.3 :

If $\{a_n\}$ is a sequence of real numbers, let $a_n = \max\{\alpha_1, \alpha_2, ...\alpha_n\}$ (ie) a_n is the greatest number among the numbers $\alpha_1, \alpha_2, ...,\alpha_n$. Here $\{a_n\}$ is an increasing sequence, and hence it has a limit A_1 (finite or $+\infty$), which is also least upper bond of the numbers α_n's. We can write $A = \lim\limits_{k\to\infty} A_k = \lim\limits_{k\to\infty} (\sup(\alpha_k))$.

Remark 7.2.4 :

If A is finite in (7.2.3) and $\in > 0$, there exists an N such that $A_{n_0} < A+\in$, and it gives that $\alpha_n \leq A_N \leq A+\in$ for $n \geq N$.

If $\alpha_n \leq A-\in$; $\forall n \geq N$, then $A_{n_0} \leq (A-\in)$ which is a contradiction.

$\therefore$ For a large `n', $\alpha_n > A-\in$. If $A = +\infty$, there are arbitrarily large α_n and $A = -\infty$, iff $\alpha_n \to (-\infty)$.

Definition 7.2.5 :

$$\underline{\lim}\,(\alpha_n) = \lim_{n\to\infty} (\inf \alpha_n)$$

$$\overline{\lim}\,(\alpha_n) = \lim_{n\to\infty} (\sup \alpha_n)$$

Then we have the following :

$$\begin{aligned}\underline{\lim}\,(\alpha_n) + \underline{\lim}\,(\beta_n) &\leq \underline{\lim}\,(\alpha_n + \beta_n)\\ &\leq \underline{\lim}(\alpha_n) + \overline{\lim}(\beta_n)\end{aligned}$$

$$\begin{aligned}\underline{\lim}(\alpha_n) + \overline{\lim}(\beta_n) &\leq \overline{\lim}(\alpha_n + \beta_n)\\ &\leq \overline{\lim}(\alpha_n) + \overline{\lim}(\beta_n)\end{aligned}$$

Theorem 7.2.6 :

Prove that a sequence is convergent iff it is a Cauchy sequence.

Proof :

Case (i)

Let $\{a_n\}$ be a convergent sequence $\Rightarrow \{a_n\} \to A$ as $n \to \infty$. $\forall \in > 0$, there exists a positive integer n_0 such that $|a_n - A| < \in/2$, $\forall n \geq n_0$.

For $m \geq n_0$; $n \geq n_0$, we get

$$\begin{aligned} |a_n - a_m| &= |a_n - a + a - a_m| \\ &\leq |a_n - a| + |a - a_m| \\ &= |a_n - a| + |a_m - a| \\ &< \in/2 + \in/2, \quad \forall\, n, m \geq n_0 \\ &= \in \end{aligned}$$

$\therefore$ $\{a_n\}$ is a Cauchy sequence.

Case (ii)

Let $\{a_n\}$ a Cauchy sequence. $\therefore$ $\forall \in > 0$, there exists a positive integer N satisfying $|a_n - a_m| < \in$, $\forall n, m \geq N$.

$\Rightarrow |a_n - a_N| < \in, \forall n \geq N$

$\Rightarrow |a_n| \leq |a_N| + \in, \forall n \geq N.$

It follows that $A = \overline{\text{Lim}}\, a_n$; $a = \underline{\text{Lim}}\, (a_n)$

Choose $\in = (A-a)/3$

If $\in > 0$, $\exists$ a positive integer N such that $a_n < a + \in$, and $a_m > a - \in$ with $n, m \geq N$.

$$\begin{aligned} A - a &= (A - a_m) + (a_m - a_n) + (a_n - a) \\ &< \in + \in + \in = 3\in \end{aligned}$$

$$(A-a)/3 < \in \text{ which is a contradiction}$$

Otherwise $\in = 0 = (A-a)/3 \Rightarrow A = a$

$\therefore$ The given sequence $\{a_n\}$ converges.

SERIES :

Definition 7.2.7 :

Let $\sum_{n=1}^{\infty} a_n$ be a series, and $s_m = a_1+a_2+.....+a_m$; $\forall$ m=1,2,3,....

Then $\sum_{1}^{\infty} (a_n)$ converges if the partial sum sequence $\{s_n\}$ converges. Here the limit of the sequence is the sum of the series.

Definition 7.2.8 :

A series $\sum_{1}^{\infty}(a_n)$ is Cauchy series iff $\forall \in > 0$, $\exists\, n_0$ such that $|a_n+a_{n+1}+......+a_{n+p}| < \in$, $\forall n \geq n_0$; $p \geq 0$. If p=0, $|a_n| < \in$ (ie) the general term of a convergent series tends to zero.

Definition 7.2.9 :

The series $\sum_{1}^{\infty} (a_n)$ is absolutely convergent if the series $\sum_{1}^{\infty} |a_n|$ is convergent.

Definition 7.2.10 :

A sequence of functions $\{f_n(x)\}$ are all defined on the same set E, and $x \in E$. Then $f_n(x)$ uniformly converges to a function f(x) if $\forall \in > 0$, and $\forall\, x \in E$, $\exists\, n_0$ such that $|f_n(x)-f(x)| < \in$, $\forall n \geq n_0$ where n_0 is independent of x.

Example 7.2.11 :

$$\lim_{n\to\alpha}\left(1+\frac{1}{n}\right)x = x$$

Solution :

$$\left|\left(1+\frac{1}{n}\right)x - x\right| = \left|x+\left(\frac{x}{n}\right)-x\right|$$

$$\leq \frac{|x|}{n}$$

$\forall \in > 0$, choose $n_0 > \dfrac{|x|}{\in}$.

Such an n_0 is exist for every fixed x, but cancel the requirement cannot be met simultaneously for all x. Therefore $\left\{\left(1+\frac{1}{n}\right)x\right\}$ converges pointwise, but not uniformly because n_0 depends on x & $\in$.

Theorem 7.2.12 :

Show that the limit function of a uniformly convergent sequence of continuous functions is again continuous.

Proof :

Assume that the function $f_n(x)$ are continuous, and $\{f_n(x)\}$ converges uniformly to f(x) on a set E. By uniformity, $\forall \in > 0$, $\exists$ n such that $|f_n(x)-f(x)| < \in/3$, for all $x \in E$.

Let $x_0 \in E$. Then $f_n(x)$ is continuous at x_0. $\exists\ \delta > 0$ such that $|f_n(x)-f_n(x_0)| < \in/3$, for all $x \in E$ with $|x-x_0| < \delta$.

We have $$\begin{aligned}|f(x)-f(x_0)| &\leq |f(x)-f_n(x)|+|f_n(x)-f_n(x_0)|+|f_n(x_0)-f(x_0)| \\ &< \in/3 + \in/3 + \in/3 = \in \text{ whenever } |x-x_0| < \delta.\end{aligned}$$

$\therefore$ f is continuous at x_0.

Hence f is continuous function on E.

Theorem 7.2.13 :

Show that the sequence $\{f_n(x)\}$ converges uniformly on E iff $\forall \in > 0$, $\exists\ n_0$ such that $|f_n(x)-f_m(x)| < \in$ for all $n, m \geq n_0$, and for all $x \in E$.

Proof :

Case (i) :

Suppose $\{f_n(x)\}$ converges uniformly to a function f(x) on E. Then if $x \in E$, then $\forall \in > 0$, $\exists\ n_0$ such that $|f_n(x)-f(x)| < \in/2$, $\forall n \geq n_0$.

Fix $m, n \geq n_0$.

Then $$\begin{aligned}|f_n(x)-f_m(x)| &= |f_n(x)-f(x)+f(x)-f_m(x)| \\ &\leq |f_n(x)-f(x)| + |f_m(x)-f(x)| \\ &< \in/2 + \in/2 = \in;\ \forall x \in E\end{aligned}$$

Case (ii) :

To prove the sufficient condition, assume that $\forall \in > 0$, $\exists\ n_0$ such that $|f_n(x)-f_m(x)| < \in$, $\forall n, m \geq n_0$, and for all $x \in E$. The limit function $f(x)$ exists by Cauchy's test. (ie) $f_m(x) \to f(x)$ as $m \to \infty$.

Keeping n as fixed, and $m \to \infty$, we obtain

$$\left| f_n(x) - \lim_{m\to\infty} f_n(x) \right| < \in, \forall n \geq n_0; \forall x \in E$$

(ie) $|f_n(x)-f(x)| < \in$; $\forall n \geq n_0$; $\forall x \in E$.

$\therefore f_n(x)$ converges uniformly to $f(x)$ on E.

SECTION–3 : POWER SERIES

Definition 7.3.1 :

A power series is a series of the form $a_0+a_1z+a_2z^2+.....+a_nz^n+..... = \sum_{n=0}^{\infty} (a_nz^n)$ where the coefficients a_n, and the variable z are complex numbers.

Now $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is a power series with respect to the center z_0.

Definition 7.3.2 :

The geometric series is $\sum_{n=0}^{\infty}(z^n) = 1+z+z^2+...$, whose partial sums can be written as

$$1+z+....+z^{n-1} = \left(\frac{1-z^n}{1-z}\right).$$

Since $z^n \to 0$ if $|z| < 1$, and $|z^n| \geq 1$ if $|z| \geq 1$. Therefore the geometric series converges to $1/(1-z)$ if $|z| < 1$, and diverges if $|z| \geq 1$.

Every power series converges inside a circle, and diverges outside the same circle.

Theorem 7.3.2 : (Abel)

For every power series $\sum_{n=0}^{\infty} a_nz^n$, $\exists$ a number R, $0 \leq R \leq \infty$, called the radius of convergence with the following properties :

1) The series converges absolutely for every z with $|z| < R$. If $0 \le \rho < R$, the convergence is uniform for $|z| < \rho$.

2) If $|z| > R$, the terms of the series are unbounded, the series is divergent.

3) If $|z| < R$, the sum of the series is an analytic function. The derivative can be obtained by termwise differentiation, and the derived series has the same radius of convergence.

Proof :

The circle $|z| = R$ is called the radius of convergence.

Choose R such that $1/R = \limsup_{n\to\infty} (|a_n|)^{1/n}$ which is said to be hadamard's formula.

Case (i) :

Let $|z| < R$. Choose ρ so that $|z| < \rho < R$. Then $1/\rho > 1/R$.

By definition of limit superior, $\exists\, n_0$ such

$$(|a_n|)1/n < 1/\rho;\ \forall n \ge n.$$

$$\Rightarrow |a_n| < (1/\rho^n);\ \forall n \ge n.$$

Thus
$$|a_n z^n| = |a_n|\,|z^n|$$
$$< 1/\rho^n\,|z|n = (|z|/\rho)^n$$

$\therefore$ The given power series has a convergent geometric series as a majorant and so it is absolutely convergent.

To verify uniform convergence for $|z| \le \rho < R$, we choose ρ' so that $\rho < \rho' < R$. Then $|a_n z^n| \le (|z|/\rho)^n \le (\rho'/\rho)^n$, $\forall n \ge n_0$. Since the majorant is convergent and has constant terms, we conclude that the given power series is uniformly convergent by Weierstrass's M-test.

Case (ii) :

Let $|z| > R$. We choose ρ with $R < \rho < |z|$. Then $1/\rho < 1/R$. For large n, $(|a_n|)^{1/n} > (1/\rho)$; $|a_n| > 1/\rho^n$. Therefore $|a_n z^n| > (|z|/\rho)^n$ for infinitely many `n'.

$\therefore$ The terms are unbounded.

The given series is $\rho(z) = \sum_0^{\infty} a_n . z^n$; $\rho'(z) = \frac{d}{dz}(\rho(z)) = \sum_0^{\infty} a_n (nz^{n-1}) = \sum_0^{\infty} n.a_n z^{n-1}$

whose radius of convergence R_1 satisfies

$$\frac{1}{R_1} = \operatorname*{Lim\,sup}_{n\to\infty}(|n\, a_n|)^{1/n} = \operatorname*{Lim\,sup}_{n\to\infty}(n^{1/n})(|a_n|)^{1/n} \text{ ----(*)}$$

Therefore subclaim

$$\lim_{n\to\infty} n^{1/n} = 1$$

Let $n^{1/n} = 1+\delta_n$. Then $\delta_n > 0$.

By the Binomial theorem, $n=(1+\delta_n)^n = 1+nc_1\delta_n+nc_2\delta_n^{\,2}+.....$

$$\therefore \quad n > 1+\frac{n(n+1)}{2}\delta_n^{\,2} \Rightarrow \delta_n^{\,2} < (2/n)$$

Thus $\delta_n \to 0$ as $n \to \infty$. $\therefore$ $n^{1/n} \to 1$ as $n \to \infty$. The equation (*) becomes

$$1/R_1 = \operatorname*{Lim\,sup}_{n\to\infty}(|a_n|)^{1/n} = \frac{1}{R}$$

$$\therefore \quad R = R_1$$

Therefore the derived series and the given power series have the same radius of convergence.

Case (ii)

Claim :

If $|z| < R$, the sum of the series is an analytic function.

Assume $|z| < R$.

$$f(z) = \sum_0^{\infty} a_n z^n = P(z) = s_n(z) + R_n(z)$$

where
$$s_n(z) = a_0+a_1z+....+a_{n-1}z^{n-1}$$
$$R_n(z) = a_nz^n+a_{n+1}z^{n+1}+.....$$
$$= \sum_{k=n}^{\infty}(a_kz^k)$$

$$f_1(z) = \sum_1^{\infty} na_nz^{n-1} = P'(z) = \lim_{n\to\infty} s_n'(z)$$

Subclaim :

$$f'(z) = f_1(z) = \lim_{n\to\infty} s_n'(z)$$

$$\frac{f(z)-f(z_0)}{(z-z_0)} - f_1(z_0) = \frac{s_n(z)+R_n(z)-[s_n(z_0)-R_n(z_0)]}{(z-z_0)} - f_1(z_0)$$

$$= \frac{s_n(z)-s_n(z_0)}{(z-z_0)} + \frac{R_n(z)-R_n(z_0)}{(z-z_0)} - f_1(z_0)$$

$$= \left[\frac{s_n(z)-s_n(z_0)}{(z-z_0)} - s_n'(z_0)\right] + \left[\frac{R_n(z)-R_n(z_0)}{z-z_0}\right]$$

$$+[s_n'(z_0)-f_1(z_0)] \quad \text{------(4)}$$

where $z \neq z_0$; $|z|, |z_0|, |z_1| < \rho < R$

The second term is $\sum\limits_{k=n}^{\infty} a_k\left(z^{k-1}+z^{k-2}z_0+.....+zz_0^{k-2}+z_0^{k-1}\right)$

It follows that

$\left|\frac{R_n(z)-R_n(z_0)}{(z-z_0)}\right| \leq \sum\limits_{k=n}^{\infty} k|a_k|\rho^{k-1}$ which is the remainder of a convergent series.

$\therefore \exists\, n_0$ so that $\left|\frac{R_n(z)-R_n(z_0)}{(z-z_0)}\right| < \epsilon/3;\ \forall n \geq n_0$ ------(1)

Since $s_n'(z_0)\to f_1(z_0)$, there exists n_1 such that

$|s_n'(z_0)-f_1(z_0)| < \epsilon/3;\ \forall n \geq n_1$ ------(2)

By definition of derivative, we can find $\delta > 0$ such that $0 < |z-z_0| < \delta$ implies

$\left|\frac{s_n(z)-s_n(z_0)}{(z-z_0)} - s_n'(z_0)\right| < \epsilon/3$ ------(3)

From (1), (2), (3) & (4), we obtain

$\left|\frac{f(z)-f(z_0)}{(z-z_0)} - f_1(z_0)\right| < \epsilon/3+\epsilon/3+\epsilon/3 = \epsilon$ whenever $0 < |z-z_0| < \delta$.

