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DIRECTORATE OF DISTANCE EDUCATION

MATHEMATICS

B.Sc., THIRD YEAR

PAPER - VIII

LINEAR PROGRAMMING AND OPERATION RESEARCH

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SYLLABUS

LINEAR PROGRAMMING AND OPERATION RESEARCH

Unit - A : LINEAR PROGRAMMING

Unit I Vector spaces over R - Linear independence - Linear combinations - Hyper plane and Hyper space - convex sets and their properties - convex hulls - Hyper planes and half space - Local extremum Global extremem.

Unit II Simplex method - Standard Maximization case - Minimization problem - Artificial variables - Big - M method - Two phase method - Degeneracy - cycling in LPP - Application of simplex method.

Unit III Concept of duality - Duality theorems - Duality and simplex methods - Dual simplex method - Integer programming - cutting plane method - (Gomarian constraint)

Unit IV Assignment model - Formulation of assignment problem - Hungarian method- Knoig;s theorem - Minimization type - Maximizatin type - Unbalanced type - Routing problem - Traveling saleman problem.

Unit V Transportation problem - Introduction and mathematical formulation of TP - Initial feasible solution - Row minima method - Column minima method - Northwest corner method - Least cost method - Vogal's approximation method - Degeneracy in TP - Loops in table - Optimum solution - Modi method - Unbalanced transportation table.

Operation Research

Unit VI Origin and development of OR - Nature and scope of OR - Models in OR - Application of OR - Limitation of OR - Decision making in OR - probability - random variable - Probability density functions - Distribution function - Distribution functions - Poission distribution - exponential distribution.

Unit VII Inventory control - Types of inventory - Division on inventory - EOQ problem for the following types - (i) No shortage and instanteous production, (ii) No shortage, production runs of unequal length, instantaneous production, (iii) No shortage, finite rate replacement (iv) shortage permitted, production rate is minite, optimum level is S (v) Shortage permitted, scheduling period constraint, production rate is infinite

Unit VIII Replacement model - Replacement of items that deteriorates with items - items that fail suddenly sequencing problem - problem of sequencing

Unit IX Game theory - Two person zero sum games - The maximin and minimax principle saddle point - Games without saddle point - Mixed strategies - Solutions of 2x2 games - Graphical method - Method of dominance principles - LP method

Unit X Introduction to Queing Theory - Type of Queue discipline

Reference Books : The M/M/1 queing system - (M/M/1 : ∝/FIFo) model

-(M/M/1 : N/FIFo) model

Operations Research by Kanthiswaroop and other.

Scheme of Lessons

LINEAR PROGRAMMING

OPERATIONS RESEARCH (O.R.)

LINEAR PROGRAMMING

UNIT - 1 : VECTOR SPACES

Throughout this chapter, R stands for the set of real numbers. For any positive integer n, consider the set

$R^n = \{ (x_1, x_2, ...x_n) / x_i \varepsilon R, i=1,2...n\}$

Any element of R^n is called an n-component vector x_i is called the i^{th} component of the vector $x = (x_1, x_2, ...x_n)$. In particular,(0,0,.. 0) is called the zero vector and it is denoted as o. The elements of R are called scalars.

Let $x = (x_1, x_2, .. \quad x_n)$ and $y = (y_1, y_2,..... y_n)$ be two vectors in R^n. Let $a \in R$. We define

$x+y = (x_1+y_2, x_2+y_2. \ ... \ x_n+y_n)$ .... (1)

$\propto x = (\propto x_1, \ \propto x_2, \ ... \ \propto x_n)$.......... ....(2)

These operations are called <u>vector addition</u> and <u>scalar multiplication.</u>

Thus addition of vectors is carried out by component wise addition. Similarly, for multiplication by scalar, each component of the vector is multiplied by the corresponding scalar.

The distance between the two vectors x and y is defined by,

$$|x-y| = \left[\sum_{i=1}^{n} (x_i-y_i)^2 \right]^{1/2}$$

<u>Definition :</u>

R^n together with addition of vectors and scalar multiplication defined by above equation in (1) and (2) is a **<u>vector space</u>** over R

This vector space together with distance concept is called n-dimensional <u>Euclidean space</u> and is denoted by <u>E^n or E_n</u>.

The scalar product or innerproduct of the vectors x and y is defined as

$x.y = x_1y_1 + x_2y_2 + \ ...... \ +x_ny_n$.

We observe that the scalar product of any two vectors is a scalar. Usually, we write xy instead of x.y. Also $\sqrt{x\,x}$. gives the distance of the point x from the origin.

Remarks :

1 The one-dimensional Euclidean space is the usual real line R

2 The two-dimensional Euclidean Space is the usual plane R^2 or RxR.

3. The three-dimensional Euclidean Space is the usual space R^3 Hereafter, we confine ourselves to the vector space R^n

Defintion :

A Subset w of R^n is called a subspace of R^n if x, y$\in$ w and $\alpha \in R \Rightarrow x+y \in w$ and $\alpha x \in w$. In other words if w is a subspace of R^n then w itself is a vector space over R.

Remark :

It can be easily verified that the intersection of any collection of subspaces of R^n is again a subspace.

1.2: Linear combination and Linear indepencence.

Definition :

Let $x_1, x_2, \ldots x_k \in R^n$. Let $\alpha_1, \alpha_2 \ldots \alpha_k \in R$. Then the vector $\alpha x_1+\alpha x_2+\ldots+\alpha_k x_k$ is called a linear combination of the vectors $x_1, x_2, \ldots x_k$.

Definition :

A finite set of vectors $x_1, x_2, \ldots x_k$ in R^n is said to be Linearly independent if $\alpha_1 x_1+\alpha_2 x_2+ \ldots \alpha_k x_k = 0 \Rightarrow \alpha_1=\alpha_2=\ldots\alpha_k=0$, where $\alpha_1,\alpha_2 \ldots \alpha_k$ are scalars.

If $x_1, x_2, \ldots x_k$ are not linearly independent then they are said to be linearly dependent.

Note :-

If $x_1, x_2, \ldots x_k$ are linearly dependent then There exist scalars $\alpha_1,\alpha_2\ldots\alpha_k$ not all zero, such that $\alpha_1 x_1+\alpha_2 x_2+\ldots \alpha_k x_k=0$

Example :- 1

In R^2 consider the vectors $x_1=(1,0)$ and $x_2=(0,1)$. Suppose α_1,α_2 are scalars such that $\alpha_1 x_1+\alpha_2 x_2=0$.

Then,

$$\alpha_1(1,0)+\alpha_2(0,1)=(0,0)$$
$$\Rightarrow (\alpha_1,0)+(0,\alpha_2)=(0,0)$$
$$\Rightarrow (\alpha_1,\alpha_2)=(0,0)$$
$$\Rightarrow \alpha_1=0, \quad \alpha_2=0.$$

Hence the vectors x_1 and x_2 are Linearly independent in R^2.

Example 2 :-

In R^3 the vectors $x_1=(1,0,0)$, $x_2=(0,1,0)$ $x_3=(0,0,1)$ are Linearly independent.

Example 3 :-

In R^2 consider the vectors $x_1=(1,0)$, $x_2=(0,1)$ and $x_3=(1,1)$

suppose $\alpha_1,\alpha_2,\alpha_3$ are scalars such that

$$\alpha_1 x_1+\alpha_2 x_2+\alpha_3 x_3=0 \ldots\ldots\ldots\ldots (1)$$
$$\alpha_1(1,0)+\alpha_2(0,1)+\alpha_3(1,1)=(0,0)$$
$$\Rightarrow(\alpha_1,0)+(0,\alpha_2)+(\alpha_3,\alpha_3)=(0,0)$$
$$\Rightarrow(\alpha_1+\alpha_3, \alpha_2+\alpha_3)=(0,0)$$
$$\Rightarrow \alpha_1+\alpha_3=0 \text{ and } \alpha_2+\alpha_3=0.$$

$\therefore\ \alpha_1=\alpha_3\ ;\ \ \alpha_2=-\alpha_3$

Hence $\alpha_3=-1$ and $\alpha_1-\alpha_2 = 1$

Apply in equation (1)

$$\alpha_1x_1+\alpha_2x_2+\alpha_3x_3 = 0.$$

$\Rightarrow\ x_1+x_2-x_3 = 0$

Hence x_1, x_2, x_3 are Linearly dependent vectors.

Definition :-

Let $S\subseteq R^n$. Then s is called a basis for R^n if every vector in R^n can be uniquely expressed as a Linear combination of vectors in S.

Example :-

In R^n, let $S = \{(1,0,0....,0), (0,1,0,....,0) ....(0,0,0,......1)\} = \{e_1, e_2,.....e_n\}$.

Then S is a basis for R^n called Standard basis.

Remarks :-

We state without proof the following important properties of a basis in a vector space

1. Any basis is a linearly independent set.
2. Any two bases of a vector space have the same number of elements.

Definition :-

The number of elements in a basis of a vector space is called the dimension of the vector space.

Example :-

R^n is a vector space of dimension n, Since $S=\{e_1,e_2,.....e_n\}$ is a basis for R^n and S has elements.

In particular the real line R is a vector space of dimension 1.

The Euclidean plane R^2 is a vector space of dimension 2.

1.3 convex sets

Defintion :-

$U=(u_1,u_2)$, $V=(v_1,v_2)$ be two points in the plane R^2. We know that any point x on the line joining u and v can be written in the form

$$\boxed{x=\lambda u+(1-\lambda)\, v \text{ where } \lambda\varepsilon R.}$$

Thus the set $\{\lambda u+(1-\lambda)v/\lambda\varepsilon R\}$ gives the set of all points lying on the line joining u and v If we restrict $\lambda\varepsilon R$ suchthat $0\leq\lambda\leq1$ then the corresponding set represents the line segment joining the points u and v.

This motivates the following definition.

Let $X=(x_1,x_2, \dots x_n)$, $Y=(y_1, y_2,....y_n)$ be any two vectors in R^n.

Defintion :-

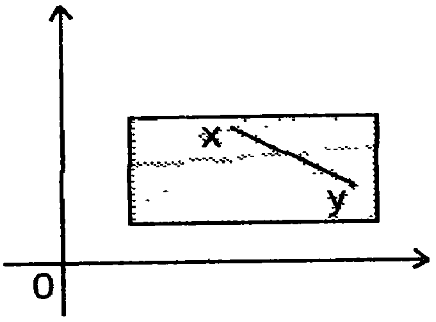
The line passing through the points X and Y is defined to the set $X=\{\lambda x+(1-\lambda)y / \lambda \varepsilon R\}$. The segment of the line joining x and y is defined as the set $\{\lambda x+(1+\lambda)y / 0 \leq \lambda \leq 1\}$.

Defintion :-

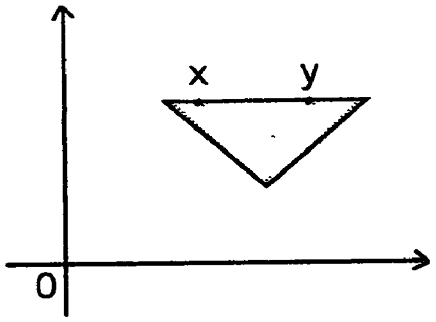
A subset S of R^n is called a convexset if, for any two elements $x, y \varepsilon s$ and for any real number λ with $0 \leq \lambda \leq 1$. The point $(\lambda x+(1-\lambda)y) \varepsilon S$.

In otherwords, S is convex if for any two points in S, the line segment joining these two points lies entirely within S.

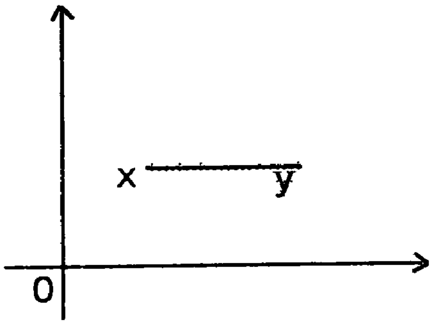
The regions in R^2 given in the following figures are convex sets.



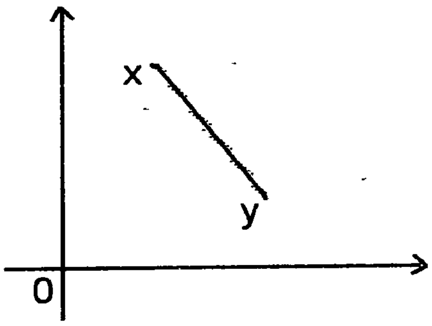
convex sets



convex set

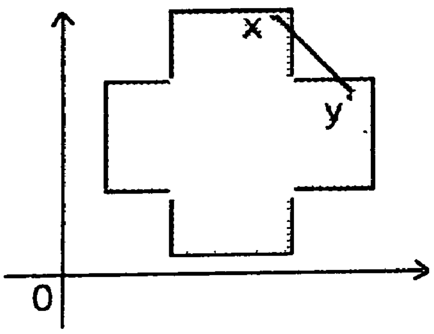


convex sets

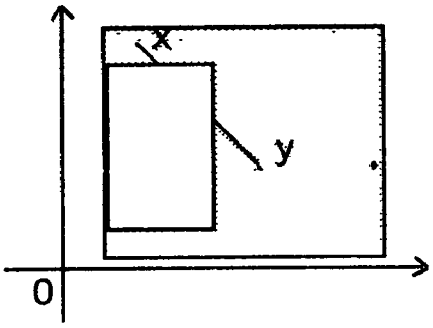


convex sets

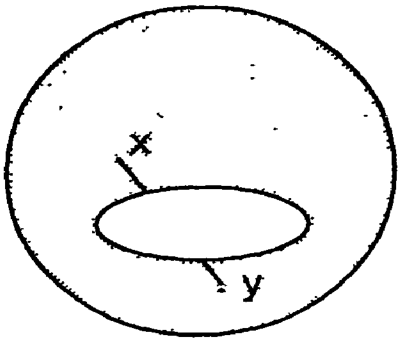
The regions in R^2 given in the following figures are not convex sets.



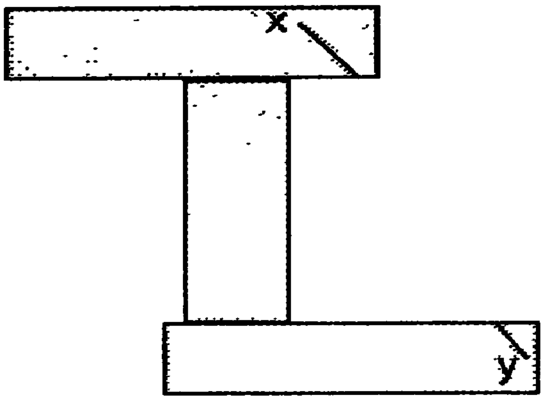
not convex set



not convex set



not convex set



not convex set

Example :-

Let $x_0 \varepsilon R^n$ and $r>0$.

Let $S=\{x / x \varepsilon R^n$ and $/x-x_0/<r\}$. Then S is a convex set.

Proof :-

Let x_1, $x_2 \varepsilon S$ and $0 \leq \lambda < 1$.

Since $x_1, x_2 \varepsilon S$, we have $|x_1-x_0| <r$ and

$/x_2-x_0/<r$ .... (1)

Claim :-

$\lambda x_1+(1-\lambda)x_2 \varepsilon S$

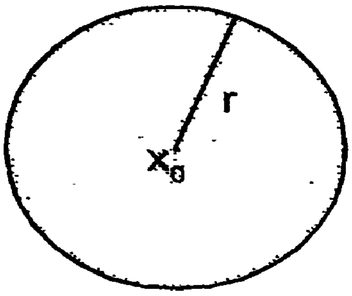
now,

$$
\begin{aligned}
|\lambda x_1+(1-\lambda)x_2-x_0| &= |\lambda x_1+(1-\lambda)\, x_2-\lambda x_0+\lambda x_0-x_0| \\
&= |\lambda(x_1-x_0)+(1-\lambda)x_2+x_0(\lambda-1)| \\
&= |\lambda(x_1-x_0)+(1-\lambda)(x_2-x_0)| \\
&\leq |\lambda(x_1-x_0)| + |(1-\lambda)\,(x_2-x_0)| \\
&\qquad \text{(using } \Delta^{le} \text{ inequality)} \\
&= |\lambda|\,|(x_1-x_0)| + |(1-\lambda)|\,|(x_2-x_0)| \\
&= \lambda\,|(x_1-x_0)| + (1-\lambda)|(x_2-x_0)| \\
&< \lambda r+(1-\lambda)r \qquad \therefore (0 \leq \lambda \leq 1) \\
&= r \qquad [\text{using } (1)]
\end{aligned}
$$

Thus

$\lambda x_1+(1-\lambda)x_2 \varepsilon S$.

$\therefore$ Sis a convex set.



Sis a convex set.

Definition :-

Let $x_0 \varepsilon R^n$ and $r>0$. Let $S=\{x/x\varepsilon R^n$ and $|(x-x_0)| <r\}$ The convex set S is called an open neighbourhood with centre x_0 and radius r and is denoted by $N(x_0,r)$.

Note :-

1) In R^2 an open neighbourhood of x_0 is a Circular disc.

2) In R^3 an open neighbourhood of x_0 is the interior of a sphere.

Theorem :-

Let $C= (c_1, c_2, ...c_n) \varepsilon R^n$. Let $Z \varepsilon R$.

Let $H= \{x/x\varepsilon R^n$ and $C.x =z\}$

(ie.) H is the set of all points $X=\{x_1,x_2,....x_n\}$

satisfying the equation $c_1x_1+c_2x_2+.....+c_nx_n=z$.

Then H is a convex set.

Proof :-

Let x_1, $x_2 \varepsilon H$ and $0\leq\lambda\leq 1$

Let $X_1=(x'_1,x'_2,.....x'_n)$ and $X_2=(x''_1, x''_2, x''_3,....x''_n)$

Since X_1, X_2, $\in H$, we have

$$\left.\begin{array}{l} c_1x'_1+c_2x'_2+.......+c_nx'_n = Z \\ c_1x''_1+c_2x''_2+........+c_nx''_n = Z \end{array}\right\}....①$$

now,

$$\begin{aligned} C[\lambda x_1+(1-\lambda)x_2] &= C.[\lambda(x_1',x_2',......x'_n) + (1-\lambda)(x''_1,x''_2,......x''_n)] \\ &= \lambda(c_1x'_1+c_2x'_2+.......+c_nx'_n)+(1-\lambda)(x'_1,x''_2,.....x''_n) \\ &= \lambda z + (1-\lambda)Z \quad \text{[using (1)]} \\ &= Z. \end{aligned}$$

Thus $\lambda x_1+(1-\lambda)\, x_2 \varepsilon H$.

∴H is a convex Set.

Definition :-

Let $C=(c_1,c_2,.....c_n) \varepsilon R^n$. Let $Z\varepsilon R$ and $H=\{x/x\varepsilon R^n$ and $C.x=z\}$. The convex set H is called a Hyper Plane in R^n.

Remarks :-

1) In R^2 the hyperplane $H=\{(x_1, x_2)\varepsilon R^2$ and $c_1x_1+c_2x_2=z\}$ is a Straight line.

2) In R3 the hyperplane $H=\{(x_1,x_2,x_3) \varepsilon R^3$ and $c_1x_1+c_2x_2+c_3x_3=z\}$ is a plane.

Any hyperplane H in R^n defined above determines. The following four subsets of R^n in a natural way. Let $C=(c_1, c_2,..............c_n) \varepsilon R^n$.

Let $Z \varepsilon R$

$H_1 = \{x/x\varepsilon R^n \text{ and } c.x<z\}$

$H_2 = \{x/x\varepsilon R^n \text{ and } c.x\leq z\}$

$H_3 = \{x/x\varepsilon R^n \text{ and } c.x>z\}$

$H_4 = \{x/x\varepsilon R^n \text{ and } c.x\geq z\}$

It can be easily verified that H_1, H_2, H_3 and H_4 are convex sets.

H1 and H3 are open half spaces determined by the hyper plane H.

H2 and H4 are closed half spaces determined by the hyper plane H.

Definition :-

Let $S \subseteq R^n$. Let $x_0 \varepsilon S$. The point x_0 is called an interior point of S if there exists a real number r>0 suchthat $N(x_0,r)\subseteq S$. Then S is called an open set if every point of S is an interior point of S.

Definition:-

Let $S \subseteq R^n$ and $x_0 \varepsilon R^n$. The point x_0 is called a boundary point of S if every neighbourhood of x_0 contains points of S and points not in S.

S is called a closed set if S contains all its boundary points.

Remark:-

Let $xy\varepsilon R^n$. Then the vector $\lambda x+(1-\lambda)y$ where $\lambda\varepsilon R$ is suchthat $0\leq\lambda\leq 1$ is called a convex combination of x and y.

More generally if $x_1,x_2, \ldots x_k \varepsilon R^n$ then any vector of the form $\lambda_1x_1+\lambda_2x_2+\ldots+\lambda_kx_k$ where λ_i are suchthat $0\leq\lambda_i\leq 1$ and $\sum_{i=1}^{k}\lambda_i=1$ is called a convex combination of $x_1,x_2,\ldots x_k$.

Thus a set $S\subseteq R^n$ is a convex set if the convex combination of any two elements of S is again an element of S.

Theorem :-

Intersection of any convex sets in R^n is again a convex set in R^n.

Proof :-

Let S_1 and S_2 be two convex sets in R^n.

Let $S=S_1\cap S_2$

Let x_1, $x_2\varepsilon S$ and $\lambda\varepsilon R$ be suchthat $0\leq\lambda\leq 1$. Since $x_1,x_2\varepsilon S = S_1\cap S_2 \Rightarrow x_1\varepsilon S_1$ and S_2 and $x_2\varepsilon S_1$ and S_2.

Since S_1 is a convex set.

we have;

$\lambda x_1+(1-\lambda)x_2\varepsilon S_1$ ($\because x_1\varepsilon S_1$ and $x_2\varepsilon S_1$)

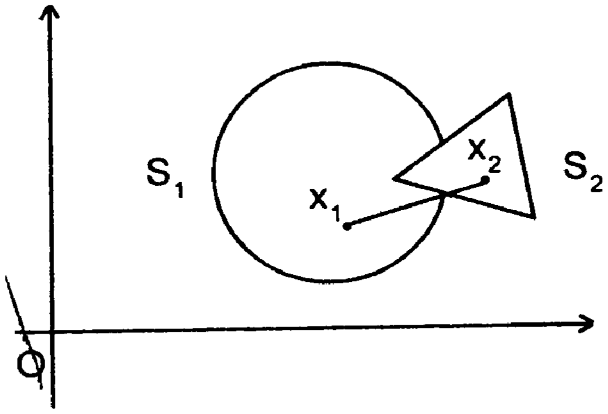
III^{ly} $\lambda x_1+(1-\lambda)x_2\varepsilon S_2$

$\because \lambda x_1+(1-\lambda)x_2\varepsilon S_1\cap S_2 = S$

Hence S is a concex Set.

Note :-

1) Union of two convex sets need not be a convex set. (Refer figure)



Theorem :-

Let $S=\{x_1,x_2,\ldots x_k\} \subseteq R^n$. Then the set of all convex combinations of $(x_1,x_2,\ldots x_k$ is a *convex set* in R^n.

Proof:-

Let $C(s) = \{\lambda_1x_1+\lambda_2x_2+\ldots\ldots\lambda_kx_k / 0\leq\lambda_i\leq1, \sum\limits_{i=1}^{k}\lambda_i=1\}$ be the set of all convex combinations of $x_1, x_2,\ldots\ldots x_k$.

Let $x, y \varepsilon C(s)$ and $0\leq \lambda \leq1$.

$\therefore$ $x=\lambda_1x_1+\lambda_2x_2+\ldots\ldots+\lambda_kx_k$ where $0\leq\lambda_i\leq1$ and $\sum\limits_{i=1}^{k}\lambda_i=1$

and $y=\mu_1x_1+\mu_2x_2+\ldots\ldots+\mu_kx_k$ where $0\leq\mu_i\leq1$ and $\sum\limits_{i=1}^{k}\mu_i=1$

Now,

$$\begin{aligned}\lambda x+(1-\lambda)y &= \lambda[\lambda_1x_1+\ldots\ldots\ldots+\lambda_kx_k] + (1-\lambda)[\mu_1x_1+\ldots\ldots+\mu_kx_k) \\ &= \sum_{i=1}^{k} [\lambda_i\lambda+(1-\lambda)\mu_i]\, x_i \\ &= \sum_{i=1}^{k}\gamma_i x_i \text{ where } \gamma_i = \lambda_i\lambda+(1-\lambda)\mu_i\end{aligned}$$

Now,

Since $0\leq\lambda\leq1$, $0\leq\lambda_i\leq1$ and $0\leq\mu_i\leq1$

we have

$0\leq\gamma_i\leq1$ for $i = 1, 2\ldots\ldots k$

Further,
$$\sum_{i=1}^{k} \gamma_i = \sum_{i=1}^{k} [\lambda_i\lambda+(1-\lambda)\mu_i]$$
$$= \lambda\sum_{i=1}^{k}\lambda_i + (1-\lambda)\sum_{i=1}^{k}\mu_i$$
$$= \lambda+(1-\lambda) \quad \left[\because \sum_{i=1}^{k}\lambda_i=1 \text{ and } \sum_{i=1}^{k}\mu_i=1\right]$$
$$= 1$$

$\therefore \lambda x+(1-\lambda)y$ is a convex combination of $x_1, x_2 \ldots\ldots x_k$.

$\therefore$ $\lambda x+(1-\lambda)\, y \in C(S)$.

$\therefore$ C(S) is a convex set.

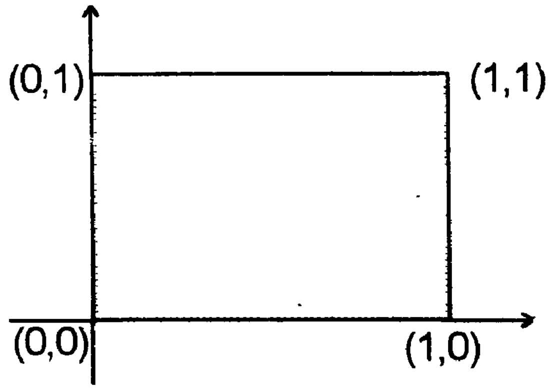
1.4. : Convex Hull

Definition :-

Let $S \subseteq R^n$. Then C(S) is called the convex hull of S. In Particular, if S is finite Then C(S) is called a convex polyhedron. Also C(S) is said to be spanned by (generated by) S.

Example :-

Let S={(0,0),(1,0),(0,1),(1,1)} The convex Polyhedron C(S) generated by the the set S is the closed unit squarl given in the figure.



Definition :-

Let S be a convex set. A point $x \in S$ is called an extreme point of S if x cannot be expressed in the form,

$x=\lambda x_1+(1-\lambda)x_2$ where $x_1, x_2 \in S$ and $x_1 \neq x_2$ and $0<\lambda<1$.

ie) x is an extreme point of S iff x is not an interior point of any line segment joining two distinct points of S.

Example : 1

Let S denote the convex set consisting of the set of all points in R^2, which lie inside or on a triangle. The verticles of the triangle are the extreme points of the convex sets.

Example : 2

Let S be the convex set given by $s=\{x/x=(x_1,x_2)/x_1^2+x_2^2\leq 1\}$ ie) S is the closed circular disc with centre origin and radius unity. All points lying on the circumference of the circle, $x_1^2+x_2^2=1$ are the extreme points of S.

Exercises:-

which of the following subsets of R^2 are convex?

(i) $\{x,y)/x^2+y^2\leq 1$ and $x^2+y^2\geq 3\}$

(ii) $\{x,y)/0\leq x\leq z$ and $0\leq y\leq 2\}$

(iii) $\{x,y)/x\geq 0\} \cup \{(x,y)y\geq 0\}$

(iv) $\{x,y)/0\leq x \leq 2\}$

(v)' $\{x,y)/2x+3y=0\}$

Answer :-

(ii), (iv) and (v) are convex.

1.5 : LOCAL AND GLOBAL EXTREMA :-

Definition 1 :- (Global minima)

A global minimum of the function f(x) is said to be attained at x_0 if $f(x_0)\leq f(x)$ $\forall x$ in the feasible solution.

Example :-

Function $f(x)=x_1^2$, subject to the constraint $x_1\geq 0$, has a minimum at $x_1=0$.

Definition 2 :(Local minima)

A local minimum $f(x_0)$ of function f(x) is said to be attained at x_0 if there exists a positive ε suchthat $f(x_0)\leq f(x)$ for all x in the feasible region which also satisfy the condition $|x_0-x|\leq\varepsilon$.

Example :

The function $f(x)=x_1^2-x_1^3$, subject to the constraint $x_1\geq 0$, has a local maximum at $x_1=0$ (with $\varepsilon=\frac{2}{3}$).

Note that f(x) has no global minimum at all.

Note :- The word extremum (plural extrema) is used to indicate either maximum or minimum.

Theorem :-

Let f(x) be a convex function on a convex set S. If f(x) has a local minimum on S Then this local minimum is also a global minimum on S.

Proof :-

Let a local minimum $f(x_0)$ be attained at x_0. Then, there exists atleast one x_1 in $S(x_1 \neq x_0)$ such that $f(x_1)<f(x_0)$.

Since f(x) is a convex function on S

we have

$$f[\lambda x_1+(1-\lambda)x_0] \leq \lambda f(x_1) + (1-\lambda)f(x_0)$$

Also

$$\lambda f(x_{1)}+(1-\lambda)f(x_0) < \lambda f(x_0) + (1-\lambda)f(x_0) = f(x_0)$$

$$\therefore \; f[\lambda x_1+(1-\lambda)x_0] \leq f(x_0)$$

Now, for any $\varepsilon>0$, we observe that

$$\left|[\lambda x_1+(1-\lambda)x_0] - x_0\right| = \left|\lambda (x_1-x_0)\right| < \varepsilon$$

so long as $\lambda<\varepsilon<|x_1 - x_0|^{-1}$

and clearly such a $\lambda <1$ does exist whenever

$$\varepsilon<|x_1-x_0|$$

$\therefore$ $\lambda x_1+(1-\lambda)x_0$ will give a smallar value for f(x) in the neighbourhood of x_0, whenever is so chosen as desired above.

This contradicts the fact that f(x) takes on a local minimum at x_0.

Hence x_0 is a global minima

Theorem :-

Let f(x) be a convex function on a convex set S. Then the set of points in sat which f(x) takes on its global minimum, is a covex set.

Proof :-

The result is obvious if the global - minimum is attained at just a single point. Let us assume that the global minimum is attained at two different points x_1 and x_2 of S. Then $f(x_1) = f(x_2)$.

Since f(x) is convex

$$f\,[\lambda x_2+(1-\lambda)x_1 \leq f\,(x_2)+(1-\lambda)f(x_1) = f(x_2),\; 0\leq\lambda\leq 1$$

$$\Rightarrow f\,[\lambda x_2+(1-\lambda)x_1] \;=\; f(x_2) = f(x_1)$$

Thus every point $x = \lambda x_2+(1-\lambda)x_1$ corresponds to a global minima

The set of all such x is, obviously, a convex set. Hence the theorem.

Introduction to L.P.P.:-

Many business and economic situations are concerned with a problem of planning activity. In each case, there are limited resources at your disposal and your problem is to make such a use of these resources so as to yield the maximum production or to minimise the cost of production, or to give the maximum profit, etc. Such problems are referred to as the problems of constrained optimisation.

Linear programming is a technique for determining an optimum schedule of interdependent activities in view of the available resoures.

Programming is just another word for 'planning' and refers to the process of determining a particular plan of action from amongst several alternatives. The word linear stands for indicating that all relationships involved in a particular problem are linear.

In the present - chapter, some simple examples of linear programming problems, their mathematical formulation and their solution by graphical methods are discussed

1.6 : Mathematical formuation of L.P.P :-

The procedure for mathematical formulation of linear programming problem consists of the following major steps:-

Step 1 :-

write down the decision variables of the problem.

Step 2 :-

Formulate the objective function to be optimised (maximised or minimised) as a linear function of the decision variables.

Setp 3 :-

Formulate the other conditions of the problem such as resource limitations, market constraints, inter-relation between variables etc. as linear equations or inequations in terms of the decision variables.

Setp 4 :-

Add the 'Non-negativity' constraint from the consideration that negative values of the decision variables do not have any valid physical interpretation.

The objective function, the set of constraints and the non-negative constraint together form a linear programming problem.

Problem 1.-

A carpenter has 100 sq. feet teak wood and 80sq. feet rose wood. He wants to make tables and book shelves utilizing these two woods only. A table requires 16sq. feet of teak wood and 8sq. feet of rose wood where as a book shelf requires 12 sq. feet teak wood and 16 sq. feet rose wood. He wants to earn Rs.25 per table and Rs. 20 per bookshelf. How many tables and bookshelves can be made to earn maximum profit out of his available stock of woods. Give a mathematical formulation of the linear programming problem.

solution:-

Let x_1 be the number of tables and x_2 be the number of bookshelves the carpenter wants to make.

The details of availability and requirement of materials and profit on each unit are given in the following table

	Table	Bookshelf	Availability
Teakwood	16 sq. feet	12 sq. feet	100 sq. feet
Rose wood	8 sq. feet	16 sq. feet	80 sq. feet
Profit	Rs. 25	Rs. 20	

Constraint on Teak wood is : $16x_1+12x_2 \leq 100$

Constraint on Rose wood is . $8x_1+16x_2 \leq 80$

Non-negative constraints are : $x_1, x_2 \geq 0$.

The objective function is : $Z = 25x_1+20x_2$.

The mathematical formulation of the L.P.P. is :

Maximize

$$Z= 25x_1+20x_2$$

Subject to

$$16x_1+12x_2 \leq 100$$

$$8x_1+16x_2 \leq 80$$

$$x_1, x_2 \geq 0$$

Problem : 2

An egg contains 6 units of vitamin A pergram and 7 units of vitamin B per gram and costs 12 paise per gram. Milk contains 8 units of vitamin A and 12 units of vitamin B pergram and costs 20 paise pergram. The daily minimum requirements of vitamin A and vitamin B are 100 units and 120 units respectively. Find the optimal product mix. Formulate a L.P. model for the above problem.

Solution :-

Let x_1 and x_2 be the number of units of egg and milk respectively. The details of the product mix of the problem are given in the following table :

	Vitamin A	Vitamin B	Cost
egg	6	7	12
Milk	8	12	20
Requirements	100	120	

The L.P.P. model of the problem is given below:

Minimize

$$Z=12x_1+20x_2$$

Subject to

$6x_1+8x_2 \geq 100$
$7x_1+12x_2 \geq 120$
$x_1, x_2 \geq 0$

Problem 3:-

An electronic company manufactures two models of toy dolls each by separate production batch. The daily capacity of production of first model is 60 toy dolls and that of the second model is 75 toy dolls. Each unit of first model uses 10 pieces of certain electronic component whereas each unit of second model uses 8 pieces of it. Maximum ready availability of the electronic component is 800 pieces. Profit per unit of first and second models are Rs. 30 and Rs. 20 respectively. What should be the daily production of each model to get maximum profit from the two models? Give a mathematical formulation to the L.P.P.

Solution :-

Let x_1, x_2 be the number of units of production of two models of toy dolls.
constraint on capacity of first model is : $x_1 \leq 60$
Constraint on capacity of second model is : $x_2 \leq 75$
Constraint on availability of electronic components for the two models is $10x_1+8x_2 \leq 800$.
Total Profit on two models is $Z=30x_1+20x_2$
The non-negative constrainsts are $x_1, x_2 \geq 0$
The mathematical formulation of the L.P.P. is Maximize $Z=30x_1+20x_2$

subject to

$10x_1+8x_2 \leq 800$
$x_1 \leq 60$
$x_2 \leq 75$
$x_1, x_2 \geq 0$

Problem 4 :-

A person requires 15, 13, 14 units of chemicals A, B, c respectively for his mango grove. There chemicals are available in liquid product in jars and dry products in bags. The liquid product contains 6 units of A, 2units of B and 2 units of C per jar and the dry product contains 2 units of A, 3 units of B and 4 units of C per bag. If the liquid product costs Rs. 8 per jar and the dry product costs Rs. 3 per bag, how many of each should he purchase in order to minimize the cost and meet the requirements? Express this problem as a L.P.P.

solution :-

Let x_1 be the number of units of liquid product to be purchased and x_2 the number of units of dry product to be purchased.

The details of the problem are given in the following table.

Chemicals	Liquid product	Dry product	Requirements
A	6	2	15
B	2	3	13
C	2	4	14
Cost	8	3	

Objective function is

$$Z=8x_1+3x_2$$

Maximum requirement for chemical A is 15 units.

Constraint on chemical A is : $6x_1+2x_2 \geq 15$

III ^{ly} constraint on chemical B is : $2x_1+3x_2 \geq 13$

III ^{ly} constraint on chemical C is : $2x_1+4x_2 \geq 14$

Obviously, $x_1, x_2 \geq 0$

Hence the mathematical form of the LPP is

Minimize $Z = 8x_1+3x_2$

Suject to

$$6x_1+2x_2 \geq 15$$
$$2x_1+3x_2 \geq 13$$
$$2x_1+4x_2 \geq 14$$
$$x_1 , x_2 \geq 0$$

Exercises :-

1) A student with self-employment motive wishes to purchase and sell video and audio cassettes. He has Rs 5,760 only to invest and wants to posses at most 20 packs at a time. A video cassette pack costs him Rs. 360 and an audio cassette pack costs him Rs. 240. His expectation is that he can sell a

video cassette at a profit of Rs. 22 per pack and that an audio cassette at a profit of Rs. 18 per pack. Assuming that he can sell all the packs that he can buy, how should he invest his money in order to maximize his profit? Formulate it as a L.P.P

Problem 2 :-

A company produces two types of pen A and B. Pen A is superior quality and pen B is a lower quality. Profit on pens A and B is Rs. 5 and Rs. 3 respectively. Raw materials required for each pen A is twice as that of pen B. The suppplly of raw materials is sufficient only for 1000 pens of type B perday. Pen A requires a special clip and only 400 such clips are available perday. For Pen B only 700 clips are available perday Find the product mix so that the company can make maximum profit. Prepare a linear programming model for this problem.

Problem 3 :-

Vitamin A and Vitamin B are found in two different foods F_1 and F_2. One unit of food F_1 contains 2 units of vitamin A and 5 units of vitamin B. One unit of food F_2 contains 4 units of vitamin A and 2 units of vitamin B. One unit of food F_1 and F_2 costs Rs. 10 and Rs. 15 respectively. The minimum daily requirements of vitamins A and B is 40 and 50 units respectively. Assuming that anything in excess of daily minimum requirement of vitamins A and B is not harmful findout the optimal units of foods F_1 and F_2 at the minimum cost that meets the daily minimum requirements of vitamin A and B. Formulate this as a L.P.P.

Problems 4 :-

A firm manufactures three products A, B and C Their profit per unit are Rs. 300, Rs. 200 and Rs. 400 respectively. The firm has two machines and the required processing time in minutes on each machine for each product is given in the following table.

Machine	Product		
	A	B	C
1	4	3	5
2	2	2	4

Machines 1 and 2 have 2000 and 2500 machine minutes respectively. The upper limit for the production volumes of the products A,B,C are 100 units, 200 units and

500 units respectively. But the firm nust produce a minimum of 50 units of the product A. Develop a L.P molel for this manufacturing situation to determine the production volume of each product such that the total profit is maximized.

1.7 Canonical form of a Linear Programming Problem :-

a) The Condition minimize z is equivalent to maximize (-z). Hence, in any L.P.P, the objective function can always be expressed in maximizing type.

The value of minimum Z=-maximum (-z) (or) equivalently

maximum Z = -minimum (-z)

b) The inequality $x \geq y$ is equivalent to $-x \leq -y$. Also, the equality $x=y$ can be replaced by $x \leq y$ and $x \geq y$.

Using these observations all the constraints of a L.P.P. can be expressed in terms of $\leq$ types.

Hence any general LPP can always be put in the form

$$\text{Maximize } Z = \sum_{j=1}^{n} c_j x_j$$

Subject to

$$\sum_{j=1}^{n} a_{ij} \; x_j \leq b_j, \; \forall i = 1,2,........m$$

$$x_i \geq 0 \quad j = 1,2,........n$$

This is called the canonical form of the LPP.

1.8 : Standard form of a Linear programming problem:-

a) consider a constraint of the form

$$a_{11}x_1 + a_{12}x_2 + ...+a_m x_n \leq b_1$$

It is of < type inequation.

This constraint can be expressed in an equation form by introducing a new variable S_1 as follows.

$$\boxed{a_{11}x_1 + a_{12}x_2 + ...+a_m x_n + S_1 = b} \qquad \text{Where } S_1 \geq 0$$

The new variable S_1 is called the slack variable.

b) Consider a constraint of the form

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n \geq b_1$

It is of > type inequation.

This constraint can be expressed in an equation form by introducing a new variable S_1 as follows.

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n - S_1 = b_1$ where $S_1 \geq 0$
The new variable S_1 is called the surplus variable.
Hence using the above observations any LPP can be expressed in the following form.
Maximize

$$Z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n + OS_1 + OS_2 + \ldots\ldots + OS_m$$

Subject to

$$\sum_{j=1}^{n} a_{ij}\, x_j \pm S_i = b_j \; ; \quad i = 1,2,\ldots\ldots m \;\; \& \; j = 1,2,\ldots\ldots n$$

$$x_j \geq 0 \; ; \; S_i \geq 0$$

This is know as standard form of the LPP.

Solved problems :-

1) Write the following Linear programming problem in standard form.
Maximize $z = x_1 + x_2 + 3x_3$
Subject to

$x_1 + 3x_2 - x_3 \leq 15$
$2x_1 + x_2 + x_3 \leq 10$
$x_1, x_2, x_3 \geq 0.$

Solution :-

Here the objective function is of maximizing type and the first two constraints are $\leq$ type

Hence introducing two slack variables
$S_1, S_2 > 0$
The LPP becomes,

Maximize

$$Z = x_1 + x_2 + 3x_3 + OS_1 + OS_2$$

Subject to

$x_1+3x_2-x_3+S_1 = 15$

$2x_1+x_2+x_3+S_2 = 10$

$x_1, x_2, S_1, S_2 \geq 0$

This is the standard form of the given LPP.

2) Write the following LPP in standard form Minimize

$Z= 2x_1+5x_2+x_3$

Subject to

$x_1+3x_2-4x_3 \leq 20$

$2x_1+x_2+x_3 \geq 10$

$x_1+4x_2+5x_3=10$

$x_1, x_2, x_3 \geq 0.$

solution :-

The objective function is of minimize type. Changing to maximizing type we can write it as

Maximize

$Z^* = -2x_1-5x_2-x_3$

Where $Z^* = -Z$

Since the first two constraints are respectively $\leq$ and $\geq$ type, we introduce a slack variable $S_1 \geq 0$ and surplus variable $S_1 \geq 0$.

The third constraint is an equation. Hence the standard form of the LPP is :

Maximize

$Z^* = -2x_1-5x_2-x_3+O.S_1+O.S_1$

Subject to

$x_1+3x_2-4x_3+S_1=20$

$2x_1+x_2+x_3-S_2=10$

$x_1+4x_2+5x_3=10$

$x_1, x_2, x_3\ S_1, S_2 \geq 0.$

3) Write the following LPP in Standard form :

Minimize

$Z = 2x_1-3x_2+x_3$

Subject to

$-x_1+3x_2 \leq -5$

$x_1+2x_2+x_3 \leq 6$

$x_1+x_2+x_3 \geq -8$

$x_1,x_2,x_3 \geq 0$

Solution :

The first constraint is of $\leq$ type with the right hand side being negative. To make the right hand side positive we multiply by -1 on both sides.

Hence we have

$$x_1-3x_2 \geq 5.$$

Introducing slack variables S_2, S_3 and surplus variable S_1 for the required constraints The standard form of the LPP is

Maximize

$Z^* = -2x_1+3x_2-x_3$

where $Z^* = -Z$

Subject to

$x_1-3x_2-S_1=5$

$x_1+2x_2+x_3+S_2 = 6$

$-x_1-x_2-x_3+S_3 = 8$

$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0.$

4) Write the following LPP in standard form Maximize

$Z = x_1+2x_2+3x_3$

Subject to

$x_1-x_2 \leq 10$

$x_1+2x_2+2x_3 \geq 15$

$3x_1-x_2+x_3 \leq -2$; $x_1, x_2 \geq 0$ and x_3 is unrestricted.

sol :- There are 3 decision variables x_1, x_2, x_3 of which only x_1 and x_2 are restricted while x_3 is unrestricted.

Let $x_3 = x'_3-x''_3$ where $x'_3 \geq 0$ and $x''_3 \geq 0$

The objective function now becomes,

$Z = x_1+2x_2+3(x'_3-x''_3)$

$Z = x_1+2x_2+3x'_3-3x''_3$

Hence the constraints now becomes,

$x_1-x_2 \leq 10$

$x_1 + 2x_2+2x'_3-2x''_3 \geq 15$

$-3x_1+x_2-x_3 \geq 2$

$\Leftrightarrow -3x_1+x_2-(x'_3-x''_3) \geq 2$

$\Leftrightarrow -3x_1+x_2-x'_3+x''_3 \geq 2$

$x_1, x_2, x'_3, x_3'' \geq 0$

Introducing slack and surplus variables the standard form of the LPP is :-

Maximize

$Z = x_1+2x_2+3x'_3+3x''_3+O.S_1+O.S_1+O.S_2$

subject to

$x_1-x_2+S_1 = 10$

$x_1+2x_2+2x_3-2x''_3-S_1 = 15$

$-3x_1+x_2-x'_3+x''_3-S_2 = 2$

$x_1, x_2, x'_3, x_3'', S_1, S_1, S_2 \geq 0.$

Exercises :-

I. Write the following LPP in standard form :-

(i) Maximize

$Z = 2x_1+3x_2$

Subject to $-x_1-2x_2 \leq 4$

$x_1 + x_2 \geq 5$

$x_1-3x_2 \leq -6$

$x_1, x_2 \geq 0.$

2. Minimize

$Z = 2x_1+3x_2+x_3$

subject to

$x_1-x_2+x_3 \geq 10$

$x_1+x_2+x_3 \leq 15$

$x_2, x_3 \geq 0$ and x_1 is unrestricted.

3) Minimize

$Z = 3x_1+2x_2+5x_3$

Subject to

$2x_1+3x_2-2x_3 \leq 40$

$4x_1-2x_2+x_3 \leq 24$

$x_1-5x_2-6x_3 \geq 2$

$x_1, x_2 \geq 0$, x_3 unrestricted.

4) Maximize

$Z = 8x_1-4x_2$

subject to

$4x_1+5x_2 \leq 20$

$x_1-3x_2 \leq 23$

$x_1 \geq 0$, x_2 unrestricted.

II. Write the following LPP in canonical form:-

1. Maximize $Z = x_1+2x_2-3x_3$,

Subject to $x_1-2x_2 \leq 5$

$x_1+3x_3 \geq -7$

$x_1+x_2+x_3 = 10$ $\qquad x_1, x_2, x_3 \geq 0$.

2) Minimize

$Z=3x_1-2x_2$

Subject to

$x_1+2x_2 < 3$

$2x_1-3x_3 \geq -7$

$x_1, x_2 \geq 0$

1.9 GRAPHICAL METHOD OF SOLVING L.P.P. :-

The optimal solution of a Linear Programming problem involving two decision variables x_1 and x_2 can be obtained by graphical method. In this method we geometrically describe the region of feasible solution of the given LPP as a subset of the Euclidean plane R^2 (x_1, x_2 plane)

Problems :-

Solve graphically the LPP :-

Maximize

$Z = 3x_1+5x_2$

Subject to

$x_1 \leq 4$

$x_2 \geq 6$

$3x_1+2x_2 \leq 18$

$x_1 >0, x_2 \leq 0.$

Solution :-

$x_1=4, x_2=6$; (4,0) and (0,6)

$3x_1+2x_2=18.$

put $x_2=0$; $3x_1+2x_2=18$

$3x_1=18$

$\Rightarrow x_1=6$

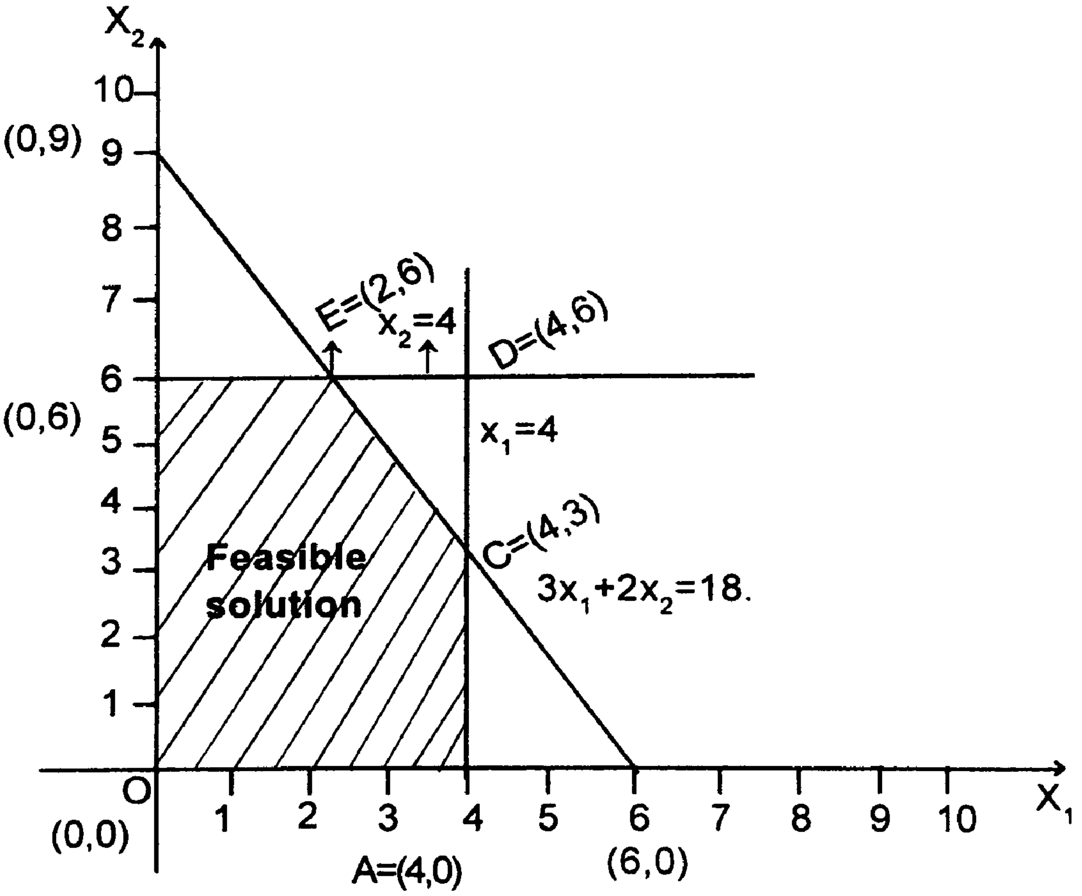
(6,0)

put $x_1=0$; $3x_1+2x_2=18$

$2x_2 = 18$

$\Rightarrow x_2 = 9$

(0,9)



O (0,0), A=(4,0), B=(6,0)

C is the point of intersection of

$x_1=4 \quad ; \quad 3x_1+2x_2=18$

$\Rightarrow 3x4+2x_2=18$

$\Rightarrow 2x_2=18-12$

$\Rightarrow x_2=3 \quad ; \quad C=(4,3)$ and D= (4,6)

E is point of intersection of $x_2=6 \quad ; \quad 3x_1+2x_2=18$

$3x_1+2(6) = 18$

$\Rightarrow 3x_1+12 = 18 \Rightarrow 3x_1 = 18-12=6 \Rightarrow 3x_1=6$

$\Rightarrow x_1=2 \quad ; \quad E = (2,6)$ and $F=(0,6)$

(i) O = (0,0)

z=0

(ii) A = (4,0) ; $Z = 3x_1+5x_2$

Z= 3.4 +5.0

Z = 12

(iii) C = (4,3)

$Z = 3x_1+5x_2$

Z = 3.(4) +5.(3) = 12+15

$\Rightarrow Z = 27$

(iv) E = (2,6)

$Z = 3x_1+5x_2$

$\Rightarrow Z = 3.2 + 5.6 \Rightarrow Z = 6+30 = 36$

$\Rightarrow Z = 36$

(v) F = (0,6)

$Z = 3x_1+5x_2$

$\Rightarrow Z = 5.6 = 30$

$\Rightarrow Z = 30$

Hence the

maximum Value Z = 36.

Hence the corresponding solution E = (2,6)

2) Solve by graphical method the LPP.

Maximize

$Z = 4x_1+3x_2$

Subject to

$2x_1-3x_2 \leq 6$

$6x_1+5x_2 \geq 30$

$x_1, x_2 \geq 0$

Solution :

The constraint equations $2x_1-3x_2 = 6$.........(1) and

$6x_1+5x_2=30$........ (2)

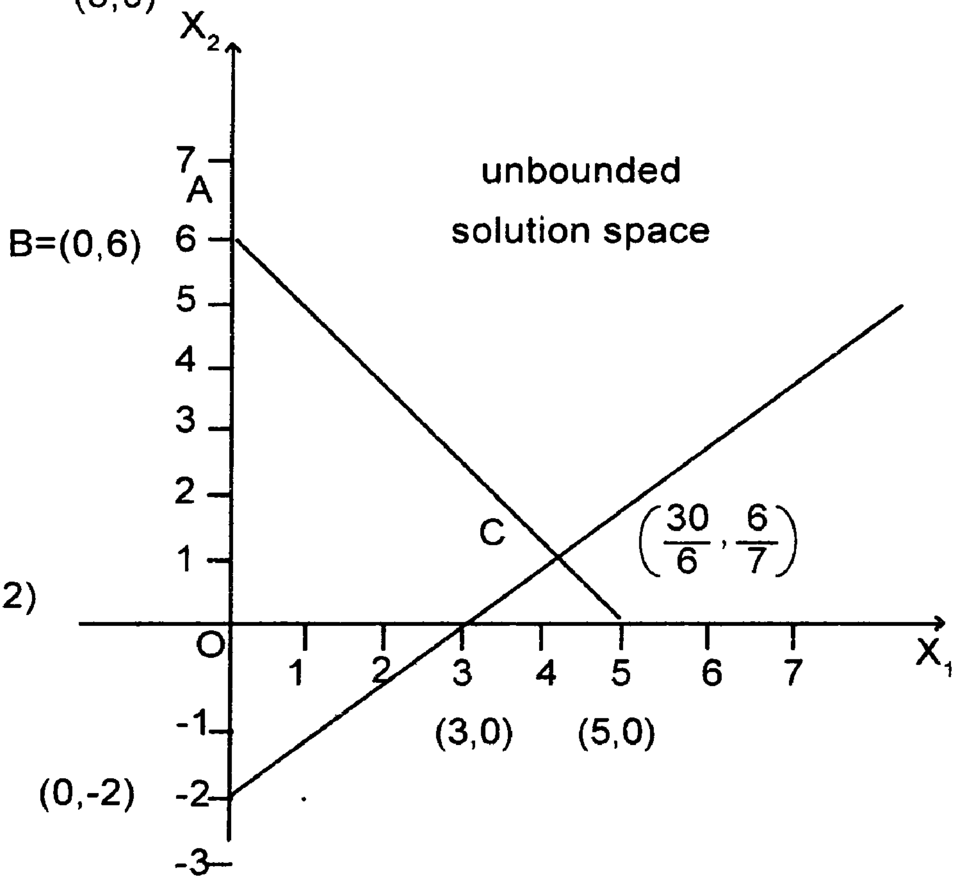
put $x_1=0$; $2x_1-3x_2=6$ put $x_2=0$; $2x_1-3x_2=6$

$-3x_2=6$ $\Rightarrow 2x_1=6$

$\Rightarrow x_2=-\frac{6}{3}= 2$ $\Rightarrow x_1=\frac{6}{2} = 3$

$\Rightarrow x_2=-2$ $\Rightarrow x_1=3$

(0,-2) (3,0)



put $x_1=0$; $6x_1+5x_2=30$

$\Rightarrow 5x_2=30$

$\Rightarrow x_2= \frac{30}{5} = 6$

(0,6)

put $x_2=0$; $6x_1+5x_2=30$

$\Rightarrow 6x_1=30$

$\Rightarrow x_1= \frac{30}{6} = 5$

$\Rightarrow$ (0,5)

solving the equation (1) and(2)

(1) x 3 - 2

$6\cancel{x}_1-9x_2=18$

$\cancel{6}x_1+5x_2=30$

$-\quad -\quad -$

$-14x_2=12$

$\Rightarrow x_2=\frac{12}{14} = \frac{6}{7}$

$x_2= \frac{6}{7}$ applying in equation (1)

$2x_1-3x_2=6$

$\Rightarrow 2x_2-3.\frac{6}{7} = 6$

$\Rightarrow 2x_1- \frac{18}{7}= 6$

$\Rightarrow 14x_1-18=42$

$\Rightarrow 14x_1=42+18$

$\Rightarrow 14x_1=60$

$\Rightarrow x_1= \frac{60}{14}= \frac{30}{7}$

$C=(\frac{30}{6}, \frac{6}{7})$

In this figure the solution space is denoted as the region which is obviously unbounded. The value of the objective function can increase to infinity.

If the objective function is of minimization type then it has values Z(A) = 21 ; Z(B) = 18, Z(c) = 19.7

The optimum solution is at B(0,6) and minimum value is 18.

3) Solve graphically

Maximize

$Z = 3x_1+4x_2$

Subject to

$4x_1 +2x_2 \leq 80$

$2x_1 +5x_2 \leq 180$

$x_1, x_2 \geq 0$

Solution:-

Consider the equation $4x_1+2x_2 \leq 80$

Let $4x_1+2x_2=80$.

when $x_2=0 \Rightarrow 4x_1=80 \Rightarrow x_1=20$

∴ point A = (20,0)

when $x_1=0 \Rightarrow 2x_2=80 \Rightarrow x_2=40$

∴ point B=(0,40)

Again consider the line $2x_1+5x_2=180$

when $x_2=0 \Rightarrow 2x_1=180 \Rightarrow x_1=90$

∴ point C=(90,0)

when $x_1=0 \Rightarrow 5x_2=180 \Rightarrow x_2=36$

∴ point D = (0,36)

To find E.

solve $4x_1+2x_2=80$ .. ...(1)

$2x_1+5x_2=180$ . ..... (2)

(1) - (2) x 2

$$\cancel{4x_1} +2x_2 = 80$$

$$\underline{\cancel{4x_1} +10x_2 = 360}$$

$$\underline{- \quad - \quad\quad -}$$

$$-8x_2 = -280$$

$$x_2 = \frac{280}{8} = 35$$

$x_2 = 35$ put in (1)

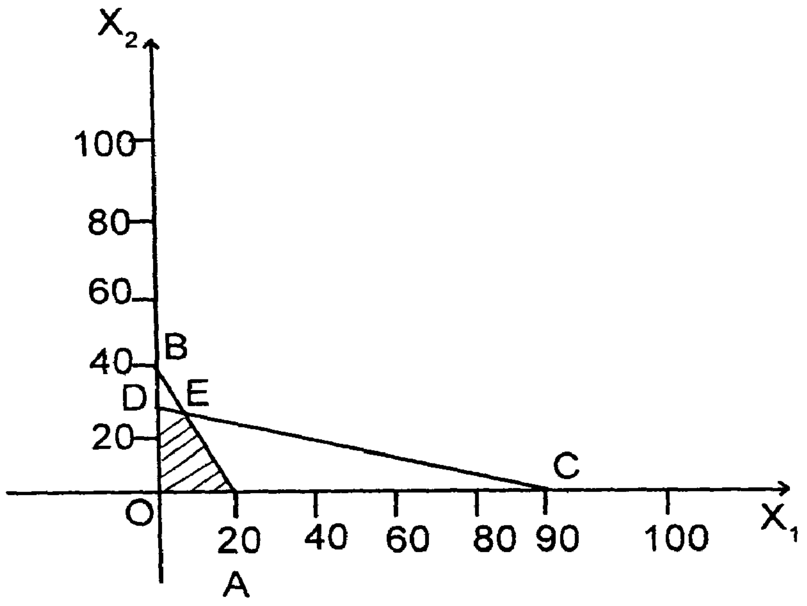
$4x_1+2 \times 35 = 80$

$\Rightarrow 4x_1 = 80-70$

$\Rightarrow 4x_1 = 10$

$\Rightarrow x_1=2.5$

E = (2.5, 35)



The solutions space OAED

where O=(0,0), A=(20, 0), E(2.5,35), D=(0,30)

$Z = 3x_1+4x_2$

$Z(A) = 3 \times 20 + 0=60$

$Z(E) = 3 \times 2.5+4 \times 35 = 147\ 5$

$Z(D) = 0+4 \times 36 = 144$

∴ At E, Z is maximum.

∴ maximum Z = 147.5 where $x_1=2.5$, $x_2=35$

4) Solve minimum $Z=x_1+15x_2$

Subject to

$x_1+x_2 \geq 1$

$2x_1+2x_2 \geq 1$

$x_1+10x_2 \geq 1$

$x_1, x_2 \geq 0$

Solution .-

consider $x_1+x_2=1$

when $x_2=0 \Rightarrow x_1=1$; when $x_1=0 \Rightarrow x_2=1$

A=(1,0) B = (0,1)

consider $2x_1+2x_2=1$

when $x_2=0 \Rightarrow 2x_1=1 \Rightarrow x_1=0.5$

C=(0.5, 0)

when $x_1=0 \Rightarrow 2x_2=1$

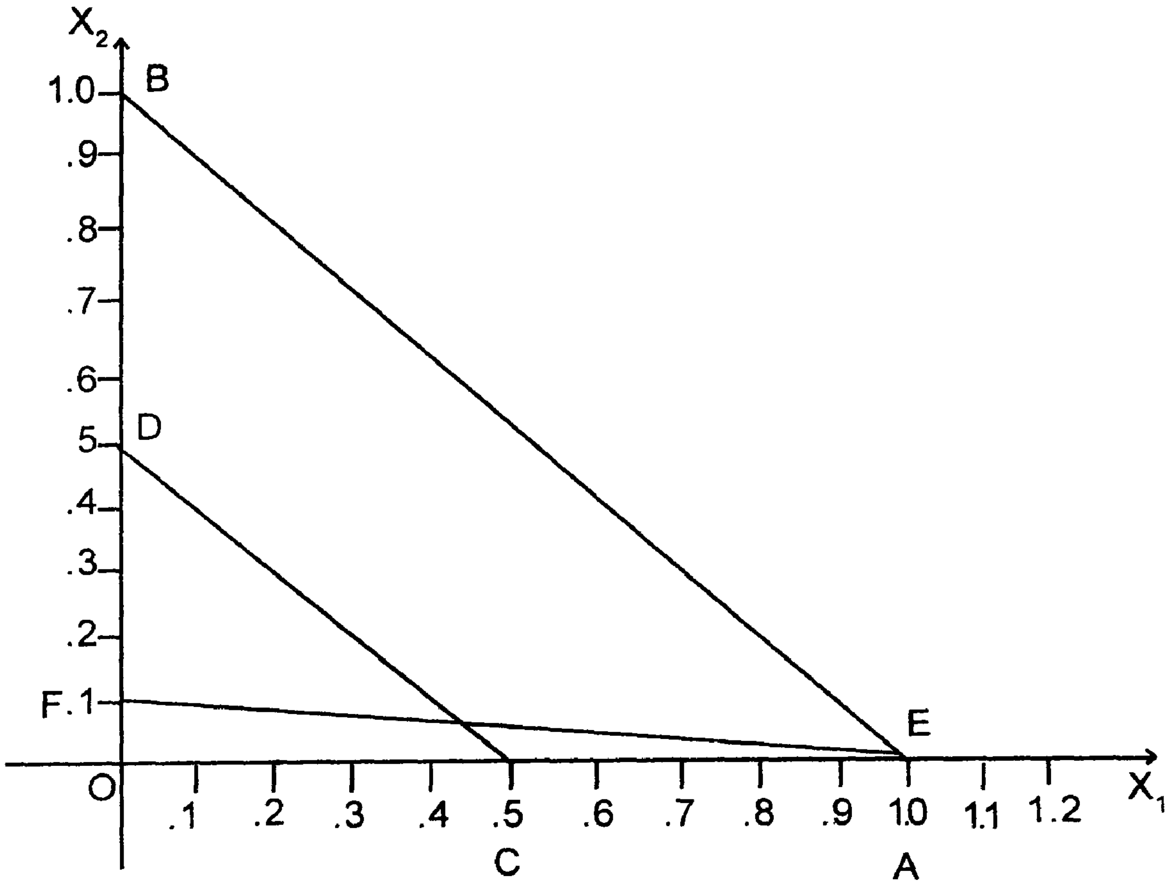
$\Rightarrow x_2=0.5$

D=(0, 0.5)

Consider $x_1+10x_2=1$; when $x_1=0 \Rightarrow x_2=0.1$

when $x_2=0 \Rightarrow x_1=0$ F = (0,0.1)

E = (1,0)



The solution space is the region above the line AB

& A= (1,0), B=(0,1)

$Z=x_1+15x_2$

At z(A) = 1+0 = 1

z(B) = 0+15 = 15

So minimum z=1

at A=(1,0)

$x_1=1,\ x_2=0$

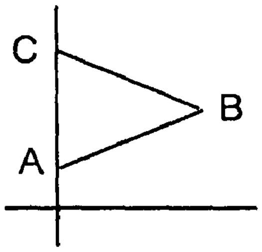
Aliter : - Graphical Procedure :-

step (i) :-

Determine the feasible region.

Setp (ii) :-

List all the extreme point and evaluate the value of objective function at each extreme point



Extreme	z
A	z_1
B	z_2
C	z_3

When the number of extreme points is large. plot the objective function as a line by taking arbitrary Z value z'. and determine a smaller set of extreme points which can be optimal points.

Problem :-

Slove graphically

1) Maximıze

$$Z=2x_1+3x_2$$

S.t $2x_1+x_2 \leq 6$ ....(1)

$x_1+2x_2 \leq 8$ .... (2)

$x_1-x_2 \leq 1$ ....(3)

$x_2 \leq 2$ .... (4)

& $x_1, x_2 \geq 0$ .... (5) & (6)

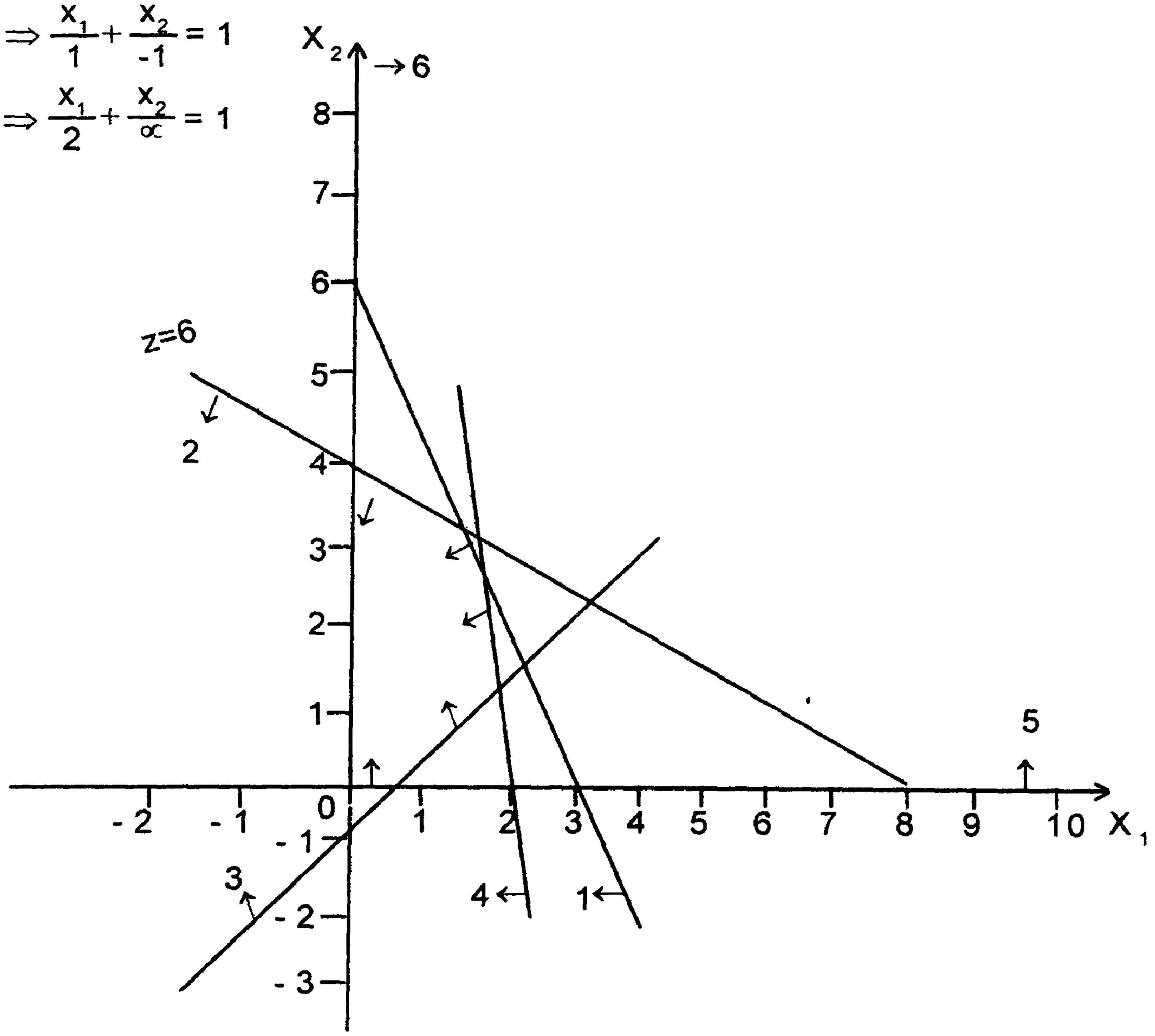
Solution :-

$$2x_1+x_2=6 \Rightarrow \frac{x_1}{3}+\frac{x_2}{6}=1$$

$$x_1+2x_2=8 \Rightarrow \frac{x_1}{8}+\frac{x_2}{4}=1$$

$$x_1-x_2 = 1 \Rightarrow \frac{x_1}{1}+\frac{x_2}{-1} = 1$$

$$x_1=2 \Rightarrow \frac{x_1}{2}+\frac{x_2}{\propto} = 1$$



Let Z=6

$2x_1+3x_2=6$

$\Rightarrow \frac{x_1}{3}+\frac{x_2}{2}=1$

To find (x_1,x_2) solve

$x_1+2x_2=8$ ....(1)

$2x_1+x_2=6$ .... (2)

(1) X (2) - (2)

$\cancel{2x_1}+4x_2=16$

$\cancel{2x_1}+x_2 = 6$

$- \quad - \quad -$

$3x_2=10$

$\Rightarrow x_2=\frac{10}{3}$

(1) - (2) x (2)

$x_1+2x_2 = 8$

$4x_1+2x_2=12$

$- \quad - \quad -$

$-3x_1=-4$

$x_1=\frac{4}{3}$

$\left(\frac{4}{3},\frac{10}{3}\right)$

$Z=2\left(\frac{4}{3}\right)+3\left(\frac{10}{3}\right)=\frac{38}{3}=12\frac{2}{3}$

Let (0,4) ; z=12

Let (2,2) ; z=10

maximum z=12

2) solve graphically

maximize

$Z=2x_1+3x_2$

Subject to

$4x_1+3x_2 \leq 12$ ....(1)

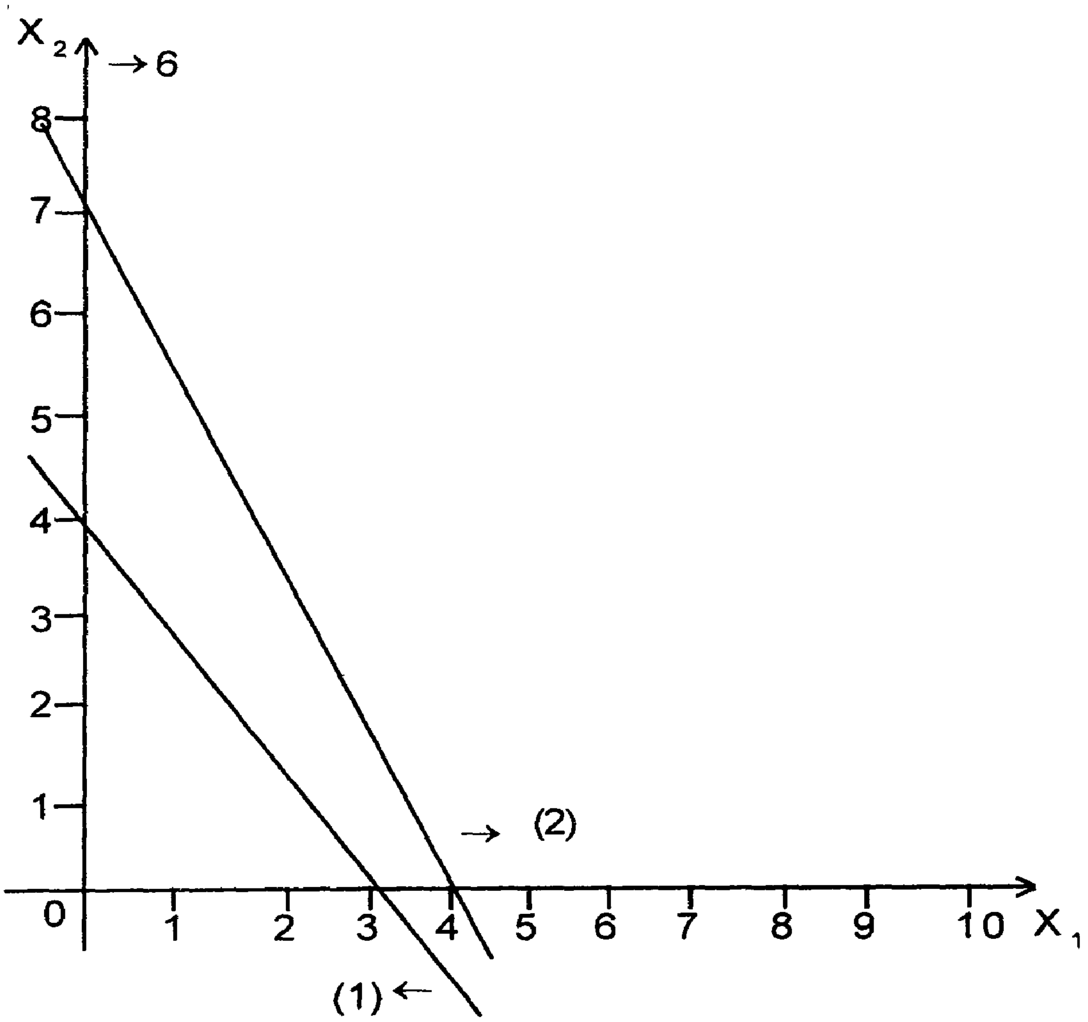
$7x_1+4x_2 \geq 28$ .... (2)

$x_1, x_2 \geq 0$ ....(3) & (4)

Solution :-

$4x_1+3x_2 \le 12 \Rightarrow \frac{x_1}{3}+ \frac{x_2}{4}= 1$

$7x_1+4x_2 \ge 28 \Rightarrow \frac{x_1}{4}+ \frac{x_2}{7}= 1$



In the graph is a feaible solution.

because of inconsistent constraints.

3) Maximum

$Z=3x_1+x_2$

S.t $2x_1+x_2 \ge 2$ .. ..(1)

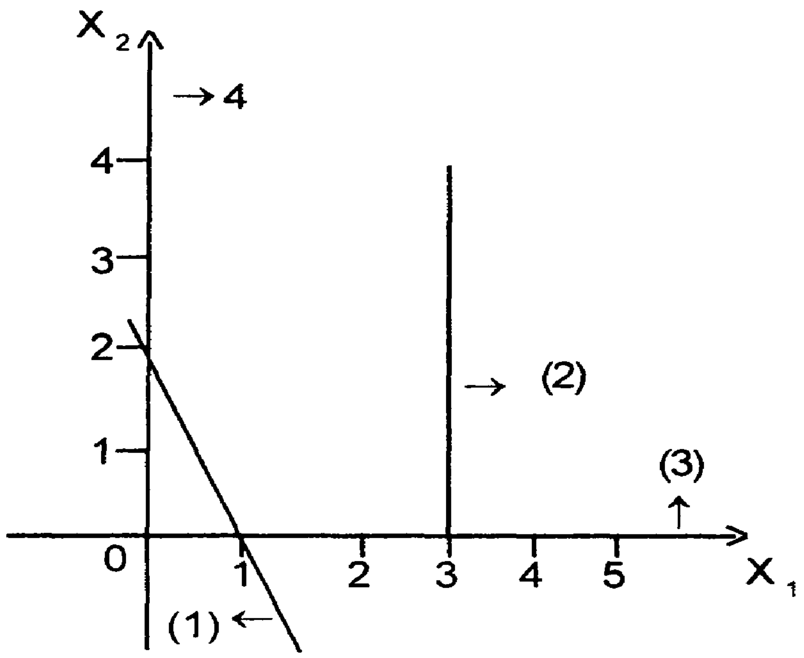
$x_1 \le 3$ ... .(2)

& $x_1, x_2 \ge 0$ ....(3) & (4)

Sol :-

$2x_1+x_2 \ge 2 \Rightarrow \frac{x_1}{1} + \frac{x_2}{2} = 1$

$x_1 \le 3 \Rightarrow \frac{x_1}{3} + \frac{x_2}{\propto} = 1$



This graph is a unbounded solution.

