MADURAI KAMARAJ UNIVERSITY

(University with Potential for Excellence)

DISTANCE EDUCATION

B.Sc. (Mathematics)

THIRD YEAR

UNIT: 1 - 5 (Volume -1)

Paper - IV

LINEAR PROGRAMMING AND OPERATIONS RESEARCH

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B.Sc., Mathematics

LINEAR PROGRAMMING AND OPERATIONS RESEARCH

Third Year

Paper - IX

Dear Student,

We welcome you as a Student of the Third year B.Sc degree Course in Mathematics.

This paper – IX deals with Linear Programming and Operations Research. The learning material for this paper will be supplemented by Contact seminars.

Learning through the Distance Education mode, as you are all aware, involves self – learning and self – assessment and in this regard you are expected to put in disciplined and dedicated effort.

On our part, we assume of our guidance and support.

With best wishes.

SYLLABUS

B.Sc., Mathematics – Third year.

Paper – IX

LINEAR PROGRAMMING AND OPERATIONS RESEARCH

UNIT – 1:

Definition – Nature and Scope – Models – Definition of a Standard of a standard linear programming problems – Definition of feasible solution – optimal solution – Optimum basic feasible solutions – Degenerate solution of a L.P.P

UNIT – 2:

Mathematical Formulation of a L.P.P – Slack and Surplus Variables – Graphical Solutions of L.P.P.

UNIT – 3:

Simple method – Standard Maximization case – Minimization Problem – Artitic Variables – Big – M method – Two Phase Method – Degeneracy – Cycling in L.P.P. Application Simplex Method.

UNIT – 4:

Concept of duality – Duality theorems – Duality and simples methods – Simplex Method – Integer programming – Cutting Plan Method – (Gomarian Constraint)

UNIT – 5:

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UNIT – 6:

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UNIT – 8:

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UNIT – 1

INTRODUCTION TO OPERATIONS RESEARCH

Introduction:

The term Operations Research, was first coined in 1940 by McClosky and Trefthen in a small town, Bowdsey, of the United Kingdom. This new science came into existence in military context. During World War II, military management called on scientists from various disciplines and organised them into teams to assist in solving strategic and tactical problems, i.e., to discuss, evolve and suggest ways and means to improve the execution of various military projects. By their joint efforts, experience and deliberations, they suggested certain approaches that showed remarkable progress. This new approach to systematic and scientific study of the operations of the system was called the **Operation Research** or **Operational Research** (abbreviated as O.R.).

This chapter provide an overall view of the subject of operations research. It covers general ideas on the subject, thus providing a perspective. The remaining chapters deals with specific ideas and specific methods of solving O.R. problems.

OBJECTIVES

After completing this unit you will be able to

1. Understand the definition and scope of O.R.
2. Understand the different types of models in O.R.

STRUCTURE

1.1. Nature and Definitions of O.R.

1.2. Scope of O.R.

1.3. Modeling in O.R.

1.4. Standard Linear programming problem

1.5. Keywords

1.6. Model Questions

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1.1 NATURE AND DEFINITIONS OF O.R.

Operations research, rather simply defined, is the research of operations. An operation may be called a set of acts required for the achievement of a desired outcome. Such complex, inter – related acts can be performed by four types of systems: Man, Machine, Man – Machine unit and any organization of men, machines, and man – machine units. OR is concerned with the operations of the last type of system.