Hence $f'(z_0) = f_1(z_0)$.

$\therefore$ The derivative can be obtained by terminoise differentiation.

A power series with positive radius of convergence has derivatives of all orders.

$$f(z) = a_0+a_1z+a_2z^n+.....$$

$$f'(z) = a_1+2a_2z+3a_3z^2+.....$$

$$f''(z) = 2a_2+6a_3z+12a_4z^2+......$$

•

•

$$f^{(k)}(z) = k!a_k + \frac{(k+1)!}{1!}a_{k+1}z + \frac{(k+2)!}{2!}a_{k+2}z^2 + ...$$

$$\therefore \quad a_k = \frac{f^{(k)}(0)}{k!};$$

Then $$f(z) = f(0)+f'(0)z+\frac{f''(0)}{2!}z^2+.....+\frac{f^{(n)}(0)}{n!}z^n+......$$

which is Taylor–Maclaurin development.

$\therefore$ f(z) has a power series development.

Every analytic function has a Taylor development and the sum of this series is analytic.

Theorem 7.3.3 : (Abel's Limit Theorem)

If $\sum_0^{\infty} a_n$ converges, then $f(z) = \sum_0^{\infty} a_n z^n$ tends to f(1) as z approaches 1 in such a way that $|1-z|/(1-|z|)$ remains bounded.

Proof :

Without loss of generality, $\sum_0^{\infty} a_n = 0$.

Let $$s_n = a_0+a_1+....+a_n$$

$$s_n(z) = a_0+a_1z+a_2z_2+......+a_nz_n$$

$$= s_0+(s_1-s_0)z+(s_2-s_1)z^2+....+(s_n-s_{n-1})z^n$$

$$= s_0(1-z)+s_1(z-z^2)+....+s_{n-1}(z^{n-1}-z^n)+s_nz^n$$

$$= (1-z)(s_0+s_1z+s_2z^2+....+s_{n-1}z^{n-1})+s_nz^n$$

By assumption, $s_nz^n \to 0$ as $n \to \infty$.

$$\therefore \quad f(z) = \sum_0^{\infty} a_n z^n = \sum_0^{\infty} s_n(z) = (1-z)\sum_0^{\infty} s_n z^n$$

Assume that $\quad |1-z| \le k(1-|z|)$

$s_n \to 0$ as $n \to \infty \Rightarrow \exists$ m such that $|s_n| < \epsilon$ for all $n \ge m$.

$$\therefore \quad \left|\sum_{n=m}^{\infty} s_n z^n\right| \le \epsilon \sum_{n=m}^{\infty} |z^n| = \epsilon \sum_{n=m}^{\infty} |z|^n$$

$$\le \frac{\epsilon |z|^m}{(1-|z|)} < \frac{\epsilon}{(1-|z|)}$$

$$\therefore \quad |f(z)| = \left|(1-z)\sum_0^{\infty} s_n z^n\right|$$

$$\le |(1-z)|\left|\sum_{k=0}^{m-1} s_k z^k\right| + k\epsilon$$

The first of RHS can be made arbitrary small since $z \to 1$.

Section 4 :

7.4.1. The Expontial & Trigonometric functions

The exponential function is the solution of the differential equation $f'(z)=f(z)$ with $f(0)=1$.

Suppose $\quad f(z) = a_0+a_1z+a_2z^2+.....+a_nz^n+....$

$$f'(z) = 0+a_1+2a_2z+3a_3z^2+....+na_nz^{n-1}+......$$

$$f'(z) = f(z) \Rightarrow na_n = a_{n-1};\ a_0 = 1$$

$$\therefore \quad a_n = 1/n! = 1/\angle n$$

Thus the solution is $e^z = 1+\frac{z}{\angle 1}+\frac{z^2}{\angle 2}+.....$ which is convergent series.

We know that $(n!)^{1/n} \to \infty$ as $n \to \infty$. We have also that e^z satisfies the following properties

$$e^{a+b} = e^a.e^b;\ \frac{d}{dz}(e^z) = e^z;$$

e^c = constant; $e^0 = 1$;

e^z is never zero; For real $x > 0$, $e^x > 1$. e^z and e^{-z} are reciprocals;

$0 < e^x < 1$ if $x < 0$; $|e^{iy}|=1$;

$|e^z| = |e^{x+iy}| = e^x$.

7.4.2. Trigonometric Functions :

They are defined by $\cos z = (e^{iz}+e^{-iz})/2$; $\sin z = (e^{iz}-e^{-iz})/2$;

$$\therefore \quad \cos z + i \sin z = e^{iz} = 1+\frac{iz}{\angle 1}+\frac{i^2z^2}{\angle 2}+\frac{i^3z^3}{\angle 3}+\ldots.$$

$$\Rightarrow \qquad \cos z = 1-\frac{z^2}{\angle 2}+\frac{z^4}{\angle 4}-\frac{z^6}{\angle 6}+\ldots\ldots.$$

$$\sin z = \frac{z}{\angle 1}-\frac{z^3}{\angle 3}+\frac{z^5}{\angle 5}-\frac{z^7}{\angle 7}+\ldots\ldots$$

which are Taylor development of cos z & sin z respectively.

Finally we get $\cos^2 z+\sin^2 z = 1$;

$$\frac{d}{dz}(\sin z) = \cos z; \quad \frac{d}{dz}(\cos z) = -(\sin z);$$

$$\cos(a+b) = (\cos a)(\cos b)+(\cos a)(\sin b)$$

$$\tan z = (-i)\frac{e^{iz}-e^{-iz}}{(e^{iz}+e^{-iz})};$$

The other trigonometric functions tan z, cot z, sec z and cosec z are found in the similar manner.

All the trigonometric functions are rational functions of (e^{iz}).

UNIT – VIII

SECTION–1 : ELEMENTARY POINT SET TOPOLOGY

Definition 8.1.1 :

A set $N \subset S$ is called a neighborhood of $y \in S$, if it contains a ball $B(y, \delta) = \{x \in S: d(x,y) < \delta\}$. Here `d' is a metric on S.

Definition 8.1.2 :

A set is open if it is a neighborhood of each of its elements.

Theorem 8.1.2 :

Every ball is open.

Proof :

Let $B(y, \delta)$ be a ball; and $z \in B(y, \delta)$.

Assume $\delta' = \delta - d(y, z)$ Then $\delta' > 0$

Let $x \in B(z, \delta') \Rightarrow d(x, z) < \delta'$.

$\therefore \quad d(x, z) < \delta' = \delta - d(y, z)$

$\Rightarrow \quad -d(x, y) - d(y, z) < \delta$

$\Rightarrow \quad d(x, y) \leq d(x, y) + d(y, z) < \delta$

$\therefore \quad x \in B(y, \delta)$

Hence $B(z, \delta') \leq B(y, \delta)$

$\therefore$ $B(y, \delta)$ is open.

Remark 8.1.3 :

In complex plane, $B(z_0, \delta)$ is an open disk with center z_0, and radius δ; It consists of all complex z such that $|z-z_0| < \delta$. Empty set and the whole space are both open and closed. We have the following properties :

1) The intersection of a finite number of open sets is open.

2) The union of any collection of open sets is open.

3) The union of a finite number of closed sets is closed.

4) The intersection of any collection of closed sets is closed.

Definition 8.1.4 :

The interior of a set A in X is the largest open set contain containing A, denoted by Int (A).

The closure of a subset A of X is the smallest closed set contained in A (or) the intersection of all closed sets (in X) contained in A, denoted by $\overline{A}$.

The boundary of A is $\overline{A}$ – Int(A). (ie) A point belongs to the boundary iff all its neighborhood intersect both A and ~A.

The exterior of A is the interior of ~A. (ie) It is $(\overline{A})^C$.

Definition 8.1.5 :

A is open iff A = Int (A) = A^0

A is closed iff A = $\overline{A}$

$X \subseteq Y \Rightarrow X^0 \subseteq Y^0$; $\overline{X} \subseteq \overline{Y}$

$x \in X$ is an isolated point of X if x has a neighborhood whose intersection with X reduces to the point x.

An accumulation point is a point of X, which is not isolated point. It follows that x is an accumulation point of X iff every neighborhood of x contains infinitely many points from X.

Definition 8.1.6 :

A subset of a metric space is connected if it cannot be represented as the union of two disjoint relatively open sets. None of which is empty.

Theorem 8.1.7 :

The non-empty connected subsets of the real line R are the intervals.

Proof :

Suppose R is disconnected space. Then $|R| = A \cup B$; $A \cap B = \{\}$; A & B are open. $A' = B$; $B' = A$; $\therefore$ A & B are both closed. If neither is empty, we choose $a_1 \in A$; $b_1 \in B$.

Assume that $a_1 < b_1$; we bisect the interval (a_1, b_1) such that one of the two halves has its left end point in A, and its right end point in B. We denote this new interval $(a_2\ b_2)$, and continue the process indefinitely.

We obtain a sequence of nested intervals $(a_n\ b_n)$ with $a_n \in A$; $b_n \in B$. Then two sequences $\{a_n\} \& \{b_n\}$ have a common limit c. Sine A & B are closed, then $c \in A$; $c \in B$

$\Rightarrow c \in A \cap B$ which is a contradiction. Otherwise either A or B is empty.

$\therefore$ R = A or B which are closed sets (so intervals).

Let E be a subset of R. Then (1) α is a lower bound of E if $\alpha \leq x$, $\forall x \in E$, (2) α is greatest lower bound of E if α is a lower bound, and it is the greatest among all lower bounds of E, (3) β is an upper bound of E if $x \leq \beta$, $\forall\ x \in E$, (4) β is least upper bound of E if it is an upper bound of E and it is the least among all upper bounds of E.

We assume that E is a connected subset of R. Let a & b be g.l.b. and l.u.b. of E respectively. Then all points of E lie between a and b.

Suppose that a point $\xi \in (a\ b) - E$. $\Rightarrow a < \xi < b$, and $\xi \notin E$.

The open sets defined by $x > \xi$ and and $x < \xi$ cover E which is connected. $\therefore$ one of them must fail to meet E. No point of E lies to the left of ξ. $\Rightarrow \xi$ is a lower bound, which is a g.l.b. of E. Hence $\xi \notin E$ is impossible $\Rightarrow \xi \in E$. It follows that E is either an open, closed, or semiclosed interval with end points a & b. The cases $a = -\infty$, and $b = \infty$ are included.

Theorem 8.18 :

Any closed and bounded non-empty set of real numbers has a maximum and a minimum.

Proof :

Let E be a closed, and bounded. It has g.l.b. and l.u.b. of E. Since E is closed, the above bounds are finite, and belonging to the set E. They are called minimum and maximum.

Theorem 8.1.9 :

A non-empty open set in the plane is connected iff any two of its points can be joined by a polygon which lies in the set.

Proof :

Case (i)

Let E be an open connected set, and choose a point $a \in A$. We denote by A_1 the subset of A whose points can be joined to a by polygons in A, and by A_2, the subset whose points cannot be so joined.

Claim :

Both A_1 & A_2 are open. Suppose $a_1 \in A_1 \Rightarrow a_1 \in A$ ($\because A_1 \leq A$); A is open. There is a neighborhood $N_1 : |z-a_1| < \in$ contained in A. All points in this neighborhood N_2 can be joined to a_1 by a line segment, and from there to a by a polygon. Therefore the whole neighborhood N_2 is contained in A_1. Thus A_1 is open.

Suppose $a_2 \in A_2 \Rightarrow a_2 \in A$. There is a neighborhood $N_2 : |z-a_2| < \in$ contained in A. If a point in N_2 can be joined to a by a polygon, then a_2 is joined to this point by a line segment, and from there to a, which is a contradication to the definition of A_2. We conclude that A_2 is open. Further A_1 & A_2 are two open subsets satisfying $A = A_1 \cup A_2$, $A_1 \cap A_2 = \{\}$. Since A is connected, it follows that either A_1 (or) A_2 must be empty. Since $a \in A_1$, then $A_2 = \{\}$. All points of A can be joined to `a'. (ie) any two points of A can be joined by way of a.

Case (ii) :

Assume that any two of its points can be joined by a polygon which lies in the set.

Calim :

Given open set A is connected. Suppose not. Then $A = A_1 \cup A_2$ as the union of two disjoint non-empty open sets. We choose $a_1 \in A_1$; $a_2 \in A_2$. Assume that a_1 & a_2 can be joined by a polygon in A.

Since one of the sides of the polygon must join a point in A_1 to a point in A_2, it is sufficient to consider that a_1 & a_2 are joined by a line segment, which has a parametric representation $z = a_1 + t(a_2 - a_1)$. Here t runs through the interval $0 \leq t \leq 1$. The subsets of the intervals $0 < t < 1$ (which correspond to points in A_1 and A_2, respectively) are open, disjoint and non-empty. This is a contradiction to the connectedness of the interval. Hence A is connected subset.

Theorem 8.1.10 :

Every set has a unique decomposition into components.

Proof :

A component is a maximal connected subset; A non-empty connected open set is called a region.

Let E be the given set, and $a \in E$. Assume that C(a) is the union of all connected subsets of E containing a. Since the single point a is connected, $a \in C(a)$.

Claim (1) :

C(a) is connected. Suppose not. Then C(a) is disconnected. Assume that $C(a)=A \cup B$; $A \cap B = \{\}$; $A \neq \{\}$; $B \neq \{\}$; A & B are open subsets of C(a); we choose $a \in A$; $b \subset B$. Thus $b \in A \cup B \Rightarrow b \in C(a)$. There exists a connected set $E_0 \subseteq E$ containing `a' & `b'.

We have also $E_0 = (E_0 \cap A) \cup (E_0 \cap B)$ which is a decomposition into relatively open subsets; $a \in E_0 \cap A$; $b \in E_0 \cap B$; They are disjoint. This is a contradiction. It follows that C(a) is connected. Then it is a maximal connected set (component) by definition.

Claim (2) :

Any two components are either disjoint or idential.

Let $c \in C(a) \cap C(b) \Rightarrow c \in C(a); C \in C(b)$.

By the definition of C(c), and the connectedness of C(a), $C(a) \subseteq C(c)$.

Since $a \in C(a)$, $a \in C(c)$. Thus $C(c) \subseteq C(a)$. Hence $C(a) = C(c)$.

The set E has only one decomposition into components.

Theorem 8.1.11 :

In R^n, the components of any open set are open.

Proof :

Let E by open in R^n, and C(a) be a component (subset) of E. Then $a \in C(a)$; $C(a) \subseteq E \Rightarrow a \in E$. But E is open. It contains a ball $B(a, \delta)$. Every ball is δ-neighbor-hood as well as connected. Thus $B(a, \delta)$ is connect subset of E containing `a'. Therefore $B(a,\delta) \subseteq E$. Hence C(a) is open in E.

Remark 8.1.12 :

(1) A space is called a locally connected, if any neighborhood of a point contains a connected neighborhood of the point. (2) Every open subset of R is a countable union of disjoint open intervals. (3) In a locally connected separable space, every open set is a countable union of disjoint regions.