Exercises:-

I. Slove grapically the following LPPs.

1) Maximize $Z=15x_1+10x_2$

Subject to

$4x_1+6x_2 \leq 360$

$x_1 \leq 60$

$x_2 \leq 40$

$x_1, x_2 \geq 0$

2) Maximize

$z=10x_1+15x_2$

Subject to

$2x_1+x_2 \leq 26$

$2x_1+4x_2 \leq 56$

$-x_1 + x_2 \leq 40$

$x_1, x_2 \geq 0$

3) Minimize

$z=1000x_1+800x_2$

Subject to

$6x_1+2x_2 \geq 12$

$2x_1+2x_2 \geq 8$

$x_1+3x_2 \geq 6$

$x_1, x_2 \geq 0$

4) Minimize

$z = 3x_1+2x_2$

Subject to $5x_1+x_2 \geq 10$; $x_1+x_2 \geq 6$; $x_1+4x_2 \geq 12$, $x_1, x_2 \geq 0$.

5) Minimize

$z=-x_1+2x_2$

subject to

$-x_1+3x_2 \leq 10$

$x_1+x_2 \leq 6$

$x_1-x_2 \leq 2$

$x_1, x_2 \geq 0$

6) Maximize

$z = 50x_1+30x_2$

Subject to

$2x_1+x_2 \geq 18$

$3x_1+2x_2 \leq 34$

$x_1+x_2 \geq 12$

$x_1, x_2 \geq 0$

7) Minimze

$z = 4x_1+3x_2$

Subject to

$2x_1+x_2 \geq 40$

$x_1+2x_2 \geq 50$

$x_1+x_2 \geq 35$,

$x_1, x_2 \geq 0$

8) Maximize

$x = 5x_1+8x_2$

Subject to

$15x_1+10x_2 \leq 180$; $x_1+2x_2 \leq 20$

$3x_1+4x_2 \leq 42$, $x_1, x_2 \geq 0$.

Unit II

2.1 Simplex method :-

A Linear programming problem in which there are two decision variables can be solved by using graphical method. However this mehtod cannot be extended for a linear programming problem, which involves more than two variables. Simplex mehtod developed by an American mathematician G.B. Dantzig in 1947 is the most efficient iterative method of solving a general L.P.P. Efficient computer algorithms are available to implement the simplex method.

In this section, we explain the simplex method to solve a general Linear programming problem and illustrate the same with several examples. First, we present the simplex method for a LPP, which has no constraints that are of equality type. i.e. All the constraints in the LPP are either $\leq$ type or $\geq$ type.

Consider the LPP :-

Maximize z=cx subject to

$Ax \ (\leq \ \geq) \ b$

$x \geq 0$ where $C = (c_1, c_2 \, \ldots\ldots\ldots c_n)$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} ; \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_m \end{pmatrix} \quad \text{and } A = \begin{pmatrix} a_{11} & a_{12} \cdots\cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots \\ a_{m1} & a_{m2} \cdots\cdots & a_{mn} \end{pmatrix}$$

Step 1 :- Standard form :-

Express the given LPP in standard form by introducing Slack / surplus variables in each of the constraints.

Then the form of the LPP is as follows.

Maximize

$Z = c_1x_1 + c_2x_2 + \ldots\ldots\ldots\ldots + c_nx_n + OS_1 + OS_2 + \ldots\ldots\ldots + OS_m$

Subject to

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n + S_1 = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n + S_2 = b_2$

....

....

....

$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mn}x_n + S_m = b_{1m}$

$x_1, x_2, \ldots\ldots\ x_n, s_1, s_2, \ldots\ldots\ldots\ldots s_m \geq 0.$

This can the represented in tabular form as follows:-

c_1	c_2	....	c_n	0	0	....	0
x_1	x_2	....	x_n	S_1	S_2	....	S_m
a_{11}	a_{12}	....	a_{1n}	1	0	....	0
a_{21}	a_{22}	....	a_{2n}	0	1	....	0
...	....	....	..	..	..	.. ..	..
..	...	....	..	..	..	....	..
a_{m1}	a_{m2}	. ...	a_{mn}	0	0	....	1

Step 2 - **Initial basic feasible solution :_**

we observe that the columns corresponding to slack / surplus variables $s_1, s_2, \ldots s_m$ forms an m x m identity matrix B and B is called the basis matrix. Hence if we take $s_1, s_2, \ldots s_m$ as basic variables, we obtain the value of the basic variables as $s_1=x_{B_1}=b_1$; $s_2=x_{B_2}=b_2$, ...$s_m=x_{B_m}=b_m$

$$\text{Let } x_B = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_m \end{pmatrix} = b$$

we observe that $x_B=B^{-1}b = Ib = b$ and b is the initial basic feasible solution The cost colfficients corresponding to the basic variables in the objective function are denoted by C_{B_1}, $C_{B_2}, \ldots\ldots . C_{B_n}$.

Step 3:- **Evaluation and net evaluation in simplex table:-**

Compute the following quantities.

$$\text{Evaluation : } Z_j = \sum_{i=1}^{m} C_{B_i} a_{ij} \quad ; \quad j=1,2, \ldots\ldots m+n$$

& net Evaluation : $Z_j - C_j$ where $j=1,2,\ldots\ldots m+n$

The information can be represented in the tabular form known as

INITIAL SIMPLEX TABLE

Basis		c_j	c_1	c_2	....	c_n	0	0	....	0
c_{B_i}	x_{B_i}	$x_B=b$	x_1	x_2	....	x_n	s_1	s_2	.	s_m
c_{B_1}	s_1	b_1	a_{11}	a_{12}		a_{1n}	1	0	..	0
c_{B_2}	s_2	b_2	a_{21}	a_{22}	...	a_{2n}	0	1	...	0
.	..	..	..	..	....	..	..	..	...	.
..	.	..	..	..	..	..	..	..	..	.
c_{B_m}	s_m	b_m	a_{m1}	a_{m2}	....	a_{mn}	0	0	...	1
		z_j	0	0	....	0	0	0	...	0
		z_j-c_j	z_1-c_1	z_2-c_2	...	z_n-c_n	0	0	. .	0

Step 4 :- Test for optimality (Entering variable)

Examine the values of z_j-c_j. There arises three

Cases :-

case (i) .-

$z_j-c_j \geq 0 \forall j$. In this case the current basic feasible solution is the required optimal solution. STOP

case (ii) :-

$z_k-c_k < 0$ for some K and $a_{ik} <0$ for i=1,2,....m In this case this LPP has unbounded solution.

Case (iii) :-

$z_j-c_j <0$ and there exists atleast one i suchthat $a_{ij} \geq 0$. In this case the objective function can be further improved we choose j suchthat z_j-c_j is minimum (most negative). If there is a tie between $z_j-c_j^s$, it is broken by arbitrary choice.

This determines a variable that is currently non-basic and becomes a basic variable in the next iteration. For this reason, x_j is called the entering variable and the corresponding j^{th} column to be entered is called the keycolumn (Pivotal column)

Step V:- Leaving variable.-

Compute the quantities (replacement ratios):-

$$\left\{ \frac{X_{B_r}}{a_{rj}} \Big/ a_{rj} > 0 \right\} ; \qquad r = 1,2, ...m$$

Then, choose the min $\left\{ \frac{X_{B_r}}{a_{rj}} \middle/ a_{rj} > 0 \right\}$ Let $\frac{X_{B_k}}{a_{kj}}$

be the minimum. Then x is the leaving variable the corresponding kth row is called the key row (Pivotal row). The intersection of the key row and the key column a_{kj} is called the key element (Pivotal element)

Step VI :- First iteration -

In the current basis, replace the leaving variable by the entering variable Divide all the elements of the pivotal row by the pivotal element. The current values of the elements in the remaining rows are obtained by applying elementary row operations, so as to make all the entries in the column corresponding to the entering varialbe zero.

(i.e.,) using the formula given below, we can get the values of the current iteration table :-

$$\boxed{\text{New value} = \text{old value} - \left(\frac{\text{Pivotal column value}}{\text{Pivotal element}}\right) \text{X pivotal row value}}$$

Thus, we get the First iteration table We observe that the columns corresponding to the current basic variables in the first iteration table form an identity matrix.

Step VII :- Go to step 2 :-

This completes the simplex algorithm for obtaining optimum solution. These are explained explicity in the following solved problems

Solved problems .-

Problem 1 :-

Use simplex method to solve the LPP:-

Maximize

$Z=30x_1+20x_2$

Subject to

$10x_1+8x_2 \leq 800$

$x_1 \leq 60$

$x_2 \leq 75$

$x_1, x_2 \geq 0.$

solution :-

The given LPP is to Maximize z=cx

Subject to $Ax \leq b$, $x \geq 0$

Since there are three constraints with $\leq$ type.

we need three slack variables $s_1, s_2, s_3 \geq 0$ to get the LPP in standard form.

Maximize

$$Z=cx$$

Subject to

$$Ax = b,\ \ x \geq 0$$

Thus the given LPP in standard form is :

Maximize

$$Z=30x_1+20x_2+OS_1+OS_2+OS_3$$

Subject to

$$10x_1+8x_2+S_1 =800$$

$$x_1+s_2=60$$

$$x_2+s_3=75$$

$$x_1,\ x_2,\ s_1,\ s_2,\ s_3 \geq 0.$$

The initial basic feasible solution is $(s_1, s_2, s_3) = (800, 60, 75)$

Initial simplex table

Basis		c_j	30	20	0	0	0	Ratio	
C_B	B	X_B	x_1	x_2	s_1	s_2	s_3		
0	s_1	800	10	8	1	0	0	$\frac{800}{10} = 80$	
0	s_2	60	$\boxed{1}$	0	0	1	0	$\frac{60}{1} = 60$	(least +ve)
0	s_3	75	0	1	0	0	1	-	
Optimality is not attained		$z_j=0$	0	0	0	0	0	Go to next iteration	
		z_j-c_j	-30	-20	0	0	0		

↑

(most(-ve))

x_1 is the entering varialbe and s_2 is the leaving variable; 1 is the Pivotal element. Here the ratio for s_3 doesnot exist and hence it is left out in ratio column.

First interation table

Basis		c_j	30	20	0	0	0	Ratio
C_B	B	x_B	x_1	x_2	s_1	s_2	s_3	
0	s_1	200	0	$\boxed{8}$	1	-10	0	$\frac{200}{8} = 25$ →
30	x_1	60	1	0	0	1	0	-
0	s_3	75	0	1	0	0	1	$\frac{75}{1} = 75$
Optimality is not attained		(60x30) z_j=1800	30	0	0	30	0	Go to next iteration
		z_j-c_j	0	-20	0	0	0	

↑

x_2 is the entering varialbe and s_1 is the leaving variable; 8 is the Pivotal element

Second interation table

Basis		c_j	30	20	0	0	0	
C_B	B	x_B	x_1	x_2	s_1	s_2	s_3	End of simple procedure
20	x_2	25	0	1	$\frac{1}{8}$	$-\frac{10}{8}$	0	
30	x_1	60	1	0	0	1	0	
0	s_3	50	0	0	$-\frac{1}{8}$	$\frac{5}{4}$	1	
Optimality is attained		20x25+60x30 z_j=2300	30	20	$\frac{5}{2}$	5	0	
		z_j-c_j	0	0	$\frac{5}{2}$	5	0	

Optimality is attained. The optimal solution is $x_1 = 60$; $x_2=25$ and the maximum value of z=2300.

Problem 2 :-

Using simplex method, solve the following LPP.

Minimize

$$Z=x_1-3x_2+2x_3$$

subject to

$$3x_1-x_2+2x_3 \leq 7$$

$$-2x_1+4x_2 \leq 12$$

$-4x_1+3x_2+8x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$

Solution .-

Simplex method can be applied for the LPP in which the objective function is maximizing type.

However the objective function, which is of minimizing type can be converted to maximizing type by taking $Z^* = -Z$

Introducing slack variables $s_1, s_2, s_3 \geq 0$

The LPP in standard form becomes

Maximize $Z^* = -x_1+3x_2-2x_3+OS_1+OS_2+OS_3$

subject to

$3x_1-x_2+2x_3+s_1=7$

$-2x_1+4x_2+S_2=12$

$-4x_1+3x_2+8x_3+S_3=10$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$

The initial basic feasible solution is $(s_1, s_2, s_3) = (7,12,10)$

Initial Simplex table

Basis		c_j	-1	3	-2	0	0	0	Ratio
C_B	B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	7	3	-1	2	1	0	0	-
0	s_2	12	-2	$\boxed{4}$	0	0	1	0	$\frac{12}{4} = 3$ →
0	s_3	10	-4	3	8	0	0	1	$\frac{10}{3} = 3.33$
Optimality is not attained		$z_j^*=0$	0	0	0	0	0	0	Go to next iteration
		$z_j^*-c_j$	1	-3	2	0	0	0	

↑

x_2 is the entering variable and s_2 is the leaving variable ; 4 is the pivotal element.

First iteration Table

Basis		c_j	-1	3	-2	0	0	0	Ratio
C_B	B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	10	$\frac{5}{2}$	0	2	1	$\frac{1}{4}$	0	$10x\frac{2}{5}=4$ →
3	x_2	3	$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0	-
0	s_3	1	$-\frac{5}{2}$	0	8	0	$-\frac{3}{4}$	1	-
Optimality is attained		$z_j^*=9$	$-\frac{3}{2}$	3	0	0	$\frac{3}{4}$	0	Go to next iteration
		$z_j^*-c_j$	$-\frac{1}{2}$	0	2	0	$\frac{3}{4}$	0	
			↑						

s is the leaving variable , x_1 is the entering variable 5/2 is the pivotal element.

Second iteration table

Basis		c_j	-1	3	-2	0	0	0	
C_B	B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	
-1	x_1	4	1	0	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0	End of simple procedure
3	x_2	5	0	1	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	0	
0	s_3	11	0	0	10	1	$-\frac{1}{2}$	1	
Optimality is attained		$z_j^*=11$	-1	3	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0	
		$z_j^*-c_j$	0	0	$\frac{12}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0	

Since all $z_j^*-c_j>0$, optimality is attained.

The optimal solution is

$x_1 = 4$

$x_2 = 5$

$x_3 = 0$ and the optimal value is Max $z^* = 11$.

Hence minimum of Z = -(maximum of z*)

= - (11)

minimum of Z = 11

Problem 3 :-

An agriculturalist in Nilgris forest has 1000 acres of land on which he can grow crops coffee, tea or potato. An acre of coffee costs Rs. 100 per preparation; requires 7 man-days of work and yields a profit of Rs. 30 per unit load. An acre of tea costs Rs. 120 per preparation; requires 10 man-days of work and yields a profit of Rs. 40 per unit load An acre of potato costs Rs. 70 for preparation; requires 8 man-days of work and yields a profit of Rs 20 per unit load The agriculturalist has Rs one lakh for preparation and can utilize 800 man-days of work. How many acres shall he allocate to each crop to maximize his profit? Formulate this problem as a LPP and slove by simplex method

Solution :-

The details of the problem can be summerized as follows.

crops	Preparation cost (in Rs.)	man days	profit (in Rs.)
coffee	100	7	30
Tea	120	10	41
Potato	70	8	20
Availability	100000	8000	

Let x_1, x_2 and x_3 be acres of land utilized for coffee, tea and potato respectively.

The LPP is ·

Maximize

$$Z = 30x_1+40x_2+20x_3$$

subject to

$$x_1+x_2+x_3 \le 1000$$

$$100x_1+120x_2+70x_3 \le 100000$$

$$7x_1+10x_2+8x_3 \le 8000 \qquad x_1, x_2, x_3 \ge 0$$

Introducing slack variables s_1, s_2 and s_3 we have the LPP in standard form as

Maximize

$$Z= 30x_1+40x_2+20x_3+OS_1+Os_2+Os_3$$

Subject to

$$x_1+x_2+x_3+S_1 = 1000$$

$$10x_1+12x_2+7x_3+S_2 = 10000$$

$$7x_1+10x_2+8x_3+S_3 = 8000$$

$$x_1, x_2, x_3, S_1, S_2, S_3 \ge 0.$$

Initial Simplex table

Basis		c_j	30	40	20	0	0	0	Ratio
C_B	B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	1000	1	1	1	1	0	0	$\frac{1000}{1}$
0	s_2	10000	10	12	7	0	1	0	$\frac{10000}{12}$
0	s_3	8000	7	$\boxed{10}$	8	0	0	1	$\frac{8000}{10}$ →
Optimality is not attained		$z_j=0$	0	0	0	0	0	0	Go to next iteration
		z_j-c_j	-30	-40	-20	0	0	0	

↑

s_3 is the leaving variable, x_2 is the entering variable; 10 is the pivotal element

First iteration table

Basis		c_j	30	40	20	0	0	0	Ratio
C_B	B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	200	$\frac{3}{10}$	0	$\frac{1}{5}$	1	0	$\frac{-1}{10}$	$\frac{2000}{3}$
0	s_2	400	$\boxed{\frac{8}{5}}$	0	$\frac{-13}{5}$	0	1	$\frac{-6}{5}$	$\frac{2000}{8}$ →
40	x_2	800	$\frac{7}{10}$	1	$\frac{8}{10}$	0	0	$\frac{1}{10}$	$\frac{8000}{7}$
Optimality is not attained		$z_j=32000$	28	40	32	0	0	4	Go to next iteration
		z_j-c_j	-2	0	12	0	0	4	

↑

s_2 leaves the basis; x_1 enters the basis and $\frac{8}{5}$ is the pivotal element :

Second iteration table

Basis		c_j	30	40	20	0	0	0	End of simple procedure
C_B	B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	125	0	0	$\frac{11}{16}$	1	$-\frac{3}{16}$	$-\frac{1}{8}$	
30	x_1	250	1	0	$-\frac{13}{8}$	0	$\frac{5}{8}$	$-\frac{3}{4}$	
40	x_2	625	0	1	$\frac{31}{16}$	0	$-\frac{7}{16}$	$\frac{5}{8}$	
Optimality is attained		z_j=32500	30	40	$\frac{115}{4}$	0	$\frac{5}{4}$	$\frac{5}{2}$	
		z_j-c_j	0	0	$\frac{35}{4}$	0	0	0	

Optimal solution is

$x_1=250$

$x_2=625$

$x_3=0$

Maximum profit

$z=30\text{x}250+40\text{x}625+0$

$z=7500+25000 = 32500$ (in Rs.)

Problem 4 :-

Solve the following LPP.

Maximize

$Z = 8x_1-4x_2$

subject to

$4x_1+5x_2 \leq 20$

$x_1-3x_2 \leq 23$

$x_1 \geq 0$, x_2 is unrestricted.

solution :-

Since the decision variable, x_2 is given to be unrestricted in sign, we introduce variables, $x'_2 \geq 0$ and $x''_2 \geq 0$ such that $x_2=x'_2-x''_2$.

Now the LPP in standard form becomes.

maximize $Z=8x_1-4(x'_2-x''_2)+OS_1+OS_2$

subject to

$4x_1+5(x'_2-x''_2)+s_1=20$

$x_1-3(x'_2-x''_2)+s_2=23$

$x_1, x_1', x_2'', S_1, S_2 \geq 0$

(i.e) Maximize

$Z= 8x_1-4x'_2+4x''_2+OS_1+OS_2$

Subject to

$4x_1+5x_2'-5x''_2+S_1=20$

$x_1-3x'_2+3x''_2+S_2=23$

$x_1, x'_2, x''_2, S_1, S_2 \geq 0$

we now solve the LPP by simplex method and find the optimum solution for the new decision variables x_1, x'_2 and x''_2.

The initial simplex table is given below :-

Initial Simplex table

Basis		c_j	8	-4	4	0	0		Ratio
C_B	B	x_B	x_1	x'_2	x''_2	s_1	s_2		
0	s_1	20	$\boxed{4}$	5	-5	1	0		$\frac{20}{4} = 5$ →
0	s_2	23	1	-3	3	0	1		$\frac{23}{1} = 23$
Optimality is not attained		$z_j=0$	0	0	0	0	0		Go to next iteration
		z_j-c_j	-8	4	-4	0	0		

↑

x_1 enters the basis and s_1 leaves the basis : 4 is the pivotal element.

First iteration table

Basis		c_j	8	-4	4	0	0		Ratio
C_B	B	x_B	x_1	x'_2	x''_2	s_1	s_2		
8	x_1	5	1	$\frac{5}{4}$	$-\frac{5}{4}$	$\frac{1}{4}$	0		-
0	s_2	18	0	$-\frac{17}{4}$	$\boxed{\frac{17}{4}}$	$-\frac{1}{4}$	1		$\frac{18}{7} \times 4$ →
Optimality is not attained		$z_j=40$	8	10	-10	2	0		Go to next iteration
		z_j-c_j	0	14	-14	2	0		

↑

x_2'' enters the basis and S_2 leaves the basis ; $\frac{17}{4}$ is the pivotal element.

Second iteration table

Basis		c_j	8	-4	4	0	0	End of simple procedure
C_B	B	x_B	x_1	x'_2	x''_2	s_1	s_2	
8	x_1	$\frac{175}{17}$	1	0	0	$\frac{3}{17}$	$\frac{5}{17}$	
4	x''_2	$\frac{72}{17}$	0	-1	1	$\frac{-1}{17}$	$\frac{4}{17}$	
Optimality is attained		$z_j = \frac{1688}{17}$	8	-4	4	$\frac{20}{17}$	$\frac{56}{17}$	
		$z_j - c_j$	0	0	0	$\frac{20}{17}$	$\frac{56}{17}$	

All $z_j - c_j \geq 0$. Hence an optimal solution is attained.

The optimal solution is $x_1 = \frac{175}{17}$

$x''_2 = \frac{72}{17}$; $x'_2 = 0$

maximum $z = \frac{1688}{17}$. Since $x_2 = x'_2 - x''_2 = -\frac{72}{17}$

The optimum solution is $x_1 = \frac{175}{17}$; $x_2 = -\frac{72}{17}$

& max $Z = \frac{1688}{17}$

Note : **Unbounded solution :**

In a LPP of maximization type, at some iteration if $z_j - c_j < 0$ for same column x_j not in the basis and if all the entries in that column are negative then unbounded sloution occurs for the LPP. In this case, the value of the objection function can be increased indestinitely.

Problem 5 :

Use simplex method to solve the following LPP.

Maximize

$Z = 3x_1 + 2x_2$

subject to

$x_1 - x_2 \leq 1$

$3x_1 - 2x_2 \leq 6$

$x_1, x_2 \geq 0$

solution :-

Introducing slack variables, the LP Problem can be expressed as:

Maximize

$Z = 3x_1+2x_2+OS_1+OS_2$

subject to

$x_1-x_2+S_1 = 1$

$3x_1-2x_2+S_2=6, \quad x_1, \ x_2, \ S_1, \ S_2 \geq 0$

Initial simplex table

Basis		c_j	3	2	0	0	Ratio
C_B	B	x_B	x_1	x_2	s_1	s_2	
0	s_1	1	$\boxed{1}$	-1	1	0	1 →
0	s_2	6	3	-2	0	1	2
Optimality is not attained		$z_j = 0$	0	0	0	0	Go to next iteration
		$z_j - c_j$	-3	-2	0	0	

↑

S_1 leaves the basis; x_1 enters the basis and 1 is the pivotal element.

First iteration table

Basis		c_j	3	2	0	0	Ratio
C_B	B	x_B	x_1	x_2	s_1	s_2	
3	x_1	1	1	-1	1	0	-
0	s_2	3	0	$\boxed{1}$	-3	1	3 →
Optimality is not attained		$z_j = 3$	3	-3	3	0	Go to next iteration
		$z_j - c_j$	0	-5	3	0	

↑

S_2 leaves the basis; x_2 enters the basis and 1 is the pivotal element.

Second iteration table

Basis		c_j	3	2	0	0	Ratio
C_B	B	x_B	x_1	x_2	s_1	s_2	
3	x_1	4	1	0	-2	1	Unbounded solution
2	x_2	3	0	1	-3	1	
Optimality is not attained		$z_j = 18$	3	2	-12	5	
		$z_j - c_j$	0	0	-12	5	

↑

Here in the second iteration table $z_3 - c_3 < 0$. and the third column has all negative entries. Hence the LPP has unbounded solution.

Exercise :- Solve follwoing problems using simplex method:-

1) Maximize

$$Z = 6x_1 + 9x_2$$

subject to

$$2x_1 + 2x_2 \leq 24;\ x_1 + 5x_2 \leq 44\ ;\ 6x_1 + 2x_2 \leq 60$$

$x_1, x_2 \geq 0$.

2) Maximize

$$Z = 10x_1 + 15x_2$$

Subject to

$$3x_1 + 3x_2 \leq 36;\ 5x_1 + 2x_2 \leq 50;\ 2x_1 + 6x_2 \leq 60$$

$x_1, x_2 \geq 0$

3) maximize

$$Z = 5x_1 + 3x_2$$

subject to

$$x_1 \leq 6\ ;\ x_2 \leq 4;\ 2x_1 + 3x_2 \leq 18$$

$x_1, x_2 \geq 0$.

4) A company makes two kinds of leather belts. Belt A is a high quality belt and belt B is of lower quality. The respective profits per belt are Rs. 4 and Rs. 3. Each belt of type A requires twice as much time as a belt of type B, and if all belts were of type B, the company could make 1000 per day. The supply of leather is sufficient for only 800 belts per day (both A and B combined). Belt A requires a fancy buckle and only 400 per day are available. There are only 700 buckles a day available for belt B. What should be the daily production of each type of belt? Formulate the linear programming problem and solve it.

5) Minimize

$Z = x_1 - 3x_2 + 2x_3$

Subject to

$3x_1 - x_2 + 2x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$

6) Maximize

$Z = 10x_1 + 15x_2 + 20x_3$

subject to

$2x_1 + 4x_2 + 6x_3 \leq 24$

$3x_1 + 9x_2 + 6x_3 \leq 30$

$x_1, x_2, x_3 \geq 0.$

7) Maximize

$z = 2x_1 + 3x_2$

subject to

$x_1 - 2x_2 \leq 0$

$-2x_1 + 3x_2 \leq -6$

x_1, x_2 unrestricted.

8) Maximize

$z = 2x_1 + 3x_2$

subject to

$-x_1 + 2x_2 \leq 4$

$x_1 + x_2 \leq 6$

$x_1 + 3x_2 \leq 9$ $\quad$ x_1, x_2 unrestricted.

9) Maximize

$z = -2x_1 + 3x_2$

subject to

$x_1 \leq 5$

$2x_1 - 3x_2 \leq 6$

$x_1, x_2 \geq 0$

10) Maximize

$z = 6x_1 - 2x_2$

subject to

$2x_1 - x_2 \leq 2$

$x_1 \leq 4$

$x_1, x_2 \geq 0.$

2.2 BIG - M METHOD :-

Artificial Variables:-

Consider a LPP which has a constraint of the form $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ where $x_j \geq 0$, but b_i^S are negative (i=1,2,.....m).

To convert this constraint to an equation form we introduce a slack variable S_i. In the intial solution we have $S_i=b_i$

Hence the non-negativity condition of the slack variable is violated

III^{ly}, if the LPP has a constraint of the form $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$ where $x_j \geq 0$ and $b_j \geq 0$, then to convert it into equation form, we add a surplus variable S_i we have $S_i = -b_i$ in the initial solution. Here also the non-negative condition for the surplus variable S_i is violated.

If an LPP involves equality constraints then, we will not get an identity matrix (unit column vectors) in the initial simplex table.

To deal with such cases, we add a new type of variable called artificial variable a_i that will enable us to get an initial basic feasible solution. These variables are included in the objective function with a colfficient -M (Penalty) where M is very large.

Example :-

Consider the LPP

Maximize

$z=3x_1+2x_2$

subject to

$2x_1+x_2 \leq 2$

$3x_1+4x_2 \geq 12$

$x_1-x_2 = 2$

$x_1, x_2 \geq 0$

solution :-

In the first constraint, we introduce slack variable s_1.

In the second constraint, we introduce a surplus variable, S_1 and artificial variable a_1.

In the third constraint, we introduce another artificial variable a_2.

Associating a big penalty M>0 to the artificial variables in the objection function the standard form of the given LPP is.

Maximize

$z = 3x_1+2x_2+os_1+OS_1-Ma_1-Ma_2$

subject to

$2x_1+x_2+s_1=2$

$3x_1+4x_2-S_1+a_1=12$

$x_1-x_2+a_2=2$

$x_1, x_2, s_1, S_1, a_1, a_2 \geq 0$

In the initial basic feasible solution to the new LPP obtained by adding slack, surplus and artificial variables, we have $a_i=b_i$, However, this does not constitute a solution to the original LPP. To get back to the original problem, the artificial variable must be eleminated.

There are two familiar methods for eliminating the artificial variables from the solution namely

1) Big - M method (Method of penalty) due to charnes.

2) Two phase method due to Dantzig.

Big-M Method :-

In this method, in the objective function Z=cx, we introduce the artificial variables with a very large negative colfficient -M called penalty. This enables us to get rid of the artificial variables in the final solution.

Algorithm for Big -M method:-

Step 1:- Experss the given LPP in standard form by introducing slack, surplus and artificial variables if needed.

In the objective function, slack and surplus variables are associated with zero coefficients and artificial variables with coefficient -M, where M is a large positive quantity Thus we get a modified LPP.

Step :2:- Solve the modified LPP by using the simplex algorithm. One of the following three cases arise.

Case 1:- No artificial variable appears in the basis and optimality condition is satisfied Then the current solution is the required optimal basic feasible solution to the given LPP STOP.

Case 2 :- At least one artificial variable appears in the basis with zero value and optimality condition is satisfied. Then the current solution is an optimal basic feasible solution. Though, it is a degenerate solution for the modified LPP, this gives a non-degenerate optimal basic feasible solution to the original linear programming problem. STOP.

Case 3:- Atleast one artificial variable appears in the basis with non-zero value and optimality condition is satisfied. Then the original problem has no feasible solution. The obtained solution satisfies the constraints but does not optimize the objective function since it contains a very large penalty M. In this case the solution obtained is called Pseudo optimal solution.

Note :-

While applying simplex method, whenever an artificial variable happens to leave the basis, it will never again enter the basis. it is because, the co-efficient in the objective function is -M where M is very large. Hence we can drop that artificial variable and omit all the entries corresponding to its column from the simplex table in the procedure.

Solved problems :-

1) Solve the following LPP using Big-M method.

Minimize

$Z=60x_1+80x_2$

Subject to

$20x_1+30x_2 \geq 900$

$40x_1+30x_2 \geq 1200$

$x_1, x_2 \geq 0$

solution:-

The given LPP is to minimize the objective function. Hence we convert the objective function into maximization type by using the relation.

min Z = -max(-Z)

Let Z* = -Z

Hence min Z=-max Z*.

We introduce surplus variables S_1, S_2 >0 to balance the constraints and artificial variables a_1, $a_2 \geq 0$ to obtain initial basic feasible solution.

The standard form of the LPP is

Maximize

$Z^* = -60x_1-80x_2+OS_1+OS_2-Ma_1-Ma_2$

Subject to

$20x_1+30x_2-S_1+a_1 = 900$

$40x_1+30x_2+S_2+a_2 = 1200$

$x_1, x_2, S_1, S_2, a_1, a_2 \geq 0$ and $M \geq 0$ is very large.

we now employ simplex method to solve this LPP. The LPP can be represented in the following initial simplex table.

Initial Simplex table

Basis		c_j	-60	-80	0	0	-M	-M	Ratio
C_B	B	x_B	x_1	x_2	S_1	S_2	a_1	a_2	
-M	a_1	900	20	30	-1	0	1	0	$\frac{900}{20}=45$
-M	a_2	1200	$\boxed{40}$	30	0	-1	0	1	$\frac{\overset{30}{\cancel{1200}}}{\underset{1}{\cancel{40}}}=30$ →
Optimality is not attained		$z_j^*=-2100M$	-60M	60M	+M	+M	-M	-M	Go to next iteration
		$z_j^*-c_j$	-60M+60	-60M+80	M	M	0	0	

↑

Artificial variable a_2 leaves the basis. x_1 enters the basis & 40 is the Pivotal element Hence the vector a_2 may be removed from the iteration table.

First iteration table

Basis		c_j	-60	-80	0	0	-M	Ratio
C_B	B	$x_{B'}$	x_1	x_2	S_1	S_2	a_1	
-M	a_1	300	0	$\boxed{15}$	-1	$\frac{1}{2}$	1	$300=\frac{300}{15}=20$ →
-60	x_1	30	1	$\frac{3}{4}$	0	$-\frac{1}{40}$	0	$30=\frac{30}{3/4}=40$ 1
Optimality is not attained		$z_j^*=-300M-1800$	-60	-15M-45	M	$\frac{-M}{2}+\frac{3}{2}$	-M	Go to next iteration
		$z_j^*-c_j$	0	-15M+35	M	$\frac{-M}{2}+\frac{3}{2}$	0	

↑

Aritificial variable a_1 leaves the basis. Hence the vector a_1 may be removed from the iteration table. x_2 enters the basis and 15 is the pivotal elememt.

Second iteration table

Basis		c_j	-60	-80	O	O	End of simplex procedure
C_B	B	x_B	x_1	x_2	S_1	S_2	
-80	x_2	20	0	1	$-\frac{1}{15}$	$\frac{+1}{30}$	
-60	x_1	15	1	0	$-\frac{1}{20}$	$\frac{-1}{20}$	
Optimality is not attained		$z_j^*=-2500$	-60	-80	$\frac{7}{3}$	$\frac{1}{3}$	
		$z^*_j-c_j$	0	0	$\frac{7}{3}$	$\frac{1}{3}$	

Hence all $z_j^*-c_j \geq 0$ Hence optimality is attained. Hence the optimum solution is $x_1=15$, $x_2=20$. Maximum $z^* = -2500$

Hence Min $Z = -\max Z^* = -(-2500) = 2500$

Minimum $Z = 2500$.

Problem 2 :-

Solve the following LPP using Big-M method.

Maximize

$z=2x_1+3x_3$

subject to

$x_1+x_2+2x_3 \leq 5$ .... (1)

$2x_1+3x_2+4x_3=12$ .... (2)

$x_1, x_2, x_3 \geq 0$.

Soultion :

We introduce a slack variable $S_1 \geq 0$ in the constraint (1) and an artificial variable $a_1 \geq 0$ in the constraint (2).

Associating a Big-M method the objective function for the artificial variable we get the LPP in the standard form as :

Maximize

$Z = 2x_1+O.x_2+3x_3+OS_1-Ma_1$

Subject to

$x_1+x_2+2x_3+S_1 = 5$

$2x_1+3x_2+4x_3+a_1=12$

$x_1, x_2, x_3, S_1, a_1 \geq 0$.

The initial basic feasible solution is $(s_1, a_1) = (5, 12)$ The given LPP can be represented in the following initial simplex table

Initial Simplex table

Basis		c_j	2	0	3	0	-M	Ratio
C_B	B	x_B	x_1	x_2	x_3	S_1	a_1	
0	s_1	5	1	1	$\boxed{2}$	1	0	$\frac{5}{2}$ →
-M	a_1	12	2	3	4	0	1	$\frac{12}{4}=3$
Optimality is not attained		$z_j=-12M$	-2M	-3M	-4M	0	-M	Go to next iteration
		z_j-c_j	-2M-2	-3M	-4M-3	0	0	

↑

x_3 enters the basis and S_1 leaves the basis and 2 is the Pivotal element.

First Iteration table

Basis		c_j	2	0	3	0	-M	Ratio
C_B	B	x_B	x_1	x_2	x_3	S_1	a_1	
3	x_3	5/2	1/2	1/2	1	1/2	0	$\frac{5}{2}/\frac{1}{2}=5$ →
-M	a_1	2	0	$\boxed{1}$	0	-2	1	$\frac{2}{1}=2$
Optimality is not attained		$z_j=-2M+\frac{15}{2}$	$\frac{3}{2}$	$\frac{3}{2}-M$	3	$2M+\frac{3}{2}$	-M	Go to next iteration
		z_j-c_j	$-\frac{1}{2}$	$-M+\frac{3}{2}$	0	$2M+\frac{3}{2}$	0	

↑

The net evaluation -M is the most negative. Hence we choose the next most negative evaluation is $-M+\frac{3}{2}$. Hence x_2 enters the basis and a_1 leaves the basis and 1 is the Pivotal element. Hence the vector a_1 may be removed from the iteration.

Seoncd iteration table

Basis		c_j	2	0	3	0	Ratio
C_B	B	x_B	x_1	x_2	x_3	S_1	
3	x_3	3/2	$\boxed{1/2}$	0	1	3/2	$\frac{3/2}{1/2}=3$ →
0	x_2	2	0	1	0	-2	-
Optimality is not attained		$z_j=+9/2$	3/2	0	3	9/2	Go to next iteration
		z_j-c_j	-1/2	0	0	9/2	

↑

x_1 enters the basis and x_3 leaves the basis and $\frac{1}{2}$ is the Pivotal element.

Third iteration table

Basis		c_j	2	0	3	0	End of simplex procedure
C_B	B	x_B	x_1	x_2	x_3	S_1	
2	x_1	3	1	0	2	3	
0	x_2	2	0	1	0	-2	
Optimality is not attained		$z_j=6$	2	0	4	6	
		z_j-c_j	0	0	1	6	

The optimum solution is $x_1=3$; $x_2=2$; $x_3=0$ and Max $z=6$

Problem 3:- Using Big-M method, solve the following LPP.

Minimize

$$z=4x_1+x_2$$

Subject to

$3x_1+x_2=3$ .... (1)

$4x_1+3x_2 \geq 6$ .... (2)

$x_1+2x_2 \leq 4$ ...... (3)

$x_1, x_2 \geq 0$.

Solution :-

The given problem is to minimize The objective function

We introduce slack variable S_1 to the constraint (3), surplus variable S_1 to the constraint (2) and artificial variables a for (1) and a_2 for (3)

Writing $Z^* = -Z$ the LPP in standard form is :

Max $Z^* = -4x_1-x_2+OS_1+OS_2-Ma_1-Ma_2$

Subject to

$3x_1+x_2+a_1=5$

$4x_1+3x_2-S_1+a_2=6$

$x_1+2x_2+S_1=4$

$x_1, x_2, s_1, S_1, a_1, a_2 \geq 0.$

Initial Simple table

Basis		c_j	-4	-1	0	0	-M	-M	Ratio
C_B	B	x_B	x_1	x_2	S_1	s_1	a_1	a_2	
-M	a_2	3	$\boxed{3}$	1	0	0	0	1	3/3 =1 →
-M	a_1	6	4	3	-1	0	1	0	6/4 =3/2
0	s_1	4	1	2	0	1	0	0	4/1=4
Optimality is not attained		$z_j^*=-9M$	-7M	-4M	M	0	-M	-M	Go to next iteration
		$z^*_j-c_j$	-7M+4	-4M+1	M	0	0	0	

↑

Artificial variable a_2 leaves the basis. Hence the vector a_2 may not be considered for further iteration and hence removed from the iteration table. x_1 enters the basis and 3 is the Pivotal elements.

First iteration table

Basis		c_j	-4	-1	0	0	-M	Ratio
C_B	B	x_B	x_1	x_2	S_1	s_1	a_1	
-4	x_1	1	1	1/3	0	0	0	3
-M	a_1	2	0	$\boxed{5/3}$	-1	0	1	6/5=1.2 →
0	S_1	3	0	5/3	0	1	0	9/5=1.8
Optimality is not attained		$z_j^*=-2M-4$	-4	-5M/3-4/3	M	0	-M	Go to next iteration
		$z^*_j-c_j$	0	-5M/3-1/3	M	M	0	

↑

Artificial variable a_1 leaves the basis. Hence the vector a_1 is removed from the iteration table x_2 enters the basis and $\frac{5}{3}$ is the Pivotal element.

Second iteration table

Basis		c_j	-4	-1	0	0	Ratio
C_B	B	x_B	x_1	x_2	S_1	s_1	
-4	x_1	3/5	1	0	1/5	0	3
-1	x_2	6/5	0	1	- 3/5	0	-
0	s_1	1	0	0	$\boxed{1}$	1	1 →
Optimality is not attained		$z_j^* = -18/5$	-4	-1	-1/5		Go to next iteration
		$z^*_j - c_j$	0	0	-1/5	0	

↑

S_1 leaves the basis. S_1 enters the basis and 1 is the Pivotal element.

Third iteration table

Basis		c_j	-4	-1	0	0	
C_B	B	x_B	x_1	x_2	S_1	s_1	End of simplex procedure
-4	x_1	2/5	1	0	0	-1/5	
-1	x_2	9/5	0	1	0	3/5	
0	S_1	1	0	0	1	1	
Optimality is not attained		$z_j^* = -17/5$	-4	-1	0	1/5	
		$z^*_j - c_j$	0	0	0	1/5	

$$\text{Min } z = \text{Max } Z^* = -\frac{17}{5}, \quad x_1 = \frac{2}{5}, \quad x_2 = \frac{9}{5}$$

$$\text{Min } z = \frac{17}{5}$$

Problem 4 :- solve

Maximize $z= 4x_1+x_2$

Subject to

$3x_1+x_2=3$

$4x_1+3x_2 \geq 6$

$x_1+2x_2 \leq 4$

$x_1, x_2 \geq 0$

Solution :- The standard form of the Lpp is

Maximize $Z=4x_1+x_2+OS_1-Ma_1-Ma_2+Os_1$

Subject to

$3x_1+x_2+a_1=3$

$4x_1+3x_2-S_1+a_2=6$

$x_1+2x_2+s_1=4$

$x_1, x_2, s_1, S_1, a_1, a_2 \geq 0$

The iteration tables are given below.

Initial simplex table

Basis		c_j	4	1	0	-M	-M	0	Ratio
C_B	B	x_B	x_1	x_2	S_1	a_1	a_2	s_1	
-M	a_1	3	$\boxed{3}$	1	0	1	0	0	3/3 =1 →
-M	a_2	6	4	3	-1	0	1	0	6/4 =1.5
0	s_1	4	1	2	0	0	0	1	4/1=4
Optimality is not attained		z_j =-9M	-7M	-4M	M	-M	-M	0	Go to next iteration
		z_j-c_j	-7M-4 ↑	-4M-1	M	0	0	0	

a_1 leaves the basis; x_1 enters the basis and 3 is the Pivotal element. Hence the vector a_1 may be removed from the iteration table.

First iteration table

Basis		c_j	4	1	0	-M	0	Ratio
C_B	B	x_B	x_1	x_2	S_1	a_2	s_1	
-4	x_1	1	1	1/3	0	0	0	3
-M	a_2	2	0	$\boxed{5/3}$	-1	1	0	6/5=1.2 →
0	s_1	3	0	5/3	0	1	1	9/5=1.8
Optimality is not attained		z_j =-2M+4	4	-3M+1/3	M	-M	0	Go to next iteration
		z_j-c_j	0	$\frac{-9M-2}{3}$ ↑	M	0	0	

a_2 leaves the basis, x_2 enters the basis and $\frac{5}{3}$ is the Pivotal element. Hence the vector a_2 may be removed from the iteration table.

second iteration table

Basis		c_j	4	1	0	0	End of simplex procedure
C_B	B	x_B	x_1	x_2	S_1	s_1	
4	x_1	3/5	1	0	1/5	0	
1	x_2	6/5	0	1	-3/5	0	
0	s_1	1	0	0	1	1	
Optimality is not attained		z_j =18/5	4	1	1/5	0	
		$z_j - c_j$	0	0	1/5	0	

↑

Optimal solution is $x_1 = \frac{3}{5}$, $x_2 = \frac{6}{5}$ &

Maximum $Z = \frac{18}{5} = \frac{18}{5}$

Problem 5 :- Solve the LPP.

Maximize

$Z = 3x_1 + 2x_2$

Subject to

$2x_1 + x_2 \leq 2$ ....(1)

$3x_1 + 4x_2 \geq 12$ ....(2)

$x_1, x_2 \geq 0$

solution :-

Introducing slack variable s_1, from equation (1) and surplus variable and an artificial variable a for the equation (2). The objective function the LPP in standard form becomes.

Maximze

$Z = 3x_1 + 2x_2 + os_1 + OS_1 - Ma_1$

Subject to

$2x_1 + x_2 + s_1 = 2$

$3x_1 + 4x_2 - S_1 + a_1 = 12$

$x_1, x_2, s_1, S_1, a_1 > 0.$

Initial Simplex table

Basis		c_j	3	2	0	0	-M	Ratio
C_B	B	x_B	x_1	x_2	s_1	S_1	a_1	
0	S_1	2	2	$\boxed{1}$	1	0	0	2/1 =2 →
-M	a_1	12	3	4	0	-1	1	12/4 =3
Optimality is		z_j =-12M	-3M	-4M	0	M	0	Go to
not attained		z_j-c_j	-3M-3	-4M-2	0	M	0	next iteration

↑

$_1$ enters the basis and s_1 leaves the basis; 1 is the Pivotal element

First iteration table

Basis		c_j	3	2	0	0	-M	End of simplex procedure
C_B	B	x_B	x_1	x_2	s_1	S_1	a_1	
2	x_2	2	2	1	1	0	0	
-M	a_1	4	-5	0	-4	-1	1	
Optimality is not attained		z_j =-4M+4	5M+4	2	4M+2	M	-M	
		z_j-c_j	5M+1	0	4M+2	M	0	

Since all z_j-c_j $\geq$0 and one artificial variable a_1 appears in the basis at non-zero level (a_1=4). The given LPP has no feasible solution.

Exercises :- solve the following problems using Big-M or Penalty method

1) Minimize $Z=4x_1+3x_2$

Subject to

$2x_1+x_2 \geq 10$

$-3x_1+2x_2 \leq 6$

$x_1+x_2 \geq 6$

$x_1, x_2 \geq 0$.

2) Minimize

$Z= 4x_1+x_2$ subject to

$3x_1+x_2=3$; $4x_1+3x_2 \geq 6$; $x_1+2x_2 \leq 3$

$x_1, x_2 \geq 0$

3) Minimize

$Z= 3x_1+4x_2$

Subject to

$20x_1+60x_2 \geq 80$; $30x_1 + 40x_2 \geq 10$; x_1 , $x_2 \geq 0$.

4) Minimize

$z = 4x_1+6x_2$

subject to

$x_1+2x_2 \geq 80$, $3x_1+x_2 \geq 75$

x_1, $x_2 \geq 0$.

5) Maximize $z = 5x_1+x_2$

subject to

$5x_1+2x_2 \leq 20$; $x_1 \geq 3$, $x_2 \leq 5$

x_1, $x_2 \geq 0$.

6) Maximize

$Z = 20x_1+10x_2$

subject to $x_1+x_2 = 1500$; $x_1 \leq 40$; $x_2 \geq 20$

x_1, $x_2 \geq 0$

7) Maximize

$z = x_1+x_2+x_4$

subject to

$x_1+x_2+x_3+x_4 = 4$

$x_1+2x_2+x_3+x_5=4$

x_1, x_2, x_3, x_4, x_5, ≥ 0.

8) Maximize

$z= x_1+2x_2+3x_3-x_4$

subject to

$x_1+2x_2+3x_3=24$

$2x_1+x_2+5x_3=20$

$x_1+2x_2+x_3+x_4=10$

x_1, x_2, x_3, $x_4 \geq 0$.

?.3 Two phase method :-

Two phase method is another method of solving a LPP that involves artificial variables. The solution is obtained in two phases. In phase1, the sum of the artificial variables is minimized subject to the given constraints of the given LPP.

It this solution contains artificial variables with positive value then the given LPP has no optimum solution. Otherwise, we go to phase 2, where the basic feasible solution obtained in phase 1 is taken as the initial basic feasible solution to the given LPP and then the simplex algorithm is applied.

Phase I:-

Step1:-

Let Z' be the new objective function obtained by replacing all the coefficients c_i of variables x_i to zero in z=cx and introducing required artificial variables a_i with coefficients -1 for each. The LPP given by maximize z' subject to the constraints of the given LPP is known as auxiliary LPP.

Step 2:-

Solve the auxiliary LPP by applying the simplex algorithm. Then the following cases may arise.

Case 1:-

Maximum z'=0 and atleast one artificial variable is present in the basis with positive values. Then the original LPP has no optimal solution.

case 2:-

Maximum z'=0 and no artificial variable is present or an artificial variable is present with zero value then we go to phase II.

Phase II :-

Consider, the original objective function z=cx. If, in the basic feasible solution obtained in phase I, the artificial variable with zero value is present then the artificial variable is added to z with zero coefficient. Then solve the LPP using simplex algorithem with the basic feasible solution obtained in phase I as the starting solution.

This method is illustrated in the following examples.

Solved problems:-

Problem 1 :- solve the following LPP using Two-phase simplex method.

Minimize $Z=60x_1+80x_2$

Subject to $20x_1+30x_2 \geq 900$

$40x_1+30x_2 > 1200$

$x_1, x_2 \geq 0$.

Solution :-

The given LPP is of minimization type. Hence it is converted to maximization type as maximize $Z^* = -60x_1-80x_2$
where $z^* = -z$
Phase I :- Solving auxiliary LPP using simplex method.

We introduce surplus variables $S_1 \geq 0$, $S_2 \geq 0$ to convert the inequalities to equalities; artificial variables $a_1 \geq 0$, $a_2 \geq 0$ to get an initial basis B (unit column vectors).

Further, we assign zero coefficients to basic variables and coefficient -1 to artificial variables to the objective function.

We get the auxiliary LPP as :

Maximize

$z' = O.x_1+O.x_2+OS_1+OS_2-a_1-a_2$

subject to

$20x_1+30x_2-S_1+a_1 = 900$

$40x_1+30x_2-S_2+a_2 = 1200$

$x_1, x_2, S_1, S_2, a_1, a_2 \geq 0.$

we now use simplex method to solve the auxiliary LPP. The initial simplex table is given as follows.

Phase I - Initial simplex table

Basis		c_j	0	0	0	0	-1	-1	Ratio
C_B	B	x_B	x_1	x_2	S_1	S_2	a_1	a_2	
-1	a_1	900	20	30	-1	0	1	0	$\frac{900}{20}=45$
-1	a_2	1200	$\boxed{40}$	30	0	-1	0	1	$\frac{1200}{40}=30$ →
Optimality is not attained		$z_j'=-2100$	-60	-60	1	1	-1	-1	Go to next iteration
		z'_j-c_j	-60	-60	1	1	0	0	

↑

x_1 enters the basis, the artificial variable a_2 leaves the basis and 40 is the Pivotal element The column vector a_2 need not be considered in further iterations.

Phase I - First iteration table

Basis		c_j	0	0	0	0	-1	Ratio
C_B	B	x_B	x_1	x_2	S_1	S_2	a_1	
-1	a_1	300	0	$\boxed{15}$	-1	$\frac{1}{2}$	1	$\frac{300}{15}=20 \rightarrow$
0	x_1	30	1	$\frac{3}{4}$	0	$-\frac{1}{40}$	0	$\frac{30\times4}{3}=40$
Optimality is not attained		$z'_j=$	0	-15	1	$-\frac{1}{2}$	-1	Go to next iteration
		z'_j-c_j	0	-15	1	$-\frac{1}{2}$	0	

↑

x_2 enters the basis; the artificial variable a_1 leaves the basis and 15 is the Pivotal element. The column vector a_1 need not be considered in further iterations.

Phase I - second iteration table

Basis		c_j	0	0	0	0	Go to Phase - II
C_B	B	x_B	x_1	x_2	S_1	S_2	
0	x_2	20	0	1	$-\frac{1}{15}$	$\frac{1}{30}$	
0	x_1	15	1	0	$\frac{1}{20}$	$-\frac{1}{20}$	
Optimality is not attained		$z'_j=0$	0	0	0	0	
		z'_j-c_j	0	0	0	0	

Here all z'_j-c_j are non-negative, max $z'_j=0$ and no artificial variable appears as basic variable Hence the optimum basic feasible solution is got for the auxiliary LPP. Hence the basic feasible solution is obtained to the given LPP. Now, we proceed to phase II to obtain optimum basic feasible solution to the given LPP, taking the final simplex table of phase I as the initial simplex table of phase II and proceed to optimize z'_j using simplex method

Phase II:- To find the optimum basic feasible solution.

The new objective function is $z^*= -3x_1-4x_2+OS_1+OS_2$. In this phase, the non-basic artificial variable columns may be deleted; actual costs corresponding to the original variables and costs O to the artificial variables are taken

Phase - II - Initial simplex table

Basis		c_j	-60	-80	0	0	End of Phase - II
C_B	B	x_B	x_1	x_2	S_1	s_2	
-80	x_2	20	0	1	$-\frac{1}{15}$	$\frac{1}{30}$	
-60	x_1	15	1	0	$\frac{1}{20}$	$-\frac{1}{20}$	
Optimality is not attained		$z_j^*=-2500$	-60	-80	$\frac{7}{3}$	$\frac{1}{3}$	
		z^*-c_j	0	0	$\frac{7}{3}$	$\frac{1}{3}$	

Since all $z^*_j-c_j \geq 0$.

Optimality is attained for Z^*.

∴ Optimal solution is $x_2 = 20$, $x_1 = 15$

Max $Z^* = -2500$

∴ Min $z = -\max z^* = -(-2500)$

Minimum $z = 2500$.

$x_1=15$

$x_2=20$.

Problem 2 :- Solve the LPP using Two phase method.

Maximize $Z=5x_1+8x_2$

Subject to $3x_1+2x_2 \geq 3$

$x_1+4x_2 \geq 4$; $x_1+x_2 \leq 5$

x_1, $x_2 \geq 0$.

Solution : - Phase - I (solving auxiliary LPP using simplex method)

We introduce surplus variables $S_1 \geq 0$, $S_2 \geq 0$ and $a_1 \geq 0$, $a_2 \geq 0$ to the first two constraints and slack variable $s_1 \geq 0$, to the third constraint. Further, we assign zero colfficients to basic variables and coefficients -1 to artificial variables to the objective function. Thus we get the auxiliary LPP as :

Maximize $z' = Ox_1+Ox_2+OS_1+OS_2-a_1-a_2-s_1$

subject to $3x_1+2x_2-S_1+a_1=3$; $x_1+4x_2-S_2+a_2=4$; $x_1+x_2+s_1=5$.

x_1, x_2, S_1, S_2, a_1, a_2, $s_1 \geq 0$

Phase - I - Initial simplex table

Basis		c_j	0	0	0	0	-1	-1	0	Ratio
C_B	B	x_B	x_1	x_2	S_1	S_2	a_1	a_2	S_1	
-1	a_1	3	3	2	-1	0	1	0	0	$\frac{3}{2}=1.5$
-1	a_2	4	1	$\boxed{4}$	0	-1	0	1	0	$\frac{4}{4}=1$ →
0	S_1	5	1	1	0	0	0	0	1	$\frac{5}{1}=5$
Optimality is not attained		$z_j'=-7$	-4	-6	1	1	-1	-1	0	Go to next iteration
		z'_j-c_j	-4	-6	1	1	0	0	0	

↑

x_2 enters the basis; the artificial variable a_2 leaves the basis and 4 is the Pivotal element. The column vector a_2 need not be considered in further iterations.

Phase - I First iteration table

Basis		c_j	0	0	0	0	-1	0	Ratio
C_B	B	x_B	x_1	x_2	S_1	S_2	a_1	S_1	
-1	a_1	1	$\boxed{\frac{5}{2}}$	0	-1	$\frac{1}{2}$	1	0	$\frac{2}{5}$ →
0	a_2	1	$\frac{1}{4}$	1	0	$-\frac{1}{4}$	0	0	4
0	S_1	4	$\frac{3}{4}$	0	0	$\frac{1}{4}$	0	1	$\frac{16}{1}$
Optimality is not attained		$z_j'=-1$	$-\frac{5}{2}$	0	1	$-\frac{1}{2}$	-1	0	Go to next iteration
		z'_j-c_j	$-\frac{5}{2}$	0	0	$-\frac{1}{2}$	0	0	

↑

x_1 enters the basis; the artificial variable a_1 leaves the basis and $\frac{5}{2}$ is the Pivotal element The column vector a_1 need not be considered in further iterations.

Phase I - Second iteration table

Basis		c_j	0	0	0	0	0	
C_B	B	x_B	x_1	x_2	S_1	S_2	s_1	
0	x_1	$\frac{2}{5}$	1	0	$-\frac{2}{5}$	$\frac{1}{5}$	0	Go to Phase - II
0	x_2	$\frac{9}{10}$	0	1	$\frac{1}{10}$	$-\frac{3}{10}$	0	
0	S_1	$\frac{37}{10}$	0	0	$\frac{3}{10}$	$\frac{1}{10}$	1	
Optimality is not attained		$z_j' = 0$	0	0	0	0	0	
		$z'_j - c_j$	0	0	0	0	0	

Hence all $z_j' - c_j$ are non-negative, Max $z' = 0$ and no artificial variable appears as basic variable. Optimum basic feasible solution is got for the auxiliary LPP. Hence the basic feasible solution is obtained to the given LPP.

Phase - II (To find optimal basic feasible solution)

Now, we proceed to phase II to obtain optimum basic feasible solution to the given LPP, taking the final simplex table of phase I as the initial simplex table of phase II and proceed to optimize z using simplex method.

Phase II- Initial simplex table

Basis		c_j	5	8	0	0	0	Ratio
C_B	B	x_B	x_1	x_2	S_1	S_2	s_1	
5	x_1	$\frac{2}{5}$	1	0	$-\frac{2}{5}$	$\boxed{\frac{1}{5}}$	0	2 →
8	x_2	$\frac{9}{10}$	0	1	$\frac{1}{10}$	$-\frac{3}{10}$	0	-
0	S_1	$\frac{37}{10}$	0	0	$\frac{3}{10}$	$\frac{1}{10}$	1	37
Optimality is not attained		$z_j' = \frac{46}{5}$	5	8	$-\frac{6}{5}$	$-\frac{7}{5}$	0	Go to next iteration
		$z'_j - c_j$	0	0	$-\frac{6}{5}$	$-\frac{7}{5}$	0	
						↑		

S_2 enter the basis; x_1 leaves the basis and $\frac{1}{5}$ is the Pivotal element.

Phase II- First iteration table

Basis		c_j	5	8	0	0	0	Ratio
C_B	B	x_B	x_1	x_2	S_1	S_2	s_1	
5	s_2	2	5	0	-2	1	0	-
8	x_2	$\frac{3}{2}$	$\frac{3}{2}$	1	$-\frac{1}{2}$	0	0	-
0	S_1	$\frac{7}{2}$	$-\frac{1}{2}$	0	$\boxed{\frac{1}{2}}$	0	1	7 →
Optimality is not attained		z_j'= 12	12	8	-4	0	0	Go to next iteration
		z'_j-c_j	7	0	-4	0	0	

↑

S_1 enters the basis; s_1 leaves the basis and $\frac{1}{2}$ is the Pivotal element.

Phase II- second iteration table

Basis		c_j	5	8	0	0	0	
C_B	B	x_B	x_1	x_2	S_1	S_2	s_1	End of Phase - II
0	S_2	10	3	0	0	1	4	
8	x_2	5	1	1	0	0	1	
0	S_1	7	-1	0	1	0	2	
Optimality is not attained		z_j'= 40	8	8	0	0	8	
		z'_j-c_j	3	0	0	0	8	

Optimal solution is x_1 =0; x_2=5

Max z = 40

Problem 3 :- Solve the following LPP.

Maximize

$$Z= 2x_1+3x_2+5x_3$$

Subject to

$$3x_1+10x_2+5x_3 \leq 15$$
$$x_1+2x_2+x_3 \geq 4$$
$$33x_1-10x_2+9x_3 \leq 33$$
$$x_1, x_2, x_3 \geq 0.$$

Solution :- we employ two-phase simplex method to solve the LPP:-

Phase I .- (Solving auxiliary LPP using simplex method)

We introduce slack variables $S_1 \geq 0$, $S_2 \geq 0$ surplus variable $S_1 \geq 0$ and artificial variable $a_1 \geq 0$. Further, we assign zero colfficients to basic variables and slack variables and colfficients -1 to artificial variable to the objective function.

Thus we get the auxiliary LPP as:-

Maximize $z' = Ox_1 + Ox_2 + Ox_3 + Os_1 + Os_2 + OS_1 - a_1$

Subject to

$3x_1 + 10x_2 + 5x_3 + s_1 = 15$

$x_1 + 2x_2 + x_3 - S_1 + a_1 = 4$

$33x_1 - 10x_2 + 9x_3 + s_2 = 33$

$x_1, x_2, x_3, s_1, s_2, S_1, a_1 \geq 0$. We use simplex method to solve the auxiliary. LPP The initial simplex table is given as follows.

Phase I - Initial Simplex table

Basis		c_j	0	0	0	0	-1	0	0	Ratio
C_B	B	x_B	x_1	x_2	x_3	s_1	a_1	S_1	s_2	
0	s_1	15	3	$\boxed{10}$	5	1	0	0	0	$\frac{15}{10} = \frac{3}{2}$ →
-1	a_1	4	1	2	1	0	1	-1	0	$\frac{4}{2} = 2$
0	s_2	33	33	-10	9	0	0	0	1	-
Optimality is not attained		$z_j = -4$	-1	-2	-1	0	0	1	0	Go to next iteration
		$z_j - c_j$	-1	-2	-1	0	0	1	0	
				↑						

x_2 enters the basis; s_1 leaves the basis and 10 is the Pivotal element.

Phase I - First iteration table

Basis		c_j	0	0	0	0	-1	0	0	Ratio
C_B	B	x_B	x_1	x_2	x_3	s_1	a_1	S_1	s_2	
0	x_2	$\frac{3}{2}$	$\frac{3}{10}$	1	$\frac{1}{2}$	$\frac{1}{10}$	0	0	0	5
-1	a_1	1	$\frac{2}{5}$	0	0	$-\frac{1}{5}$	1	-1	0	$\frac{5}{2}$
0	S_2	48	$\boxed{36}$	0	14	1	0	0	1	$\frac{4}{3}$ →
Optimality is not attained		$z_j = -1$	$-\frac{2}{5}$	0	0	$\frac{1}{5}$	-1	1	0	Go to next iteration
		$z_j - c_j$	$-\frac{2}{5}$	0	0	$\frac{1}{5}$	0	1	0	
			↑							

x_1 enters the basis; s_2 leaves the basis and 36 is the Pivotal element.

Phase I - second iteration table

Basis		c_j	0	0	0	0	-1	0	0	End of Phase - I
C_B	B	x_B	x_1	x_2	x_3	s_1	a_1	S_1	s_2	
0	x_2	$\frac{11}{10}$	0	1	$\frac{23}{60}$	$\frac{11}{120}$	0	0	$\frac{-1}{120}$	
-1	a_1	$\frac{7}{15}$	0	0	$\frac{-7}{45}$	$\frac{-19}{90}$	1	-1	$\frac{-1}{90}$	
0	x_1	$\frac{4}{3}$	1	0	$\frac{7}{18}$	$\frac{1}{36}$	0	0	$\frac{1}{36}$	
Optimality is not attained		$z_j = \frac{-7}{15}$	0	0	$\frac{7}{45}$	$\frac{19}{90}$	-1	1	$\frac{1}{90}$	
		$z_j - c_j$	0	0	$\frac{7}{45}$	$\frac{19}{90}$	0	1	$\frac{1}{90}$	

Hence all $z_j - c_j \geq 0$. But the artificial variable a_1 is present as basic variable with positive value, $a_1 = \frac{7}{15}$

Hence, the problem has infeasible

Solution :-

Exercises:- Use two-phase simplex method to solve the following. LPP

1) Minimize

$z = x_1 + x_2$

Subject to

$2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \leq 7$

$x_1, x_2 \geq 0.$

2) Minimize

$z = 4x_1 + 3x_2$

Subject to

$2x_1 + x_2 \geq 40$

$x_1 + 2x_2 \geq 50$

$x_1 + x_2 \geq 35$

$x_1, x_2 \geq 0$

3) Maximize

$z = 9x_1 + 10x_2$

Subject to

$x_1 + 2x_2 \geq 25$

$4x_1+3x_2 \geq 24$

$3x_1+2x_2 \geq 60$

$x_1, x_2 \geq 0.$

4) Maximize

$z=5x_1-4x_2+3x_3$

Subject to

$x_1+5x_2-3x_3 \geq 15$

$5x_1-6x_2+10x_3 \leq 20$

$x_1+x_2+x_3 = 5$

$x_1, x_2, x_3 \geq 0.$

5) Minimize

$z= 12x_1+18x_2+15x_3$

Subject to $4x_1+8x_2 \geq 64$; $3x_1+6x_2+12x_3 \leq 96$

$x_1, x_2, x_3 \geq 0.$

2.4 Applications of simplex method :-

I. Solution of simultaneous linear equations by simplex method:-

Consider a system of n simultaneous linear equations in n variables given by Ax=b where A is a nxn real matrix and x and b are nx1 real matrices.

To solve this system of equations by using simplex method, we introduce artificial variables $a_1, a_2, \ldots\ldots a_n$ and an objective function.

$z = Ox_1+Ox_2+ \ldots\ldots+Ox_n-a_1-a_2\ldots\ldots-a_n.$

since each x_i is an unrestricted variable, we introduce non-negative variables x'_i and x''_i suchthat $x_i=x'_i - x''_i$

Thus we obtain the LPP.

Maximize

$z = O(x'_1-x''_1)+O(x'_2-x''_2)+\ldots\ldots+O(x'_n-x''_n)-a_1-a_2\ldots\ldots-a_n$

Subject to the constraints $A(x'-x'')+a=b$

where $x', x'', a \geq 0$.

Here $x = x'-x''$ and $a=(a_1, a_2, \ldots\ldots a_n)^T$

we solve this LPP by simplex method. If we obtain a basic feasible solution to this LPP suchthat no artificial variable appears in the solution, then this solution satisfies the constraints of the LPP $A(X'-X'') +a = b$.

Since no artificial variable appears in the solution it gives a solution to the given system of simultaneous equations.

solved problems :-

1) Solve the following system of equations using simplex method.

$2x_1+x_2=3$

$x_1+x_2=1$

solution :-

we consider the LPP given by

Maximize

$z = 0x_1+0x_2-a_1-a_2.$

Subject to

$2x_1+x_2+a_1=3$

$x_1+x_2+a_2=1$

x_1, x_2 unrestricted and $a_1, a_2 \geq 0$. since x_1 and x_2 are unrestricted in sign, we introduce variables $x'_1, x''_1, x'_2, x''_2 \geq 0$.

Suchthat $x_1 = x'_1-x''_1$; $x_2=x'_2 - x''_2$.

Now the LPP becomes,

Maximize

$$z = 0(x'_1 - x''_1) +0(x'_2 - x''_2)-a_1-a_2$$

Subject

$2(x'_1-x''_1) + (x'_2-x''_2) + a_1 = 3$

$(x'_1-x''_1) + (x'_2-x''_2) + a_2 =1$

$x'_1, x''_1, x'_2, x''_2, a_1, a_2 \geq 0.$

we solve the above LPP for the variables x'_1, x''_1, x'_2, x''_2 using simplex method.

Initial simplex method

Basis		c_j	0	0	0	0	-1	-1	Ratio
C_B	B	x_B	x'_1	x''_1	x''_2	x'_2	a_1	a_2	
-1	a_1	3	2	-2	1	-1	1	0	$\frac{3}{2}$ =1.5
-1	a_2	1	$\boxed{1}$	-1	1	-1	0	1	$\frac{1}{1}$ =1 $\rightarrow$
Optimality is not attained		z_j=-4	-3	3	-2	2	-1	-1	Go to next iteration
		z_j-c_j	-3	3	-2	2	0	0	
			$\uparrow$						

a_2 leaves the basis; x'_1 enters the basis and 1 is the Pivotal element.

First iteration table

Basis		c_j	0	0	0	0	-1	-1	Ratio
C_B	B	x_B	x'_1	x''_1	x'_2	x''_2	a_1	a_2	
-1	a_1	1	0	0	-1	$\boxed{1}$	1	-2	$\frac{1}{1}=1 \rightarrow$
0	x_1	1	1	-1	1	-1	0	1	-
Optimality is not attained		$z_j=-1$	0	0	1	-1	-1	2	Go to next iteration
		z_j-c_j	0	0	1	-1	0	3	

↑

a_1 leaves the basis; x''_2 enters the basis and 1 is the Pivotal element.

Second iteration table

Basis		c_j	0	0	0	0	-1	-1	End of simple produre
C_B	B	x_B	x'_1	x''_1	x'_2	x''_2	a_1	a_2	
0	x''_2	1	0	0	-1	1	1	-2	
0	x'_1	2	1	-1	0	0	1	-1	
Optimality is not attained		$z_j=0$	0	0	0	0	0	0	
		z_j-c_j	0	0	0	0	1	1	

all $z_j-c_j \geq 0$, Hence optimality is obtained.

∴ $x'_1=2$; $x''_2=1$, $x''_1=0$, $x'_2=0$

Hence $x_1=x'_1-x''_1 = 2 - 0 = 2$

$\Rightarrow x_1 = 2$; $x_2 = x'_2-x''_2=0-(+1) = -1$

$\Rightarrow x_2 = -1$

∴ The solution of the given equation is $x_1 = 2$; $x_2 = -1$

Problem 2 :- Using simplex method. Prove that the following system of equations has no solution $x_1+2x_2=3$; $2x_1+4x_2=8$.

Solution :- we consider the LPP given by

Maximize $z=Ox_1+Ox_2-a_1-a_2$ subject to

$x_1+2x_2+a_1=3$; $2x_1+4x_2+a_2=8$; x_1, x_2 unrestricted and a_1, $a_2 \geq 0$.

Since x_1 and x_2 are unrestricted in sign, we introduce variables x'_1, x''_2, x'_1, $x''_2 \geq 0$ suchthat $x_1=x'_1-x''_1$; $x_2=x'_2-x''_2$

Now, the LPP becomes,

Maximize $z = O(x'_1-x''_1) +O(x'_2-x''_2) -a_1-a_2$

subject to

$$(x'_1-x''_1)+2(x'_2-x''_2)+a_1=3$$

$$2(x'_1-x''_1)+4(x'_2-x''_2)+a_2=8$$

$$x'_1,\ x'_2,\ x''_1,\ x''_2\ a_1,\ a_2 \geq 0.$$

we solve the above LPP for the variables x', x'_2, x''_1, x''_2 using simplex method

The initial simplex table is as follows.

Initial simplex table

Basis		c_j	0	0	0	0	-1	-1	Ratio
C_B	B	X_B	x'_1	x''_1	x'_2	x''_2	a_1	a_2	
-1	a_1	3	1	-1	$\boxed{2}$	-2	1	0	$\frac{3}{2}=1.5\rightarrow$
-1	a_2	8	2	-2	4	-4	0	1	$\frac{8}{4}=2$
Optimality is not attained		$z_j=-11$	-3	3	-6	6	-1	-1	Go to next iteration
		z_j-c_j	-3	3	-6	6	0	0	

↑

a_1 leaves the basis; x'_2 enters the basis and 2 is the Pivotal element.

First iteration table

Basis		c_j	0	0	0	0	-1	-1	End of simple procedure
C_B	B	X_B	x'_1	x''_1	x'_2	x''_2	a_1	a_2	
0	x'_2	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	-1	$\frac{1}{2}$	0	
-1	a_2	2	0	0	0	0	-2	1	
Optimality is not attained		$z_j=-2$	0	0	0	0	2	-1	
		z_j-c_j	0	0	0	0	3	0	

In the final simplex table optimality is obtained but artificial variable a_2 appears at non-zero level. Hence the original LPP has no solution. Hence the system has no solution (inconsistent)

Exercises:-

I) Solve the following system of equations by simplex method:-

a) $x_1+2x_2=4$
$2x_1-3x_2=1$

b) $x_1-2x_2=5$
$2x_1+3x_2=3$

II Using simplex method, Prove that the following system of equations is inconsistent.

a) $x_1+3x_2 = 5$

$3x_1+9x_2 = 18$

II. Inverting a non-sigular matrix by simplex method:-

Let A be a non-singular matrix. Let I denote the nxn identity matrix. The matrix A ca be reduced to I by applying a sequence of elementary row operations. The matrix B obtaine from I by applying the same sequence of elementary row operations is the inverse of th martix A. Since each iteration in the simplex method involves an elementary row operation the inverse of A can be computed by applying simplex method.

Since A is non-singular, for any nx1 real matrix b the system of simultaneous linea equations $Ax = b$ ....(1)

has a unique solution for x namely $x = A^{-1}b$. We choose the column vector b suchthat, (1 has a unique solution. $x = (x_1, x_2, \ldots x_n)^T$ with $x_i \geq 0$ for each i.

We introduce artificial variables $a_1, a_2, \ldots a_n$ in the equations (1).

Let $a = (a_1, a_2, \ldots a_n)^T$.

Consider the LPP

Maximize

$$z = Ox_1+Ox_2+\ldots+Ox_n-a_1-a_2-\ldots-a_n$$

subject to

$$Ax+a=b, \quad x, a \geq 0.$$

In the initial simplex table for this LPP the columns corresponding to the variables a_1, $a_2, \ldots a_n$ form the identity matrix and the columns corresponding to the variables x_1, x_2 $\ldots x_n$ form the matrix A. The final simplex table gives the unique solution x of the system of equations (1) and the column corresponding to the basic variables form the identity matrix. Hence the column vectors in the final simplex table corresponding to the initial basic variables $a_1, a_2, \ldots a_n$ costitute the inverse of the matrix A

Solved Problems :-

1) Use simplex method to find the inverse of the matrix $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$

Solution :-

since $|A| = (4-3) = 1 \neq 0$

$\therefore$ A is non-singular matrix

Consider the system of equations

$2x_1+x_2=3$ and .... (1)

$3x_1+2x_2=5$ .... (2)

The unique solution to the system of equations is given by :-

(1) X 2 - (2) $4x_1 + \cancel{2x_2} = 6$;

$-3x_1 + \cancel{2x_2} = 5$

$x_1 = 1$

$x_1=1$ applying in equation (1)

$2.1+x_2=3$

$\Leftrightarrow x_2=3-2$

$\Leftrightarrow x_2=1$

consider the LPP.

Minimize

$z = 0x_1+0x_2$

Subject to

$2x_1+x_2=3$

$3x_1+2x_2=5$

$x_1, x_2 \geq 0$

Hence the LPP in standard form is

Maximize

$z = 0x_1+0x_2-a_1-a_2$

Subject to

$2x_1+x_2+a_1=3$

$3x_1+2x_2+a_2=5$

$x_1, x_2, a_1, a_2 \geq 0$.

Then we have the following simplex tables.

Basis		c_j	0	0	-1	-1	Ratio
C_B	B	x_B	x_1	x_2	a_1	a_2	
-1	a_1	3	$\boxed{2}$	1	1	0	$\frac{3}{2}=1.5$ →
-1	a_2	5	3	2	0	1	$\frac{5}{3}=1.6$
Optimality is not attained		$z_j' = -8$	-5	-3	-1	-1	
		z'_j-c_j	-5	-3	0	0	

↑

x_1 enters the basis; a_1 leaves the basis and 2 is the Pivotal element.

First iteration table

Basis		c_j	0	0	-1	-1	Ratio
C_B	B	x_B	x_1	x_2	a_1	a_2	
0	x_1	$\frac{3}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$ x2 =3
-1	a_2	$\frac{1}{2}$	0	$\boxed{\frac{1}{2}}$	$-\frac{3}{2}$	1	$\frac{1}{2}$ x2 = 1 →
Optimality is not attained		$z_j' = -\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{3}{2}$	-1	
		$z_j' - c_j$	0	$-\frac{1}{2}$	$\frac{5}{2}$	0	

↑

x_2 enters the basis; a_2 leaves the basis and $\frac{1}{2}$ is the Pivotal element.

second iteration table

Basis		c_j	0	0	-1	-1	End of simple procedure .
C_B	B	x_B	x_1	x_2	a_1	a_2	
0	x_1	1	1	0	2	-1	
0	x_2	1	0	1	-3	2	
Optimality is not attained		$z_j' = -\frac{1}{2}$	0	0	0	0	
		$z_j' - c_j$	0	0	1	0	

The initial basic variables are a_1 and a_2 in initial simplex table. From the final simplex table the colums corresponding to these vectors give the matrix

$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$. Hence $A^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

Exercise :-

Cases :- Using simplex method find the inverse of the following matrices.

1) $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

2) $\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$

3) $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix}$

4) $\begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix}$

2.5 Degenceacy and cycling in LPP.

In this section, we discuss some problems encountered in applying the simplex method and the method of dealing with the same.

I. **Tie for entering basic variable:-**

Simple method is an iterative procedure, which consists of selecting a new basis set at each iteration by removing one current basis vector and introducing one current non-basis vector in its place. The selection of anon-basis vector that enters the basis is decided by the net evaluation z_j-c_j that is most negative. A situation may arise in which the most negative z_j-c_j is found in more than one column. Hence we have atie in the choice of a variable to enter the new basis.