Many definitions of OR have been suggested from time to time. On the other hand are put forward a number of arguments as to why it cannot be defined. Perhaps the subject is too young to be defined in an authoritative way. Some of the different definitions suggested are:

(1) OR is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control. – *Morse & Kimball*

(2) OR, in the most general sense, can be characterized as the application of scientific methods, tools and techniques to problems involving the operations of systems so as to provide those in control of the operations with optimum solutions to the problems – *Churchman, Ackoff, Arnoff.*

(3) Operations research is applied decision theory. It uses any scientific, mathematical or logical means to attempt to cope with the problems that confront the executive when he tries to achieve a thorough going rationality in dealing with his decision problems. – *Miller and Starr*

(4) Operations research is a scientific approach to problem solving for executive management. – *H.M. Wagner*

(5) Operations research is the art of giving bad answers to problems, to which, otherwise, worse answers are given. – *Thomas L. Saaty*

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(6) Operations research is an aid for the executive in making his decisions by providing him with the needed quantitative information based on the scientific method of analysis. – *C. Kittel*

(7) Operations research is the systematic, method – oriented study of the basic structure, characteristics, functions and relationships of an organization to provide the executive with a sound, scientific and quantitative basis for decision – making. – *E.L. Arnoff & M.J. Netzorg*

(8) Operations research is the application of scientific methods to problems arising from operations involving integrated systems of men, machines and materials. It normally utilizes the knowledge and skill of an interdisciplinary research team to provide the managers of such systems with optimum operating solutions. – *Fabrycky and Torgersen*

(9) Operations research is an experimental and applied science devoted to observing, understanding and predicting the behaviour of purposeful man – machine systems; and operations research workers are actively engaged in applying this knowledge to practical problems in business, government and society. – *Operations Research Society of America*

(10) Operations research is the application of scientific method by interdisciplinary teams to problems involving the control of organized (man – machine) systems so as to provide solutions which best serve the purpose of the organization as a whole. – *Ackoff and Sasieni*

(11) Operations research utilizes the planned approach (updated scientific method) and an interdisciplinary team in order to represent complex functional relationships as mathematical models for the purpose of providing a quantitative basis for

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decision – making and uncovering new problems for quantitative analysis. – *Thierauf and Klekamp*

(12) O.R. is the application of modern methods of mathematical science to complex problems involving management of large systems of men, machines, materials and money in industry, business, government and defence. The distinctive approach is to develop a scientific model of the system incorporating measurement of factors such as chance and risk to predict and compare the outcomes of alternative decisions, strategies or controls. – J.O.R. Society, U.K.

1.2 SCOPE OF O.R.

Having known the definition of OR, it is easy to visualize the scope of operations research. Whenever there is a problem for optimization, there is scope for the application of OR. When we broaden the scope of OR, we find that really it has been practiced for hundreds of years before World War II.

In the field of industrial management, there is a chain of problems starting from the purchase of raw material to the dispatch of finished goods. The management is interested in having an overall view of the method of optimizing profits. In order to take decision on scientific basis, OR team will have to consider various alternative methods of producing the goods and the return in each case. OR study should also point out the possible changes in the overall structure like installation of a new machine, introduction of more automation, etc. OR has been successfully applied in industry in the fields of production, blending, product mix, inventory control, demand forecast, sale and purchase, transportation, repair and maintenance, scheduling and sequencing, planning, scheduling and control of projects and scores of other associated areas.

OR has a wide scope for application in defence operations. In modern warfare the defence operations are carried out by a number of

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different agencies, namely airforce, army and navy. The activities performed each of them can be further divided into sub – activities viz. Operations, intelligence, administration, training and the like. There is thus a need to coordinate the various activities involved in order to arrive at optimum strategy and to achieve consistent goals. Operations research, conducted by team of experts from all the associated fields, can be quite helpful to achieve the desired results.

In both developing and developed economies, OR approach is equally applicable. In developing economies, there is a great scope of developing an OR approach towards planning. The basic problem is to orient the planning so that there is maximum growth of per capita income in the shortest possible time, by taking into consideration the national goals and restrictions imposed by the country. The basic problem in most of the countries in Asia and Africa is to remove poverty and hunger as quickly as possible. There is, therefore, a great scope for economists, statisticians, administrators, technicians, politicians and agriculture experts working together to solve this problem with an OR approach.