Definition 8.1.13 : (Complete)

A metric space is called complete if every Cauchy sequence is convergent.

Then (1) A complete subset of a metric space is closed. (2) a closed subset of a complete space is complete.

Definition 8.1.14 :

(1) An open covering of a set of X is a collection of open sets in X such that X is contained in the union of the open sets. (2) A subcovering is a subcollection with the same (above) property. (3) A finite covering is an open covering consisting of a finite number of sets.

Definition 8.1.15 :

A set X is compact iff every open covering of X contains a finite subcovering.

Definition 8.1.16 :

A set X is totally bounded if $\forall \in > 0$, X can be covered by finitely many balls of radius $\in$.

(ie) $X \subset B(x, \in) \cup \ldots \cup B(x_n, \in)$ for some $x_1, x_2, \ldots, x_n \in X$.

Theorem 8.1.17 :

Every totally bounded set is bounded.

Proof :

Let X be totally bounded. The the collection of all balls of radius $\in$ is an open covering. But since X is totally bounded, X is covered by a finite number of open balls of radius $\in$.

(i.e.) $X \subseteq B(x_1, \in) \cup \ldots \cup B(x_n; \in)$

Let x, y, $\in$ X. Then $x \in B(x_i, \in)$; $y \in B(x_j, \in)$ for some i, $j \in \{1,2,......,n\}$

Thus $d(x_i, x) < \in$; $d(y, x_j) < \in$.

$$\therefore \quad d(x, y) \le d(x, x_i)+d(x_i, x_j)+d(x_j, y)$$

$$< \in+d(x_i, y_i)+\in$$

$$< 2\in+\max\{d(x_i, x_j)\}; \quad \forall i, j \in \{1,2,3,......,n\}$$

Hence X is bounded.

Theorem 8.1.18 :

A set X is compact iff it is complete and totally bounded.

Proof :

Case (i)

Suppose X is compact.

Claim (1) :

X is complete.

Let $\{x_n\}$ be a Cauchy sequence in X. Assume thai $\{x_n\}$ is not convergent. Then every point $y \in X$ is not a limit of $\{x_n\}$.

Let $y \in X$. Then y is not a limit point of $\{x_n\}$. There exists an $\in > 0$ such that $d(x_n, y) > 2\in$ for infinitely many n. (ie) The open ball $B(y, 2\in)$ contains only finitely many x_n.

We know that $\{x_n\}$ is a Cauchy sequence.

$\therefore$ Choose n_0 such that $d(x_m, x_n) < \in$, $\forall$ m, $n \ge n_0$. we choose again a fixed $n \ge n_0$ for which $d(x_n, y) > 2\in$.

$$\text{Then} \quad d(x_n, y) \le d(x_n, x_m)+d(x_m, y)$$

$$\therefore \quad d(x_m, y) \ge d(x_n, y)-d(x_m, x_n)$$

$$> 2\in-\in = \in$$

Thus $\in$-neighborhood $B(y, \in)$ contains also only finitely many x_n -----(*)

We now consider the collection of all open sets U that contain only finitely many x_n. Since (*) holds for each $\in > 0$, and for each point $y \in X$, this collection is an open covering of X.

X is compact implies that this (collection) cover contains a finite subcovering, namely $U_1, U_2, \ldots, U_N$. Note that each U_i contains only finitely many x_n; $\forall$ i=1,2,....,N. Therefore $\{x_n\}$ is finite, which is a contradiction.

Hence $\{x_n\}$ is convergent and so X is complete.

Claim (2) :

X is totally bounded. The collection of all balls of radius $\in$ is an open covering. By the compactness, this collection contains a finite subcovering. (ie) $X \subset B(x_1, \in) \cup \ldots \cup B(x_m, \in)$ for some points $x_1, x_2, \ldots, x_m \in X$.

$\therefore$ X is totally bounded.

Case (ii) :

X is complete and totally bounded.

Claim (3) :

X is compact. Suppose not. Therefore there exists an open covering which does not contain any finite subcovering. Fix $\in_n = 1/2^n$.

We know that X is covered by finitely many $B(x, \in_1)$. If each had a finite subcovering, the same holds in X. Otherwise there is a $B(x_1, \in_1)$ that does not admit a finite subcovering. Since $B(x_1, \in_1)$ is itself totally bounded, we choose an $x_2 \in B(x_1, \in_1)$ such that $B(x_2, \in_2)$ has no finite subcovering. Every subset of totally bounded is again totally bounded. If we continue our process, we obtain a sequence x_n with the property that $B(x_n, \in_n)$ has no finite subcovering and $x_{n+1} \in B(x_n, \in_n)$. Thus $d(x_n, x_{n+1}) < \in_n$.

$\therefore d(x_n, x_{n+p}) < \in_n + \in_{n+1} + \ldots + \in_{n+p-1} < 1/2^{n-1}$

$\Rightarrow \{x_n\}$ is a Cauchy sequence in X.

But X is complete. Thus $\{x_n\}$ converges to a limit y, and y belongs to one of the open sets U in this covering. U is open; $y \in U \Rightarrow$ it contains a ball $B(y, \delta)$; Choose n so that $d(x_n, y) < \delta/2$ and $\in_n < \delta/2$. Then $B(x_n, \in_n) \subset B(y, \delta)$.

[For this, $x \in B(x_n, \epsilon_n) \Rightarrow d(x, x_n) < \epsilon_n$

$\Rightarrow d(x, y) \leq d(x, x_n)+d(x_n, y) < \epsilon_n + \delta/2 < (\delta/2)+(\delta/2)=\delta$]

$\therefore$ Thus $B(x_n, \epsilon_n)$ admits a finite subcovering, namely U itself, which is a contraction.

Hence X is compact.

Corollary 8.1.19 :

A subset of R or C is compact iff it is closedand bounded.

We know that R or C is complete; complete subset of a complete space is closed. Every bounded subset in R or C is totally bounded. For this, if $X \subseteq$ R or C is bounded, then it is countained in a disk, and so in a square, which can be divided into a finite number of squares with arbitrarily small side, and the squares can then be covered by disks with arbitarily small radius. Therefore X is totally bounded.

Theorem 8.1.20 : (Bolzano-Weierstrass)

A metric space is compact iff every infinite sequence has a limit point.

Proof :

Let X be compact, and $\{x_n\}$ a sequence in X.

Suppose $\{x_n\}$ does not converge. No point $y \in X$ is a limit of $\{x_n\}$. Then y has a neighborhood that contains only finitely many x_n. Therefore for each point $y \in X$, there is such open set. All these neighborhoods is an open cover for X, which is compact by assumption.

This open cover has a finite subcover in which each open set contains only finitely many x_n. Thus the sequence $\{x_n\}$ is finite, which is a contradiction. $\therefore$ $\{x_n\}$ converges, and every infinite set has a limit.

Case (ii) :

Every infinite set in X has a limit.

Claim :

X is compact. By assumption, every Cauchy sequence has a limit (so is convergent).

$\therefore$ X is complete.

To prove totally bounded, we assume that X is not totally bounded. Then there is an $\in > 0$ such that the space X cannot be covered by finitely many $\in$-neighborhoods.

We construct a sequence $\{x_n\}$ as follows : Let x_1 be an arbitrary point of X. After selecting $x_1, x_2, \ldots\ldots\ldots, x_n \in X$, we choose $x_{n+1} \in X$ so that it does not lie in $B(x_1, \in) \cup \ldots \cup B(x_n, \in)$. This is always possible because these neighborhoods do not cover the whole space X. But it is clear that $\{x_n\}$ has no convergent subsequence ($\because d(x_m, x_n) > \in$, for all m & n) which is a contradiction.

$\therefore$ X is totally bounded, and it is compact.

Definition 8.1.21 : (Continuous)

f is continuous at a point $a \in X$ if, $\forall \in > 0$, there is a $\delta > 0$ such that $d(x, a) < \delta$ implies $d'(f(x), f(a)) < \in$.

Definition 8.1.22 :

A function $f : X \to Y$ is continuous if the inverse of every open set in Y, is open in X.

Thus f is continuous from X into Y if the inverse of each closed set in Y, is closed in X.

Theorem 8.1.23 :

Under a continuous map, the image of a compact set is compact (so closed).

Proof :

Suppose that $f : X \to Y$ is a continuous map, and X is compact.

Claim :

f(X) is compact. Clearly $f(X) \subseteq Y$.

Consider an open covering T_1 of f(X) by open sets U. since f is continuous, $f^{-1}(U)$ is open in Y, for every open set U in the cover. Thus $\{f^{-1}(U)\}$ is an open cover T for X. But X is compact. This implies that there exist $U_1, \ldots.. U_m$ of the open sets in T_1 such that

$$X \subseteq f^{-1}(U_1) \cup \ldots \cup f^{-1}(U_m)$$

$\Rightarrow f(X) \subseteq U_1 \cup \ldots \cup U_m$

$\therefore$ T_1 has finite subcover, and f(X) is compact.

Corollary 8.1.24 :

A continuous real-valued function on a compact set has a maximum and a minimum.

Proof :

The image of the given compact set is compact in R. Thus the image is closed and bounded.

Theorem 8.1.25 :

Under a continuous map, the image of a connected set is connected.

Proof :

Assume that $f : X \to Y$ is a continuous map; X is connected.

Claim :

f(X) is connected. Suppose f(X) is disconnected. Then $f(X) = A \cup B$, where A & B are non-empty, disjoint, open sets in f(X).

Since f is continuous, then $f^{-1}(A)$ and $f^{-1}(B)$ are open sets in X.

$$A \cap B = \{\} \Rightarrow f^{-1}(A) \cap f^{-1}(B) = \{\}$$

$$A, B \neq \{\} \Rightarrow f^{-1}(A), f^{-1}(B) \neq \{\}$$

Further $\quad X = f^{-1}(A) \cup f^{-1}(B)$

$\Rightarrow$ X is disconnected which is a contradiction.

Hence f(X) is connected in Y.

Theorem 8.1.26 :

If $f : X \to Y$ is continuous, and X is compact, then f is uniformly continuous.

Proof :

Given that $f : X \to Y$ is continuous; X is compact; For every $y \in X$, f is continuous at y. $\forall \epsilon > 0$, there is a ball $B(y, \rho)$ such that $d'(f(x), f(y)) < \epsilon/2$, for $x \in B(y, \rho)$

((ie) $d(x, y) < \rho$; $\forall\, y \in X$)

By continuity of f, ρ depends on y.

Consider the open covering of X by the smaller balls $B(y, \rho/2)$. X is compact $\Rightarrow$ This cover has a finite subcovering such that

$X \subset B(y_1, \rho_1/2) \cup B(y_2, \rho_2/2) \cup \ldots \cup B(y_m, \rho_m/2)$

Put $\quad \delta = \min\{\rho_1/2, \rho_2/2, \ldots\ldots, \rho_m/2\}$

Fix $x, y \in X$ such that $d(x, y) < \delta$. $x \in X \Rightarrow x \in B(y_k, \rho_k/2)$ for some k.

$$\Rightarrow d(x, y_k) < \rho_k/2$$

$$\therefore \quad d(y, y_k) \leq d(x, y) + d(x, y_k)$$

$$< \delta + \rho_k/2 \leq \delta k$$

$$\therefore \quad d'(f(x), f(y_k)) < \epsilon/2, \text{ and so}$$

$$d'(f(y), f(y_k)) < \epsilon/2$$

$$\Rightarrow \quad d'(f(x), f(y)) < \epsilon.$$

ϵ does not depend on x & y. Hence f is uniformly continuous.

SECTION 2 : CONFORMALITY

Definition.8.2.1 :

The equation of an arc γ in a plane is in parametric form $x=x(t)$, $y=y(t)$, where t runs through an interval $\alpha \leq t \leq \beta$, and $x(t)$, $y(t)$ are both continuous.

The initial and terminal point of an arc remain the same after a change of parameter. The curve $z(t)=x(t)+iy(t)$ has the derivative $z'(t)=x'(t)+iy'(t)$ (exists and $\neq 0$). Then curve has a tangent whose direction is determined by $\arg z'(t)$. It is differentiable if $z'(t)$ exists and is continuous. If $z'(t) \neq 0$, the arc is said to be regular.

Definition 8.2.2 :

A complex-valued function $f(z)$, defined on open set Ω is called analytic (holomorphic) in Ω if it has a derivative at each point of Ω.

A function f(z) is analytic on an arbitrary point set A if it is the restriction to A of a function that is analytic in some open set containing A.

Theorem 8.2.3 :

An analytic function in a region Ω whose derivative vanishes identically, is constant. The same holds if either the real part, the imaginary part, the modulus or the argument is constant.

Proof :

Let f(z) be an analytic function, such that its derivative vanishes.

$$\therefore \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x} \text{ and } \frac{\partial v}{\partial y} \text{ are all zero.}$$

$$0 = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \Rightarrow u \text{ is constant}$$

$$0 = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow v \text{ is constant}$$

We conclude that u+iv is constant. If u or v is constant,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 0 \quad \text{or}$$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = 0$$

$\therefore$ f(z) is constant.

If modulus is constant, then $u^2+v^2=0$

$$\text{Differenting, } 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$$\Rightarrow u.u_x + v.v_x = 0$$

$$\text{Similarly} \quad 0 = u.u_y + v.v_y$$

$$= -u.v_x + v.u_x$$

We conclude that $\frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial x}$.

[Otherwise $u^2+v^2 = 0 \Rightarrow f(z) \neq 0$ for some z]

Therefore f(z) is constant.

If arg(f(z)) is constant, then u/v = constant

$\Rightarrow$ u = kv with constant k (unless v=0)

$\Rightarrow$ u–kv=0, which is real part of (1+ik)f(z)

Therefore (1+ik)f(z) is constant

$\Rightarrow$ f(z) is constant.

Linear Transformation :

It is a function of the form $w = S(z) = (az+b)/(cz+d)$, with $ad-bc \neq 0$ having an inverse.

$z = S^{-1}(w) = (dw-b)/(-cw+a)$

Note that $S(\infty) = a/c$; $S(-d/c) = \infty$.

Definition 8.2.4 :

The cross ratio (z_1, z_2, z_3, z_4) of four complex numbers z_1, z_2, z_3 & z_4 is defined as the image of z_1 under the linear transformation, which carries from z_2, z_3, z_4 into $1, 0, \infty$.

Remark 8.2.5 :

Given three distinct points z_2, z_3, z_4 in the extended plane, there exists a linear transformation S that carries them into $1, 0, \infty$ in this order.

If none of points is ∞, then S is given by $S(z) = \left(\dfrac{z-z_3}{z-z_4}\right)\left(\dfrac{z_2-z_4}{z_2-z_3}\right)$.

If z_2, z_3 or z_4 is ∞, the transformation reduces to $\left(\dfrac{z-z_3}{z-z_4}\right), \left(\dfrac{z_2-z_4}{z-z_4}\right), \left(\dfrac{z-z_3}{z_2-z_3}\right)$ respectively. If T is another linear transformation with the same property, then ST^{-1} leave $1, 0, \infty$ as invariant. This holds for identity transformation. $\therefore$ S = T, and we conclude that S is uniquely determined.

Theorem 8.2.6 :

If z_1, z_2, z_3, z_4 are distinct points in the extended plane, and T any linear transformation, then $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.

Proof :

Let $S(z) = (z, z_2, z_3, z_4)$

Then $S.T^{-1}$ takes Tz_2; Tz_3, Tz_4 into $1, 0, \infty$.