In order to break this tie, the selection of entering variable can be made arbitrarily. However, if we adopt the following rules for breaking the tie, the number of iterations can be minimized

a) If there is a tie between two decision variables then the selection can be made arbitrarily

b) If there is a tie between a decision variable and a slack / surplus variable then priority is given to the decision variable to enter into the basis.

c) If there is a tie between two slack variables then selection can be made arbitrarily.

II. Tie for leaving basic variable - Degeneracy and cycling.

To select the basis vector that has to leave the basis at any iteration of the simplex procedure, we choose r, for which replacement ratio $\left\{ \frac{X_{Br}}{a_{r_j}} \Big/ a_{r_j} > 0 \right\}$; r = 1,2, ...m is minimum If the minimum ration is same for two or more current basis vectors then there is a tie for the selection of the leaving variable. To resolve this tie, we usually select any of the tied vectors arbitrarily. However, this arbitrary choice leads to one or more of the basic variables to become zero in the next iteration. Hence the new solution is degenerate. In such a situation, there is no assurance that the value of the objective function will improve, since the solution in subsequent iteration may remain degenerate. As a result it may happen that the same sequence of simplex iterations get repeated endlessly without improving the solution. This is know as cycling or circling. The process of cycling occurs very rarely in practical problems. For reaching optimal solution in the simplex algorithm, we need not worry about degeneracy as long as cycling is avioded.

Following is the procedure to avoid cycling (perturbation rule)

(i) Indentify rule
(ii) Write the unit matrix first and then the body matrix.
(iii) Divide each element in the tied rows by the colfficient of the key column in each of th rows.
(iv) Compare the resulting rows, column by column, first in the unit matrix and then in th body matrix from left to right.
(v) The row which first contains the smallest algebraic ratio determines the leaving variable

Example :- Use simplex method to solve the LPP.

maximize $z = 5x_1+3x_2$ subject to $x_1+x_2 \leq 2$, $5x_1+2x_2 \leq 10$

$-2x_1-8x_2 \geq -12$, $x_1, x_2 \geq 0$.

Solution :-

The standard form of the given LPP is :

Maximize $z=5x_1+3x_2+OS_1+OS_2+OS_3$

Subject to $x_1+x_2+S_1 = 2$

$5x_1+2x_2+S_2=10$

$2x_1+8x_2+S_3=12$

$x_1, x_2, S_1, S_2, S_3 \geq 0$ where s_1, s_2, s_3 are slack variables. The initial basic feasible solution is $(s_1,s_2,s_3) = (2, 10, 12)$.

Initial simplex table

Basis		cost c_j	5	3	0	0	0	Ratio
C_B	B	x_B	x_1	x_2	s_1	s_2	s_3	
0	s_1	2	1	1	1	0	1	$\frac{2}{1}=2$
0	s_2	10	$\boxed{5}$	2	0	1	0	$\frac{10}{5}=2 \rightarrow$
0	s_3	12	2	8	0	0	0	$\frac{12}{2}=6$
Optimality is not attained		$z_j=0$	0	0	0	0	0	Go to next iteration
		z_j-c_j	-5	-3	0	0	0	

↑

From the last column of the initial simplex table we observe that there is atie between the leaving variables S_1 and S_2 we break the tie arbitrarily by choosing S_2 as the leaving variable in the basis. The Pivotal element is 5.

First iteration table

Basis		cost c_j	5	3	0	0	0	Ratio
C_B	B	x_B	x_1	x_2	s_1	s_2	s_3	
0	s_1	0	0	$\boxed{\frac{3}{5}}$	1	$-\frac{1}{5}$	0	$\frac{8}{5}=1.6$
5	x_1	2	1	$\frac{2}{5}$	0	$\frac{1}{5}$	0	$\frac{10}{5}=2 \rightarrow$
0	s_3	8	0	$\frac{36}{5}$	0	$-\frac{2}{5}$	1	$\frac{26}{5}=5.2$
Optimality is not attained		$z_j=10$	0	-1	0	1	0	Go to next iteration
		z_j-c_j	0	-1	0	0	0	

↑

There is atie between the leaving variables S_1 and x_1 the tie is arbitrarily broken by choosing S_1 as the leaving variables.

Second iteration table

Basis		cost c_j	5	3	0	0	0	
C_B	B	x_B	x_1	x_2	s_1	s_2	s_3	End of simple procedure.
3	x_2	0	0	1	$\frac{5}{3}$	$-\frac{1}{3}$	0	
5	x_1	2	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	
0	s_3	8	0	0	12	2	1	
Optimality is not attained		$z_j=10$	5	3	$\frac{5}{3}$	$\frac{2}{3}$	0	
		z_j-c_j	0	0	$\frac{5}{3}$	$\frac{2}{5}$	0	

Here the optimal solution is $x_1=2$; $x_2=0$ & mamimum z=10

Note :- If our arbitrary choic in the simplex table in breaking the tie were 1 instead of S_2 we would have got the following simplex tables.

Initial simplex table

Basis		cost c_j	5	3	0	0	0	Ratio
C_B	B	x_B	x_1	x_2	s_1	s_2	s_3	
0	s_1	2	$\boxed{1}$	1	1	0	0	$\frac{2}{1}=2$ →
0	s_2	10	5	2	0	1	0	$\frac{10}{5}=2$
0	s_3	12	2	8	0	0	1	$\frac{12}{2}=6$
Optimality is not attained		$z_j=0$	0	0	0	0	0	Go to next iteration
		z_j-c_j	-5	-3	0	0	0	

↑

First iteration table

Basis		cost c_j	5	3	0	0	0	
C_B	B	x_B	x_1	x_2	s_1	s_2	s_3	End of simple procedure
5	x_1	2	1	1	1	0	0	
0	s_2	0	0	-3	-5	1	0	
0	s_3	8	0	6	-2	0	1	
Optimality is not attained		$z_j=10$	5	5	5	0	0	
		z_j-c_j	0	2	5	0	0	

↑

Here also the optimal solution is

$x_1 = 2$; $x_2 = 0$ and maximum $z = 10$.

UNIT - III

3.1 concept of Duality :-

Duality in Linear Programming problem:-

The concept of "dulaity" plays an important role in linear programming.

Formation of dual LPP :-

Definition :-

Consider the following Linear Programming problem in canonical form

Maximize

$z = c_1x_1+c_2x_2+ \dots\dots +c_nx_n$

Subject to

$a_{11}x_1+a_{12}x_2+\dots \dots\dots +a_{1n}x_n \leq b_1$

$a_{21}x_1+a_{22}x_2+\dots \dots\dots +a_{2n}x_n \leq b_2$

. . . .

. . . .

. . . .

$a_{m1}x_1+a_{m2}x_2+\dots\dots\dots\dots +a_{mn}x_n \leq b_m$

$x_1, x_2, \dots\dots x_n \geq 0.$

This is called a primal problem and the variables $x_1, x_2 \dots\dots x_n$ are called primal variables and the constraints are called primal constraints. The dual of the given primal problem is defined to be

Minimize

$W=b_1w_1+b_2w_2+\dots\dots\dots +b_mw_m$

subject to

$a_{11}w_1+a_{21}w_2+\dots \dots\dots +a_{m1}w_m \geq c_1$

$a_{12}w_1+a_{22}w_2+\dots\dots\dots\dots +a_{m2}w_m \geq c_2$

. . . .

. . . .

. . . .

$a_{1n}w_1+a_{2n}w_2+\dots\dots\dots\dots +a_{mn}w_m \geq c_n$

$w_1, w_2, \dots\dots\dots\dots w_m \geq 0.$

The variables $w_1, w_2, \dots\dots w_m$ are called dual variables and the constraints are called dua constraints.

Matrix form of primal and its dual:-

The primal problem in matrix form is given by

maximize

$f(x) = z= cx$

subject to

$Ax \leq b$

$x \geq 0$.

The corresponding dual problem in matrix form is given by

Minimize

$f(w) = w = b^T w$

Subject to $A^T w \geq CT$

$w \geq 0$.

Note :-

The relationship between the primal and its dual problem can be represented in a single table as shown below:-

$(x, \ x, \ \ldots\ldots\ldots x)$

$$\begin{pmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ w_m \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \ldots\ldots a_{1n} \\ a_{21} & a_{22} & \ldots\ldots a_{2n} \\ \ldots & \ldots\ldots & \ldots\ldots \\ \ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & \ldots\ldots\ldots a_{mn} \end{pmatrix} \leq \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{pmatrix}$$

$$\geq$$

$$(c_1, c_2 \ldots\ldots\ldots\ldots c_n)$$

The constraints of primal problem can be obtained by reading the table horizontally and the constraints of the dual problem canbe obtained by reading the table vertically.

Note 2:-

To find the dual of a given LPP, it must be first expressed in canonical form.

Remark 1:-

If the primal problem is of maximizing type then the dual problem is of minimizing type and vice versa

Remark 2:-

If the the primal problem has n variables and n constraints then the dual problem has am variables and n costraints.

Remark 3:-

The colfficients of the objective functions of the primal problem become the right hand side of the constraints of the dual problem and vice versa.

Remark 4:-

Each constraint in the primal problem gives rise to a dual variable. Hence the number of constraints in the primal problem is equal to the number of dual variables in the dual problem.

Remarks:- 5

All the constraints in the primal problem are $\leq$ type and all the constraints in the dual problem are $\geq$ type.

Thm :- Dual of the dual is the Primal.

Proof :- Let the primal LPP are

maximize

$$f(x) = z = cx$$

subject to

$$Ax \leq b$$

$$x \geq 0.$$

definition its dual is given by

minimize

$f(w) = w = b^Tw$

subject to

$A^Tw \geq c^T$

$w \geq 0.$

The canonical form of the dual is given by

Maximize

$f(w) = -b^Tw$

subject to

$-A^Tw \leq -c^T$

$W \geq 0.$

Hence the dual of the dual is given by

Minimize

$h(y) = (-c^T)^Ty$

subject to

$(-A^T)^Ty \geq (-b^T)^T$

$$y \geq 0.$$

(i.e)

Maximize $h(y) = -(-cy)$

Subject to

$$-Ay \geq -b$$

$$y \geq 0.$$

(i.e) Maximize $h(y) = cy$

subject to

$Ay \leq b$

$y \geq 0.$

This LPP that is the dual of the dual problem is same as the primal problem we started with.

Hence the result

Example :-

consider the LPP.

Maximize

$$z = 2x_1+3x_2+10x_3$$

Subject to

$$x_1-2x_2+3x_3 \leq 5$$

$$x_1+3x_2-5x_3 \leq 18$$

$$x_1, x_2, x_3 \geq 0.$$

Primal is an canonical form.

The dual of the given LPP is.

Minimze

$$w = 5w_1+18w_2$$

Subject to

$$w_1+w_2 \geq 2$$

$$-2w_1+3w_2 \geq 3$$

$$3w_1-5w_2 \geq 10$$

$$w_1, w_2 \geq 0.$$

Remarks :-

Suppose a primal LPP involves an equality constraint, say,

$$a_{11}x_1+a_{12}x_2+.............+a_{1n}x_n = b_1$$

In canonical form this constraint is replaced by the two constraints.

$$a_{11}x_1+a_{12}x_2+.............+a_{1n}x_n \leq b_1$$

$$-(a_{11}x_1+a_{12}x_2+...........a_{1n}x_n) \leq -b_1$$

If w_1, w_2 are the dual variables corresponding to the above two constraints then the corresponding terms in the objective function are given by $b_1w_1-b_1w_2 = b_1(w_1-w_2)$

Here w_1-w_2 can be taken as the single dual variable say $w=w_1-w_2$ corresponding to given equality constraint.

Since $w_1, w_2 \geq 0$, w' is an unrestricted variable.

III^{ly},

if any variable of the primal problem is unrestricted in sign then the corresponding constraint in the dual problem will be of equality type.

Maximize

$z = 3x_1+10x_2+2x_3$

Subject to

$2x_1+3x_2+2x_3 \leq 7$ ....(1)

$3x_1-2x_2+4x_3 = 3$ .... (2)

$x_1, x_2, x_3 \geq 0$

2) can be written as

$3x_1-2x_2+4x_3 \leq 3$ ....(3)

$-3x_1+2x_2-4x_3 \leq -3$ ....(4)

Now the dual of the primal is

Minimize

$f(w) = w = 7w_1+3w_2-3w_3$

subject to

$2w_1+3w_2-3w_3 \geq 3$

$3w_1-2w_2+2w_3 \geq 10$

$2w_1+4w_2-4w_3 \geq 2$

$w_1, w_2, w_3 \geq 0.$

Minimize

$w = 7w_1+3(w_2-w_3)$

Subject to

$2w_1+3(w_2-w_3) \geq 3$

$3w_1-2(w_2-w_3) \geq 10$

$2w_1+4(w_2-w_3) \geq 2$

$w_1, w_2, w_3 \geq 0.$

Taking $w_2-w_3 = w_4$.

The dual is written as

ninimize

$w = 7w_1+3w_4$

ubject to

$2w_1+3w_4 \geq 3$

$3w_1-2w_4 \geq 10$

$2w_1+4w_4 \geq 2$

lved problems:-

Form the dual of the following primal LPP.

ximize

$z = 4x_1+10x_2+25x_3$

subject to $2x_1+4x_2+8x_3 \leq 25$

$4x_1+9x_2+8x_3 \leq 30$

$x_1, x_2, x_3 \geq 0.$

Solution :-

Given primal problem is

Maximize $z = cx$

subject to $Ax \leq b$, $x \geq 0$.

Where

$$A=\begin{pmatrix} 2 & 4 & 8 \\ 4 & 9 & 8 \end{pmatrix} \;;\; b=\begin{pmatrix} 25 \\ 30 \end{pmatrix} \;;\; X=\begin{pmatrix} x \\ x \\ x \end{pmatrix}$$

$c = (4, 10, 25)$

Since the primal problem has 2 constraints and 3 decision variables the dual problem has 3 constraints and 2 decision variables say w_1, w_2. Also the objective function of the primal problem is of maximizing type.

Hence the dual is of minimizing type.

Hence the dual of the given LPP is.

Minimize $f(w) = w = b^T w$

subject to

$A^T w \geq c^T \;;\; w \geq 0$

(i.e.) Minimize

$$w = (25 \;\; 30)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 25w_1+30w_2$$

subject to

$$\begin{pmatrix} 2 & 4 \\ 4 & 9 \\ 8 & 8 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \geq \begin{pmatrix} 4 \\ 10 \\ 25 \end{pmatrix} \quad \text{and } w_1, w_2 \geq 0.$$

Hence the dual problem is

Minimize

$f(w) = w= 25w_1+30w_2$

subject to

$2w_1+4w_2 \geq 4;$

$4w_1+9w_2 \geq 10;$

$8w_1+8w_2 \geq 25;$

$w_1, w_2 \geq 0.$

Problem 2:-

Write the dual of the LPP.

Minimize

$z = 4x_1+6x_2+18x_3$

subject to

$x_1+3x_2 \geq 3$

$x_1+2x_3 \geq 5$

$x_1, x_2, x_3 \geq 0.$

Solution :-

Given primal problem is

Minimize

$z = cx$

subject to

$Ax \geq b,\ x \geq 0.$

where

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix} ;\ b = \begin{pmatrix} 3 \\ 5 \end{pmatrix} ;\ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$c = (4, 6, 18)$

Since the primal problem has 2 constraints and 3 decision variables the dual problem has 3 constraints and 2 decision variables say w_1, w_2.

Also the objective function of the primal problem is of minimizing type. Hence in the dual problem, the objective function is of maximizing type.

Hence the dual of the given LPP is :

Maximize

$w = b^T w$

subject to

$A^T w \leq c^T ;\ w \geq 0$

(i.e.) Maximize

$$f(w) = w = (3\ \ 5) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 3w_1+5w_2$$

subject to

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \leq \begin{pmatrix} 4 \\ 6 \\ 18 \end{pmatrix} \quad \text{and } w \geq 0.$$

Hence the dual problem is :

Maximize

$w = 3w_1+5w_2$

subject to

$w_1 \leq 4$

$3w_1+w_2 \leq 6$

$2w_2 \leq 18$

$w_1, w_2 \geq 0.$

Problem 3:-

write the dual of the LPP.

Maximize

$z = x_1+2x_2+3x_3$

subject to

$4x_1+5x_2+4x_3 \leq 9$

$6x_1-x_2+5x_3 = 10$

$x_1, x_2, x_3 \geq 0.$

solution :-

Since the primal problem has 2 constraints and 3 decision variables the dual problem has 3 constraints and 2 decision variables say w_1, w_2.

Since the second constraint in the primal is equality the corresponding second dual variable w_2 is unrestricted in sign. Also the objective function of the primal is of maximizing type and hence in the dual problem it is of minimizing type.

Hence the dual of the given LPP is :

Minimize

$w = 9w_1+10w_2$

subject to

$4w_1+6w_2 \geq 1$

$5w_1-w_2 \geq 2$

$4w_1+5w_2 \geq 3$; $w_1 \geq 0$; w_2 is unrestricted.

Problem 4 :-

Write the dual of the LPP :

Maximize

$z = 10x_1+11x_2$

subject to

$-x_1+x_2 \leq -5$

$x_1-2x_2 \geq -7$

$x_1, x_2 \geq 0.$

solution :-

The given primal problem can be written in the canonical form

Maximize

$z = 10x_1 + 11x_2$

subject to

$-x_1 + x_2 \leq -5$

$-x_1 + 2x_2 \leq 7$

$x_1, x_2 \geq 0.$

Since the primal problem has 2 constraints and 2 decision variables the dual problem has 2 constraints and 2 decision variables say w_1, w_2.

Also the objective function of the primal is of maximizing type and hence the dual is of minimizing type.

Hence the dual of the given LPP is

Minimize

$w = -5w_1 + 7w_2$

subject to

$-w_1 - w_2 \geq 10$

$+w_1 + 2w_2 \geq 11$

$w_1 \geq 0;\ w_2 \geq 0$

Problem 5:-

write the dual of the LPP.

Minimize $z = x_1 + 3x_2 - 4x_3$

subject to

$2x_1 + x_2 - 2x_3 \leq 14$

$2x_1 - 4x_2 \geq 13$

$-4x_1 + x_2 - 7x_3 = 10$ $\quad x_1, x_2 \geq 0$; x_3 is unrestricted

Solution :-

Since x_3 is an unrestricted variable it can be expressed as $x_3 = x'_3 - x''_3$ where $x'_3, x''_3 \geq 0$.

The given primal LPP now becomes

Minimize

$z = x_1 + 3x_2 - 4(x'_3 - x''_3)$

subject to

$-2x_1 - x_2 + 2(x'_3 - x''_3) \geq -14$ .... (1)

$2x_1 - 4x_2 \geq 13$ .... (2)

$-4x_1 + x_2 - 7(x'_3 - x''_3) = 10$ .... (3)

Taking

$x'_3 - x''_3 = x_4$

The given Primal becomes

Minimize

$z = x_1 + 3x_2 - 4x_4$

subject to

$-2x_1 - x_2 + 2x_4 \geq -14$ .... (1)

$2x_1 - 4x_2 \geq 13$ .... (2)

$-4x_1 + x_2 - 7x_4 = 10$ .... (3)

Since the primal problem has 3 constraints and 3 primal decision variables the dual problem has also 3 constraints and 3 decision variables say w_1, w_2 and w_3.

Since the third constraint in the primal is of equality type the corresponding third dual variable w_3 is unrestricted in sign. Also the objective function of the primal is of minimizing type.

Hence the dual problem it is of minimizing type. Hence the dual of the given LPP is:-

Maximize

$w = -14w_1 + 13w_2 + 10w_3$

subject to $-2w_1 + 2w_2 - 4w_3 \leq 1$

$-w_1 - 4w_2 + w_3 \leq 3$

$2w_1 - 7w_3 = -4.$

$w_1, w_2 \geq 0$ and w_3 is unrestricted.

Exercise :-

write the dual of the following LPPs.

1) Maximize $z = 4x_1 + 2x_2$

subject to $-x_1 - x_2 \leq -3$; $-x_1 + x_2 \geq -2$;

$x_1, x_2 \geq 0.$

2) Maximize $z = x_1 + 2x_2 + x_3$

subject to $2x_1 + x_2 - x_3 \leq 2$; $-2x_1 + x_2 - 5x_3 \geq -6$

$4x_1 + x_2 + x_3 \leq 6$; $x_1, x_2, x_3 \geq 0.$

3) Minimize $z = 4x_1 + 6x_2 + 18x_3$

subject to $x_1 + 3x_2 \geq 2$; $x_2 + 2x_3 \geq -5$;

$x_1, x_2, x_3 \geq 0.$

4) Minimize $z = x_1 + x_2$

subject to $2x_1 + x_2 \geq 2$; $-x_1 - x_2 \geq 1$

$x_1, x_2 \geq 0$

5) **Maximize $z = 4x_1+10x_2+25x_3$**

subject to

$2x_1+4x_2+8x_3 = 25$

$4x_1+9x_2+8x_3 = 30$

$6x_1+8x_2+2x_3 = 40$

$x_1, x_2, x_3 \geq 0.$

6) **Maximize**

$z = 2x_1+3x_2$

subject to

$5x_1+2x_2 \leq 40$

$6x_1+12x_2 \leq 80$

x_1, x_2 unrestricted

II. Verify the statement "dual of a dual is primal" with the following LPP :-

7) Maximize $z = 40x_1+50x_2$ subject to

$2x_1+3x_2 \leq 3$; $4x_1+2x_2 \leq 15$; $x_1, x_2 \geq 0.$

8) Minimize $z = 3x_1+4x_2$

subject to $x_1-x_2 \geq 1$; $x_1+x_2 \geq 4$; $x_1-3x_2 \leq 3$

$x_1, x_2 \geq 0.$

3.2 Primal and dual :-

We now proceed to show that the optimal solution to a dual problem can be obtained directly from the final simplex table of the primal problem.

Lemma 1:-

Let x_0 be a feasible solution to the primal problem.

maximize

$f(x) = cx$ subject to $Ax \leq b,\ x \geq 0$

where $c \in R^n$, $x^T \in R^n$, A is mxn matrix, $b^T \in R^m$. Let w_0 be a feasible solution to the dual of the above primal problem namely.

Minimize

$f(w) = b^T w$ subject to $A^T w \geq c^T$, $w \geq 0$ then $cx_0 \leq b^T w_0$.

proof :-

Since x_0 is a feasible solution to the primal problem, we have

$Ax_0 \leq b,\ x_0 \geq 0$ .... (1)

Since w_0 is the feasible solution to the dual problem, we have

$A^T w_0 \geq c^T,\ w_0 \geq 0$ .... (2)

Now, taking transpose on both sides in (2), we

$c \leq w_0^T A$

$\therefore cx_0 \leq w_0^T (Ax_0)$

$\therefore cx_0 \leq w_0^Tb$ [using (1)]

$\Leftrightarrow (cx_0)^T \leq (w_0^Tb)^T = b^T w_0$

$\therefore cx_0 \leq b^Tw_0$

[Since cx_0 is a real number $(cx_0)^T = cx_0$].

Hence the lemma.

Lemma 2 :-

Let x_0 be a feasible solution to the primal problem.

Maximize $f(x) = cx$ subject to $Ax \leq b$, $x \geq 0$ where $c \in R^n$, $x^T \in R^n$, A is mxn matrix, $b^T \in R^m$. Let w_0 be a feasible solution to the dual of the above primal problem. If $cx_0 = b^Tw_0$,

Then (i) x_0 is an optimal solution to the primal.

(ii) w_0 is an optimal solution to the dual.

Proof :- Let x_0 be any feasible solution to the primal problem.

$\therefore$ By Lemma 1,

$cx^*_0 \leq b^Tw_0$

$cx^*_0 \leq cx_0$ [$\because$ $cx_0 = b^Tw_0$]

$\therefore x_0$ is an optimal solution to its primal.

Now,

Let w_0^* be any feasible solution to the dual problem.

Then by Lemma 1,

$cx_0 \leq b^Tw_0^*$

(i.e.) $b^T w_0 \leq b^T w_0^*$ [$\because$ $cx_0 = b^Tw_0$]

$\therefore w_0$ is an optimal solution to the dual problem.

Remarks :-

Consider a primal problem given by maximize $f(x) = cx$ subject to $Ax \leq b$, $x \geq 0$, where $c \in R^n$, $x^T \in R^n$, A is mxn matrix, $b^T \in R^m$.

Its corresponding dual is given by minimize $f(w) = b^Tw$ subject to $A^Tw \geq c^T$, $w \geq 0$ where $c \in R^n$, A is mxn matrix, $b^T \in R^m$.

Then $cx_0 \leq b^T w_0$, Let x_0 be an optimal solution to the primal problem. Then by Lemma 2, There exists an optimal solution w_0 to the dual problem suchthat $cx_0 = b^Tw_0$.

Hence it follows that the primal problem has optimum solution if and only if its dual also has an optimum solution.

This is summarized in the following theorem known as fundamental theorem of duality, which we state without proof.

Fundamental theorem of Duality:-

Let the primal problem be given by Maximize $f(x) = cx$ subject to $Ax \leq b$, $x \geq 0$ where $c \in R^n$, $x^T \in R^n$, A is mxn matrix, $b^T \in R^m$. Then its corresponding dual is given by minimize

$f(w) = b^Tw$ subject to $A^Tw \geq c^T$, $w \geq 0$ where $c \in R^n$, A is mxn matrix, $b^T \in R^m$. Let x_0 be a feasible solution to the primal problem.

(i) Then x_0 is an optimal solution if and only if There exists a feasible solution w_0 to the dual problem such that $cx_0 = b^T w_0$.

(ii) If w_0 is a feasible solution to the dual problem then w_0 is an optimal solution if and only if there exists a feasible solution x_0 to the primal problem suchthat $cx_0=b^Tw_0$.

In bothx cases x_0 is an optimal solution to the primal and w_0 is an optimal solution to its dual.

Remark 1:-

The following are the results relating to the existence of solution of the primal and its dual.

1) An LPP has an optimal solution, if and only if there exists a feasible solution to the primal and its dual.

2) If there doesn't exist any feasible solution to the dual problem and there exists at least one feasible solution to the primal problem, then the primal has no optimal solution.

3) If there doesn't exist any feasible solution to the primal problem and there exists at least one feasible solution to the dual problem, then the dual has no optimal solution.

4) If there is no optimal solution to the primal problem then there doesn't exist a feasible solution to the dual problem.

5) If there is no optimal solution to the dual problem then there doesn't exist a feasible solution to the primal problem.

Remark 2 :- Since any LPP can be solved by using simplex method we can solve the primal as well as its dual by applying simplex method.

An optimal solution to the dual of an LPP can be obtained from the final simplex table of the primal and vice versa.

An optimal solution to one of the problems can be obtained from the final simplex table of the other by using the following general rules.

Rule 1:-

If the primal (dual) variable corresponds to a slack/surplus variable in the dual (primal) problem, its optimal value indirectly read off from the net evalutaion row of the optimum dual (primal) simplex table as the net evaluation corresponding to this slack/surplus variable

Rule 2:-

If the primal (dual) variable corresponds to an artificial starting variable in the dual (primal) problem, its optimal value is directly read off from the net evaluation row of the optimum dual (primal) simplex take as the net evaluation corresponding to this artificial variable after deleting the constant M.

Problems :-

1) Solve by simplex method using the dual of the following LPP:-

Minimize

$z = 2x_1+3x_2$

subject to

$x_1+x_2 \geq 5$

$x_1+2x_2 \geq 6$

$x_1, x_2 \geq 0.$

solution :-

The dual of the given primal problem is

maximize

$w = 5w_1 +6w_2$

subject to

$w_1+w_2 \leq 2$

$w_1+2x_2 \leq 3$

$w_1, w_2 \geq 0$

writing in standard form we have

$w = 5w_1+6w_2+OS_1+OS_1$

subject to $w_1+w_2+s_1 = 2$

$w_1+2w_2+s_2=3$

$w_1, w_2, s_1, s_2 \geq 0$.Then we have the following simplex tables.

Initial simplex table

Basis		c_j	5	6	0	0	Ratio
C_B	B	w_B	w_1	w_2	s_1	s_2	
0	s_1	2	1	1	1	0	$\frac{2}{1}$ =2
0	s_2	3	1	$\boxed{2}$	0	1	$\frac{3}{2}$ = 1.5 →
Optimality is not attained		z_j=0	0	0	0	0	Go to next iteration
		z_j-c_j	-5	-6	0	0	

↑

w_2 enters the basis; s_2 leaves the basis and 2 is the Pivotal element.

First iteration table

Basis		c_j	5	6	0	0	Ratio
C_B	B	w_B	w_1	w_2	s_1	s_2	
0	s_1	$\frac{1}{2}$	$\boxed{\frac{1}{2}}$	0	1	$-\frac{1}{2}$	$\frac{1}{2}$ x2 =1 →
6	w_2	$\frac{3}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{3}{2}$ x2= 3
Optimality is not attained		z_j=9	3	6	0	3	Go to next iteration
		z_j-c_j	-2	0	0	3	

↑

w_1, enters the basis; s_1 leaves the basis and $\frac{1}{2}$ is the Pivotal element.

second iteration table

Basis		c_j	5	6	0	0	End of simplex procedure
C_B	B	w_B	w_1	w_2	s_1	s_2	
5	w_1	1	1	0	2	-1	
6	w_2	1	0	1	-1	1	
Optimality is attained		z_j=11	5	6	4	1	
		z_j-c_j	0	0	4	1	

From the final simplex table, the net evaluation corresponding to the columns of the initial basic variables S_1 and S_2 are 4 and 1 respectively and the net evaluation is z= 11.

Hence the optimum solution of the primal problem is x_1 = 4 and x_2 = 1 and minimum of z = 11.

Problem 2 :-

Solve the following primal LPP finding its dual LPP.

Minimize

$z = 20x_1 + 24x_2 + 18x_3$

subject to

$2x_1+x_2+x_3 \geq 30$

$x_1+x_2+x_3 \geq 20$

$x_1+2x_2+x_3 \geq 24$

$x_1, x_2, x_3 \geq 0.$

Solution :-

The dual of the given primal problem is :-

maximize $w = 30w_1+20w_2+24w_3$

subject to $2w_1+w_2+w_3 \leq 20$

$w_1+w_2+2w_3 \leq 24$

$w_1+w_2+w_3 \leq 18, \quad w_1, w_2, w_3 \geq 0.$

Then we have the following simplex tables :

Initial simplex table

Basis		c_j	30	20	24	0	0	0	Ratio
C_B	B	w_B	w_1	w_2	w_3	s_1	s_2	s_3	
0	s_1	20	$\boxed{2}$	1	1	1	0	0	$\frac{20}{2}=10 \rightarrow$
0	s_2	24	1	1	2	0	1	0	$\frac{24}{1}=24$
0	s_3	18	1	1	1	0	0	1	$\frac{18}{1}=18$
Optimality is not attained		$z_j=0$	0	0	0	0	0	0	Go to next iteration
		z_j-c_j	-30	-20	-24	0	0	0	

↑

w_1 enters the basis; s_1 leaves the basis and 2 is the Pivotal element.

First iteration table

Basis		c_j	30	20	24	0	0	0	Ratio
C_B	B	w_B	w_1	w_2	w_3	s_1	s_2	s_3	
30	w_1	10	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$10\times\frac{2}{1}=20$
0	s_2	14	0	$\frac{1}{2}$	$\boxed{\frac{3}{2}}$	$-\frac{1}{2}$	1	0	$\frac{14\times2}{3}=\frac{28}{3}=9.3 \rightarrow$
0	s_3	8	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	1	$8\times\frac{2}{1}=16$
Optimality is not attained		$z_j=300$	30	15	15	15	0	0	Go to next iteration
		z_j-c_j	0	-5	-9	0	0	0	

↑

w_3 enters the basis; s_2 leaves the basis and $\frac{3}{2}$ is the Pivotal element.

seoncd iteration table

Basis		c_j	30	20	24	0	0	0	Ratio
C_B	B	w_B	w_1	w_2	w_3	s_1	s_2	s_3	
30	w_1	$\frac{16}{3}$	1	$\frac{1}{3}$	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{16}{3}$ x3 =16
24	w_3	$\frac{28}{3}$	0	$\frac{1}{3}$	1	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{28}{3}$ x3 = 28
0	s_3	$\frac{10}{3}$	0	$\boxed{\frac{1}{3}}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{10}{3}$ x3 = 10 →
Optimality is not attained		z_j=384	30	18	24	12	6	0	Go to next iteration
		z_j-c_j	0	-2	0	12	6	0	

↑

w_2 enters the basis; s_3 leaves the basis and $\frac{1}{3}$ is the Pivotal element.

Third iteration table

Basis		c_j	30	20	24	0	0	0	End of simplex procedure
C_B	B	w_B	w_1	w_2	w_3	s_1	s_2	s_3	
30	w_1	2	1	0	0	1	0	-1	
24	w_3	6	0	0	1	0	1	-1	
20	w_2	10	0	1	0	-1	-1	3	
Optimality is attained		z_j=404	30	20	24	10	4	6	
		z_j-c_j	0	0	0	10	4	6	

From the final simplex table, the net evaluations corresponding to the columns of the initial basic variables S_1, S_2 and S_3 are 10, 4 & 6 respectively and the evaluation is $z = 404$.

Hence the optimum solution of the primal problems's

$x_1 = 10$, $x_2 = 4$, $x_3 = 6$ and minimum of $z = 404$.

Problem 3 :-

Write down the dual of the following LPP and solve it. Hence write down the optimum solution of the primal.

Maximal $Z = 4x_1+2x_2$

subject to $x_1+x_2 \geq 3$

$x_1-x_2 \geq +2$ $\qquad$ $x_1, x_2 \geq 0.$

Solution :

The canonical form of the given primal problem is

Maximize $w = 4x_1+2x_2$

subject to $-x_1-x_2 \leq -3$

$-x_1+x_2 \leq -2$

$x_1, x_2 \geq 0.$

∴ The dual of the given primal problem is

Minimize

$w = -3w_1-2w_2$

subject to

$-w_1-w_2 \geq 4$

$-w_1+w_2 \geq 2$

$w_1, w_2 \geq 0.$

writing the dual in standard form we have

Maximize $w = 3w_1+2w_2+OS_1+OS_2-Ma_1-Ma_2$

subject to $-w_1-w_2-s_1+a_1 = 4$

$-w_1+w_2-S_2+a_2 = 2$

$w_1, w_2, w_3, s_1, s_2, a_1, a_2 \geq 0.$

Then we have the following simplex tables.

Initial simplex table

Basis		c_j	3	2	0	0	-M	-M	Ratio
C_B	B	w_B	w_1	w_2	s_1	s_2	a_1	a_2	
-M	a_1	4	-1	-1	-1	0	1	0	-
-M	a_2	2	-1	$\boxed{1}$	0	-1	0	1	$\frac{2}{1}$ =2 →
Optimality is not attained		z_j=-6M	+2M	0	M	M	-M	-M	
		z_j-c_j	2M-3	-2 ↑	M	M	0	0	

w_2 enters the basis; a_2 leaves the basis and 1 is the Pivotal element.

First iteration table

Basis		c_j	3	2	0	0	-M	End of simplex procedure
C_B	B	w_B	w_1	w_2	s_1	s_2	a_1	
-M	a_1	6	-2	0	-1	-1	1	
2	w_2	2	-1	1	0	-1	0	
Optimality is attained		z_j=-6M+4	-2M-2	2	M	M-2	-M	
		z_j-c_j	2M-5	0	M	M-2	0	

Here all $z_j-c_j \geq 0$ and hence optimality is attained But an artificial variable a_1 is present in the basis at non-zero level Thus the dual problem does not posses any optimum basic feasible solution Hence there exists no finite optimum solution to the given LPP. (i.e.) The solution of the given LPP is unbounded.

Problem 4.-

Write down the dual of the following LPP and solve it Hence write down the optimum solution of the primal if exists

Maximize $z= 3x_1+2x_2$

subject to

$x_1-x_2 \leq 1$

$x_1+x_2 \geq 3$

$x_1, x_2 \geq 0.$

Solution:-

The dual of the given primal problem is

Minimize $w = w_1-3w_2$

Subject to $w_1-w_2 \geq 3$

$-w_1-w_2 \geq 2$; $w_1, w_2 \geq 0$

Initial simplex table

Basis		c_j	-1	3	0	0	-M	-M	End of simplex procedure
C_B	B	w_B	w_1	w_2	s_1	s_2	a_1	a_2	
-M	a_1	3	1	-1	-1	0	1	0	
-M	a_2	2	-1	-1	0	-1	0	1	
Optimality is attained		z_j=-5M	0	2M	M	M	-M	-M	
		z_j-c_j	1	2M-3	M	M	0	0	

Here all $z_j - c_j \geq 0$ and hence optimality is attained. But artificial variables are present in the basis at non-zero level. Thus the dual problem does not posses any optimum basic feasible solution. Hence there exists no finite optimum solution to the given LPP (i.e.) The solution of the given LPP is unbounded.

Problem 5:-

Find the optimal solution for the following LPP and its dual by simplex method.

Maximize $z = 25x_1 + 20x_2$

Subject to $16x_1 + 12x_2 \leq 100$

$8x_1 + 16x_2 \leq 80$

$x_1, x_2 \geq 0$.

Solution :-

Initial simplex table

Basis		c_j	25	20	0	0	Ratio
C_B	B	x_B	x_1	x_2	s_1	s_2	
0	s_1	100	$\boxed{16}$	12	1	0	$\frac{100}{16} = 6.25 \rightarrow$
0	s_2	80	8	16	0	1	$\frac{80}{8} = 10$
Optimality is not attained		$z_j = 0$	0	0	0	0	Go next iteration
		$z_j - c_j$	-25	-20	0	0	

↑

s_1 leaves the basis; x_1 enters the basis and 16 is the Pivotal element

First iteration table

Basis		c_j	25	20	0	0	Ratio
C_B	B	x_B	x_1	x_2	s_1	s_2	
25	x_1	$\frac{25}{4}$	1	$\frac{3}{4}$	$\frac{1}{16}$	0	$\frac{25}{4} \times \frac{3}{4} = \frac{25}{3}$
0	s_2	30	0	$\boxed{10}$	$-\frac{1}{2}$	1	$\frac{30}{10} = 3 \rightarrow$
Optimality is not attained		$z_j = \frac{625}{4}$	25	$\frac{75}{4}$	$\frac{25}{16}$	0	Go To next iteration
		$z_j - c_j$	0	$-\frac{5}{4}$	$\frac{25}{16}$	0	

↑

s_2 leaves the basis; x_2 enters the basis and 10 is the Pivotal element.

Second iteration table

Basis		c_j	25	20	0	0	End of simplex procedure
C_B	B	x_B	x_1	x_2	s_1	s_2	
25	x_1	4	1	0	$\frac{1}{10}$	$-\frac{3}{40}$	
20	x_2	3	0	1	$-\frac{1}{20}$	$\frac{1}{10}$	
Optimality is attained		$z_j=$	25	20	$\frac{3}{2}$	$\frac{1}{8}$	
		z_j-c_j	0	0	$\frac{3}{2}$	$\frac{1}{8}$	

In the second iteration table all $z_j-c_j \geq 0$. Hence optimality is reached, and the optimal solution is $x_1=4$; $x_2=3$ and the value of the objective function is $z = 160$.

Now the dual of the given primal is :-

Minimize $w = 100w_1+80w_2$

subject to

$16w_1+8w_2 \geq 25$

$12w_1+16w_2 \geq 20$

$w_1, w_2 \geq 0.$

The standard form of the dual LPP is .

Maximize

$w = -100w_1-80w_2+OS_1+OS_2-Ma_1-Ma_2$

subject to

$16w_1+8w_2-s_1+a_1 = 25$

$12w_1+16w_2-s_2+a_2 = 20$

$w_1, w_2, s_1, s_2, a_1, a_2 \geq 0.$

where s_1, s_2 are surplus variables and a_1, a_2 are artificial variables

we have the following simplex tables.

Initial simplex table

Basis		c_j	-100	-80	0	0	-M	-M	Ratio
C_B	B	w_B	w_1	w_2	s_1	s_2	a_1	a_2	
-M	a_1	25	$\boxed{16}$	8	-1	0	1	0	$\frac{25}{16}=1.5625$ →
-M	a_2	20	12	16	0	-1	0	1	$\frac{20}{12}=1.666$
Optimality is not attained		$z_j=-45M$	-28M	-24M	M	M	-M	-M	Go next iteration
		z_j-c_j	-28M+100	-24M+80	M	M	0	0	

↑

w_1 enters the basis; a_1 leaves the basis and 16 is the Pivotal element.

Basis		c_j	-100	-80	0	0	-M	-M	Ratio
C_B	B	w_B	w_1	w_2	s_1	s_2	a_1	a_2	
-100	w_1	$\frac{25}{16}$	1	$\frac{1}{2}$	$\frac{1}{16}$	0	$\frac{1}{16}$	0	
-M	a_2	$\frac{5}{4}$	0	$\boxed{10}$	$\frac{3}{4}$	-1	$-\frac{3}{4}$	1	→
Optimality is not attained		$z_j=\frac{-5M}{4}+\frac{2500}{16}$	-100	-10M-50	$\frac{-3M}{4}+\frac{100}{16}$	M	$\frac{-100}{16}+\frac{3M}{4}$	-M	Go To next iteration
		z_j-c_j	0	-10M+30	$\frac{-3M}{4}+\frac{100}{16}$	M	$\frac{7M}{4}-\frac{100}{16}$	0	

↑

w_2 enters the basis; a_2 leaves the basis and 10 is the Pivotal element.

Basis		c_j	-100	-80	0	0	-M	-M	
C_B	B	w_B	w_1	w_2	s_1	s_2	a_1	a_2	End of simplex procedure
-100	w_1	$\frac{3}{2}$	1	0	$-\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{10}$	$-\frac{1}{20}$	
-80	w_2	$\frac{1}{8}$	0	1	$\frac{3}{40}$	$-\frac{1}{10}$	$-\frac{3}{40}$	$\frac{1}{10}$	
Optimality is attained		$z_j=-160$	-100	-80	4	3	-4	3	
		z_j-c_j	0	0	4	3	-4+M	-3-M	

↑

In the final simplex table (second iteration table) for the primal problem, the starting basic variables are the slack variables S_1 and S_2 and the net evaluations corresponding to these columns are $\frac{3}{2}$ and $\frac{1}{8}$ respectively

The optimal solutions are $x_1=4$ and $x_2=3$ and optimal value (evalution) is 160 (Refer table) :-

Primal Problem :- Primal basic variables : S_1 S_2

optimum net evaluation : $\frac{3}{2}$ $\frac{1}{8}$

Dual variables : w_1 w_2

Thus an optimum solution to the primal gives directly an optimum solution to the dual Actually the optimum solution of the dual is the absoluate value of the net evaluations of the starting primal basic variables.

conversely, an optimum solution to the dual gives directly an optimum solution to the primal. From the final simplex table (second iteration table) for the dual problem we have :-

Dual problem :- Dual basic variables : a_1 a_2

Optimum net evaluation : -4 -3

Absolute value of net evaluation deleting M :- 4 3

Primal variables . x_1 x_2

Hence from the final dual simplex table we can read the optimal solution of the primal as $x_1 = 4$ and $x_2 = 3$.

Max Z = 160 = -Min (-w)

Maz = 160 = -Min (-w)

Exercise :

I) Use simplex method to solve the following LPPs by using its dual:-

1) Minimize $z = 2x_1+3x_2$

subject to $x_1+x_2 \geq 5$; $x_1+2x_2 \geq 6$; $x_1, x_2 \geq 0$

2) Minimize $z = 60x_1+80x_2$

subject to $2x_1+3x_2 \geq 90$; $4x_1+3x_2 \geq 120$; $x_1 , x_2 \geq 0$.

II) Write down the dual of the following LPPs and solve them. Hence write down the optimal solution of the primal problem :-

1) Maximize

$z = 3x_1+2x_2$

subject to

$2x_1+x_2 \leq 5$; $x_1+x_2 \leq 3$; $x_1, x_2 \geq 0$

2) Maximize

$z = 5x_1-2x_2+3x_3$

subject to

$2x_1+2x_2-x_3 \geq 2$; $3x_1-4x_2 \leq 3$; $x_1+3x_2 \leq 3$; $x_1, x_2, x_3 \geq 0$.

III) Prove using duality that the following LPP is feasible but has no optimum solution:-

1) Minimize $z = x_1-x_2+x_3$

subject to $x_1-x_3 \geq 4$; $x_1-x_2+2x_3 \geq 3$ $\quad x_1, x_2, x_3 \geq 0$.

3.3 Dual simplex method :-

The primal problem starts with a feasible but non-optimal solution and continues to be feasible until optimum is reached. But the dual problem starts with an infeasible but better than optimal solution and continues to be infeasible until the optimal solution is reached Thus the primal problem seeks optimality and dual problem seeks feasibility. This is the major difference between primal and dual problems.

This observation leads the possibility of constructing a procedure similar to regular simplex method for solving a linear programming problem. This procedure is known as dual simplex method.

we now summarize the iterative procedure for

dual simplex algorithm:-

Dual simplex Algorithm.

Step 1 :-

Convert the given problem to maximization type if it is in minimizatioin type.

Step 2 :-

Convert the $\geq$ type constraints, if any, to $\leq$ constraints by multiplying both sides of such constraints by -1

Step 3 :-

Introduce slack variables and obtain an initial basic feasible solution and write the initial simplex table as in the regular simplex method.

Step 4 :- (Optimality condition)

Test the nature of z_j-c_j in the initial simplex table.

(i) If all $z_j-c_j \geq 0$ and all $x_{Bi} \geq 0$ for all i and j then the current solution is an optimum basic feasible solution STOP

(ii) If all $z_j-c_j \geq 0$ and atleast one basic variable $x_{Bi} \leq 0$ the go to step 5.

(iii) If atleast one $z_j-c_j \leq 0$ then this method cannot be applied to the given problem. STOP

Step 5 :- (Feasiblity condition)

(i) (Leaving variable) Select the most negative of x_{Bi} then the corresponding basis vector leaves the basis. Let x_{Bk} be the leaving variable

(ii) (Entering variable) If all $a_{kj} \geq 0$ then there is no feasible solution to the given problem STOP.

(iii) If at least one $a_{kj} < 0$ then compute the replacement ratios $\left\{\frac{z_j - c_j}{a_{kj}}, a_{kj} < 0\right\}$; j = 1, 2, .. n choose the maximum of these ratios. Let it be say x_r. Then x_r is the entering variabe. Pivotal element is the intersection of the Pivotal row and pivotal column.

Step 6:-

Form the new iteration table and repeat the procedure until either an optimum feasible solution is reached or there is an indication of the non existence of a feasible solution.

Note :-

From the two algorithms on simplex method and dual simplex method we understand that the regular simplex method starts with a basic feasible but non optimal solution and works towards optimality, whereas the dual simplex method starts with a basic infeasible but optimal solution and works towards feasiblity. Also in regular simplex method we first determine the entering variable and then the leaving variable while in the case of dual simple method we first determine the leaving variable and then the entering variable

Solved problems :-

1) solve the following LPP by Dual simplex method.

Minimize $z = 60x_1 + 80x_2$

subject to

$20x_1 + 30x_2 \geq 900$ ....(1)

$40x_1 + 30x_2 \geq 1200$ .... (2)

$x_1, x_2 \geq 0.$

Solution :-

The given LPP is of minimizing type. Hence, it can be converted into maximizing type as

maximize $z^* = -60x_1 - 80x_2$

where $z^* = -z$

The constraints (1) and (2) are of $\geq$ type.

They are converted into $\leq$ type as follows.

$-20x_1 - 30x_2 \leq -900$ .... (3)

$-40x_1 - 30x_2 \leq -1200$ .... (4)

Introducing slack variables $s_1 \geq 0$ and $s_2 \geq 0$ to the contraints (3) and (4) respectively and assciating zero costs in the objective function corresponding to the slack variables we have the given LPP in standard form as :

Maximize

$z^* = -60x_1 - 80x_2 + OS_1 + Os_2$

subject to

$-20x_1-30x_2+s_1 = -900$

$-40x_1-30x_2+s_2 = -1200$

$x_1, x_2, s_1, s_2 \geq 0$.

Note that $b_1 = -900$; $b_2 = -1200$ are negative

Hence we can use dual simplex method. The given LPP can be represented in the following initial dual simplex method.

Initial simplex table

Basis		c_j	-60	-80	0	0	
C_B	B	x_B	x_1	x_2	s_1	s_2	
0 →	s_1	-900	-20	-30	1	0	Go to next iteration
0	s_2	-1200	$\boxed{-40}$	-30	0	1	
Optimality is attained		$z^*_j=0$	0	0	0	0	
		$z^*_j-c_j$	60	80	0	0	
Feasibility is not attained		Ratio	$\frac{-60}{40}$	$\frac{-80}{30}$	-	-	

↑

since all $z^*_j-c_j \geq 0$, optimal solution is obtained However, since all x^s_{Bi} are negative, the solution is optimum but not feasible.

To obtain an optimum feasible solution we go for a new basis.

Leaving variable :-

We choose the most negative baisc variable.

min {-900, -1200} = -1200 = s_2 which corresponds to the second row. Hence s_2 is the leaving basic variable from the basis.

Entering variable :-

To obtain the entering variable in the basis we find

$$\max\left\{\frac{z^*_j-c_j}{a_{rj}} \Big/ a_{rj}<0\right\} \quad r = 2;\ j = 1,2,3,4$$

$$=\max\left\{\frac{60}{-40}, \frac{80}{-30}\right\} = \frac{-60}{40}$$

This corresponds to the variable x_1. Hence x_1 is the entering variable in the new basis. Hence the Pivotal element is -40.

First iteration table

Basis		c_j	-60	-80	0	0	
C_B	B	x_B	x_1	x_2	s_1	s_2	Go to next iterationt
0 →	s_1	-300	0	$\boxed{-15}$	1	$-\frac{1}{2}$	
-60	x_1	30	1	$\frac{3}{4}$	0	$-\frac{1}{40}$	
Optimality is attained		z^*_j=-1800	-60	-45	0	$\frac{3}{2}$	
		$z^*_j-c_j$	0	35	0	$\frac{3}{2}$	
Feasibility is not attained		Ratio	-	$\frac{-35}{15}$	-	$\frac{3}{2} x \frac{-2}{1} =-3$	

↑

x_1 enters the basis; s_1 leaves the basis and -15 is the Pivotal element.

Second iteration table

Basis		c_j	-60	-80	0	0	
C_B	B	x_B	x_1	x_2	s_1	s_2	End of Dual simplex procedure
-80	x_2	20	0	1	$-\frac{1}{15}$	$-\frac{1}{30}$	
-60	x_1	15	1	0	$\frac{1}{20}$	$-\frac{1}{20}$	
Optimality is attained		z^*_j=-2500	-60	-80	$\frac{7}{3}$	$\frac{17}{3}$	
		$z^*_j-c_j$	0	0	$\frac{7}{3}$	$\frac{17}{3}$	

∴ Max z* = -2500

Hence min z = 2500

The optimum feasible solution is x_1 = 15, x_2 = 20 and min z = 2500

Problem 2 :- using Dual simplex method solve the LPP.

maximize $z = -x_1 - x_2$

subject to $2x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq 1$

$x_1, x_2 \geq 0$

solution :-

The given LPP can be written as

Maximize $z = -x_1 - x_2$

subject to

$-2x_1 - x_2 \leq -2$

$x_1 + x_2 \leq -1$

$x_1, x_1 \geq 0.$

Introducing slack variables $s_1 \geq 0$ and $s_2 \geq 0$ we get the LPP in standard form as

Maximize $z = -x_1 - x_2 + OS_1 + OS_2$

subject to

$-2x_1 - x_2 + s_1 = -2$

$x_1 + x_2 + s_2 = -1$

$x_1, x_2, s_1, s_2 \geq 0.$

The initial simplex table is as follows.

Initial simplex table

Basis		c_j	-1	-1	0	0	Go to next iteration
C_B	B	x_B	x_1	x_2	s_1	s_2	
0 →	s_1	-2	$\boxed{-2}$	-1	1	0	
0	s_2	-1	1	1	0	1	
Optimality is attained		$z_j=0$	0	0	0	0	
		z_j-c_j	1	1	0	0	
Feasibility is not attained		Ratio	$\frac{-1}{2}$	$\frac{-1}{1}$	-	-	

↑

s_1 leaves the basis; x_1 enters the basis and -2 is the Pivotal element.

First iteration table

Basis		c_j	-1	-1	0	0
C_B	B	x_B	x_1	x_2	s_1	s_2
-1	x_1	1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0
0 →	s_2	-2	0	$\frac{1}{2}$	$\frac{1}{2}$	1
Optimality is attained		$z_j=-1$	-1	$\frac{-1}{2}$	$\frac{1}{2}$	0
		z_j-c_j	0	$\frac{+1}{2}$	$\frac{1}{2}$	0
Feasibility is not attained		Ratio	Ratio with negative denominator does not exist			

Since all $z_j-c_j \geq 0$, optimality is attained and one of the basic variables $s_2 = -2 < 0$. since there is no $a_{2j} < 0$, for $j = 1, \ldots\ldots 4$ there is no variable to enter the new basis. Hence the given LPP has no feasible solution.

Problem 3:-

Use Dual simplex method to solve the following LPP.

Minimize

$z = 4x_1+x_2$

subject to

$3x_1+x_2 \geq 3$ ....(1)

$4x_1+3x_2 \geq 6$ ....(2)

$x_1, x_2, \geq 4$ ......(3)

$x_1, x_2 \geq 0$

Sol :-

The given LPP is of minimizing type. Hence, it can be converted into maximizing type as :

maximize $z^* = -4x_1-x_2$

where $z^* = -z$

Maximize $z^* = -4x_1-x_2+OS_1+OS_2+OS_3$

subject to

$-3x_1-x_2+s_1 = -3$

$-4x_1-3x_2+s_2 = -6$

$x_1+x_2+s_3 = 4$;

$x_1, x_2, s_1, s_2, s_3 \geq 0$ where s_1, s_2, s_3 are slack variables.

Initial Dual simplex table

Basis		c_j	-4	-1	0	0	0	
C_B	B	x_B	x_1	x_2	s_1	s_2	s_3	Go to next iterationt
0	s_1	-3	-3	-1	1	0	0	
0 →	s_2	-6	-4	$\boxed{-3}$	0	1	0	
0	s_3	4	1	1	0	0	1	
Optimality is attained Feasibility is not attained		$z^*_j=0$	0	0	0	0	0	
		$z^*_j-c_j$	4	1	0	0	0	
		Ratio	$\frac{-4}{4}=-1$	$\frac{-1}{3}$	-	-	-	

↑

x_2 enters the basis; s_2 leaves the basis and -3 is the Pivotal element.

First iteration table

Basis		c_j	-4	-1	0	0	0	
C_B	B	x_B	x_1	x_2	s_1	s_2	s_3	Go to next iterationt
0	s_1 →	-1	$\boxed{-\frac{5}{3}}$	0	1	$\frac{1}{3}$	0	
-1	x_2	2	$\frac{4}{3}$	1	0	$-\frac{1}{3}$	0	
0	s_3	2	$-\frac{1}{3}$	0	0	0	1	
Optimality is attained Feasibility is not attained		$z^*_j=-2$	$-\frac{4}{3}$	-1	0	0	0	
		$z^*_j-c_j$	$\frac{8}{3}$	0	0	$\frac{1}{3}$	0	
		Ratio	$\frac{-8}{5}$	-	-	-	-	

↑

x_1 enters the basis; s_1 leaves the basis and $\frac{-5}{3}$ is the Pivotal element.

Second iteration table

Basis		c_j	-4	-1	0	0	0	
C_B	B	x_B	x_1	x_2	s_1	s_2	s_3	Go to next iterationt
-4	x_1	$\frac{3}{5}$	1	0	$-\frac{3}{5}$	$\frac{1}{3}$	0	
-1	x_2	$\frac{6}{5}$	0	1	$\frac{4}{5}$	$-\frac{1}{15}$	0	
0	s_3	$\frac{11}{5}$	0	0	$-\frac{1}{5}$	$\frac{1}{5}$	1	
Feasibility is attained		$z^*_j = -\frac{18}{5}$	-4	-1	$\frac{8}{5}$	$\frac{4}{15}$	0	
		$z^*_j - c_j$	0	0	$\frac{8}{5}$	$\frac{13}{15}$	0	

since all $z^*_j - c_j \geq 0$ and all $x_{Bi} > 0$. *Optimum feasible solution is obtained.*

Maximum $z^* = -\frac{18}{5}$

Optimum solution is $x_1 = \frac{3}{5}$

$x_2 = \frac{6}{5}$ and minimum $z = \frac{18}{5}$

Exercise :-

Solve the following linear programming problems using dual simplex method:-

1) maximize

$z = -3x_1 - x_2$

subject to

$x_1 + x_2 \geq 1$

$2x_1 + 3x_2 \geq 2$

$x_1, x_2 \geq 0$

2) Minimize

$z = 5x_1 + 6x_2$

subject to

$x_1 + x_2 \geq 2$

$4x_1 + x_2 \geq 4$

$x_1, x_2 \geq 0$

3) Minimize

$z = 6x_1 + x_2$

subject to

$2x_1 + x_2 \geq 3$

$x_1 - x_2 \geq 0$

$x_1, x_2 \geq 0$

4) minimize

$z = x_1+x_2$

subjct to

$2x_1+x_2 \geq 2$

$-x_1-x_2 \geq 1$

$x_1, x_2 \geq 0$.

5) Minimize

$z = 2x_1+x_2$

subject to

$3x_1+x_2 \geq 3$

$4x_1+3x_2 \geq 6$

$x_1+2x_2 \geq 3$

$x_1, x_2 \geq 0$.

6) Maximize

$z = -2x_1 - x_3$

subject to

$x_1+x_2-x_3 \geq 5$

$x_1-2x_2+4x_3 \geq 8$

$x_1, x_2, x_3 \geq 0$.

3.4 Integer programming cutting plane technique (Gomarian Constraint)

Gomory's cutting plane method:-

We discuss a systematic method by which we can generate new constraints so as to ensure integral solution of the given I.P.P. (integer programming problem). These additional constraint do not cut off that portion of the original feasible region which contains a feasible integer point. Also it cuts off the current non-integer optimal solution of the L.P.P.

First we solve the given L.P.P. in the usual manner without considering the integer requirement. If the optimal solution contain integer values for the decision variables then it is itself the solultion of the I.P.P. If some of the variables have fractional values then choose the variable having the largest fractional part say X_r. Then the r^th^ constraint can be written as

$$b_r = x_r+a_{r_1}x_1+a_{r_2}x_2+\ldots\ldots\ \ldots.$$

$$= x_r+\sum a_{rj}x_j \qquad (j \neq r)$$

This can be written as

$$[b_r]+f_r = x_r+\sum\left\{[a_{rj}]+f_{rj}\right\}x_j$$

where $[b_r]$ denotes the greatest integer less than or equal to b_r. which $0<f_r<1$.

III^{ly} $[a_{rj}]$ denotes the integral part of $[a_{rj}]$ where $0 \leq f_{rj} \leq 1$. Thus we get

$f_r+\{ [b_r] - x_r - \sum[a_{rj}] x_j \} = \sum f_{rj}x_j \quad (j \neq r)$

we note that $\{ [b_r] - x_r + \sum[a_{rj}] x_r \}$ is always non-negative.

Hence $f_r \leq \sum f_{rj}x_j$

we write it as an equality by introducing a slack variable s.

$f_r+s = \sum f_{rj}x_j$

(i.e.) $s-\sum f_{rj}x_r = -f_r$

This is a new constraint called Gomory's constraint which represents a cutting plane.

II) Step by step procedure of Gomory's algorithm:-

Step 1 :-

Solve the given LPP using simplex method ignoring the integer requirement of the variables.

Step 2 :-

If the optimal values of the variables are au integers it is the solution of the IPP(step)

Step 3:-

If some variables assume fractional values choose the variable having the largest fractional part, say x_r. write the Gomory's constraint in the form

$$S - \sum f_{rj} x_j = -f_r \qquad \begin{cases} 0<f_r<1 \\ 0<f_{rj}<1 \end{cases}$$

Step 4:-

Add the new constraints at the bottom of the simplex table and find a new optimal solution using dual simplex method. If the new solultion gives integer values to the decision variables it is the solution of the IPP otherwise go to step 3. The process is repeated until all the variables assume non-negative integer values.

Solved problem :-

1) Consider the IPP

maximize

$z = 7x_1+9x_2$

s.t $-x_1+3x_2 \leq 6$

$7x_1+x_2 \leq 35$

$x_1, x_2 \geq 0$are integers.

Solution :-

Introducing the slack variables x_3, x_4 and we solve problem by regular simplex method.

Basic	7	9	0	0		
	x_1	x_2	x_3	x_4	b	Q
x_3 →	-1	$\boxed{3}$	1	0	6	$\frac{6}{3}$ =2
x_4	1	1	0	1	35	$\frac{35}{1}$ = 35
z_j-c_j	-7	-9 ↑	0	0	0	
x_2	$-\frac{1}{3}$	1	$\frac{1}{3}$	0	2	
x_4 →	$\boxed{\frac{22}{3}}$	0	$-\frac{1}{3}$	1	33	
z_j-c_j	-10↑	0	3	0	18	
x_2	0	1	$\frac{7}{22}$	$\frac{1}{22}$	$\frac{7}{2}$	$(3+\frac{1}{2})$
x_1	1	0	$-\frac{1}{22}$	$\frac{3}{22}$	$\frac{9}{2}$	$(4+\frac{1}{2})$
z_j-c_j	0	0	$\frac{28}{11}$	$\frac{15}{11}$	63	

$x_1 = 4\frac{1}{2}$; $x_2 = 3\frac{1}{2}$

$x_2 + (0+\frac{7}{22})x_3 + (0+\frac{1}{22})x_4 = 3 + \frac{1}{2}$

The new constraint is,

$$(-\frac{7}{22})x_3 + (-\frac{1}{22})x_4 + y_1 = -\frac{1}{2}$$

Dual simplex table :-

Basic	x_1	x_2	x_3	x_4	y_1	
x_2	0	1	$\frac{7}{22}$	$\frac{1}{22}$	0	$\frac{7}{2}$
x_1	1	0	$-\frac{1}{22}$	$\frac{3}{22}$	0	$\frac{9}{2}$
y_1 →	0	0	$\boxed{-\frac{7}{22}}$	$\frac{-1}{22}$	1	$-\frac{1}{2}$
	0	0	$\frac{28}{11}$	$\frac{15}{11}$	0	63
			↑			

we apply the dual simplex method.

(i) The leaving variable is y_1

(ii) The entering variable is associated with

$$Q = \max \left\{ \frac{\frac{28}{11}}{\frac{-7}{22}} , \frac{\frac{15}{11}}{\frac{-1}{22}} \right\}$$

$= \max \{-8, -30\} = -8$ which is x_3.

The new simplex table is :-

Basic	x_1	x_2	x_3	x_4	y_1	b	
x_2	0	1	0	0	1	3	
x_1	1	0	0	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{32}{7}$	$4+\frac{4}{7}$
x_3	0	0	1	$\frac{1}{7}$	$\frac{-22}{7}$	$\frac{11}{7}$	$1+\frac{4}{7}$
	0	0	0	1	8	59	

current solution is

$x_1 = 4+\frac{4}{7}$; $x_2 = 3$

$x_3 = 1+\frac{4}{7}$ is not an integer solution.

$$x_1+(0+\frac{1}{7})x_4 + (-1+\frac{6}{7})\, y_1 = 4+\frac{4}{7}$$

The new constraint is

$$(-\frac{1}{7})x_4 + (-\frac{6}{7})y_1 + y_2 = -\frac{4}{7}$$

Dual simplex table:-

Basic	x_1	x_2	x_3	x_4	y_1	y_2	
x_2	0	1	0	0	1	0	3
x_1	1	0	0	$\frac{1}{7}$	$\frac{-1}{7}$	0	$\frac{32}{7}$
x_3	0	0	1	$\frac{1}{7}$	$\frac{-22}{7}$	0	$\frac{11}{7}$
y_2	0	0	0	$\boxed{\frac{-1}{7}}$	$\frac{6}{7}$	1	$\frac{4}{7}$ →
	0	0	0	1 ↑	8	0	59

we apply the dual simplex method.

(i) the leaving variable y_2

(ii) the entering variable is associated

with $Q = \max \left\{ \dfrac{1}{\frac{-1}{7}}, \dfrac{8}{\frac{-6}{7}} \right\} = \{-7, -9\} = -7$

simplex table :-

Basic	x_1	x_2	x_3	x_4	y_1	y_2	
x_2	0	1	0	0	1	0	3
x_1	1	0	0	0	-1	1	4
x_3	0	0	1	0	-4	1	1
x_4	0	0	0	1	6	-7	4
z_j-c_j	0	0	0	0	2	7	55

Hence $x_1 = 4$; $x_2 = 3$; $x_3 = 1$, $x_4 = 4$

The maximum z value is 55

maxi z = 55

Solved problems :-

I) Consider the ILP problem using Gomory's cutting plane method :-

1) Maximum

$z = 2x_1+20x_2-10x_3$

subject to

$2x_1+20x_2+4x_3 \leq 15$

$6x_1+20x_2+4x_3 = 20$

$x_1, x_2, x_3 \geq 0$ as integers.

2) maximize

$z = x_1+x_2$

subject to

$3x_1+2x_2 \leq 5$

$x_2 \leq 2$

$x_1, x_2 \geq 0$ and are integers.

3) Maximize

$z= 3x_1+12x_2$

subject to

$2x_1+4x_2 \leq 7$

$5x_1+3x_2 \leq 15$

$x_1, x_2, \geq 0$ and are integers.

4) Maximize

$z = 2x_1+x_2$

subject to

$4x_1+x_2 \leq 7$

$x_1+3x_2 \leq 4$

$x_1, x_2 \geq 0$ and are integers.

5) Maximize

$z = x_1+2x_2$

subject to

$x_1+2x_2 \leq 12$

$4x_1+2x_2 \leq 14$

$x_1, x_2 \geq 0$ and are integers.

Unit - 4

4.1 Assignment problems :-

Introduction :-

An assignment problem is a particular case of transportation problem in which a number of operations are to be assigned to an eqoual number of operators, where each operat or performs only one operation. The objective is to maximize over all profit (or) minimize overall cost for a given assignment schedule.

An assignment problem is a completely degenerate form of a transporatation problem The units available at each orgin and units demanded at each destination are all equal to one. That menas exactly one occupied cell in each. row and each column of the transportation table. (ie.) only n occupied cells in place of the required n+n-1 = (= 2n-1)

4.2 Mathematical formulation of an AP:-

Consider an assignment problem of assigning n jobs (operations) to n persons (operators). Let c_{ij} be the cost incurred in assigning i^{th} operation to jth. operator and let

$$x_{ij} = \begin{cases} 1 \text{ if} & i^{th} \text{ operation is assigned to } j^{th} \text{ operator.} \\ 0 & \text{otherwise.} \end{cases}$$

Then the assignment problem is simply the following LPP.

$$\text{Minimize } z = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} c_{ij}$$

subject to the constraints

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1........n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad i = 1..........n$$

with $x_{ij} = 0$ (or) 1

An A.P could thus solved by simplex method. It also happens to be an nxn transportation problem with each $a_i = b_j = 1$ However as an assignment problem is highly degenerate. It will be frustrating to attempt to solve it by simplex method (or) transporation method - Infact a very convenient iterative procedure is available for solving an A.P.

Tnm the optimal assignment schedule remains unaltered if we add (or) subtract a constant o/from all the elements of the row (or) column of the assignment cost matrix.

Pr : Let the assginment have a cost matrix $c = (c_{ij})$ i, j = 1.......n.

Let $c^*_{ij} = c_{ij} \pm u_i \pm v_j$ (for all i & j)

That is we add (or) subtract a constant u_i, v_j to /from all the elements of the i^{th} row and j^{th} column of c respectively. Then for the new cost matrix c*, the objective function is

$$z^* = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} c^*_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} (c_{ij} \pm u_i \pm v_j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} c_{ij} \pm \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} u_i \pm \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} v_j$$

$$= x \pm U \pm v$$

where $U = \sum_{i=1}^{n} u_i$ $\qquad v = \sum_{j=1}^{n} v_j$

$$\left(\because \sum_{i=1}^{n} x_{ij} = \sum_{j=1}^{n} x_{ij} = 1 \right)$$

Since U and V are constants, an assignment (x_{ij}) which minimze Z will also minimize z*

4.3 Hungarian Assignment method.

Assignment Algorithm :-

Various steps of the computational procedure for obtaining an optimal assignment may be summarized as follows.

Step 1:- Check whether the cost matrix is square. If not make it square by adding suitable number of dummy rows (or columns) with 0 cost elements.

step 2 :- Locate the smallest cost elements in each row of the cost matrix. Subtract this smallest element from each element in that row. As a result there shall be atleast one zero in each row of the reduced cost matrix.

Step 3:- In the reduced cost matrix obtained cosider each column and locate the smallest element in it. subtract the smallest value from every other entry in the column. As a result there would be column As a result there would be atleast one zero in each of the rows and columns of the second reduced cost matrix.

Step 4:- In the above reduced cost matrix search for an optimum assignment as follows.

(i) Examine the rows succesively until a row with exactly one zero is found. Enrectangle this zero as and □ cross out all the other o's in its column. Proceed in this manner until all the rows have been examined. If there are more than one o in any row, then do not touch that row and pass on to the next row.

ii) Repeat the procedure for the columns of the reduced cost matrix, If there is no single zero in any row (or) column of the reduced matrix, then arbitarily choose a row (or) column having the minimum number of O's. Arbitarily select and enrectangle any one 0 in the row (or) column. Thus chosen and cross all other o's in its row and column. Repeat steps (i) and (ii) until all the o's have been either assigned (or) crossed.

(iii) If each row and each column of the reduced matrix has one and only one assigned zero, the optimum assignment is made in the cells of enrectangled zero's. Otherwise go to next step.

Step 5:- Draw the minimum number of horizontal and / or vertical lines. though all the 0's as follows.

a) Mark (√) the rows in which assignment has not been made.

b) Mark (√) columns which have zero's (o's) in the marked rows.

c) Mark (√) rows (not already marked) which have assignments in marked columns.

d) Repeat (b) and (c) until the chain of marking is completed.

e) Draw straight lines through all unmarked rows and marked columns.

Step 6 :- If the minimum number of lines passing through all the zero's is equal to the number of rows (or) columns, the optimum solution is attained by an arbitary allocation in the positions of the zeros not crossed in step 3 otherwise go to next step.

Step 7:- Revise the costs matrix as follows.

(i) Find the smallest element not covered by any of the lines of step 4.

(ii) Subtract this from all the uncrossed elements and add the same as the point of intersection of the two lines.

(ii) Other elements crossed by the lines remain unchanged.

Step 8 :- Go to step (4) and repeat the procedure till an optimum solutions is attained.

The above iterative method to determine an assignment schedule is known as Hungarian Assignment method.

4.4. : Special cases in Assignment problems.

Maximisation case in Assignment problems.

In some cases the pay off elements of the assignment problem may represent revenues (or) profits instead of costs so that the objective will be to maximise the total revenue (or) profit. The Hungarian method explained. Earlier can also be used for maximisation case The problem of maximisation case can be converted into minimisation case by selecting the largest element among all elements of the profit matrix and then subtracting it from all other elements in the matrix. We can then proceed as usual and obtain the optimal solution by adding the original values of these cells to which the assignments have been made.

2) Multiple optimal solutions:-

Sometimes it is possible to have two (or) more ways to cross out all zero elements in the final reduced matrix for a given problem. This implies that there are more than the required number of independent zero elements. In such cases there will be multiple optimal solutions with the same total cost of assignment. In such type of situations, management may exercise their judgement and select the set of optimal assignments which is more suited to their requirement

4.5 : Unbalanced assignment problem:-

Whenever the pay off matrix of an assignment problem is not a square matrix ((ie.) the number of rows are not equal to the number of column the assignment problem is called unbalanced assignment problem. In such case dummy rows and / or columns are added in the matrix to make it a square matrix. Then we can apply the Hungarian method to this resulting balanced (square matrix) assignment problem. For example if four workers are to be assigned to for machines, a dummy row is simply added to transform the assignment problem in to square (5x5) matrix. Creating dummy rows (or) columns will give us a matrix of equal dimensions and allow us to solve the problem as discussed earlier. The cost (or time) associated with this dummy row or column is assigned zero element in the matrix.

Remarks : i) There are situations when a particular assignment may not be permissible In such situations we assign a very high cost (say M) for such an assignment and proceed as usual

2) If the assignment problem involves maximization convert the effective matrix to an opportunity loss matrix by subtracting each element from the highest element of the matrix, Minimization of the resulting matrix is the same as of the resulting matrix is the same as the maximization of the original matrix.

Solved problems:-

1) A department head has four subordinates and four tasks to be performed. The subordinates differ in efficiency and tasks differ in their intrinsic difficulty. His estimate of the time each man would take to perform each task is given in the matrix below.

Men Tasks

	E	F	G	H
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

How should the tasks be allocated one to a man so as to minimize the total man-hours?

Solution :-

Step 1:- Subtracting the smallest element of each row from every element of the corresponding row, we get the reduced matrix;

$$\begin{pmatrix} 7 & 15 & 6 & 0 \\ 0 & 15 & 1 & 13 \\ 23 & 4 & 3 & 0 \\ 9 & 16 & 14 & 0 \end{pmatrix}$$

Step 2:- Subtracting the smallest element of each column of the reduced matrix from every element of the corresponding column, we get the following reduced matrix.

$$\begin{pmatrix} 7 & 11 & 5 & 0 \\ 0 & 11 & 0 & 13 \\ 23 & 0 & 2 & 0 \\ 9 & 12 & 13 & 0 \end{pmatrix}$$

Step 3:- Starting with row 1, we enrectangle □ ((ie.) make assignment) a single zero, if any and cross (x) all other zeros in the column so marked. Thus we get

$$\begin{pmatrix} 7 & 11 & 5 & \boxed{0} \\ \boxed{0} & 11 & 0 & 13 \\ 23 & \boxed{0} & 2 & \cancel{0} \\ 9 & 12 & 13 & \cancel{0} \end{pmatrix}$$

In the above matrix we arbitrarily enrectangled a zero in column 1, because row 2 had two zeros.

It may be noted that column 3 and row 4 do not have any assignment so we move on to the next step.

Step 4:- (i) Since row 4 does not have any assignment we tick this row (√)

(ii) Now there is a zero in the fourth column of the ticked row, so, we tick fourth column (√)

(iii) Further there is an assignment in the first row of the ticked column. So we tick first row (√)

(iv) Draw straight lines through all unmarked rows and marked columns. Thus we have

$$\begin{array}{cc} \begin{pmatrix} 7 & 11 & 5 & \boxed{0} \\ \boxed{0} & 11 & \cancel{0} & 13 \\ 23 & \boxed{0} & 2 & \cancel{0} \\ 9 & 12 & 13 & \cancel{0} \end{pmatrix} & \begin{array}{c} \surd \\ \\ \\ \surd \end{array} \\ \begin{array}{cccc} & & & \surd \end{array} & \end{array}$$

Step 5:- In step 4, we observe that the minimum number of lines so drawin is 3, which is less than the order of the cost matrix, indicating that the current assignment is not optimum. To increase the minimum number of lines, we generate new zeros in the modified matrix

Step 6 :- The smallest element not covered by the lines is 5. Subtracting this element from all the uncovered elements and adding the same to all the elements lying at the intersection of the lines. We obtain the following new reduced cost matrix.

$$\begin{pmatrix} 2 & 6 & 0 & 0 \\ 0 & 11 & 0 & 18 \\ 23 & 0 & 2 & 5 \\ 4 & 7 & 8 & 0 \end{pmatrix}$$

Step 7: Repeating step 4 on the reduced matrix, we get

$$\begin{pmatrix} 2 & 6 & \boxed{0} & \cancel{0} \\ \boxed{0} & 11 & \cancel{0} & 18 \\ 23 & 0 & 2 & 5 \\ 4 & 7 & 8 & \boxed{0} \end{pmatrix}$$

Now since each row and each column has one and only one assignment an optimal solution is reached. The optimum assignment is

A → G, B → E, C → F, and D → H

The minimum total time for this assignment scheduled is 17+13+19+10 = 59 men hours.

Problem 2:- A marketing manager has 5 salesman and 5 districts. Considering the capabilities of the salesmen and the nature of districts, the marketing manager estimates that sales permonth (in hundred rupees) for each salesman in each district would be follows.

Job	Machine				
	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

Find the assignment of salesmen to districts that will result in maximum sales.

Solution:- Convert the profit matrix into oppurtunity loss matrix by subtracting all the entries from the highest element 41 of the given matrix.