OR approach needs to be equally developed in agriculture sector on national or international basis. With population explosion and consequent shortage of food, every country is facing the problem of optimum allocation of land to various crops in accordance with climatic conditions and available facilities. The problem of optimal distribution of water from the various water resources is faced by each developing country and a good amount of scientific work can be done in this direction.

OR approach is equally applicable to big and small organizations. For example, whenever a departmental store faces a problem like employing additional sales girls, purchasing an

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additional van, etc., techniques of OR can be applied to minimize cost and maximize benefit for each such decision.

OR methods can also be applied in big hospitals to reduce waiting time of out – door patients and to solve the administrative problems.

Monte Carlo methods can be applied in the area of transport to regulate train arrivals and their running times. Queuing theory can be applied to minimize congestion and passengers' waiting time.

OR is directly applicable to business and society. For instance, it is increasingly being applied in L.I.C. offices to decide the premium rates of various policies. It has also been extensively used in petroleum, paper, chemical, metal processing, aircraft, rubber, transport and distribution, mining and textile industries.

Thus we find that OR has a diversified and wide scope in the social economic and industrial problems of today.

1.3 MODELLING IN O.R.

A model in O.R. is a simplified representation of an operation or a process in which only the basic aspects or the most important features of a typical problem under investigation and considered.

The objective of a models is to provide a means for analyzing the behaviour of the system for the purpose of improving its performance.

There are several models in each area of business, or industrial activity. For instance, an account model is a typical budget in which business accounts are referred to with the intention of providing measurements such as rate of expenses, quantity sold, etc, a mathematical equation may be considered to be a mathematical model in which a relationship between constants and variables is represented. A model which has the possibility of measuring observations may be called a quantitative model; a product, a device or any tangible thing used for experimentation may represent a physical model.

Following are the main characteristics that a good model for **Operations Research study should have:**

1. A good model should be capable of taking into account new formulations without having any significant change in its frame.
2. Assumptions made in the model should be as small as possible.
3. It should be simple and coherent. Number of variables used should be less.
4. It should be open to parametric type of treatment.
5. It should not take much time in its construction for any problem.

However, besides the above characteristics, a model has the following limitations:

(i) Models are only an attempt in understanding operations and should never be considered as absolute in any sense.

(ii) Validity of any model with regard to corresponding operation can only be verified by carrying the experiment and relevant data characteristics.

Classification of Models

Although the classification of models is a subjective problem, they may be distinguished as follows:

Models by degree of abstraction: These models are based on the past data/information of the problems under consideration and can be categories into (a) language models, and (b) case studies.

A book may be regarded as an example of a language model.

Models by function: These models consist of (a) Descriptive models, (b) Predictive models, and (c) Normative models.

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(a) Descriptive models: A descriptive model simply describes some aspects of a situation based on observation, survey, questionnaire results, or other available data. The result of an opinion poll represents a descriptive model.

(b) Predictive models: Such models can answer 'what if' type of questions, i.e., they make predictions regarding certain events. For example, based on survey results, televisions networks attempt to explain and predict the election outcome before all the votes are actually counted.

(c) Normative models: Finally, when a predictive model has been repeatedly successful it can be used to prescribe a source of action. Linear programming is a normative or prescriptive model, because it prescribes what the managers ought to do.

Models by structure: These models are represented by (a) Iconic models, (b) Analogue models, and (c) Symbolic models.

Iconic or Physical models are pictorial representation of real systems and have the appearance of the real thing. Examples of such models are: city maps, houses blueprints, globe, and so on. An iconic model is said to be 'scaled – down' or 'scaled – up' according as the dimensions of the model are smaller or greater than those of the real item. For instance, in biology, the structure of a cell may be illustrated by an enlarged (scaled – up) iconic model for teaching purposes.

Iconic models are easy to observe, build and describe, but are difficult to manipulate and not very useful for the purposes of prediction. Commonly, these models represent a static event.