By definition, $(Tz_1, Tz_2, Tz_3, Tz_4) = (ST^{-1})(Tz_1) = S(z_1) = (z_1, z_2, z_3, z_4)$

Theorem 8.2.7 :

The cross ratio (z_1, z_2, z_3, z_4) is real iff the four points lie on a circle or on a straight line.

Proof :

By elementary geometry, $\arg(z_1, z_2, z_3, z_4)$.

$$= \arg\left(\frac{z_1 - z_3}{z_1 - z_4}\right) - \arg\left(\frac{z_2 - z_3}{z_2 - z_4}\right)$$

If the points lie on a circle, then this difference of angles is either 0 (or) $\pm \pi$, depending on the relative location.

For another proof, we need only show that the image of the real axis under any linear transformation is either a circle or a striaght line. Infact, $Tz = (z, z_2, z_3, z_4)$ is real on the image of the real axis under the transformation T^{-1}, and nowhere else.

The values of $w=T^{-1}(z)$ for real z satisfy the equation $T(w)=\overline{T(w)}$. This condition is of the form

$$\frac{aw+b}{cw+d} = \frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}.$$

By cross multiplication, we obtain

$(a\bar{c}-c\bar{a})|w^2|+(a\bar{d}-c\bar{b})w+(b\bar{c}-d\bar{a})\bar{w}+(b\bar{d}-d\bar{b})=0$

If $a\bar{c}-c\bar{a} \neq 0$, we divide by this coefficient and complete the square.

After simplication, we get

$$\left|w+\left(\frac{\bar{a}d-\bar{c}b}{\bar{a}c-\bar{c}a}\right)\right| = \left|\frac{ad-bc}{\bar{a}c-\bar{c}a}\right|$$

which is the equation of a circle.

Corollarly 18.2. 8 :

A linear transformation carries circles into circles.

8.2.9 Definition (Symmetry) :

The points z and z* are called symmetric with respect to the circle C passing through z_1, z_2, z_3 iff $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$

The points on C, and only those are symmetric to themselves. The map that takes z into z*, is a one-to-one correspondence, and is called reflection with respect to C.

We wish to check the geometric significance of symmetry.

Assume that C is a striaght line.

If $z_3 = \alpha$, symmetric requirement becomes

$$\frac{(z^*-z_2)}{(z_1-z_2)} = \frac{(\overline{z}-\overline{z_2})}{(\overline{z_1}-\overline{z_2})}$$

$\Rightarrow |z^*-z_2| = |z-z_2|$. z_2 is a point on C and so we conclude that z & z* are equidistant from all points on C.

Also $$\text{lm}\left(\frac{z^*-z_2}{z_1-z_2}\right) = -\text{lm}\left(\frac{z-z_2}{z_1-z_2}\right)$$

$\therefore$ z & z* are in different half-plane determined by C.

Theorem 8.2.10 : (The Symmetric Principle)

If a linear transformation carries a circle C_1 into a circle C_2, then it transforms any pair of symmetric points with respect to C_1 into a pair of symmetric points with respect to C_2.

Proof :

We know that a linear transformation preserves symmetry.

If C_1 or C_2 is the real axis, the principle follows from the definition of symmetry. In the general case, the conclusion follows by use of a linear transformation that takes C_1 into the real axis.

The principle of symmetry is useful to identify, and find the linear transformations which carry a circle C into a circle C'. There is a linear transformation carring z_1, z_2, z_3

on C into w_1, w_2, w_3 on C'. The transformation is given by $(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$. Then the transformation is determined, if (i) point z_1 on C maps to a point w_1 on C' (ii) point z_2 not on C maps to a point w_2 not on C'. Thus z_2^* maps w_2. Hence the transformation is obtained from the equation $(w, w_1, w_2, w_2^*) = (z, z_1, z_2, z_2^*)$.

8.2. 11 Elementary Conformal Mapping :

Suppose $f(z) = w$ is an analytic function. We investigate all informations about the specific geometric properties of the map. If we wish to study the image of curves of the striaght lines $x = x_0$ and $y = y_0$, we concentrate the following :

If $f(z) = u(x, y)+i(x, y)$, the image of $x = x_0$ under $f(z)$ is given by the parametric equation $u = u(x_0, y)$; $v = v(x_0, y)$; y acts as a parameter, and can be eliminated.

The image of $y = y_0$ is determined in the same way (ie) the parametric equation $u = u(x, y_0)$; $v(x, y_0)$; x acts as a parameter, and can be eliminated.

The curves form an orthogonal net in the w-plane, and consider the curves in z-plane as $u(x, y) = u_0$ & $v(x, y) = v_0$. They are orthogonal and called the level curves of u & v.

If the consider a linear transformation $w = z^\alpha$ (α is real), then $|w| = |z^\alpha| = \alpha|z|$ and $\arg(w) = \alpha \arg(z)$.

This transformation carries concentric circles about the origin into circles of the same family: half lines from the origin correspond to other half lines. The map is unformal at all points $z \neq 0$, but an angle θ at the origin is transformed into an angle $(\alpha\theta)$. For $\alpha \neq 1$, the transformation of the whole plane is not one-to-one, and if α is fractional, z^α is not even single-valued. It maps one sector into another.

Example 8.2.12 :

Consider a map $w = z^2$. The $u+iv = (x+iy)^2 = (x^2-y^2)+i(2xy)$

$\therefore u = x^2-y^2$; $v = 2xy$

We recognize that the level curves $u = u_0$; $v = v_0$ are equilateral hyperbolas with the diagonal to each other.

The image of $x = x_0$ is $v^2 = 4x_0^2(n_0^2-u)$.

The image of $y = y_0$ is $v^2 = 4y_0^2(y_0^2+4)$.

Both families represent parabolas with the focus at the origin whose axes are pointed in the negative and positive direction of U-axis. The images curves are given by $|w-1| = k$ in the w-plane. (ie) $(x^2+y^2) = 2(x^2-y^2)+(k^2-1)$, and represents a family of lemniscates with the focus points ± 1.

The orthogonal family is represented by $x^2-y^2 = 2hxy+1$, and consists of all equilateral hyperbolas with center at the origin, which pass through the points ± 1.

SECTION–1 : FUNDAMENTAL THEOREMS

An indefinite integral is a function whose derivative equals a given analytic function in a region. In many cases, they can be obtained by inversion of known derivation formulas. The definite integrals are taken over differentiable or piecewise differentiable arcs, and are not limited to analytic functions.

Definition 9.1.1 :

If f(t) = u(t) + iv(t) is a continuous function defined in an interval (a, b), then we define

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

This integral has similiar properties of the real integral.

(i) If $c = \alpha+i\beta$ is complex number, than

$$\int_a^b (cf)dt = c\int_a^b (f)dt$$

(ii) $$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|.dt \quad \text{for any complex function } f(t).$$

(iii) $$\text{Re}\left[e^{-i\theta}\int_a^b f(t)dt\right] = \int_a^b \text{Re}\left(e^{-i\theta}f(t)\right)dt$$

$$\leq \int_a^b |f(t)|dt$$

(iv) If f(z) is defined and continuous on a curve γ which is a piecewise differentiable arc γ with the equation z = z(t); $a \leq t \leq b$, then f (z(t)) continuous, and

$$\int_\gamma f(z)dz = \int_a^b f(z(t)).z'(t).dt$$

This is the definition of the complex line integral of f(z) extended over the arc γ.

(v) A change of parameter is identified by an increasing function t = t(T) that maps an interval $\alpha \leq T \leq \beta$ onto $\alpha \leq t \leq \beta$. We assume that t(T) is piecewise differentiable. Then

$$\int_a^b f(z(t)).z'(t).dt = \int_a^b f(z(t(T))).z'(t(T)).t'(T)dt$$

(vi) If r is z = z(–t) with $-b \leq t \leq -a$, then

$$\int_{-\gamma} f(z)dz = \int_{(-a)}^{(-b)} f(z(t))z'(t).dt$$

$$\left(\text{(ie)} \int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz\right)$$

(vii) $$\int_{\gamma_1+\gamma_2+\ldots..+\gamma_n} (f)dz = \int_{\gamma_1} (fd_2) + \ldots... + \int_{\gamma_n} (fd_2)$$

(viii) $$\int_{\gamma} \overline{f}(z).dz = \int_{\gamma} f.\overline{dz}$$

(ix) $$\int_{\gamma} fdx = \frac{1}{2}\left(\int_{\gamma} fdz + \int_{\gamma} f\overline{dz}\right)$$

$$\int_{\gamma} fdz = \frac{1}{2i}\left(\int_{\gamma} fdz - \int_{\gamma} f\overline{dz}\right)$$

(x) If f = u+iv; z = x+iy, then

$$\int_{\gamma} f(z).dz = \int_{\gamma} (u.dx - v.dy) + i\int_{\gamma} (u.dy + v.dx)$$

(xi) $$\int_{\gamma} (fdz) = \int_{\gamma} f.|dz| = \int_{\gamma} f(z(t)).|z'(t)|.dt$$

where s is arc length.

Definition 9.1.2 :

The length of an arc is defined as the least upper bound of all sums $|z(t_1)-z(t_0)| + |z(t_2)-z(t_1)| + \ldots.. + |z(t_n)|-z(t_{n-1})|$ where $a = t_0 < t_1 < t_2 < \ldots... < t_n = b$

If this least upper bound is finite, then the arc is rectifiable.

Theorem 9.1.3 :

The line integral $\int_{\gamma} p\,dx + q\,dy$, defined in Ω depends only on the end points of γ iff there exists a function U(x, y) in Ω with the partial derivatives $\frac{\partial u}{\partial x} = p$; $\frac{\partial u}{\partial y} = q$.

Proof :

If sufficient condition holds, then

$$\int_{\gamma}(p\,dx + q\,dy) = \int_a^b \left[\frac{\partial u}{\partial x}.x'(t) + \frac{\partial u}{\partial y}.y'(t)\right]dt$$

$$= \int_a^b \frac{d}{dt}[u(x(t), y(t))].dt$$

$$= [u(x(t), y(t))]_a^b$$

$$= u(x(b), y(b)) - u(x(a), y(a))$$

and the integral value depends only on the end points.

To verify the necessary condition, we fix a point $(x_0, y_0) \in \Omega$, joining it to (x, y) by a polygon $r \subseteq \Omega$ whose sides are parallel to the coordinate axes.

Define a function $u(x, y) = \int_{\gamma} p\,dx + q\,dy$.

Since the integral depends only on the end points, the function is well defined. If we select a horizontal line segment in γ, then x varies, and keeping y as constant, and we obtain.

$$v(x, y) = \int^x p(x, y)dx + \text{constant}$$ where the lower limit is irrelevant.

$$\therefore \quad \frac{\partial u}{\partial x} = p$$

In the similar way, $\frac{\partial u}{\partial y} = q$ if we select vertical line segment.

Note 9.1.4 :

$\int_{\gamma}(f\,dz)$ with continuous function f depends on the end points of γ iff f is the derivative of an analytic function in Ω.

If $n \geq 0$, we find $\int_{\gamma} (z-a)^n dz = 0$ -----(1) for all closed curves γ. In fact, $(z-a)^n$ is the derivative of $(z-a)^{n+1}/(n+1)$, which is analytic function in the whole plane. If n is negative, and $\neq -1$ the same result holds for all closed curves that do not pass through $z = a$. In the complemetary region of the point a, the indefinite integral is also analytic, and single-valued.

For $n = -1$, (1) does not hold.

Consider a circle C with the center a written in the equation $z = a+ie^{it}$; $0 \leq t \leq 2\pi$.

Then we obtain $\int_{c} \frac{dz}{(z-a)} = \int_{0}^{2\pi} i.dt = (2\pi i)$. This conclusion gives that it is impossible to define a single-valued branch of $\log(z-a)$ in an annulus $\rho_1 < |z-a| < \rho_2$.

On the other hand, if the closed curve γ is contained in a half plane that does not contain `a', the integral vanishes, for in such a plane a single-valued and analytic branch of $\log(z-a)$ can be defined.

Definition 9.15 :

A rectangle R is defir ed b; inequalities $a \leq x \leq b$; $c \leq y \leq d$.

Theorem 9.1.6 : (Cauchy theorem for Rectangle)

If the function f(z) is analytic on a rectangle R, then $\int_{\partial R} f(z)dz = 0$ ------(1).

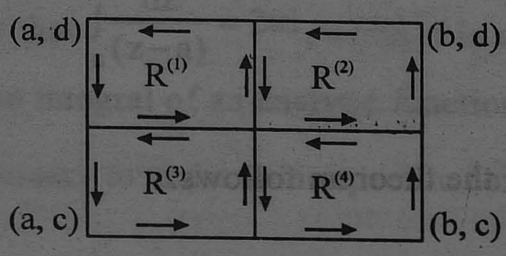
Proof :

∂R is the boundary curve or contour of the given rectangule R.

Given that f is analytic on R. Let $\eta(R) = \int_{\partial R} f(z)dz$.

If R is divided into four angruent rectangules $R^{(1)}$, $R^{(2)}$, $R^{(3)}$ & $R^{(4)}$, we may know that $\eta(R) = \eta(R^{(1)})+\eta(R^{(2)})+\eta(R^{(3)})+\eta(R^{(4)})$ --------(2)

where the integrals over the common sides cancel each other.



• Bisection of rectangle

From (2), at least one of the rectangles $R^{(1)}$, $R^{(2)}$, $R^{(3)}$ & $R^{(4)}$ satisfies the condition

$$|\eta(R^{(k)})| \geq \frac{1}{4}|\eta(R)|$$

We denote this rectangle by R_1. This procedure can be repeated indefinitely. Finally we get a sequence of nested rectangles $R \supset R_1 \supset R_2 \supset \ldots. \supset R_n \ldots$ with the condition $|\eta(R_n)| \geq \frac{1}{4}|\eta(R_{n-1})|$

$$\therefore \qquad |\eta(R_n)| \geq \frac{1}{4^n}|\eta(R)| \qquad \text{------(3)}$$

The rectangles R_n converges to a point $z^* \in R$ in such a way that R_N is contained in a neighborhood $|z-z^*| < \delta$ for a large n.

Choose δ so small such that f(z) is defined and analytic function in $|z-z^*| < \delta$.

$\therefore\ \forall \in > 0, \exists\ \delta > 0$ such that $\left|\frac{f(z)-f(z^*)}{(z-z^*)} - f'(z^*)\right| < \in$ whenever $0 < |z-z^*| < \delta$.

(ie) $|f(z)-f(z^*)-(z-z^*)f'(z^*)| < \in|z-z^*|$ Whenever $0 < |z-z^*| < \delta$. -------(4)

We consider that δ satisfies both conditions and that R_n is contained in $|z-z^*| < \delta$. Then $\int_{\partial R_n} dz = 0$; $\int_{\partial R_n} (z\, dz) = 0$ which are trivial.

Thus $\qquad \eta(R_n) = \int_{\partial R_n} |f(z)-f(z^*)-(z-z^*)f'(z^*)|dz$

By (4), $\qquad |\eta(R_n)| \leq \in \int_{\partial R_n} |z-z^*|.|dz|$

In the above integral, $|z-z^*|$ is at most equal to the length d_n of the diagonal of R_n. 'L_n' denotes the length of the perimeter of R_n.

$$\therefore \qquad |\eta(R_n)| \leq \in d_n.L_n.$$

If d & L are the corresponding quantities for the original rectangle R, then $d_n=2^{-n}d$; $L_n=2^{-n}L$, and so

$$|\eta(R)| \leq \in d.L.$$

Since $\in$ is arbitrary, $\eta(R_n)=0$, and the theorem follows.