Initial iteration:-

Reduce the opportunity loss matrix sothat there is atleast one zero each row and each column make the proper assignments in rows and columns. Also draw the minimum number of lines to cover all the zeros of the reduced matrix see table 1 1.-

$$\begin{array}{|ccccc|c} 8 & \boxed{0} & \cancel{0} & 7 & \cancel{0} & \\ \boxed{0} & 14 & 12 & 14 & 4 & \checkmark \\ \cancel{0} & 12 & 8 & 6 & 4 & \checkmark \\ 19 & 1 & \cancel{0} & \boxed{0} & 5 & \\ 11 & 5 & \boxed{0} & \cancel{0} & 1 & \\ \hline \checkmark & & & & & \end{array}$$

Table 1.1

Talbe : - 1.2

12	$\boxed{0}$	$\cancel{0}$	7	$\cancel{0}$
$\boxed{0}$	10	8	10	$\cancel{0}$
$\cancel{0}$	8	4	2	$\boxed{0}$
23	1	$\boxed{0}$	$\cancel{0}$	5
15	5	$\cancel{0}$	$\boxed{0}$	1

Final iterations :- Modify table 1.1 by subtracting the element '4' from all the elements not covered by all lines and adding the same at the intersection of two lines. Thus we get table 1.2

The optimum assignment is

1 → B, 2 → A, 3 → E, 4 → C, 5 → D

(or)

1 → B, 2 → E, 3 → A, 4 → C, 5 → D (or)

1 → B, 2 → A, 3 → E, 4 → D, 5 → C (or)

1 → B, 2 → E, 3 → A, 4 → D, 5 → C

Maximum profit is Rs. 191

Problem 3:-

A department head has four task to be performed and three subordinate the subordinates differ in efficiency. The estimates of time each subordinate would take to perform is given below in the matrix. How should he allocate the tasks one to each man so as to minimize the total man hours?

Task	Men		
I	9	26	15
I	13	27	6
II	35	20	15
V	18	30	20

Solution :- Since the problem is unbalanced we add a dummy column with all the entries as zero and use assignment methods for optimum solution.

Now reduce the balanced cost matrix and make assignments in rows and columns having single zeros. Thus we have.

$$\begin{array}{|cccc|}
\boxed{0} & 6 & 9 & \cancel{0} \\
4 & 7 & \boxed{0} & \cancel{0} \\
26 & \boxed{0} & 9 & \cancel{0} \\
26 & 10 & 14 & \boxed{0}
\end{array}$$

The optimum assignment is I ⟶ 1 II ⟶ 3 and III ⟶ 2 while task IV should be assigned to a dummy man (ie)

The minimum time is $\boxed{35}$ hours.

4.6 : Routing problem:-

Network scheduling is a technique for the planning and scheduling of large. Projects It has succeessfully been applied in the transportation and communication problems. A typical network problem consists of finding route from one node (origin) to another (destinat) between which alternative paths are available at various stages of the journey. The problem is to select the route that yields mininum cost. A number of different constraints ma, placed on acceptable routes. For example not returning to the node already passed through or passing through everynode once and only once. Problems of such type are called routing problems.

Although a large variety of problems other than the routing one may be develped in connection with the construction and utilization of networks here we shall consider only the special type of routing problem, that occurs most frequently in O R. the travelling salesman problem.

4.7 : The travelling salesman problem:-

Suppose a saleman has to visit the wises to start from a particular city, visit each city once, and then return to his starting point. The objectiveis to select the sequence in which the cities are visited in such a way that his total travelling time is minimized Clearly starting from a given city, the salesman will have a total of (n-1)! different sequences (possible round trips). Further since the saleman has to visit all the n cities, the optimal solution remains independent of selection of the starting point. The problem can be represents network where the nodes and arcs represent the cities and distance between them repectively let in a five city problem, a round trip of the salesman be given by the following arcs

(3,1), (1,2), (2,4) (4,5) (5,3)

These arcs taken in order are called the first,second, third, fourth and fifth directed arcs for the trip. In gerneal the K^{th} directed arc represents the K^{th} leg of the trip (ie) on leg

K the siaesman travels from city i to city j (i, j= 1,2,........n, i≠j) To formulate the problem whose solution will yield the minimum travelling time, let the variables x_{ijk} be defined as

$$x_{ij} = \begin{cases} 1, & \text{if k directed arc is} \\ & \text{from city i to city j} \\ 0 & \text{otherwise} \end{cases}$$

where i, j and k are integers that vary between 1 and n.

Following are the constraints of the problem.

a) Only one directed arc may be assigned to a specific K, this

$$\sum_{j}\ \sum_{\substack{k \\ i\neq j}} x_{ijk} = 1 \qquad k = 1,2, \ldots\ldots\ldots n$$

b) Only one other city may be reached from a specific city i, thus

$$\sum_{j}\ \sum_{k} x_{ijk} = 1 \qquad i = 1,2, \ldots\ldots\ldots n$$

c) only one other city can initiate a directly arc to a specified city j, thus

$$\sum_{i}\ \sum_{k} x_{ijk} = 1 \qquad j = 1,2, \ldots\ldots\ldots n$$

d) Given the Kth directed arc ends at some specific city j, the (k+1)th, directed arc must start at the same city j, Thus

$$\sum_{i\neq j} x_{ijk} = \sum_{r\neq j} x_{jr}(k+1) \text{ for all j and k}$$

These constraints ensure that the remandtrip will consist of connected directed ares, the objective function is to minimize

$$z = \sum_{i}\ \sum_{j}\ \sum_{k} d_{ij}\ x_{ijk} \quad ; \ i \neq j$$

where d_{ij} is the distance from city i to city j.

Formulation of Travelling salesman problem as an Assignment problem.

The travelling salesman problem is very similar to the assignment problem except that in the former, there is an additional restriction.A similar problem arises when n items say A_i i = 1.........n are to be processed on a machine in a scheduled time. The problem then becomes of choosing the sequence of these items. Le the setup cost of the machine when item A_i is followed by A_j by c_{ij}. Also let x_{ij} =1 if item A_i is followed by A_j directly and 0 otherwise. It is to be noticed that $c_{ij} = \propto$ when i = j (ie) the item A_i is not processed again after A_i. It is important to note that only one n_{ij} = 1 for each value of i and for each value of j

In view of the above, the assignment problem can be solved and one may hope that the solution satisfies the additional restriction also.

If the solution to the assignment problem does not satisfy the additional restriction,then after solving the problem by assignment technique, we use the method of enumeration The procedure is best illustrated with the help of the following sample problems.

problem 4:-

A machine operator processes five types of items on his machine each week and must choose a sequence for them. The set-up cost per change depends on the item presently on the machine and the set-up to be made according to the following table.

From item :- To item

	A	B	C	D	E
A	∝	4	7	3	4
B	4	∝	6	3	4
C	7	6	∝	7	5
D	3	3	7	∝	7
E	4	4	5	7	∝

If the processes each type of them once and only once each week, how should be sequence the items on his machine in order to minimize the total set-up cost?

Solution :- Reduce the cost matrix and make assignments in rows and colums having single zeros.

Initial Iteration :- Draw the minimum number of lines to cover all the zeros see the below table.

∝	1	3	$\boxed{0}$	1	√
1	∝	2	$\cancel{0}$	1	√
2	1	∝	2	$\boxed{0}$	
$\cancel{0}$	$\boxed{0}$	3	∝	4	
$\boxed{0}$	$\cancel{0}$	0	3	∝	

First iteration :- Modify the above table by subtracting the lowest element from all the elements not covered by lines and adding the same at the intersection of two lines. See the below table

$$\begin{array}{ccccc}
\propto & \xcancel{0} & 2 & \boxed{0} & \xcancel{0} \\
\boxed{0} & \propto & 1 & \xcancel{0} & \xcancel{0} \\
2 & 1 & \propto & 3 & \boxed{0} \\
\xcancel{0} & \boxed{0} & 3 & \propto & 4 \\
\xcancel{0} & \xcancel{0} & \boxed{0} & 4 & \propto
\end{array}$$

The optimum assignment is

A → D, B → A, C → E, D → B, D → B and

E → C with the minimum cost of 20.

The assignment schedule does not provide us the solultion of travelling salesman problem as it gives A → O, D → B, B → A while B is not allowed to follow A unless C and E are processed.

Second iteration :- Now we try to find the next best solution which satisfies this extra restriction. The next minimum (non-zero) element in the matrix, is 1, so we try to bring 1 into the solution. But the element 1 occurs at two places. We shall consider all the cases separatively until the acceptable solution is reached.

We start with making an assignment at (2,3) instead of zero assignment at (2,1). The resulting feasible solution then will be.

A → D, D → B, B → C, C → E, E → A when an assignment is made at (3,2) instead of zero assignment at (3,5) the resulting feasible solution will be A → E, E → C, C → E, C → B, B → D, D → A. The total set-up cost in both the programmes comes out to be 21.

Problem 5 :- Solve the following travelling salesman problem so as to minimise the cost per cycle.

	To				
From	A	B	C	D	E
A	-	3	6	2	3
B	3	-	5	2	3
C	6	5	-	6	4
D	2	2	6	-	6
E	3	3	4	6	-

solution :- Reduce the cost matrix and make assignment in rows and columns having single zeros.

Initial Interation :- Draw the minimum number of lines to cover all the zeros. See the below table

$$\begin{array}{|ccccc|c}
\hline
M & 1 & 3 & \boxed{0} & 1 & \surd \\
1 & M & 2 & \cancel{0} & 1 & \surd \\
2 & 1 & M & 2 & \boxed{0} & \\
\boxed{0} & \cancel{0} & 3 & M & 4 & \\
\cancel{0} & \cancel{0} & \boxed{0} & 3 & M & \\
\hline
 & & & \surd & &
\end{array}$$

First Iteration :- Modify the above table by subtracting the the lowest element 1' from all the elements not covered by lines and adding the same at the intersection of two lines. See below table.

$$\begin{array}{|ccccc|}
\hline
M & \boxed{0} & 2 & \cancel{0} & \cancel{0} \\
\cancel{0} & M & 1 & \boxed{0} & \cancel{0} \\
2 & 1 & M & 3 & \boxed{0} \\
\boxed{0} & \cancel{0} & 3 & M & 4 \\
\cancel{0} & \cancel{0} & \boxed{0} & 4 & M \\
\hline
\end{array}$$

The optimum assignment is

A → B, B → D, C → E, D → A and E → B with minimum cost is 15.

Since this assignment schedule does not provide us the solution of travelling salesman problem. We try to find the next best solution which satisfies the extra condition also.

Second iteration :- Make an assignment at (2,3) instead o zero at (2,4)

The resulting table is shown below.

$$\begin{array}{|ccccc|}\hline M & \cancel{0} & 2 & \boxed{0} & \cancel{0} \\ \cancel{0} & M & \boxed{1} & \cancel{0} & \cancel{0} \\ 2 & \cancel{0} & M & 3 & \boxed{0} \\ \cancel{0} & \boxed{0} & 3 & M & 4 \\ \boxed{0} & \cancel{0} & \cancel{0} & 4 & M \\ \hline\end{array}$$

A sımilar argument gives on alternate optimal assignment shown below table.

$$\begin{array}{|ccccc|}\hline M & \cancel{0} & 2 & \cancel{0} & \boxed{0} \\ \cancel{0} & M & 1 & \boxed{0} & \cancel{0} \\ 2 & \boxed{1} & M & 3 & \cancel{0} \\ \boxed{0} & \cancel{0} & 3 & M & 4 \\ \cancel{0} & \cancel{0} & \boxed{0} & 4 & M \\ \hline\end{array}$$

The optimum assignment schedule is
A → D → B → C → E → A (or) A → E → C → B → D → A
Total minimum cost per cycle in both the cases will be $\boxed{16}$

Problem : 6 Given the following matrix the set up costs show how to sequence production. So as to minimize set up cost per cycle.

From	To A	 B	 C	 D	 E
A	∝	2	5	7	1
B	6	∝	3	8	2
C	8	7	∝	4	7
D	12	4	6	∝	5
E	1	3	2	8	∝

Solution : Reduce the cost matrix and make assignments in rows and columns having single zeros. Thus we get

$$\begin{array}{|ccccc|}\hline \propto & 1 & 3 & 6 & \boxed{0} \\ 4 & \propto & \boxed{0} & 6 & \cancel{0} \\ 4 & 3 & \propto & \boxed{0} & 3 \\ 8 & \boxed{0} & 1 & \propto & 1 \\ \boxed{0} & 2 & \cancel{0} & 7 & \propto \\ \hline\end{array}$$

The optimum assignment is
A → E, E → A, B → C, C → D, D → B with minimum cost is 13.

First iteration :- For the travelling salesman problem, we make assignment at (1,2) instead of zero assignment at (1,5) see the below table.

$$\begin{array}{ccccc}
\propto & \boxed{1} & 3 & 6 & \cancel{0} \\
4 & \propto & \boxed{0} & 6 & 0 \\
4 & 3 & \propto & \boxed{0} & 3 \\
8 & \cancel{0} & \cancel{0} & \propto & \boxed{1} \\
\boxed{0} & 2 & \cancel{0} & 7 & \propto
\end{array}$$

The optimum assignment schedule to the travelling salesman problem is

A $\rightarrow$ B $\rightarrow$ C $\rightarrow$ D $\rightarrow$ E $\rightarrow$ A with total minimum cost is $\boxed{15.}$

Problem 7:- Solve the travelling salesman problem given by the following

$c_{12} = 20$ $c_{13} = 4$ $c_{14} = 10$ $c_{23} = 5$ $c_{34} = 6$

$c_{25} = 10$ $c_{35} = 6$ $c_{45} = 20$ where $c_{ij} = c_{ij}$

and there is no route between cities i and j if a value for c_{ij} is not shown.

solution:- We are given that there is no route when i=j. So we take $c_{ij} \propto =$ for i=j The given problem therefore can be expressed in the form of an assignment problem.

$$\begin{array}{c|ccccc}
 & I & II & III & IV & V \\
\hline
I & \propto & 20 & 4 & 10 & \propto \\
II & 20 & \propto & 5 & \propto & 10 \\
III & 4 & 5 & \propto & 6 & 6 \\
IV & 10 & \propto & 6 & \propto & 20 \\
V & \propto & 10 & 6 & 20 & \propto
\end{array}$$

Initial Iteration :-

Following the Procedure of assignment method, optimum assignment schedule is obtained in the below table.

	I	II	III	IV	V
I	∝	11	$\boxed{0}$	$\cancel{0}$	∝
II	12	∝	1	∝	$\boxed{0}$
III	$\cancel{0}$	$\cancel{0}$	∝	$\boxed{0}$	$\cancel{0}$
IV	$\boxed{0}$	∝	$\cancel{0}$	∝	8
V	∝	$\boxed{0}$	1	9	∝

The optimum assignment is
I → III, III → IV, IV → I, II → V, V → II

Since the above assignment schedule does not provide as the solution of travelling salesman problem. We try to find the next best solution which satisfies the extra restriction also.

First Iteration:- Make assignment at (2,3) instead of zero assignment at (2,5) The revised assignment schedule is given in the below table.

∝	11	$\cancel{0}$	$\boxed{0}$	∝
12	∝	$\boxed{1}$	∝	$\cancel{0}$
$\cancel{0}$	$\cancel{0}$	∝	$\cancel{0}$	$\boxed{0}$
$\boxed{0}$	∝	$\cancel{0}$	∝	8
∝	$\boxed{0}$	$\cancel{0}$	9	∝

The revised optimum assignment is
I → IV, IV → I, II → III, III → V, V → II

This assignment also does not provide us the solution to travelling salesman problem as it again gives the route I → IV, IV → I, so we try find the next best solution satisfying the additional restriction.

Second restriction :- Make assignment at (4,5) instead of zero assignment at (4,1) see the below table.

∝	11	$\cancel{0}$	$\boxed{0}$	∝
12	∝	$\boxed{1}$	∝	$\cancel{0}$
$\boxed{0}$	$\cancel{0}$	∝	$\cancel{0}$	$\cancel{0}$
$\cancel{0}$	∝	$\cancel{0}$	∝	$\boxed{8}$
∝	$\boxed{0}$	$\cancel{0}$	9	∝

From the above table the optimum sequence to travelling salesman problem is

I → IV → V → II → III → I

The total setup cost according to above assignment schedule is 49.

Problem 8:- A complete centre has got three expert programmers. The centre needs three application programme to be developed. The Head of the computer centre after studying carefully the programmes to be develped estimates the computer time in minutes required by the experts to the application programmes as follows.

Programmes

$$\begin{array}{r|ccc|} & A & B & C \\ \hline \text{Programmers } 1 & 120 & 100 & 80 \\ 2 & 80 & 90 & 110 \\ 3 & 110 & 140 & 120 \\ \hline \end{array}$$

Assign the programmers to the programmes in such a way that the total computer time is least.

Solution :- Using Hungarian Assignment method we subtract the smallest element of each row and each column from the corresponding row and column elements. Thus we get at least one zero in each row and each column in the reduced matrix. Now we make an assignment in the row having a single zero by enrectangling $\boxed{}$ it and cross (x) all other zeros in its column. Repeating the process for columns also, we have the following table

$$\begin{array}{|ccc|} \hline 40 & 10 & \boxed{0} \\ \cancel{0} & \boxed{0} & 30 \\ \boxed{0} & 20 & 10 \\ \hline \end{array}$$

As each row and each column has one and only one assignment an optimal assignment has been made Thus the optimum solution is

assign 1 to C, 2 to B and 3 to A.

Minimum computer time will be

80+90+110 = 280 minutes.

Problem 9 :-Solve the following assignment problem which minimises the total man hours.

$$\begin{array}{ll} & \begin{array}{cccc} A & B & C & D \end{array} \\ \text{Jobs} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 10 & 25 & 15 & 20 \\ 15 & 30 & 5 & 15 \\ 35 & 20 & 12 & 24 \\ 17 & 25 & 24 & 20 \end{pmatrix} \end{array}$$

Solution :- Reduce the effectiveness matrix by subtracting smallest element of each row (column) from the corresponding row (column) elements. In the reduced matrix make assignments in rows and colums that have single zeros. Thus we have an optimum assigment table.

$$\begin{array}{|cccc|}
\hline
\boxed{0} & 7 & 5 & 7 \\
10 & 17 & \boxed{0} & 7 \\
23 & \boxed{0} & \cancel{0} & 9 \\
\cancel{0} & \cancel{0} & 7 & \boxed{0} \\
\hline
\end{array}$$

optimum solution is to assign.

Job 1 to A, Job 2 to c, Job 3 to B and Job 4 to D.

Total minimum time will be $\boxed{55}$ hours.

Problem 10 :- solve the following assignment problem.

	1	2	3	4
A	10	12	19	11
B	5	10	7	8
C	12	14	13	11
D	8	15	11	9

Soultion :- Reduce the cost matrix by subtracting smallest element of each row (column) from the corresponding row (column) elements. In the reduced matrix make assignments in rows and columns that have single zeros. Thus we have the optimum assignment table.

$$\begin{array}{|cccc|}
\hline
\cancel{0} & \boxed{0} & 7 & 1 \\
\cancel{0} & 3 & \boxed{0} & 3 \\
1 & 1 & \cancel{0} & \boxed{0} \\
\boxed{0} & 5 & 1 & 1 \\
\hline
\end{array}$$

Following optimum assignment schedule results

A → 2, B → 3, C → 4 and D → 1

The minimum cost for this assignment comes out to be 38.

Problem 11 :-

The following is the cost matrix of assigning 4 clerks to 4 key punching jobs. Find the optimal assignment if clerk I cannot be assinged to job 1

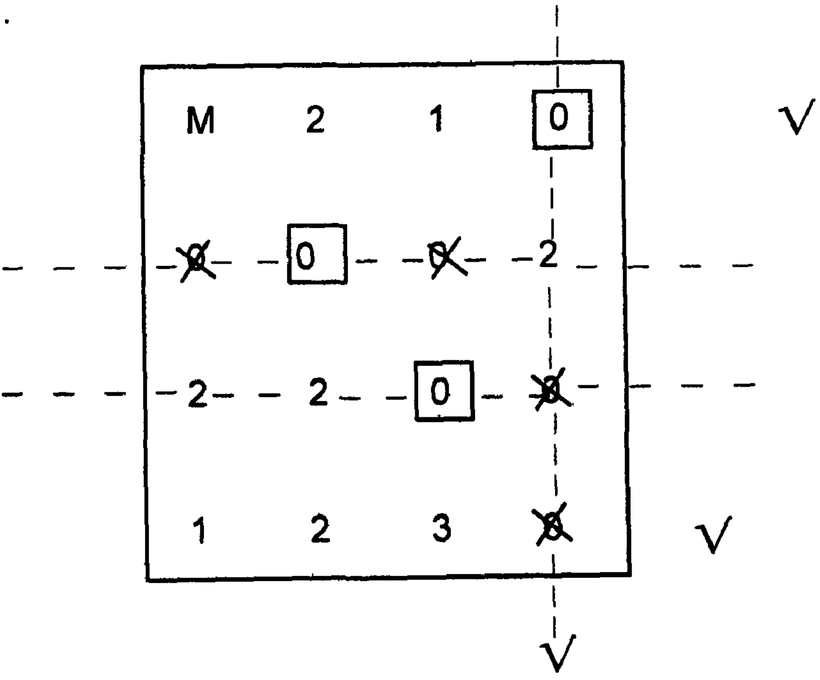
	Job			
clerk	1	2	3	4
1	-	5	2	0
2	4	7	5	6
3	5	8	4	3
4	3	6	6	2

What is the minimum total cost?

Solution :- Reduce the cost matrix by subtracting smallest element of each row (column) from the corresponding row (column) elements. In the reduced matrix make assignments in rows and columns that have single zeros. Thus we have

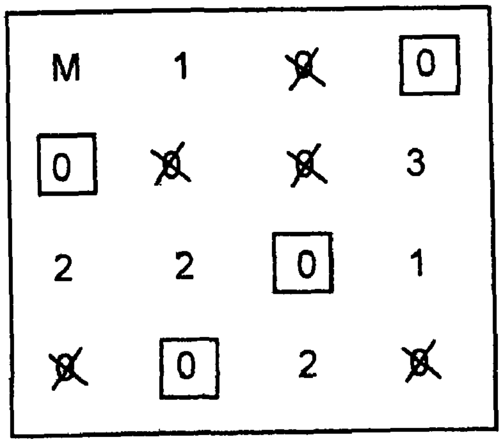
Initial Iteration :-

Draw the minimum number of lines to cover all the zeros of the reduced matrix. See the below table.



$$\begin{array}{cccc|c} M & 2 & 1 & \boxed{0} & \surd \\ \cancel{0} & \boxed{0} & \cancel{0} & 2 & \\ 2 & 2 & \boxed{0} & \cancel{0} & \\ 1 & 2 & 3 & \cancel{0} & \surd \\ \hline & & & \surd & \end{array}$$

Final Iteration :- Modify the reduced cost matrix by subtracting element 1 from all the elements not covered by the lines and adding the same at the intersection of two lines. See the below table.



$$\begin{array}{cccc} M & 1 & \cancel{0} & \boxed{0} \\ \boxed{0} & \cancel{0} & \cancel{0} & 3 \\ 2 & 2 & \boxed{0} & 1 \\ \cancel{0} & \boxed{0} & 2 & \cancel{0} \end{array}$$

Since the number of assignments is equal to the order of the matrix an optimum solution is reached. Also since there are at least two zeros for assignment in row 2 and row 4 as well as in column 1 and column 2 there exists an alternative assignment schedule. The optimum solution is Assign clerk 1 to job 4, clerk 2 to job 1, clerk 3 to job 3 and clerk 4 to job 2 (or) clerk 1 to Job 4, clerk 2 to job 2, clerk 3 to job 4 and clerk 4 to job 1.

Total minimum cost will be $\boxed{14.}$

Exercise problem :-

1) Four professors are each capable of teaching any one of four different courses. Class preparation time in hours for different topics varies from professor to professor and is given in the table below. Each professor is assigned only one course so as to minimise the total course prepartion time for all courses.

	1	2	3	4
A	2	10	9	7
B	15	4	14	8
C	13	14	16	11
D	4	15	13	9

Answers :- A → 3, B → 2, C → 4, D → 1 minimum total time will be 28 hours.

(2) Consider the problem of assigning five jobs to five persons. The assignment costs are given as follows.

(10)

	1	2	3	4	5
A	8	4	2	6	1
B	0	9	5	5	4
C	3	8	9	2	6
D	4	3	1	0	3
E	9	5	8	9	5

Determine the optimum assignment schedule.

[Answer :- A → 5, B → 1, C → 4, D → 3, E → 2

The minimum cost of assignment is 9]

3) The head of the department has five jobs A, B, C, E, E and five subordintes v, w, x, y and z. The number of hours each man would take to perform each job is as follows.

$$\begin{array}{c|ccccc|} & v & w & x & y & z \\ A & 3 & 5 & 10 & 15 & 8 \\ B & 4 & 7 & 15 & 18 & 8 \\ C & 8 & 12 & 20 & 20 & 12 \\ D & 5 & 5 & 8 & 10 & 6 \\ E & 10 & 10 & 15 & 25 & 10 \end{array}$$

How should the jobs be allocated to minimize the total time?

Answer :

A → x, B → w, C → v, D → y, E → z

minimum total time is $\boxed{45}$

(4) Solve the following travelling salesman problem.

	1	2	3	4	5	6	7
1	-	6	12	6	4	8	1
2	0	-	10	5	4	3	3
3	8	7	-	11	3	11	8
4	5	4	11	-	5	8	6
5	5	2	7	8	-	4	7
6	6	3	11	5	4	-	2
7	2	3	9	7	4	3	-

Answer : 1 → 7 → 2 → 3 → 5 → 6 → 4 → 1

Minimum cost will be $\boxed{31.}$

UNIT - 5

5.1: Transportation Problem :

Introduction :-

The Transportation problem is one of the subclasses of L.P.Ps in which the objective is to transport various quantities of single homogeneous commodity that are initially stored at various origins to different destinations in such a way that the total transportation cost is minimum. To achieve this objective we must know the amount and location of available supplies and the qantities demand. In addition we must know the costs that result from transporting one unit of commodity from various origins to various destinations.

5.2 Mathematical formulation of T.P. :-

A transportation problem can be stated mathematically as a linear programming problem as below :-

$$\text{Minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\, c_{ij}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i \quad i = 1......m$$

$$\sum_{j=1}^{n} x_{ij} = b_j \quad j - 1......n$$

$$x_{ij} \geq 0 \text{ for all } i \text{ and } j$$

where a_i = quantity of commodity available at origin i.

b_j = quantity of commodity needed at distination j.

c_{ij} = cost of transporting one unit of commodity from origin i to destination j

and x_{ij} = quantity transported from origin i to destination j.

The above transportation problem can also be portrayed in a tabular form.

	1	2	--------	-		n	supply
1	x_{11} c_{11}	x_{12} c_{12}	--------	--------	---	x_{1n} c_{1n}	a_1
2	x_{21} c_{21}	--------	--------		.	x_{2n}	-
Origin	-----	--------	------	------	------		-
m	x_{m1} c_{m1}	x_{m2} c_{m2}	--------	--------		x_{mn} c_{mm}	a_m
Demand	b_1	b_2	--------	--------		b_n	

The transportation table represents a matirx with in a matrix. The one is the cost matrix representing unit transportation costs c_{ij} indicating the cost of shipping a unit from the ith origin to the jth destination super imposed on this matrix is the matrix of transportation variables x_{ij} indicating the amount shipped from ith source to the jth destination. Right and bottom sides of the transportation table point out the amounts of supples a_i available at source i and the amount demanded b_j at destination j.

5.3 : Finding Initial Basic feasible solution:-

The triangularity of all basis in a T.P makes it possible to assign an initial basic feasible solution to a transportation problem in such a manner that all the rim requirements are satisfied with exactly (m+n-1) allocations in independent positions. This can be achieved either by inspection (or) by following some simple rules. We describe the three most commonly used methods.

5.4 : North corner Rule :- (NWC method)

Step 1:- Starting with the cell at the upper left (north-west) corner of the transportation matrix we allocate as much as possible. So that either the capacity of the first row is exhausted (or) the destination requirement of thr first column is satisfied.

(ie) $x_{11} = \min (a_1, b_1)$

Step 2 : If $b_1 > a_1$ we make down vertically to the second row and make the second allocation of magnitude. $x_{21} = \min (a_2, b_1 - x_{11})$ in the cell (2,1)

If $b_1 < a_1$ we move right horizontally to the second column and make the second allocation of magnitude $x_{12} = \min (a_1 - x_{11}, b_2)$ in the cell (1,2)

If $b_1 = a_1$ there is a tie for the second allocation. One can make the second allocation of magnitude.

$x_{12} = \min (a_1 - a_1, b_1) = 0$ in the cell (1,2)

(or) $x_{21} = \min (a_2, b_1 - b_1) = 0$ in the cell (2,1)

Step 3 :- Repeat steps 1 and 2 moving down towards the lower right corner of the transportation table unitl all the rim requirements are satisfied.

Illustration :- Obtain an initial basic feasible solution to the following transportation problem

	D	E	F	G	Available
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Requirements	200	225	275	250	

Soution : Since $\sum a_i = \sum b_i$ = 950 there exists a feasible solution to the transportation problem. We obtain initial feasible solution as follows.

The transportation table of the given problem has 12 cells. Following norht west corner method the first allocation is made in the cell (1,1) the magnitude being x_{11} = min (250,200) = 200. The second allocation is made in the cell (1,2) and the magnitude of the allocation is given by x_{12} = min (250-200, 225) = 50

The third allocation is made in the cell (2,2) the magnitude being x_{22} =min (300, 225-50) = 175. The magnitude of fourth allocation in the cell (2,3) is given by x_{23} = min(300-175, 275) 125. The fifth allocation is made in the cell (3,3) the magnitude being x_{33} = min (400, 275-125) = 150 and the sixth allocation is made in the cell (3,4) with magnitude x_{34} = min(400-150, 250) 250. Hence an inital basic feasible solution to given T.P has been obtained and is displayed uin Table (below).

200 11	50 13	17	14	250
16	175 18	125 14	10	300
21	24	150 13	250 10	400
200	225	275	250	

The transportation cost according to the above route is given by

z = (200 x 11) + (50 x 13) + (175 x 18) + (128 x 14) + (150 x 13) + (250 x 10)

z = 12,200 Ans.

5.5. : Least cost method :-

(Intuitively Best method)

Step 1:- Determine the smallest cost in the cost matrix of the transportation table. Let it be c_{ij} Allocate x_{ij} = min (a_i, b_j) in the cell (i, j)

Step 2 :- If $x_j = a_i$ cross off the i^{th} row of the transportation value and decrease b_j by a_i Go to step (3)

If $x_{ij} = b_j$. cross off the j^{th} column of the transportation table and decrease a_i by b_j Go to setp 3:-

If $x_{ij} = a_i = b_j$ cross off either the i^{th} row (or) the j^{th} column but not both.

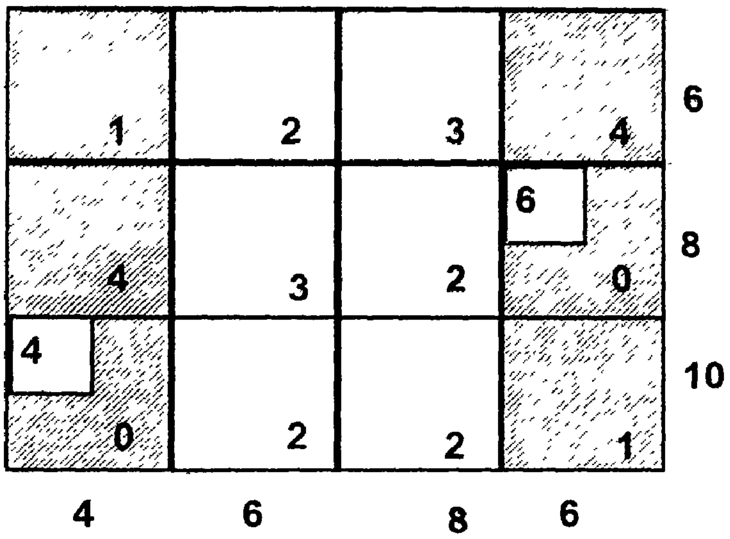
Step 3 :- Repeat stpes 1 and 2 for the resulting reduced transportation table until all the rim requirements are satisfied. Whenever the minimum cost is not unique make an ordinary choice among the minima.

Illustration :- Obtain an initial basic feasible solution to the following T.P using the matrix minima method.

	D_1	D_2	D_3		Supply
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
	4	6	8	6	

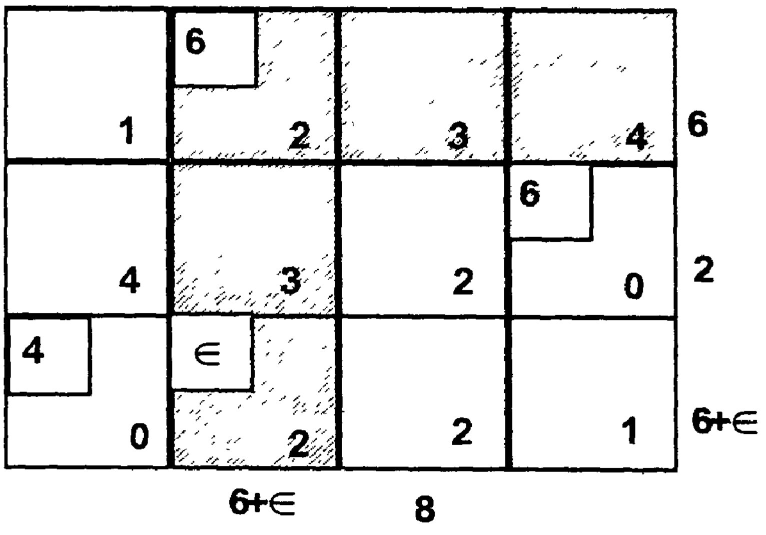
Demond

Solution :- Since $\sum a_i = \sum b_j = 24$ there exists feasible solution to the transportation problem. The transportation table has 12 cells. Following the matrix minima method. The first allocation is made in the cells (3,1), the magnitude being $x_{31} = 4$. This satisfies the requirement at destination D_1 and thus we cross off the first column from the table the second allocation is made in the cell (2,4) magnitude $x_{24} = \min(6,8) = 6$ cross off the fourth column of the table.



1	2	3	4	6
4	3	2	$\boxed{6}$ 0	8
$\boxed{4}$ 0	2	2	1	10
4	6	8	6	

Table - 1



1	$\boxed{6}$ 2	3	4	6
4	3	2	$\boxed{6}$ 0	2
$\boxed{4}$ 0	$\boxed{\in}$ 2	2	1	$6+\in$
	$6+\in$	8		

Table - 2

This yields the first table. There is again a tie for the third allocation we choose arbitarily the cell (1,2) and allocate $x_{12} = \min(6,6) = 6$ there cross off either the second column (or) first row. We choose to cross off the first row of the table. The next allocation of magnitude. $x_{33} = 0$ (or) $\in$ where $\in \rightarrow 0$ is made in the cell (3,2) cross off the second column getting in table-2.

We choose arbitarily again to make the next allocation in cell (2,3) of magnitude $x_{23} = \min(2,8) = 2$ cross off the second row. This gives table 3. The lost allocation of magnitude. $x_{33} = \min(6,6) = 6$ is made in the cell (3,3)

1	6 2	3	4	
4	3	2 2	6 0	2
4 0	∈ 2	6 2	1	6

Table - 3

1	6 2	3	4
4	3	2 2	6 0
4 0	∈ 2	6 2	1

Table - 4

Now all the rim - requirements have been satisfied and hence an initial feasible solution has been determined. This solution is displayed in Transportation Table - 4.

Since the cells do not form a loop, the solution is basic one. Moreover the solution is degenerate also. The transportations cost according to the above route is givne by

z = (6x2) + (2x2) + (6x0) + (4x0) + (2xt∈) + (6x2)

= 28 + 2 = 2 ∈28 As ∈→ 0

Ans = 28

5.6 Vogel's approximation Method ;- (VAM (or) penalty method)

Step 1 :- Calculate penalties by taking differences between the minimum and next to minimuim unit transportatin costs in each row and each column.

Step 2:- Circle the largest row Difference (or) column Difference. In the event of a tie choose either.

Step3 :-

Allocate as much as possible in the lowest cost cell of the row (Column) having a circled Row (column) difference.

Step 4 :- In case the allocation is made fully to a row (or column) ignore the row (or column) for further consideration by crossing it.

Step 5 :- Revise the differences given again and cross out the earlier figures Go to step (2).

Step 6:- continue the Procedure unitl all rows and columns have been crossed out (ie) distribution is complete.

Illustration :- Use vogel's Approximation method to obtain an initial basic feasible solution of the transportation problem.

$$\begin{array}{c|cccc|c} & & & & & \text{Available} \\ \hline & 11 & 13 & 17 & 14 & 250 \\ & 16 & 18 & 14 & 10 & 300 \\ & 21 & 24 & 13 & 10 & 400 \\ \hline \text{Demand} & 200 & 225 & 275 & 250 & \end{array}$$

Solution :- since there exists a feasible solution we obtain an initial BFS using vogel's Approximation method. The differences between the smallest and next to the smallest costs in each row and each column are first computed and displayed inside the parenthesis against the respective rows and columns. The leagest of these difference is (5) and is associated with the first column of the transportation table.

See the below table - 1

Since the minimum cost in the first column is $c_{11} = 11$. we allocate x_{11} = mnin (250,200) = 200 in the cell (1,1). This exhausts the requirements of the first column and therefore we cross off the first column. The row and column differences are now computed for the results reduced transportation table. See the below table - 2. The largest of these is (5) which is associated with the second column. Since $c_{12} = 13$ is the minimum cost we allocate x_{12} = min(50,225) = 50

Table - 1

200 11	13	17	14	250 (2)
16	18	14	10	300 (4)
21	24	13	10	400 (3)
200	225	275	250	
(5)	(5)	(1)	(0)	

Table - 2

50 13	17	14	50 (2)
18	14	10	300 (4)
24	13	10	400 (3)
225	275	250	
(5)	(1)	(0)	

This exhausts the availability of first row and therefore we cross off the first row continuing in this manner The subsequent reduced transportation tables and the differences for the surviving rows and columns are shown below :

175 18	14	10	300 (4)
24	13	10	400 (3)
175	275	250	
(6)	(1)	(0)	

14	125 10	125 (4)
13	10	400 (3)
275	250	
(1)	(0)	

275 13	125 10	400
275	125	

Eventually the basic feasible solution is shown in the below table :-

200 11	50 13	17	14
16	175 18	14	125 10
21	24	275 13	125 10

The transportation cost according to this route is given by

$$z = 200x11 + 50x13 + 175x18 + 120x10 + 275x13 + 125x10 = 12,075$$

5.7 : Degenercy in Transportation Problem :-

We have seen that for an mxn transportation table, the nuber of basic cell must be m+n-1. The basic solution will degenerate whenever number of basic cells is less than m+n-1. Degeneracy can occur in the initial solution (or) it may arise in some subsequent iterations. We now discuss a procedure to deal with the problem of degeneracy.

Case 1:- Degeneracy in the initial solution:-

To resolve degeneracy at the initial solution a very small quantity $\in$ (>0) is allocated in an unoccupied cell so as to get m+n-1 number of unoccupied cells. In a minimization transportation problem it is better lowest transportation costs. In some cases $\in$ must be added in one of those unoccupied cell which make possible the determination of u_i and v_j uniquely.

The quantity $\in$ is considered to be so small that if it is transferred to an occupied cell it does not change the quantity of allocation. That is $x_{ij} + \in = x_{ij} - \in = x_{ij}$ but $\in - \in = 0$. Also does not affect the total transportation cost of the allocation. Hence the quantity $\in$ is used to evaluate unoccupied cells and once the purpose is over, $\in$ must be removed from the scene.

Case 2 :- Degenercy at subsequent iterations :-

To resolve degeneracy which occur during optimality test the quantity $\in$ may be allocated to one or more cells which have become unoccupied recently to have m+n-1 number of occupied cells in the new solution. It may be removed once the purpose is over.

5.8 : Transportation Algorithm :- (MODI method)

Various steps involved in solving any transportation problem may be summarised in the following iterative procedure.

Step 1 :- Find the initial basic feasible solution by using any of the three methods discussed above.

Step 2 :- Check the number of occupied cells. If there are less than m+n-1, there exists degeneracy and we introduced a very small positive assignment of ($\in$=0) in suitable independent positions. So that the number of occupied cell is exactly equal to m+n-1

Step 3 :- For each occupied cells in the current solution solve the system of equations

$$u_i + v_j = c_{ij}$$

Starting initially with some $u_i=0$ (or) $v_j=0$ and entering the successively the values of u and v in the transportation table margins.

Step 4 :- Compute the net evaluations

$z_{ij} = c_{ij} = u_i + v_j = u_i + v_j = c_{ij}$ for all unoccupied basic cells and enter them in they upper right corners of the corresponding cells.

step 5 :- Examine the sign of each $z_{ij} - c_{ij}$ If all $z_{ij} - c_{ij} \leq 0$ then the current basic feasible solution is an optimum one. If at least one $z_{ij} - c_{ij} \geq 0$ select the unoccupied cells having the largest positive net evaluation to enter the basis.

<u>Step 6 :-</u> Let the unoccupied cell ((r,s)) enter the basis. Allocate an unknown quantity say (0) to the cell (r,s). Identify a loop that starts and ends at the cell (r,s) and connects some of the basic cells. Add and subtract interchangeably O to and from the transition cells of the loop in such a way that the rim requirements remain satisfied.

<u>Step7 :-</u> Assign a maximum value to O in such a way that the value of one basic variable becomes zero and the other basic variables remain non-negative. The basic cell whose allocation has been reduced to zero, leaves the basis.

<u>Step 8.-</u> Return to step 3 and repeat the process until an optimum basic feasible solution has been obtained

Problem 1 :- Consider the following transportation

Problem : Godowns

Factory	1	2	3	4	5	6	Stock available
A	7	5	7	7	5	3	60
B	9	11	6	11	-	5	20
C	11	10	6	2	2	8	90
D	9	10	9	6	9	12	50
Demand	60	20	40	20	40	40	

It is not possible to transport any quantity from factory B to Godown 5.

<u>Determine :-</u>

a) Initial solution by vogel's Approximation method

b) Optimum basic feasible solution.

c) Is the optimum solution unique?

If not, find the alternative optimum basic feasible solution.

<u>solution :-</u> a) Since it is not possible to transport any quantity from Factory B to Godown 5, we assign a very high cost to the cell (2,5) say M. Then using vogel's Approximation method to find the initial basic feasible solution. We obtain the below table - 1.
column differences:-

Since the number of occupied cells is 8. (ie) less than (4+6-1) there is degenercy in the initial solution. To overcome degenercy we allocated a small quantity ($\in$)>0 in the cell (1,5) being the unoccupied cell having the lowest transportation cost.

						Row Difference			
7	20 5	7	∈ 7	40 5	3	(2)	(2)	(4)	(6)
10 9	11	10 6	11	M	5	(1)	(1)	(1)	(3)
11	10	30 6	20 2	40 2	8	(0)	(4)	(2)	-
50 9	10	9	6	9	12	(3)	(0)	(0)	(0)
(2)	(5)	(0)	(4)	(3)	(2)				

b) To test the initial solution for optimality, we use MODI method:-

We calculate the unknown numbers u_i (i=1,2,3,4) and $v_j c_j$=1,2,3,4,5,6) starting with u_i =0 and using the relations $c_{ij} = u_i+v_j$ for occupied cells as shown below.

$c_{12} = u_1+v_2 \longrightarrow 5 = 0+v_2 \longrightarrow v_2 = 5$

$c_{15}=u_1+v_5 \longrightarrow 5 = 0+v_5 \longrightarrow v_5=5$

$c_{16} = u_1+v_{16} \longrightarrow 3 = 0+ v_6 \longrightarrow v_6= 3$

$c_{21} = u_2+v_1 \longrightarrow 9 = -3+ v_1 \longrightarrow v_1= 12$

$c_{23} = u_2+v_3 \longrightarrow 6 = u_2+ 9 \longrightarrow u_2= -3$

$c_{33} = u_3+v_3 \longrightarrow 6 = -3+ v_3 \longrightarrow v_3= 9$

$c_{34} = u_3+v_4 \longrightarrow 2 = -3+ v_4 \longrightarrow v_4= 5$

$c_{35} = u_3+v_5 \longrightarrow 2 = u_3+ 5 \longrightarrow u_3= -3.$

$c_{41} = u_4+v_1 \longrightarrow 9 = u_4+ 12 \longrightarrow u_4= -3.$

The net evaluation for the unoccupied cells are vown calculated as below.

$z_{11} - c_{11} = u_1 + v_1 - c_{11} = 0 + 12 - 7 = 5$

$z_{13} - c_{13} = u_1 + v_3 - c_{13} = 0 + 9 - 7 = 2$

$z_{14} - c_{14} = u_1 + v_4 - c_{14} = 0 + 5 - 7 = -2$

$z_{22} - c_{22} = u_2 + v_2 - c_{14} = -3 + 5 - 11 = -9$

$z_{24} - c_{24} = u_2 + v_4 - c_{24} = -3 + 5 - 11 = -9$

$z_{25} - c_{25} = u_2 + v_5 - c_{25} = -3 + 5 - M = 2-M$

and so on.

Making use of the above information. We get initial iteration:-

Initial Iteration : - Inroduce the cell (1,1) and

Drop the cell (1,5)

(5) θ 7	20 5	(2) 7	(-2) 7	$\in$ $-\theta$ 5	40 3
10 $-\theta$ 9	(-9) 11	10 $+\theta$ 6	(-9) 11	2-M M	(-5) 5
(-2) 11	(-8) 10	30 $-\theta$ 6	20 2	40 $+\theta$ 2	(-8) 8
50 9	(-8) 10	(-3) 9	(-4) 6	(-7) 9	(-12) 12

Since $z_{11} - c_{11}$ (=5) is the most positive cell (1,1) enters the basis. We allocate an unknown quantity θ to this cell and identify a closed loop involving basic cellsd around this entering cell making $\pm\theta$ adjustments' in the corner cells of the loop. We observe that the maximum value that θ can admit $\in$ is . Thus the present occupied cell (1,5) becomes unoccupied in the next iteration

Final Iteration : - Optimum solution is obtained and shown in the below table.

						u_i
$\in$ 7	20 5	(-3) 7	(-7) 7	(-5) 5	40 3	0
10 9	(-4) 11	10 6	(-9) 11	(2-M) M	(0) 5	2
(-2) 11	(-3) 10	30 6	20 2	40 2	(-3) 8	2
50 9	(-3) 10	(-3) 9	(-4) 6	(-7) 9	(-7) 12	2
7	5	4	0	0	3	v_j

In the above transportation table. Since all the net evaluations are non-positive, an optimum solution has been obtained. Hence the optimum solution is $x_{11} = 0$, $x_{12} = 20$, $x_{16} = 40$. $x_{21} = 10$, $x_{33} = 30$, $x_{34} = 20$, $x_{35} = 40$, and $x_{41} = 50$.

The minimum transportation cost is

20x5 + 40x3 + 10x9 + 30x + 20x2 + 40x2 + 50x9 + 7 x $\in$

(ie) 1120 + 7

$\boxed{1120}$ as $\in \to 0$

(c) Since $z_{26} - \overline{c_{26}} = 0$ there exists an alternative optimum solution. Letting the unoccupied cell (2,6) enter the basis it is easily seen that the currently occupied cell (1,1) becomes unoccupied in the next iteration. In the revised transportation table, the following alternate optimum solution is obtained.

$x_{12} = 20$ $x_{16} = 40$ $x_{21} = 10$ $x_{23} = 10$ $x_{26} = 0$ x_{33} 30 $x_{34} = 20$ $x_{35} = 40$ $x_{41} = 50$ with minimum transportation cost as $\boxed{1120}$

Problem - 2:-

Consider four bases of operations B_i and three targets T_j. The tons of bombs per aircraft from any base that can be delivered to any target are given in the following table.

$$\text{Base } (B_1) \qquad \begin{array}{c|ccc|} & T_1 & T_2 & T_3 \\ \hline B_1 & 8 & 6 & 5 \\ B_2 & 6 & 6 & 6 \\ B_3 & 10 & 8 & 4 \\ B_4 & 8 & 6 & 4 \\ \hline \end{array}$$

The daily sortie capability of each of the four bases is 150 sorties per day. The daily requirement in sorties over each individual target is 200. Find the allocaiton of sorties from each base to each target which maximizes the total tonnage over all the three targets explaining each step.

Solution : Since the problem is to find the maximum of the total tonnage over the three targets, we replace each entry of the transportation value by the difference between the maximum of all these (=10) and the corresponding element. Then by using vegel's approximation method for initial basic feasible solution and MODI method for optimum solution. We have.

				u_i
	$\boxed{50}$ 2	$\boxed{50}$ 4	$\boxed{50}$ 3	0
	(-3) 4	(-1) 4	$\boxed{150}$ 4	-1
	$\boxed{150}$ 0	(0) 2	(-3) 6	2
	(0) 2	$\boxed{150}$ 4	(-1) 6	0
v_j	2	4	5	

Hence an optimum solution to the original problem is

$x_{11} = 50 \quad x_{12} = 50 \quad x_{23} = 150 \quad x_{31} = 150 \quad x_{42} = 150$

Total maximum tonnage will be

$z = 50x8 + 50x6 + 50x5 + 150x6 + 150x10 + 150x6$

$= \boxed{4250.}$

5.9 : Unbalanced Transportation problems :-

For a feasible solution to exist in a transportation problem it is necessary that the total supply must equal total demand. That is $\sum_{i=1}^{m} a_i = \sum_{j=1}^{m} b_j$. But a situation may arise. When the total available supply is not equal to the total available supply is not equal to the total requirement. Such type of T.P's are called unbalanced transportation problem.

Case 1:- When the supply exceeds demand the constraints of the transportation problem will appear as

$$\sum_{j=1}^{m} x_{ij} \leq a_i \quad i = 1,\ldots\ldots\ldots\ldots m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad j = 1\ldots\ldots\ldots\ldots\ldots n$$

and $x_{ij} \geq 0$ for all i and j

Introducing slack vairables x, n+i (i=t.......m) in the first m constraints. We get

$$\sum_{i=1}^{n} x_{ij} + x_{i,\, n+1} = a_i$$

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_{ij} + x_{i,\, n+1} \right) = \sum_{i=1}^{m} a_i$$

$$\sum_{i=1}^{m} x_{i,\, n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j =$$

excess of availability.

Thus if we denote this excess availability by b_{n+1} the modified general transportation problem can be restated as

$$\text{Minimize} \quad z = \sum_{i=1}^{m} \sum_{j=1}^{n+1} x_{ij} c_{ij}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} + x_{i,\,n+1} = a_i \quad i = 1,\ldots\ldots.m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad j= 1\ldots\ldots\ldots\ldots n+1$$

$x_{ij} \geq 0$ for all i and j

where $c_{i,\,n+1} = 0$ for $i = 1,\ldots\ldots\ldots.m$

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_j + b_{n+1}$$

This form of the problem is nothing but that of any balanced T.P and thus can be solved by MODI method.

Remarks :

When $\sum a_i \geq \sum b_j$ introduce a dummy destination in the transportation table the costs of transporting to this destination are all set equal to zero. The requirement at this dummy destination is assumed to be equal to $\sum a_i - \sum b_j$

Case 2 :-

When the demand exceeds supply, the constraints of the transportation problem will be

$$\sum_{j=1}^{n} x_{ij} = a_i \quad (i = 1\ldots\ldots.m) \text{ and } \quad \sum_{j=1}^{m} x_{ij} \leq b_j \quad (j = 1,\ldots\ldots..n)$$

In the second constraint the introduction of slack variables $x_{m+1,\,j}$ (j = 1......n) yields

$$\sum_{i=1}^{m} x_{ij} + x_{m+1,\,j} = b_j$$

$$\sum_{j=1}^{n} \left[\sum_{i=1}^{m} x_{ij} + x_{m+n,\,j} \right] = \sum_{j=1}^{n} b_j$$

$$\sum_{j=1}^{n} x_{m+1,\,j} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i = a_{m+1} \text{ (say) } (a_{m+1})$$

The modified general T.P in this case also is a balanced T.P and can be solved by MODI method.

It follows that if $\sum b_j > \sum a_i$ then a dummy source row can be added to the transportation table to account for required demand quantity. The unit transportation cost here also for the cells in the dummy row is set equal to zero.

Problem :-

A company has three plants at locations A, B and C which supply to warehouse located at D, E, F, G and H monthly plant capacities are 800,500 and 900 units respectively.

Monthly warehouse requirements are 400, 400, 500, 400 and 800 units repectively. Unit transportation costs (in Rupees) are given below.

		D	E	F	G	H
	A	5	8	6	6	3
Form	B	4	7	7	6	5
	C	8	4	6	6	4

Determine an optimum distribution for the company in order to minimize the total transportation cost.

Solution :- Since the total warehouse requirements (= 2,500 unit is greater than the total plan capacity (2200 units) the given problem is an unbalanced T.P we introduce a dummy plant having all transportation costs equal to zero and having the plant a vailability equal to 2500 - 2200 = 300 units using vogel's Approximation method for initial baisc feasible solution and MODI method for optimum solution we have.

Initial Iteration :- Introduce the cell (4,3) and drop the cell (2,5) See the below table

					u_i
(-3) 5	(-5) 8	500 $-\theta$ 6	(-2) 6	300 $+\theta$ 3	-2
400 4	(-2) 7	(1) 7	100 $+\theta$ 6	$\in$ $-\theta$ 5	0
(-5) 8	400 4	(1) 6	(-1) 6	500 4	-1
(-2) 0	(-1) 0	θ 0	300 $-\theta$ 0	(-1) 0	-6
4	5	8	6	5	v_j

First Iteration :- Introduce the cell (3,4) and drop the cell (4,4). See the below table.

					u_i
(-1) 5	(-5) 8	500 $-\theta$ 6	(0) 6	300 $+\theta$ 3	0
400 4	(-4) 7	(-1) 7	100 6	(-2) 5	0
(-3) 8	400 4	(1) 6	(1) 6	500 $-\theta$ 4	1
(-2) 0	(-3) 0	$\in$ $+\theta$ 0	300 $-\theta$ 0	(-3) 0	-6
4	3	6	6	3	v_j

Second Iteration :- Introduce the cell (3,3) and drop cell (3,5) see the below table.

						u_i
	(-2) 5	(-5) 8	200 - θ 6	(-1) 6	600 + θ 3	-1
	400 4	(-3) 7	(0) 7	100 6	(-1) 5	0
	(-4) 8	400 4	θ (1) 6	300 (1) 6	200 - θ 4	0
	(-3) 0	(-3) 0	300 0	(-1) 0	(-3) 0	-7
v_j	4	4	7	6	4	

Final Iteration : - Optimum solution is obtained and as shown below :-

						u_i
	5	8	6	6	3	0
	4	7	7	6	5	0
	8	4	6	6	4	0
	0	0	0	0	0	-6
v_j	4	4	6	6	3	

The optimum transportation schedule is A ⟶ 80 units to H ; B ⟶ 400 units to D and 100 units to G i C ⟶ 400 units to E, 200 units to F and 300 units to G. The transportation cost according to this optimum route will be Rs. 9200.

UNIT 6

OPERATIONS RESEARCH (O.R.)

6.1 : Origin and Development of O.R. :-

The term operations Research was first coined in 1940 by Mcclosky and Trefthen in a small town Bowdesy, of the united kingdom. This new science came in to existence in military context During world war II, military, management called on scientists from various disciplines and organised them into teams to assist in solving strategie and tactical problems (ie) to discuss evolve and suggest ways and means to improve the execution of various military projects By their joint efforts experience and deliberations they suggested certain approaches that showed remarkable progress. This new approach to systematic and scientific study of the operations of the system was called the operations Research (or) operational Research

6.2: Nature and charactersitic features of O.R. :-

After tracing the process of establishment and growth of operations Research, we can consider it as a source to other new sciences. Literally the word "operation" may be defined as some action that we applyto some problems (or) hypotheses and the word "Research" is organised process of seeking out facts about the same. In fact it is very difficult to define O R. mainly because of the fact that its boundaries are not clearly marked O.R. has been variouosly described as the "science of use" quantitative common sense" "scientific approach to descision making problems" etc. But only a few are commonly used and widely accepted namely

(i) O R. is the application of scientific methods techniques and tools to problems involving the operations of a system so as to provide those in control of the system with optimum solutions to the problem

- C W Churchman, R.L. Ackoff$ E.L. Arno

(ii) O.R. is the art of giving bad answers to problems which otherwise have worse answers.

(iii) O.R. is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control.

- Morse $ Kimball.

(iv) O R is applied decision theory. It was many scientific, mathematical or logical means to attempt to cope with the problems that confront the executive when he tries to achieve a thorough - going nationality in dealing with his decision problems.

(v) O.R is a scientific approach to problems solving for executive management D.W. Miller $ M K Stars. - H.M. Wagner.

(vi) OR is a Scientific knowledge through inter disciplinary team effort for the purpose of determining the best utilization of limited Resources.

Some significant features of O.R. are highlighted below:-

a) Decision making :-

Primarily O.R. is addressed to managerial decision making (or) problem solving. A major premise of O.R. that decision making, irrespective of the situation involved can be considered as a general systematic process.

(b) Scientific Approach :- O.R. employs scientific methods for the purpose of solving problems. It is a formalised process of reasoning.

(c) Objective : O.R.attempts to locate the best (or) optimal solution to the problem under consideration. For this purpose it is necessary that a measure of effectiveness is defined which is based on the goals of the organisation. This measure is then used as the basis to compare the alternative coures of action.

(d) Inter - disciplinary Team Approach :-

O.R. is inter-disciplinary in nature and requires a team approach to a solution of the problem. Managerial problems have economic physical, psychological, biological sociological and engineering aspects. This requires a blend of the people with expertise in the areas of mathematics statisties engineering, economics, mangaement, computer science and soon.

(e) Digital computer

Use of a digital computer has become an integral part of the O.R approach to decision making. The computer may be required due to the complexity of the model, volume of data required and the computations to be made.

6.3 : Models in O.R. :-

Operations Research makes the structure of its algebraic reasoning robust in the face of an inevitable incertainty in the values of the unknown parameters it has to empty.

A model in OR is a simplified representation of an operation (or) a process in which only the basic aspects (or) the most important features of a typical problem under investigation are considered constructing a model aids in putting the complexities and possible uncertainties attending a decision - making. Problem into a logical framework amenable to comprehensive analysis. Such a model clarifies the decision alternatives, their anticipated effects, indicates the relevant data for analysing the alternatives and leads to informative conclusions. In short, the model is a vehicle used to arrive at a well-structured view of reality. There are several models in each area of business (or) industrial activity. For instance, an account model is a typical budget in which business accounts are referred, to with the intention of providing measurements such as rate (or) expenses quantity sold etc a mathematical equation may be considered to be a mathematical model in which a relationsship between constants and variables is represented.A model which has the possibility of measuring observations may be called a quantitative model, a product, a drvice (or) any tangible thing used for experimentation may represent a physical model.

The word "model" has several shades of meaning all of which are relevant to OR. For instance "a model" may act as a substitute for representing reality such as small scale model locomotive may imply some sort of idealization, such as model plan for exployment scheme etc. Following are the main charactersitics that a good model for operations Research study should have

1. A good model should be capable of taking into account new formulations without having any significant change in its frame.
2) Assumptions made in the model should be as small as possible
3) It should be simple and coherent. Number of variables used should be less.
4) It should be open to prarmetric type of treatment.
5) It should not take much time in its construction fo any problem.

Advantages of a Model :-

The cheif advantages of a Model are ;

a) Through a model the problem under consideration becomes controllable
b) It provides some logical and systematic approach to the problem.
c) It indicates the limitations and scope of an activity.
d) Models help in incorporating useful tools that eleminate duplication of methods applied to solve any specific problem.
e) Models help in finding avenues for new Research and improvements in a system.
f) It provides economic descriptions and explanations of the operations of the system they represent.

Operations Research And Decision - Making :-

Operations Research uses the method of science to understand and explain the phenomena of operating systmems. It divses the theories (models) to explain these phenomena, uses these theories to describe what takes place under altered conditions and checks these predictions against new observations.

Thus

"Operations Research is a tool employed to increase the effectivenes of managerial decisions as an objective supplement to the subjective feeling of the decision maker".

For instance in distribution (or) allocation areas OR may suggest the best locations for agencies, warehouses as well as the most economical kind of transportation in marketing areas it may aid in indicating the most profitable type use and size of advertising compaigns in regard to available financial limit O R. may suggest alternative courses of action when a problem is analysed and a solution is attempted. However the study of complex problems by OR techniques becomes useful only when a choice between two (or) more courses of action is possible.

O.R. may be regarded as a tool that enables the decision maker to be objective in creating alternatives and choosing an alternative which is best from among these. Decision making is not only the headache of management rather all of us make decisions.We daily decide about many minor and major issues. The essential characteristics of all decisions are

1) Objective
2) alternatives at the disposal.
3) Influencing factors

Once these characteristics are known one can think of improving the characteristics so as to improve upon the decision itself.

Let us consider a situation where a decision concerns spending summer vocations at a hill resort. The next problem may be to decide the mode of conveyance from amongst the alternatives, i train bus and a taxi;

At the first level of decision making, bus is chosen as the mode of conveyence just by intution (may be at random). At the second level of decision making the three conveyances are compared and it is decided qualitatively that the bus will be preferred since it is less time consuming compared to the train and cheaper as compared to taxi. At the third level of decision making the three alternatives are compared and it is suggested that the bus will be chosen as it will be taking only half the time taken by train and shall be 40% less costiler than the taxi.

Although outcome of all these decisions is the same one can easily judge the quality of each decision. We may brand the first decision as "bad" Since it is highly emotional while we may call the second decisions as "good" since it is scientific though qualitative. The third decision is droubtlessly the best as it is scientific as well as quantitative

It is this scientific quantifications used in OR that helps management to make better decisions.

Advantages of O.R. Approach in Decisions Making :-

Following are the salient advantages of an operations Research study approach in decision making:-

(i) Better decisions:- O.R models frequently yield actions that do improve on intuitive decision making. A situation may be complex so that the human mind can never hope to assimilate all the significant factors without the aid of O.R guided computer analysis.

(ii) Better co-ordination :-

sometimes operations Research has been instrumental in bringing order out of chaos. For instance, OR - oriented planning model becomes a vehicle for co-ordinating marketing decisions with in the limitations imposed on manufacturing capabilities.

(iii) Beter control :- The managements of large organizations recognize that it is extremely costly to require continuous executive supervision over routine decisions. An OR approach thereby gained new freedom to the excutives to devote their attention to more pressing matters. The most frequently adopted application in this category deals with production scheduling and inventory replemishment.

(iv) Better systems:- Often an OR study is initiated to analyse a particular decision problem. Such as whether to open a new warehouse. After words the approach is further developed into a system to be employed repeatedly thus the cost of undertaking the first application may produce benefits.

6.4 : Applications of OR :-

OR is mainly concerned with the techniques of applying scientific knowledge besides the developement of science. It provides an understanding which gives the expert / manager new insights and capabilities to determine better solutions in his decision making problems with great speed competence and confidence In recent years OR has successfully entered many different areas of research in defence, Government, Service, organisations and Industry. We briefly describe some applications, of O.R in the functional areas of management.

Finance, Budgeting and Investments :-

(i) Cash flow analysis, long range capital requirements, dividend policies investment portfolios.

(ii) Credit, policies, credit risks and delinquent account procedures

(iii) Claim and complaint procedure.

Marketing.

(i) Product, selection, timing competitive actions.

(ii) Advertising media with respect to cost and time.

(iii) Number of salesman, frequency of calling of account etc.

(iv) Effectiveness of market Research.

Physical Distribution :

(i) Location and size of warehouses, distribution centres retail outlets etc.

(ii) Distribution policy.

Purchasing Procurement and exploration :-

(i) Rules for buying

(ii) Determining the quantity and timing of purchase.

(iii) Bidding policies and vendar analysis

(iv) Equipment replacement policies.

Personnel :-

(i) Forecasting the man power requirement recruitment policies and assignment of jobs.

(ii) Selection of suitable personnel with due consideration for age and skills etc.

(iii) Determination of optimum, number of persons for each service centre.

Production :-

(i) Scheduling and sequencing the production run by proper allocation of machines.

(ii) calculating the optimum product mix.

(iii) Selection location and design of the sites for the production plant.

Research and Developement :-

(i) Reliability and evaluation of alternative designs.

(ii) Control of developed projects.

(iii) Co-ordination of multiple research project.

(iv) Determination of time and cost requirements.

Besides the above mentioned applications of O.R. in the context of modern management, its use has now extended to a wide Range of problems such as the problems of communication and information, socio-economic fields and national planning.

Uses and Limitations of O.R.:-

Formulation of industrial problems may be generalised into different groups of classical problems, the package programme,for which is available for mechanisation and for manual solutions. Various problems of optimization can be brought to the model of linear progromme for which solution is available. While formulating the problem, the class of the problem is to be decided and the parameters are to be defined accordingly.

Inventory control production planning product mix, transportation problem etc are very common to the industries. The cost reduction with the help of these tools is very much powerful in comparison to any other conventional method we can enumerate the advantages of these techniques as.

(i) Optimum use of production factors :-

Linear programming techniques indicate how a manager can most effectively empty his production factors by more efficiently selecting and distributing thse elements

(ii) Improved quality decision :-

The computation table gives a clear picture of the happenings with in the basic restrictions and the possibilities of compound behaviour of the elements involved in the problem. The effect on the profitability due to changes in the production pattern will be clearly indicated in the table

eg. simplex table.

(iii) Preparation of fature managers :- These methods substitute a means for improving the knowledge and skill of young managers.

(iv) Modificaiton of mathematical solution :-

O.R. presents a possible practical solution when one exists, but if is always a responsibility of the manager to accept (or) modify the solution before its use. The effect of these modification may be evaluated from the computational steps and tables.

(v) Alternative solutions :- O.R. techniques will suggest all the alternative solution available for the same profit so that the management may decide on the basis of its stategies.

6.5 : Limitations of operations Research :-

OR has certain limitaions. However these limitations are mostly related to the time and money factors involved in its applications rather than its practical utility. These limitations are as follows.

(a) Magnitude of computations :- O.R tries to find out the optimal solution taking all the factors into account. In the modern society these factors are numberous and expressing them in quantity and establishing relationship among these, requires huge calculations. All these calculations cannot be handled manually and require electronic computers which bear very heavy cost.Thus the use of OR is limited only to very large organisations.

(b) Absence of a quantification :-

OR provides solution only when all the elements related to a problem can be quantified. The rangible factors such as price, product etc Can be expressed interms, of quantity, but intangible factors such as human relations etc cannot be quantified. Thus these intangible elements of the problem are excluded from the study though these might be equally or more important than quantifiable intangible factors as far as possible.

(c) Distance between managers and operations Research :-

O.R being specialists job requires a mathematician or a statistician who might not be aware of the business problems Similarly a manager may fail to understand the complex working of OR Thus there is a gap between one who provides the solution and one who uses the solution. Thus the manages becomes suspicious about the optimal solution. This problem is mainly of training.Both the persons should have a working knowledge of each other's jobs to have better understanding of insights of the problem and its optimal solution.

6.6 : Probability

Introduction :-

The operations Reserach techiniques discussed in the preceding chapters assume that all the data are known with certainty. This assumption is not true always. For example, the demand for electric power during the summer months can vary from year to year depending on weather conditions.

In such cases we observed or historical data to discribe the demand by a probability distribution.

Sample space and probability :-

Definition :-

The set of all possible outcomes of a random experiment is called the sample space associated with the experiment. The possible outcomes are called sample points.

Example :-

Consider the random experiment of tossing a coin once. There are three thinkable outcomes for the coin, viz to land on :

(i) Head (ii) Tail and (iii) Edge

Definition 2 :-

A sample space that consists of a finite or an infinite but countable number of sample points is called a discrete sample space. A sample space which is not discrete is called a continuous sample space

Example :-

Consider the number of customers arriving at a service winter through one hour. The sample space associated with this random experiment is

$S = \{ 0, 1, 2, 3 \ldots..\}$

Definition 3 :- An event is a subset of the sample space. An event consisting of only single sample point is calle & elementory event.

Ex:- Consider the random experiments of rolling a dice once. The associated sample space is S = { 1,2,3,4,5,6}

Then, the sunsets.

A ={1,2} B ={3,4,5,6} and C = {4} etc. all represent events of s, clearly, C is an elementary event.

Definition 4 -

Outcomes of a experiment are said to be equally likely if taking into consideration all the relevant evidences, these is no reason to expect one is preference to other

Ex · As the result of drawing a card from a well-suffled pack, any card may appear in a drawn. Thus the 52 possible outcomes are equally likely.

Definition 5 - The outcomes of a random experiment which entaie the occurence of an event a are said to be outcomes favourable to A.

Ex :- In tossing a dice, the number of outcomes favourable to the event.

E = { multiple of 3 appears}

is two namely 3 & 6

Definition 6 :-

The outcomes of a random experiment are said to be mutually exclusive if they cannot occur simultaneously.

Ex - In the case of tossing a coin H & T are mutally exclusive.

Classical Definition of probability :-

Definition 7 : - If there are n exhausive pairwise mutually exclusive and equally likely outcomes of a random experiment and if m out of these are favourable to an event A, then the probability of the happening of A, denoted by P(A) is defined by

$$P(A) = \frac{m}{n}$$

Ex .-A dice is tossed once and the dots on the face turned up observed. The sample space associated with the experiment is

S = {1,2,3,4,5,6}

If A = the event that an even number occurs, then clearly n = 6 and m = 3 Thus we have P (an even number occurs)

$$P = \frac{3}{6} = \frac{1}{2}$$

Algebra of events :-

Definition 1 :- Two events A and B are said to be equal if and only if they consist of exactly the same sample points. We denote it by writing A = B.

Ex :- Consider the random experiment of throwing a dice once. Then the two events A and B defined by A = An even number comes face up and

B = {2,4,6}

are equal.

Definition 2 :-

Let A and B be two events of some sample space Then the set of all sample points that belong to either event B is called the union of A and B and is denoted by $A \cup B$.

In other words, the occurrence of the event $A \cup B$ means that either event A occurs or event B occurs or both A and B occur ie at least one of A and B occurs.

Ex :- Consider the events.

A ={1,2} and B = {2,3,4} of

S = {1,2,3,4,5,6} Then $A \cup B$ = {1,2,3,4}

Definition 3:- Let A and B be two events of some sample space. Then the set of all sample points that belong to both the events A & B is called the intersection of A & B and is denoted by $A \cap B$.

In other words, the occurrence of the event $A \cap B$ means that both the events A&B occurs simultaneously. This is also called the joint occurrence of A & B

Example :- In the preceding example

$A \cap B$ = {2}

Definition 4 :- An event A that contains no sample point is called an impossible event and is usually denoted by

A = ϕ or A = { }

In other words an impossible event cannot occurs.

Ex: A ball is drawn at random from a bag that contains 10 red and 5 black balls then the event.

A = A green ball is drawn

is an impossible event

Definition 5:- The event consisting of all sample points not containded in the event A, is called the complementary event of A & is denoted by A'.

In other words A'. represents the event that A does not occur.

Ex :- For any sample space. S' = ϕ

Definition 6 :- Two events A andB are said to be mutually exclusive if

$A \cap B = \phi$

In other words 2 mustually exclusive events cannot occur simultaneously

Ex :- For the random experiment of tossing a coin, the 2 events defined by

A = Head appears

&

B = tail appears

are mutually exclusive.

The events being the subsets of the sample space satisfy certain laws of algebra of sets these include commutative laws.

$A \cup B = B \cup A$; $A \cap B = B \cap A$ and

associative laws,

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

These laws allow us to ectended the depinitions of union and intersection of events to cover more than two events laws of probablity :-

The following laws of probability are easily established from the classical definition of probability

1. $P(\phi) = 0$ (The probability of an impossible event if zero)
2. P(A') = 1-PA) (The sum of the probability of an evehnt and its complement is unity)
3. $P(A \cup B)=P(A)+P(B)-P(A \cap B)$ (The Addition theorems of probablitiy)

An event a such that P(A) = 1 is called a certain event thus it follows that sample space is a certain event.

Conditional probability :-

Deinition : If P(B) $\geq$, then the conditional probability of the occurrence of an event A given that the eventB has already occurred, is defined by

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Example :- An examination is Queuing was given to a management class of 50 students of which only 20 had studied probability earlier. The following result was obtained.

	passed	Failed	Total
Students who studied probability earlier	17	3	20
Students who did not study probability	19	11	30
Total	36	14	50

A student was then selected at random from the class. If

A = the student had studied probability earlieer

B = the student passed the examination,

Then $A \cap B$ = the students had studied probability earlier and passed the examination.

clearly, $P(B) = 36/50$, $P(A \cap B) = 17/50$

Thus, $P(A/B) = P(A \cap B) / P(B) = 17/36$ is probability. That a student selected at random had studied probability given that he has passed the examination in Queuing.

Laws of conditional probability :

The following laws of conditional probability are easily established for any events A,B,C with $P(B) > 0$

1. $P(\phi/B) = 0$
2. $P(A'/B) = 1 - P(A/B)$
3. $P(A \cup C/B) = P(A/B) + P(C/B) - P(A \cap C/B)$
4. $P(A \cap B) = P(A) . P(B/A)$

Independence of events :

Definition : Two events A and B of the same sample space are said to be independent if the occurrence of one does not influence the probability of occurrence or non-occurrence of the other.

Thus if A and B are two independent events, we must have

$$P(A/B) = P(A), (B) < 0$$

and $$P(B/A) = P(B), P(A) > 0$$

This leads us to the following:

Criterion if Independence :

Two events A and B well be independent if

$$P(AB) = P(A) . P(B)$$

Example : Let two coins be tossed in the air. Then

$S = \{HH, HT, TH, TT\}$

If we defined A = the event that a tail appears on the First coin; and B = the event that a head appears on the second coin, then

A = {TH, TT} and B = {TH, HH}

Clearly P(A) = 2/4 = P(B) Also AB = {TH) ; P(AB) = 1/4

Since P(AB) = P(A) . P(B) therefore, A and B are independent

6.7 : Random variables :-

Discrete Random variable

Definition 1 :- A random variable x is a rule defined over the sample space S that assigns to every $e_i \in s$, a real number $x(e_i)$

The set of all values $x(e_i)$ thus obtained is called the range space of x and its denoted by R_x.

Example : The sample space associated with the tossing of two coins is

S = {HH, HT, TH, TT}

If we define the random variable x by

x = the number of heads obtained, then we have

x (HH) = 2, x(HT) = 1, X(TH) =1 and X(TT) =0

Thus the set of all possible value of x, the range space is given by

$R_x = \{0,1,2\}$

Note :- It is quite possible that the outcomes of an experiment are themselves numerical. For example in the case of die rolling, the corresponding random variable is represented by the set of outcomes {1,2,3,4,5,6}

Definition 2:-

Let x be a random variable, with range space R_x. If R_x is finite or countably infinite, then x is called a discribe random variable.That is possible values of x may be listed as $x_1, x_2 ....x_n$. The list terminates in the finite case, and in the countably infinite case, the list continues indefinitly.

Example :- Consider the random experiment of tossing a coin untill the head appears. The sample space is

S = {H, TH, TTH, TTTH, ...}

If x = number of throws required to obtain a head, then the range space of X is given by $R_x = \{1,2,3,4,......\}$

Thus x is a discrete random variable

Definition 3 :

Let x be a random variable with

$R_x = \{x_1, x_2, \ldots\ldots\}$ The function $P_x(.)$ defined over R_x by

$P_x(x_j) = P(x=x_j);$

is called the point probaility function x.

The collection of pairs $\{x_j, P_x(x_j)\}$ i = 1,2,...... is called the probability distribution of x. will be within certain interval can be computed if its probability distribution is known

Example :

Let x be a random variable with its probability distribution given by

x_i	-1	0	1/4	1/2	2	5/2	4
$P_x(x_j)$	0.1	0.5	.2	.1.5	.2	.2	.1

Then

(i) $P(x \geq 2) = P_x(2) + P_x(5/2) + P_x(4) = .5;$

(ii) $P(0 \leq x \leq 2) = P_x(0) + P_x(1/4) + P_x(1/2) + P_x(2) = 6;$ and

(iii) $P_x(0 \leq x \leq 1) = P_x(1/4) + P_x(1/2) = .35$

Note :- we must have

$$P_x(x_j) \geq 0 \text{ for all } x_j \in R_x \text{ and } \sum_{x_j \in R_x} P_x(x_j) = 1.$$

continuous random variables , Distribution functions.

To Take into account the 'continuous type' variables, "the life of an electric bulb", the height of an individual," etc ...., we require the following extension of the definition 1 above.

Definition 4:- A random variable x is a function whose do main is the sample space S and whose range R_x is non-empty set of real number such that for every real number x, the set of sample points e for which $x(e) \leq x$ is an event of S.

This definition is a more general one and covers our earlier definition of a random variables.

Definition 5:- Let x be a random variable. The function $F_x(.)$ whose value for each real number x given by

$F_x(x) = P(x \leq x)$

is called cumulative distribution function of the random variable x and is denoted by cdf. Thus if x is a discrete random variable then

$F_x(x) = \sum P(x_j)$

The real number x defines an event of R_x, namely the set

$A = \{y : Y \in R_x, Y \leq x\}$

By extending R_x to all real numbers, if necessary the set A becomes the interval and the event A has a probability $F_x(x)$ attached to it. Consequently, the domain of F is the set of all real numbers while the real numbers between 0 and 1 constitue its range space.

Definition 6 : A random variable x is said to be continuous if its cdf $f_x(x)$ is a continuous function of x.

Sometimes for the sake of simplicity, a continuous random variable is considered to be one whose range space an interval or a collection of intervals on a real line. This certainly is not equivalent to the above definition, but works equally well. For example, if

x = Life of an electric bulb before its

Failure,

then $R_x = [0, \propto]$ x is a continuous random variable that can assume any value between $0 \leq x \leq \propto$. It can be shown that x satisfies the condition of definition 6.