Analogue models are more abstract than the iconic ones for there is no 'look – alike' correspondence between these models and real life items. They are built by utilizing one set of properties to represent another set of properties. For instance, a network of pipes through which water is running could be used as a parallel for understanding the distribution of electric currents. Graphs and maps in various colours are analogue models, distribution of electric currents. Graphs and maps in various colours are analogue models, in which different colours correspond to different characteristics. A flow process chart is an analogue model which represents the order of occurrence of various events to make a product.

Mathematical or Symbolic models are most abstract in nature. They employ a set of mathematical symbols to represent the components (and relationships between them) of the real system. These models are most general and precise. However, it is not always possible to depict a real system in mathematical formulaiton, sometimes it is easier to use mathematical symbols for describing the relationship of the components, and sometimes an analogue model may express the pattern of its relationship in a better way.

Models by nature of the environment: These models can be classified into (a) Deterministic models, and (b) Probabilistic models.

In deterministic models, all the parameters and functional relationship are assumed to be known with certainty when the decision is to be made. Linear Programming and Break – even models are the examples of deterministic models.

On the other hand, models in which at least one parameter or decision variable is a random variable are called probabilistic or stochastic models. These models reflect, to some extent, the complexity of the real world and the uncertainty surrounding it.

Models by the extent of generality: These models can be categorised into (a) Specific models, and (b) General models.

When a model presents a system at some specific time, it is known as a specific model. In these models if the time factor is not considered, then they are termed as static models, and dynamic models otherwise. An inventory problem of determining economic

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order quantity for the next period, assuming that the demand in planning period would remain same as that of today, is an example of a static model. Dynamic Programming may be considered as an example of dynamic model.

Simulation and Heuristic models fall under the category of general models. These models are mainly used to explore alternative strategies (courses of action) which have been overlooked previously. These models do not yield any optimum solution to the problem, but give a solution to a problem depending on assumptions based on the past experience.

1.4 STANDARD LINEAR PROGRAMMING PROBLEMS

General Linear Programming Problem

The linear programming involving more than two variables may expressed as follows:

Maximize (or) Minimize $Z = c_1x_1 + c_2x_2 + c_3x_3 + \dots\dots\dots + c_nx_n$

subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots\dots\dots + a_{1n}x_n \le \text{ or } = \text{ or } \ge b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots\dots\dots + a_{2n}x_n \le \text{ or } = \text{ or } \le b_2$$

$$a_{31}x_1 + a_{32}x_2 + \dots\dots\dots\dots\dots + a_{3n}x_n \le \text{ or } = \text{ or } \le b_3$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots\dots\dots\dots\dots + a_{mn}x_n \le \text{ or } = \text{ or } \ge b_m$$

and the non – negativity restrictions.

$$x_1, x_2, x_3, \dots\dots\dots\dots\dots x_n \ge 0.$$

Note:

Some of the constraints may be equalities, some others may inequalities of $(\le)$ type and remaining ones inequalities of $(\ge)$ type or of them are of same type.

Definition:

A set of values $x_1, x_2, \dots\dots\dots, x_n$ which satisfies the constraints of

LPP is called its **solution.**

Definition:

Any solution to a LPP which satisfies the non negativity restrictions of the LPP is called its **feasible solution.**

Definition:

Any feasible solution which optimizes (maximizes (or) minimizes) the objective function of the LPP is called its **optimum solution** or **optimal solution.**

Definition:

A basic solution is said to be a non – degenerate basic solution if none of the basic variables is zero.

Definition:

A basic solution is said to be the degenerate basic solution if one or more of the basic variables are zero.

Definition:

A feasible solution which is also basic is called a basic feasible solution.

1.5 KEY WORDS

Operations Research, Scientific Methods, Decision Theory, Industrial Management, Monte Carlo Methods.

1.6 MODEL QUESTIONS

1. Define O.R. and discuss its scope.
2. What are the applications of O.R?
3. "Model building is the essence of the operations research approach" Discuss.
4. Give any three definitions of operations research and explain.
5. Explain the nature of operations research and its limitation.
6. "Operation Research is a bunch of Mathematical Techniques" comment.