Theorem 9.1.7 :

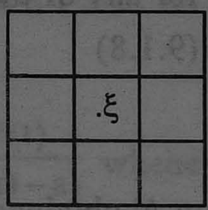
Let f(z) be analytic function on the set R' obtained from a rectangle R by omitting a finite number of interior points ξ_j. If $\lim_{z \to \xi_j} (z-\xi_j)f(z) = 0$ for all j, then $\int_{\partial R} f(z)dz = 0$.

Proof :

It is sufficient to consider the case of a single exceptional point ξ (For this, R can be divided into smaller rectangules containing at most one ξ_j).

We divide now R into nine rectangles and apply the previous theorem to all but the rectangle R_0 in the center.

Hence $\int_{\partial R} (f\,dz) = \int_{\partial R_0} (f\,dz)$ ------(1)



(Since other integral vanishes by the previous resuit).

If $\in > 0$, we choose the rectangle R_0 so small that $|f(z)| \leq \in/(|z-\xi|)$ on ∂R_0.

By (1), $\left| \int_{\partial R} (f\,dz) \right| \leq \in \int_{\partial R_0} \frac{|dz|}{|z-\xi|}$ ------(2)

Since R_0 is a square of center ξ, and (2), we obtain

$$\left| \int_{\partial R} (f\,dz) \right| \leq \in(8)$$

Since $\in$ is arbitrary, the theorem is verified.

We have that $\int_c \frac{dz}{(z-a)} = 2\pi i$, where c is a circle with center `a'. (ie) It does not hold always that the integral of an analytic function over a closed curve, is zero.

It is necessary to make some specific conditions in the given region.

Theorem 9.1.8 : (Cauchy theorem for disk)

If f(z) is analytic in an open disk Δ, then $\int_\gamma f(z)dz = 0$ ------(*) for every closed curve γ in Δ.

Proof :

We define a function f(z) by $F(z) = \int_\sigma (f\,dz)$, where σ consists of the horizontal line segment from the center (x_0, y_0) to (x, y_0), and the veritcal segment from (x, y_0) to (x, y). Thus $\partial F/\partial y = i\, f(z)$; By (9.1.7); σ can be replaced by a path consisting of a vertical segment followed by a horizontal segment. $\therefore$ We get $\partial F/\partial x = f(z)$. Thus F(z) is analytic in Δ with derivative f(z), and f(z)dz is an exact differential.

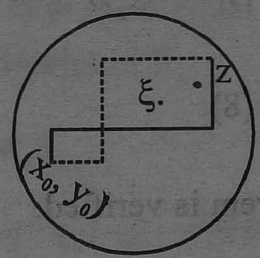
The same proof follows from any region that contains the rectangle with the opposite vertices z_0 & z. A rectangle, a half plane; or the inside of an ellipse all have the property, and the theorem holds for any of these region. The conclusion is followed under weaker conditions of theorem (9.1.8)

Theorem 9.1.9:

Let f(z) by analytic in the region Δ' obtained by omitting a finite number of points ξ_j from an open disk Δ. If f(z) satisfies the condition $\lim_{z \to \xi_j} (z-\xi_j)f(z) = 0$ for all j, then (*) in (9.1.8) holds for any closed curve γ in Δ'.

Proof :

Assume that number ξ_j lies on the lines $x = x_0$ & $y = y_0$.



It is possible to avoid the exceptional points by letting σ, consisting of three segments. By (9.1.7), the value of F(z) in (*) (9.1.8) is independent of the choice of the middle segment. The last segment may be either vertical (or) horizontal. It follows that F(z) is an indefinite integral of f(z), and the theorem is showed.

SECTION–2 : CAUCHY'S INTEGRAL FORMULA

Lemma 9.2.1 :

If the piecewise differentiable closed curve γ does not pass through the point a, then the value of the integral $\int_\gamma \frac{dz}{(z-a)}$ is a multiple of $(2\pi i)$.

Proof :

$$\int_\gamma \frac{dz}{(z-a)} = \int_\gamma d(\log(z-a))$$

$$= \int_\gamma d(\log|z-a| + i\arg(z-a))$$

$$= \int_\gamma d(\log|z-a|) + i\int_\gamma d(\arg(z-a))$$

when z describes a closed curve, log |z–a| returns to its initial value, and arg (z–a) increases or decreases by a multiple of "2π".

If the equation of r is $z = z(t)$, $\alpha \le t \le \beta$, let $h(t) = \int_\alpha^t \frac{z'(t)}{z(t)-a}dt$, which is defined and continuous on $[\alpha, \beta]$.

It has the derivative $h'(t) = \frac{z'(t)}{(z(t)-a)}$ whenever $z'(t)$ is continuous. Therefore the derivative of $e^{-h(t)}(z(t)-a)$ vanishes except at a finite number of points. Since the given function is continuous, it is a constant function.

Thus $e^{h(t)} = \frac{(z(t)-a)}{(z(\alpha)-a)}$. Since $z(\beta)=z(\alpha)$, we have $e^{h(\beta)}=1$ and so $h(\beta)$ is a multiple of `$2\pi i$'.

Definition 9.2.2 :

The index of a point `a' with respect to the curve γ by $n(\gamma, 1) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{(z-a)}$

Note that $n(-\gamma, a) = -n(\gamma, a)$.

Corollary 9.2.3 :

If γ lies inside of a circle, then $n(\gamma, a)=0$ for all points `a' outside of the same circle.

Proof :

It is obvious from (9.1.9).

Corollary 9.2.4 :

As a function of `a', the index $n(\gamma, a)$ is constant in each of the regions determined by γ, and zero in the unbounded region.

Proof :

Any two points in the same region determined by γ can be joined by a polygon that does not meet γ. It remains to check that $n(r, a) = n(r, b)$ if γ does not meet the line segment from a into b. Outside of this segment, the function $\frac{(z-a)}{(z-b)}$ is never real and ≤ 0. The principle branch of $\log\left[\frac{(z-a)}{(z-b)}\right]$ is analytic in the complement of the segment. It is derivative is equal to $\frac{1}{(z-a)} - \frac{1}{(z-b)}$ and γ does not meet the segment.

$$\therefore \int_\gamma \left(\frac{1}{(z-a)} - \frac{1}{(z-b)}\right) dz = 0$$

$$\int_\gamma \frac{dz}{(z-a)} = \int_\gamma \frac{dz}{(z-b)}$$

$$\Rightarrow \quad n(\gamma, a) = n(\gamma, b).$$

If $|a|$ is large, then γ is contained in a disk $|z| < \rho < |a|$, and it follows that, by (9.2.3), $n(\gamma, a) = 0$. This verifies $n(\gamma, a) = 0$ in the unbounded region.

Lemma 9.2.5 :

Let z_1 & z_2 be two points on a closed curve γ that does not pass through the origin. Denote the subarc from z_1 to z_2 in the direction of the curve by γ_1, and the subarc from z_2 to z_1 by γ_2. Assume that z_1 lies in the lower half plane, and z_2 in the upper half-plane. If γ_1 does not meet the negative real axis and γ_2 does not meet the possible real axis, then verify that $n(\gamma, 0) = 1$.

Proof :

We draw the half-lines L_1 & L_2 from the origin through z_1 & z_2. Assume ξ_1 & ξ_2 are the point in which L_1 & L_2 intersect a circle C about the origin. If C is described in the positive sense, then arc C_1 from ξ_1 to ξ_2 does not intersect the negative axis, and the arc C_2 from ξ_2 to ξ_1 does not intersect the positive axis.

Denote the directed line segments from z_1 to ξ_1 and z_2 to ξ_2 by ∂_1 & ∂_2. Introducing the closed curves $\sigma_1 = r_1+\delta_2-c_1-\delta_1$; $\sigma_2 = r_2+\delta_1-c_2-\delta_2$, it follows that $n(r, 0)= n(c, 0)+n(\sigma_1, 0)+n(\sigma_2,0)$. σ_1 does not meet the negative axis. Therefore the origin belongs to the unbounded region determined by σ_1, and we get $n(\sigma_1, 0) = 0$; similarly $n(\sigma_2, 0)=0$. It follows that $n(\gamma, 0) = n(c, 0) = 0$.

Theorem 9.2.6 : (Cauchy's integral formula)

Suppose that f(z) is analytic in an open disk Δ, and γ a closed curve in Δ. For any point `a' not on γ,

$$n(\gamma, a) . f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{(z-a)}$$

where $n(\gamma, a)$ is the index of a with respect to γ.

Proof :

Let f(z) be given analytic function in an open disk Δ, γ a closed curve in Δ, and a point $a \in \Delta$ what does not lie on γ.

Consider the function $F(z) = \dfrac{f(z)-f(a)}{(z-a)}$

Then F is analytic for $z \neq a$. For $z = a$, it is not defined, but it satisfies the condition

$$\lim_{z\to a} (z-a)F(z) = \lim_{z\to a} (f(z)-f(a))$$

$$= \lim_{z\to a} f(z)-f(a) = f(a)-f(a) = 0$$

By (9.1.9), $\displaystyle\int_\gamma \frac{f(z)-f(a)}{(z-a)} dz = 0$

$$\Rightarrow \int \frac{f(z)dz}{(z-a)} = f(a).\int_\gamma \frac{dz}{(z-a)}$$

But $\displaystyle\int_\gamma \frac{dz}{(z-a)} = (2\pi i)\, n(\gamma, a)$

$$n(\gamma, a).f(a) = \left(\frac{1}{2\pi i}\right)\int_\gamma \frac{f(z)dz}{(z-a)}$$

Remark 9.2.7 :

If $n(\gamma, a)=1$, then $f(a) = \left(\frac{1}{2\pi i}\right)\int_\gamma \frac{f(z)dz}{(z-a)}$, that is called a representation formula.

If a is arbitrary point taking different values provided that the order of a with respect to r remains equal to 1, then `a' is a variable, and

$$f(z) = \left(\frac{1}{2\pi i}\right)\int_\gamma \frac{f(\xi)d\xi}{(\xi-a)}$$

From the assumption that f(z) is analytic in an arbitrary region Ω, and $a \in \Omega$, $\exists$ a δ-neighborhood Δ contained in Ω and a circle C with center `a' in Δ. Theorem 9.2.6 is applied to f(z) in Δ. Since $n(C, a)=1$, we have $n(C, z)=1$ for all points z inside of C. For such z, we get $f(z) = \frac{1}{(2\pi i)}\int_C \frac{f(\xi)d\xi}{(\xi-z)}$.

If the differentiations are obtained, then

$$f'(z) = \frac{1}{(2\pi i)}\int_C \frac{f(\xi)d\xi}{(\xi-z)^2}$$

$$\therefore \quad f^{(n)}(z) = \left(\frac{n!}{2\pi i}\right)\int_C \frac{f(\xi)d\xi}{(\xi-z)^{n+1}}$$

provided that all derivatives exist at the points inside of C.

Theorem 9.2.8 :

Suppose that $\varphi(\xi)$ is continuous on the arc γ. Then the function $F_n(z) = \int_\gamma \frac{\varphi(\xi)}{(\xi-z)^n}d\xi$ is analytic in each of the regions determined by γ, and its derivative is $F_n'(z) = n.F_{n+1}(z)$.

Proof :

Assume that $\varphi(\xi)$ is continuous on the arc γ.

Claim (1) :

$F_1(z)$ is continuous. Let $z_0 \notin \gamma$, and choose the neighborhood $|z-z_0| < \delta$ so that it does not meet γ. By restricting z to the smaller neighborhood $|z-z_0| < \delta/2$, we obtain that $|\delta-z| > \delta/2$ for all $\xi \in \gamma$.

Now $\quad F_1(z)-F_1(z_0) = \int_\gamma \frac{\varphi(\xi)d\xi}{(\xi-z)(\xi-z_0)}$

$$\Rightarrow \quad |F_1(z)-F_1(z_0)| < |z-z_0|\cdot\left(\frac{2}{\delta^2}\right)\int_\gamma |\varphi|.|d\xi|$$

whenever $|z-z_0| < \delta$, $|F_1(z)-F_1(z_0)| < \in$.

$\Rightarrow F_1(z)$ is continuous at z_0.

Using this conclusion to the function $\frac{\varphi(\xi)}{(\xi-z_0)}$, it follows that the difference quotient is

$$\frac{[F_1(z)-F_1(z_0)]}{(z-z_0)} = \int_\gamma \frac{\varphi(\xi)d\xi}{(\xi-z)(\xi-z_0)} \text{ tends to the limit } F_2(z_0) \text{ as } z \to z_0.$$

Hence $F_1'(z) = F_2(z)$.

The general case is verified by induction.

Assume that $F_{n-1}'(z) = (n-1)F_n(z)$.

Now $F_n(z)-F_n(z_0)$

$$= \int_r \frac{\varphi(\xi)d\xi}{(\xi-z)^n} - \int_r \frac{\varphi(\xi)d\xi}{(\xi-z_0)^n}$$

$$= \int_\gamma \frac{\varphi d\xi}{(\xi-z)^{n-1}(\xi-z_0)} - \int_\gamma \frac{\varphi d\xi}{(\xi-z_0)^n} + (z-z_0)\int_\gamma \frac{\varphi d\xi}{(\xi-z)^n(\xi-z_0)}$$

$F_n(z)$ is continuous function;

The induction hypothesis is applied to the function $\frac{\varphi(\xi)}{(\xi-z_0)}$. Then the first second term, the factor of $(z-z_0)$ is bounded in a neighborhood of z_0. If we divide the identity by $(z-z_0)$, and $z \to z_0 \Rightarrow$ the quotient in the first term tends to a derivative that is equal to $(n-1)\ F_{n+1}(z_0)$ by the induction hypothesis. The remaining factor in the second term is continuous by what we proved and has the limit $F_{n+1}(z_0)$. Thus $F_n'(z_0)$ exists, and equals $n.F_{n+1}(z)$.

Corollary 9.2.9 : (Morera's Theorem)

If f(z) is defined and continuous in a region Ω, and $\int_\gamma (f\ dz) = 0$ for all closed curves γ in Ω, then f(z) is analytic in Ω.

By (9.1.3) and (9.1.4), f(z) is the derivative of some analytic function F(z), and so f(z) is itself analytic.

Theorem 9.2.10 : (Liouville's Theorem)

A function which is analytic and bounded in the whole plane, is a constant function.

Proof :

Let the radius of a circle be r, and $|f(\xi)| \leq M$ on C.

We have $f^{(n)}(z) = \left(\frac{n!}{2\pi i}\right) \int_C \frac{f(\xi)d\xi}{(\xi - z)^{n+1}}$

It $z = a$, we get $|f^{(n)}(a)| \leq M.n!.(r^{-n})$ ------(1)

(Cauchy's estimate). We consider the case $n = 1$. Then $|f(\xi)| \leq M$ on all circles. As $r \to \infty$, (1) $\to f'(a) = 0$, for all `a'. Thus f is constant.

Theorem 9.2.11 : (Fundamental theorem of algebra)

Suppose that P(z) is a polynomial of degree > 0. It has a root.

Proof :

Assume that $P(z) \neq 0$ for any z.

Then 1/P(z) is analytic in the whole plane. We also have $P(z) \to \infty$ as $z \to \infty$.