Remarks 1 : For a continuous random variable x, the point probability $P(x = x_j)$, x_i being a specified value vanish, thus we have

$P(a \leq x \leq b) = P(a \leq x \leq b) = P(b \leq x \leq b)$

$= P(a<x<b)$, fo any.

real a <b

2: The connection between the probability of the event $\{a<x<b\}$ and then cdf is given by

$P(a<x<b) = F_x(b) - F_x(a)$

Definition 7:-

Let x be a continuous random variable. A non-negative function f defined for all real numbers x such that

$$F(x) = \int_{\propto}^{x} f(t)\, dt$$

is called a probability density function denoted by pdf, for the random variable x whose cdf is $f_x(.)$

The function f must have enough properties for the existence of the imporper integral. In fact, it can be proved that any continuous function f satisfying.

Probability :

$$\text{(a() } f(x) \geq 0 \text{ and (b) } \int_{R_x} f(x)\, dx = 1$$

For all x in R_x, qualifies as an pdf for x.

Example :

Let fbe a function defined by

$$f(x) = \begin{cases} 2x, 0<x<1 \\ 0, \text{ otherwise} \end{cases}$$

Clearly we observe that

$$R_x = (-\infty<x<\infty),$$

f(x) is defined and continuous for all $x \in R_{xj}$ and

(a) $f(x) \geq 0$ for all $x \in R_x$

(b) $f(x) \int_{R_x} dx = \int_{-\infty}^{\infty} f(x)\, dx$

$$= \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)\, dx + \int_{1}^{\infty} f(x)\, dx$$

$$= 0 + \int_{0}^{1} 2xdx + 0 = 1$$

Thus f(x) represents a pdf.

Remark . since $F_x(b) = \int_{-\infty}^{b} f(t)\, dt$ and $f_y(a) = \int_{-\infty}^{a} f(t)dt,$

probability $P(a<x\leq b) = f_x(b) - F_x(a)$ can be writen in terms of the integrals as follows :

$$P(a<x\leq b) = \int_{-\infty}^{b} f(t)\, dt - \int_{-\infty}^{a} f(t)\, dt = \int_{-\infty}^{b} f(t)\, dt$$

Since $f\geq 0$, we can also say that $P(a<x\leq b)$ is the area under the curve Y=f(x) bounded by the ordintes at x = a and x = b, as shown in fig 14.1

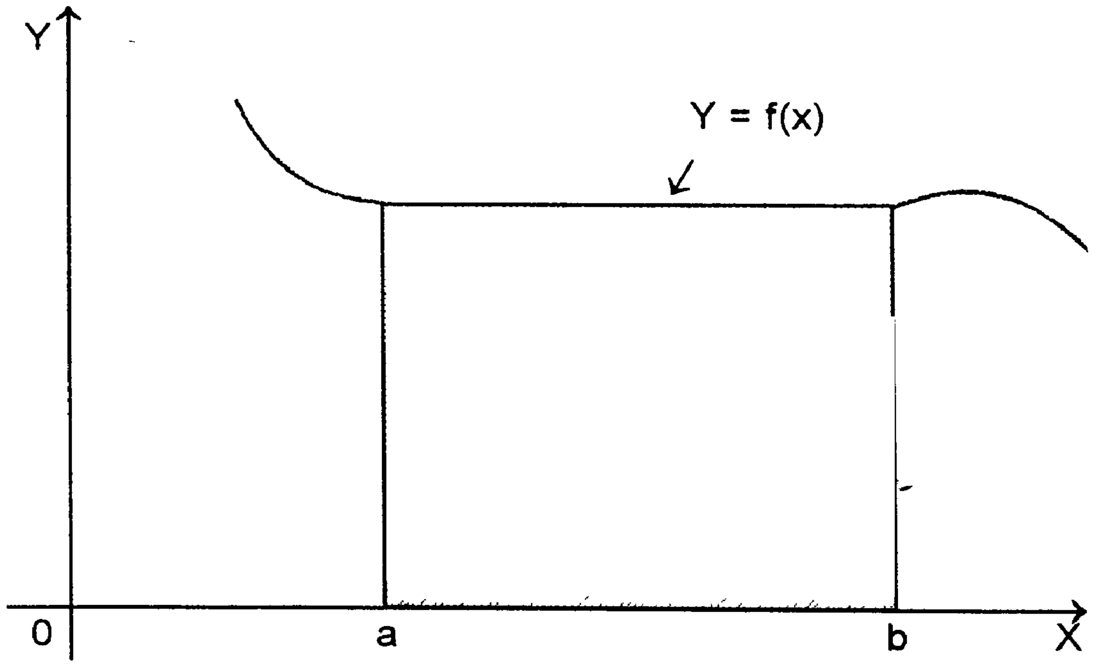


Fig 14.1 Area under the curve y = f(x)

Theorem :-

(a) Let x be a continuous random variables with cdf F and pdf f, then

$$f(x) = \frac{dF(x)}{dx}$$

For all x at which derivative exists

(b) Let x be a discrete random variable with cdf F and

$$R_x = \{x_1, x_2 \ldots\ldots / x_1 < x_2 < \ldots\}$$

Then $P_x(x_j) = f(x_j) - F(x_j - 1)$ for $j = 1, 2, \ldots$

Proof :- Beyond the scope of this book.

Example : The waiting time x for customer arrival at some service counter has been found to be a continuous random variable with cdf.

$$F_x(x) = \begin{cases} 0 & x < 0 \\ 1-e^{-\propto} & \text{otherwise} \end{cases}$$

the pdf of x is obtained by,

$$f(x) = \frac{df_x(x)}{dx}$$

$$\text{Thus } f(x) = \begin{cases} 0 & \text{if } x < 0 \\ e^{-\propto} & \text{if } x > 0 \end{cases}$$

Expection of a Random variables :

Definition :

Let x be a random variable with range space R_x. The expectation of x, denoted by E(x) is defined by

$$E(x) = \begin{cases} \sum_{i=1}^{n} g(x_j)\, P(x_i) & \text{if x is a discrete with } p(x_i) = P(x=x_i) \\ \int_{-\propto}^{\propto} x\, f(x)dx & \text{if x is a continuous with } R_x = (-\propto, \propto),\ \text{pdf } f(x) \end{cases}$$

E(x) is said to exist if the above sum converges.

Example : Let a discrete random variable x have the probability distribution.

x_i	0	1	2	3
$P(x_i)$	.2	1	.4	.3

The expectation of x is given by

$$E(x) = \sum_{i=1}^{4} x_i p(x_i) = 0(.2) + 1(.1) + 2(4) + 3(.3) = 1.8$$

Note : E(x) is generally called the 'expected value' or the 'mean value of x.

Remarks 1 :- The concept of expectation of a random variable x can be generalised to include functions of x also In particular, if x is a random variable and g is a single valued function of x, then g(x) is also a random variable and

$$E[g(x)] = \begin{cases} \sum_{i=1_{i=1}}^{n} g(x_i)\, p(x_i) & \text{if x is discrete} \\ \int_{-\infty}^{\infty} g(x)\, f(x)\, dx & \text{if x is continuous} \end{cases}$$

For example, if $g(x) = \log x$, then $E[\log x] = \sum (\log x)\, p(x=x_i)$

2. If x and Y are random variables defined over the sample space, then the combinations of x and y such as

$x \pm y$, xy, bx, $a \pm bx$

where a, b are constants, are also random variables

Theorem 14-2 :

Let X and Y be two random variables defined over the same sample space, then,

1. $E(b) = b$; (b is a constant)
2. $E(bx) = b(x)$
3. $E(a \pm bx) = a \pm bE(x)$; (a,b are constants)
4. $E(x \pm y) = E(x) \pm E(y)$
5. $E(xy) = E(x)\, E(y)$ if x and y are independent.

proof : Excercise for the reader

Note : The laws of expectation for two random variables may easily be generalised to three or more variables.

Some probability Distributions :-

We are now in a positions to consider some important probability laws that shall be used in the sequel.

The probability distributions generated by the probability laws shall be disignated by some commonly known names.

6.8 : Poisson Distribution :

Definition : The probability distribution of a discrete random variable x, with the range space R = {0,1,2,......} that obey the poisson probability law.

$$P(x=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad \text{for all } x \in R_x \;;\ \lambda > 0,$$

is called a poisson distribution with parameter λ .

An important property of poisson distribution is that its mean and variance coincide. In fact, for a poisson variable with parameter λ, the mean and variance both are equal to λ.

Note : The 'Parameter λ' is generally reffered to as 'mean λ'.

A probability chart for a poisson distribution has been shown in Fig 14.2 $\lambda=2$

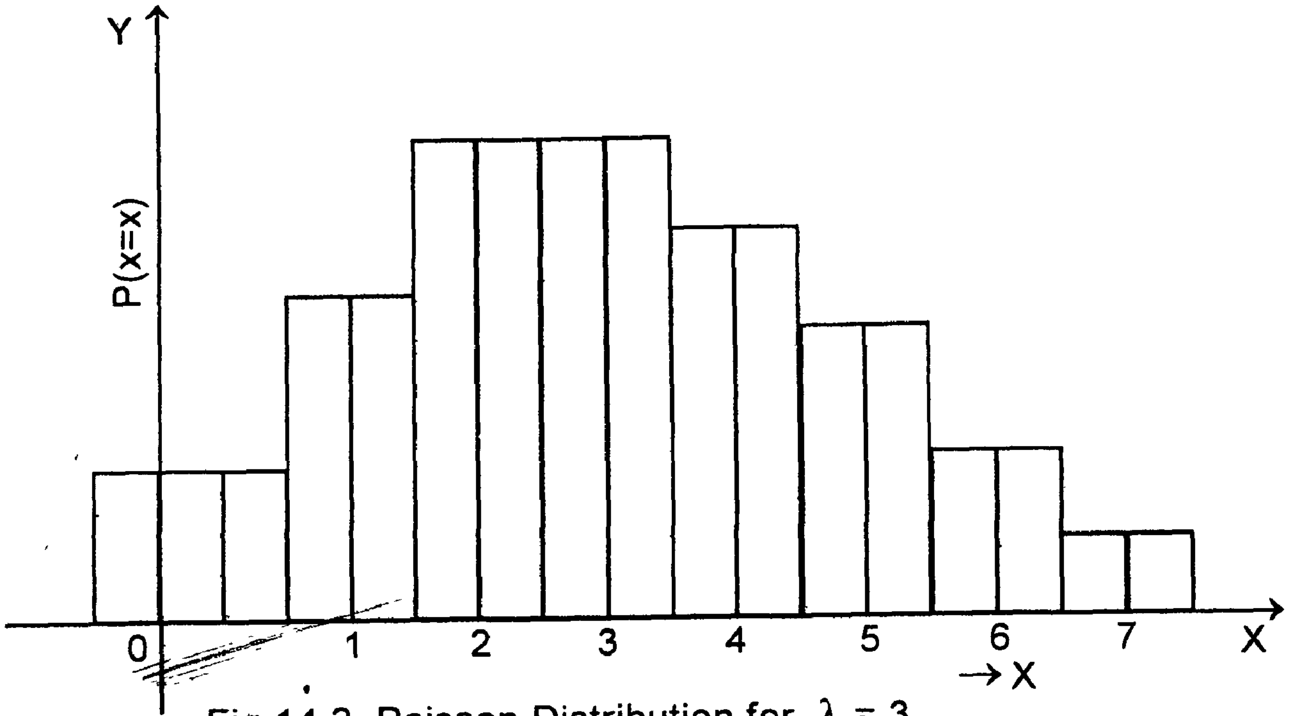


Fig 14.2 Poisson Distribution for $\lambda = 3$

The poisson distribution can be applied to situations where the chance of an event's happening is very very small and a very large population is exposed to it. For example, the whole population in a certain city is exposed to committing suicide, but the chance is very small that a particular individual will commit suicide.

A typical application of the poisson distribution occurs in analysing Queuing problems. Where customers arrive randomly at a service counter. In certain siutations, the number of arrivals and departures to and form the service counter can be described by the poisson distribution.

Another important properly of poisson distribution is its additivity. Namely, if X and Y are two independent poisson random variables with mean λ and μ respectively, the x+y is also a poisson random variable with mean $\lambda+\mu$.

Example :-

The number of customers arriving at a facility for service between 10 A.M. and 11 A.M. is a random variable, say x_1 with poisson distribution with mean 2. Similarly, the number of customers arriving between 11 A.M. and 12 noon, say x_2 has a poisson distribution with mean 6. If will come between 10 A.M. and 12 noon.

Let $x = x_1+x_2$. Then by additivity property of poisson distribution, x has a poisson distribution with mean 2+6 (=8).

Thus the probability that there will be x customers between 10A.M and 12 noon is given by

$$P(x=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-8}8^{x}}{x!} \quad ; x = 0,1,2,...........$$

probability that more than 5 customers will arrive between 10 A.M and 12 noon is given by

$$P(x>5) = 1-P(x\leq 5) = 1- \sum_{x=0}^{5} \frac{e^{-8}8^{x}}{x!}$$

$$= 1-.1912 = .8088$$

6.9 : Exponential Distribution :

Definition 4 : A continuous random variable x is said to have an expontential distribution with prrameter $\propto$ is its pdf is given by

$$f(x) = \begin{cases} \propto e^{-\propto x}, & \text{for } 0<x<\propto \\ 0, & \text{otherwise} \end{cases}$$

Where $\propto$ is a positive constant and

$e = 2.7187.$

The graph of the pdf of the exponental distribution is given if Fig 15.10 :

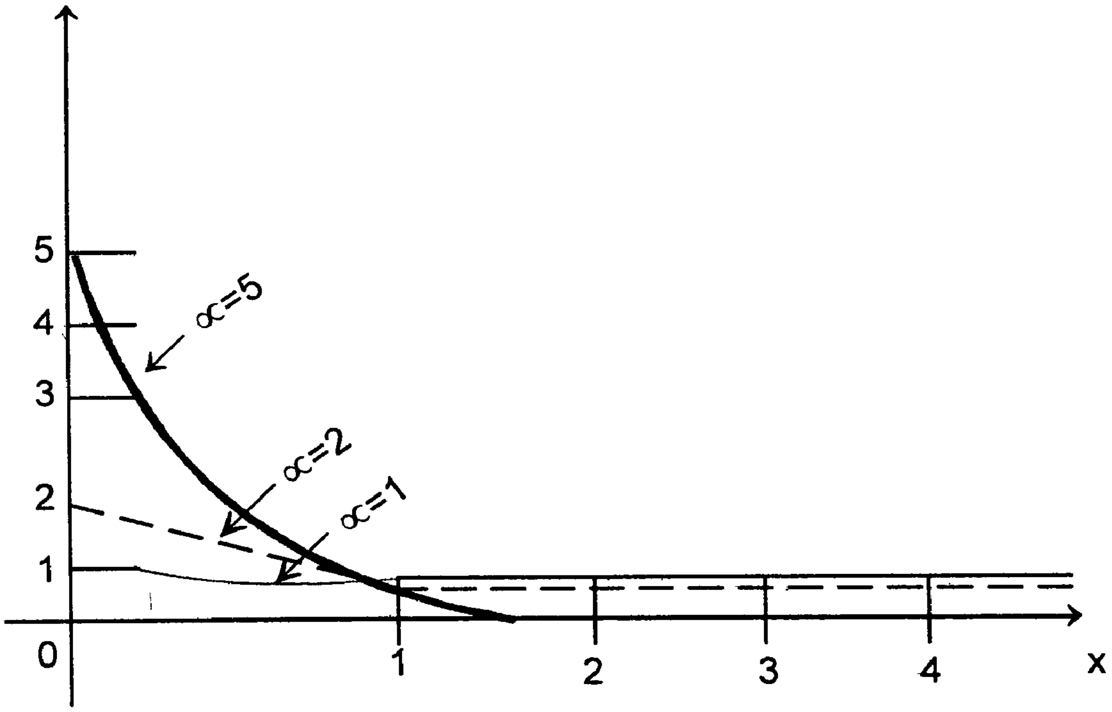


Fig 14 :10. Exponential Distribution

Probability

It is easy to see that (i) the exponential density function is an exponential decreasing function with a maximum at x =0, the maximum value being $\propto$; (ii) the mean and variance of the distribution are given by

$$E(x) = 1/\propto \text{ and } V(x) = 1/\propto^{2}$$

It is an interesting to note that exponential distribution is the continuous analog to the geometric distribution in the discrete case. For example, if the geomatric random variable represents the number of trials before the first failure occurs, its equivalence in the exponential distribution would be the waiting time for failure. In fact as $P \longrightarrow 0$ and the inter - trial time $\longrightarrow 0$, the geometric distribution tends to exponential distribution.

Another important relationship exists between the poisson distribution and the exponential distribution. If the poisson distribution discribes the number of failure per unit time, the exponential distribution will represent the time between two successive failures.

The exponential distribution applies to events that are subject to a constant chance of failure, e.g., life testing electronic components.

Exponential Distribution in Queuing problems :-

Exponential distribution is often encountered in waiting line problems as a probability model for service time. Its mean $\propto$ represents 'the rate of service' ie., The average number of customers served per unit of time the random variable x represents the arrival time and is the $\propto$ parameter whose value rate depends on the rate of performance of the service $(0<x<\propto)$.

Let T be the arrival time that follows an exponential distribution with parameter $\propto$. The distribution generates the probability distribution of inter-arrival times. The probability of the next arrival time t, and t_2 is given by

$$P(t_1 < T < t_2) = f(t_2) - F(t_1) = e^{-\propto t_1} - e^{-\propto t_1}$$

since $F(t) = P(T \leq t) = \int_0^t e^{-\propto x} dx = 1 - e^{-\propto t}$

Like geometric distribution, exponential distribution alos lacks in memory. That is,

$$P(t \leq T \leq t + k / T \geq t) = P(0 \leq T \leq k)$$

It means that the probability of completing a service during time interval $(t, t+k)$ given that the service has been in progress for time t is the same as the unconditioned probability of completing the service in time interval (O,k).

Example 1: At standard, one of the largest restaurants in New Delhi, it normally takes 10 minutes to serve the order after receiving it. If the service time is exponentially distributed, what is the probability that the customer waiting time is (i) more than 8 minutes (ii) 12 minutes or less, and (iii) between 5 and 10 minutes.

Solution :- Let T be the customer waiting time for the service. Then T is exponential with avarage serving rate = $\frac{1}{20}$ = 0.1 per minutes. The required probabilities are

(i) $P(T>8) = e^{-\propto t} = e^{-1 \times 8} = e^{-0\ 8} = 0.449$

(ii) $P(T \leq 12) = 1 - e^{-12\propto} = 1 - e^{-1\ 2} = 1 - 0.301 = 0.699$

(iii) $P(5 \leq T \leq 10) = e^{-5\propto} - e^{-10\propto} = e^{-0\ 5} - e^{-1} = 0.607 - .368 = 0.239$

Example 2:-

Customers arrive at a booking office window being manned by a single individual at a rate of 30 per hour. The inter-arrival time of customers is exponentially distributed. For a time span of 10 minutes calculate various probabilties of next arrival in a given time interval. Also determine the mean inter-arrival time.

Soultion : We are given the average number of customers arriving at the booking office window, per minutes.

$d = 30/60 = 0.5$

The probability of next arrival between a time interval t_1-t_2 is given by

$P(t_1<T<t_2) = e^{-\propto t_1} - e^{-\propto t_2}$

Considering time intervals of 1 minute duration each, the probability distribution of inter-arrival time and the mean inter-arrival time are calculated in the following table.

Inter-arrival time t_1-t_2 minutes	Midvalue of inter-arrival time (ii)	Inter-arrival probability $e^{-\propto t_1} - e^{-\propto t_2}$ (iii)	(ii) X (iii)
0-1	0.5	0.393	0.1965
1-2	1.5	0.239	0.3585
2-3	2.5	0.145	0.3625
3-4	3.5	0.088	0.3080
4-5	4.5	0.053	0.2385
5-6	5.5	0.032	0.1760
6-7	6.5	0.200	0.1300
7-8	7.5	0.012	0.0900
8-9	8.5	0.007	0.0595
9-10	9.5	0.004	0.0380

mean inter-arrival time = 1.9575 minutes

UNIT 7

Inventory control

The term is generally used to indicate raw materials in process, finished, product, packaging spares and others - stocked in order to meet an expected demand (or) distribution in the future. Though inventory of materials is an idle resource - It is not meant for immediate use - it is not meant for immeidate use - it is almost essential to maintain some inventories for the smooth functioning of an enterprise.

7.1 : Types of inventories :-

Inventories may be held for a variety of purpose, but in general there are following five types of inventories that an enterprise can use for serving these purposes.

Transportation inventories :- These also called transit or pipeline inventories arise due transportation of inventory items to distribution entres and customers from various production centres. The amounts of transportation inventory depend on the time consumed in transportation and the nature of demand.

Buffer inventories :-

These are maintained to meet uncertainties of demand and supply. Such "buffer" inventories which are in excess of those necessary to just meet the average demand during the lead time (the time clasping between placing and order and having the goods in stock ready for use held for protecting against the fluctuations in demand and lead time are also termed as safety stocks.

Anticipation inventories:-

These are built up in advance for a big selling season, a promotion programme, (or) a plant shut down period production of specialized items like crackers well before Diwali, electric fans or coolers while summers are approaching are some examples of anticipation inventories.

Decoupling inventories:- If various production stages operate successively then in the event of breakdown of one or any disturbance at some stage can affect the entire system. This kind of inter-dependence is not only costly but also disruptive for the entrire system. The inventories used to reduce the interdependence of various stages of production system are known as decoupling inventories. Decoupling inventories are not only in the form of in-process inventories to decouple successive production stages but also in the form of raw material which are used to decouple the producers from the suppliers, and finished goods to decouple the consumer from producer inventories may also be carried to take advantage of quantity discounts (or) raw material price in the event of an expected rise in price

Lot size inventories :-

These are held for the resons that purchases are usually made in lots rath than for the exact amounts which may be needed at a point of time. Lot-size inventorie are also called cycle inventories.

7.2 : Inventory costs:-

Various costs associated with inventory control are often classified as follows.-

Set - up - cost :-

This is the cost associated with the setting up of machinery before starting production. Set-up-cost is generally assumed to be independent of the quantity ordered for (or) produced.

Ordering cost :-

This is a cost associated with ordering of raw material for production purposes. Advertisements, consumption of stationery and postage telephone charges, telegrams, rent for space used by the purchasing department travelling expenditures incurred etc, constitue, the ordering cost.

Purchase (or Production cost) :- The cost of purchasing ((or) Producing) a unit of an item is known as purchase (or production) cost. The purchase price will become important when quantity discounts are allowed for purchases above a certain quantity (or) when economics of scale suggest that the perunit production cost can be reduced by a larger production run.

Carrying ((or) holding cost) :- The carrying cost is associated with carrying ((or) holding) inventory. This cost generally includes the costs such as rent for space used for storage, interest on the money locked-up, insurance of stored equipment production taxes depreciation of equipment and furniture used etc.

Shortage (or stockout) cost :-

The penalty cost for running out of stock (ie) when an item cannot be supplied on the customer's demand) is known as shortage cost. This cost includes the loss of potential profit through sales of items and loss of goodwill, interms of permanent loss of customers and its associated lost profit in future sales.

Salvage cost (or selling price) :-

When the demand for certain commodity is affected by the quantity stocked decision problem is based on a profit maximization. Criterion that includes the revenue rom selling. Salvage value may be combined with the cost of storage and hence is generally eglected.

Revenue cost :-

When it is assumed that both the price and the demand of the product are not under control of the organisation, the revenue from the sales is independent of the company's inventory policy and may be neglected except for the situation when the organisation cannot meet the demand and the sale is lost. Therefore the revenue cost may (or) may not be included in the study of inventory policy.

7.3 : Other factors involved in Inventory Analysis:-

Besides the costs that determine the profitability, other factors which play an important role in the study of inventory problems are the following.

Demand : Demand is the number of units required per period and may be either known exactly (or) known in terms of probabilities (or) be completely unknown. Further if the demand is known it may be either fixed (or) variable per R unit time. Problems in which demand is known and fixed are called deterministic problems. Whereas those problems in which demand is assumed to be a random variable are called stochastic (or) probabilistic problems

Lead time :-

The time gap between placing of an order and its actual arrival in the inventory is known as leadtime. Lead time is of fundamental importance is dertmining inventory levels. The level of inventory of an item depends upon the length of its lead time. The longer the lead time, the higher is the average inventory. Lead time has two components, namely the administrative lead time - from initiation of procurement action until the placing of an order and the delivery lead time - from placing of an order until the delivery of the ordered material

Order cycle :-

The time period between placement of two successive orders is referred to as an order cycle The order may be placed on the basis of following two types of inventory review systems

(a) Continuous review -

The record of the inventory level is checked continuously until a specified point (called reorder point) is reached where a new order is placed. This is often referred to as the two-bin system This divides the inventory into two parts and places it physically (or) on paper in two bins. Items are drawn from only one bin and when it is empty a new order is placed Demand is then satisfied from the second bin until the order is received upon receipt of the order enough items are placed in the second bin to make up the earlier total. The remaining items are placed in the first bin. This procedure is then repeated.

(b) Periodic review :-

In this system the inventory levels are viewed at equal time intervals and orders are placed at such intervals. The quantity ordered each time depends on the available inventory level at the time of review

Stock replenishment :-

Although an inventory problem may operate with lead time; the actual replacement of stock may occur instantaneously or uniformly instantaneous replenishment occurs in case the stock s purchased from outside sources whereas the uniform replenishment may occur when the product is manufactured by the company.

Time horizon :- The time period over which the inventory level will be controlled is called the time horizen. This horizon may be finite (or) infinite depending upon the nature of the demand for the commodity.

Re-order level :-

The level between maximum and minimum stock at which purchasing (or) manufacturing activities must start for replenishment is known as re-order level

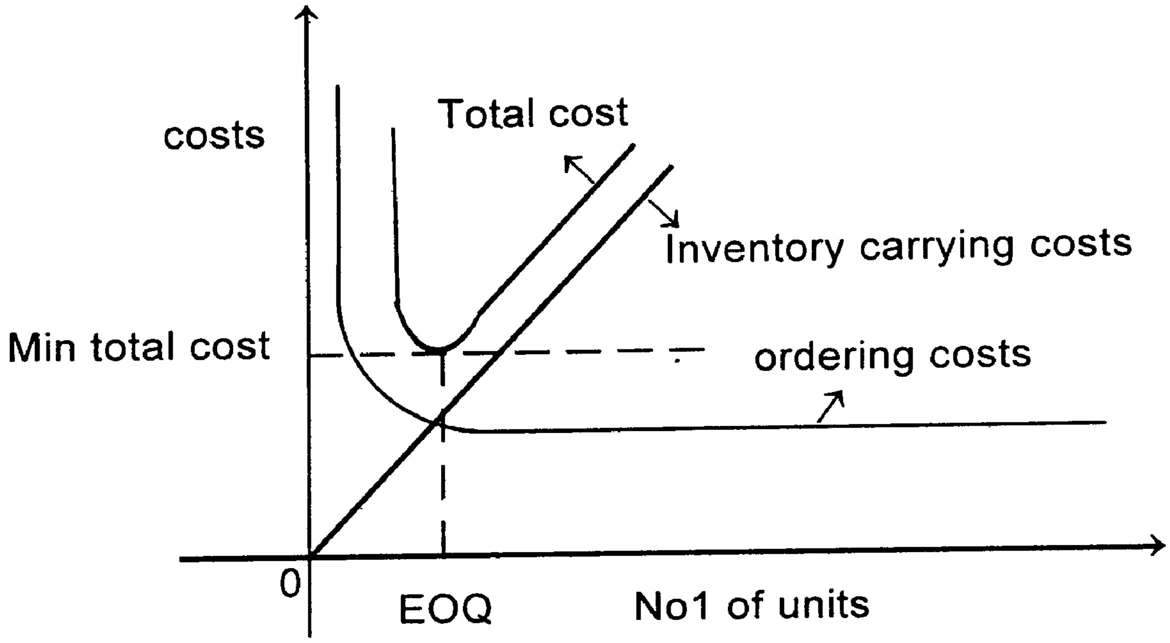
Re-order quantity :-

This is the quantity of replacement order. In certain cases it is the "Economic order Quantity".

7.4 : Economic order quantity :-

The inventory problems in which demand is assumed to be fixed and completely pre-determined are usually referred to as the "Economic ofer quantity".(EOQ) (or) Lot size problems. We mean the quantity produced (or) procured during one production cycle. This is also termed as reorder quantity. When the size of order increases the ordering costs (costs of purchasing, inspection etc) will decrease whereas the inventory carrying costs (cost of storage insurance etc) will increase

Thus in the production process there are two opposite costs, one encourages the increase in the order size and the other discourages. "Economic order quantity" is that size of order which minimizes total annual (or any other time period as specified by individual firms) costs of carrying inventory and cost of ordering. Two opposite costs can be shown graphically by plotting them against the order size as shown below :-



Graph of EOQ

It is evident from above that the minimum total cost occurs at the point where the ordering costs and inventory costs are equal.

7.5: Deterministic inventory problems :-

In this section we shall deal with inventory problems in which demand is assumed to be fixed and completely pre-determined. Such problems are usually referred to as the Economic Lot size problems (or) Economic order quantity (EOQ problems).

Case 1 : EOQ problem with no shortages :-

The objective of the study of this problem is to determine an optimum order quantity (EOQ) such that the total inventory cost is minimized. We illustrate the problem under consideration after making the following assumptions.

(i) Demand is known and uniform.

(ii) Q denotes the lot size in each production run and D denotes the total number of units purchased / produced (or) supplied per time period.

(iii) shortages are not permitted (ie) as soon as the level of the inventory reaches zero, the inventory is replenished thus the cost of shortage is assumed to be infinite.

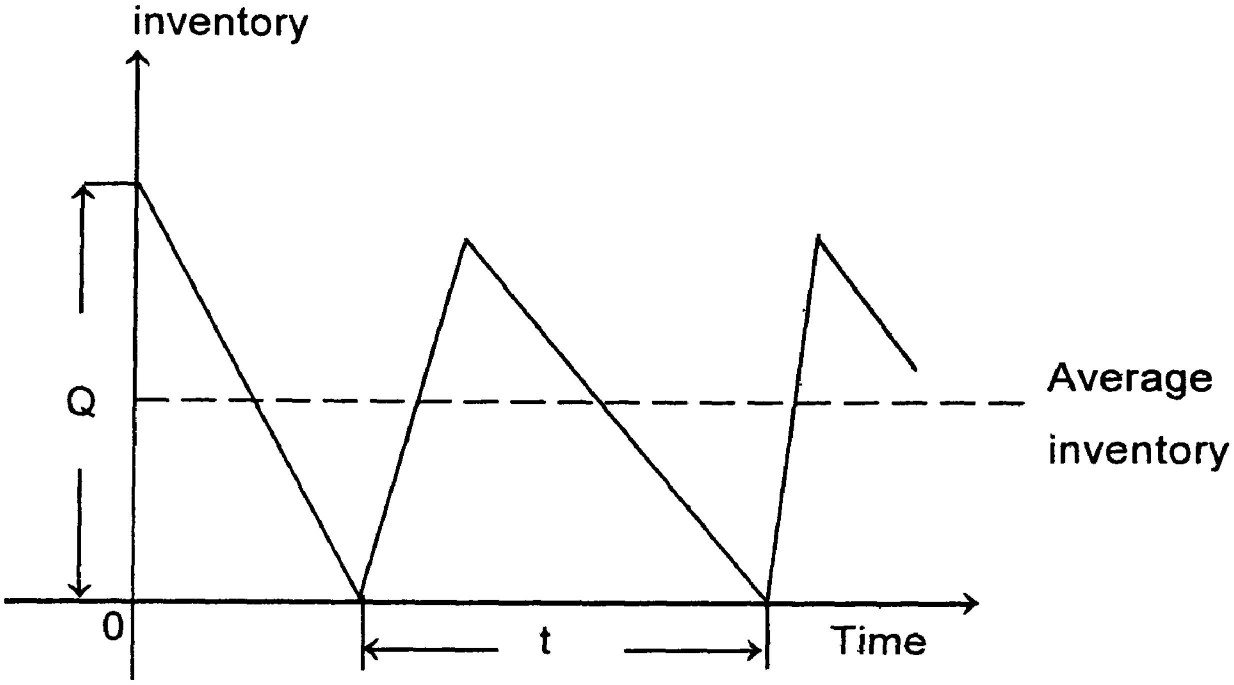
(iv). Production (or) supply of commodity is instantaneous (Abundant Availability)

(v) Lead time id zero.

(vi) Set-up -cost per production run on procurement cost is C_s (or A)

(vii) Holding cost is c_1 per unit in inventory for a unit. (ie) c_1 = IC where c is the unit cost, I is called inventory carrying chage expressed as a % of the value of the average inventory.

This fundamental situation can be shown on an inventory - time diagram with Q on the vertical axis and time on the horizontal axis. The total time period (one year) is divided into 'n' parts.



EOQ problem with uniform demand

Here it is assumed that after each time t, the quantity Q is produced / purchased (or) supplied throughout the entire time period say one year. Now if n denotes the total number of runs of the quantity produced (or) purchased during the year, then clearly we have I = nt and D = nQ. It may be clear that the average amount of inventory at hand on any day is then ½Q as shown by dotted line in the above fig:- Total inventory over the time period t days is clearly the area of the first triangle (=½Qt). Thus the average inventory at any time on any

given day in the t period is $\frac{1}{2}\frac{Qt}{t} = \frac{1}{2}Q$.

Now since each of the triangles in the above fig:- over a year period looks the same ½Q remains the average amount of inventory in each intervel of length t during the entire period. Annual inventory holding cost is therefore given by

$$f(Q) = \frac{1}{2}Qc_1$$

Annual costs associated with runs of size Q are given by

$$g(Q) = nc_s = \frac{D}{Q}c_s$$

Thus the total annual cost is given by

$$\begin{aligned} TC &= f(Q) + g(Q) \\ &= \frac{1}{2}Qc_1 + \frac{D}{Q}c_s \end{aligned}$$

Now we observe that $\frac{d}{dQ}(TC) = 0$

$$\Rightarrow Q = \sqrt{\frac{2Dc_s}{c_1}}$$

$$\frac{d^2}{dQ^2}(TC) > 0 \text{ for } Q > 0.$$

The optimum value of Q has thus been obtained and is given by

$$Q^0 = \sqrt{\frac{2Dc_2}{c_1}}$$

This is known as the economic (optimum) lot size formula due to R.H. Wilson.

The above EOQ formula can also be expressed in terms of the economic order value terms as follows.

$$Q^0 = \sqrt{\frac{2AD}{IC}}$$

Characteristics of case 1 :-

1) Optimum number of orders placed per year

$$n^0 = D/Q^0 = \sqrt{\frac{Dc_1}{2c_2}}$$

2) Optimum length of time between orders

$$t^0 = T/n^0 = T\sqrt{\frac{2Dc_s}{Dc_1}} \quad (\text{or}) \quad \sqrt{\frac{2c_s}{Dc_1}}$$

where T (total time horizon) is one year.

3) Minimum total annual inventory cost

$$TC^0 = \tfrac{1}{2}Q^0C_1 + \frac{Dc_s}{Q^0} = \sqrt{2Dc_1c_s}$$

Corollary :- In the EOQ problem discussed above, if the setup cost is c_s+bQ instead of being fixed (where b is set up cost per unit item produced) then there is no change in the optimum order quantity produced due to change in the set-up-cost

Pf :- In this case, the annual cost is given by $C_A = \tfrac{1}{2}Qc_1 + \frac{D}{Q}(C_s+bQ)$

For the optimum value of Q we see that

$$\frac{d}{dQ}(TC) = 0 \quad \Rightarrow \quad Q = \sqrt{\frac{2Dc_s}{c_1}}$$

$\frac{d^2}{dQ^2}(TC) > 0$ for $Q > 0$

Hence $Q^0 = \sqrt{\frac{2Dc_s}{Dc_1}}$

This shows that there is no change in Q^0 inspite of change inthe set-up-cost.

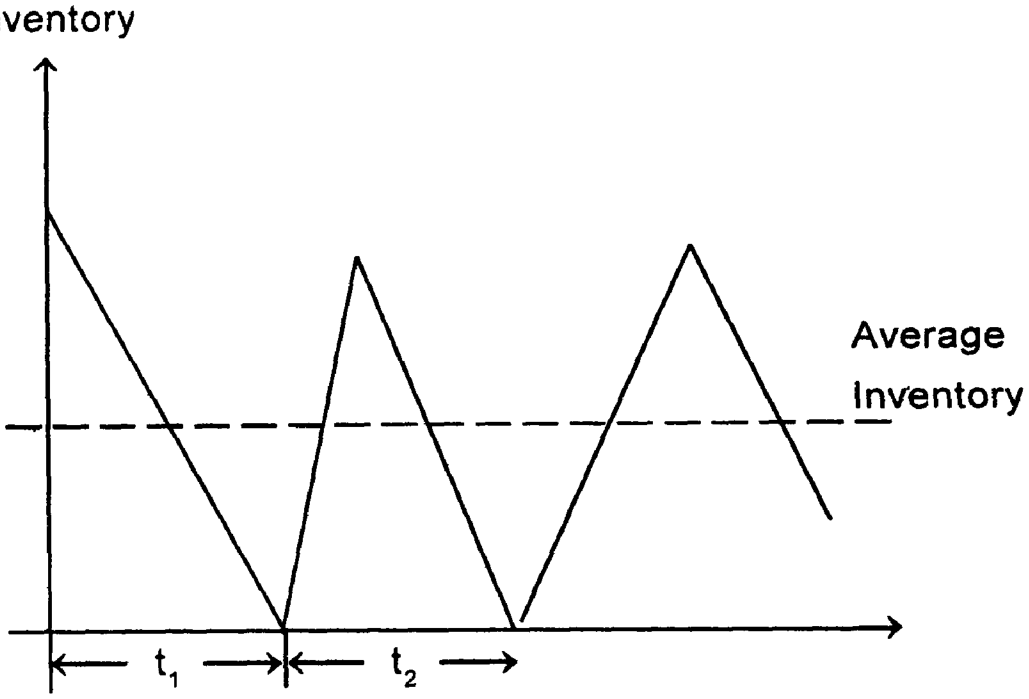
Case 2 :- EOQ problem with no shortages and several production runs of unequal length.

In this problem all the assumptions are same as in case 1 except that the demand is uniform and the production runs differ in units.

Let t_1, t_2, t_3, ... t_n denote the times of successive production runs such that

$t_1+t_2+.......+t_n = 1$ year

Thus the fundamental situation can be represented graphically as shown in the below Fig ·



obviously, the annual inventory hoding cost is given by

$$f(Q) = (\tfrac{1}{2} Qt_1) C_1 + (\tfrac{1}{2} Qt_2) c_1 + ...+(\tfrac{1}{2} Qt_n) c_1$$
$$= \tfrac{1}{2} Q(t_1+..........+t_n) c_1$$
$$= \tfrac{1}{2} Q c_1$$

and the setup costs associated with runs of size Q are given by

$g(Q) = \frac{D}{Q} c_s$ since $nQ = D$

Total annual costs is

$$TC = f(Q) + g(Q)$$
$$= \tfrac{1}{2}Qc_1 + \frac{D}{Q}c_s$$

This cost is the same as was obtained in case 1 and hence the optimum Quantities are

$$Q^0 = \sqrt{\frac{2Dc_s}{c_1}} \quad \text{and } TC^0 = \sqrt{2Dc_1c_s}$$

Remark :- If the total time period is T instead of one year, then the optimum order quantity becomes $Q^c = \sqrt{\frac{2c_sD}{c_1T}}$ and the minimum cost becomes $TC^0 = \sqrt{\frac{2c_1c_sD}{T}}$

Thus the uniform rate of demand is replaced by average rate of demand (ie) D is replaced by D/T

Sample Problems :-

1) An Oil engine manufacturer purchases lubricants at the rate of Rs 42 per piece from a vendor. The requirement of these lubricants is 1,800 per year. What should be the order quantity per order If the cost per placement of an order is Rs 16 and inventory carrying charge per rupee per year is only 20 pause.

Solution :-

We are given

D = Annual requirement of an order in rupees

= 1,800 x 42 = 75,600

Cs = 16 and $c_1 = 0.20$

$Q^0 = \sqrt{2x75,600 \text{ x } 16/.02} = 34, 776$

Thus at the price of Rs 42 per lubricant, the optimum inventory quantity of lubricant is 34,776 / 42 = 83 lubricants.

Case 3 :- Production problem with no shortages :-

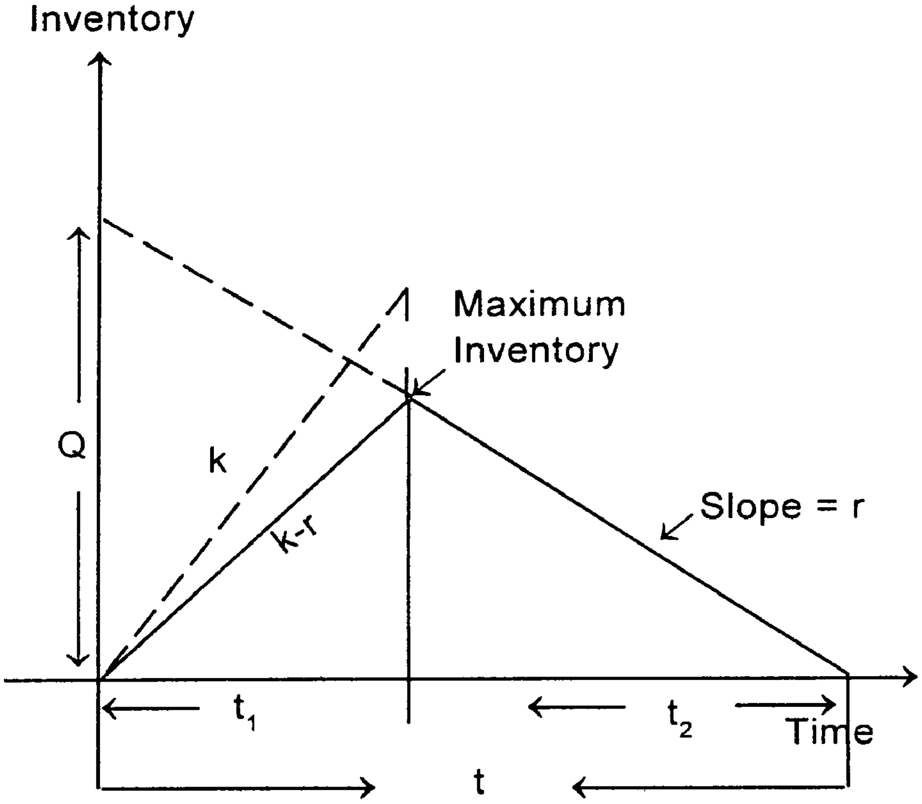
In this problem all the assumptions are the same as in case 1 except that of instantaneous replenishment This is because of the fact that in many situations the amount ordered is not delivered all at once but available at a finite rate.((ie) given supply rate) per unit of time. This situation can arise in case of an order being filled by a machine having finite production rate. Assume that each production run of length t consists of two parts say t_1 and t_2 such that

(i) The inventory is building up at a constant rate of (k-r) units per unit of time during t_1, k>r

(ii) There is no replenishment (or production) during time t_2 and the inventory is decreasing at the rate of r per unit of time.

The graphical representation of the situation is shown in below fig.:

Here the total (order) quantity Q is producedover a period t_1 which is defined by the production rate k. Since the inventory does not pile up in one shot but rather continuously over a time period and is also consumed simultaneously, the average inventory level would be determined not only by the lot size Q but also be affected by the production rate k and depletion (demand) rate r :



To determine the average inventory, we proceed as follows :-

Since t_1 is the time required to produce Q at a rate k, we shall have

$Q = kt_1$ (or) $t_1 = Q/k$

During production period t_1, inventory is increasing at the rate of k and simultaneously drcreasing at the rate of r. Thus inventory accumulates at the rate of (k-r) units. Therefore the maximum inventory level shall be equal to $t_1(k-r)$

$$\text{Average inventory} = \tfrac{1}{2} t_1 (k-r)$$
$$= \tfrac{1}{2} Q (1-r/k)$$

Since $t_1 = Q/k$

Now with c_1 as the holding cost per unit per year, the total annual holding cost is given by

$$f(Q) = \text{Avarage inventory} \times c_1$$
$$= \tfrac{1}{2} Qc_1 (1-r/k)$$

Annual ordering cost is given by

$$g(Q) = c_s \times \frac{D}{Q}$$

Since D is the total demand in a year.

Thus total inventory cost is

$$TC = f(Q) + g(Q)$$
$$= \tfrac{1}{2} Qc_1 (1-r/k) + c_s \frac{D}{Q}$$

Now $\frac{d}{dQ}$ (TC) = 0

$$\Rightarrow Q = \sqrt{\frac{2Dc_s}{c_1}} \sqrt{\frac{k}{k-r}}$$

$$\frac{d^2}{dQ^2} (TC) > 0 \text{ For } Q > 0$$

$$\text{Thus } Q^0 = \sqrt{\frac{2Dc_s}{c_1(1-r/k)}}$$

Characteristics of case 3 :-

1) Optimum number of production runs per year

$$n^0 = D/Q^0 = \sqrt{\frac{Dc_1}{2c_s} \left(1 - \frac{r}{k}\right)}$$

2) Optimum length of each lot size production run

$$t_1{}^0 = Q^0/k = \sqrt{\frac{2Dc_s}{c_1 k(k-r)}}$$

Total minimum production inventory cost

$$TC^0 = \frac{D}{Q^0} + \frac{1}{2} Q^0 (1-r/k)\, c_1$$

$$= \sqrt{2Dc_s c_1(1-r/k)}$$

Note : If $k \to r$, then $c^0 \to 0$, this shows that there will be no holding cost and no set - up cost.

If $k \to \infty$ (ie) when the production rate becomes infinite, the above problem reduces to the one considered in case 1 the inventory holding cost per unit of time is reduced from the cost discussed in case 1 in the ratio

(1-r/k) 1 for minimum cost, although the set-up cost remains the same

Solved problems :-

A contractor has to supply 10,000 bearings perday to an automobile manufacturer. He finds that when he starts a production run, he can produce 25,000 bearings perday. The cost of holding a bearing in stock for one year is 2 paise and the setup cost of a production run is Rs 18. How frequently should production run he made?

solution - we are given

c_1 = Rs 0 02 per bearing per year.

= Rs 0.000055 per bearing per day.

c_s = Rs 18 00 per production run.

r = 10,000 bearing per day.

k = 25,000 bearings per day.

$$Q^0 = \sqrt{\frac{2 \times 18}{0.000055}} \sqrt{\frac{10,000 \times 25,000}{25,000 - 10,000}} = 1,04,447 \text{ bearings}$$

$$t^0 = \frac{1}{r} \times Q^0 = \frac{1}{10,000} \times 1,04,447 \text{ days}$$

$$= 10.4 \text{ days (approximately)}$$

Length of production cycle

$$= \frac{1,04,447}{25,000} = 40 \text{ days (approx)}$$

Thus the production cycle starts at an interval of 10.4 days and production continues for 4 days so that in each cycle a batch of 1,04,447 bearings is produced.

Case 4 :- EOQ problem with shortages :-

In a business concern, if shortages occur then these can be classified in to the following two categories.

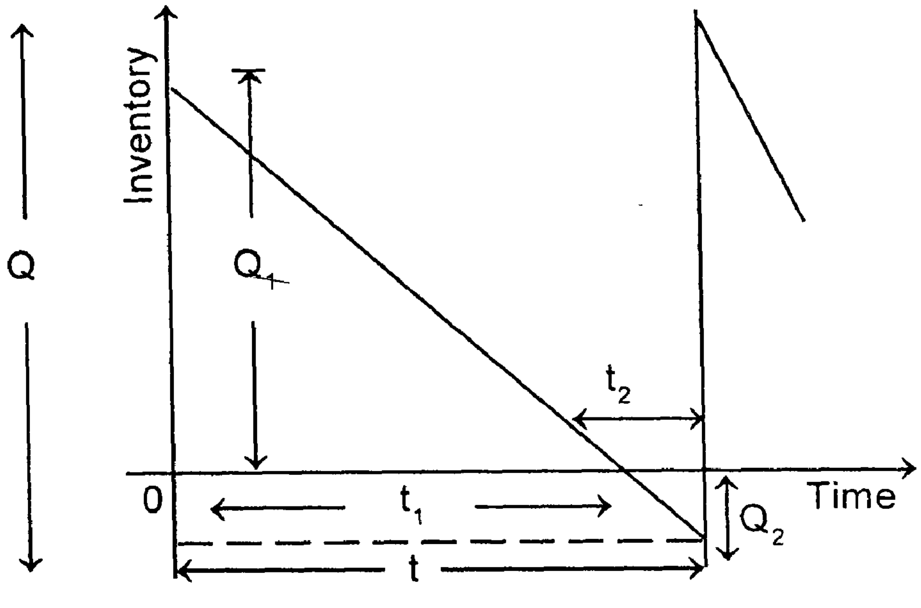
(a) as soon as the desired units of a certain commodity arrive in inventory, the back/orders are satisfied.

(b) shortages are lost sales.

In the first category demand of the customer is met in the beginning of new production run whereas in the second category the customer moves to some other firm to fulfil his requirements. This case deals with those problems of shortages where back orders are entertained.

EOQ problem with Instantaneous production and variable order cycle Time :-

The problem that we now discuss is same as was discussed in above cases with the difference that the shortages are now permitted. Let c_2 be the shortage cost per unit of time per unit quantity. This inventory situation can also be illustrated graphically as shown inthe below fig :-



Here the total time period is one year and is divided into equal parts say of interval t. Further, this time interval t. is divided into two parts t_1 and t_2 such that $t = t_1+t_2$

During the interval t_1, the items are drawn from the inventory as needed and during t_2 orders for the item are being accumulated but not filled. Then at the end of the interval t an amount Q is produced (or delivered). The amount Q has been divided into Q_1 and Q_2 such that $Q = Q_1+Q_2$ where Q_1 denotes the amount which goes into inventory and Q_2 denotes the amount which is immediately taken to satisfy past orders or unfilled demand.

The problem now is concerned with the areas of triangles above the time axis (representing items in inventory) and below the same axis (representing items in shortage).

Now Total inventory over the time period $t = \frac{1}{2}Q_1t_1$

Average inventory at any time $= \frac{1}{2}Q_1t_1 /t$

Annual inventory holding cost $= C_1(\frac{1}{2}Q_1t_1) /t$

Similarly Total amount of shortage over time period $t = \frac{1}{2}Q_2t_2$

Annual shortage costs $= C_2(\frac{1}{2} Q_2t_2) /t$

Annual costs associated with runs of size

$Q = nc_s = \frac{D}{Q}c_s$

Since $\frac{D}{Q}$ runs are produced in each year.

$\therefore$ Total annual cost is given by

$$TC = \frac{[C_1C\,\frac{1}{2}\,(Q_1T_1) + c_2\,\frac{1}{2}(\,Q_2\,t_2)]}{t} + \frac{D}{Q}c_s$$

Now using the relationship for similar triangles we have $\frac{t_1}{t} = \frac{Q_1}{Q}$ and $\frac{t_2}{t} = \frac{Q_2}{Q}$

$$t_1 = \frac{Q_1}{Q}t \qquad t_2 = \frac{Q_2}{Q}t$$

$$\therefore Tc = \tfrac{1}{2}c_1\left(\frac{Q_1^{\,2}}{Q}\right) + \tfrac{1}{2}c_2\left[\frac{(Q-Q_1)^2}{Q}\right] + c_s\left(\frac{D}{Q}\right)$$

Since $Q_2 = Q-Q_1$

For determining the optimum values of Q_1 and Q so as to optimize TC, we have

$$\frac{\partial\,(TC)}{\partial\,Q_1} = 0 \Rightarrow Q_1 = \frac{c_2Q}{c_1+c_2}$$

$$\frac{\partial\,(TC)}{\partial\,Q} = 0 \Rightarrow Q = \sqrt{\frac{2c_sD+c_1Q_1^{\,2}}{c_2} + Q_1^{\,2}}$$

and $\frac{\partial^2\,(TC)}{\partial\,Q_1^{\,2}} > 0$; $\frac{\partial^2\,(TC)}{\partial\,Q^2} > 0$ for these

vaules of Q_1 and Q_2

Thus the optimum quantities are given by (on simplification)

$$Q^0 = \sqrt{\frac{2c_sD}{c_1}} \quad \sqrt{\frac{C_1+C_2}{c_2}}$$

$$Q^0 = \left(\frac{c_2}{c_1+c_2}\right) \qquad Q^0 = \sqrt{\frac{c_2}{c_1+c_2}} \quad \sqrt{\frac{2c_sD}{c_1}}$$

Characteristics of case 4 :

1) Time between receipt of orders (when to order)

$$t^0 = Q^0/D = \sqrt{\frac{2c_s}{Dc_1}} \quad \sqrt{\frac{C_1+C_2}{c_2}}$$

2) Total optimum inventory cost

$$TC^0 = \sqrt{2Dc_sc_1}\sqrt{\frac{c_2}{c_1+c_2}}$$

3) Maximum inventory level

$$Q^0 - Q^0_2 = Q^0\left(1 \frac{c_1}{c_1+c_2}\right)$$

$$= \sqrt{\frac{2c_sD}{c_1}} \quad \sqrt{\frac{c_2}{c_1+c_2}}$$

Remarks :-

1) If $c_1>0$ and $c_2 = \propto$ shortages are prohibited. In this case $Q_1^0 = Q^0 = \sqrt{\frac{2c_sD}{c_1}}$ and each batch Q^0 is used entirely for inventory.

2) If $c_1 = \propto$ and $c_2 > 0$ inventories are prohibited. In this case $Q_1^0 = 0$.

$$Q^0 = \sqrt{\frac{2c_sD}{c_2}}$$

and each batch is used only to fill back orders.

3) If shortages costs are negligible then $c_1 > 0$ and $c_2 \longrightarrow 0$

In this case $Q_1^0 \longrightarrow 0 \qquad Q^0 \longrightarrow \propto$.

4) If inventory costs are negligible then $c_1 \longrightarrow 0$ $c_2>0$. In this case $Q^0 \longrightarrow \propto$ and $Q_1^0 \longrightarrow \propto$ (ie) $Q_1^0 \longrightarrow Q^0$.

Thus as inventory costs become very small, increasingly large batches should be produced and used entirely as inventory for future demands.

5) When inventories and shortages are equally costly (ie) when

$$c_1 = c_2 \qquad \frac{c_2}{c_1+c_2} = \frac{1}{2}$$

Thus in this case

$$Q^0 = \sqrt{2}\sqrt{\frac{2c_sD}{c_1}} = (1.414)\sqrt{\frac{2c_sD}{c_1}}$$

This shows that the lot size .414 times as large as earlier when no shortages were allowed.

Case 5 :- EOQ problem with Instantaneous production and Fixed order cycle :-

Let r be fixed (ie) inventory is to be replenished after every time period t. Also let us assume that items are being supplied (or) produced at the rate of r units per unit of time during this fixed time period.

Here total inventory over the time

$$\text{Period } t_1 = \tfrac{1}{2}Q_1t_1$$

and total amount of shortages over time

$$\text{Period } t_2 = \tfrac{1}{2}Q_2t_2$$

Total production cost is given by

$$TC = \tfrac{1}{2}\,Q_1t_1c_1 + \tfrac{1}{2}\,Q_2t_2c_2$$

using now the relationship of similar triangles ; (Viz)

$$\frac{t_1}{t} = \frac{Q_1}{Q} \quad \text{and} \quad \frac{t_2}{t} = \frac{Q_2}{Q}$$

the cost equation reduces to

$$TC = \frac{1}{2Q}\,c_1Q_1^2t + \frac{1}{2Q}\,c_2Q_2^2t + \frac{1}{2r}\,c_1Q_1^2 + \frac{1}{2r}\,C_2(rt-Q_1)^2$$

Since $Q_2 = Q - Q_1$ and $Q = rt$.

The optimum value Q_1^0 is obtained as follows :-

$$\frac{\partial(TC)}{\partial Q_1} > 0 \Rightarrow c_1Q_1 + c_2(Q_1 - rt) = 0$$

$$\Rightarrow Q_1 = \frac{rtc_2}{c_1+c_2}$$

$$\frac{\partial^2(TC)}{\partial Q_1^2} > 0 \text{ for all values of } Q_1$$

therefore $Q_1^0 = rt\left(\frac{c_2}{c_1+c_2}\right)$

Note :- In this problem set up cost is not considered, because of it being fixed as the time of one production run is fixed substituting the value of Q_1^0 in the total production cost equation the optimum inventory cost is $TC^0 = \left(\frac{c_1c_2}{c_1+c_2}\right) rt$

Case 6 :- **Production problem with shortages :-**

EOQ problem with finite Replenishment :-(production)

In this problem all the assumptions are same in case 4 except that the rate of replenishment of inventory is finite, say k units permit of time.

Assume that each production run of length t consists of two parts t_1 and t_2 which are further sub - divided into two parts say t_{11} and t_{12} t_{21} and t_{22} where

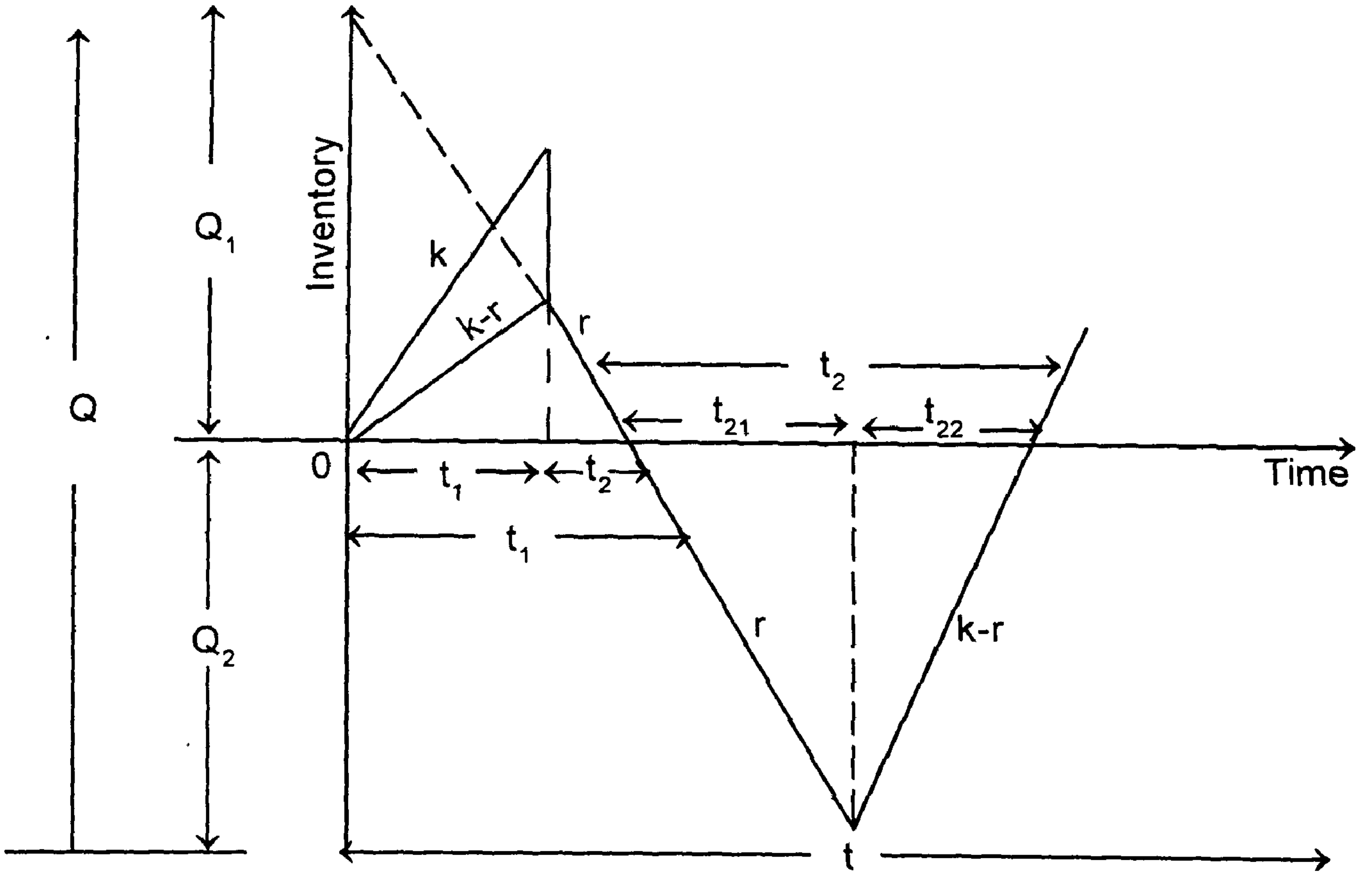
(i) inventory is building up at a constant rate of (k-r) units per unit of time during time t_{11}

(ii) no replenishment during time t_{12} and inventory is decreasing at the rate r per unit of time.

(iii) shortage is building up at a constant rate of r per unit of time during time t_{21}

(iv) Shortages are being, filled immediately at the rate of (k-r) units per unit of time during time t_{22}

The graphical representation of the situation is as follows :-



From the above fig we see that at the end of t_{11}, the level of inventory is Q_1 and at the end of period t_{12} inventory becomes Nil Now shortages start and suppose that the shortages build up of quantity Q_3 up to time t_{21} and let then these shortages be filled up during time t_{22}

Then obviously $Q_1 = t_{11}(k-r)$

$Q_1 = t_{12}r$

$Q_2 = t_{21}r$ and $Q_2 = t_{22}(k-r)$

Now if Q is the lot size, then

$Q_1 = Q-Q_2 - rt_{11} - rt_{22}$

Eliminating t_{11} and t_{22} from this we have

$$Q_1 = Q - Q_2 - r\left(\frac{Q_1}{k-r} + \frac{Q_2}{k-r}\right)$$

(or) $Q_1 + Q_2 = (k-r)\,Q/k$

Production cycle is

$$t = t_{11} + t_{12} + t_{21} + t_{22}$$

$$= \frac{Q_1}{k-r} + \frac{Q_1}{r} + \frac{Q_2}{r} + \frac{Q_2}{k-r}$$

$$= k\,(Q_1+Q_2)\,/\,r(k-r)$$

substituting the value of Q_1+Q_2

we get $t = Q/r$.

The average inventory and amount of shortage during production cycle time are the average inventoy

$$= \tfrac{1}{2}\, Q_1(t_{11}+t_{12})\,/\,t \text{ and}$$

Average shortage $= \tfrac{1}{2}Q_2\,(t_{21}+t_{22})\,/\,t$

The total inventory cost is

$$TC = \tfrac{1}{2}\, Q_1c_1\left(\frac{t_{11}+t_{12}}{t}\right) + \tfrac{1}{2}\, Q_2c_2\left(\frac{t_{21}+t_{22}}{t}\right) + rc_s/Q$$

$$= \frac{1}{2Q} \times \frac{k}{k-r}\left[c_1\left(\frac{k-r}{k}Q - Q_2\right)^2 + c_2Q_2^2\right] + \frac{r}{Q}\,c_s$$

Since $Q_1+Q_2 = \dfrac{r(k-r)}{k}\,t = \dfrac{r(k-r)}{k} \times \dfrac{Q}{r}$

$$= \frac{k-r}{k}Q$$

Now $\dfrac{\partial}{\partial Q_2}(TC) = 0$

$$\Rightarrow Q_2 = \frac{c_1Q}{c_1+c_2}\,(1-r/k)$$

and $\dfrac{\partial}{\partial Q}(TC) = 0 \Rightarrow Q = \sqrt{\dfrac{2C_s(C_1+C_2)}{C_1C_2}}\sqrt{\dfrac{kr}{k-r}}$

since $\dfrac{\partial^2 TC}{\partial Q^2} > 0$ and $\dfrac{\partial^2 TC}{\partial Q_2^2} > 0$ for all values of Q and Q_2 the optimum values of Q and Q_2 so as to minimize Tc are given by

$$Q^0 = \sqrt{\frac{2C_s(C_1+C_2)}{C_1C_2}}\sqrt{\frac{kr}{k-r}}$$

$$Q_2{}^0 = \sqrt{\frac{2C_sC_1r}{(C_1+C_2)\ c_2} \quad \frac{k-r}{k}}$$

Characteristics of case 3 :-

1. Production cycle time is

$$r^0 = \frac{Q^0}{r} \quad = \sqrt{\frac{2C_s(c_1+c_2)}{C_1C_2\ r(1-r/k)}}$$

2) Maximum inventory level is

$$Q_1{}^0 = \frac{k-r}{k} Q^0 - Q_2{}^0$$

$$= \sqrt{\frac{2C_2C_sr}{C_1(C_1+C_2)} \left(1-\frac{r}{k}\right)}$$

3) Total minimum production inventory cost is

$$TC^0 = \frac{1}{2Q^0} \times \frac{k}{k-r} (c_1Q_1{}^{02} + c_2Q_2{}^{02}) + \frac{r}{Q^0} c_s$$

$$= \sqrt{\frac{2c_1c_2c_sr}{C_1+C_2} \quad (1-r/k)}$$

UNIT 8

REPLACEMENT MODEL

8.1 INTRODUCTION :-

The study of replacement is concerned with situations that arise when some items such as machines, men, electric - light bulbs, etc., need replacement due to their deteriorating efficiency, failure or breakdown. The deteriorating efficiency or complete breakdown may be either gradual or all of a sudden. For example a machine becomes more and more expensive to maintain after a number of years, a railway time table gradually becomes more and more out of date, an electric light bulb fails all of a sudden, pipeline is blocked, or an employee loses his job and so like. In all such situations, these is a need to formulate a most economic replacement policy for replacing faulty units or to take some remedial special action to restore the efficiency of deteriorating units.

Following are the situations when the replacement of certain item needs to be done :

(i) An old item has failed and does not work at all, or the old item is expected to fail shortly.

(ii) The old item has deteriorated and works badly, or requires expensive maintenance.

(iii) A better design of equipment has been developed.

Replacement problems can be broadly classified into the following two categories :

(a) When the equipment / assets deteriorate with time and the value of money.

(i) does not change with time.

(ii) changes with time

(iii) when the items / units fail completely all of a sudden.

8.2 Replacement of items that Deteriorates with times.

Generally, the cost of maintenance and repair of certain items (equipments) increases with time and a stage may come when these costs become so high that it is more economical to replace the item by a new one.

At this point, a replacement is justified.

Case 1 :- Value of money does not change with time.

The aim here is to determine the optimum replacement age of an equipment / item whose running / maintenance cost increases with time and the value of money remains static during that period. Let

C : capital cost of equipment

S : Scrap value of equipment

n : number of years that equipment would be in use.

f(t) : Maintenance cost function, and

A(n) : Average total annual cost.

When t is a continuous variable :

If the equipment is used for 'n' years, thenthe total cost incurred during this period is given by

TC = Capital cost - scrap value + maintenance cost

$$= C - S \int_0^n f(t)\, dt$$

Average annual total cost, therefore is

$$A(n) = \frac{1}{n} TC = \frac{C-S}{n} + \frac{1}{n} \int_0^n f(t) dt$$

For minimum cost, we must have $\frac{d}{dn}$ [A-(n)] = 0

$$(\text{or})\ - \frac{(C-S)}{n^2} - \frac{1}{n^2} \int_0^n f(t)\, dt + \frac{1}{n}\ f(n) = 0$$

$$(\text{or})\ f(n) = \frac{C-S}{n} + \frac{1}{n} \int_0^n f(t)\, dt \equiv A(n)$$

Clearly,

$$\frac{d^2}{dn^2}\ [A(n)] \geq 0 \text{ at } f(n) = A(n)$$

This suggests that equipment should be replaced when maintenance cost equals the average annual total cost.

When t is a discrete variable

Here, the period of time is considered as fixed and n,t take the values 1,2,3, .

$$\text{Then } A(n) = \frac{C-S}{n} + \frac{1}{n} \sum_{t=1}^{n} f(t)$$

Now, A(n) will be a minimum for that value of n, for which

$A(n+1) \geq A(n)$ and $A(n-1) \geq A(n)$

(or) $A(n+1) - A(n) \geq 0$ and $A(n) - A(n-1) \leq 0$

For this, we write

$$A(n+1) = \frac{C-S}{n+1} + \frac{1}{n+1} \sum_{t=1}^{n+1} f(t)$$

$$= \frac{1}{n+1} \left[C-S+\sum_{i=1}^{n} f(t)\right] + \frac{1}{n+1}\ f(n+1)$$

$$= \frac{1}{n+1}\left[A(n) + f(n+1)\right]$$

$$A(n+1) - A(n) = \frac{1}{n+1}\ [f(n+1) - A(n)]$$

Thus $A(n+1) - A(n) \geq 0$

$\Rightarrow f(n+1) \geq A(n)$

similarly, it can be shown that

$$A(n) - A(n-1) \leq 0 \Rightarrow f(n) \leq A(n-1)$$

This suggests the optimal replacement policy :

(i) Replace the equipment at the end of n years, if the maintenance cost in the $(n+1)^{th}$ year is more than the average total cost in the n^{th} year.

(ii) not replace the equipment, if the current year's maintenance cost is less than the previous year's average total cost.

SAMPLE PROBLEMS :-

(1) A firm is considering replacement of a machine, whose cost price is Rs 12,200 and the scrap value, Rs. 200. The running (maintenance and operating) costs in Rs. are found from experience to be as follows.

year	:	1	2	3	4	5	6	7	8
Running cost	:	200	500	800	1200	1800	2500	3200	4000

when should the machine be replaced?

solution :-

We are given the running cost, f(n), the scrap value S = Rs 200 and the cost of the machine, C = Rs. 12,200. In order to determine the optimal time n when the machine should be replaced, we calcuate an average total cost per year during the life of the machine as shown in table given below :

Year of service n	Running cost (Rs.) f(n)	cumulative running cost (Rs.) $\sum f(n)$	Depreciation cost (Rs.) C-S	Total cost (Rs.) TC (3)+(4)	Average cost (Rs) A(n) (5)/(1)
(1)	(2)	(3)	(4)	(5)	(6)
1	200	200	12,000	12,200	12,200
2	500	700	12,000	12,700	6,350
3	800	1,500	12,000	13,500	4,500
4	1,200	2,700	12,000	14,700	3,675
5	1,800	4,500	12,000	16,500	3,300
6	2,500	7,000	12,000	19,000	3,167*
7	3,200	10,200	12,000	22,200	3,171
8	4,000	14,200	12,000	26,200	3,275

From the table it is noted that the average total cost per year, A(n) is minimum in the 6th year (Rs. 3,167). Also the average cost in 7th year (Rs. 3,171) is more than the cost in the 6th year. Hence the machine should be replaced after every 6 years.

(2) The cost of a machine is Rs 6,100 and its scrap value is only Rs. 100. The maintenance costs are found from experience to be :

Year	1	2	3	4	5	6	7	8
Maintenance cost :	100	250	400	600	900	1,250	1,600	2,000

When should the machine be replaced?

Solution :-

Given, cost of the machine C = Rs. 6,100

the scrap value S = Rs. 100

Year of service n	Running cost (Rs.) f(n)	cumulative running cost (Rs.) $\sum f(n)$	Depreciation cost (Rs.) C-S	Total cost (Rs) TC (3)+(4)	Average cost (Rs) A(n) (5)/(1)
(1)	(2)	(3)	(4)	(5)	(6)
1	100	100	6000	6100	6100
2	250	350	6000	6350	3175
3	400	750	6000	6750	2250
4	600	1350	6000	7350	1837.5
5	900	2250	6000	8250	1650
6	1250	3500	6000	9500	1583.3*
7	1,600	5100	6000	11100	1585.7
8	2000	7100	6000	13100	1637.5

From the table it is noted that the average total cost per years, A(n) is minimum is the 6th year (Rs 1583.3). Also the average cost in 7th year(Rs1585.7) is more than the cost in the 6th year.

The machine should be replaced after every 6 years.

(ie) End of 6th year.

(3) A firm is considering replacement of a mechine whose cost price is Rs 17,500 and the scrap value is Rs 500. The maintenance costs (in Rs) are found from experience to be as follows :

Year :	1	2	3	4	5	6	7	8
Maintenance cost (in Rs) :	200	300	3500	1,200	1,800	2,400	3,300	4,500

when should the machine be replaced?

Solution :-

Given, cost of the machine C = Rs. 17,500

the scrap value S = Rs 500.

Year of service n	Running cost (Rs.) f(n)	cumulative running cost (Rs.) $\sum f(n)$	Depreciation cost (Rs.) C-S	Total cost (Rs.) TC (3)+(4)	Average cost (Rs) A(n) (5)/(1)
(1)	(2)	(3)	(4)	(5)	(6)
1	200	200	17,000	17,200	17,200
2	300	500	17,000	17,500	8,750
3	3500	4000	17,000	21,000	7,000
4	1200	5200	17,000	22,200	5,550
5	1800	7000	17000	24,000	4800
6	2400	9400	17000	26,400	4400
7	3300	12700	17000	29,700	4242.8*
8	4500	17200	17000	34,200	4275

From the table it is noted that the average total cost per year, A(n) is minimum in the 7th year (Rs 4242.8). Also, the average cost in 8th year(Rs. 4275) is more than the cost in the 7th year. Hence the machine should be replaced after every 7 years.

(ie) End of 7th year.

(4) Following table gives the running costs per year and resale price of a certain equipment whose purchase price is Rs 5,000.

year :	1	2	3	4	5	6	7	8
Running cost (in Rs) :	1,500	1,600	1,800	2,100	2,500	2,900	3,400	4,400
Resale value (in Rs) ·	3,500	2,500	1,700	1,200	800	500	500	500

At what year is the replacement due?

Solution :-

Year of service n	Running cost (Rs.) f(n)	cumulative running cost (Rs.) $\sum f(n)$	Resale value S	Depreciation cost (Rs.) C-S	Total cost (Rs.) TC (3)+(5)	Average cost (Rs) A(n) (6)/(1)
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	1500	1500	3500	1500	3,000	3000
2	1600	3100	2500	2500	5,600	2800
3	1800	4900	1700	3300	8,200	2733.3
4	2100	7000	1200	3800	10,800	**2700** *
5	2500	9500	800	4200	13,700	2740
6	2900	12400	500	4500	16,900	2816.6
7	3400	15800	500	4500	20,300	2900
8	4400	20200	500	4500	24,700	3087.5

From the table it is noted that the average total cost per year, A(n) is minimum in the 4th year (Rs. 2700). Also the average cost in 5th year (Rs 2740) is more than the cost in the 4th year. Hence the machine should be replaced after every 4th years. (ie) End of 4th year.

(5) (a) Machine A costs Rs. 9,000. Annual operating costs are Rs. 200 for the first year, and then increase by Rs. 2,000 every year. Determine the best age at which to replace the machine. If the optimum replacement policy is followed, what will be the average yearly cost of owning and operating the machine?

(b) Machine B costs Rs. 10,000. Annual operating costs are Rs. 400 for the first year, and then increase by Rs. 800 every year. You now have a machine of type A which is one year old. Should you replace it with B, if so when?

Solution :-

(a) Let the machine have no resale value when replaced Then, for machine A, the average total annual cost ATC(n) is computed as follows :

Year (n)	f(n)	$\sum f(n)$	C-S	TC	A(n)
(1)	(2)	(3)	(4)	(5)	(6)
1	200	200	9,000	9,200	9,200
2	2,200	2,400	9,000	11,400	5,700
3	4,200	6,600	9,000	15,600	**5,200** *
4	6,200	12,800	9,000	21,800	5,450
5	8,200	21,000	9,000	30,000	6,000

This table shows tnat tne best age for the replacement of machine A is 3rd year. The average yearly cost of owning and operating for this period is Rs. 5,200.

(b) For machine B the average cost per year can similarly be computed as given in the following table.

Year (n)	f(n)	$\sum f(n)$	C-S	T	A(n)
(1)	(2)	(3)	(4)	(5)	(6)
1	400	400	10,000	10,400	10,400
2	1,200	1,600	10,000	11,600	5,800
3	2,000	3,600	10,000	13,600	4,533
4	2,800	6,400	10,000	16,400	4,100
5	3,600	10,000	10,000	20,000	**4,000***
6	4,400	14,400	10,000	24,400	4,066

Since the minimum average cost for machine B is lower than that for machine A, machine should be replaced by machine A.

To decide the time of replacement, we should compare the minimum average cost for B (Rs 4,000) with yearly cost of maintaining and using the machine A. Since these is no salvage value of the machine, we shall consider only the maintenance cost. We would keep the machine a so long as the yearly maintenance cost is lower than Rs 4,000 and replace when it exceeds Rs. 4000.

On the one-year old machine A, Rs. 2,200 would be required to be spent in the next year; while Rs 4,200 would be needed in year following. Thus, we should keep machine A for one year and replace it thereafter.

Exercise :

1. A truck owner finds from his past records that the maintenance costs per year, of a truck whose purchase price is Rs. 8,000 are as given below.

Year :	1	2	3	4	5	6	7	8
Maintenance cost (Rs)	1,000	1,300	1,700	2,200	2,900	3,800	4,800	6,000
Resale price :	4,000	2,000	1,200	600	500	400	400	400

Determine at which time it is profitable to replace the truck.

2. A truck with first cost of Rs 80,000 has the depreciation and service pattern shown below :

Year :	1	2	3	4	5	6
Depreciation during year :	28000	20000	14000	5000	4000	4000
Annual service cost :	18000	21000	25000	29000	34000	40000

Assume no interest charges are necessary for the evaluation. How many year should the truck be kept in service before replacement?

3. Fleet cars have increased their costs as they continue in service due to increased direct operating cost (gas and oil) and increased maintenance (repairs, tyres, batteries,etc). The initial cost is Rs. 3,500, and the trade in value drops as time passes until it reaches a constant value of Rs. 500.

Given the cost of operating, maintaining and the trade-in value. Determine the proper length of service before cars should be replaced.

Years of service :	1	2	3	4	5
Year end trade in value :	1,900	1,050	600	500	500
Annual operating Cost :	1,500	1,800	2,100	2,400	2,700
Annual maintaining cost :	300	400	600	800	1,000

Answers :-

(1) End of 5th year

(2) After 6 years

(3) End of 3rd year

Case 2 : Value of Money changes with time :-

when the time value of money is taken into consideration, we shall assume that (i) the equipment in question has no salvage value, and (ii) the maintenance costs are incurred in the beginning of the different time periods.

Since it is assumed that the maintenance cost increases with time and each cost is to be paid just in the start of the period, let the money carry a rate of interest r per year. Thus a rupee invested now will be worth $(1+r)$ after a year, $(1+r)^2$ after two years and so on. In this way a rupee invested to day will be worth $(1+r)^n$, n years hence, or, in other words, if we have to make a payment of one rupee in n years time, it is equipment to making a payment of $(1+r)^{-n}$ rupees today. The quantity $(1+r)^{-n}$ is called the present worth factor (pwf) of one rupee spent in n years time from now onwords. The expression $(1+r)^n$ is known as the payment compound amount factor (caf) of one rupee spent in n years time.

Let the initial cost of the equipment be C and let R_n be the operating cost in year n. Let ν be the rate of interest in such a way that $v = (1+r)^{-1}$ is the discount rate. Then the present value of all future discounted costs V_n associated with a policy of replacing the equipment at the end of each n years is given by

$$V_n = \{(C+R_0) + \nu R_1 + \nu^2 R_2 + \dots\dots + \nu^{n-1} R_{n-1}\} +$$

$$\{(C+R_0)\,\nu^n + \nu^{n+1} R_1 + \nu^{n+2} R_2 + \dots\dots + \nu^{2n-1} R_{n-1}\} +$$

....

....

$$= \left[C + \sum_{k=0}^{n-1} \nu^{k}R_{k}\right] \times \sum_{k=0}^{\infty} (\nu^{n})^{k}$$

$$= \left[C + \sum_{k=0}^{n-1} \nu^{k}R_{k}\right] (1-\nu^{n})$$

Now, v_n will be a minimum fpr that value of n, for which

$$V_{n+1} - V_{n} > 0 \text{ and } v_{n-1} - v_{n} > 0$$

For this we write

$$V_{n+1} - V_{n} = \left[C + \sum_{k=0}^{n} \nu^{k}R_{k}\right] (1-\nu^{n+1})^{-1} - V_{n}$$

$$= \nu^{n}\left[R_{n} - (1-\nu)V_{n}\right] / (1-\nu^{n+1})$$

and $V_{n} - V_{n-1} = \nu^{n-1}\,[R_{n-1} - (1-\nu)V_{n}] / (1-\nu^{n-1})$

Since ν is the depreciation value of money, it will always be less than 1 and therefore $1-\nu$ will always be positive. This inflies that $\nu^{n} / (1-\nu^{n+1})$ will always be positive.

Hence, $V_{n+1} - V_{n} > 0 \Rightarrow R_{n} > (1-\nu)\, V_{n}$

and $V_{n} - V_{n-1} < 0 \Rightarrow R_{n-1} < (1-\nu)\, V_{n}$

Thus, $R_{n-1} < (1-\nu)\, V_{n} < R_{n}$

$$\text{(or) } R_{n-1} < \frac{C+R_{0} + \nu R_{1} + \nu^{2}R_{2} + \ldots\ldots + \nu^{n-1}R_{n-1}}{Hv + v^{2} + \ldots\ldots v^{n-1}} < R_{n}$$

$$\text{Since } (1-\nu^{n})\,(1-\nu)^{-1} = \sum_{k=0}^{n-1} \nu^{k}$$

The expression which lies between R_{n-1} and R_{n} is called the "weighted average cost" of all the previous n years with weight $1, \nu, \nu^{2}, \ldots\ldots \nu^{n-1}$ respectively.

Hence, the optimal replacement policy of the equipment after n periods is :

(a) Do not replace the equipments if the next period's cost is less than the weighted average of previous costs.

(b) Replace the equipments if the next period's cost is greater than the weighted average of pervious costs.

Remark :-

Procedure for determining the weighted average of costs (annualized cost) may be summerized in the following steps :

Step 1 .- Find the present value of the maintenance cost for each of the years,

(ie) $\sum Rv^{n-1}$ (n=1,2, ...); where $v=(1-r)^{-1}$

Step 2 :- Calculate cost plus the accumulated present values obtained in step 1;

(ie) $C+\sum R\nu^{n-1}$

Step 3 :- Find the cumulative present value factor up to each of the years 1,2,3,... ..,

(ie) $\sum \nu^{n-1}$

Step 4 :- Determine the annualized cost w(n), by dividing the entries obtained in stpe 2 by the corresponding entries obtained in step 3 (ie) $\left[C+\sum \nu^{n-1}\right] \Big/ \sum \nu^{n-1}$

Corollary :-

When the time value of money is not taken into consideration, the rate of interest becomes zero and hence ν approaches to unity. Therefore, as $\nu \to 1$.

$$\text{we get } R_{n-1} < \frac{C+R_0 + R_1+ \ldots\ldots +R_{n-1}}{1+1+\ldots\ldots \text{ n times}} < R_n$$

(or) $R_{n-1} < W(n) < R_n$.

Note :- It may be noted that the above result is in complete agreement with the result that was obtained in case I.

Selection of the best equipment Amongst Two

Following is the procedure for determining a policy for the selection of an economically best item amongst the available equipments.

Step 1 :- Considering the case of two equipments, say A and B, we first find the best replacement age for both the equipments by making use of $R_{n-1} < (1-\nu)\, V_n < R_n$

Let the optimum replacement age for A and B comes out to be n_1 and n_2 respectively

Step 2 :- Next, compute the fixed annual payment (or weighted average cost) for each equipments by using formula

$$W(n) = \frac{C+R_0 + \nu R_1+\nu^2 R_2 \ldots\ldots +\nu^{n-1}R_{n-1}}{1+\nu+\ldots\ldots \nu^{n-1}}$$

and substitute $n = n_1$ for equipment A and $n = n_2$ for equipment B in it.

Step 3 :- (i) If $W(n_1) < W(n_2)$, choose equipment A.

(ii) If $W(n_1) > W(n_2)$, choose equipment B.

(iii) If $W(n_1) = W(n_2)$, both equipments are equally good.

SAMPLE PROBLEMS.

1. Let the value of money be assumed to be 10% per year and suppose that mechine A is replaced after every 3 years whereas machine B is replaced after every six years. The yearly costs of both the machines are given below.

Year	:	1	2	3	4	5	6
Machine A	:	1,000	200	400	1,000	200	400
Machine B	:	1,700	100	200	300	400	500

Determine which machine should be purchased.

Solution :-

Since the money carries the rate of interest, the present worth of the money to the spent over in a period of one year is

$$v = \frac{100}{100+10} = \frac{10}{11} = 0.9091$$

∴The total discounted cost (present worth) of A for 3 years is

$1000+200 \times (0.9091) + 400 \times (0.9091)^2$ = Rs. 1512 approx.

Again, the total discounted cost of B for six years is

$$1{,}700+(100 \times 0.9091)+200 \times (0.9091)^2+300 \times (0.9091)^3+400 \times (0.9091)^4 + 500 \times (0.9091)^5 = \text{Rs. } 2{,}765$$

Average yearly cost of machine A = Rs. 1512 /3 = Rs 504.

Average yearly cost of Machine B = Rs. 2765 / 6 = Rs 461

This shows that the apparent advantage is with machine B. But, the comparison is unfair since the periods for which the cost are considered are different. So, it we consider 6 years period for machine A also then the total discounted cost of A will be

$$1000+200 \times (.9091)+400 \times (.9091)^2+1000 \times (0.9091)^3+200 \times (0.9091)^4+400 \times (0.9091)^5.$$

After simplification this comes out to be Rs 2,647 which is Rs. 118 less costlier than machine B over the same period.

Hence machine A should be perchased.

2. The initial cost of an item is Rs 15,000 and maintenance or running costs (in Rs) for different years are given below :-

Year	:	1	2	3	4	5	6	7
Running cost	:	2,500	3,000	4,000	5,000	6,500	8,000	10,000

What is the replacement policy to be adopted if the capital is worth 104 and no salvage value?

Solution :-

Since the money is worth 10% per year, the discount rate will be $v = 100/(100+10)$ (ie) 0.9091

We compute the following table by using pwf of one rupee to be spent in n years time :

Year (n)	Running cost R_{n-1}	ν^{n-1} (pwf)	$R_{n-1}\nu^{n-1}$	$C+\sum_{k}\nu^{k-1}R_{k-1}$	$\sum_{k}\nu^{k-1}$	W(n) (Rs)
(1)	(2)	(3)	(4)	(5)	(6)	(7)= (5)/(6)
1	2,500	1.000	2500	17500	1.0000	1,7500
2	3,000	0.9091	2727	20227	1.9091	1,0595
3	4,000	0.8264	3306	23533	2.7355	8603
4	5,000	0.7513	3756	27289	3.4868	7826
5	6,500	0.6830	4440	31729	4.1698	**7609** *
6	8,000	0.6209	4967	36696	4.7907	7660
7	10,000	0.5645	5645	42341	5.3552	7906

From the above table, we observe that $R_4 < W(5) < R_5$ (ie) Rs 6,500 < Rs. 7609 < Rs 8,000.

∴ The optimum replacement period is 5th year.

(3) A machine costs Rs 10,000. Operating costs are Rs 500% per year for the fist five years. In the sixth and succeeding years operating cost increases by Rs 100% per year. Assuming a 10% discount rate of money per year, find the optimum length of time to hold the machine before we replace it.

solution :-

The discount rate per rupee will be 0.1. Hence the depreciation ratio will be given by

$$\nu = \frac{100}{100+10} = \frac{10}{11} = 0.9091.$$

and C = 10,000.

Now for the solution of the problem we construct the following table to calculate W(n). The expression for w(n) is given by

$$W(n) = \frac{C+\sum_{k}\nu^{k-1}R_{k-1}}{\sum_{k}\nu^{k-1}}$$

Year (n)	Running cost R_{n-1}	v^{n-1} (pwf)	$R_{n-1} v^{n-1}$	$C+\sum_k v^{k-1}R_{k-1}$	$\sum_k v^{k-1}$	W(n) (Rs)
(1)	(2)	(3)	(4)	(5)	(6)	(7)= (5)/(6)
1	500	1.000	500	10,500	1.000	10,500
2	500	0.9091	456	10,956	1.9091	5738.8
3	500	0.8264	413	11,369	2.7355	4156.3
4	500	0.7513	376	11,745	3.4868	3368.4
5	500	0.6830	342	12,087	4.1698	2898.7
6	600	0.6209	373	12,460	4.7907	2600.8
7	700	0.5645	395	12,855	5.3552	2400.4
8	800	0.5132	411	13,266	5.8684	2260.5
9	900	0.4665	420	13,686	6.3349	2160.4
10	1000	0.4241	424	14,110	6.7590	2087.5
11	1100	0.3856	424	14.534	7.1446	2034.2
12	1200	0.3506	421	14,955	7.4952	1995.2
13	1300	0.3187	414	15,369	7.8139	1966.8
14	1400	0.2897	406	15,775	8.1036	1946 6
15	1500	0.2637	396	16,171	8.3673	1932.6
16	1600	0.2397	384	16,555	8.6070	1923.4
17	1700	0.2179	370	16,925	8.8249	1917.8
18	1800	0.198	357	17,282	9.0230	1915.3
19	1900	0.1801	342	17,624	9.2031	**1915.0***
20	2000	0 1637	327	17,951	9.3668	1916.4

From the above table, we observe that

$R_{20} > W(19)$ and $R_{19} < W(18)$

The optimum replacement period is 19 years.

4. A manufacturer is offered two machines A and B. A is priced at Rs 5,000 and running costs are estimated at Rs 800 for each of the first five years, increasing by Rs 200 per year in the sixth and subsequent years. Machine B, which has the same capacity as A, costs Rs 2,500 but will have running costs of Rs 1,200 per year for six years, increasing by Rs 200 per year thereafter.

If money is worth 10% per year, which machine should be purchased? (Assume that the machines will eventually be sold for scrap at a negligible price).

Solution :-

Since the money is worth 10% per year, the discount rates for both the machines is given by

$$\nu = \frac{1}{1+0.10} = 0.9091$$

For the solution of this problem we compute the following tables for machines A and B separately, by using pwf table value

For machine A.

Year (n)	Running cost R_{n-1}	ν^{n-1} (pwf)	$R_{n-1}\nu^{n-1}$	$C+\sum_k \nu^{k-1}R_{k-1}$	$\sum_k \nu^{k-1}$	W(n) (Rs)
(1)	(2)	(3)	(4)	(5)	(6)	(7)= (5)/(6)
1	800	1.000	800	5,800	1.000	5800 00
2	800	0.9091	727	6,527	1.9091	3418.88
3	800	0.8264	661	7,188	2.7355	2627.67
4	800	0.7513	601	7,789	3.4868	2233.85
5	800	0.6330	546	8,335	4.1698	1998.89
6	1,000	0.6209	621	8,956	4.7907	1896.45
7	1,200	0.5645	677	9,633	5.3552	1798.81
8	1,400	0.5132	718	10,351	5.8684	1763.85
9	**1,600**	0.4665	746	11,097	6.3349	**1751.72 ***
10	1,800	0.4241	763	11,860	6.7590	1754.70

For machine B

Year (n)	Running cost R_{n-1}	ν^{n-1} (pwf)	$R_{n-1}\nu^{n-1}$	$c+\sum_k \nu^{k-1}R_{k-1}$	$\sum_k \nu^{k-1}$	W(n) (Rs)
(1)	(2)	(3)	(4)	(5)	(6)	(7)= (5)/(6)
1	1,200	1.00000	1200.00	3700.00	1.0000	3700.00
2	1,200	0.9091	1090.91	4790.91	1.9091	2509.51
3	1,200	0.8264	991.98	5782.59	2.7353	2113.91
4	1,200	0.7513	901.56	6684.15	3.4868	1916.99
5	1,200	0.6830	819.60	7503.75	4.1698	1799.55
6	1,200	0.6209	745.08	8248.83	4.7907	1721.84
7	1,400	0.5645	790.30	9039.13	5.3532	1687.92
8	1,600	0.5132	821.12	9860.25	5.8684	1680.23
9	1,800	0.4665	839.70	10699.95	6.3349	**1689.05***
10	2,000	0.4241	848.20	11548.15	6.7590	1708.56

From the above tables we observe that for machine A,

1,600 < 1,751.72 < 1800

Now since the running cost of 9th year is 1,600 and that of 10th year is 1,800 and since 1,800 > 1,751.72, it is better to replace the machine A after 9th year.

Similarly, for machine B since 1,800 >1680.23 it is better to replace the machine B after 8th year.

Further since the weighted average cost in 9 years of machine A is Rs 1751.72 and the weighted average cost in 8 years of machine B is Rs. 1,680.23, it is advisable to perchase machine B.

(5) A person is considering to purchase a machine for his factory. The related data about the alternative machine are as follows.

	Machine A	Machine B	Machine C
Present investment Rs	10,000	12,000	15,000
Total annual cost Rs	2,000	1,500	1,200
Life years	10	10	10
Salvage value Rs	500	1,000	1,200

As an advisor of the company, you have been asked to select best machine considering 12% normal rate of return per year.

Given PWF 12% for 10 years = 5.650

Caf 12% for 10 years 0.322

Solution :-

Here present value of total cost of each machine for a period of 10 years is to be calculated and the machine for which the present value is least to be recomended. The calculations are given below.

	Machine A	Machine B	Machine C
Present value of			
(1) Present investment	10,000	12,000	15,000
(2) Total annual cost	2000 x 5.650	1500 x 5.650	1200x5.650
(3) Salvage value	500x0.322	1,000x0.322	1200x0.322
Total value Rs	21,283.90	Rs 20,153. 00	Rs 21,393.60

= [present investment+
total annual cost -
Salvage value]

The machine B, having least present value of total cost, should be purchased.

Exercise :-

1. What is replacement? Describe some important replacement situations.

2. Let v = 0.9 and initial price is Rs. 5,000. Running cost varies as follows.

Year	:	1	2	3	4	5	6	7
Running cost (in Rs)	.	400	500	700	1,000	1,300	1,700	2,100

What would be the optimum replacement interval?

3. A truck has been purchased at a cost Rs 1,60,000. The value of the truck is depreciated in the first three years by Rs 20,000 each year and Rs 16,000 per year thereafter Its maintenance and operating costs for the first three years are Rs 16,000, Rs 18,000 and Rs 20,000 in that order and increase by Rs 4,000 every year. Assuming an interest rate of 10% find the economic life of the truck.

4. A machine costs Rs 10,000. Operating costs are Rs 500 per year for the first five years. Operating costs increase by Rs 100 per year in the sixth and succeeding years Assuming a 10 percent discount rate of money per year, find the optimum length of time to hold the machine before it is replaced.

5. A Pipeline is due for repairs. It will cost Rs 10,000 and last for 3 years Alternatively, a new pipeline can be laid at a cost of Rs 30,000 and lasts for 10 years. Assuming cost of capital to be 10% and ignoring salvage value, which alternative should be chosen?

Answers :-

(2) six years

(3) Seven years

(4) Nineteen years.

(5) Existing pipeline should be continued.

8.3 Replacement of items that fail suddenly :

It is difficult to predict that a particular equipment will fail at a particular time. This difficulty can be overcome by determining the probability distribution of failures. Here it is assumed that the failurers occur only at the end of the period, say t. Thus the objective becomes to find the value of t which minimizes the total cost invovled for the replacement.

We shall consider the following two types of replacement policies,

Individual Replacement policy .- Under this policy, an item is replaced immediately after its failure.

Group replacement policy : Under this policy, we take decisions as to when all the items must be replaced, irrespective of the fact that items have failed or have not failed, with a provision that it any item fails before the optimal time, it may be individually replaced.

Mortality Tables :

These are used to derive the probability distribution of the life span of an equipment, let,

$M(t)$ = number of survivors at any time t

$M(t-1)$ = number of survivors at any time t-1, and

N = initial number of equipments.

Then the probability of failure during time period t is given by

$$p(t) = [M(t-1) - M(t)] / N$$

The probability that an equipment survived till age (t-1), wili fail during the interval (t-1) to t can be defined as the conditional probability of failure. It is given by

$$Pc(t) = [M(t-1) - M(t)] / M(t-1)$$

The probability of survival till age t is given by

$$ps(t) = M(t) / N$$

Theorem 8.1 (Mortality)

A large population is subject to a given mortality law for a very long period of time. All deaths are immediately replaced by births and there are no other entries or exits. Then the age distribution ultimately becomes stable and that the number of deaths per unit time becomes constant (which is equal to the size of the total population divided by the mean age of death)

Proof :-

Let k be a constant such that no item of the system can survive upto and beyond time k+1 (ie) the life span of any item lies between t=0 and t=k. We define

f(t) : the number of births at time t, and

p(x) : the probability that an equipment will die (fail) just before achieving the age x+1, (ie) at age x.

It is easy to note that $\sum_{x=0}^{k} p(x) = 1$. Now f(t-x) births take place at time t-x, t=k , k+1,.... such newly born items attain the age x at time t. Therefore the expected number of deaths of such alive numbers at time t is p(x)f(t-x). Thus the mathematical expectation of the number of deaths before time t+1 is $\sum_{x=0}^{k} f(t-x)p(x)$, t=k, k+1, ...

Moreover, since deaths are immediately replaced by births. We must have

$$f(t+1) = \sum_{x=0}^{k} f(t-x)\, p(x), \quad t = k, k+1, \ldots \ldots$$

The solution to this difference equation in t can be obtained by making use of

$f(t) = A\alpha^1$, where A is some constant and $|\alpha|<1$

$$\therefore A\alpha^{t+1} = \sum_{x=0}^{k} A\alpha^{t-x}\, p(x) = A[\alpha^t p(0)+\alpha^{t-1} p(1)+\ldots+\alpha^{t-k} p(k)]$$

(or) $\alpha^{k+1} = \alpha^k\left[\sum_{x=0}^{k} \alpha^{-x}p(x)\right]$

$$= \alpha^k[p(0)+\alpha^{-1}p(1)+......+\alpha^{-k}p(k)]$$

Thus $\alpha^{k+1}\ [\alpha^k p(0)+\alpha^{k-1}p(1)+.....+p(k)]=0$

This is a linear homogeneous difference equation of degree k+1 and thus has exactly k+1 roots. Let roots be $\alpha_0, \alpha_1, ... \alpha_k$ For α =1, the equaiton yields.

L.H.S. = $1- \sum_{x=0}^{k} p(x) = 1-1 =$ R.H.S.

Thus $\alpha=1$ is a root of the above equation. Let us denote it by $\alpha_0=1$. The most general solution of the difference equation will be of the form.

$$f(t) = A_0\alpha_0^t + A_1\alpha_1^t+....+A_k\alpha_k^t$$
$$= A_0+A_1\alpha_1^t+.....+A_k\alpha_k^t$$

where $A_0, A_1,......A_k$ are constants whose values are to be determined. We observe that,

since $|\alpha_t|<1$ as $t \to \alpha$, $\lim_{t\to\propto} f(t) = A_0$.

Thus under our assumption for a long period t, the number of deaths per unit time is equal to A_0.

Now the problem is to determine the value of the constant A_0.

let g(x) = probability of survivor for more than x years.

(or) g(x) = 1-P (survivor will die before attaining the age x)

$$= 1-[p(0)+p(1)+.......+p(x-1)\}$$

Obviously, it can be assumed that g(0) =1.

Since the number of births as well as deaths have become constant, each equal to A_0, therefore expected number of survivors of age x is given by $A_0 \cdot g(x)$

As deaths are immediately replaced by births and therefore size N of the population remains constant.

Thus, we must have

$$N= A_0 \sum_{x=0}^{k} g(x) \text{ or } A_0 = N\Big/\sum_{x=0}^{k} g(x)$$

From finite differences, we know that

$$\Delta(x) = (x+1) -x = 1$$

and $\sum_{x=0}^{b} f(x)\ \Delta h(x) = f(b+1)h(b+1) - f(a)h(a) - \sum_{x=a}^{b} h(x+1)\Delta f(x)$

Therefore, we can write,

$$\sum_{x=0}^{k} g(x) = \sum_{x=0}^{k} g(x)\ \Delta(x) = (k+1)g(k+1) - \text{O}xg(0) -\sum_{x=0}^{k} (x+1)\Delta g(x)$$

$$= (k+1)\, g(k+1) - \sum_{x=0}^{k} (x+1)\Delta g(x)$$

But $g(k+1) = 1-[p(0)+p(1)+\ldots+p(k)] = 0$

and $\Delta g(x) = g(x+1) - g(x)$

$$= [1-p(0) - p(1) - \ldots -p(x)] - [1-p(0) - \ldots -p(x+1)]$$

$$= -p(x)$$

$$\therefore \sum_{x=0}^{k} g(x) = (k+1)\, g(k+1) - \sum_{x=0}^{k} (x+1)\, [-p(x)]$$

$$= \sum_{x=0}^{k} (x+1)\, p(x);$$

which happens to be the mean (expected age at death)

Hence, A_0=N/ Mean age at death.

Theorem 8.2 : (Group replacement)

Let all the items in a system be replaced after a time interval 't' with provisions that individual replacements can be made if and when any item fails during this time period. Thus (a) Group replacement must be made at the end of t^th^ period if the cost of individual replacement for the period is greater than the average cost per unit time period thorugh the end of t periods.

(b) Group replacement is not advisable at the end of period t if the cost of individual replacement at the end of period t-1 is less than the average cost per period through the end of period t.

Proof : Let,

N = total number of items in the system,

C_2 = Cost of replacing an individaul item,

C_1 = Cost of replacing an item in group,

C(t) = total cost of group replacement after time period t,

f(t) = number of failurers during time period t.

Then, clearly

$$C(t) = NC_1+C_2 \sum_{x=1}^{t-1} f(x)$$

The average cost of group replacement per period of time during a period t, is thus given by

$$\frac{C(t)}{t} = \left[NC_1+C_2 \sum_{x=0}^{t-1} f(x)\right] \Big/ t$$

We shall determine the optimum t so as to minimize C(t) / t.

Note that whenever $\frac{C(t-1)}{t-1} > \frac{C(t)}{t}$ and

$\frac{C(t+1)}{t+1} > \frac{C(t)}{t}$, it is better to replace all the items after time period t.

$$\text{Now, } \frac{C(t+1)}{t+1} - \frac{C(t)}{t} > 0 \Rightarrow C_2 f(t) > C(t)/t;$$

$$\text{and } \frac{C(t-1)}{t-1} - \frac{C(t)}{t} > 0 \Rightarrow C_2 f(t-1) < C(t)/t$$

$$tC_2 f(t-1) < C(t) < t.C_2 f(t)$$

$$\text{(or) } t\, f(t-1) - \sum_{x=0}^{t-1} f(x) < \frac{NC_1}{C_2} < tf(t) - \sum_{x=0}^{t-1} f(x)$$

Sample Problems:-

1. The following failure rates have been observed for a certain type of transistors in a digital computer :-

End of the weak :	1	2	3	4	5	6	7	8
Probability of failure to date :	0.05	0.13	0.25	0.43	0.68	0.88	0.96	1.00

The cost of replacing an individual failed transistor is Rs 1 25. The decision is ma replace all these transistors simultaneously at fixed intervals, and to replace the individual transistors as they fail in service. If the cost of group replacement is 30 paise per transistor, what is the best interval between group replacements? At what group replancement price per transistor would a policy of strictly individual replacement become preferable to the adopted policy?

Solution :-

Suppose these are 1,000 trnasistor in use. Let p_i be the probability that a transistor, which was new when placed in position for use, fails during the ith week of its life. Thus, we have

$p_1 \equiv 0.05$ $\qquad$ $p_2 \equiv 0.13 = 0.5 = 0.08$

$p_3 \equiv 0.25 - 0.13 = 0.12$ $\qquad$ $p_4 \equiv 0.43 - 0.25 = 0.18$

$p_5 \equiv 0.68 - 0.43 = 0.25$ $\qquad$ $p_6 \equiv 0.88 - 0.68 = 0.20$

$p_7 \equiv 0.96 - 0.88 = 0.08$ $\qquad$ $p_8 \equiv 1.00 - 0.96 = 0.04$

Let N_i denote the number of replacements made at the end of the ith week.

Then, we have

N_0 = Number of transistors in the beginning = 1,000

$N_1 = N_0p_1$ = 1,000 x 0.05 = 50

$N_2 = N_0p_2+N_1p_1$ = 1,000x0.08+50x0.05 = 82

$N_3=N_0p_3+N_1p_2+N_2p_1$=1,000x0.12+50x0.08+82x0.05= 128

$N_4 = N_0p_4+N_1p_3+N_2p_2+N_3p_1$ = 199

$N_5 = N_0p_5+N_1p_4+N_2p_3+N_3p_2+N_4p_1$ = 289

$N_6 = N_0p_6+N_1p_5+N_2p_4+N_3p_3+N_4p_2+N_5p_1$ = 272

$N_7 = N_0p_7+N_1p_6+N_2p_5+N_3p_4+N_4p_3+N_5p_2+N_6p_1$ = 194

$N_8 = N_0p_8+N_1p_7+N_2p_6+N_3p_5+N_4p_4+N_5p_3+N_6p_2+N_7p_1$ = 195

From the above calculations, we observe that the expected number of transistors failing each week increased till 5th week and then starts decreasing and later again increasing from 8th week.

Thus, Ni will oscillate till the system acquires a steady state. The expected life of each transistor is

1x0.5+22x0.08+3x.12+4x.18+5x0.25+6x0.2+7x0 08+8x0.04 = 4.62

Average number of failures per week = 1,000 / 4.62 = 216 approximately.

Therefore, the cost of individual replacement

= 216 x 1.25 = Rs 270 00

Now, since the replacement of all the 1,000 transistors simultaneously cost 30 paise per transistors and the replacement of an individual transistor on failure cost Rs 1.25, the average cost for different group replacement policies is given as under :-

End of Week	Individual replacement	Total cost (Rs) Individual+Group	Average cost (Rs)
1	50	50x1.25+1000x0.30=363	363
2	132	132x1.25+1000x0.30=465	232.50
3	260	260x1.25+1000x0.30=625	**208.30***
4	459	459x1.25+1000x0.30=874	218.50

Since the average cost is lowest against week 3, the optimum interval between group replacements is 3 weeks. Further, since the average cost is less than Rs 270 (For individual replacement), the policy of group replacement is better.

2. At time zero all items in a system are new. Each item has a probability p of failing immediately before the end of the first month of life, and a probablity q = 1-p of failing immediately before the end of the second month (ie, all items fail by the end of the second month). If all items are replaced as they fail, show that the expected number of failures f(x) at the end of month x is given by $f(x) = \frac{N}{1+q}[1-(-q)^{x+1}]$,

where N is the number of items in the system.

If the cost per item of individual replacement is C_1, and the cost per item of group replacement is C_2 find the condition under which

(a) A group replacement policy at the end of each month is the most profitable.

(b) No group replacement policy is better than a policy of purse individual replacement.

Solution :-

Let N = Number of items in the system in the beginning.

N_1 = number of items expected to fail at the end of 1st month

$= N_0p = N(1-q)$, since $p = 1-q$

N_2 = number of items expected to fail at the end of 2nd month

$= N_0q+N_1p = Nq+N(1-q)^2 = N(1-q+q^2)$

N_3 = number of items expected to fail at the end of 3rd month

$= N_1q+N_2p = N(1-q)q+N(1-q+q^2)(1-q)$

$= N(1-q+q^2-q^3)$

and so on. In general,

$N_k = N[1-q+q^2-q^3+......+(-q)^k]$

$N_{k+1} = N_{k-1}q+N_kp$

$= N[1-q+q^2+.....+(-q)^{k-1}]q + N[1-q+q^2+....+(-q)^k](1-q)$

$= N[1-q+q^2+.....+(-q)^{k+1}]$.

Hence by mathematical induction, the expected number of failurers at the end of month x will be given by

$f(x) = N[1-q+q^2+.....+(-q)^x]$

$= N[1-(-q)^{x+1}] / (1+q)$.

The value of f(x) at the end of month x will vary for different values of $(-q)^{r+1}$ and it will reach the steady - state as $x \to \propto$.

Hence, in the steady state case, the expected number of failures will be

$$\lim_{x\to\propto} f(x) = \lim_{x\to\propto} N[1-(-q)^{x+1}] / 1+q \ .$$

$= N/(1+q)$; since $q<1$ and $(-q)^{x+1} \to o$ as $x \to \propto$.

= Total number of items in the system / Mean age.

Now, since C_1 is the cost of replacement per item individualy and C_2 is the cost of an item in group, therefore,

(i) If we have a group replacement at the end of each month, then the cost of replacement is NC_2.

(ii) If we have a group replacement policy at the end of the every other month, then the cost is. $NC_2+N_pC_1$

The average cost per month, therefore, is $(NC_2+N_pC_1)/2$ and

(iii) Average life of an item

$= 1xp+2xq = 1x(1-q)+2q = 1+q$

Therefore, the average number of failurers is $N/(1+q)$ and hence the cost of individual replacement is $NC_1/(1+q)$

(a) A group replacement at the end of first month will be better than individual replacement, of total cost of group replacement is less than the average monthly cost of individual replacement.

Thus, $N(1-q)C_1+NC_2 < NC_1 / (1+q)$

(ie) $C_2<C_1q^2 /1+q$

For a group replacement at the end of every second month, the total cost of replacement will be

$(N_1+N_2)C_1 + NC_2 = N(2-2q+q^2)C_1 + NC_2$

Average monthly cost of group replacement at the end of second month is

$[N(2-2q+q^2)C_1+NC_2] /2$

In this case, the group replacement policy will be better than the individual replacement policy, if

Average monthly cost of group replacement < Average monthly cost of individual replacement

(or) $[N(1-q+q^2/2)C_1 + NC_2 /2] < NC_1 / (1+q)$

(or) $C_2 < q^2(1-q)C_1 / (1+q)$

(b) For the individual replacement policy to be beter than any of the group replacement policies discussed above, we must have,

$C_2 > C_1q^2 / (q+1)$ and $C_2 \geq C_2q^2 (1-q) / (q+1)$

(or) $C_1 < C_2(1+q) / q^2$ and $C_1 < C_2(1+q) / [q^2(1-q)]$

But $q<1$, therfore $(1+q) / q^2 < (1+q) / q^2(1-q)$

Hence, $C_1 < (1+q) C_2 / q^2$

(3) A computer contains 10,000 resistors. When any resistor fails, it is replaced. The cost of replacing a resistor individually is Rs 1 only. If all the resistors are replaced at the same time, the cost per resistor would be reduced to 35 paise. The percent serviving say S(t) at the end of month t and P(t) the probability of failure during the month t, are

t :	0	1	2	3	4	5	6
S(t) :	100	97	90	70	30	15	0
P(t) :	-	0 03	0.07	0.20	0.40	0.15	0.15

What is the optimum replacement plan?

solution :-

The whole problem can be divided into two parts :-

(1) There is a policy of individual replacement.

(2) There is a policy of group replacement.

It should be noted that no resistor survives for more than 6 months. Thus a resistor which has survived for 5 months is sure to fail during sixth month. We assume that the resistors failing during a month are replaced just at the end of the month.

Let Ni denote the number of resistors replaced at the end of i^{th} month. The different values of Ni can be calculated in the following way:-

N_0 = Number of resistors in the beginning = 10,000.

N_1 = Number of resistors being replaced by the end of 1st month.

= Number of resistors during the 1st month x probability that a resistor fails during 1st month of installation = 10,000 x 0.03 = 300

N_2 = Number of resistor replaced by the end of second month

= (Number of resistors in beginning x probability that these fail in 2nd month)+ Number of resistors replaced in first month X probability that these fail during second month)

$= N_0p_2+N_1p_1$

$= (10,000\text{x}0.07)+(300\text{x}0.03)= 709$

$N_3 = N_0p_3+N_1p_2+N_2p_1$

$= (10,000\text{x}0.20)+(300\text{x}0.07)+(709\text{x}0.03)$

$= 2042$

$N_4 = N_0p_4+N_1p_3+N_2p_2+N_3p_1$

$= (10,000\text{x}0.40)+(300\text{x}0.20)+(709\text{x}0.07)+(2042\text{x}0.03)$

$= 4171$

$N_5 = N_0p_5+N_1p_4+N_2p_3+N_3p_2+N_4p_1$

$= (10,000\text{x}0.15)+(300\text{x}0.40)+(709\text{x}0.20)(2042\text{x}0.07)+(4171\text{x}0.03)$

$= 2030$

$N_6 = N_0p_6+N_1p_5+N_2p_4+N_3p_3+N_4p_2+N_5p_1$

$= (10,000\text{x}0.15)+(300\text{x}0.15)+(709\text{x}0.40)(2042\text{x}0.20)+(4171\text{x}0.07)+(2030\text{x}0.03)$

$= 2590$

It can be seen from the above calculaltions that Ni increased upto fourth month and then decreases. It can also be seen that Ni will later tend to increase and the value of Ni will osciallate till the system acquires a steady state.

The expected life of each resistor is given by

$= \sum x_i P_i$, where x_i is the month and P_i is the corresponding probability of failure.

= (1x0.03)+(2x0.07)+(3x0.20)+(4x0.40)+(5x0.15)+(6x0.15)

= 0.02 months.

∴ Average number of replacements every month

$$= \frac{N}{(\text{mean age})} = \frac{10000}{4.02}$$

$$= 2487.5 \simeq 2488 \text{ resistors.}$$

Here average cost of monthly individual replacement policy = Rs. 2488 (the cost being Rs.1 per resistor) Now let us consider the policy of group replacement

End of month	Total cost of group replacement in Rs	cost/month in Rs
1	300x1+10,000x0 35 = 3800	3800
2	(300+709)x1+10,00x0.35=4509	2254.50
3	(300+709+2042)x1+10,000x0.35 = 6551	2183.66
4	(300+709+2042+4171)x1 +(10,000x0.35) = 10722	2680.50
5	(300+709+2042+4171+2030)x1 +10,000x0.35 = 12752	2550.40
6	15342	2557.00

Hence the minimum cost per month is obtained by group replacement of all resistors after three months with an average cost of Rs 2183.66 per month.

(4) The management of a large hotel is considering the periodic replacement of light bulbs fitted in its rooms. There are 500 rooms in the hotel and each room has 6 bulbs. The management is now following the policy of replacing the bulbs as they fail at a total cost of Rs.3 per bulb. The management feels that this cost canbe reduced to Rs.1 by adopting the periodic replacement method. On the busis of the information given below, evaluate the alternatives and make a recomendation to the management.

Months of use	Percent of bulbs failing by that month	prob-of faillure during the month
1	10	.10
2	25	.15
3	50	.25
4	80	.30
5	100	.20

Solution :-

The average time of failure is

= (1x0.10)+(2x0.15)+(3x0.25)+(4x0.30)+(5x0.20)

= 3.35 months.

These are 6x500 = 3000 bulbs in all the rooms.

Hence average number of replacement per month

$=\frac{3000}{3.35}=$ 895.5 bulbs

Thus cost of individual replacement will be Rs(895.5x3) = Rs. 2686.5 per month.

The group replacement strategy can be determined with the following calculations With an initial lot of 3000 bulbs, the number of bulbs failing at the end of each month from the initial lot will be

Number of bulbs failing each month from initial lot of 3000 bulbs and those replaced during each period on failure.

Initially at period zero all bulbs are working.

$N_0 = 0$

N_1 = Failure after one month

= No. of bulbs in order at time t = 0.

x prob. of failure after one month

= 3000 x 0.10 = 300

N_2 = Failure after two months

= (No. of bulbs installed at t =0 x prob. of failure after 2 months)

+(No. of bulbs replaced on failure after one month x prof of failure after one month)

= (3000 x 0.15) +(300 x 0 10)

= 480

N_3 = Failure after three months

= $N_0p_3+N_1p_2+N_2p_1$

= (3000x0.25) + (300x0.15) + (480x0.10)

= 843

N_4 = Failure after four months

= $N_0p_4+N_1p_3+N_2p_2+N_3p_1$

= (3000x0.30)+(300x0.25)+(480x0.15)+(843x0.10)

= 1131.30

N_5 = Failure after five months

= $N_0p_5+N_1p_4+N_2p_3+N_2p_2+N_4p_1$

= (3000x0.20)+(300x0.30)+(480x0.25)+(843x0.15)+(1131.30x0.10)

= 1049.58

Period in months	Total cost of Group Replacement		Total cost in Rs.	Average cost / period in Rs =total cost / No of months
	Cost of group Replacement of 3000 bulbs in Rs. 1- per bulb	Cost of individual replacement on failure in Rs. 3 per bulb		
1	3000	300x3=900	3900	3900
2	3000	(300+480)3=2340	5340	5340/2=2670
3	3000	(300+480+843)x2 = 4869	7869	7869/3=2623
4	3000	(300+480+843+ 1131.3)x3 = 8262.9	11262.9	11262.9/4 = 2815.73
5	3000	(300+480+843 +1131.3 +1049.58) x3 = 11411.64	14411.64	11411.64/5 = 2882.38

It can be observed from the above table that total average cost of group replacement is minimum after a period of three months (being Rs. 2623) which is also less than the cost of individual replacement policy (Rs 2686.50). Hence the optimum period for group replacement is three months.

Exercise :-

1. These are 1,000 bulbs in the system. Survival rate is given below

Week	:	0	1	2	3	4
Bulbs in operation at the end of week	:	1000	850	500	200	00

The group replacement of 100 bulbs costs Rs 100 and individual replacement is Rs. 0.50 per bulb. Suggest suitable replacement policy.

2. The following mortality rates have been observed for a certain type of light bulbs.

Week	:	1	2	3	4	5
percent failing by end of week	:	10	25	50	80	100

There are 1,000 bulbs in use, and it costs Rs.2 to replace an individual bulb which has burnt out. If all bulbs were replaced simultaneously it would cost 50 paise per bulb. It is proposed to replace all bulbs at fixed intervals, whether or not they have burnt out, and to continue replacing burnt out bulbs as they fail. At what interval should all the bulbs be repalced?

3) A computer contains 10,000 resistors. When any resister fails, it is replaced. The cost of replacing a resistor individually is Rs1. If all the resistors are replaced at the same time, the cost per resistor would be reduced to 35 paise. The percent surviving say S(t) at the end of month t and P(t) the probability of failure during the month t, are

t	0	1	2	3	4	5	6
S(t)	100	97	90	70	30	15	0
P(t)	-	0.03	0.07	0.20	0.40	0.15	0.15

What is the optimum replacement plan?

Answers :

1) Group replacement after a week.
2) Group replacement after second week.
3) Group replacement after 3 months.

8.4. : Sequencing Problems

Introduction :

In this chapter, we determine an appropriate order (sequence) for a series of jobs to be done on a finite number of service facilities, in some pre-assigned order, so as to optimize the total involved cost (time). A practical situation may correspond to an industry producing a number of products, each of which are to be processed through different machines, of course, finite in number.

Problem of sequencing

Consider a problem of machine operator who has to perform three operations, namely (i) turning, (ii) threading and (iii)knurling on a finite number of different jobs. Let there be six jobs the time required to perform these operations (in minutes) for each job is known Also it is given that each job first goes for turning, then for threading, and lastly for knurling The problem of the machine operator is to decide which job should be processed first, which to process next, and so on, ie. the order (sequence) of the jobs for the above mentioned operations in order to minimize the total time required to turn out all the jobs. This is an example of six job and three machine sequencing problem. We now consider the general case.

Let there be n jobs, each of which has to be processed one at a time, on each of the m different machines. The order in which these machines are to be used for processing each job as well as the expected or actual processing time of each job on each of the machines is known.

The sequencing problem then is to collect from the $(n!)^m$ theoretical feasible alternatives, the one that is both technologically feasible and minimizes the total elapsed time (ie. the time from the start of the first job to the completion of the last job as well as idle time of machines). A technologically feasible sequence is one which satisfies the constraints (if any) or the order in which each job must be performed through m machines.

Remark :- Theoretically a solution by enumeration is always possible, but in practice, it is impossible because of the large number of cumputations involved even for moderate values of m and n. For example, it there are five jobs to be processed at each of the four machines (ie, n =5 and m=4), the total number of different theoretically possible different sequences will be $(5!)^4$ = 207,360,000.

In the present chapter, we shall discuss some procedures that help in solving a sequencing problem, without actually considering all the possible sequences.

Terminology, Notations and Assumptions

The following terminology and notations shall be used :

M_{ij} = processing time for job i on machine j.

x_{ij} = idle time on machine j from the end of (i-1)th job to the start of job i.

T = total elapsed time for processing all the jobs including idle time, if any.

Following assumptions are usually made while dealing with sequencing problems:-

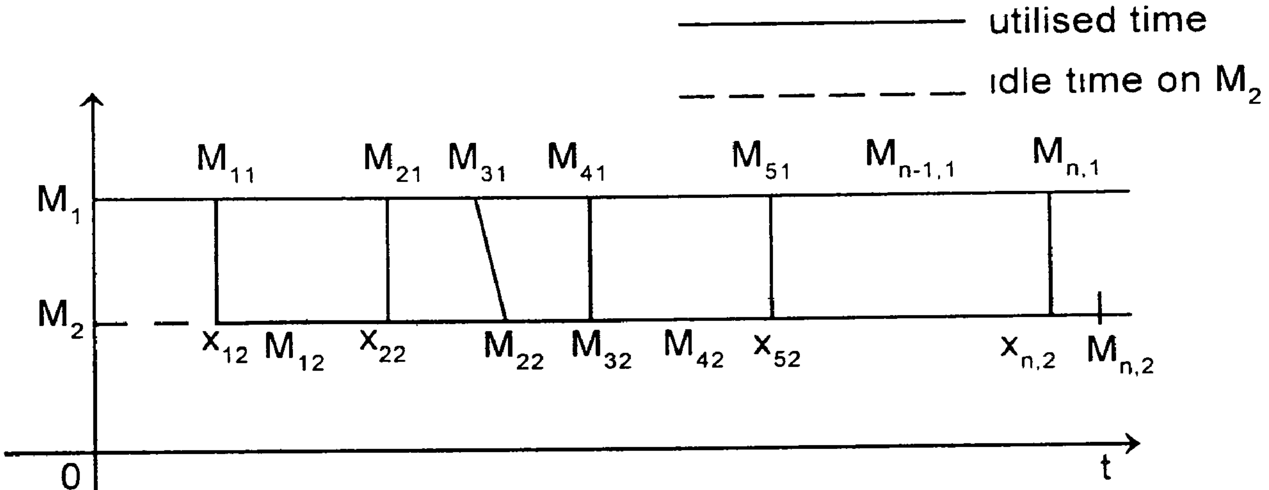
(i) only one operation is carried out on a machine at a time
(ii) Processing times are known and do not change.
(iii) Processing time Mij's (i=1,2,. .....n) (j = 1,2,.. ...m) are independent of order of processing the jobs.
(iv) The time involved in moving jobs from one machine to another is neligible.
(v) Each operation, once started, must be completed.
(vi) An operation, must be completed before its succeeding operation can start.
(vii) Only one machine of each type is available
(viii)A job is processed as soon as possible, but only in the order specified
(ix) No passing rule is followed strictly, ie, the same order of jobs is maintained over each machine

8.5 :Problems with n jobs and two machines

Let there be n jobs, each is to be processed through two machines, say M_1 and M_2 in order M_1M_2 That is, each job will go to machine M_1 first and then to M_2; or in other words, passing is not allowed. Let Mij (i=1,2,....... .n; j =1,2) be the time required for processing ith job on the jth machine. (i=1,2,... ....n). Since passing is not allowed, it is obvious that all n jobs are to be processed on machine M_1 without any idle time for it On the otherhand, machine M_2 is subject to its remaining idle at various stages Let X_{i2} be the time for which machine M_2 remains idle after finishing (i-1)th job and before starting processing the ith job Clearly, the total elapsed time T is given by

$$T = \sum_{i=1}^{n} M_{i2} + \sum_{i=1}^{n} x_{i2}$$ Where some of the X_{i2}'s may be zeros.

The problem now is to minimize T. However, Since $\sum M_{i2}$ is the total time for which machine M_2 has to work and is thus fixed, it does not form a part of the optimization problem. Thus the problem reduces to that of minimizing $\sum X_{i2}$. A very convenient procedure for obtaining a sequence of performing jobs to minimize $\sum X_{i2}$ is well illustrated by the Ganttchart



Time taken in hours

Gantt chart

From the chart, it is apparent that

$$X_{12} = M_{11}$$

$$X_{22} = \begin{cases} M_{11}+M_{21}-X_{12}-M_{12} & \text{if } M_{11}+M_{21} > X_{12}+M_{12} \\ 0 & \text{otherwise} \end{cases}$$

The expression for X_{22} may be rewritten as

$$X_{22} = \text{Max. }\{M_{11}+M_{21}-X_{12}-M_{12},\ 0\}$$

Thus

$$X_{12}+X_{22} = \text{Max. }\{M_{11}+M_{21}-M_{12}, M_{11}\} \text{ since } X_{12} = M_{11}$$

similary

$$X_{32} = \text{Max. }\{M_{11}+M_{21}+M_{31}-M_{12}-M_{22}-X_{12}-X_{22},\ 0\}$$

This gives

$$X_{12}+X_{22}+X_{32} = \text{Max. }\left\{\left(\sum_{i=1}^{3} M_{i1} - \sum_{i=1}^{2} M_{i2}\right), \sum_{i=1}^{2} X_{i2}\right\}$$

$$= \text{Max}\left\{\left(\sum_{i=1}^{3} M_{i1} - \sum_{i=1}^{2} M_{i2}\right), \left(\sum_{i=1}^{2} M_{i1} - M_{i2}\right), M_{11}\right\}$$

In general, we have

$$\sum_{i=1}^{n} X_{i2} = \text{Max.}\left\{\left(\sum_{i=1}^{n-1} M_{i1} - \sum_{i=1}^{n-1} M_{i2}\right), \left(\sum_{i=1}^{n-1} M_{i1} - \sum_{i=1}^{n-2} M_{i2}\right), \ldots\ldots M_{11}\right\}$$

$$= \underset{1 \leq u \leq n}{\text{Max.}}\left\{\sum_{i=1}^{n} M_{i1} - \sum_{i=1}^{n} M_{i2}\right\}$$

Now, if we denote $\sum_{i=1}^{n} X_{i2}$ by $D_n(S)$, then the problem becomes that of finding the sequence <S*> for processing the jobs 1,2,........,n. so as to have the inequality $D_n(S^*) < D_n(S_0)$ for any sequence $<S_0>$ other than <S*>. In other words, one has to determine an optimal sequence <S> so as to minimize $D_n(S)$. This can be achieved iteratively by successively interchanging the consecutive jobs. Each such interchange of jobs gives a value of $D_n(S)$smaller than or equal to its value before the change.

Optimal sequence Algorithm

The iterative procedure for determining the optimal sequence for n jobs and two machines M_1 and M_2 in the order M_1M_2 can be summarized as follows :-

Step 1 :- Examine M_{is}'S and M_{i2}'S for i - 1,2,.......n and select the minimum of these, ie, min.$\{M_{i1}, M_{i2}\}$. Let this minimum occur for some i = k.

Step 2 :- If the samllest processing time is for the machine M_1. Process (do) that (kth) job first and place it at the beginning of the sequence. If it is for the machine M_2, Process (do) the kth job in the last and place it at the end of the sequence.

Step 3:- When there is a tie in selecting the minimum processing times, then there may be three situations (say for i = k and j = r)

(i) If there are equal minimal entries (processing times) one for each machine, then place the place job in the machine M_1, First and the one in the machine M_2 last in the job sequence.

(ii) If the equal minimal values occur only for the machine values M_1, select the job with the larger processing time in M_2 for placing in the job sequence first.

(iii) If the equal minimal values occurs only for machine M_2 select the job with the larger processing time in M_1 to be placed in the job sequence last. The next to largest processing time in M_1 determine the penultimate job.

Step 4 :- Cross off the jobs already assigned. If all the jobs have been assigned, go to the next step. Otherwise, repeat steps 1 to 3.

Step 5 : Calculate the time at which each job in the sequence will be processed on machine M_1. This time can be calculated as follows :

Time at which the ith job in a sequence finishes on machine M_1.

= time when the (i-1)th job in a sequence finishes on machine M_1 plus the time of processing the ith job on machine M_1 ; (i=1,2,.......n) and the time for start of first job on machine M_1 is 0.

Step 6 :- Calculate the time at which each job in the sequence will start and finish on machine M_2 as follows :-

(i) time when the first job in a sequence start on machine M_2 = time when the first job in a sequence finishes on machine M_1.

(ii) time the i^{th} job in a sequence finishes on M_2 = time when the i^{th} job in a sequence start on machine M_2 + the processing time of i^{th} job on machine M_2.

(iii) time at which the $(i+1)^{th}$ job in a sequence finishes on machine M_2 = Max.{time when the $(i+1)^{th}$ job in a sequence finishes on machine M_1, time when the i^{th} job in a sequence finishes on machine M_2}

Step 7 :- Calculate the total elapsed time to process all jobs through machines, ie, time when the n^{th} job in a sequence finishes on machine M_2.

Step 8 :- Compute the idle time for machines M_1 and M_2 as follows.

(i) Idle time for machine M_1 = time when the n^{th} jobs in a sequence finishes on machine M_2 minus the time when the n^{th} job in a sequence finishes on machine M_1.

(ii) Idle time for machine M_2 = time at which the first job in a sequence finishes on machine M_1 plus $\sum_{i=2}^{n}$ (time when the i^{th} job in a sequence starts on machine M_2 - time when the $(i-1)^{th}$ job in a sequence finishes on machine M_2}.

Note :- the procedure out lined above for the processing of n jobs on 2-machines gives us the minimum total elapsed time.

Sample Problems :-

1. In a factory, there are six jobs to perform, ech of which should go through two machines A and B, in the order A,B. The processing timings (in hours) for the jobs are given here. You are required to determine the sequence for performing the jobs that would minimize the total elapsed time, T. what is the value of T?

Job :	J_1	J_2	J_3	J_4	J_5	J_6
machine A :	1	3	8	5	6	3
machine B :	5	6	3	2	2	10

Solution : The smallest processing time in the given problem is 1 on machine A. So, perform J_1 in the beginning as shown below.

J_1					

The reduced set of processing times becomes

Job :	J_2	J_3	J_4	J_5	J_6
Machine A :	3	8	5	6	3
Machine B :	6	3	2*	2*	10

The mininmum processing time in this reduced problem is 2 which corresponds to J_4 and J_5 both on machine B. Since the corresponding processing time of J_5 on machine A is larger than the corresponding processing time of J_4 on A, J_5 will be processed in the last and J_4 shall be penultimate. The updated job sequence is

J_1				J_4	J_5

The remaining processing times are

Job :	J_2	J_3	J_6
Machine A :	3*	8	3*
Machine B :	6	3*	10

Now there is a tie among 3 jobs for the smallest processing time in this reduced problem. These correspond to J_2 and J_6 on machine A, and to J_3 on machine B. As the corresponding processing time of J_6 on machine B is larger than the corresponding processing time of J_2 on machine B, J_6 will be processed next to J_1. Now step 3(1) applies and J_2 should be placed next. The updated job seqence is

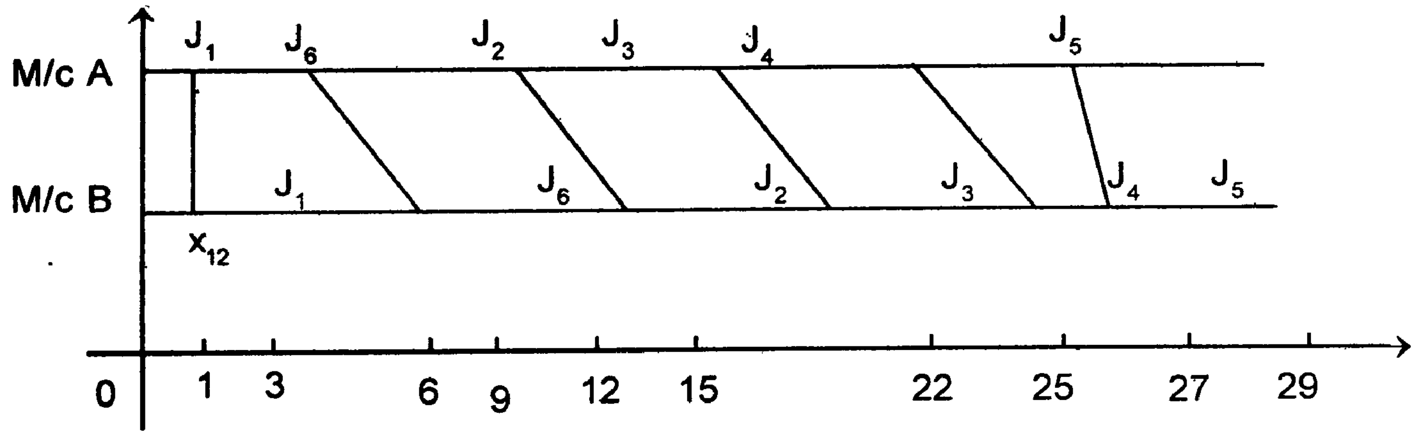
J_1	J_6	J_2	J_3	J_4	J_5

The sequence is the optimum one. The total elapsed is calculated below.

Job	Machine A		Machine B		Idle time on B.
	In	Out	In	Out	
J_1	0	1	1	6	1
J_6	1	4	6	16	-
J_2	4	7	16	22	-
J_3	7	15	22	25	-
J_4	15	20	25	27	-
J_5	20	26	27	29	-

From the above information we get T = 29 hours. Idle time of machine A is (29-26) =3 hours and that for machine B is one hour.

The Gantt chart for the above problem can be illustrated as below :-



(2) A book binder has one printing press, one binding machine, and the manuscripts of a number of different books. The time required to perform the printing and binding operations for each book ar shown below. Determine the order in which books should be processed, in order to minimize the total time required to turn out all the books :-

Book	:	1	2	3	4	5	6
Printing time(hrs)	:	30	120	50	20	90	110
Printing time(hrs)	:	80	100	90	60	30	10

Solution :- Here, the books will first go to the printing press and then on the binding machine. If P_i (i=1,2,.......6) denotes the time in hours on printing press and B_i=(i=1,2,. 6) the binding time for books, then since min.$\{P_i,B_i\}$=10 correspoding to B_6, book 6 will be processed in the last. The problem then reduces to the following five and two machine

Book	:	1	2	3	4	5
P_i	:	30	120	50	20*	90
B_i	:	80	100	90	50	30

Now, min.$\{P_i,B_i\}$=20 which corresponds to P_4. Therefore, book 4 will be processed just in the beginning. The entries are shown in the sequence cells as below :

4					6

After assigning books 4 and 6, we are now left with 4 books and two machines with their processing times as follows :

Book	:	1	2	3	4
p_i	:	30*	120	50	90
B_i	:	80	100	90	30*

Now, the minimum of P_1 and B_1 is 30 which corresponds to P_1 and B_5, ie, there is a tie for the minima. So, we place book 1 next to the first and book 5 next to last, yielding us the sequence.

4	1			5	6

We are now left with the problem of 2 jobs and 2 machines with their respective processing time as follows :-

Book	:	2	3
P_1	:	120	50*
B_1	:	100	90

Here, since samllest printing time is 50 hours for book 3, we place book 3 in the third cell and remaining book 2 in the fourth cell and get the following optimal sequence :

4	1	3	2	5	6

The minimum elapsed time from the start of the first book to the completion of the last book corresponding the optimal sequence is computed as shown in the following table.

Book	Printing machine		Binding machine		Idle time of binding machine
	Time in	Time out	Time in	Time out	
4	0	20	20	80	20
1	20	50	80	160	0
3	50	100	160	250	0
2	100	220	250	350	0
5	220	310	350	380	0
6	310	420	420	430	40

From the above table it is clear that minimum elapsed timed is 430 hrs. Idle time for printing machine is 10 hours (from 420 hrs to 430 hrs) and for binding machine is 20+40=60 hrs.

Remarks 1 : It may be noted that the total elapsed time is equal to the sum of the idle time of binding machine and the total processing time on binding machine.

2) The total elapsed time can also be calculated by using Gantt chart.

3) A company has 3 jobs on hand. Each of these must be processed through two department, the sequential order for which is :

Department A :- Press shop

Department B :- Finishing

The table below lists the number of days required by each job in each department.

	Job I	Job II	Job III
Department A	8	6	5
Department B	8	3	4

Find the sequence in which the 3 jobs should be processed so as to take minimum time to finish all the 3 jobs.

Solution :-

Min $\{A_{ij}, B_{ij}\} = 3$

The smallest processing time in the given problem is 3 on machine B. So II will be processed in the last as shown below.

		II

	Job I	Job III
A	8	5
B	8	4

Min $\{A_{ij}, B_{ij}\} = 4$

The minimum processing time in this reduced problem is 4 with corresponds to [illegible] on machine B. ∴ III will be processed in the last is shown below.

	III	II

The optimal sequence is

I	III	II

The total elapsed time is calculated below :

Job	Department A		Department B		Idle
	Time In	Time out	Time In	Time out	Time
I	0	8	8	16	8
II	8	13	16	20	-
III	13	19	20	23	-
					8

Total elapsed time = 23 days.

Ideal time for department A is (23-19) 4 days.

Ideal time for department B is 8 days.

(4) We have five jobs, each of which has to go through the machines A and B in the order AB. Processing times are given in the table below.

Processing - times in hours

Job	A_i	B_i
1	5	2
2	1	6
3	9	7
4	3	8
5	10	4

Determine a sequence of these jobs that will minimize the total elapsed time T.

Solution :- The minimum time in the above table is 1 which is A_2. Hence we shall do the 2nd job first. We list the jobs as shown below.

2				

Now we are left with four jobs with the processing times as shown in the below given table.

Job i	A_i	B_i
1	5	2
3	9	7
4	3	8
5	10	4

Again as the minimum time in this table is 2 which is B_1, we shall do the first job in last.

2				1

Now the times for the remaining jobs are as shown in the following table.

Job i	A_i	B_i
3	9	7
4	3	8
5	10	4

Similary using the prescribed criterion, we conclude that the optimal sequence of jobs is

2	4	3	5	1

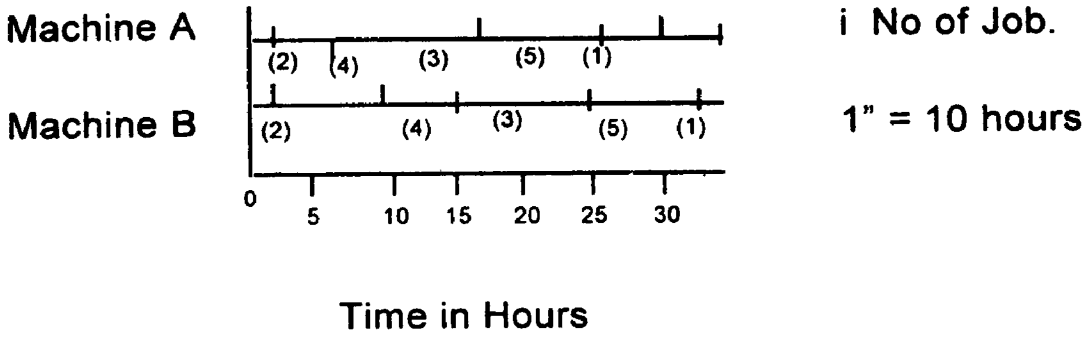
Further the minimum elapsed time can be calculated as follows :

	Machine A		Idle time	machine B		Idle time
Job	Time in	Time out	of A	Time in	time out	of B
2	0	1		1	7	1
4	1	4		7	15	
3	4	13		15	22	
5	13	23		23	27	1
1	23	28	2	28	30	1

From the above table it is clear that the total time elapsed is 30 and the idle time for the machine B is 3. Note that the total elapsed time is equal to the sum of the idle time of B and the total processing time on machine B.

The total elapsed time can also be calculated by using Gantt chart as follows.

From the following figure it can be seen that the total elapsed time is 30 hours and the idel time of the machine B is 3 hours :



In the above problem it is noted that a job may be held in inventory before going to the machine. For instance 4th job will be free on machine A after 4th hour and will start on machine B at 7th hour. Therefore it will be kept in inventory for 3 hours. So it is assumed that the storage space is available and the cost of holding the inventory for each job is either same or negligible. For short duration process problems generally it is negligible. Second general assumption is that the order of completion of jobs has no significane (ie.) no job claims the priority.

Exercise :-

1. We have five jobs, each of which must go through the two machines A and B in the order AB. Processing times (in hours) are given in the table below.

Job	1	2	3	4	5
Machine A	5	1	9	3	10
Machine B	2	6	7	8	4

Determine a sequence for the five jobs that will minimize the elapsed time.

2. Six jobs go first over machine I and then over machine II. The order of the completion of jobs has no significance. The following table gives the machine times in hours for six jobs and the two machines:

Job No	1	2	3	4	5	6
Time on machine I(A_i)	5	9	4	7	8	6
Time on machine II (B_i)	7	4	8	3	9	5

Find the sequence of jobs that minimizes the total elapsed time to complete the jobs. Find the minimum time by using Gantt's chart or by any other method

(3) Find the sequence that minimizes the total elapsed time (in hours) required to complete the following tasks on two machines:-

Task :	A	B	C	D	E	F	G	H	I
machine I :	2	5	4	9	6	8	7	5	4
machine II :	6	8	7	4	3	9	3	8	11

(4) A company has six jobs on hand, coded 'A' to 'F' all the jobs have to go through two machines 'MI' and 'MII'. The time required for the jobs on each machine, in hours, is given below :

	A	B	C	D	E	F
M I :	1	4	6	3	5	2
M II :	3	6	8	8	1	5

Draw a seqence table scheduling the six jobs on the two machines.

Answers :

1. 2→4→3→5→1; Minimum time is 30 hours.
2. 3→1→5→6→2→4; minimum time is 42 hours.
3. A→I→C→B→H→F→D→E→G (or)
   A→I→C→H→B→F→D→E→G;
   minimum time is 61 hours.
4. A→F→D→B→C→E; minimum time is 32 hours.

8.6 Problems with n jobs and k machines

There is no general method available by which we can obtain optimal sequence (s) in problems involving processing of n jobs on k machines. They canbe handled only by enumeration, which is a very lengthy and time consuming exercise because a total of $(n!)^k$ different sequence would require consideration in such a case. However, we do have a method applicable under the conditilon that no passing of jobs is permissible and if either or both of the condition stipulated below is / are satisfied.

Let there be n jobs, each of which is to be processed through k machines, say M_1, M_2 ...M_k in the order M_1,M_2,........M_k. *The iterative procedure of obtaining an optimal sequence* is as follows :

Step 1 : Find min M_{i1}, min M_{ik}, and maximum of each of M_{i2}, M_{i3},.....M_{ik-1} for i = 1,2,..n

Step 2 : Check whether

(i) min M_{i1} > max M_{ij} for i = 2,3, ...k-1 or

(ii) min M_{ik} > max M_{ij} for j = 2,3,........k-1.

Step 3 :- If the inequations of step 2 are not satisfied, method fails. otherwise go to nex step.

Step 4 :- Convert the k machine problem into two machine problem by introducing two fictitious machines G and H, such that.

$$M_{iG} = M_{i1} + M_{i2} + \ldots + M_{ik-1}$$

and $$M_{iH} = M_{i2}+M_{i3}+\ldots+M_{ik}$$

Step 5 :- Determine the optimal *sequence* of *performance* of the jobs on G and H, in accordance with the rules given earlier in respect of processing n jobs and 2 machines *The resulting sequence shall be optimal for the given problem.*

Remark 1 : In addition to conditions given in step 4, if

$$M_{i2}+M_{i3}+\ldots+M_{ik-1} = c,$$

is a fixed positive constant for all i=1,2,........n then determine the optimal sequence for n jobs and two machines M_1 and M_k in the order M_1, M_k by using the optimal sequence algorithm.

2. In addition to the conditions given is step 4, if

$M_{i1} = M_{im}$ and $M_{iG} = M_{iH}$ *for i = 1,2, ...n*

then there will be n! optimal sequences, each of which will yield minimum total elapsed time.

3) The above - mentioned procedure of solving othe *sequencing* problems is not a general procedure. The method is applicable only to those sequencing problems in which the minimum cost (time) of processing the jobs through first and / or last machine is greater than or equal to the cost of processing the jobs through mediocre machines. There are many industrial operations, in which the machines are set in some order which does not obey this rule.

Sample problems :

1) Determine the optimal sequence of jobs that minimizes the total elapsed time based on the following information processing time on machines is given in hours and passing is not allowed.

Job	:	A	B	C	D	E	F	G
machine M_1		3	8	7	4	9	8	7
machine M_2	:	4	3	2	5	1	4	3
machine M_3	:	6	7	5	11	5	6	12

Solution : We are given 7 jobs each of which is to be processed through 3 machines M_1, M_2 and M_3 in order $M_1 M_2 M_3$.

Therefore, for n = 7 and k = 3; we observe that min.M_{i1}=3, min.M_{i3} = 5 and max. M_{i2} = 5. since min. M_{i3}> max. M_{i2} is satified, the problem can be converted into that of 7 jobs and 2 machines.

Thus, if G and H are the two machines, such that

$G_i = M_{i1} + M_{i2}$

and $H_i = M_{i2} + M_{i3}$, for i = 1,2,.........7

Then the problem can be rewritten as the following 7 jobs and 2 machines problem.

Job	A	B	C	D	E	F	G
G	3+4=7	8+3=11	7+2=9	4+5=9	9+1=10	8+4=12	7+3=10
H	4+6=10	3+7=10	2+5=7	5+11+16	1+5=6	4+6=10	3+12=15

using the optimal sequence algorithm, the following optimal sequence can easily be obtained.

A	B	G	B	F	C	E

For total elapsed time, we have

job

		A	D	G	B	F	C	E
M_1 In	:	0	3	7	14	22	30	37
M_1 out	:	3	7	14	22	30	37	46

		A	D	G	B	F	C	E
M_2 In	:	3	7	14	22	30	37	.46
M_2 out	:	7	12	17	25	34	39	47

		A	D	G	B	F	C	E
M_3 In	:	7	13	24	36	43	49	54
M_3 out	:	13	24	36	43	49	54	59

This table indicates that the minimum total elapsed times is 59 hours Idle time is 13 hours for M_1,37 hours for M_2 and 7hours for M_3.

2. We have 4 jobs each of which has to go through the machines M_j j = 1,2, ...6 in the order M_1, M_2, ...M_6. Processing time (in hours) is given below :

	Machines					
	M_1	M_2	M_3	M_4	M_5	M_6
Job A	18	8	7	2	10	25
Job B	17	6	9	6	8	19
Job C	11	5	8	5	7	15
Job D	20	4	3	4	8	12

Determine a sequence of these four jobs that minimizes the total elapsed time T.

Solution :-

Here min. M_{i1} = 11 and min. M_{i6} = 12; max. of M_{i2}, M_{i3}, M_{i4} and M_{i5} are 8, 9, 6 and 10 respectively. Since the conditions

min $M_{i1} \geq$ max. M_{ij} and min. $M_{i6} \geq$ max. M_{ij} for i = 1,2,3,4,5 are satified, the given problem can be written as

Job	:	A	B	C	D
machine G	:	45	46	36	39
machine H	:	52	48	40	31

Where $G_i = \sum_{j=1}^{5} M_{ij}$ and $H_i = \sum_{j=2}^{6} M_{ij}$.

Using the optimal sequence algorithm, the following optimal sequence is easily obtained.

C	A	B	D

The total elapsed time is given in the following table.

	machines					
Job	M_1	M_2	M_3	M_4	M_5	M_6
C	0-11	11-16	16-24	24-29	29-36	36-51
A	11-29	29-37	37-44	44-46	46-56	56-81
B	29-46	46-52	56-61	61-67	67-75	81-100
D	46-66	66-70	70-73	73-77	77-85	100-112

This table shows the minimum total elapsed time is 112 hours.

3. solve the following seauencing problem when passing out is not allowed.

Item	machine (processing time in hours)			
	A	B	C	D
I	15	5	4	15
II	12	2	10	12
III	16	3	5	16
IV	17	3	4	17

solution : Here, min A_i=12 and max.B_i = 5 and max. C_i = 10 since min.$A_i \geq$ max.B_i, maxC_i, the given problem can be rewritten as.

Item :	I	II	III	IV
machine G :	24	24	24	24
machine H :	24	24	24	24

where $G_i = A_i + B_i + C_i$ and $H_i = B_i + C_i + D_i$;

Since $G_i = H_i$ and $A_i = D_i$; are satisfied. Therefore using the optimal sequence algorithm, we get 41(=24) sequence, each giving us an optimal sequence :

I II III IV, I II IV III, I III II IV,
I III IV II, I IV II III, etc.

Each of the above optimal sequence will yield us the same total elapsed time. For the elapsead time, we have.

	Item		I	II	III	IV
machine A	In	:	0	15	27	43
	out	:	15	27	43	60
machine B	In	:	15	27	43	60
	out	:	20	29	46	63
machine C	In	:	20	29	46	63
	out	:	24	39	51	67
machine D	In	:	24	39	61	67
	out	:	39	51	67	84

Total elapsed time is 84 hours.

(4) Determine the optimal sequence of jobs which minimizes the total elapsed time based on the following information.

Processing times on the machines A, B, C

Job	A_i	B_i	C_i
1	3	3	5
2	8	4	8
3	7	2	10
4	5	1	7
5	2	5	6

Solution :

Here min $A_i = 2$, max $B_i = 5$, min $C_i = 5$

To solve this type of problem, we replace the three machines by two fictitious machines say G and H with corresponding processing times given by

$G_i = A_i+B_i$, $H_i = B_i+C_i$, $i = 1,2,\ldots\ldots.n$

since max $B_i \leq$ min C_i,

The times for the fictitious machines G and H are given by the following table.

	Processing time	
Job	$G_i = A_i+B_i$	$H_i = B_i+C_i$
1	6	8
2	12	12
3	9	12
4	6	8
5	7	11

Note that here minimum time is 6, which is both G_1 and G_4. As there is a tie, and of the jobs first and fourth can be performed in the starting. Thus the optimal sequence may be formed in any of the two ways.

1	4	5	3	2

4	1	5	3	2

Total elapsed time associated to the first sequence is calculated below :

	A		B		C		Idle time
Job	Time in	Time out	Time in	Time out	Time in	Time out	of C.
1	0	3	3	6	6	11	6
4	3	8	8	9	11	18	0
5	8	10	10	15	18	24	0
3	10	17	17	19	24	34	0
2	17	25	25	29	34	42	0

Hence the total elapsed time is 42.

(5) Find the optimal sequence for processing 4 jobs A,B,C,D on four machines A_1, A_2, A_3, A_4, in the orderA_1, A_2, A_3, A_4. Processing times are as given below.

Processing times (a_{ij}) in hours.

Job/machine	$A_1(a_{i1})$	$A_2(a_{i2})$	$A_3(a_{i3})$	$A_4(a_{i4})$
A	15	5	4	14
B	12	2	10	12
C	13	3	6	15
D	16	0	3	19

Solution :-

From the above table we get that for extreme machines

$\min(a_{i1})$ = min. processing time on first machine = 12

$\min(a_{i4})$ = min. processing time on last machine = 12 and for intermediate machines.

max.(a_{i2}) = max. processing time on 2nd machine = 5

max.(a_{i3}) = max. processing time on 3rd mechine = 10.

Since min a_{i1}>max a_{i2} and max a_{i3} (both) the problem can be reduced to a problem involving only two mechines G and H with processing times as

Processing times

Jobs	G_i	H_i
A	15+5+4=24	5+4+14=23
B	12+2+10=24	2+10+12=24
C	13+3+6=22	3+6+15=24
D	16+0+3=19	0+3+19=22

Using the algorithm for solving a sequencing problem of n jobs and 2 mechines, we get the optimal sequence as

D	C	B	A

Total elapsed time can be calculated as follows :

Job/machine	A_1		A_2		A_3		A_4	
	in	out	In	out	In	out	In	out
D	0	16	19	16	16	19	19	38
C	16	29	29	32	32	38	38	53
B	29	41	41	43	43	53	53	65
A	41	56	56	61	61	65	65	79

Total elapsed time = 79 hours.

Exercise :

1) We have five jobs, each of which must go through machines A, B,and C in the order A,B,C, Processing times (in hours) are given in the following table.

Job		1	2	3	4	5
machine A (A_i)	:	8	10	6	7	11
machine B (B_i)	:	5	6	2	3	4
machine C (C_i)	:	4	9	8	6	5

2. Find the sequence that minimizes that total time required in performing the following jobs on three machines in the order ABC.

Processing time (in hours) on	Job					
	1	2	3	4	5	6
machine A	8	3	7	2	5	1
machine B	3	4	5	2	1	6
machine C	8	7	6	9	10	9

3. Solve the following sequencing problem when passing is not allowed :-

Item	machine				
	A	B	C	D	E
I	9	7	4	5	11
II	8	8	6	7	12
III	7	6	7	8	10
IV	10	5	5	4	8

Answers :-

(1) 3→ 2→4→1→5; minimum time is 51 hours.

(2) 4→5→6→2→1→3; minimum time is 53 hours.

(3) 1→3→2→4; minimum time is 66 hours.

8.7 : PROBLEMS WITH 2 JOBS AND K MACHINES

Let there be two jobs 1 and 2 each of which is to be processed on k machines say M_1, M_2, ...M_k in two different orders. The technological ordering of each of the two jobs through k machines is known in advance. Such ordering may not be same for both the jobs. The exact or expected processing times on all the given machines are known. Each machine can perform only one job at a time. The objective is to determine an optimal sequence of processing the jobs so as to minimize total elaspsed time.

The optimal sequence in this case can be obtained by making use of graph. The solution procedure can be summerised in the following steps.

Step 1: Draw two perpendicular lines, horizontal one representing the processing time job 1 while job 2 remains idle, and the vertical one representing the processing time for job 2 while job 1 remains idle.

Step 2. Mark the processing time for jobs 1 and 2 on the horizontal and vertical lines respectively according to the given order of machines.

step 3 : Construct various blocks starting from the origin (starting point) by pairing the same machines until the end point.

Step 4 : Draw the line starting from origin to end point by moving horizontally, vartically and diagnally along a line which makes an angleof $45^0$with the horizontal line (base). The horizontal segment of this line indicates that first job is under process while second job is idle. Similarly, the vertical segment of the line indicates that the second job is under process while first job is idle. The diagonal segment of the line shows that both the jobs are under process simultaneously.