$\therefore$ $1/P(z) \to 0$ as $z \to \infty$.

$\Rightarrow$ P(z) is bounded (the absolute value is continuous on the Riemann sphere and has a finite maximum).

By (9.2.10), 1/P(z) is constant which is a contradiction.

Hence P(z) = 0 for some z.

$\therefore$ P(z) has a root.

SECTION–3 : LOCAL PROPERTIES OF ANALYTIC FUNCTIONS :

We discuss about analytic functions having derivatives of all orders. We include here a classification of the isolated singularities of analytic functions.

Theorem 9.3.1 :

Suppose f(z) is analytic in the region Ω' obtained by omitting a point `a' from a region Ω. A necessary and sufficient condition that there exists an analytic function in Ω which coincides with f(z) in Ω' is that $\lim\limits_{z\to a}(z-a)f(z)=0$. The extended function is uniquely determined.

Proof :

Since the extended function is analytic at z = a, then it is continuous at `a', and so the necessity and the uniqueness are trivial.

To verify the sufficiency, assume that $\lim\limits_{z\to a}(z-a)f(z)=0$. We draw a circle C with center `a' so that C and its inside are contained in Ω. By Cauchy's formula,

$$f(z) \;=\; \frac{1}{2\pi i}\int_C \frac{f(\xi)d\xi}{(\xi-z)}, \text{ for all } z \neq a \text{ inside of C.}$$

But right-hand side represents an analytic function of z throughout the inside of C.

The function, which is equal to f(z) for z ≠ a, and which has the value $\left(\frac{1}{2\pi i}\right)\int_C \frac{f(\xi)d\xi}{(\xi-a)}$ for z = a, is analytic function in Ω. This is required extension and uniquely identified.

Taylor Theorem 9.3.2 :

If f(z) is an analytic function in a region Ω, then

$$f(z)=f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+\ldots+\frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1}+f_n(z).(z-a)^n \text{ --(*)}$$

where $f_n(z)$ is analytic in Ω.

Proof :

Given that f(z) is analytic in Ω containing `a'. Consider the function

$$F(z) \;=\; \frac{[f(z)-f(a)]}{(z-a)}.$$

Then F is not defined for z=a, but it satisfies the condition $\lim\limits_{z\to a}(z-a)F(z)=0$. As $z\to a$, the limit of F(z) is f′(a). By (9.3.1), there exists an analytic function $f_1(z)$ that is equal to F(z) for z ≠ a, and equal to f′(a) for z = a.

Repeating the process, we get an analytic function $f_2(z)$ which equals $(f_1(z)-f_1(a))/(z-a)$ for $z \neq a$; and $f_1'(a)$ for $z = a$, and so on.

Then we obtain

$$
\begin{aligned}
f(z) &= f(a)+(z-a)f_1(z)\\
f_1(z) &= f_1(a)+(z-a)f_2(z)\\
&\ldots\ldots\\
&\ldots\ldots\\
f_{n-1}(z) &= f_{n-1}(a)+(z-a)f_n(z)
\end{aligned}
$$

For z = a, we have

$$
\begin{aligned}
f(z) &= f(a)+(z-a)f_1(z)\\
&= f(a)+(z-a)[f_1(a)+(z-a)f_2(z)]\\
&= f(a)+(z-a)f_1(a)+(z-a)^2f_2(z)\\
&\ldots\ldots\\
&\ldots\ldots\\
&= f(a)+(z-a)f_1(a)+(z-a)^2f_2(z)+\ldots\\
&\qquad +(z-a)^{n-1}f_{n-1}(a)+(z-a)^nf_n(z).
\end{aligned}
$$

Differentiating `n' times, and putting z = a, we get $f^{(n)}(a) = n!.[f_n(a)]$

Hence we obtain (*).

Zeros and Poles 9.3.3 :

If f(a) and all derivatives $f^{(r)}(a)$ vanish, then $f(z) = (z-a)f_n(z)$ for all `n' ------ (1)

in Taylor theorem.

The expression for $f_n(z)$ is obtained as

$$f_n(z) = \left(\frac{1}{2\pi i}\right)\int_C \frac{f(\xi)d\xi}{(\xi-a)^n(\xi-z)}$$

The disk with the circumference C has to be contained in the region Ω in which f(z) is defined and analytic. The absolute value |f(z)| has a maximum M on C. If the radius of C is denoted by R, then

$$|f_n(z)| \leq \frac{M}{R^{n-1}(R-|z-a|)} \text{ for } |z-a| < R.$$

By (1), we have $|f(z)| \le \left(\frac{|z-a|}{R}\right)^n \left(\frac{MR}{R-|z-a|}\right)$.

Since $\left(\frac{|z-a|}{R}\right)^n \to 0$ as $n \to \infty$ ($\because$ $|z-a| < R$), we conclude that $f(z) = 0$ inside of C.

We verify that f(z) is identically zero in all of Ω. Let E_1 be the set on which f(z) and all derivatives vanish, and E_2 the set on which the function or one of the derivatives is different from zero. Then E_1 & E_2 are both open sets.

Therefore either E_1 or E_2 is empty set. If E_2 is empty, we are done. Otherwise E_1 is empty, and so f(z) can never vanish together with all its derivatives.

Assume that f(z) is not identically zero. If $f(a) = 0$, there exists a first derivative $f^{(h)}(a)$ which is different from zero.

Then we say that a is a zero of order `h' and there is no zero of infinite order.

Definition 9.3.4 :

Suppose $f(z) = (z-a)^h f_h(z)$ where $f_h(z)$ is analytic, and $f_h(a) \neq 0$.

Then $f_h(a)$ is continuous $\Rightarrow$ $f_h(z) \neq 0$ in a neighborhood of a; and f = a is the only zero of f(z) in this neighborhood. The zeros of a analytic function that do not vanish identically are isolated.

This property can be formalated in the uniqueness theorem as follows :

If f(z) & g(z) are analytic in Ω, and f(z) = g(z) on a set that has an accumulation point in Ω, then f(z) is identically equal to g(z).

$\therefore$ A point `a' is said to be isolated singularity of f(z) if f(z) is analytic in a region $0 < |z-a| < \delta$.

Definition 9.3.5 : (Removable singularity)

It is a point `a' for which f(a) is defined, and analytic function in the disk $|z-a| < \delta$ including `a'.

If $\lim_{z\to a} f(z) = \infty$, then the point a is said to be a pole of f(z), and $f(z) = \infty$. Then $f(z) \neq 0$ in some neighborhood of a, at which $1/f(z)$ is defined and analytic. Therefore g(z) at a is removable, g(z) has analytic extension with g(a)=0 and g(z) does not vanish identically. If g has a zero of order h, then $g(z) = (z-a)^h\, g_h(z)$ with $g_h(z) \neq 0$.

Thus $f(z)=(z-a)^{-h} f_h(z)$, where $f_h(z)=1/g_h(z)$ is analytic and different from zero in some neighborhood of a.

Definition 9.3.6 :

A function f(z) that is analytic in a region Ω except for poles, is said to be meromorphic in Ω.

(ie) For every $a \in \Omega$, there exists a neighborhood $|z-a| < \delta$, contained in Ω, such that f(z) is analytic either in the whole neighborhood $|z-a| < \delta$, or in $0 < |z-a| < \delta$.

Thus f(z)/g(z) of two analytic functions in Ω is a meromorphic function in Ω provided that g(z) is not identically zero.

Weierstrass Theorem 9.3.7 :

An analytic function comes arbitrarily close to any number value in every neighborhood of an essential singularity.

Proof :

Let f(z) be an analytic function in a region Ω. Assume `a' is given essential singularity. Suppose the assertion does not hold. Then there exists a complex number A and a neighborhood $0 < |f(z)-A| < \delta$ in this neighborhood of a (except for a).

For $\alpha < 0$, $\lim\limits_{z\to a} |z-a|^{\alpha} |f(z)-A| = \infty$. Thus `a' is not an essential signularity of (f(z)–A). For essential singularity; there exist a $\beta > 0$ satisfying $\lim\limits_{z\to a} |z-a|^{\beta} |f(z)-A| = 0$.

Also $\lim\limits_{z\to a} |z-a|^{\beta} |A| = 0$. Hence $\lim\limits_{z\to a} |z-a|^{\beta} |f(z)| = 0 \Rightarrow$ a is not essential singularity which is a contradiction.

Therefore the conclusion follows.

Theorem 9.3.8 : (The Local Mapping)

Let z_j be the zeros of a function f(z) that is analytic in a disk Δ, and does not vanish identically, each zero being counted as many times as its order indicates. For every closed curve γ in Δ which does not pass through a zero

$$\sum_j n(r, z_j) = \left(\frac{1}{2\pi i}\right) \int_\gamma \frac{f'(z)}{f(z)} dz \qquad \text{------(1)}$$

where the sum has only a finite number of terms $\neq 0$.

Proof :

Let f(z) be analytic, and not identically zero in an open disk Δ. Assume z_j are the zeros of f(z), j = 1 to n, and γ is a closed curve in Δ satisfying f(z) ≠ 0 on γ.

By repeated applications of (9.3.2), It is clear that $f(z) = (z-z_1)(z-z_2).....(z-z_n)g(z)$ where g is analytic, and ≠ 0 in Δ.

Taking `logarithmic, we obtain $\log(f(z)) = \log(z-z_1) + ... + \log(z-z_n) + \log(g(z))$

Differentiaing, we get

$$\frac{f'(z)}{f(z)} = \frac{1}{(z-z_1)} + \frac{1}{(z-z_2)} + ... + \frac{1}{(z-z_n)} + \frac{g'(z)}{g(z)}$$

for all $z \neq z_j$, and on γ.

Since g(z)≠0 in Δ, $\int_\gamma \frac{g'(z)}{g(z)} dz = 0$

By the definition of $n(r, z_j)$,

$$\left(\frac{1}{2\pi i}\right) \int_\gamma \frac{f'(z)}{f(z)} dz = n(\gamma, z_1) + ... + n(\gamma, z_n) \qquad \text{------(*)}$$

This holds whenever f(z) has infinitely many zeros in Δ. It is clear that γ is contained in a concentric disk Δ' smaller than Δ. Unless f(z) is identically zero, a case that must obviously be excluded, it has only a finite number of zeros in Δ'. This is followed by Bolzano - Weierstrass theorem, for there are infinitely many zeros, they have an accumulation point in the closure of Δ', and this is not possible.

We apply the equation (*) to the disk Δ'. The zeros outside of Δ' satisfy $n(\gamma, z_j)=0$, and so do not give the sum in the equation (*).

Theorem 9.3.9 :

Suppose that f(z) is analytic at z_0; $f(z_0) = w_0$, and that $f(z)-w_0 = 0$ has a zero of order n at z_0. If ∈ is sufficiently smal, there is a δ > 0 such that ∀a with $|a-w_0| < \delta$, the equation f(z) = a has exactly n roots in the disk $|z-z_0| < \epsilon$.

Proof :

The function w = f(z) maps γ in disk Δ onto a closed curve Γ in the w-plane. Then

$$\int_\Gamma \frac{dw}{w} = \int_\gamma \left(\frac{f'(z)}{f(z)}\right) dz$$

From the equation (1) in (9.3.8), $n(\Gamma, 0) = \sum_j n(\gamma, z_j)$ ------(**)

Each $n(\gamma, z_j)$ is either 0 or 1, and the equation (1) in (9.3.8) yields a formula for the total number of zeros enclosed by γ. This holds when γ is a circle.

Assume that a is an arbitrary complex value. Then theorem (9.3.8) to the function $(f(z)-a)$, whose zeros are the roots of the equation $f(z) = a$; we denote them by $z_j(a)$.

By (1) in (9.3.8), we obtain

$$\sum_j n(\gamma, z_j(a)) = \left(\frac{1}{2\pi i}\right)\int_\gamma \frac{f'(z)}{f(z)-a} dc$$

(**) takes the for $n(\Gamma, a) = \sum_j n(\gamma, z_j(a))$

Assume then that $f(z) \neq a$ on γ. If a & b are in the same region determined by Γ, we know that $n(\Gamma, a) = n(\Gamma, b)$. Therefore $\sum_j n(\gamma, z_j(a)) = \sum_j n(\gamma, z_j(b))$. If γ is a circle, then f(z) takes the values a & b equally many times inside of γ.

Choose $\in > 0$ such that f(z) is defined and analytic for $|z-z_0| \leq \in$, and z_0 is the only zero of $f(z)-w_0 = 0$in this above disk; let γ be the circle $|z-z_0| = \in$; Γ its image under the mapping $w = f(z)$. Since $w_0 \in$ the complement of the closed set Γ, there is a neighborhood $|w-w_0| < \delta$ that do not intersect Γ. It follows that all values `a' in this neighborhood are taken the same number of times inside of γ.

The equation $f(z)=w_0$ has exactly n coinciding roots inside of γ, and so every value a is taken n-times. The multiple roots are counted due to their multiplicity, but whether $\in$ is sufficiently small implies that all roots of the equation $f(z) = a$ are simple for $a \neq w_0$. It is then choosen $\in$ so that $f'(z)$ does not vanish for $0 < |z-z_0| < \in$.

This completes the proof.

Corollary 9.3.10 :

A non-constant analytic function maps open sets onto open sets.

Corollary 9.3.11 :

If f(z) is analytic at z_0 with $f'(z_0) \neq 0$, it maps a neighborhood of z_0 conformally and topologically onto a region.

Theorem 9.3.11 : (The maximum principle)

If f(z) is non-constant analytic in a region Ω, then its absolute value $|f(z)|$ has no maximum in Ω.

Proof :

Let $w_0 = f(z_0) \in \Omega$. Then there is a neighborhood $|w-w_0| < \epsilon$ contained in the image of Ω. In this neighborhood, there are points of modulus $> |w_0|$ and so $|f(z_0)|$ is not the maximum of $|f(z)|$. Thus the proof is clear.

Theorem 9.3.12 :

It f(z) is defined and continuous on a closed bounded set E, and analytic on the interior of E, then the maximum of $|f(z)|$ on E is assumed on the boundary of E.

Proof :

E is closed bounded set $\Rightarrow$ E is compact. Then $|f(z)|$ has a maximum on E. Assume it is z_0 at which a maximum exists. If z is an interior point, then $|f(z_0)|$ is the maximum of $|f(z)|$ in a disk $|z-z_0| < \delta$ contained in E. But this is impossible unless f(z) is constant in the component of the interior of E containing z_0. By the continuity, $|f(z)|$ is equal to its maximum on the whole boundary of that component. The boundary is not empty, and it is contained in the boundary of E. Therefore the maximum is attained at a point on the boundary of E.

Schwarz's Lemma 9.3.13 :

If f(z) is analytic for $|z| < 1$, and satisfies the conditions $|f(z)| \leq 1$; $f(0) = 0$, then $|f(z)| \leq |z|$, and $|f'(0)| \leq 1$. If $|f(z)| = |z|$ for some $z \neq 0$, or if $|f'(0)| = 1$ then $f(z) = Cz$ with a constant C of absolute value 1.

Proof :

Given that f(z) is analytic for $|z| < 1$, and satisfies $|f(z)| \leq 1$. Define $f_1(z) = f(z)/z$ for $z \neq 0$; and $f'(0)$ for $z = 0$. We apply the maximum principle to the function $f_1(z)$. On the circle $|z| = \alpha < 1$, it is of absolute value $\leq (1/r)$, and so $|f_1(z)| \leq 1$ for $|z| \leq r$. As $r \to 1$, we obtain $|f_1(z)| \leq 1$ for all z.