Step 5:- As optimum path is one that minimizes the idle time for both the jobs. Thus, we must choose the path on which diagnoal movement is maximum.

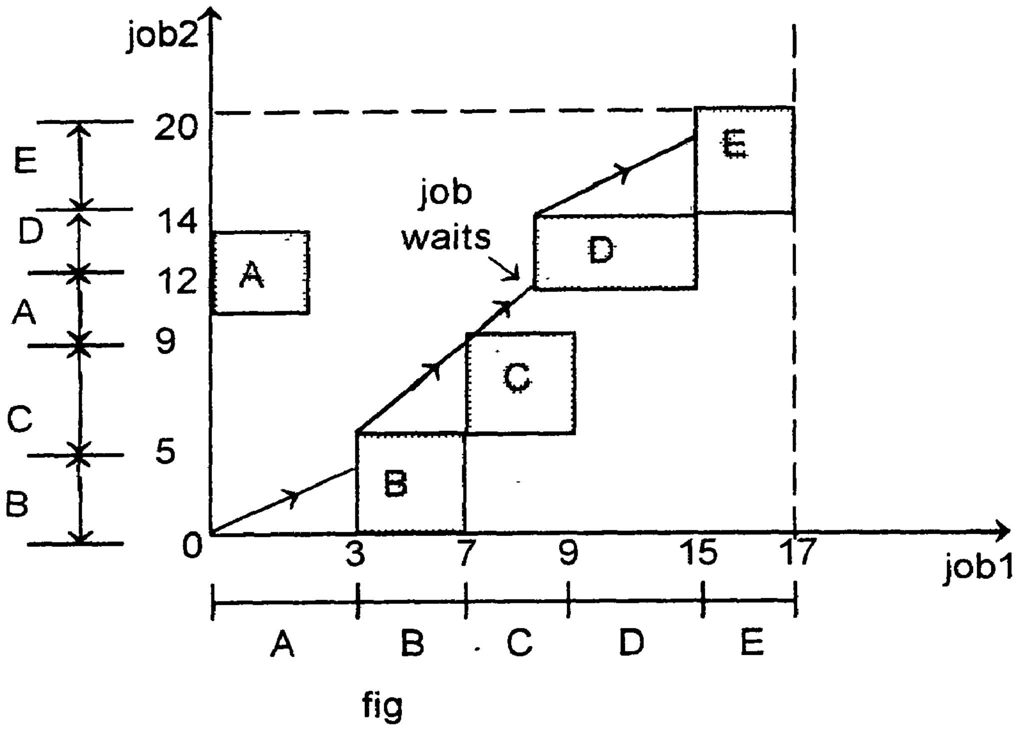
step 6:-The total elapsed time is obtained by adding the idle time for either job to the processing time for that job.

Sample problem :

1 Use grpahical method to minimize the time added to process the following jobs on the machines shown (ie) for each machine find the job which should be done first. Also calculate the total time elapsed to complete both the jobs:

Job 1	sequence	A	B	C	D	E
	time	3	4	2	.6	2
Job 2	sequence	B	C	A	D	E
	time	5	4	3	2	6

solution : The given information is depicted in the following Fig in which shaded blocks represent the overlaps which are to be avoided.



fig

An optimal path (programme) is one that minimizes the idle time for job 1 (horizontal movement). Similarly, an optimal path is one that minimizes the idle time for job 2 (vertical movement)

For the elapsed total time, we shall add the idle time for either of the two jobs to the processing time of that job. Now since the idle time for the choosen path is 5 hours for job 1 and 2 hours for job 2. The total elapsed time is obtained as follows :

Processing time of job 1 + idle time for job 1

= 17+ (2+3) =22 hours.

(or) processing time of job 2 + idle time for job 2

= 20+2=22 hours.

Exercise :

1) a machine shop has four machine A,B,C,D Two jobs must be processed through each of these machines. The time (in hours) taken on each of the machines and the necessary sequence of jobs through the shop are given below.

Job 1	sequence	A	B	C	D
	time	2	4	5	1
Job 2	sequence	D	B	A	C
	time	6	4	2	3

Use graphic method to obtain the total minimum elapsed time.

2) Use graphic method to find the minimim elapsed total time sequence of 2 jobs and 5 mechines when we are given the following information.

		machines				
Job 1	sequence :	A	B	C	D	E
	time (in hours) :	2	3	4	6	2
Job 2	sequence :	C	A	D	E	B
	time(in hours) :	4	5	3	2	6

3) Two jobs are to be processed on four machines a, b, c and d. The technological order for these jobs on machines is as follows.

Job 1 : a, b, c, d

Job 2 : d, b, a, c

Processing times are given in the following table :

	Machine			
Job	a	b	c	d
1	4	6	7	3
2	4	7	5	8

Find the optimal sequence of jobs on each of machines.

4) A machine shop has six machines A,B,C,D,E&F. Tow jobs must be processed through each of machines. The times on machines and the necessary sequence of the jobs through the shop are given below.

Order	1	2	3	4	5	6
Job I	A-20	C-10	D-10	B-30	E-25	F-16
Job II	A-10	C-30	B-15	D-10	F-15	E-20

Determine the optimum sequence for the job in order to minimize the total time necessary to finish the jobs.

Answers :

1. Total elapsed time is 15 hours.
2. Total elapsed time is 20 hours.
3 Total elapsed time is 24 hours.

. Minimum total time is 150 hours.

Unit 9
GAME THEORY

9.1 Introduction :

In many practical problems,it is required to take decision in a situation where there are two(or more) opposite parties with conflicting interests and the action of one depends upon the action which the opponent takes. The outcome of the situation is controlled by the decisions of all the parties involved. Such a situation is termed as a **"Competitive situation"**. Such problems occur frequently in economic and political and social, military fields. For example, in military operations, any situation is a competitive situation because each party takes decision in a way which assures that it is least advantageous to the opponent.

In a competitive situation the courses of action (alternatives) for each competitor may be either finite or infinite. A competitive situation will be called a "Game". if it has the following properties.

(i) There are a finite number of competitors (participants) called players.
(ii) Each player has a finite number of strategies (alternatives) available to him.
(iii) A play of the game takes place when each player employs his strategy.
(iv) Every game resultsin an outcome,
e.g : loss or gain or a draw, usually called payoff, to some player.

9.2 Two person zero - sum games :

A game in which the gains of one player are the losses of other players, is called a **zero sum game** (ie) in a zero sum game the algebraic sum of the gains to all players after a play is bound to be zero.

Zero-sum games with two players are called two-person zero-sum (or) rectangular games. In this case the lass (gain) of one player is exactly equal to the gain (loss) of the other.

The gains resulting from to two-person zero-sum game can be represented in the matrix form, usually called a 'pay-off matrix".

Example 1 :

To illustrate the defintions of a two-person zero-sum game, consider a non-matching situation in which each of the two players A and B selects a head (H) or a tail (T). If the outcomes match (that is. H and H or T and T) player A wins Rs 1 from player B. Otherwise, A loses Rs 1 to B.

In this game each player has two strategies (H or T). This yields the following 2x2 game matrix expressed in terms of the pay off to A.

$$\begin{array}{cc|cc} & & \multicolumn{2}{c}{\text{Player B}} \\ & & H & T \\ \hline & H & 1 & -1 \\ \text{player A} & & & \\ & T & -1 & 1 \end{array}$$

The "optimal" solution to such a game may require each player to play a pure strategy (for example either T or H) or a mixture of pure strategies The latter case is known as mixed strategy selection.

9.3 The maximim - minimax principle :

We shall now explain the so-called maximin-minimax principle for the selection of the optimal strategies by the two players. We assume that both the players are conservative (ie) While employing his strategy A_1, plyaer A believes that his opponent knows that he is going to employ A_1, and similarly the player B believes so about player A while employing his moves.

To illustrate the maximin - minimax principle let us consider a two-person zero-sum game with the following 3x2 pay off matrix for player A.

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ & \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{pmatrix} 9 & 2 \\ 8 & 6 \\ 6 & 4 \end{pmatrix} \end{array}$$

player A

Let the pure strategies of the two players be designated by $S_A = \{A_1, A_2, A_3\}$ and $S_B=\{B_1,B_2\}$

Suppose that player A starts the game knowing fully well that whatever strategy he adopts, B will select that particulars counter stragegy which will minimize the payoff to A Thus if A selects the strategy A_1, the B will reply by selecting B_2, as this corresponds to the minimum payoff to A in the first row corresponding to A_1. Similarly, if A chooses the strategy A_2, he may gain 8 or 6 depending upon the strategy chosen by B. However, A can guarantee a gain of atleast min{8,6}=6 regardless of the strategy chosen by B. In otherwords, whatever strategy A may adopt he can guarantee only the minimum of the correspoding row payoffs. Naturally, A would like to maximise his minimum gain. In the above example the selection of strategy A_2 gives the maximum of the minimum gain to A. We shall call this gain as the maximin value of the game and the corresponding strategy as the maximin strategy. The maximin value is indicated in bold type with a star.

On the otherhand, player B wishes to minimize his losses. If the plays strategy B_1, his loss is at the most max {9,8,6}=9 regardless of what strategy A has selected. He can lose no more than max {2,6,4}=6 if he plays B_2. This minimum of the maximum losses will be called the minimax value of the game and the corresponding strategy the minimax strategy. The minimax value is indicated in bold type marked with [†]. We observe that in the present example the maximum of row minima is equal to the minimum of the column maxima In symbols,

$$\max_{i} \{r_i\} = \min_{j}.\{c_j\} = 6$$

(or) $\max_{i} [\min_{j}.\{a_{ij}\}] = 6 = \min_{j}.[\max_{i}\{a_{ij}\}]$,

where i = 1,2,3 and j = 1,2

The selection of maximin and minimax strategies by A and B was based upon the so-called maximin minimax principle which guarantees the best of the worst results.

In such cases, where the maximin value of the game is equal to the minimax value of the game, the corresponding pure strategies are called "optimum" strategies.

Example :

Consider the following pay off matrix which represents player A's gain. The computations of the minimax and maximin values are shown on the matrix.

		Player B					
		1	2	3	4	row minimum	
player A	1	8	2	9	5	2	
	2	6	$\boxed{5}$	7	18	$\boxed{5}$	maximin
	3	7	3	-4	10	-4	
column maximum		8	$\boxed{5}$	9	18		
			minimax				

When player A plays his first strategy, he may gain 8,2,9 or 5 depending on player B's selected strategy. He can guarantee, however, a gain of atleast min{8,2,9,5} - regardles of B's selected strategy. Similarly, it A plays his second strategy, he guarantees an income of atleast min {6,5,7,18} = 5; done plays his third strategy, he guarantees an income of atleast min{7,3, -4,10}=-4; Thus, the minimum value in each row represents the minimum gain guaranteed A if he plays his pure strategies. These are indicated in the above matrix by "Row minimum". Now, player A, by selecting his second strategy, is maximizing his minimum gain. This gain is given by max {2,5,-4}=5. Player A's selection is called the maximin strategy, and his corresponding gain is called the maximin (or lower) value of the game.

Player B, on the otherhand, wants to minimize his losses. He realizes that, if he plays his first pure strategy, he can lose no more than max{8,6,7} =8 regardless of A's selections

A similar argument can also be applied to the three remaining strategies. The corresponding results are thus indicated in the above matrix by "Column maximum". Player B will then select the strategy that minimizes his maximum losses. This is given by the second strategy and his corresponding loss is given by min{8,5,9,18} =5. Player B's selection is called the minimax strategy and his corresponding loss is called the minimax (or upper) value of the game.

In this example, maximin value = minimax value = 5. This implies that the game has a saddle point which is given by the entry (2,2) of the matrix. The value of the game is thus equal to 5.

9.4 Saddle point

Definition :-

A saddle point (or Equilibrium point) of a pay off matrix is that position in the payoff matrix where the maximum of row minima coincides with the minimum of the column maxima. The payoff at the saddle point is called the value of the game and is obviously equal to the maximum and minimax values of the game.

Thus (k.r)th position of the pay off matrix (a_{ij}) will be a saddle point if and only if,

$$a_{kr} = \max_{i}.\ [\min_{j}\{a_{ij}\}] = \min_{i} [\max_{j}\{a_{ij}\}]$$

The saddle point, and hence the value of the game, need not be unique. We shall denote a value of the game by v. The importance of the saddle point arises from the fact that, in general, the optimum play consists in sticking to the strategies which correspond to the saddle point. To solve a game we therefore merely need to look for the saddle point of the payoff matrix. If it exists, the game is solved. But unfortunately, most payoff matrices do not posses any saddle point. In great ℓ (theorem 9.1), the value of the game v satisfies

maximum value $\leq v \leq$ minimax value.

We shall denote the maximin value of the game by $\underline{v}$ and the minimax value of the game by $\overline{v}$. These values are also called the lower vlaue and the upper value of the game, respectively. A game is said to be a fair game if $\underline{v} = 0 = \overline{v}$. A game is said to be strictly determinable if $\underline{v} = v = \overline{v}$.

Theorem 9.1 :

Let (a_{ij}) be the mxn pay off matrix for a two-person zero-sum game. If $\underline{v}$ denotes the maximin value and $\overline{v}$ the minimax value of the game, then $\overline{v} \geq \underline{v}$. That is,

$$\min_{1\leq j\leq n} [\max_{1\leq i\leq m}\{a_{ij}\}] > \max_{1\leq i\leq m} [\min_{1\leq j\leq n}\{a_{ij}\}]$$

Proof : We have

$$\max_{1\leq i\leq m}\{a_{ij}\} \geq a_{ij} \qquad \text{for all } j = 1,2,\ldots\ldots\ldots..n$$

and $$\min_{1 \le j \le n} \{a_{ij}\} \le a_{ij} \qquad \text{for all } i = 1,2,\ldots\ldots\ldots m$$

Let the above maximum be attained at i = i' and the minimum be attained at j=j',

(ie) $\max\limits_{1 \le i \le m} \{a_{ij}\} = a_{i'j}$ and $\min\limits_{1 \le j \le n} \{a_{ij}\} = a_{ij'}$

Then we must have

$$a_{i'j} \ge a_{ij} \ge a_{ij'} \qquad \text{for all } i = 1,2,\ldots\ldots m \ ; j = 1,2, \ldots\ldots n$$

From this, we get

$$\min_{1 \le j \le n} \{a_{i'j}\} \ge a_{ij} \ge \max_{1 \le i \le m} \{a_{ij'}\} \text{ for all } i = 1,2,\ldots\ldots m$$

$$\min_{1 \le j \le n} \left[\max_{1 \le i \le m} \{a_{ij}\} \right] \ge \max_{1 \le i \le m} \left[\min_{1 \le j \le n} \{a_{ij}\} \right]$$

(or) $\overline{\gamma} \ge \underline{\gamma}$

A Rule for determining a saddle point

We may now summarize the procedure of locating the saddle point of a pay-off matrix as follows :

Step 1 : Select the minimum element of each row of the pay off matrix and mark them *.

Step 2 : Select the greatest element of each column of the payoff matrix and mark them +.

Step 3 : If these appears an element in the payoff matrix marked * and + both, the position of that element is a saddle point of the pay-off matrix.

Solved problems :

1. Determine which of the following two-person zero-sum games are strictly determinable and fair. Give optimum strategies for each player in the case of strictly determinable games.

$$\begin{array}{cc} & \text{Player B} \\ \text{(a) Player A} & \begin{bmatrix} -3 & -2 & 6 \\ 2 & 0 & 2 \\ 5 & -2 & -4 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \text{Player B} \\ \text{(b) player A} & \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \text{Player B} \\ \text{(c) player A} & \begin{bmatrix} 0 & 2 \\ -1 & 4 \end{bmatrix} \end{array}$$

Solution :-

(a) The payoff matrix for player A is

player B

		B_1	B_2	B_3	Row minima
Player A	A_1	-3*	-2	6†	-3
	A_2	2	0†*	2	0 (maxmin)
	A_3	5†	-2	-4*	-4
column maxima		5	0 (minimax)	6	

Since the pay-offs marked with * represents the minimum pay-off in each row and those marked with + the maximum pay off in each column of the pay-off matrix, we have

$\underline{\nu}$(maximin value) = 0 and $\overline{\nu}$(minimax value) = 0

As $\underline{\nu} = \overline{\nu} = 0$, the game is strictly determinable and fair. Optimum strategies for players A and B are given by $S_0 = (A_2, B_2)$.

(b) The pay-off matrix for player A is

player B

		B_1	B_2	Row minima
player A	A	5†	0*	0
	A	0*	2†	0
Column maxima		5	2	

The pay-off marked with * represent the minmum pay-off in each row and those marked with (+) represent the maximum pay-off in each column of the pay-off matrix. The largest component of row minima represents $\underline{\nu}$(maximin value) and the smallest component of column maxima represents $\overline{\nu}$(minimax value).

Thus obviously, we have

$\underline{\nu} = 0$ and $\overline{\nu} = 2$.

Since $\underline{\nu} \neq \nu$, the game is not strictly determinable.

(c) The pay-off matrix for player A is

player B

		B_1	B_2	Row minima
Player A	A_1	0*+	2	0
	A_2	-1*	4 +	-1
Column maxima		0	4	

Since the pay-off marked with * represent the minimun pay-off in each row and those marked with + the maximum pay-off in each column of the pay-off matrix, we have

$\underline{\nu}$ (maximin value) =0 and

$\overline{\nu}$ (minimax value) = 0

As $\underline{\nu} = \overline{\nu} = 0$, the game is strictly determinable and fair. Optimum strategies for players A and B are given by

$S_0 = (A_1, B_1)$

2. Determine the range of value of pand q that will make the pay off element a_{22} a saddle point for the game whose payoff matrix (a_{ij}) is given below.

(a)

$$\begin{array}{cc} & \text{Player B} \\ \text{player A} & \begin{bmatrix} 2 & 4 & 5 \\ 10 & 7 & q \\ 4 & p & 8 \end{bmatrix} \end{array}$$

(b)

$$\begin{array}{cc} & \text{player B} \\ \text{player A} & \begin{bmatrix} 1 & q & 3 \\ p & 5 & 10 \\ 6 & 2 & 3 \end{bmatrix} \end{array}$$

Solution :-

(a) Let us first of all ignore the values of pand q and determine the maximin and minimax values of the pay-off matrix. For this, we have

$$\begin{array}{c|ccc|c} & B_1 & B_2 & B_3 & \text{Row minima} \\ \hline A_1 & 2 & 4 & 5 & 2 \\ A_2 & 10 & 7 & q & 7 \\ A_3 & 4 & p & 8 & 4 \\ \hline \text{column maxima} & 10 & 7 & 8 & \end{array}$$

obviously, the maximin value $\underline{\nu}$ is 7 and the minimax value $\underline{\nu}$ is also 7. Thus there exists a saddle point at position (2,2).

This imposes the condition on p as $p \leq 7$, and on q as $q \geq 7$.

Hence, the required range of values of p and q is $7 \leq q$, $p \leq 7$.

(b)

$$\begin{array}{cc|ccc|c} & & & \text{Player B} & & \\ & & B_1 & B_2 & B_3 & \text{Row minima} \\ \hline & A_1 & 1 & q & 3 & 1 \\ \text{Player A} & A_2 & p & 5 & 10 & 5 \\ & A_3 & 6 & 2 & 3 & 2 \\ \hline \text{column maxima} & & 6 & 5 & 10 & \end{array}$$

Obviously, the maximin value $\underline{\nu}$ is 5 and the minimax value $\overline{\nu}$ is also 5. Thus there exists a saddle point at position (2,2).

This imposes the condition on p as $p \geq 5$, and on q as $q \leq 5$. Hence, the required range of values of p and q is $5 \leq p$, $q \leq 5$

3. For what value of λ, the game with following pay-off matrix is strictly determinable?

$$\begin{array}{lc|ccc|}
 & & & \text{player B} & \\
 & & B_1 & B_2 & B_3 \\
\text{player A} & A_1 & \lambda & 6 & 2 \\
 & A_2 & -1 & \lambda & -7 \\
 & A_3 & -2 & 4 & \lambda
\end{array}$$

Solution :-

Let us first of all ignore the value of λ and determine the maximin and minimax values of the pay-off matrix For this, we have

$$\begin{array}{c|ccc|c}
 & B_1 & B_2 & B_3 & \text{Row minima} \\
A_1 & \lambda & 6 & 2 & 2 \\
A_2 & -1 & \lambda & -7 & -7 \\
A_3 & -2 & 4 & \lambda & 2 \\
\text{column maxima} & -1 & 6 & 2 &
\end{array}$$

Obviously, maximin value $\underline{\nu}$ is 2. and minimax value $\overline{\nu}$ is -1
This imposes the condition on λ as $-1 \le \lambda \le 2$.

The game is strictly determinable when the value of λ is $-1 \le \lambda \le 2$

(4) Solve the game whose pay-off matrix is given below.

$$\begin{bmatrix}
9 & 3 & 1 & 8 & 0 \\
6 & 5 & 4 & 6 & 7 \\
2 & 4 & 3 & 3 & 8 \\
5 & 6 & 2 & 2 & 1
\end{bmatrix}$$

Solution :

The pay-off matrix for player A is

$$\begin{array}{c|ccccc|c}
 & B_1 & B_2 & B_3 & B_4 & B_5 & \text{Row minima} \\
\hline
A_1 & 9 & 3 & 1 & 8 & 0 & 0 \\
A_2 & 6 & 5 & 4^{*\dagger} & 6 & 7 & 4 \\
A_3 & 2 & 4 & 3 & 3 & 8 & 2 \\
A_4 & 5 & 6 & 2 & 2 & 1 & 1 \\
\hline
\text{column maxima} & 9 & 6 & 4 & 8 & 8 &
\end{array}$$

$\underline{\nu}$ (maximin value) = 4

$\overline{\nu}$ (mimimax value) = 4

As $\underline{\gamma} = \overline{\gamma} = 4$, the game is strictly determinable optimum strategies for player A & B is given by

$S_0 = (A_2, B_3)$

(ie) S_0 = Row 2 column 3.

∴The value of the game $\nu = 4$.

(5) Solve the game whose pay - off matrix is given below :-

$$\begin{bmatrix} -2 & 0 & 0 & 5 & 3 \\ 3 & 2 & 1 & 2 & 2 \\ -4 & -3 & 0 & -2 & 6 \\ 5 & 3 & -4 & 2 & -6 \end{bmatrix}$$

Solution :-

Marking * for the row minimums and putting + for thr column maximums, we get

$$\begin{array}{cc|ccccc|c} & & & & B & & & \\ & & B_1 & B_2 & B_3 & B_4 & B_5 & \text{Row minima} \\ & A_1 & -2^* & 0 & 0 & 5_+ & 3 & -2 \\ A & A_2 & 3 & 2 & 1^*_+ & 2 & 2 & 1 \\ & A_3 & -4^* & -3 & 0 & -2 & 6_+ & -4 \\ & A_4 & 5_+ & 3_+ & -4 & 2 & -6^* & -6 \\ \text{Column maxima} & & 5 & 3 & 1 & 5 & 6 & \end{array}$$

Obviously the matrix has a saddle point at (A_2, B_3)

$S_0 = (A_2, B_3)$

= Row 2 column 3

$\bar{\nu} = 1 = \underline{\nu}$

The value of the game = 1.

Exercise problems :-

1) Determine which of the following two-persons zero-sum games are strictly determinable and fair. Give the optimum strategies for each player in the case of strictly determinable games.

(a)

$$\begin{array}{cc|cc|} & & \text{Player B} & \\ \text{player A} & & B_1 & B_2 \\ & A_1 & -5 & 2 \\ & A_2 & -7 & -4 \end{array}$$

(b)

$$\begin{array}{cc|cc|} & & \text{player B} & \\ & & B_1 & B_2 \\ \text{player A} & A_1 & 1 & 1 \\ & A_2 & 4 & -3 \end{array}$$

2. Consider the game G with the following pay off :

$$\text{player A}\quad \begin{array}{c|cc} & \multicolumn{2}{c}{\text{player B}} \\ & B_1 & B_2 \\ \hline A_1 & 2 & 6 \\ A_2 & -2 & \mu \end{array}$$

(a) Show that G is strictly determinable whatever μ may be

(b) Determine the value of G.

3. For the game with pay-off matrix

$$\begin{array}{c} \text{player A} \\ \text{player B}\quad \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix} \end{array}$$

determine the best strategies for players A and B and also values of the game for them. Is this game (i) fair? (ii) strictly determinable?

4. Solve the game whose pay-off matrix is given by

$$\begin{array}{c} \text{Player B} \\ \text{player A}\quad \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} \end{array}$$

Answers :

1. a) not fair , $S_0 = (A_1, B_1)$; $\nu = -5$
   b) not fair , $\nu = 1$
2. $S_0 = (A_1, B_1)$, $\nu = 2$
3. $S_0 = (B_1, A_3)$, $\nu = -2$ Game is strictly determinable and not fair.
4. $S_0 = (A_1, B_1)$ (or) (A_1, B_3) ; $\nu = 1$

9.5 Games without saddle points - mixed strategies

As determining the minimum of column maxima and the maximum of row minima are two different operations, there is no reason to expect that they should always lead to unique pay off position - the saddle point.

In all such cases to solve games, both the players must determine an optimal mixture of strategies to find a saddle (equilibrium) point. The optimal strategy mixture for each player may be determined by assigning to each strategy its probability of being chosen. The strategies so determined are called mixed strategies because they are probabilistic combination of available choices of strategy.

The value of game obtained by the by use of mixed strategies represents which least player A can expect to win and the least which player B can lose. The expected pay off to a player in a game with arbitrary pay off matrix (a_{ij}) of order mxn is defined as :

$$E(p, q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j = p^T A q$$

where $p \in S_m = \{p=(p_1, p_2, \ldots\ldots\ldots p_m) \,/\, p_i \geq 0 \text{ and } \sum^{n} p_i = 1\}$ is a mixed strategy of the row player and

$q \in S_n = \{q = (q_1, q_2, \ldots. q_n) \,/\, q_j \geq 0 \text{ and } \sum q_j = 1\}$

is a mixed strategy of the column player.

we shall call E(p,q) the expectation function of the rectangular game, where $p \in S_m$, $q \in S_n$. Observe that if p is kept fixed at some value and q is varied, then E(p,q) will be a minimum for some value of q. Let this minimum value be $\phi = \min\limits_{q \in S_n} E(p,q)$

When p is given some other fixed value, a different value of ϕ is obtained. Thus by assigning different value to p, a set of values of ϕ is obtained assuming that ϕ exists in every case. This implies that ϕ is a function of p, and therefore we can write

$$\phi(p) = \min_{q \in S_n} E(p,q)$$

Now, if we assume that $\phi(p)$ is a maximum for some value of p, then we can write

$$\max_{p \in S_m} \phi(p) = \max_{p \in S_m} . \min_{q \in S_n} E(p,q)$$

Similarly, we can interpret the expression

$$\min_{q \in S_n} , \max_{p \in S_m} E(p,q)$$

by first finding the maximum value of E(p,q) with respect to p keeping q fixed, and then finding the minimum of the function so obtained with respect to q.

Theorem 9.2 :

Let E(p,q) be such that both

$\max\limits_{p \in S_m} . \min\limits_{q \in S_n} E(p,q)$ and $\min\limits_{q \in S_n} \max\limits_{p \in S_m} E(p,q)$ exist,

then, $$\min_{q \in S_n} . \max_{p \in S_m} E(p,q) \geq \max_{p \in S_m} \min_{q \in S_n} E(p,q)$$

Proof:

Let p^0 and q^0 be some arbitrarily chosen points in S_m and S_n respectively. Then we must have

$$\max_{p \in S_m} E(p,q^0) \geq E(p^0,q^0) \text{ and } \min_{q \in S_n} E(p^0,q) \leq E(p^0, q^0)$$

These two inequalities yield

$$\max_{p \in S_m} . \ E(p,q^0) \geq \min_{q \in S_n} E(p^0, q)$$

But q^0 is arbitrarily chosen and could have been any point in S_n, and for every one of them the inequality holds. Even if we had chosen q^0 to be that point for which $\max_{p \in S_m} E(p,q)$ has the minimum value, $\max_{p \in S_m}$ the inequlity remains true. Therefore

$$\min_{q \in S_n} . \max_{p \in S_m} E(p,q) \geq \min_{q \in S_n} E(p^0, q)$$

Again, since p^0 is any point in S_m, the inequality holds even if we choose that p^0 which gives the maximum value of $\min_{q \in S_n} E(p,q)$

Hence $\min_{q \in S_n} . \max_{p \in S_m} E(p,q) \geq \max_{p \in S_m} \min_{q \in S_n} E(p,q)$

Corollary : Let (a_{ij}) be the pay off matrix for a two-person zero-sum game. If $\underline{\nu}$ denotes the maximin value and $\bar{\nu}$ the minimax value of the game, then $\bar{\nu} \geq \underline{\nu}$ That is

$$\min_{j} . [\max_{i} \{a_{ij}\}] \geq \max_{i} [\min_{j} \{a_{ij}\}]$$

Definition :- (Saddle point of a function).

The function $E(p,q)$ is said to have a saddle point at (p^0,q^0) iff $E(p^0,q) \geq E(p^0,q^0) \geq E(p,q^0)$ for all $p \in S_m$ and $q \in S_n$. Moreover, these strategies p^0, q^0 are said to be optimal.

Theorem 9.3 (Existence of saddle point)

Let $E(p,q)$ be such that both

$$\min_{q \in S_n} . \max_{p \in S_m} E(p,q) \text{ and } \max_{p \in S_m} . \min_{q \in S_n} E(p,q) \text{ exist.}$$

Then a necessary and sufficient condition for the existence of a saddle point (p^0, q^0) of $E(p,q)$ is that $E(p^0,q^0) = \min_{q \in S_n} . \max_{p \in S_m} E(p,q) = \max_{p \in S_m} . \min_{q \in S_n} E(p,q)$

Proof :-

Necessity of the condition. Let the point (p^0,q^0) be a saddle point of $E(p,q)$. Then, since

$E(p^0,q) \geq E(p^0, q^0) \geq E(p, q^0)$, for all $q \in S_n$, $p \in S_m$.

we have

$$\min_{q \in S_n} E(p^0, q) \geq E(p^0,q^0) \geq \max_{p \in S_m} E(p,q^0) \qquad \text{..........(1)}$$

Now $\max\limits_{p \in S_m} [\min\limits_{q \in S_n}. E(p,q)] \geq \min\limits_{q\in S_n}. E(p^0, q)$

and $\min\limits_{q\in S_n} [\min\limits_{p\in S_m}. E(p,q)] \leq \max\limits_{p\in S_m}. E(p, q^0)$

Thus (1) gives,

$$\max_{p\in S_m}. \min_{q\in S_n}. E(p,q) \geq E(p^0, q^0) \geq \min_{q\in S_n} \max_{p\in S_m} E(p,q) \qquad \text{..........(2)}$$

Also from theorem 9.2, we have

$$\max_{q\in S_m}. \min_{q\in S_n} E(p,q) \leq \min_{q\in S_n}. \max_{p\in S_m} E(p,q) \qquad \text{.......... (3)}$$

It now follows from (2) and (3) that,

$$\min_{q\in S_n}. \max_{p\in S_m} E(p,q) = \max_{p\in S_m}. \min_{q\in S_n} E(p,q) = E(p^0, q^0)$$

Which establishes the necessity of the condition.

(sufficiency of the condition)

Let the point (p^0,q^0) satisfy

$$\min_{q\in S_n}. \max_{p\in S_m} E(p,q) = \max_{p\in S_m}. \min_{q\in S_n}. E(p,q) = E(p^0, q^0)$$

Let $\min\limits_{q\in S_n}. E(p,q) = E(p,q^0)$ for all $p\in S_m$

and $\max\limits_{p\in S_m} E(p,q) = E(p^0,q)$ for all $q\in S_n$

The given condition then is

$$\min_{q\in S_n} E(p^0, q) = \max_{p\in S_m}. E(p,q^0) = E(p^0, q^0) \qquad \text{..........(4)}$$

But by the definition of minima, we have

$$E(p^0,q^0) \geq \min_{q\in S_n} E(p^0, q)$$

Therefore, using (1) we get $E(p^0,q^0) \geq \max\limits_{p\in S_m} E(p, q^0)$

which gives $E(p^0, q^0) \geq E(p, q^0)$ for all $p\in S_m$

Again, by the definition of maxima

$$\max_{p\in S_m} E(p, q^0) \geq E(p^0,q^0)\ldots\ldots\ldots\ldots(5)$$

Using (4) and (5), we have

$$\min_{q\in S_n} E(p^0,q) \geq E(p^0,q^0)$$

which further gives

$E(p^0,q) \geq E(p^0, q^0)$ for all $q\in S_n$

Hence, we obtain

$E(p^0,q) \geq E(p^0,q^0) \geq E(p,q^0)$ for all $p\in S_m$, $q\in S_n$

This shows that (p^0,q^0) is a saddle point of $E(p,q)$

This completes the proof.

Corollary :-

Let (a_{ij}) be an mxn pay off matrix for a two-person zero-sum game. Then a necessary and sufficient condition for (a_{ij}) to have a saddle point at i=k and j=r is that

$$a_{kr} = \max_{i}[\min_{j} \{a_{ij}\}] = \min_{j}. [\max_{i} \{a_{ij}\}]$$

Note :- A saddle point of a payoff matrix is also sometimes called the equilibrium point of the payoff matrix.

Definition : (Value of the game)

If (p^0,q^0) be a saddle point of $E(p,q)$, then the value of the game is $\nu = E(p^0,q^0)$

Theorem 9.4 :

Let ν be the value of an mxn game. Then, a necessary and sufficient condition for $p^0\in S_m$ to be an optimal strategy for the row player is that $\nu < E(p^0,q)$, for all $q\in S_n$. Similarly, a necessary and sufficient condition for $q^0\in S_n$ to be an optimal strategy for the column player is that $E(p,q^0) \leq \nu$ for all $p\in S_m$.

Proof :

Let ν be the value of the game.

(ie) $E(p^0,q^0) = \nu$.

We know that $\nu = \max_{p}. \min_{q} E(p,q) = \min_{q} E(p^0,q)$

Therefore $E(p^0,q) \geq \nu$. for all $q\in S_n$.

Again, $\nu = \min_{q} \max_{p} E(p,q) = \max_{p} E(p,q^0)$

$\therefore E(p,q^0) \leq \nu$ for all $p\in S_m$.

9.6 Solution of 2x2 Rectangular Games

A game without saddle point can be solved by various solution methods.

In most of the situations,the given rectangular game can be reduced to a much smaller 2x2 game. It is, therefore, worthwhile to determine formulae for the optimal strategies and the value of the game in the case of 2x2 game. The following theorem gives these formulae.

Theorem 9.5

For any 2x2 two-person zero-sum game without any saddle point having the pay off matrix for player A

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{array}$$

the optimum mixed strategies

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \quad \text{and} \quad S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

are determined by

$$\frac{p_1}{p_2} = \frac{a_{22}-a_{21}}{a_{11}-a_{12}}, \qquad \frac{q_1}{q_2} = \frac{a_{22}-a_{12}}{a_{11}-a_{21}}$$

where $p_1+p_2 = 1$ and $q_1+q_2=1$. The value ν of the game to A is given by

$$\nu = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}+a_{22} - (a_{12}+a_{21})}$$

Proof :- Let a mixed strategy for player A be given by

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$$

where $p_1+p_2=1$. Thus, if player B moves B_1,the net expected gain of A will be

$E_1(p) = a_{11}p_1+a_{21}p_2$

and if B moves B_2, the net expected gain of A will be

$E_2(p) = a_{12}p_1+a_{22}p_2$.

Similarly, it B plays his mixed strategy

$$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

where $q_1+q_2 =1$,

then B's net expected loss will be

$E_1(q) = a_{11}q_1+a_{12}q_2$

if A plays A_1 and

$E_2(q) = a_{21}q_1+a_{22}q_2$ if A plays A_2.

The expected gain of player A, when B mixes his moves with probabilities q_1 & q_2 is therefore given by

$E(p,q) = q_1 [a_{11}p_1+a_{21}p_2] + q_2 [a_{12}p_1+a_{22}p_2]$

player A would always try to mix his moves with such probabilities so as to maximize his expected gain.

$$\text{Now, } E(p,q) = q_1 [a_{11}p_1+a_{21}(1-p_1] +(1-q_1) [a_{12}p_1+a_{22}(1-p_1)]$$

$$= [a_{11}+a_{22}-(a_{12}+a_{21})]\, p_1q_1+(a_{12}-a_{22})p_1 + (a_{21}-a_{22})q_1 + a_{22}$$

$$= \lambda \left(p_1 - \frac{a_{22}-a_{21}}{\lambda}\right)\left(a_1 - \frac{a_{22}-a_{12}}{\lambda}\right) + \frac{a_{11}a_{22}-a_{12}a_{21}}{\lambda},$$

where $\lambda = a_{11}+a_{22} - (a_{12}+a_{21})$

We see that if A chooses $p_1 = \dfrac{a_{22}-a_{21}}{\lambda}$,

he ensures an expected gain of atleast $(a_{11}a_{22}-a_{12}a_{21})/\lambda$. Similarly if B chooses $q_i = \dfrac{a_{22}-a_{21}}{\lambda}$ then B will limit his expected loss to atmost $(a_{11}a_{22}-a_{12}a_{21})/\lambda$. These choices of p_1 & q_1 will thus be optimal to the two players.

Thus we get

$$p_1 = \frac{a_{22}-a_{21}}{\lambda} \quad \text{and} \quad p_2 = 1-p_1 = \frac{a_{11}-a_{12}}{\lambda} \;;$$

$$q_1 = \frac{a_{22}-a_{12}}{\lambda} \quad \text{and} \quad q_2 = 1-q_1 = \frac{a_{11}-a_{21}}{\lambda} \;;$$

$$\text{and } \nu = \frac{a_{11}a_{22} - a_{12}a_{21}}{\lambda}$$

Hence, we have

$$\frac{p_1}{p_2} = \frac{a_{22}- a_{21}}{a_{11}- a_{12}}, \qquad \frac{q_1}{q_2} = \frac{a_{22}- a_{12}}{a_{11}- a_{21}}$$

$$\text{and } v = \frac{a_{11}a_{22}- a_{12}a_{21}}{a_{11}+ a_{22} - (a_{12}+ a_{21})}$$

Note : The above formula valued only for 2x2 games with out saddle point.

Solved problems :

1) Solve the following game and determine the value of the game

$$A = \begin{array}{c} B \\ \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \end{array}$$

Solution :

Clearly, the given matrix is without a saddle point. So the mixed strategies of A & B

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix} \quad \begin{array}{l} p_1+p_2 = 1 \text{ and} \\ q_1+q_2 = 1 \end{array}$$

If E[p,q] denotes the expected pay off function, then

$$E(p,q) = \lambda \left(p_1 - \frac{a_{22}-a_{21}}{\lambda} \right) \left(q_1 - \frac{a_{22}-a_{12}}{\lambda} \right) + \frac{a_{11}a_{22}-a_{12}a_{21}}{\lambda}$$

where $\lambda = a_{11}+a_{22} - (a_{12}+a_{21})$

$$= 16 \left(p_1 - \frac{8}{16} \right) \left(q_1 - \frac{8}{16} \right) + \frac{0}{16}$$

$$= 16 \left(p_1 - \frac{1}{2} \right) \left(q_1 - \frac{1}{2} \right)$$

If A choose $p_1 = ½$, he ensures an expected gain of atleast 0. Similarly if B chooses $q_1 = ½$ then B will limit his expected loss to at most 0. These choice of p_1 & q_1 will thus be optimal to the two players.

Thus we get,

$p_1 = ½$ and $p_2 = 1 - ½ = ½$

$q_1 = ½$ and $q_2 = 1 - ½ = ½$

and $\nu = 0$

Hence the optimum strategies for A and B are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ ½ & ½ \end{bmatrix} \text{ and } S_B = \begin{bmatrix} B_1 & B_2 \\ ½ & ½ \end{bmatrix}$$

and the value of the game is $\nu = 0$.

(2) For the game with the following pay-off matrix, determinethe optimum strategies and the value of the game.

$$\begin{array}{cc} & p_2 \\ p_1 & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

solution : Clearly, the given matrix is without a saddle point. So the mixed strategies of P_1 & P_2 are

$$SP_1 = \begin{bmatrix} 1 & 2 \\ p_1 & p_2 \end{bmatrix} \quad \text{and} \quad SP_2 = \begin{bmatrix} 1 & 2 \\ q_1 & q_2 \end{bmatrix} \; ; \; p_1 + p_2 = 1 \text{ and } q_1 + q_2 = 1 .$$

If E(p,q) dentoes the expected pay-off function,

then $E(p,q) = 5p_1q_1 + 3.1(1-p_1)q_1 + p_1(1-q_1) + 4(1-p_1)(1-q_1)$

$= 5p_1q_1 - 3p_1 - q_1 + 4$

$= 5(p_1 - \frac{1}{5})(q_1 - \frac{3}{5}) + \frac{17}{15}$

If P_1 chooses $P_1 = \frac{1}{5}$ he ensures that his expectation is atleast $\frac{17}{5}$. He cannot be sure of more than $\frac{17}{5}$,because by choosing $q_1 = \frac{3}{5}$, p_2 can keep $E(p_1, q_1)$ down to $\frac{17}{5}$. So P_1 might as well settle for $\frac{17}{5}$ and P_2 reconcile to $\frac{17}{5}$

Hence the optimum strategies for P_1 and P_2 are

$$SP_1 = \begin{bmatrix} 1 & 2 \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix} \quad \text{and} \quad SP_2 = \begin{bmatrix} 1 & 2 \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

and the value of the game is $\nu = \frac{17}{5}$

(3) Find out the optimum strategies for the following 2x2 games without saddle point

i) $$\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

(ii) $$\begin{array}{cc} & B \\ A & \begin{bmatrix} 6 & -3 \\ -3 & 0 \end{bmatrix} \end{array}$$

(iii) $$\begin{array}{cc} & B \\ A & \begin{bmatrix} 2 & 5 \\ 7 & 3 \end{bmatrix} \end{array}$$

(iv) $$\begin{array}{cc} & B \\ A & \begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix} \end{array}$$

(v) $$\begin{array}{cc} & B \\ A & \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \end{array}$$

(v) $$\begin{array}{cc} & B \\ A & \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$$

Solution :

(i)
$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

$$p_1 = \frac{a_{22} - a_{21}}{a_{11}+a_{22} - (a_{12}+a_{21})} = \frac{4-3}{5+4-(1+3)}$$

$$= \frac{1}{9-4} = \frac{1}{5}$$

$$p_2 = \frac{a_{11} - a_{12}}{a_{11}+a_{22} - (a_{12}+a_{21})} = \frac{5-1}{5+4-(1+3)} = \frac{4}{5}$$

(or) $p_2 = 1-p_1 = 1- \frac{1}{5} = \frac{4}{5}$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11}+a_{22} - (a_{12}+a_{21})} = \frac{4-1}{5+4-(1+3)} = \frac{3}{5}$$

$$q_2 = \frac{a_{11} - a_{21}}{a_{11}+a_{22} - (a_{12}+a_{21})} = \frac{5-3}{5+4-(1+3)} = \frac{2}{5}$$

(or) $q_2 = 1-q_1 = 1- \frac{3}{5} = \frac{2}{5}$

value of the game $\nu = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}+a_{22} - (a_{12}+a_{21})}$

$$= \frac{5.4 - 3.1}{(5+4)-(3+1)} = \frac{20-3}{9-4} = \frac{17}{5}$$

Hence the optimun strategies for A & B are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix} \quad \text{and} \quad S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

and the value of the game is $\nu = \frac{17}{5}$

(ii)
$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 6 & -3 \\ -3 & 0 \end{bmatrix} \end{array}$$

$$p_1 = \frac{0+3}{6+0-(-3-3)} = \frac{3}{6+6} = \frac{3}{12} = \frac{1}{4}$$

$$p_2 = 1-p_1 = 1-\frac{1}{4} = \frac{3}{4}$$

$$q_1 = \frac{a_{22}-a_{12}}{a_{11}+a_{22}-(a_{12}+a_{21})} = \frac{0+3}{6+6} = \frac{3}{12} = \frac{1}{4}$$

$$q_2 = 1-q_1 = 1-\frac{1}{4} = \frac{3}{4}$$

$$\nu = \frac{a_{11}a_{22}-a_{21}a_{12}}{a_{11}+a_{22}-(a_{12}+a_{21})} = \frac{0-9}{6+6} = -\frac{3}{4}$$

Hence the optimum strategies for A&B are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \text{ and } S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

and the value of the game is $\nu = -\frac{3}{4}$

(iii)
$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 2 & 5 \\ 7 & 3 \end{bmatrix} \end{array}$$

$$p_1 = \frac{a_{22}-a_{21}}{a_{11}+a_{22}-(a_{12}+a_{21})} = \frac{3-7}{(2+3)-(7+5)} = \frac{-4}{5-12} = \frac{-4}{-7} = \frac{4}{7}$$

$$p_2 = 1-p_1 = 1-\frac{4}{7} = \frac{3}{7}$$

$$q_1 = \frac{a_{22}-a_{12}}{a_{11}+a_{22}-(a_{12}+a_{21})} = \frac{3-5}{5-12} = \frac{-2}{-7} = \frac{2}{7}$$

$$q_2 = 1 - \frac{2}{7} = \frac{5}{7}$$

$$\nu = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}+a_{22} - (a_{12}+a_{21})}$$

$$= \frac{6-35}{5-12} = \frac{-29}{-7} = \frac{29}{7}$$

Hence the optimum strategies for A and B are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{4}{7} & \frac{3}{7} \end{bmatrix} \quad ; \quad S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{2}{7} & \frac{5}{7} \end{bmatrix}$$

The value of the game $\nu = \frac{29}{7}$

(iv)

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix} \end{array}$$

$$p_1 = \frac{a_{22}-a_{21}}{a_{11}+a_{22} - (a_{12}+a_{21})} = \frac{-3-2}{(-4-3) - (6+2)} = \frac{-5}{-15} = \frac{1}{3}$$

$$p_2 = 1 - p_1 = 1 - \frac{1}{3} = \frac{2}{3}$$

$$q_1 = \frac{a_{22}-a_{12}}{a_{11}+a_{22} - (a_{12}+a_{21})} = \frac{-3-6}{-15} = \frac{-9}{-15} = \frac{3}{5}$$

$$q_2 = 1 - \frac{3}{5} = \frac{2}{5}$$

$$\nu = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}+a_{22} - (a_{12}+a_{21})}$$

$$= \frac{12-12}{-15} = 0$$

Hence the optimum strategies for A and B are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \quad ; \quad S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

The value of the game $\nu = 0$

$$\text{(v)} \qquad \begin{array}{cc} & B \\ A & \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \end{array}$$

$$p_1 = \frac{a_{22}-a_{21}}{a_{11}+a_{22} - (a_{12}+a_{21})} = \frac{3+2}{(3+3) - (-2-2)} = \frac{5}{6+4} = \frac{5}{10} = \frac{1}{2}$$

$$p_2 = 1-p_1 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$q_1 = \frac{a_{22}-a_{12}}{a_{11}+a_{22} - (a_{12}+a_{21})} = \frac{3+2}{10} = \frac{5}{10} = \frac{1}{2}$$

$$q_2 = 1-q_1 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\nu = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}+a_{22} - (a_{12}+a_{21})}$$

$$= \frac{9-4}{10} = \frac{5}{10} = \frac{1}{2}$$

Hence the optimum strategies for A and B are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

and the value of the game is $\nu = \frac{1}{2}$

(vi)
$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$$

$$p_1 = \frac{a_{22}-a_{21}}{a_{11}+a_{22}-(a_{12}+a_{21})} = \frac{1-4}{(2+1)-(5+4)} = \frac{-3}{3-9} = \frac{-3}{-6} = \frac{1}{2}$$

$$p_2 = 1-p_1 = 1-\frac{1}{2} = \frac{1}{2}$$

$$q_1 = \frac{a_{22}-a_{12}}{a_{11}+a_{22}-(a_{12}+a_{21})} = \frac{1-5}{-6} = \frac{-4}{-6} = \frac{2}{3}$$

$$q_2 = 1-\frac{2}{3} = \frac{1}{3}$$

$$\nu = \frac{a_{11}a_{22}-a_{21}a_{12}}{a_{11}+a_{22}-(a_{12}+a_{21})}$$

$$= \frac{2-20}{-6} = \frac{-18}{-6} = 3$$

Hence the optimum strategies for A and B are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad ; \quad S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

and the value of the game is $\nu = 3$

(4) Consider a modified form of "matching biased coins" game problem. The matching player is paid Rs 8.00 if the two coins turn both heads and Rs 1.00 if the coins turn both tails. The non-matching player is paid Rs 3.00 when the two coins do not match. Given the choice of being the matching or non-matching player, which one would you choose and what would be your strategy?

Solution :

The pay off matrix for the matching player is given by

Non matching player

matching player

$$\begin{array}{c|cc|} & H & T \\ \hline H & 8 & -3 \\ T & -3 & 1 \\ \hline \end{array}$$

Clearly the payoff matrix does not posses any saddle point. The players will use mixed strategies. The optimum mixed strategy for matching player is determined by

$$p_1 = \frac{1-(-3)}{(8+1)-(-3-3)} = \frac{4}{9+6} = \frac{4}{15}$$

$$p_2 = 1-p_1 = 1 - \frac{4}{15} = \frac{11}{15}$$

and for the non matching player, by

$$q_1 = \frac{1-(-3)}{(8+1)-(-3-3)} = \frac{4}{15}$$

$$q_2 = 1 - \frac{4}{15} = \frac{11}{15}$$

The expected value of the game (corresponding to the above strategies) is given by

$$\nu = \frac{8-(-3)(-3)}{8+1-(-3-3)} = \frac{1}{15}$$

Thus the optimum mixed strategies for matching player and non-matching player are given by

$$S_{\text{match}} = \begin{bmatrix} H & T \\ \frac{4}{15} & \frac{11}{15} \end{bmatrix}$$ and

$$S_{\text{non-match}} = \begin{bmatrix} H & T \\ \frac{4}{15} & \frac{11}{15} \end{bmatrix}$$

Clearly, we would like to be the non-matching player.

(5) In a game of matching coins with two players, suppose A wins one unit of value, when there are two heads, wins nothing when there are two tails, and loses ½ unit of value when there are one head and one tail. Determine the pay-off matrix, the best strategies for each player and the value of the game to A.

Solution :

The pay off matrix is $\begin{array}{cc} A & \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \end{array}$

$$p_1 = \frac{a_{22}-a_{21}}{a_{11}+a_{22}-(a_{12}+a_{21})} = \frac{0+\frac{1}{2}}{(1+0)-(-\frac{1}{2}-\frac{1}{2})} = \frac{\frac{1}{2}}{1+1} = \frac{1}{4}$$

$$p_2 = 1-p_1 = 1-\frac{1}{4} = \frac{3}{4}$$

$$q_1 = \frac{a_{22}-a_{12}}{a_{11}+a_{22}-(a_{12}+a_{21})} = \frac{0+\frac{1}{2}}{2} = \frac{1}{4}$$

$$q_2 = 1-q_1 = 1-\frac{1}{4} = \frac{3}{4}$$

$$\nu = \frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}+a_{22}-(a_{12}+a_{21})}$$

$$= \frac{0-\frac{1}{4}}{2} = -\frac{1}{8}$$

Hence the optimum strategies for player A and B are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \quad ; \quad S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

and the value of the game is $\nu = -\frac{1}{8}$

(6) Two players A and B match coins. If one coin match then A wins one unit of value. If the coin do not match then B wins one unit of value. Determine the optimum strategies for the players and the value of the game.

Solution :-

The payoff matrix is

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$p_1 = \frac{a_{22}-a_{21}}{a_{11}+a_{22}-(a_{21}+a_{12})} = \frac{1+1}{(1+1)-(-1-1)} = \frac{2}{2+2} = \frac{2}{4} = \frac{1}{2}$$

$$p_2 = 1-p_1 = 1-\frac{1}{2} = \frac{1}{2}$$

$$q_1 = \frac{a_{22}-a_{12}}{a_{11}+a_{22}-(a_{21}+a_{12})} = \frac{1+1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$q_2 = 1-q_1 = 1-\frac{1}{2} = \frac{1}{2}$$

$$\nu = \frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}+a_{22}-(a_{21}+a_{12})}$$

$$= \frac{1-1}{4} = 0$$

Hence the optimum strategies for player A and B are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \; ; \; S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$ and the value of the game is $\nu = 0$

Exercise :-

1. A and B play a game in which each has three coins : a penny, a nickel, and a dime. Each selects a coin without the knowledge of the other's choice. If the sum of the coins is an odd amount, A wins B's coin; if the sum is even, B wins A's coin Find the best strategy for each player and the value of game.

2. Two players A and B match coins. If the coins match, then A wins two units of value, if the coins do not match, then B wins 2 units of value. Determine the optimum strategies for the players and the value of the game.

3. A and B each take out one or two matches and guess how many matches opponent has taken. If one of the players guesses correctly then the loser has to pay him as many rupees as the sum of the number held by both players. Otherwise, the payout is zero Write down the pay-off matrix and obtain the optimal strategies of both players.

Answers :-

1. $S_A = \begin{bmatrix} \text{Penny} & \text{Nickel} & \text{Dime} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$; $S_B = \begin{bmatrix} \text{Penny} & \text{Nickel} & \text{Dime} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}$ $\nu = 0$

2. $S_A = S_B = \begin{bmatrix} H & T \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$; value of game $\nu = 0$

3. $S_A = \begin{bmatrix} 1 & 2 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$; $S_B = \begin{bmatrix} 1 & 2 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$ value of game $\nu = \frac{4}{3}$

9.7 Graphical mathod

Graphical solutions of 2xn and mx2 games :

The graphical method is useful for the game where the payoff matrix is of the size 2xn or mx2 (ie) the game with mixed strategies that has only two pure strategies for one of the players in the two-person zero-sum game.

Optimal strategies for both the players assign non-zero probabilities to the same number of pure strategies. Therefore, if one player has only two strategies, the other will also use the same number of strategies. Hence this method is useful in finding out which of the two strategies can be used.

Consider the following 2xn payoff matrix of a game without saddle point :

$$\begin{array}{ll} & \text{Player B} \\ & \begin{array}{cc} & \begin{array}{ccccc} B_1 & B_2 & B_3 & \cdots\cdots & B_n \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \end{pmatrix} \end{array} \end{array}$$

player A

Let the mixed strategy for player A be given by

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$$, where $p_1+p_2=1$ and $p_1 \geq 0$, $p_2 \geq 0$.

Now for each of the pure strategies available to B, expected pay-off for player A would as follows.

B's pure move	A's expected payoff E(p)
B_1	$E_1(p) = a_{11}p_1+a_{21}p_2$
B_2	$E_2(p) = a_{12}p_1+a_{22}p_2$
.	
.	
.	
B_n	$E_n(p) = a_{1n}p_1+a_{2n}p_2$

The player B would like to choose that pure move B_j against S_A for which E_j (p) is a minimum for j = 1,2,....n. Let us denote this minimum expected pay off for A by

ν = min $\{E_j(p), j = 1,2,......n\}$

The objective of player A is to select p_1 and (hence) p_2 in such a way that γ is as large as possible. This may be done by plotting the straight lines.

$E_j(p) = a_{1j}p_1+a_{2j}p_2 = (a_{1j}-a_{2j})p_1+a_{2j}$ (j = 1,2,.........n)

as linear functions of p_1.

The highest point on the lower boundery of these lines will give maximum expected pay off among the minimum expected payoffs on the lower boundary (lower envelpe) and the optimum value of probability p_1 and p_2.

Now the two strategies of player B corresponding to those lines which pass through the maximin point can be determined. It helps in reducing the size of the game to (2x2).

The (mx2) games are also treated in the same way except that the upper boundary (upper envelope) of the stratight lines corresponding to B's expected payoff will give the maximum expected payoff to player B and the lowest point on this boundary will then give the minimum expected payoff (minimax value) and the optimum value of probability q_1 and q_2.

Solved problems :

(1) Slove the following 2x3 game graphically

player B

$$\text{Player A} \quad \begin{bmatrix} 1 & 3 & 11 \\ 8 & 5 & 2 \end{bmatrix}$$

Solution :-

Since the problem does not possess any saddle point, let the player A play the mixed strategy

$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ aganist player B, $p_2 = 1-p_1$,

The A's expected pay offs aganist B's pure moves are given by

B's pure move	A's expected payoff $E(p_1)$
B_1	$E_1(p_1) = p_1+8(1-p_1) = -7p_1+8$
B_2	$E_2(p_1) = 3p_1+5(1-p_1) = -2p_1+5$
B_3	$E_3(p_1) = 11p_1+2(1-p_1) = 9p_1+2$

These expected pay off equations are then plotted as functions of p_1 as shown in the following figure 9.1 which shows the pay offs of each column represented as points on two vertical axes 1 and 2 unit distance apart.

Lines joining the payoffs on axis 1 with the payoffs on axis 2, then represents each of B's strategies, eg. to represent B's 1st strategy we join the element 1 on axis 2 with the element 8 on axis 1.

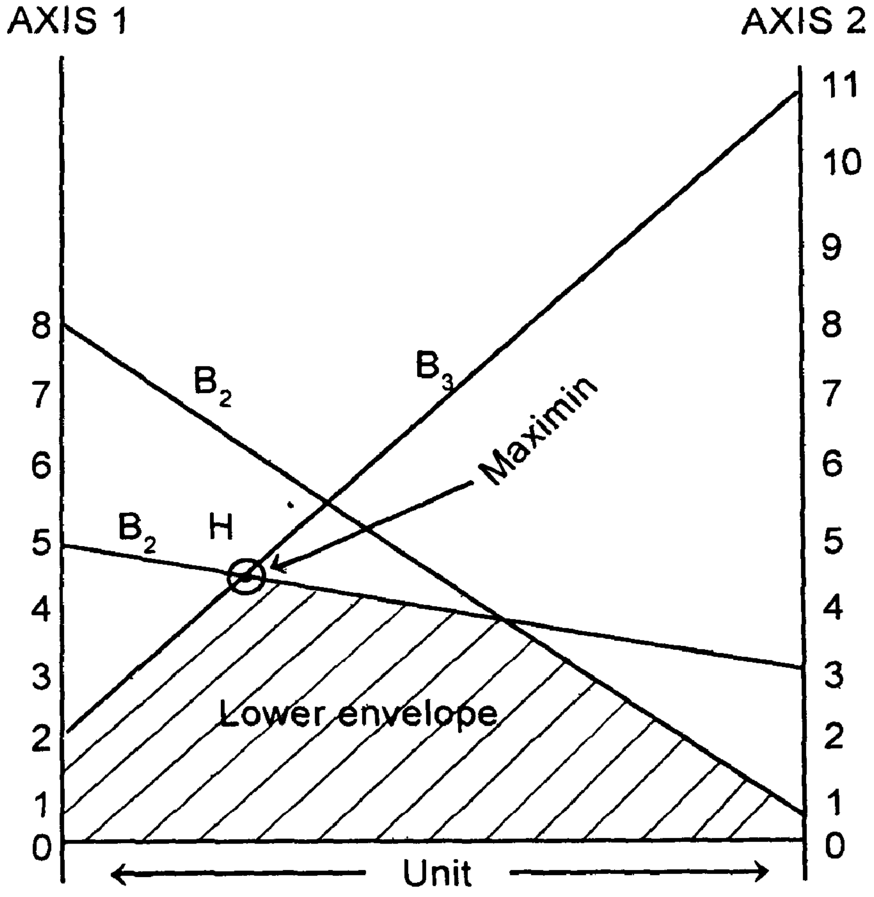


fig. 9.1

Now since the player A wishes to maximize his minimum expected payoff, we consider the highest point of intersection H on the lower envelope of A's expected payoff equations. This point H represents the maximim expected value of the game for A. The lines B_2 and B_3, passing through H_1 define the relavent moves B_2 and B_3 that alone B needs to play. The solution to the original 2x3 game, therefore, reduces to that of the simpler game with the 2x2 payoff matrix.

$$\begin{array}{c|cc|} & B_2 & B_3 \\ A_1 & 3 & 11 \\ A_2 & 5 & 2 \end{array}$$

Now, if $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$, $p_1+p_2=1$

and $S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & q_1 & q_2 \end{bmatrix}$, $q_1+q_2=1$

then using the usual method of solution for 2x2 games, the optimum strategies can easily be obtained as

$$S_A = \begin{bmatrix} A_1 & A_2 \\ 3/11 & 8/11 \end{bmatrix} \quad \text{and} \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 2/11 & 9/11 \end{bmatrix}$$

And the value of game as $\nu = 49/11$.

2. Solve the following 2x4 game graphically.

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \text{Player A} & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & 3 & 2 \end{pmatrix} \end{array}$$

Solution :

Since the problem does not passes any saddle point, let the player A play the mixed strategy $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ aganist player B, $p_2 = 1-p_1$

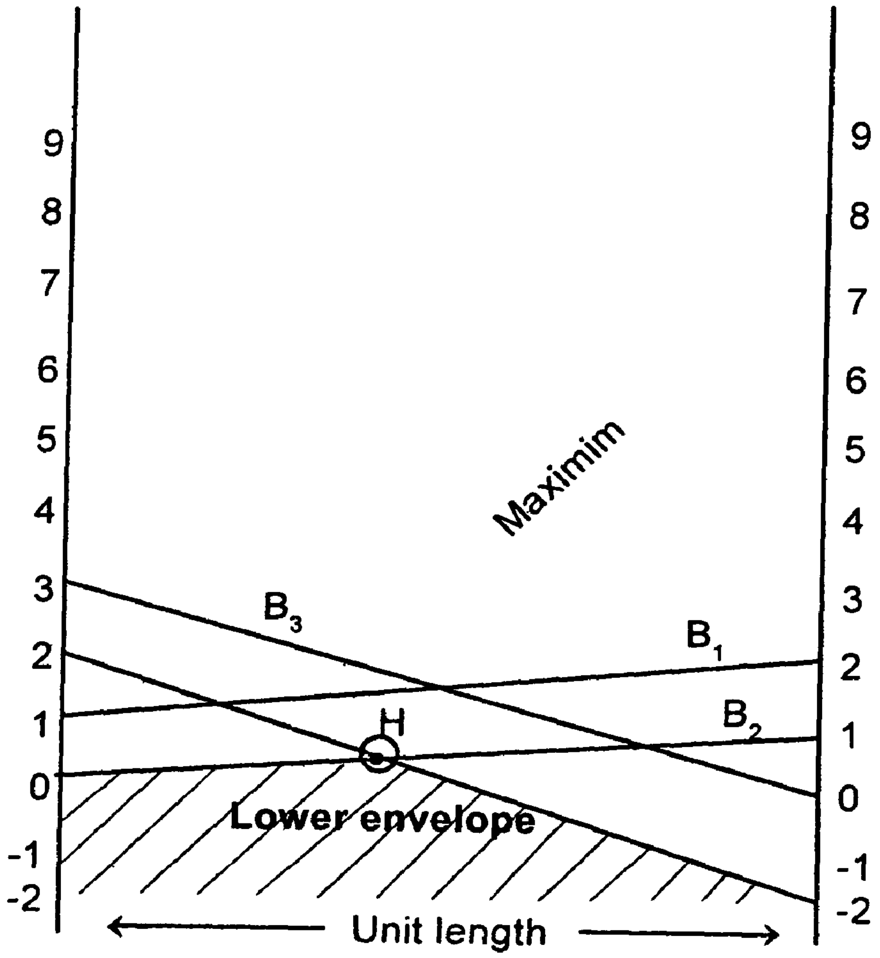
The A's expected payoffs against B's pure moves are given by

B's pure move	A's expected payoff $E(p_1)$
B_1	$E_1(p_1) = 2p_1+p_2= 2p_1+1-p_1=p_1+1$
B_2	$E_2(p_1) = p_1+0 = p_1$
B_3	$E_3(p_1) = 0+3(1-p_1) = 3-3p_1$
B_4	$E_4(p_1) = -2p_1+2(1-p_1)$ $= -2p_1+2-2p_1 = 2-4p_1$.

These expected pay-off equations are then plotted as functions of p_1 as shown in fig 9.2 which shows the pay offs of eah column represented as points on two vertical axes 1 and 2, unit distance apart.

Lines joining the payoff on axis 1 with the payoffs on axis 2, then represents each of B's strategies. eg. to represent B's 1st strategy we join the element 2 on axis 2 with the element 1 on on axis 1.

Similary lines B_2, B_3 and B_4 jion the corresponding representation of pay off elements in the second, third and fourth columns.



Now since the player A wishes to maximize his minimum expected payoff, we consier the highest point of intersection H on the lower envelope of A's expected payoff equations. This point H represents the maximin expected value of the game for A. The lines B_2 and B_4, passing through H, define the relevant moves B_2 and B_4 that alone B needs to play. The solution to the original 2x3 game, thesefore reduces to that of the simpler game with the 2x2 payoff matrix

$$\begin{array}{c c} & \begin{array}{cc} B_2 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix} \end{array}$$

Now if $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ $p_1 + p_2 = 1$

and $S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & q_1 & 0 & q_2 \end{bmatrix}$ $q_1+q_2=1$

$$p1 = \frac{a_{22} - a_{21}}{a_{11}+a_{22} - (a_{12}+a_{21})} = \frac{2-0}{(1+2) - (0-2)} = \frac{2}{3+2} = 2/5$$

$p_2 = 1 - 2/5 = 3/5$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11}+a_{22} - (a_{12}+a_{21})} = \frac{2+2}{5} = 4/5$$

$q_2 = 1-q_1 = 1 - 4/5 = 1/5$

$$\nu = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}+a_{22} - (a_{12}+a_{21})} = \frac{2-0}{5} = 2/5$$

The optimum strategies are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ 2/5 & 3/5 \end{bmatrix} \quad \text{and} \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 4/5 & 0 & 1/5 \end{bmatrix}$$

and the value of the game $\nu = 2/5$

3) Obtain the optimal strategies for both persons and the value of the game for zero-sum two-person game whose payoff matrix is as follows.

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix}$$

Soultion :

Clearly, the given problem does not posses any saddle point. So, let the player B play the mixed strategy $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ with $q_2 = 1-q_1$ against player A. Then B's expected pay-offs against A's pure moves are given by

A's pure move	B's expected pay-off $E(q_1)$
A_1	$E_1(q_1) = q_1-3q_2 = q_1-3(1-q_1) = 4q_1-3$
A_2	$E_2(q_1) = 3q_1+5(1-q_1) = -2q_1+5$
A_3	$E_3(q_1) = -q_1+6(1-q_1) = -7q_1+6$
A_4	$E_4(q_1) = 4q_1+(1-q_1) = 3q_1+1$
A_5	$E_5(q_1) = 2q_1+2(1-q_1) = 2$
A_6	$E_6(q_1) = -5q_1+0q_2 = -5q_1$

The expected payoff equations are the plotted as functions of q_1 as shown in fig. 9.3

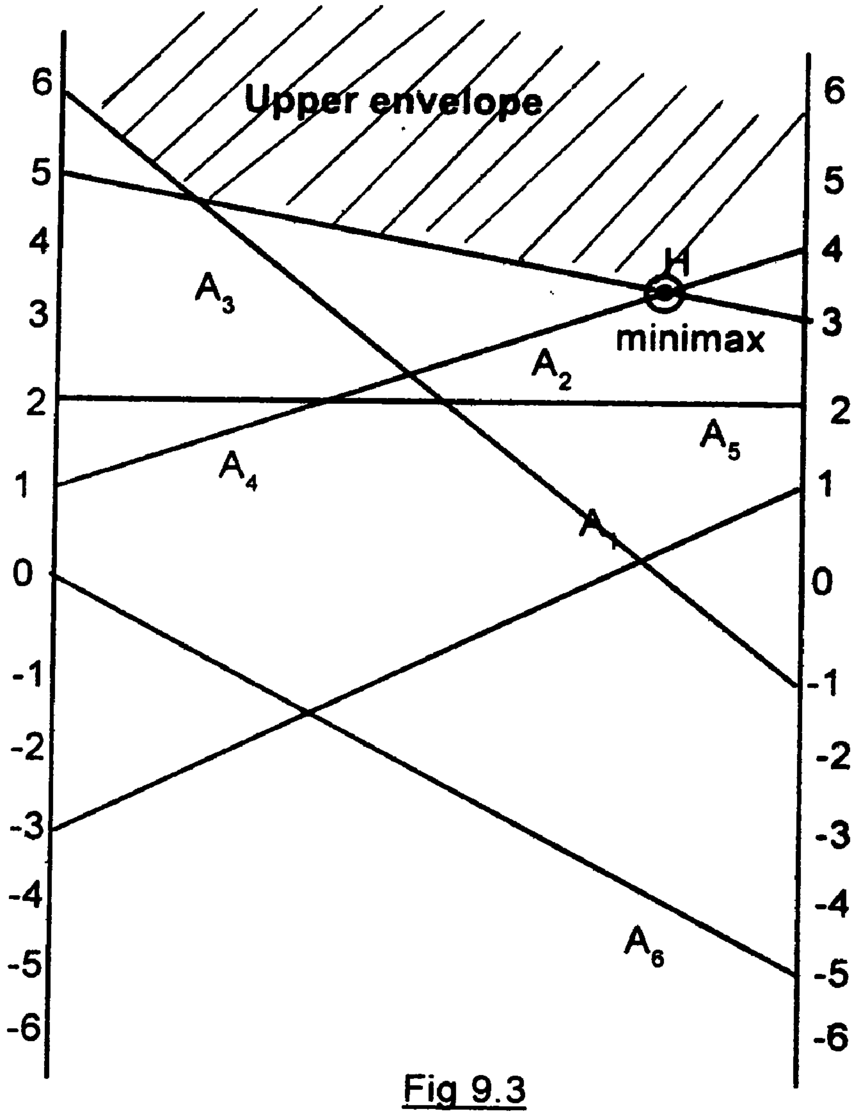


Fig 9.3

Since the player B wishes to minimize his maximum expected pay-off, we consider the lowest point of intersection H on the upper envelope of B's expected payoff equations. This point H represent the minimax expected value of the game for player B. The lines A_2 and A_4 passing through H, define the two relevant moves A_2 and A_4 that alone the player A needs to play. The solution to the orginal 6x2 game therefore reduces to that of the simpler game with 2x2 pay-off matrix.

$$\begin{array}{cc} & \text{Player B} \\ \text{player A} & \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$$

If we now let

$$S_A = \begin{bmatrix} A_2 & A_4 \\ p_1 & p_2 \end{bmatrix}, p_1+p_2 = 1\ ;\ S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}, q_1+q_2 = 1 \quad p_1, p_2$$

Then using the usual method of solution for 2x2 games, the optimum strategies can easily be obtained as

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ 0 & 3/5 & 0 & 2/5 & 0 & 0 \end{bmatrix}$$

$$S_B = \begin{bmatrix} B_1 & B_2 \\ 4/5 & 1/5 \end{bmatrix}.$$

and the value of the game as $\nu = 17/5$.

(4) Solve the 5x2 game graphically

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{pmatrix} -2 & 5 \\ -5 & 3 \\ 0 & -2 \\ -3 & 0 \\ 1 & -4 \end{pmatrix} \end{array}$$

solution :

Clearly the given problem does not have a saddle point So, let the player B play the mixed strategy $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ with $q_2 = 1-q_1$

against player A. Then B's expected pay-offs against A's pure moves are given by

A's pure move	B's expected pay off $E(q_1)$
A_1	$E_1(q_1) = -2q_1+5(1-q_1) = 5-7q_1$
A_2	$E_2(q_1) = -5q_1+3(1-q_1) = 3-8q_1$
A_3	$E_3(q_1) = 0q_1-2(1-q_1) = 2q_1-2$
A_4	$E_4(q_1) = -3q_1+0(1-q_1) = -3q_1$
A_5	$E_5(q_1) = q_1-4(1-q_1) = 5q_1-4$

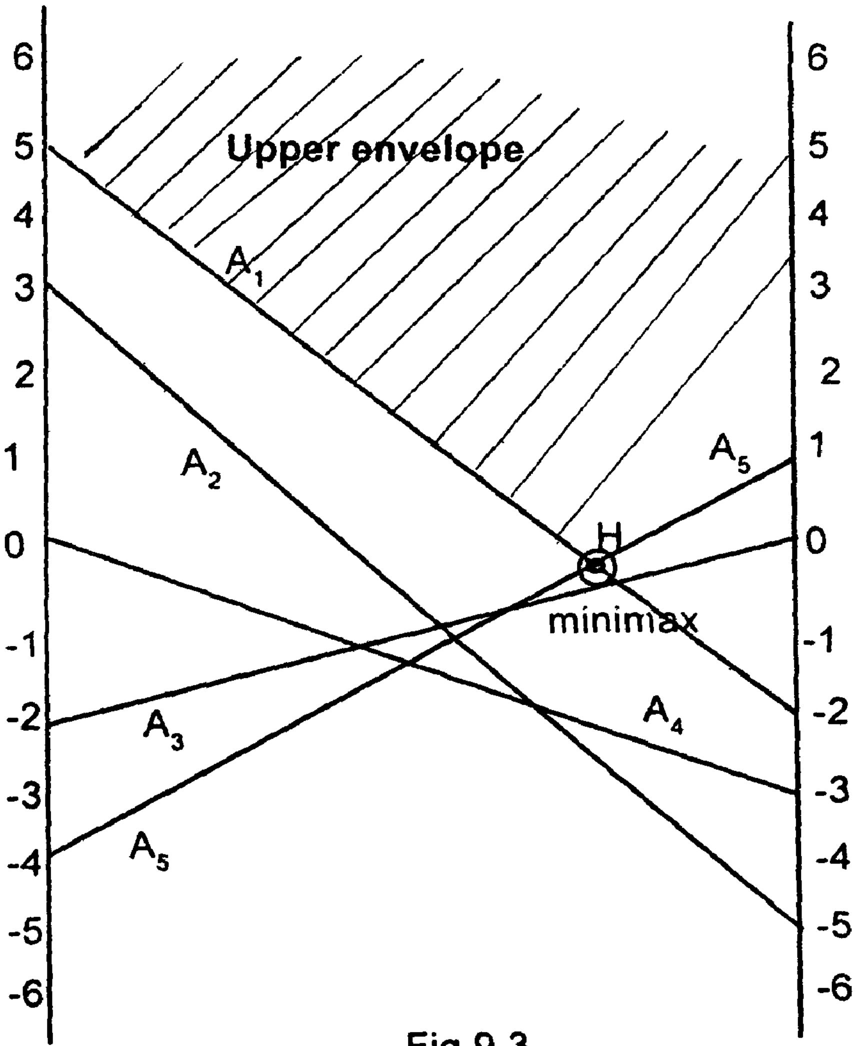


Fig 9.3

The expected payoff equations are then plotted as functions of q_1 as shown in Fig 9.4 Since the player B whises to minimize his maximum expected payoff, we consider the lowest point of intersection H on the upper envelope of B's expected pay off equations. This point H represents the minimax expected value of the game for player B.

The lines A1 and A5 passing through H define two relevant moves A1 and A5 that alone the player A needs to play. The solution to the orginal 5x2 game therefore reduces to that of the simpler game with 2x2 pay off matrix.

Player B

$$\text{player A} \quad \begin{array}{c} \\ A_1 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} -2 & 5 \\ 1 & -4 \end{bmatrix} \end{array}$$

If we, now let $S_A = \begin{bmatrix} A_1 & A_5 \\ p_1 & p_2, \end{bmatrix}$, $p_1+p_2 = 1$

$$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}, \quad q_1+q_2 = 1$$

$$p_1 = \frac{-4-1}{-6-6} = -5/-12 = 5/12$$

$$p_2 = 1-p_1 = 1-5/12 = 7/12$$

$$q_1 = \frac{-4-5}{-12} = -9/-12 = 3/4$$

$$q_2 = 1-3/4 = 1/4$$

$$\nu = \frac{8-5}{-12} = -3/12 = -1/4$$

The optimum strategies for A & B are

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 5/12 & 0 & 0 & 0 & 7/12 \end{bmatrix}$$

$$S_B = \begin{bmatrix} B_1 & B_2 \\ 3/4 & 1/4 \end{bmatrix}$$

and the value of game $\nu = -1/4$

(5) Solve the game whose pay off matrix is

$$\begin{array}{ccc} & & B \\ & B_1 & B_2 \end{array}$$

$$\begin{array}{cc} & \\ A & \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \end{array} \begin{bmatrix} -6 & 7 \\ 4 & -5 \\ -1 & -2 \\ -2 & 5 \\ 7 & -6 \end{bmatrix}$$

Solution :-

Clearly the given problem does not have a saddle point. So, let the player B play the mixed strategy $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ with $q_2 = 1-q_1$

against player A. Then B's expected pay-offs against A's pure moves are given by

A's pure move	B's expected pay off $E(q_1)$
A_1	$E_1(q_1) = -6q_1+7q_2 = 6q_1+7(1-q_1) = q_1+7$
A_2	$E_2(q_1) = 4q_1-5(1-q_1) = 9q_1-5$
A_3	$E_3(q_1) = -q_1-2(1-q_1) = q_1-2$
A_4	$E_4(q_1) = -2q_1+5(1-q_1) = 3q_1+5$
A_5	$E_5(q_1) = +7q_1-6(1-q_1) = 13q_1-6$

The expected pay off equations are then plotted as functions of q as shown in Fig 9.5

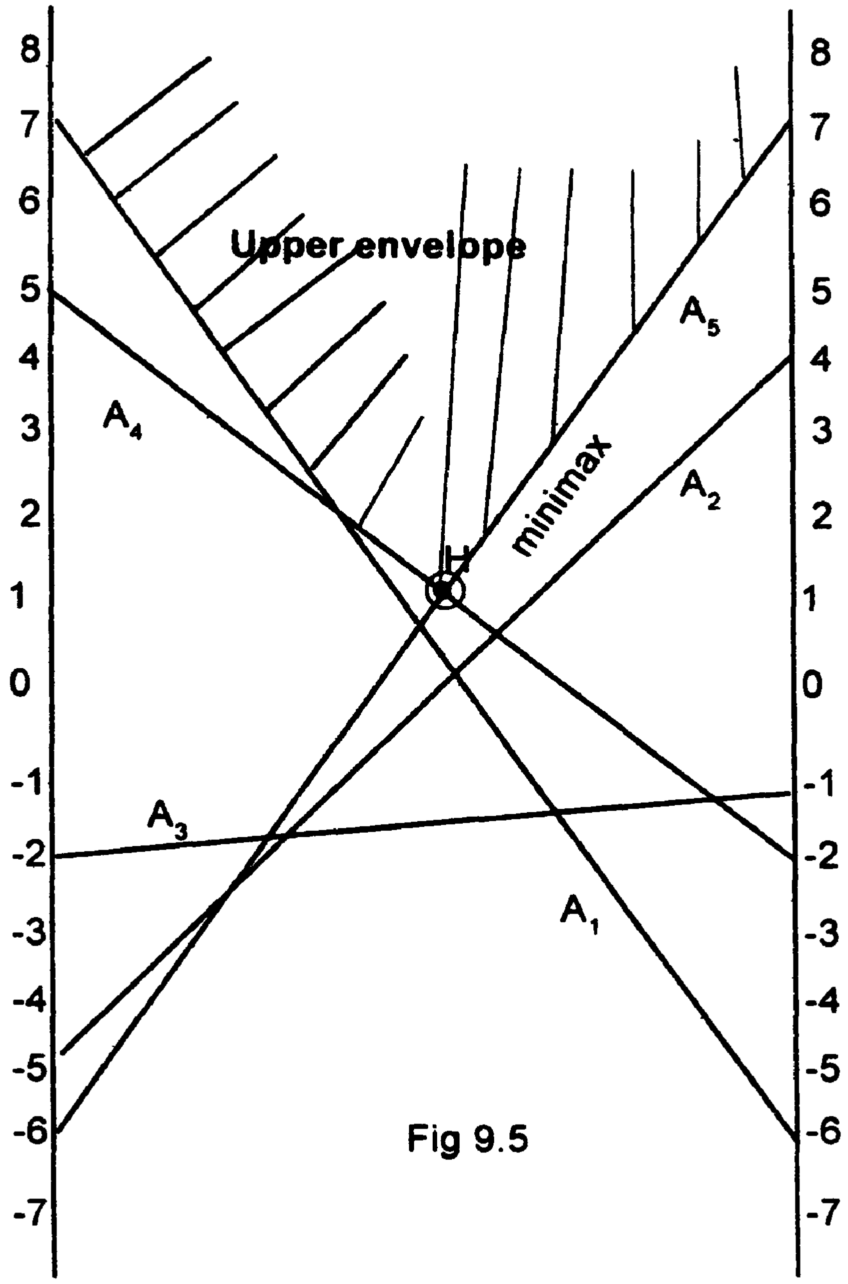


Fig 9.5

Since the player B wishes to minimize his maximum expected pay-off, we consider the lowest point of intesection H on the upper evelope of B's expected pay off equations. This point H represents the minimax expected value of the game for player B. The lines A_4 and A_5 passing through H, define the two relevant moves A_4 & A_5 passing through that alone the player A needs to play. The solution to the original 5x2 game therefore reduces to that of simpler game with 2x2 pay-off matrix.

$$\begin{array}{cc} & \text{Player B} \\ \text{player A} & \begin{bmatrix} -2 & +5 \\ 7 & -6 \end{bmatrix} \end{array}$$

If we now let, $S_A = \begin{bmatrix} A_4 & A_5 \\ p_1 & p_2 \end{bmatrix}$ $p_1 + p_2 = 1$

$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ $q_1 + q_2 = 1$

then using the usual method of solution for 2x2 games, the optimum strategies can easily be obtained as

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 0 & 0 & 0 & 13/20 & 7/20 \end{bmatrix} \qquad S_B = \begin{bmatrix} B_1 & B_2 \\ 11/20 & 9/20 \end{bmatrix}$$

and the value of the game is $\nu = 23/20 = 1.15$

Exercises :-

1) Solve the following problem graphically

$$\begin{array}{cc} & \text{player B} \\ \text{player A} & \begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix} \end{array}$$

2) Use graphical method in solving the following game

$$\begin{array}{cc} & \text{player A} \\ \text{player B} & \begin{bmatrix} 2 & 2 & 3 & -2 \\ 4 & 3 & 2 & 6 \end{bmatrix} \end{array}$$

3) Solve the following games

$$\begin{array}{cc} & A \\ B & \begin{pmatrix} -6 & 0 & 6 & -3/2 \\ 7 & -3 & -8 & 2 \end{pmatrix} \end{array}$$

4) Solve the following game graphically

$$\begin{array}{cc} & \text{player B} \\ \text{player A} & \begin{pmatrix} 1 & 3 & -3 & 7 \\ 2 & 5 & 4 & -6 \end{pmatrix} \end{array}$$

5) Solve the following game graphically

$$\begin{array}{cc} & \text{player B} \\ \text{player A} & \begin{bmatrix} 1 & 2 \\ 5 & 4 \\ -7 & 9 \\ -4 & -3 \\ 2 & 1 \end{bmatrix} \end{array}$$

6) The companies A and are competing for the same product - Their different strategies are given in the following pay off matirx

$$\begin{array}{cc} & \text{Company B} \\ \text{Company A} & \begin{bmatrix} 4 & -3 & 3 \\ -3 & 1 & -1 \end{bmatrix} \end{array}$$

Determine the best strategies for the two compaines.

7. Solve the following game graphically.

$$\begin{array}{cc} & \text{Player B} \\ \text{player A} & \begin{bmatrix} 2 & -1 & 5 & -2 & 6 \\ -2 & 4 & -3 & 1 & 0 \end{bmatrix} \end{array}$$

Answers :

1. $S_A = \begin{bmatrix} A_1 & A_2 \\ 4/11 & 7/11 \end{bmatrix}$; $S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 7/11 & 4/11 \end{bmatrix}$; $\nu = -5/11$

2. $S_B = \begin{bmatrix} B_1 & B_2 \\ 4/9 & 5/9 \end{bmatrix}$; $S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & 8/9 & 1/9 \end{bmatrix}$; $\nu = 22/9$

3. $S_A = \begin{bmatrix} A_1 & A_2 \\ 5/8 & 3/8 \end{bmatrix}$; $S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 3/16 & 13/16 & 0 & 0 \end{bmatrix}$; $\nu = 18/16$

4. $S_A = \begin{bmatrix} A_1 & A_2 \\ 1/2 & 1/2 \end{bmatrix}$; $S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & 13/20 & 7/20 \end{bmatrix}$; $\nu = 1/2$

5. $S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 0 & 16/17 & 1/17 & 0 & 0 \end{bmatrix}$; $S_B = \begin{bmatrix} B_1 & B_2 \\ 5/17 & 12/17 \end{bmatrix}$; $\nu = 73/17$

6. $S_A = \begin{bmatrix} A_1 & A_2 \\ 4/11 & 7/11 \end{bmatrix}$; $S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 4/11 & 7/11 & 0 \end{bmatrix}$; $\nu = -5/11$

7. $S_A = \begin{bmatrix} A_1 & A_2 \\ 3/7 & 4/7 \end{bmatrix}$; $S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 & B_5 \\ 3/7 & 0 & 0 & 4/7 & 0 \end{bmatrix}$; $\nu = -2/7$

9.8 The linear programming method :

A two-person zero-sum game can also be solved by linear programming approach The major advantage of using linear programming technique is that it solves mixed strategy game of any size.

To illustrate the connection between a game problem and a linear programming problem, let us consider an mxn pay off matrix (a_{ij}) for player A.

Let

$$S_m = \begin{bmatrix} A_1, \dots\dots\dots A_m \\ p_1 \dots\dots\dots p_m \end{bmatrix} \quad \text{and} \quad Sn = \begin{bmatrix} B_1, \dots \dots B_n \\ q_1, \dots \dots q_n \end{bmatrix}$$

where $\sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j = 1$ be the mixed strategies for the two players respectively.

Then the expected gains $g_j (j = 1,2 \dots\dots n)$ of player A against B's pure strategies will be

$g_1 = a_{11}p_1 + a_{21}p_2 + \dots\dots + a_{m1}p_m$

$g_2 = a_{12}p_1 + a_{22}p_2 + \dots\dots + a_{m2}p_m \dots\dots \dots\dots g_n = a_{2n}p_1 + a_{1n}p_2 + \dots . a_{mn}p_m$

and the expected losses l_i $(i = 1,2,\dots\dots m)$ of player B against A's pure strategies will be

$l_1 = a_{11}q_1 + a_{21}q_2 + \dots \dots + a_{1n}q_n$

$l_2 = a_{21}q_1 + a_{22}q_2 + \dots\dots + a_{2n}q_n$

$l_m = a_{m1}q_1 + a_{m2}q_2 + \dots\dots + a_{mn}q_n$

The objective of player A is to select p_i $(i=1,2,\dots\dots m)$ such that he can maximize his mimimum expected gains; and the player B desires to select $q_j (j = 1,2,\dots\dots n)$ that will minimize his expected losses.

Thus if we let $u = \min\limits_{j} \sum_{i=1}^{m} a_{ij}\, p_i$ $(j=1,2,\dots n)$ and

$$\nu = \max_{i} \sum_{j=1}^{n} a_{ij}\, q_j \ (i=1,2\dots m)$$

the problem of two players could be written as :

Player A

maximize u = minimize $1/u = \sum_{i=1}^{m} pi/u$

subject to the constraints

$\sum_{i=1}^{m} a_{ij}p_i \geq u$ and $\sum p_i = 1$, $p_i \geq 0$ $(i=1,2,......m)$

Player B

Minimize ν = maximize $1/\nu = \sum_{j=1}^{n} q_j/\nu$

Subject to the constraints

$\sum_{j=1}^{n} a_{ij}q_j \leq \nu$ and $\sum q_j = 1$, $q_j \geq 0$ $(j=1,2,.....n)$

Assuming that $u>0$ and $\nu>0$, we introduce new variables defined by $p_i' = p_i/u$ and $q'_j = q_j/\nu$

$(i = 1,1,.......m; j = 1,2,....n)$

Then the pair of linear programming problems can be re-written as :

Player A minimize $p_0=p'_1+p_2'+......+p'_m$

subject to the constraints

$a_{1j}P'_1 + a_{2j}p'_2+.. ..+a_{mj}p'_m \geq 1$

$p'_i \geq 0$ $(i = 1,2,..m; j = 1,2,...n)$

Player B maximize $q_0 = q'_1+q'_2+....+q'_n$

subject to the constraints

$a_{i1}q'_1+a_{i2}q'_2+...+a_{in}q'_n \leq 1$

$q'_j \geq 0$ $(i=1,2,.....m \; ; j = 1,2,.......n)$

It is easy to note that the L.P.P. 's of two players represent a primal - dual pair. Therefore by fundamental theorem of duality one can read off the optimal solution of one player,just from the optimum simplex table of the opponent. That is, we need to solve just one player's LPP by simplex method.

Remarks :-

Linear programming technique requires all variables to be non-negative and therefore to obtain a non-negatvie value V of the game, the date to the problem, (ie) aij in the payoff table should all be non-negative. If these are some negative elements in the pay off table, a constant to every element in the payoff table must be added so as to make the smallest element zero; the solution to this new game will given an optimal mixed strategy for the original game. The value of the original game then equals the value of the new game minus the constant.

SAMPLE PROBLEM

1) Solve the following game is linear programming technique :

player B

$$\text{player A} \quad \begin{pmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{pmatrix}$$

Solution :-

Since some of the entries in the pay-off matrix are negative, we add a suitable constant to each of the entries to ensure them all positive. Thus, adding a costant $c = 4$ to each element, we get the following revised pay off matrix.

Player B

$$\text{player A} \quad \begin{pmatrix} 5 & 3 & 7 \\ 7 & 9 & 1 \\ 10 & 6 & 2 \end{pmatrix}$$

Let the strategies of the two players be

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{bmatrix} \quad \text{and} \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{bmatrix}$$

where $p_1+p_2+p_3=1$ and $q_1+q_2+q_3 = 1$.

The linear programming formulation for the two player's problems are :-

For player A

maximize ν = minimize $1/\nu = x_1 + x_2 + x_3$

Subject to the constraints :

$5x_1+7x_2+10x_3 \geq 1$

$3x_1+9x_2+6x_3 \geq 1$

$7x_1+x_2+2x_3 \geq 1$

and $x_j \geq 0$ $(j=1,2,3)$

For player B

minimize $\mu \equiv$ maximize $1/\mu = y_1+y_2+y_3$

suject to the constraints :

$5y_1+3y_2+7y_3 \leq 1,$

$7y_1+9y_2+y_3 \leq 1$

$10y_1+6y_2+2y_3 \leq 1$

and $y_j \geq 0$ $(j=1,2,3)$

where $x_j = p_j/u$ $(j = 1,2,3)$ and $y_j=q_j/\nu$ $(j=1,2,3)$

u = minimum expected gain to A and

ν = mimium expected loss to B.

Let us now solve the problem for player B. By introducing slack variables $s_1 \geq 0$, $s_2 \geq 0$ and $s_3 \geq 0$. The iterative simplex tables are;

Initial Itertion : Introduce y_3 and drop y_4

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	
0	y_4	1	5	3	7*	1	0	0	
0	y_5	1	7	9	1	0	1	0	
0	y_6	1	10	6	2	0	0	1	
	1/v	0	-1	-1	-1	0	0	0	z_i-c_j

First iteration : - Introduce y_2 and drop y_5

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	
1	y_3	1/7	5/7	3/7	1	1/7	0	0	
0	y_5	6/7	44/7	60/7*	0	-1/7	1	0	
0	y_6	5/8	60/8	36/7	0	-2/7	0	1	
	$1/\nu$	1/7	-2/7	-4/7	0	1/7	0	0	z_i-c_j

Final Iteration :optimum solution :-

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	
1	y_3	1/10	2/5	0	1	3/20	-1/20	0	
1	y_5	1/10	11/5	1	0	-1/60	7/60	0	
0	y_6	1/5	24/5	0	0	-1/5	-3/5	1	
	1/v	1/5	2/5	0	0	2/15	-1/15	0	z_i-c_j

The expected value or game, therefore, is obtained as ν^0 = v-4 = 5-4 =1.

Optimum strategies for player B are

$q^0_1 = 0$, $q^0_2 = 1/10 \times 5 = 1/2$ and $q^0_3 = 1/10 \times 5 = 1/2$

Making use of duality, the optimum strategies for player A are obtained as $p^0_1 = 2/15 \times 5 = 2/3$, $p^0_2 = 1/15 \times 5 = 1/3$ and $p^0_3 = 0$.

Hence the optimum solution to the given problem is

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 2/3 & 1/3 & 0 \end{bmatrix} \qquad S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

and $\nu = 1$.

2) Solve the game whose pay-off matrix is

$$\begin{pmatrix} 3 & 2 & 4 & 0 \\ 3 & 4 & 2 & 4 \\ 4 & 2 & 4 & 0 \\ 0 & 4 & 0 & 8 \end{pmatrix}$$

Solution :- Given pay - off matrix is

$$\begin{array}{cc} & \text{playerB} \\ \text{player A} & \begin{pmatrix} 3 & 2 & 4 & 0 \\ 3 & 4 & 2 & 4 \\ 4 & 2 & 4 & 0 \\ 0 & 4 & 0 & 8 \end{pmatrix} \end{array}$$

Let the strategies of the two players be

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ p_1 & p_2 & p_3 & p_4 \end{bmatrix} \text{ and}$$

$$S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ q_1 & q_2 & q_3 & q_4 \end{bmatrix}$$ Where $p_1+p_2+p_3+p_4 = 1$ and $q_1+q_2+q_3+q_4 = 1$

The linear programming formulation for the two player's problemss are :

For player A :

Maximize ν = Minimize $1/\nu = x_1+x_2+x_3+x_4$

suject to the constraints

$3x_1+3x_2+4x_3 \geq 1$

$2x_1+4x_2+2x_3+4x_4 \geq 1$

$4x_1+2x_2+4x_3 \geq 1$

$4x_2+8x_4 \geq 1$ and $x_j \geq 0$, $(j=1,2,3,4)$

For player B :

Minimize ν = maximize $1/\nu = y_1+y_2+y_3+y_4$

subject to the costraints

$3x_1+2x_2+4x_3 \leq 1$

$3x_1+4x_2+2x_3+4x_4 \leq 1$

$4x_1+2x_2+4x_3 \leq 1$

$4x_2+8x_4 \leq 1$ and $y_j \geq 0$, $(j=1,2,3,4)$

where $X_j = p_j/u$ $(j=1,2,3,4)$ and

$Y_j = q_j/v$ (j=1,2,3,4)

u= minimum expected gain to A and

v = minimum expected loss to B.

Let us now solve the problem for player B.

By introducing slack variables $S_1 \geq 0$, $S_2 \geq 0$ $S_3 \geq 0$ & $S_4 \geq 0$

The iterative simplex talbes are

Initial Iteration

			1	1	1	1	0	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
0	y_5	1	3	2	4	0	1	0	0	0
0	y_6	1	3	4	2	4	0	1	0	0
0	y_7	1	4	2	4	0	0	0	1	0
0	y_8	1	0	4	0	8*	0	0	0	1 →
$z_j - c_j$			-1	-1	-1	-1	0	0	0	0
						↑				

Introducue y_4 and drop y_8.

First Iteration

			1	1	1	1	0	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
0	y_5	1	3	2	4	0	1	0	0	0
0	y_6	1/2	3	2	2	0	0	1	0	-1/2
0	y_7	1	4	2	4*	0	0	0	1	0 →
1	y_4	1/8	0	1/2	0	1	0	0	0	1/8
$z_j - c_j$			-1	-1/2	-1	0	0	0	0	1/8
					↑					

Introducue y_3 and drop y_7.

Final Iteration

			1	1	1	1	0	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	
0	y_5	0	-1	0	0	0	1	0	-1	1	
0	y_6	0	1	1	0	0	0	1	-1/2	-1/2	
1	y_3	1/4	1	1/2	1	0	0	0	1/4	0	
1	y_4	1/8	0	1/2	0	1	0	0	0	1/8	
		3/8	0	0	0	0	0	0	1/4	1/8	z_j-c_j

Since all $z_j-c_j \geq 0$, the solution at this iteration is optimal.
The values of the variables y_1, y_2, y_3, y_4 of B's problem are $y_1=0$, $y_2=0$, $y_3=1/4$, $y_4=1/8$
and the maximum value of $1/\nu = 1\text{x}1/4 + 1\text{x}1/8$
$= 3/8$.

minimum value of $\nu = 8/3$

Therefore from the relations $y_j = y_j/\nu$, we get

$y_1 = 0\text{x}8/3 = 0$, $y_2=0\text{x}8/3 = 0$, $y_3=1/4\text{x}1/8 = 2/3$

$y_4 = 1/8\text{x}8/3= 1/3$.

Hence B's optimal strategy is [0,0,2/3, 1/3]

Now if A's optimal strategy is (x_1,x_2,x_3,x_4),

then values of x_1, x_2, x_3, x_4, where $x_i = x_i/\nu$, i = 1,2,3,4

can be read from the z_j-c_j row in the final iteration under the columns y_5, y_6, y_7 and y_8 respectively because A's problem is the dual of B's problem.

Thus $x_1 = 0$, $x_2 = 0$, $x_3 = 1/4$, $x_4 = 1/8$

From $X_i = x_i/v$,

$x_1 = 0 \text{ x } 8/3 = 0$, $x_2 = 0 \text{ x } 8/3 =0$

$x_3 = 1/4 \text{ x } 8/3 = 2/3$, $x_4 = 1/8 \text{ x } 8/3 = 1/3$

Hence A's optimal strategy is [0,0 2/3, 1/3]. ∴Hence the optimum solution to the given problems is

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & 2/3 & 1/3 \end{bmatrix}; \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & 2/3 & 1/3 \end{bmatrix}$$

and v = 8/3.

3) Two players A and B play the following game; A has a bag containing three coins, one worth 1 unit, one 3 units and the rest worth 6 units. A takes one coin from the bag

and before it is exposed, B guesses what it is? If he, B, is right he takes the coin, if he is wrong he gives to A a coin of the same worth.

(i) Is this a fair game?

(ii) What is the value of the game to A?

(iii) What are A's and B's optimal strategies?

Solution :-

The payoff to A may be represented by the following matrix.

$$\begin{array}{cc|ccc} & & & B & \\ & & 1 & 3 & 6 \\ \hline & 1 & -1 & 1 & 1 \\ A & 3 & 3 & -3 & 3 \\ & 6 & 6 & 6 & -6 \end{array}$$

Since the maximum for A is -1 and minimax for B is 3, which are not identical, the game has no saddle point and the value of the game lies between -1 and 3. Also there is no dominance. We shall solve this by L.P. Method.

To be sure that $\nu > 0$, we shall add a number greater than or equal to 1 to all the elements of the matrix. Adding 1 to each element of this game, we form n new game as follows.

$$\begin{array}{c|ccc} & 1 & 3 & 6 \\ \hline 1 & 0 & 2 & 2 \\ 3 & 4 & -2 & 4 \\ 6 & 7 & 7 & -5 \end{array}$$

Let B's strategy by $[y_1, y_2, y_3]$.Then B's problem is

Max $1/\nu = y_1 + y_2 + y_3$

s.t. $2y_2+2y_3 \leq 1$

$4y_1-2y_2+4y_3 \leq 1$

$7y_1+7y_2-5y_3 \leq 1$,

$y_1, y_2, y_3 \geq 0$,,

where $y_j = y_j/\nu$, j = 1,2,3.

Initial Iteration

			1	1	1	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	1	0	2	2	1	0	0
0	y_5	1	4	-2	4	0	1	0
0	y_6	1	7*	7	-5	0	0	1 →
z_j-c_j			-1	-1	-1	0	0	0
			↑					

Entering variable is y_1, leaving element is y_6. Pivot element 7.

First Iteration :

			1	1	1	0	0	0	c_j
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	
0	y_4	1	0	2	2	1	0	0	
0	y_5	3/7	0	-6	48/7*	0	1	-4/7 →	
1	y_1	1/7	1	1	-5/7	0	0	1/7	
			0	0	-12/7	0	0	1/7	z_j-c_j
					↑				

Entering y_3, leaving y_5. Pivot 48/7.

Second iteration :

			1	1	1	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	7/8	0	15/4*	0	1	-7/24	1/6 →
1	y_3	1/16	0	-7/8	1	0	7/48	-1/12
1	y_1	3/16	1	3/8	0	0	5/48	1/12
			0	-3/2	0	0	1/4	0
				↑				

Entering y_2, leaving y_4. Pivot 15/4.

Final iteration :

			1	1	1	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	
1	y_2	7/30	0	1	0	4/15	-7/90	-2/45	
1	y_3	4/15	0	0	1	7/30	7/90	-2/45	
1	y_1	1/10	1	0	0	-1/10	2/13	1/15	
			0	0	0	2/5	2/15	1/15	z_j-c_j

The solution at this iterative is optimal. Hence the values of y_1, y_2, y_3 involved in the B's problem are given by

$y_1 = 1/10, \quad y_2 = 7/30, \quad y_3 = 4/15$

Also the max. Value of $(1/\nu)$ is $(3/5)$

Therefore the value of the new game = $\nu = 5/3$

(new game means game after adding the fixed number 1).

Hence $y_1 = y_1 \nu = 5/3 \times 1/10 = 1/6$

$y_2 = y_2\nu = 5/3 \times 7/30 = 7/18,$

$y_3 = y_3\nu = 5/3 \times 4/15 = 9/4$

Hence the B's optimal strategy is [1/6, 7/18, 4/9]. (check that $y_1+y_2+y_3 = 1$).

Now if $[x_1, x_2, x_3]$ be the optimal strategy of A, then the values of x_1, x_2, x_3, where $x_i = \frac{x_i}{\nu}$, $i = 1,2,3$ can be read from the z_j-c_j row under the columns y_4, y_5, y_6 in the forht tableau; because A's problem is the dual of the B's problem. Thus

$x_1 = 2/5, x_2 = 2/15, x_3 = 1/15$

$x_1 = 5/3 \times 2/5 = 2/3, \quad x_2 = 2/15 \times 5/3 = 2/9, \quad x_3 = 1/15 \times 5/3 = 1/9$

Hence the A's optimal stategy is [2/3, 2/9, 1/9]

Also note that $x_1+x_2+x_3 = 1$.

Finally the value of the original game is $v = 5/3-1 = 2/3$

Since $v \neq 0$, the game is not fair.

Exercise :

1) Two companies A and B are competing for the same product. Their different strategies are given in the following pay off matrix :

$$\begin{array}{cc} & \text{Company A} \\ \text{Company B} & \begin{bmatrix} 2 & -2 & 3 \\ -3 & 5 & -1 \end{bmatrix} \end{array}$$

Use linear programming to determine the best strategies for both the players.

2) For the following pay-off table, transform the zero-sum game into an equivalent linear programming problem and solve it by simplex method :-

$$\begin{array}{cc|ccc|} & & \multicolumn{3}{c}{\text{Player Q}} \\ & & Q_1 & Q_2 & Q_3 \\ \text{player P} & p_1 & 9 & 1 & 4 \\ & p_2 & 0 & 6 & 3 \\ & p_3 & 5 & 2 & 8 \end{array}$$

3) Solve the following game by linear programming

$$\begin{array}{cc} & \text{player B} \\ \text{player A} & \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 2 \\ 3 & 3 & -3 \end{pmatrix} \end{array}$$

4) A and B play game in which each has three coins a 5p; 10p and a 20p Each selects a coin without the knowledge of other's choice. If the sum of the coins is an odd amount, A wins B's coin, if the sum is even, B wins A's coin. Find the best strategy for each player and the value of the game.

Anwsers :-

1) $S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 7/12 & 5/12 & 0 \end{bmatrix}$; $S_B = \begin{bmatrix} B_1 & B_2 \\ 2/3 & 1/3 \end{bmatrix}$; $\nu = 1/3$

2) $S_P = \begin{bmatrix} P_1 & P_2 & P_3 \\ 3/8 & 13/24 & 1/12 \end{bmatrix}$; $S_Q = \begin{bmatrix} Q_1 & Q_2 & Q_3 \\ 7/24 & 5/9 & 11/72 \end{bmatrix}$; $\nu = 91/24$

3) $S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 6/11 & 3/11 & 2/11 \end{bmatrix}$; $S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 5/22 & 4/11 & 9/22 \end{bmatrix}$; $\nu = 6/11$

4) $S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 1/2 & 1/2 & 0 \end{bmatrix}$; $S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 2/3 & 1/3 & 0 \end{bmatrix}$; $\nu = 0$

9.9 Dominance Principle :

Sometimes. it is observed that one of the pure strategies of either player is always inferior to atleast one of the remaining ones. The superior strategies are said to dominate the inferior ones. Clearly, a player would have no incentive to use inferior strategies which are dominated by the superior ones. In such cases of dominance, we can reduce the size of the payoff matrix by deleting those strategies which are dominated by the others. Thus if each element in one row, say kth of the pay off matrix (a_{ij}) is less than or equal to the corresponding elements in some other row, say rth, then player A will never choose kth strategy

In other words, probability p_k=P (choosing the kth strategy) is zero, if $a_{kj} \leq a_{rj}$ for all $j = 1,2,\ldots\ldots\ldots n$

The value of the game and the non-zero choice of probabilities remain unchanged even after the deletion of kth row from the payoff matrix. In such a case the kth strategy is said to be dominated by the rth one.

<u>General rules for dominance are :-</u>

(a) If all the elements of a row, say kth, are less than or equal to the corresponding elements of any other row, say rth, then kth row is dominated by the rth row.

(b) If all the elements of a column, say k^{th} are greater than or equal to the corresponding elements of any other column, say rth, then k^{th} column is dominated by the r^{th} column.

(c) Dominated rows or columns may be deleted to reduce the size of payoff matrix, as the optimal stratgies will remain unaffected.

The modified Dominance property.

The dominance property is not always based on the superiority of pure strategies only. A given strategy can also be said to be dominated if it is inferior to an average of two or more other pure strategies. More generally, if some convex linear combination of some rows dominates the i^{th} row, then i^{th} row will be deleted. Similar arguments follow for columns.

Sample problem :-

1) Is the following two-person, zero-sum game stable? (the pay off is for player A). Solve the game :

player B

$$\text{Player A} \quad \begin{bmatrix} 5 & -10 & 9 & 0 \\ 6 & 7 & 8 & 1 \\ 8 & 7 & 15 & 1 \\ 3 & 4 & -1 & 4 \end{bmatrix}$$

Solution :-

It is eaily verified that the game has no saddle point and hence is not stable, Let us consider player A's moves first. Clerarly all the elements in the third row are greater than or equal to the corresponding elements of first as well as second rows. Thus player A will never choose either of the first two moves regardless of player B's strategy. This indicates that the first two moves of A are dominated by his third strategy and can be deleted, thereby reducing the payoff matrix to the following.

Player B

$$\text{player A} \quad \begin{bmatrix} 8 & 7 & 15 & 1 \\ 3 & 4 & -1 & 4 \end{bmatrix}$$

In the modified payoff matrix, let us consider player B's strategies. We observe that all the elements of his second strategy are greater than or equal to the corresponding elements of his fourth strategy. Therefore, second strategy of player B can be deleted as it is dominated by his fourth. The pay-off matrix reduces to the following :-

Player B

$$\text{player A} \quad \begin{pmatrix} 8 & 15 & 1 \\ 3 & -1 & 4 \end{pmatrix}$$

Now, we observe that no row (or column) dominates another row (or column).

The 2x3 game could now be solved by graphical method. However, we note that a convex combination of B's second and third strategies dominates his first strategy, is

$-15 \times 1/2 + 1 \times 1/2 = 8 \leq 8$

$-1 \times 1/2 + 4 \times 1/2 = 1.5 \leq 3$

Therefore, B's first move can be deleted yielding the 2x2 payoff matrix.

$$\begin{array}{cc} & \text{player B} \\ \text{player A} & \begin{pmatrix} 15 & 1 \\ -1 & 4 \end{pmatrix} \end{array}$$

If we, now, let

$$S_A = \begin{bmatrix} A_3 & A_4 \\ p_1 & p_2 \end{bmatrix}, \; p_1+p_2 = 1; \qquad S_B = \begin{bmatrix} B_3 & B_4 \\ q_1 & q_2 \end{bmatrix}; \; q_1+q_2=1$$

then by using the solution method for 2x2 games, the optimum strategies can easily be obtained as

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & 5/19 & 14/19 \end{bmatrix}; \qquad S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & 3/19 & 16/19 \end{bmatrix}$$

and the value of game is $v = 61/19$

(2) Solve the game whose payoff matrix is

$$\begin{array}{ccc} & & \text{B} \\ & & \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \text{A} & \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} & \begin{bmatrix} -1 & -2 & 8 \\ 7 & 5 & -1 \\ 6 & 0 & 12 \end{bmatrix} \end{array}$$

<u>Solution :</u>

The game has no saddle point and hence is not stable. Let us consider player A's moves first.

By dominance rule (a) Every element of 3rd row is greater than the corresponding elements of the 1st row. Then we get the following reducing pay-off matrix.

$$\begin{array}{cc} & \begin{array}{ccc} \text{I} & \text{II} & \text{II} \end{array} \\ \begin{array}{c} \text{II} \\ \text{III} \end{array} & \begin{bmatrix} 7 & 5 & -1 \\ 6 & 0 & 12 \end{bmatrix} \end{array}$$

Again every element of the I column of the above matrix is greater the corresponding elements of the II column.

∴By dominance rule (b), from player B's point of view of the pure strategy I dominated by strategy II. Thus we get the following reduce matrix

$$\begin{array}{cc|cc|} & & \text{II} \quad B & \text{III} \\ \hline & \text{II} & 5 & -1 \\ A & \text{III} & 0 & 12 \\ \hline \end{array}$$

If we, now let

$S_A = \begin{bmatrix} A_2 & A_3 \\ p_1 & p_2 \end{bmatrix}$; $p_1+p_2=1$; $\qquad S_B = \begin{bmatrix} B_2 & B_3 \\ q_1 & q_2 \end{bmatrix}$, $q_1+q_2=1$

then by using solution method for 2x2 games, the optimum strategies can easily be obtained as

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & 12/18 & 6/18 \end{bmatrix}$$

$$S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 13/18 & 5/18 \end{bmatrix}$$

and the value of the game is $\nu = 60/18$.

(3) Solve the game whose pay off matrix is given by

$$\begin{array}{c|cccc|} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \hline \text{I} & 3 & 2 & 4 & 0 \\ \text{II} & 2 & 4 & 2 & 4 \\ \text{III} & 4 & 2 & 4 & 0 \\ \text{IV} & 0 & 4 & 0 & 8 \\ \hline \end{array}$$

Solution :-

There is no saddle point in these game,.

Hence we try to reduce the size of the pay off matrix by using dominance rules.

$$\begin{array}{c|cccc|} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \hline \text{I} & 3 & 2 & 4 & 0 \\ \text{II} & 2 & 4 & 2 & 4 \\ \text{III} & 4 & 2 & 4 & 0 \\ \text{IV} & 0 & 4 & 0 & 8 \\ \hline \end{array}$$

Every element of 3rd row is greater or equal to the corresponding elements of the first ow.

The first row is dominated by the third row from A's point of view.

Deleting first row, we get the following reduced matrix.

$$\begin{array}{c|cccc|} & I & II & III & IV \\ \hline I & 2 & 4 & 2 & 4 \\ III & 4 & 2 & 4 & 0 \\ IV & 0 & 4 & 0 & 8 \end{array}$$

Also from B's point of view first column of the avbove reduced payoff matrix is dominated y III column. Thus we get the reduced pay off matrix as follows.

$$\begin{array}{c|ccc|} & II & III & IV \\ \hline II & 4 & 2 & 4 \\ III & 2 & 4 & 0 \\ IV & 4 & 0 & 8 \end{array}$$

Now, we seek that the average of B's third and fourth strategy of the above matrix. We get;

$$a_{11} > \frac{a_{12}+a_{13}}{2} \Rightarrow 4 > \frac{2+4}{2} = 3$$

$$a_{21} = \frac{a_{22}+a_{23}}{2} \Rightarrow 2 = 1/2(4+0) = 2$$

$$a_{31} = \frac{a_{32}+a_{33}}{2} \Rightarrow 4 = 1/2\ (0+8) = 4$$

By rule (c) of dominance property the strategy II of player B may be dominated by player B will not use this second pure stratagy.

∴Deleting the second strategy of player B (The first column of the above matrix) We get the following reduced matrix.

$$\begin{array}{c|cc|} & III & IV \\ \hline II & 2 & 4 \\ III & 4 & 0 \\ IV & 0 & 8 \end{array}$$

Again we see that the average of A's third and fourth strategies of the above matrix

$$2 > \frac{4+0}{2} = 2 \Rightarrow 2 = 2$$

$$4 = \frac{8+0}{2} = 4 \Rightarrow 4 = 4$$

which is the same as A's second strategy.

Thus A's second strategy may be deleted as player A will gain the same amount even if he never use his second strategy. Deleting A's second strategy. We obtain the following 2x2 payoff matrix.

$$\begin{array}{ccc} & \multicolumn{2}{c}{\text{player B}} \\ & \text{III} & \text{IV} \\ \text{player A} \quad \text{III} & 4 & 0 \\ \text{IV} & 0 & 8 \end{array}$$

If we, now, let

$$S_A = \begin{bmatrix} \text{III} & \text{IV} \\ p_1 & p_2 \end{bmatrix} \qquad S_B = \begin{bmatrix} \text{III} & \text{IV} \\ q_1 & q_2 \end{bmatrix}$$

$$p_1+p_2=1, \qquad q_1+q_2=1$$

Then by using the solution method for 2x2 games, the optimum strategies can easily be obtained as

$$S_A = \begin{bmatrix} \text{I} & \text{II} & \text{III} & \text{IV} \\ 0 & 0 & 2/3 & 1/3 \end{bmatrix}$$

$$S_B = \begin{bmatrix} \text{I} & \text{II} & \text{III} & \text{IV} \\ 0 & 0 & 2/3 & 1/3 \end{bmatrix}$$

and the value of game is $v = 8/3$

Exercise :-

1) Use dominance property to reduce the following game to 2x2 game and hence find the optimal strategies and the value of the game.

$$\begin{array}{cc} & \text{player B} \\ \text{playerA} & \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix} \end{array}$$

2) Solve the following game after reducing it to 2x2 game

$$\begin{array}{cc} & \text{player B} \\ \text{player A} & \begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{bmatrix} \end{array}$$

3) Consider the game :

$$\begin{array}{cc} & \text{player A} \\ \text{player B} & \begin{bmatrix} 5 & 1 & 10 \\ 50 & 1 & 1 \\ 50 & 0.1 & 10 \end{bmatrix} \end{array}$$

obtain the optimal strategies for player A and B, and find the value of the game.

4) Solve the game whose pay off matrix is given below by graphical method.

$$\begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ A_1 & 4 & -2 & 3 & -1 \\ A_2 & -1 & 2 & 0 & 1 \end{array}$$

$A_3 \quad -2 \quad 1 \quad -2 \quad 0$

5) Use the notation of dominance to simplify the rectangular game with the following pay off, and then solve it graphically

$$\begin{array}{ll} & \text{player K} \\ \text{player L} & \begin{array}{c|cccc} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \hline 1 & 18 & 4 & 6 & 4 \\ 2 & 6 & 2 & 13 & 7 \\ 3 & 11 & 5 & 17 & 3 \\ 4 & 7 & 6 & 12 & 2 \end{array} \end{array}$$

Answers :

1) $$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & 0 & 1 \end{bmatrix} ; \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 2/5 & 3/5 & 0 \end{bmatrix} ; \quad \nu = 2$$

2) $$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 2/5 & 3/5 & 0 \end{bmatrix} ; \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 1/2 & 1/2 & 0 \end{bmatrix} ; \quad \nu = 4$$

3) $$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & 1 & 0 \end{bmatrix} ; \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 1 & 0 \end{bmatrix} , \quad \nu = 1$$

4) $$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 2/7 & 5/7 & 0 \end{bmatrix} ; \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 2/7 & 0 & 0 & 5/7 \end{bmatrix} ; \quad \nu = 3/7$$

5) $$S_L = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 4/9 & 0 & 5/9 \end{bmatrix} ; \quad S_K = \begin{bmatrix} \text{I} & \text{II} & \text{III} & \text{IV} \\ 0 & 5/9 & 0 & 4/9 \end{bmatrix} , \quad \nu = 38/9$$

Unit 10

Queueing Theory

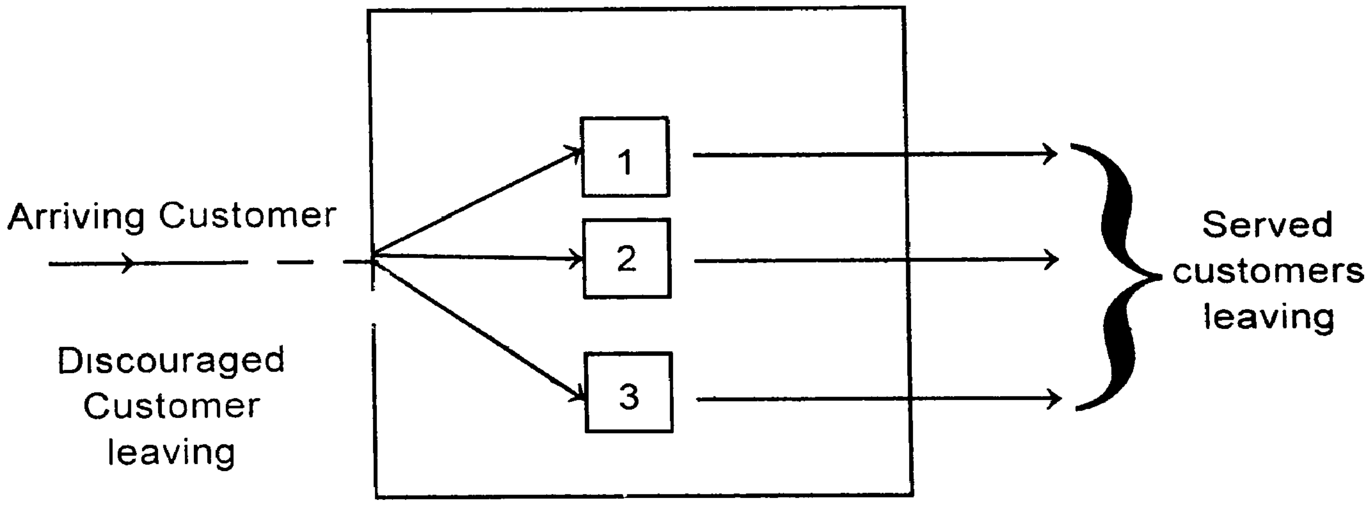
10.1 : Introduction

In everyday life there is a flow of customers to avail some service facility at some service station. The rate of flow depends on the nature of the service and the servicing capacity of the station. In many situations there is a congestion of items arriving for service, because an item cannot be seviced immediately on arrival and each new arrival has to wait for some time before it is attended This situation occurs where the tital number of customers requiring service exceeds the number of facilities. A group of customers / items waiting at some place to receive attention / service including those receiving the service, is known as queue.

The queues may be of persons waiting to be at a doctor's clinic or at railway booking office, these may be of machines waiting to be repaired or of ships in the harbour waiting to be unloaded or of letters arriving at a typist's desk. In the absence of a perfect balance between the service facilities and the customers, waiting is required either of the service facilities or for the customer's arrival.

By the term 'customer' we mean the arriving unit that requires some service to be performed. The customers may be of persons, machines, vehicles, parts etc. Queues (waiting line) stands for a number of customers waiting to be serviced. The queue does not include the customer being serviced. The process or system that performs the services to the customer to the termed by service channel or service facility

The machanism of a queueing process is very simple. Customers arrive at service counter and are attended by cne or more of the servers. As soon as a customer is served, he departs from the system. Thus a queueing system can be described as composed of customers arriving for service, waiting for service if it is not immediate, and if having waited for service, leaving the system after being served.



Queueing system

SYMBOLS AND NOTATIONS:

The following symbols and notation will be used in connection with the queuing system.

n = number of customers in the system both waiting and in service.

λ = average number of customers arriving per unit of time.

μ = average number of customers being served per unit of time.

$\lambda/\mu = p$ = traffic intensity.

C= number of paralled service channels (servers)

E(n) = average number of customers in the system, both waiting and in service.

E(m) = average number of customers waiting in the queue.

E(v) = average waiting time of a customer in the system, both waiting and in service

E(w) = average waiting time of a customer in the queue.

$p_n(t)$ = probability that there are n customers in the sytem at any time t, both waiting and in service.

P_n = Time independent probability that there are n customers in the system, both waiting and in service.

10.2 Types of Queue discipline :-

It is a rule according to which customers are selected for service when a queue has been formed. The most common discipline is the "first come, first served"(FCFS) or "First in first out" (FIFO) rule under which the customers are serviced in the strict order of their arrivals. Other queue disciplines include : "last in, first out" (LIFO) rule according to which the last arrival in the system is serviced first, "selection for service in random order"(SIRO) rule according to which the arrivals as serviced randomly irrespective of their arrivals in the system; and a variety of priority schemes according to which a customer's service is done in preference ove some the customer's service.

Under priority discipline, the service is of two types. In the first, which is called pre-emptive, the customers of high priority are given service over the low priority customer. In the second type, called the non-pre-emptive, a customer of low priority is serviced before a customer of high priority is entertained for service.

In the case of parallel channels "fastest server rule (FSR) is adopted. For its discussion we suppose that the customers arrive before parallel service channels. If only one service channel is free, then incoming customer is assigned to free service channel. But it will be more efficient to assume that an incoming customer is to be assigned a server of largest service rate among the free ones.

0.3 : Classification of Queues :

Generally queueing model may be completely specified in the following symbolic form,

(a/b/c) : (d/e)

he first and second symbols denotes the type of distributions of inter-arrival times and of nter-service times, respectively. Third symbol specifies the number of servers, whereas ourth symbol denotes the queue discipline.

f we specify the following letters as

M ≡ poisson arrival or departure distribution.

E_k ≡ Erlangian or Gamma inter - arrival for service time distribution.

GI ≡ General input distribution.

G ≡ General service time distribution.

Then ($M/E_k/1$) : (∝/FIFO) defines a queueing system in which arrivals follow poisson distribution, service times are Erlangian, single server, infinite capacity and "first in, first out" queue discipline.

Definition : of transient and steady states

A queueing system is said to be in "transient state when its operating characterstic (like input, output, mean, queue length etc) are dependent upon time.

If the characteristic of the queueing ststem becomes independent of time, then at steady - state condition is said to prevail.

If $P_n(t)$ denotes the probability that these are n customers in the system at time t, then in the steady state case,

We have $\lim_{t \to ∝} p_n(t) = P_n$ (independent of (t)) this implies that $\lim_{t \to ∝} \frac{d}{dt} p_n(t) = 0$

10.4 : The M/M/1 Queuing System :

This queueing system deals with the process in which arrivals and departures (services) occur randomly over time. Arrivals can be considered as births to the system, since if the system is in state E_n and an arrival occurs, the state is changed to E_{a+1}. On the otherhand, a departure occurring while the system is in state E_{th} sends the system down one to E_{n-1} and can be looked upon as a death. This type of process is generally referred to as a birth - death process.

Model 1 (M/M/1) : (∝/FIFO)

This model deals with a queueing system having single service channel, poisson input, Exponential services and there is no limit on the system capacity while the customers are served on a "first in, frst out" basis.

The solution procedure of this Model may be summarized in three steps.

Step 1: Obtain the system of differential - difference equations.

If P_n (t) be the probability that there are n customers in the system at time t, then in order to write the difference equation for $P_n(t)$ We first consider how the system can get to state E_n at time $t+\Delta t$. To be in state E_n at time $t+\Delta t$; the system could have been in state E_n at time t and have no arrivals or service completions in Δt, or be in state E_{n-1} at time t and have, during Δt, one arrival and no service completions, or finally, the system can be in state E_{n+1} at time t and have, during Δt, one service completion and no arrivals. By assuming, for the tme being $n \geq 1$ and since arrivals and service are both independent of each other, We can easily see that

$P_n(t+\Delta t) = P_n(t)$. P[no arrival in Δt]

. P [no service completion in Δt]

$+P_{n-1}$ (t) P[one arrival in Δt]. P completion service [inΔt] $+P_{n+1}(t)$ P[one service completion in Δt].P[no arrival in Δt] $+P_{n-1}(t)$P [one arrival in Δt]. P[no service in Δt] $+0(\Delta t)$; $n \geq 1$

which may be re written as

$$\begin{aligned}P_n(t+\Delta t) &= P_n(t)\,[1-\lambda(\Delta t)+0(\Delta t)]\,[1-\mu\Delta t+0(\Delta t)]\\ &+P_n(t)\,[\lambda\Delta t+0(\Delta t)]\,[\mu\Delta t+0(\Delta t)]\\ &+P_{n+1}(t)\,[(\mu\Delta t)+0(\Delta t)]\,[1-\lambda\Delta t+0(\Delta t)]\\ &+P_{n-1}(t)\,[\lambda\Delta t+0(\Delta t)]\,[1-\mu\Delta t+0(\Delta t)]+0(\Delta t) \quad \ldots\ldots\ldots (1)\end{aligned}$$

combining all $o(\Delta t)$ terms and neglecting terms with $o(\Delta t)^2$ the difference equation becomes

$$\begin{aligned}P_n(t+\Delta t) &= P_n(t)\,[1-\lambda\Delta t-\mu\Delta t]+P_{n+1}(t)\,[\mu\Delta t]\\ &+P_{n-1}(t)\,[\lambda\Delta t]+0(\Delta t)\ ;\ n\geq 1.\end{aligned}$$

Now since $P_{n-1}(t)$ does not exist for n=0, this equation is invalid for n=0 and therefore this state be considered separately. The system can be in state E_0 at time $t+\Delta t$ if were in E_0 at t and no arrival during Δt, or the system can be in E_1 at time t and have no arrivals but one service completion in Δt, thus,

$$\begin{aligned}P_0(t+\Delta t) = {} & P_0(t)\,[1-\lambda\Delta t+0(\Delta t)]\\ & +P_1(t)\,[1-\lambda\Delta t+0(\Delta t)]\,[\mu\Delta t]\\ & [\mu\Delta t+0(\Delta t)]+0\,(\Delta t)\\ = {} & P_0(t)\,[1-\lambda\Delta t]+P_1(t)\,\mu\Delta t+0(\Delta t)\ldots\ \ldots.\ (2)\end{aligned}$$

The difference equations given in (1) and (2) may be rewritten as follows.

$P_n(t+\Delta t) - P_n(t) = -(\lambda+\mu)\,\Delta t\,P_n(t) + \mu\Delta P_{n+1}(t) + \lambda\Delta t P_{n-1}(t)+0(\Delta t)\ n\geq 1$

and $P_0(t+\Delta t)-P_0(t) = -\lambda\Delta t P_0(t)+\mu\Delta t P_1(t)+0(\Delta t)$

If we now divide throughout by Δt and then take the limit at $\Delta t \to 0$, thse equations reduce to

$$\frac{d}{dt}P_n(t) = -(\lambda+\mu)\,P_n(t)+\mu P_{n+1}(t) + \lambda P_{n-1}(t);\ n\geq 1$$

and $$\frac{d}{dt}\,P_0(t) = -P_0(t) + \mu P_1(t)$$

These equations are called the differential difference equations.

Step 2 · obtain system of steady - state equations

To get the steady - state solution for P_n, the probability of n customers in the system at an anbitrary point of tme after steady state is reached, we take the limit as $t \to \propto$ in (3).

Now if the steady - state solution exists ($\lambda < \mu$ when $t \to \propto$), then $p_n(t) \to p_n$ and $\frac{d}{dt} P_n(t) \to 0$ as $t \to \propto$. If $\lambda = \mu$ then there is no queue and if $\frac{\lambda}{\mu} > 1$, then the state in called the explosive state.

Throughout this chapter we shall focus our attention on steady state conditions.

Using the conditions of the steady-state solution equ (3)reduces to

$0 = -(\lambda+\mu) P_n + \mu P_{n+1} + \lambda P_{n-1}$; $n \geq 1$

and $0 = -\lambda p_0 + \mu p_1$ .... (4)

Step 3 :- <u>Solve the system of difference equations.</u> For the solution of difference equations (4) there exists three methods, namely, the iterative method, use of generating functions and the use of linear operators. Out of these three the first one is the most straightforward and, therefore, the solutions of the above equations will be presented here by using this iterative method.

Using iteratively, the difference - equation (4), yield

$$P_1 = \frac{\lambda}{\mu} P_0$$

$$P_2 = \left(\frac{\lambda+\mu}{\mu}\right) P_1 - \frac{\lambda}{\mu} P_0 = \left(\frac{\lambda}{\mu}\right)^2 P_0.$$

$$P_3 = \left(\frac{\lambda+\mu}{\mu}\right) P_2 - \frac{\lambda}{\mu} P_1 = \left(\frac{\lambda}{\mu}\right)^3 P_0$$

and in general

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 \qquad \text{.............. (5)}$$

To prove that this general formula for P_n is true for all values of n, we make use of mathematical induction. Let equ (5) hold for n =n,n-1 and shall show that it also holds for n=n+1

Now

$$P_{n+1} = \frac{\lambda+\mu}{\mu} P_n - \frac{\lambda}{\mu} P_{n-1} \quad ; n \geq 1 \quad \text{................ ..... (6)}$$

using the value of P_n and P_{n-1} equation (6) reduces to

$$p_{n+1} = \left(\frac{\lambda+\mu}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{n} P_0 - \frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}\right)^{n-1} p_0$$

$$= \left(\frac{\lambda^{n+1}+\mu\lambda^{n}}{\mu^{n+1}} - \frac{\lambda^{n}}{\mu^{n}} \right) P_0$$

$$= \left(\frac{\lambda}{\mu}\right)^{n+1} P_0$$

This show that using the principle of mathematical Induction, the general formula for p_n equ(6) is valid for $n \geq 0$

To obtain the value of P_0 we make use of the boundary condition $\sum_{n=0}^{\propto} P_n = 1$
Using equ (5) this condition yields

$$= \sum_{n=0}^{\propto} \left(\frac{\lambda}{\mu}\right)^{n} P_0 = P_0 \sum_{n=0}^{\propto} \left(\frac{\lambda}{\mu}\right)^{n}$$

$$= P_0\left(\frac{1}{1-\lambda/\mu}\right) \quad \text{since } \frac{\lambda}{\mu} < 1$$

this gives $p_0 = 1 - \dfrac{\lambda}{\mu}$ ... .(7)

Hence using equs (5) & (7) the steady state solution is

$$P_n = \left(\frac{\lambda}{\mu}\right)^{n} \left(1 - \frac{\lambda}{\mu}\right)$$

$$= P^{n}(1-p) \; ; \; P = \frac{\lambda}{\mu} < 1 \text{ and } n \geq 0.$$

This expression gives us the probability distribution of queue length.

characteristics of Model I :

(i) probability of queue size being greater than or equal to n (the number of customers) in given by

$$P(\geq n) = \sum_{k=n}^{\propto} P_k = \sum_{k=n}^{\propto} (1-p)\,p^{k} = (1-p)p^{n} \sum_{k=n}^{\propto} p^{k-n}$$

$$= (1-p)p^{n} \sum_{k-n=0}^{\propto} p^{k-n} = \frac{(1-p)p^{n}}{1-p} = p^{n}$$

(ii) Average number of customers in the system in given by

$$E(n) = \sum_{n=0}^{\propto} nP_n = \sum_{n=0}^{\propto} n(1-p)p^{n}$$

$$= (1-p) \sum_{n=0}^{\propto} np^n = p(1-p) \sum_{n=0}^{\propto} n^{n-1}$$

$$= p(1-p) \sum_{n=0}^{\propto} \frac{d}{dp} p^n = p(1-p) \frac{d}{dp} \sum_{n=0}^{\propto} p^n \text{ ; since } p<1$$

$$= p(1-p) \left(\frac{1}{(1-p)^2}\right) = \frac{p}{1-p} = \frac{\lambda}{\mu-\lambda}$$

(iii) Average queue length in given by

$$E(m) = \sum_{m=0}^{\propto} mP_n$$

Where m = n-1 being the number customers in the queue, excluding the customers which in serviced. Therefore

$$E(m) = \sum_{n=1}^{\propto} (n-1)P_n = \sum_{n=1}^{\propto} nP_n - \sum_{n=1}^{\propto} P_n$$

$$= \sum_{n=1}^{\propto} nP_n - \left[\sum_{n=1}^{\propto} P_n - P_0 \right]$$

$$= \frac{p}{1-p} - [1-(1-p)] = \frac{p}{1-p} - p$$

$$= \frac{p^2}{1-p} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

iv) Average length of non-empty queue is given by.

$$E(m/m>0) = \frac{E(m)}{P(m>0)} = \frac{\lambda^2}{\mu(\mu-\lambda)} \frac{1}{(\lambda/\mu)^2} = \frac{\lambda}{\mu-\lambda}$$

$$\text{Since } P(m>0) = P(n>1) = \left[\sum_{n=0}^{\propto} P_n - P_0 - P_1 \right] = \left(\frac{\lambda}{\mu}\right)^2$$

(v) The fluctuation (variance) of queue length in given by

$$V(n) = \sum_{n=0}^{\propto} [n-E(n)]^2 P_n = \sum_{n=0}^{\propto} n^2 P_n - [E(n)]^2$$

Using some algebraic transformations and the value of P_n, the result reduces to

$$V(n) = (1-p)\frac{p+p^2}{(1-p)^3} - \left(\frac{p}{1-p}\right)^2 = \frac{p}{(1-p)^2} = \frac{\lambda\mu}{(\mu-\lambda)^2}$$

Sample problems

(1) A T.V. repairman finds that the time spent on his jobs has an exponential distribution with mean 30 minutes. If he repairs sets in the order in which they came in, and if the arrival of sets is approximately poisson with an average rate of 10 per 8 hour day, what is repairman's expected idle time each day? How many jobs are ahead of the average set just brought in?

Solution : We are given,

λ = 10 sets per day, and μ = 16 sets per day.

p = λ/μ = 10/6 = 5/8 = 0.625

The probability for the repairman to be idle is

P_0 = 1-p = 1-0.625 = 0.375

Expected idle time per day = 8x 0.375 = 3 hours.

and E(n) = p/1-p = 0.625/1-0 625 = 5/3 jobs.

(2) Arrivals at a telephone booth are considered to be following poisson law of distribution with an average time of 10 minutes between one arrival and the next. Length of a phone call is assumed to be distributed exponentially with mean 3 minutes.

(i) What is the probability that a person arriving at the booth will have to wait?

(ii) What is the average length of queue that from time to time?

(iii) The telephone department will install a second booth when convinced that an arrival would expect to wait at least three minutes for the phone. By how much must the flow arrivals be increased in order to justify a second booth?

Solution :-

Since the time interval between two successive arrivals is given to be 10 minutes the mean arrival rate is (1/10)

(ie) λ = 1/10 per minute.

Similarly μ = 1/3 = 0.33 units per minute.

(i) Now the probability that an arrival does not wait on arriving P_0 = p

Probability that an arrival has to wait

= 1-prob [an arrival does not wait on arriving]

= 1-P_0

= 1-(1-p) = p = λ/μ =0.10/0.33 = 0.3

(ii) Average length of non-empty queue

= 1/1-p = 1/1-0.3 = 1.43 persons.

(iii) The installation of second booth will be justified if the waiting time is greater than or equal to three. If the new arrival rate is λ' , then for μ = 0.33

putting $\lambda^1/\mu = p$, we get

$$E(w) = \frac{\lambda^1}{\mu(\mu-\lambda^1)} \geq 3$$

$$\lambda^1 \geq 3\mu^2 - 3\lambda\mu\mu^1$$

$$\lambda^1 \geq \frac{3\mu^2}{1+3\mu}$$

$\lambda^1 \geq 0.16$ persons per minute.

The arrival rate should be atleast 0.16 persons per minute or say one arrival in every six minutes (10 arrivals per hour) to justify the second booth.

3) In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter-arrival time follows an exponential distribution and the service time distribution is also exponential with an average 36 minutes.

Calculate the following :-

(i) the mean queue size (line length),

(ii) the probability that the queue size exceeds 10.

If the input of trains increases to an average 33 per day. What will be the change in (i), & (ii) ?

Solution :

Here we have

$$\lambda = \frac{30}{60\text{x}24} = 1/48 \quad \& \quad \mu = 1/36 \text{ trains per minute.}$$

$\therefore\ p = 1/\mu = 36/48 = 0.75$

(i) $E(m) = \frac{p}{1-p} = \frac{0.75}{1-0.75} = 3$ trains

and (ii) $p(\geq 10) = p^{10} = (0.75)^{10} = 0.06$

When the input increases to 33 trains per day.

We have $\lambda = \frac{33}{60\text{x}24} = \frac{11}{480}$ and $\mu = 1/36$ trains per minute.

$\therefore p = \lambda/\mu = 11/480 \times 36 = 0.83$

Then, we get

(i) $E(n) = \frac{p}{1-p} = \frac{0.83}{1-0.83} = 4.8$ (or) 5 trains

(ii) $p(\geq 10) = p^{10} - (0.83)^{10} = 0\ 2$ (approx)

Exercise :-

1) Let on an average 96 patients per 24 hour day require the service of an emergency clinic. Also on an average, a patient requires 10 minutes of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it costs the clinic Rs 100 per patient treated to obtain an average servicing time of 10 patients, and that each minute of decrease in this average time would cost Rs 10 per patient treated. How much would have to be budgeted by the clinic to decrease the average size of the queue from one and one third patients to half a patient.

2) The mean arrival rate to a service centre is 3 per hour. The mean service time is found to be 10 minutes per service. Assuming poisson arrival and exponential service time, find

(a) Utilisation factor for this service facility,

(b) Probability of two units in the system,

(c) Expected number of units in the queue,

(d) Expected time in minutes that a customer has to spend in the system.

Answers :-

1) Average rate of treatment is 7.5 minutes, cost is Rs 125 per patient.

2) (a) $\frac{1}{2}$ b) $\frac{1}{8}$ c) 1 d) $\frac{1}{3}$

Waiting time distribution for Model I

Let w be the waiting time of an arrival in the system before it enters service chair, its value will depend on the number of units already wiating in the system and the time taken in service by each unit.

Let $P_n(w)$ denote the probability that the waiting time of a particular unit lies between w and w+dw when there are already n units in the system (n-1) units waiting and 1 unit getting served) The waiting time distribution can be considered in two ways.

(a) As a discrete distribution when n=0, (ie) system is empty and the new arrival is immediately taken into service. Waiting time in this case will be zero.

$P(w=0) = P(\text{no customer in the system upon arrival})$

$= P_0 = 1-\rho$

(b) As a continuous distribution, when the queue length is greater than zero.

In FIFO system any new arrival can be taken into service only when all the units waiting in the system are serviced. If these ar already n units in the system, then a new arrival has to wait for a time between w and w+dw, when (n-1) units are serviced in time w with one unit in service at time w being completely served in time dw.

$P_n(w)\ dw$ = P(one unit is serviced in time dw) x P(n-1 units are serviced in time w)

$$= \mu dw \times \frac{(\mu\ w)^{n-1}\ e^{-\mu w}}{(n-1)!}$$

The probability distribution of waiting time is therefore given by

$$P(w)\ dw \quad = \quad \sum_{n=1}^{\propto} P_n(w)dw \times P[\text{n units in the system}]$$

$$= \quad \sum_{n=1}^{\propto} \mu\ dw \quad \times \quad \frac{(\mu\ w)^{n-1}\ e^{-\mu w}}{(n-1)!} \times P_n$$

$$= \mu\rho(1-P)\ e^{-\mu w} \sum_{n=1}^{\propto} \frac{(\mu\rho w)^{n-1}}{(n-1)!} dw,$$

$$[\text{since } P_n = (1-p)\ (p^{n-1}]$$

$$= \mu p(1-p)\ e^{\mu w}\ e^{\mu p w}\ dw.$$

$$= \mu p(1-p) e^{\mu w}\ (1-p)\ dw$$

$$\text{Hence, } P(w) = \begin{cases} 1-p, & \text{for } w = 0 \\ \mu p(1-p)\ e^{\mu w}\ (1-p), & \text{for } w>0 \end{cases}$$

Remark : For the busy period distribution, let the random variable w denote the total time (waiting & service) that unit has to spend in the system. Then the probability density function for the busy period is given by

$$P(w/w>0) = \frac{P(w)}{P(w>0)}$$

$$= \frac{\mu p(1-p)\ e^{-\mu w(1-p)}}{p} \qquad \text{since } P(w>0) = \int_0^{\propto} P(w)\ dw$$

$$= \mu\rho(1-\rho) \int_0^{\propto} e^{-\mu w(1-\rho)} dw$$

$$= -p\left[e^{-\mu w(1-\rho)}\right]_0^{\propto} dw = p$$

$$\text{Hence, } \quad P(w/w>0) = \mu(1-p)\ e^{-\mu w(1-p)}$$

$$= (\mu-\lambda)\ e^{-(\mu-\lambda)w}$$

clearly,

$$\int_0^{\propto} P(w/w>0)dw = \int_0^{\propto} \mu(1-P)e^{-\mu\omega(1-P)}dw\ \frac{\mu(1-\rho)}{\mu(1-\rho)} = 1.$$

Characteristics of waiting time Distribution for Model I

(i) Average waiting time of an arrival (in the queue) is given by

$$E(w) = \int_0^{\propto} w.\ p(w)dw = \int_0^{\propto} w.\mu p(1-p)\ e^{-\mu(1-p)w}\ dw$$

$$= p\int_0^{\propto} \frac{xe^{-x}}{\mu(1-\mathbf{P})}\ dx, \quad \text{for } \mu(1-\mathbf{P})\ w = x$$

$$= \frac{p}{\mu(1-p)}\ \Gamma(2) = \frac{p}{\mu(1-p)}, \qquad \text{since } \Gamma(2) = \Gamma(1) = 1$$

(iii) Average waiting time of an arrival who has to wait is given by

$$E(\omega/\omega/0) = \frac{E(w)}{P(w>0)} = \frac{1}{\mu(1-p)} \qquad \text{or} \qquad \frac{1}{\mu-\lambda}$$

(iv) Average waiting time that on arrival spends in the system is given by

$$E(v) = \int_0^{\propto} w.P\ (w/w>0)\ dw = \int_0^{\propto} w.\mu\ (1-p)\ e^{-\mu(1-p)w} dw$$

$$= \frac{1}{\mu(1-p)} \int_0^{\propto} xe^{-x}\ dx\ , \quad \text{for } \mu(1-p)\ w = x$$

$$= \frac{1}{\mu(1-p)} \text{ or } \frac{1}{\mu-\lambda}$$

sample problems

1. Arrivals at a telephone booth are considered to be poisson, with an average time of 10 minutes between one arrival and the next. The lenth of a phone call is assumed to be distributed exponential, with mean 3 minutes

(a) what is the probability that a person arriving at the booth will have to wait?

(b) The telephone department will install a second booth when convinced that an arrival would expect waiting for atleast 3 minutes for phone. By how much should the flow of arrivals increase in order to justify a second booth.

(c) Find the average number of units in the system?

(d) Estimate the fraction of a day that the phone will be in use

(e) What is the probability that it will take him more than 10 minutes altogether to wait for phone and complete his call?

Solu :

Here $\lambda = 1/10 = 0.10$

$\mu = 1/3 = 0.33$ persons per minutes

(a) $P(w>0) = 1-p_0 = 1-(1-\lambda/\mu) = \lambda/\mu$

$$= \frac{0.10}{0.33} = 0.3$$

(b) The installtion of second booth will be justified if the arrival rate is greater than the waiting time. Then the length of queue will go no increasing.

Now, $E(w) = \dfrac{\lambda'}{\mu(\mu-\lambda)}$

$$3 = \frac{\lambda^1}{0.33(0.33-\lambda)}$$

Where $E(w) = 3$ & $\lambda = \lambda^1$(say) for second booth.

On simplification this yields $\lambda^1 = 0.16$

Hence the arrival rate should become 0.16 person per minute to justify the second booth.

(c) Average number of units in the system is given by

$$E(n) = \frac{p}{1-p} = \frac{0.3}{1-0.3} = 0.43 \text{ customers}$$

(d) The fraction of a day that the phone will be busy = traffic intensity p

$= \lambda/\mu = 0.3$

(e) $p(w \geq 10) = \displaystyle\int_0^{\propto} (\mu-\lambda)\, e^{-(\mu-\lambda)w}\, dw$

$= e^{-10(\mu-\lambda)} = e^{-2.3}$

$= 0.10$

2. In the production shop of a company the breakdown of the macines is found to be poisson with an average rate of 3 machines perhour. Braekdown time at one machine costs Rs. 40 per hour to the company. There are two choices before the company for hiring the repairman. One of the repairmen is slow but cheap, the other fast but expensive. The slow-cheap repairman demands Rs.20 per hour and will repair the broken down machines exponentially at the rate of 4 per hour. The fast expensive repairman demands Rs. 30 per hour and will repair machines exponentially at an average rate of 6 per/hr. Which repairman should be hired?

Sol : In this problem, we compare the total expected daily cost for both the repairmen. This would equal the total wages paid plus the down - time cost.

case 1 : Slow - cheap repairman

λ = 3 machines per /hr

μ = 4 machines per/hr

∴Average down. time of a machine $= \frac{1}{\mu - \lambda}$

$= \frac{1}{4-3} = 1$hour

∴The down-time of 3 machines that arrive in an hour 1x3 = 3 hours.

Down-time cost = Rs. 40x3 = Rs. 120 charges paid to the repairman

= Rs. 20x3 = Rs. 60

Total cost = Rs. 120 + Rs. 60 = Rs. 180

Case 2 Fast - expensive repairmen

λ= 3 machines per hour and

μ = 6 machines per hour

Average downtime of machine $= \frac{1}{\mu - \lambda} =$ 1/3 hrs.

The down time of 3 machines that arrive in a hour = 1/3 x 2 hour = 1 hour

Down-time cost = Rs. 40x1 = Rs. 40 charges paid to the repairman

= Rs. 30 x1 = Rs. 30

Total cost = Rs. 40+Rs.30 = Rs. 70.

From the above cases, the decision of the company should be to engage the fast expensive repairman

Exercise Problems.

1. People arrive at a theatre ticket booth in a poisson distributed arrival rate of 25 per hour. Service time is constant at 2 minutes .

Calculate

(i) The mean number in the waiting line

(ii) The mean waiting time

2. Customers arrive at an office window, being manned by single individual according to the poisson input process with a mean rate of 30 per hour. The time required to serve a customer has an exponential distribution with as mean of 90 sec. Find the average waiting time of a customer?

3) At what average rate must a clerk at a super market work in order to ensure a probability of 0.90 that the customers will not have to wait longer than 12 mins? It in assumed that there is only one counter, to which customers arrive in a poisson fashion at an average rate of 15 per hour. The length of service by the clerk has an exponential distribution.

<u>Answers :-</u>

1. (i) 5 peopele (ii) 10 minutes (iii) 0.83
2. 4.5 minutes per customer
3. $p(\geq 2) = 0.10$; 2.48 minutes per service.

Model II (M/M/I) : (N/FIFO)

This model differs from that of model I in the sense that the maximum number of customer in the system is limited to N. Therefore, the difference equations of model I are valid for this model as long as $n<N$.

Now, if the system is in state E_N the probability of an arrival in to the system is zero. The additional difference equation for $n = N$ then is

$$P_N(t+\Delta t) = P_N(t)\,[1-\mu\Delta t] + P_{N-1}(t)\,.\,[\lambda\Delta t]\ \ [1-\mu\Delta t] + 0(\Delta t)$$

This gives, after simplification, the differential difference equation.

$$\frac{d}{dt}P_N(t) = -\mu P_N(t) + \lambda P_{N-1}\,(t)$$

from which the resultant steady - state difference equation is

$$0 = -\mu P_N + \lambda P_{N-1}$$

The complete set of stedy - state difference equations for this model, therefore can be written as

$$\mu P_1 = \lambda P_0$$

$$\mu P_{n+1} = (\lambda+\mu)\,P_n - \lambda P_{n-1} \qquad 1\le n\le N-1$$

$$\mu P_N = \lambda P_{N-1}$$

and

Using the iterative procedure, the first two difference equation give

$$P_n = (\lambda/\mu)^n\,P_0 \qquad n \le N-1$$

Also, we see that for this value of Pn, the third difference equation holds for n = N

We have

$$P_n = (\lambda/\mu)^n\ \ p_0 = p^nP_0$$

For obtaining the value of p_0, we make use of the boundary conditions $\sum\limits_{n=0}^{N} P_n = 1$.

$$\therefore 1 = P_0\sum_{n=0}^{N} P^n = \begin{cases} p_0\left(\dfrac{1-p^{N+1}}{1-p}\right) & (p \ne 1)\\[2ex] P_0\,(N+1) & (p = 1)\end{cases}$$

$$\text{Thus}\quad P_0 = \begin{cases} \dfrac{1-p}{1-p^{N+1}} & (p \ne 1)\\[2ex] \dfrac{1}{N+1} & (p = 1)\end{cases}$$

Hence

$$P_n = \begin{cases} \dfrac{(1-p)\,p^n}{1-p^{N+1}}, & p \neq 1 \quad ; 0 \leq n \leq N \\ \dfrac{1}{N+1} & p = 1 \end{cases}$$

Characteristic of Model II

(i) Average number of customers in the system is given by

$$E(n) = \sum_{n=0}^{N} nP_n = P_0 \sum_{n=0}^{N} np^n = P_0 p \sum_{n=0}^{N} \frac{d}{dp}\, p_n$$

$$\text{(or) } E(n) = P_0\, p\, \frac{d}{dp}\sum_{n=0}^{N} p^n\, P_0\, p \frac{d}{dp}\left(\frac{1-p^{N+1}}{1-p}\right)$$

$$= P_0\, p \, \frac{[1-(N+1)\, p^N + N\, p^{N+1}]}{(-p^2)}$$

$$= P_0 p \, \frac{[1-(N+1)\, p^N + N\, p^{N+1}]}{(1-p)\,(1-p^{N+1})}$$

(iii) Average queue lenght is given by

$$E(m) = \sum_{n=1}^{N} (n-1)\, P_n = E(n) - \sum_{n=1}^{N} P_n = E(n) - (1-p^N)$$

$$= E(n) - \frac{p(1-p^N)}{1-p^{N+1}}$$

$$= p^2 \, \frac{[1-N\, p^{N+1} + (N-1)\, p^N]}{(1-p)\,(1-p^{N+1})}$$

(iii) The average waiting time in the system can be obtained by using little's formula, that is, $E(v) = \{E(n)\}/\lambda^1$, where λ^1 is the mean rate of customers entering the system and is equal to $\lambda(1-P_N)$

The average waiting time in the queue can be obatined by using the relation

$E(w) = E(v) - 1/\mu$ (or) $E(w) = \{E(m)\}/\lambda^1$

Sample problem

1) At a railway station, only one train is handled at a time. The railway yard is sufficient only for two trains to wait while other is given signal to leave the station. Trains arrive at the station at on average late of 6 per/hr and the railway station can handle them on an average of 12 per hour. Assuming poisson arrivals and exponential service distribution,

find the steady - state probabilities for the various number of trains in the stystem. Also find the average waiting time of a new train coming into the yard.

Sol :

Here $\lambda = 6$, $\mu = 12$ so that $p = 6/12 = 0.5$ the maximum queue length is 2, (ie).The maximum number of trains in the system is 3(=N)

The probability that there is no train in the system (both waiting and in service) is given by

$$P0 = \frac{1-p}{1-p^{N+1}} = \frac{1-0.5}{1-(0.5)^{3+1}} = 0.53$$

Now, since $P_n = p_0 p^n$ therefore

$P_1 = (0.53)(0.5) = 0.27$

$P_2 = (0.53)(0.5)^2 = 0.13$

$P_3 = (0.5)(0.5)^3 = 0.07$

and hence, we get

$E(n) = 1(0.27) +(2(0.13) +3(0.07)$

$= 0.74$

Thus the average number of trains in the system is 0.74 and each train takes on an average 1/12(=0.085) hours for getting service. As the arrival of new train expects to find an average of 0.74 trains in the system before it.

$E(w) = (0.74)(0.85)$ hours = 0.0629 hours

or 3.8 minutes.

2) Assume that the goods trains are coming in the yard at the rate of 30 trains perday and suppose that the inter-arrival times follow an exponential distribution. The service time for each train is assumed to be exponential with an average of 36 minutes. If the yard can admit 9 trains at a time, there being 10 lines, one of which is reserved for shunting purposes. Calculate the probability that the yard is empty and find the average queue length.

sol :

$$\lambda = \frac{30}{60 \times 24} = 1/48$$

$\therefore P = \mu = 1/16$ trains per minute

$\therefore P = \lambda/\mu = 36/48 = 0.75$

The probability that the yard is empty is given by

$$P_0 = \frac{1-p}{1-p^{N+1}} = \frac{1-0.75}{1-(0.75)^{10}} \text{ , since } N = 9$$

$$= \frac{0.25}{0.90} = 0.28$$

Average queque length is given by

$$E(m) = \frac{p^2[1-N\,p^{N-1} + (N-1)\,p^N}{(1-p)\,(1-p^{N+1})}$$

$$= \frac{(0.75)^2[1-9(0.75)^8 + 8(0.75)^9]}{0.25\,[(0.75)^{10}]}$$

$$= 2.22\,\frac{(1-0.303)}{(1-0.005)}$$

$$= (2.22)\,(0.70)$$

$$= 1.55$$

Exercise problems

1) If for a period of 2 hours in a day (8 to 10a.m) trains arrive at the yard every 20 mins but the service time continues to remain 36 minutes, then calculate for this period

(a) The Probability that the yard is empty

(b) Average number of trains in the system, on the assumption that the line capacity of the yard is limited to 4 trains only

2) Patients arrive at a clinic according to a poisson distribution at a rate of 30 patients per/hr. The waiting room does not accommodate more than 14 patients. Examine time per patient is exponential with mean rate 20 per/hr.

(a) Find the effective arrival rate at the clinic

(b) What is the probability that an arriving patient will not wait?

(c) what is the expected waiting time until a patient is discharged from the clinic?

Answers :-

1) (a) 0.04 (b) 3 trains

2) (a) 19.98 (b) 0.67 (c) 0.65 hours.