(ie) $|f(z)/z| \leq 1 \Rightarrow |f(z)| \leq |z|$, and $|f'(0)| \leq 1$.

This is the assertion of the theorem. If the equality holds at a single point, then $|f_1(z)|$ attains its maximum and so $f_1(z)$ reduces to a constant. (ie) $f(z) = Cz$ for some constant `C'.

SECTION–4 : THE GENERAL FORM OF CAUCHY'S THEOREM.

Definition 9.4.1 :

The equation

$$\int_{\gamma_1+\gamma_2+...+\gamma_n} (f\,dz) = \int_{\gamma_1} (f\,dz)+....+\int_{\gamma_n} (f\,dz)$$

is valid, whenever $\gamma_1, \gamma_2, ...\gamma_n$ form a subdivision of an arc γ. Since RHS has a meaning for any finite collection, nothing prevents us from considering an arbitrary formal sum $\gamma_1+\gamma_2+...+\gamma_n$ (this is not an arc). Then we define the corresponding integral by means of the above equation. Such formal sums of arcs are called chains.

Definition 9.4.2 :

A chain is a cycle if it can be represented as a sum of closed curves.

The integral of an exact differential over any cycle is zero.

Definition 9.4.3. :

A region is simply connected if its complement with respect to the extended plane is connected.

Theorem 9.4.4 :

A region Ω is simply connected iff $n(\gamma, a) = 0$ for all cycles γ in Ω, and all points a which do not belong to Ω.

Proof :

The necessary condition is trivial. For this, let γ be any cycle in Ω. The complement of Ω is connected implies that it is contained in one of the regions determined by γ, and so ∞ belongs to the complement, and this is unbounded region.

$\therefore$ $n(\gamma, a) = 0$ for all finite points in the complement.

To show the sufficiency condition, we assume that $n(\gamma, a) = 0$ for all cycles γ in Ω and all points a that do not belong to Ω.

Claim :

The complement of Ω is connected.

Assume that the complement of Ω is disconnected. The complement of Ω is $A \cup B$ where A & B are two disjoint non-empty closed sets. One of these sets contains ∞, and the other is bounded. Without loss of generality, let A be the bounded set. Let δ be the shortest distance of A and B. We cover the whole plane with a net of squares Q of side $(\delta / \sqrt{2})$.

Choose the net so that a certain point $a \in A$ lies at the center of a square. The boundary curve of Q is denoted by ∂Q. We assume that the squares Q are closed, and that the interior of Q lies to the left of the directed line segments that make up ∂Q.

Consider the cycle $\gamma = \sum_j (\delta Q_j)$ ----(1) where the sum ranges over all square Q_j in the net that have a point in common with A. Since a is contained in one and only one of these squares, it follows that $n(\gamma, a)=1$. Also γ does not meet B. If the cancellations are carried out, then γ does not meet A. Any side that meets A, is a common side of two squares included in the sum (1). Since the directions are opposide, the side does not appear in the reduced expression of γ. Hence γ is untained in Ω, and the theorem is showed.

Definition 9.4.5 :

A cycle γ in an open set Ω is said to be homologous to zero with respect to Ω, if $n(\gamma, a)=0$ for all points a in the complement of Ω. We write $\gamma \sim 0 \pmod{\Omega}$.

Corollary 9.4.6 :

If f(z) is analytic in a simply connected region Ω, then $\int_\gamma f(z)dz=0$ for all cycles γ in Ω.

The conclusion of this corollary holds for all closed curves γ in a region means that the line integral of (f dz) is independent of the path or that (f dz) is an exact differential. Then there is a single-valued analytic function F(z) such that F'(z)=f(z). In a simple connected region, every analytic function is a derivative.

Corollary 9.4.7 :

If f(z) is analytic, and $\neq 0$ in a simply connected region Ω, then it is possible to define single valued analytic branches of log (f(z)) and $[f(z)]^{1/n}$ in Ω.

There exists an analytic function F(z) in Ω such that $F'(z) = f'(z)/f(z)$. The function $f(z)e^{-F(z)}$ has the derivative zero, and is so constant.

Choose a point $z_0 \in \Omega$, and one of infinitely many values $\log(f(z_0))$. We find

$$e^{F(z)-F(z_0)+\log(f(z_0))} = f(z)$$

$$\Rightarrow \quad \log(f(z)) = F(z)-F(z_0)+\log(f(z_0))$$

Define $(f(z))^{1/n} = \exp(\log(f(z))/n)$ we follow the conclusion.

Theorem 9.4.8 : (General Statement of Cauchy

If f(z) is analytic in a region Ω, then $\int_\gamma f(z)dz = 0$ for every cycle γ which is homologons to zero in Ω.

Proof :

Case (i)

Assume that Ω is bounded, but otherwise arbitrary. Given $\delta > 0$, we cover the whole plane by a net of squares of side δ, and we denote by Q_j ($j \in J$, the natural set), the closed squares in the net that are contained in Ω. Since Ω is bounded, the set J is finite. If δ is sufficiently small, it is non-empty. The union of the squares $Q_j (j \in J)$ consists of closed regions whose oriented boundaries make up the cycle $\Gamma_\delta = \sum_{j \in J} \partial Q_j$ which is a sum of oriented line segments that are sides of exactly one Q_j. We denote by Ω_δ, the interior of the union $\left(\bigcup_j Q_j\right)$.

Let γ be a cycle which is homologous to zero in Ω. Choose δ so small that γ is contained in Ω_δ. Consider a point $\xi \in (\Omega - \Omega\delta)$. It belongs to at least Q which is not a Q_j. There is a point $\xi_0 \in Q$ that is not in Ω. It is possible to join ξ & ξ_0 by a line segment that lies in Q, and so does not meet Ω_δ.

γ (considered as a point) is contained in Ω_δ implies that $n(\gamma, \xi) = n(\gamma, \xi_0) = 0$. In particular $n(\gamma, \xi) = 0$ for all ξ on Γ_δ.

Assume that f is analytic in Ω. If z lies in the interior of Q_{j_0}, then

$$\left(\frac{1}{2\pi i}\right)\int_{\partial Q_j} \frac{f(\xi)d\xi}{(\xi - z)} = \begin{cases} f(z) : j = j_0 \\ 0 \quad : j \neq j_0 \end{cases}$$

$$\therefore \qquad f(z) = \left(\frac{1}{2\pi i}\right)\int_{\Gamma_\delta} \frac{f(\xi)d\xi}{(\xi - z)}$$

Since both sides are continuous functions of z, this equation holds $\forall\ z \in \Omega_\delta$.

Finally, we obtain

$$\int_\gamma f(z)dz = \int_\gamma \left(\frac{1}{2\pi i}\right)\left(\int_{\Gamma_\delta} \frac{f(\xi)}{(\xi - z)}\right)dz \quad \text{------(1)}$$

This integrand is a continuous function of both variables, can be reversed.

(ie) $\displaystyle\int_\gamma \left(\frac{1}{2\pi i}\right)\left(\int_{\Gamma_\delta} \frac{f(\xi)}{(\xi - z)}\right)dz = \int_{\Gamma_\delta} \left(\frac{1}{2\pi i}\right)\left(\int_r \frac{dz}{(\xi - z)}\right)f(\xi)d\xi$

Inside integral that of RHS is $-n(\gamma, \xi) = 0$. The integral (1) is zero.

Case (ii) :

If Ω is bounded, we replace it by its intersection Ω' with a disk $|z| < R$ that is large enough to contain γ. Any point `a' in the complement of Ω' is either in the complement of Ω (or) lies outside the disk. In either case, $n(\gamma, a) = 0 \Rightarrow \gamma \sim 0 \pmod{\Omega'}$. Then we apply the theorem to Ω' and we conclude that the theorem is valid for arbitrary Ω.

Theorem 9.4.9 :

If pdx+qdy is locally exact in Ω, then $\int_\gamma (pdx + qdy) = 0$ for every cycle $\gamma \sim 0$ in Ω.

Proof :

Case (1) :

γ can be replaced by a polygon σ with horizontal and vertical sides such that every locally exact differential has the same integral over σ as over γ. Then $n(\sigma, a) = n(\gamma, a)$ for a in the complement of Ω, and so $\sigma \sim 0$. It is sufficient to verify the theorem for polygons with sides parallel to the axes.

We consider σ as approximation of γ. Let ρ be the distance from γ to the complement of Ω. If γ is given by $z = z(t)$, the function $z(t)$ is uniformly continuous on the closed interval [a b]. We find $\delta > 0$ so that $|z(t)-z(t')| < \rho$ whenever $|t-t'| < \delta$, and then divide [a b] into subintervals of length $< \delta$.

The corresponding subarcs γ_i of γ have the property that each is contained in a disk of radius ρ that lies entirely in Ω. The end points of γ_i can be joined within that disk by a polygon σ_i consisting of a horizontal and a vertical segment. Since the differential is exact in the disk,

$$\int_{\sigma_i}(pdx+qdy) = \int_{\gamma_i}(pdx+qdy)$$

If $\sigma = \Sigma\, \sigma_i$, we obtain

$$\int_{\sigma}(pdx+qdy) = \int_{\gamma}(pdx+qdy)$$

Case (2) :

We extend all segments that make up σ to infinite lines. They divide the plane into some finite rectangles, R_i and some unbounded regions R_j' which is regarded as infinite rectangules.

Choose a point `a_i' from the interior of each R_i, and form the cycle $\sigma_0=\sum_i n(\sigma,a_i)\delta R_i$, where the sum ranges over all finite rectangles. The coefficients $n(\sigma, a_i)$ are found for no a_i lies on σ. We make use of points a_j' chosen from the interior of each R_i'.

It is clear that $n(\partial R_i, a_k) = 1$ if $i = k$, and 0 if $i \neq k$. Thus $n(\partial R_i, a_i') = 0$ for all j. Then $n(\sigma_0, a_i) = n(\sigma, a_i)$ and $n(\sigma_0, a_j') = 0$. Also $n(\sigma, a_j') = 0$.

Therefore $n(\sigma-\sigma_0, a) = 0, \forall\, a=a_i; \forall\, a=a_i'$.

We follow that σ_0 is identical with σ up to segments that cancel against each other. Assume σ_{ik} is the common side of two adjacent rectangules R_i ; R_k so that we select the orientation with the assumption that R_i lies to the left of σ_{ik}. Suppose the reduced expression of $(\sigma-\sigma_0)$ contains the multiple $(c.\ \sigma_{ik})$. Then the cycle $\sigma-\sigma_0-(C\ \partial R)$ does not contain σ_{ik}. It follows that a_i & a_k have the same index with respect to this cycle. These indices are $-C$ and 0 respectively. Thus $C = 0$. The same reasoning holds if σ_{ij} is the common side of a finite rectangule R_i and an infinite rectangule R_j'. Therefore every side of a finite rectangle occurs with coefficient zero in $(\sigma-\sigma_0)$ proving that

$$\sigma = \sum_i n(\sigma,a_i)\partial R_i \qquad \text{------(2)}$$

We assume now that all the R_i whose corresponding coefficient $n(\sigma, a_i)$ is different from zero are contained in Ω. Assume that a point `a' in the closed rectangles. R_i(not in Ω). Then $n(\sigma, a) = 0$ becomes $\sigma \sim 0 \pmod{\Omega}$. Then the line segment between a and a_i does not intersect σ, and so $n(\sigma, a_i)$ that the integral of $(pdx+qdy)$ over any ∂R that occurs in (2) is zero, $\int_\sigma (pdx + qdy) = 0$.

The Calculus of Residues :

The Residue theorem 9.4.10 :

Let f(z) be analytic except for isolated singularities a_i in a region Ω. Then

$$\left(\frac{1}{2\pi i}\right)\int_\gamma [f(z)dz] = \sum_j n(r,a_j) Res_{z=a_j}(f(z))$$

for any cycle γ which is homologous to zero in Ω, and does not pass through any of the points a_j.

Proof :

We introduce a definition: The residue of f(z) at an isolated singularity `a' is the unique complex number R that makes $f(z) - R/(z-a)$, the derivative of a single-valued analytic function in an annulas $0 < |z-a| < \delta$.

By Cauchy's integral formula, $n(\gamma, a)\, f(a) = \left(\frac{1}{2\pi i}\right)\int_\gamma \frac{f(z)dz}{(z-a)}$ -----(*)

for every cycle γ that is homologous to zero in Ω.

Assume that $a_1, a_2, \ldots, a_n$ are only singular points of f(z). The region obtained by excluding the points a_j is denoted by Ω'. To each a_j, there is a $\delta_j > 0$ such that the doubly connected region. $0 < |z-a_j| < \delta_j$ is contained in Ω'. Draw a circle C_j about a_j of radius $< \delta_j$. Let $P_j = \int_{C_j} [f(z)dz]$ be the corresponding period of f(z). The function $1/(z-a_j)$ has the period $(2\pi i)$. If $R_j = P_j/(2\pi i)$, then $f(z) - R_j/(z-a_j)$ has a vanishing period. The constant R_j are called the residue of f(z) at a_j.

Assume that γ be a cycle in Ω' that is homologous to zero with respect to Ω. Then `γ' satisfies the homology $\gamma \sim \sum_j n(r,a_j)C_j$ with respect to Ω'. Then the points a_j as well as all points outside of Ω have the same order with respect to both cycles.

By (*), we obtain

$$\int_\gamma (f(z)dz) = \sum_j n(\gamma,a_j)P_j$$

$$P_j = (2\pi i)R_i \Rightarrow \left(\frac{1}{2\pi i}\right)\int_\gamma (fdz) = \sum_j n(r, a_j)R_j$$

Definition 9.4.11 :

A cycle γ is said to bound the region Ω iff $n(\gamma, a)$ is defined and equal to 1 for all points $a \in \Omega$, and either undefined or equal to zero for all points `a' not in Ω.

Remark 9.4.12 :

If γ bounds. Ω, and $\Omega+\gamma$ is contained in a larger region Ω', then it is clear that γ is homologous to zero with respect to Ω'. If f(z) is analytic on the set $(\Omega+\gamma)$, then

$$\int_\gamma f(z)dz = 0 \quad \text{and } f(z) = \left(\frac{1}{2\pi i}\right)\int_\gamma \frac{f(\xi)d\xi}{(\xi - z)}, \text{ for all } z \in \Omega.$$

$\therefore$ If is analytic on $(\Omega+\gamma)$ except for isolated singularities in Ω, then

$$\left(\frac{1}{2\pi i}\right)\int_\gamma [f(z)dz] = \sum_j \mathrm{Res}_{z=a_j}(f(z))$$

where the sum ranges over the singularities $a_j \in \Omega$.

Note 9.4.13 :

If f(z) is meromorphic in Ω with the zeros a_j, and the poles b_k, then

$$\left(\frac{1}{2\pi i}\right)\int_\gamma \frac{f'(z)dz}{f(z)} = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k)$$

for every cycle γ which is homologous to zero in Ω, and does not pass through any of the zeros or poles.

We know that $f(z)/(z-a)$ has a simple pole at $z = a$ with the residue f(a). Applying the residue theorem, we first investigate the number of zeros of an analytic function. For a zero of order h, we have $f(z) = (z-a)^h f_h(z)$ with $f_h(a) \neq 0$.

$$\therefore \quad f'(z) = f(z-a)^{h-1} f_h(z) + (z-a)^h f_h'(z)$$

$$\Rightarrow \quad \frac{f'(z)}{f(z)} = \frac{h}{(z-a)} + \frac{fa'(z)}{f_h(z)}$$

$\Rightarrow$ f'/f has a simple pole with the residue `h'.

Hence f has a pole of order h, then f'/f has the residue (–h).

This principle is called argument principle.

Rouche's Theorem 9.4.14 :

Let γ be homologous to zero in Ω, and such that $n(\gamma, z)$ is either 0 (or) 1 for any point z not on γ. Suppose that $f(z)$ & $g(z)$ are analytic in Ω, and satisfy the inequality $|f(z)-g(z)| < |f(z)|$ on γ. Then $f(z)$ & $g(f)$ have the same number of zeros enclosed by γ.

Proof :

$f(z)$ & $g(z)$ are zero-free on γ. They satisfy $\left|\frac{g(z)}{f(z)}-1\right|$ on γ.

$\therefore$ The values of $F(z) = g(z)/f(z)$ on γ are contained in the open disk of center 1 and radius 1. By (9.4.13), $n(\Gamma, 0) = 0$ and the assertion follows.

To find the number of zeros of a function $f(z)$ in the disk $|z| \le R$, let us write $f(z)=P_{n-1}(z)+z^n f_n(z)$

where P_{n-1} is a polynomial of degree (n–1) by Taylor's theorem. For a suitable n, $R^n|f_n(z)| < |P_{n-1}(z)|$ on $|z| = R$. Then $f(z)$ has the same number of zeros in $|z| \le R$ as $P_{n-1}(z)$, and this number can be identified by solution of the polynomial $P_{n-1}(z) = 0$.

By (9.4.13), if $g(z)$ is analytic in Ω, then $g(z)\, f'(z)/f(z)$ has the residue $h[g(a)]$ at a zero `a' of order h, and the residue $-h(g(a))$ at a pole. Thus we obtain (*) :

$$\left(\frac{1}{2\pi i}\right)\int_\gamma \frac{g(z)f'(z)}{f(z)}dz = \sum n(\gamma,a_j)g(a_j)-\sum_k n(\gamma,b_k)g(b_k)$$

By (9.3.9), the equation $f(z)=w$, $|w-w_0| < \delta$ has n-roots $z_j(w)$ in the disk $|z-z_0| < \epsilon$. We apply (*) with $g(z) = z$ implies that

$$\sum_{j=1}^{n} z_j(w) = \left(\frac{1}{2\pi i}\right)\int_{|z-z_0|=\epsilon}\left(\frac{f'(z)}{f(z)-w}\right)(zdz)$$

For n=1, the inverse function $f^{-1}(w)$ is represented as

$$f^{-1}(w) = \left(\frac{1}{2\pi i}\right)\int_{|z-z_0|=\epsilon}\left(\frac{f'(z)}{f(z)-w}\right)(z^m dz)$$

The RHS represents an analytic function of w for $|w-w_0| < \delta$. Therefore the power sums of the roots $z_j(w)$ are single-valued analytic functions of w.

The elementary symmetric functions can be expressed as polynomials in the power sums. So they are also analytic and $z_j(w)$ are the roots of a polynomial $z^n+a_1(w)z^{n-1}+\ldots+a_{n-1}(w)z+a_n(w) = 0$ whose coefficients are analytic functions of w in the neighborhood $|w-w_0| < \delta$.

SECTION–1 : SERIES AND PRODUCT DEVELOPMENTS

Definition 10.1.1 : (Uniformly Convergence)

Let $f_n(z)$ be a sequence of functions of Ω, and $f(z)$ be a function on Ω. Then $f_n(z)$ convergent uniformly to $f(z)$ in Ω if $\forall \in > 0$, $\forall z \in \Omega$, $\exists$ a positive N such that $|f_n(t)-f(z)| < \in$, $\forall n \geq N$.

Weierstrass's Theorem 10.1.2 :

Suppose $f_n(z)$ is analytic in the region Ω_n, and the sequence $f(z)$ in a region Ω uniformly on every compact subset of Ω. Then $f(z)$ is analytic in Ω. Moreover, $f_n(z)$ converges uniformly to $f'(z)$ on every compact subset of Ω.

Proof :

Let $|z-a| \leq r$ be a closed disk contained in Ω. By assumption, this disk lies in Ω_n for all n greater than a certain n_0. If γ is closed curve contained in $|z-a| < r$, then we have $\int_\gamma f_n(z)dz = 0$ for all $n \geq n_0$.

[The region Ω_n form an open covering of $|z-a| \leq r$. The disk is compact and it has a finite subcovering. Therefore it is contained in a fixed Ω_n by Cauchy's theorem].

By uniform convergence on γ, we get

$$\int_\gamma f(z)dz = \lim_{n\to\infty} \int_\gamma f_n(z).dz = 0$$

By Morera's theorem, $f(z)$ is analytic in $|z-a| < \gamma$, and finally it is analytic in the whole region Ω.

By integral formula, $f_n(z) = \left(\frac{1}{2\pi i}\right)\int_C \frac{f_n(\xi)d\xi}{(\xi - z)}$ where C is the circle $|\xi-a| = \gamma$ and $|z-a| < \gamma$.

As $n\to\infty$, $$f(z) = \left(\frac{1}{2\pi i}\right)\int_C \frac{f(\xi)d\xi}{(\xi - z)}$$ by uniform convergence.

$\therefore$ f(z) is analytic in the disk.

We have $$f_n'(z) = \left(\frac{1}{2\pi i}\right)\int_C \frac{f_n(\xi)d\xi}{(\xi - z)^2}$$

$$\Rightarrow \quad \lim_{n\to\infty} f_n'(z) = \left(\frac{1}{2\pi i}\right)\int_C \frac{f(\xi)d\xi}{(\xi-z)^2} = f'(z)$$

$\therefore$ The above convergence is uniform in $|z-a| \le p < r$. Any compact subset of Ω can be covered by a finite number of such closed disks, and so the convergence is uniform on every compact subset. It follows that $f_n^{(k)}(z)$ converges uniformly to $f^{(k)}(z)$ on every compact subset of Ω.

Corollary 10.1.3 :

If a series with analytic terms $f(z) = f_1(z)+f_2(z)+\ldots+f_n(z)+\ldots$ converges uniformly on every compact subset of a region Ω, then the sum $f(z)$ is analytic in Ω, and the series can be differentiated term by term.

Hurwitz's Theorem 10.1.4 :

If the function $f_n(z)$ are analytic and $\neq 0$ in a region Ω, and $f_n(z)$ converges to $f(z)$ uniformly on every compact subset of Ω, then $f(z)$ is either identically zero (or) never equal to zero in Ω.

Proof :

Suppose that $f(z) \neq 0$ identically. The zeros of $f(z)$ are isolated. For any point $z_0 \in \Omega$, there is a number $r > 0$ such that $f(z)$ is defined, and $\neq 0$ for $0 < |z-z_0| \neq r$. In particular, $|f(z)|$ has a positive minimum on the circle $|z-z_0| = r$, which is denoted by C. This implies that $1/f_n(z)$ converges uniformly to $1/f(z)$ on C. Also $f_n'(z) \to f'(z)$ uniformly on C

$$\Rightarrow \quad \lim_{n\to\infty}\left(\frac{1}{2\pi i}\right)\int_C \frac{f_n'(z)}{f_n(z)}dz = \left(\frac{1}{2\pi i}\right)\int_C \frac{f'(z)}{f(z)}dz$$

Here all the integrals in LHS are zero. ($\because$ they give the number of roots of the equation $f_n(z) = 0$ inside of C).

Therefore the integral in RHS is zero and so $f(z_0) \neq 0$. Since z_0 is arbitrary the conclusion is completed.

SECTION–2 : TAYLOR'S THEOREM

Theorem 10.2.1 (Taylor series) :

If $f(z)$ is analytic in the region Ω containing z_0, then

$$f(z) = f(z_0)+\frac{f'(z_0)}{1!}(z-z_0)+\ldots+\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n+\ldots$$

is valid in the largest open disk of center z_0 contained in Ω.

Proof :

We verify that every analytic can be developed in a convergent Taylor series. By the earlier studies, and if f(z) is analytic in a region contain z_0, we obtain

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \ldots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + f_{n+1}(z)[z-z_0]^{n+1}$$

with $$f_{n+1}(z) = \left(\frac{1}{2\pi i}\right)\int_C \frac{f(\xi)d\xi}{(\xi-z_0)^{n+1}(\xi-z)}$$

where C is the circle $|z-z_0| = \rho$ such that the closed disk $|z-z_0| \leq \rho$ is contained in the region Ω.

Let M be the maximum of $|f(z)|$ on C. Then we obtain at once the estimate

$$|f_{n+1}(z)[z-z_0]^{n+1}| \leq \frac{M|z-z_0|^{n+1}}{P^n(P-|z-z_0|)}$$

We conclude that the remainder term tends uniformly to zero in each disk $|z-z_0| \leq r < \rho$. But ρ is chosen arbitrary close to the shortest distance from z_0 to the boundary of Ω. Therefore the theorem follows :

Remark 10.2.2 :

1) The radius of convergence of the Taylor series is equal to the shortest distance from z_0 to the boundary of Ω.

2) We get the following developments :

$$e^z = 1 + z + \frac{z^2}{2!} + \ldots\ldots + \frac{z^n}{n!} + \ldots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6} + \ldots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots$$

3) Every convergent power series is its own Taylor series.

4) We got a direct proof that the power series can be differentiated term by term.

5) $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots$ (if $|z| < 1$).

If the logarithmic series had a radius of convergence greater then 1, then log(1+z) is bounded for $|z| < 1$. Since this is not the case, the radius of convergence is exactly 1.

6) If the binomial series is convergent in a circle of radius > 1, the function $(1+z)^n$, and all its derivatives are bounded in $|z|<1$. Unless n is a positive integer, one of the derivates is a negative power of (1+z), and so unbounded. The radius of convergence is 1 except when the binomial series reduces to a polynomial

7) $1/(1+z^2) = 1-z^2+z^4-z^6+\ldots.$

8) $\arctan z = z-\frac{z^3}{3}+\frac{z^5}{5}-\frac{z^7}{7}+\ldots..$ where $\arctan z = \int_0^z \frac{dz}{(1+z^2)}$ for any path inside the unit circle.

9) $\frac{1}{\left(\sqrt{1-z^2}\right)} = 1+\frac{z^2}{2}+\frac{1}{2}\frac{3}{4}z^4+\frac{1.3.5}{2.4.6}z^6+\ldots.$ (for $|z| < 1$).

10) $\arcsin z = z+\frac{1}{2}\frac{z^3}{3}+\frac{1.3}{2.4}\frac{z^5}{5}+\frac{1.3.5}{2.4.6}\frac{z^7}{7}+\ldots...$

11) If $f(z) = a_0+a_1z+\ldots..+a_nz^n+\ldots..$; and

$g(z) = b_0+b_1z+\ldots+b_nz^n+\ldots.,$

then $f(z)g(z) = a_0b_0+(a_0b_1+a_1b_0)z+\ldots.+(a_0b_n+a_1b_{n-1}+\ldots+a_nb_0)z^n+\ldots$

12) $\tan z = z+\frac{z^3}{3}+\frac{2(w^3)}{15}+\ldots...$

SECTION–3 : THE LAURENT SERIES

A series of the form $b_0+b_1z^{-1}+b_2z^{-2}+\ldots..+b_nz^{-n}+\ldots..$ ------(1)

is an ordinary power series in the variable 1/z. It converges outside of some circle $|z| = R$ (except $R=\infty$).

The convergence is uniform in every region $|z| \geq P > R$.

If the series (1) is combined with an ordinary power series, we get a general series of the form

$$\sum_{n=-\infty}^{\infty} a_n(z^n) \qquad \text{------(2)}$$

It is also convergent, whenever the parts consisting of non-negative powers and negative powers are separately convergent. Since the part of non-negative powers converges in a disk $|z| < R_2$, and the part of negative powers converges in a region $|z| > R_1$, then there is a common region of convergence only if $R_1 < R_2$. Further (2) is an analytic function in $R_1 < |z| < R_2$.

Conversely, suppose that f(z) is an analytic function whose region contains an annuals $R_1 < |z| < R_2$ (or) $R_1 < |z-a| < R_2$. Then we claim that such a function can be developed in a power series of the general form as

$$f(z) = \sum_{n=-\infty}^{\infty} A_n(z-a)^n$$

For this, it remains to check that $f(z) = f_1(z)+f_2(z)$, where $f_1(z)$ is analytic for $|z-a| < R_2$, and $f_2(z)$ is analytic for $|z-a| > R_1$ with a removable singularity at ∞.

With these assumptions, $f_1(z)$ can be developed in non-negative powers of $(z-a)$, and $f_2(z)$ can be developed in non-negative powers of $1/(z-a)$.

To find $f_1(z)$ & $f_2(z)$ from f(z), define the following :

$$f_1(z) = \left(\frac{1}{2\pi i}\right)\int_{|\xi-a|=r} \frac{f(\xi)d\xi}{(\xi-z)} \qquad \text{for } |z-a| < r < R_2.$$

$$f_2(z) = \left(-\frac{1}{2\pi i}\right)\int_{|\xi-a|=r} \frac{f(\xi)d\xi}{(\xi-z)} \qquad \text{for } R_1 < r < |z-a|.$$

In both integrals, r is irrelevant as long as the inequlity is fulfilled. By Cauchy's theorem, the integral does not change with r provided that the circle does not pass over the point z. Thus $f_1(z)$ & $f_2(z)$ are uniquely defined and represent analytic functions in $|z-a| > R_1$ respectively. Further $f(z) = f_1(z)+f_2(z)$.

The Taylor development of $f_1(z)$ is $\sum_{n=0}^{\infty} A_n(z-a)^n$,

where
$$A_n = \left(\frac{1}{2\pi i}\right)\int_{|\xi-a|=r} \frac{f(\xi)d\xi}{(\xi-a)^{n+1}} \qquad \text{-----(2)}$$

To find the development of $f_2(z)$, let $\xi = a+1/\xi'$; $z = a+1/z'$. This transformation carries $|\xi-a|=r$ into $|\xi'| = (1/r)$ with negative orientation.

Then
$$f_2(a+1/z') = \left(\frac{1}{2\pi i}\right)\int_{|\xi'|=1/r} \left(\frac{z'}{\xi'}\right)\frac{f(a+1/\xi')}{(\xi'-z')}d\xi'$$

$$= \sum_{n=1}^{\infty} B_n (z')^n \text{ where}$$

$$B_n = \left(\frac{1}{2\pi i}\right) \int_{|\xi'|=1/r} \frac{f(a+1/\xi')d\xi'}{(\xi')^{n+1}}$$

$$= \left(\frac{1}{2\pi i}\right) \int_{|\xi-a|=r} f(\xi)(\xi-a)^{n-1} d\xi$$

Then $$f(z) = \sum_{-\infty}^{\infty} A_n (z-a)^n, \text{ where each } A_n \text{ is found from (2).}$$

It follows that the integral in (2) is independent of r in $R_1 < r < R_2$.

If $R_1=0$, the point a is an isolated singularity, and $A_{-1} = B_1$ is the residue of a ($\because$ $f(z)-A_{-1}(z-a)^{-1}$ is the derivative of a single-valued function in $0 < |z-a| < R_2$).