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UNIT – 2

MATHEMATICAL FORMULATION AND GRAPHICAL SOLUTION OF L.P.P.

Introduction:

Many business and economic situations are concerned with a problem of planning activity. In each case, there are limited resources at your disposal and your problem is to make such a use of these resources so as to yield the maximum production or to minimise the cost of production, or to give the maximum profit, etc. Such problems are referred to as the problems of constrained optimisation. Linear Programming is a technique for determining an optimum schedule of interdependent activities in view of the available resources. Programming is just another word for 'planning and refers to the process of determining a particular plan of action from amongst several alternatives. The word linear stands for indicating that all relationships involved in a particular problem are linear.

In the present chapter mathematical formulation of the linear programming problem (LPP) and Graphical solutions of linear programming problem are discussed.

Objectives:

After working through examples and exercises of this unit you will be able to

1. Formulate the L.P.P
2. Graph the feasible region for a L.P.P with two decision variables.
3. Understand how basic and non – basic variables relate to the graph of problem.

Structure:

2.1 Mathematical formulation of L.P.P

2.2 Slack and Surplus variables

2.3 Graphical solutions of a L.P.P

2.4 Keywords

2.5 Answers to check your progress questions

2.6 Model Questions

2.1 Mathematical formulation of L.P.P

If $x_j (j = 1,2,........,n)$ are the n decision variables of the problem and if the system is subject to m constraints, the general mathematical model can be written in the form

Optimize $Z = f(x_1, x_2,......., x_n)$

Subject to $g_i(x_1, x_2,........, x_n) \leq, =, \geq b_i, (i = 1,2,........,m)$

(called structural constraints)

and $x_1, x_2,........, x_n \geq 0$,

(called the non – negativity restrictions or constraints)

Procedure for Mathematical formulation of L.P.P

Step: 1

Study the given situation to find the key decisions to be made.

Step: 2

Identify the variables involved and designate them by symbols $x_j (j = 1,2,........,)$.

Step: 3

State the feasible alternatives which generally are $x_j \geq 0$, for all j.

Step: 4

Identify the constraints in the problem and express them as linear inequalities or equations LHS of which are linear functions of the decision variables.

Step: 5

Identify the objective function and express it as a linear function of the decision variables.

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Example: 2.1.1

A company has three operational departments (Waving, Processing and Packing) with capacity to produce three different types of clothes namely suiting's, shirting's and woollens yielding a profit of Rs. 2, Rs. 4 and Rs. 3 per metre respectively. One metre of suiting requires 3 minutes in weaving, 2 minutes in processing and 1 minute in packing. Similarly one metre of shirting requires 4 minutes in weaving, 1 minute in processing and 3 minutes in packing. One metre of woolen requires 3 minutes in each department. In a week, total run time of each department is 60, 40 and 80 hours for weaving, processing and packing respectively.

Formulate the linear programming problem to find the product mix to maximize the profit.

Solution:

The data of the problem is summarized below:

	Department			
	Weaving (in minutes)	**Processing (in minutes)**	**Packing (in minutes)**	**Profit (Rs. Per meter)**
Suiting's	3	2	1	2
Shirting's	4	1	3	4
Woolens	3	3	3	3
Availability (min)	60 x 60	40 x 60	80 x 60	

Step: 1

The key decision is to determine the weekly rate of production for the three types of clothes.

Step: 2

Let us designate the weekly production of suiting's, shirting's

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and woolens by x_1 meters, x_2 meters and x_3 meters respectively.

Step: 3

Since it is not possible to product negative quantities, feasible alternatives are set of values of x_1, x_2 and x_3 satisfying $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

Step: 4

The constraints are the limited availability of three operational departments. One meter of suiting requires 3 minutes of weaving. The quantity being x_1 meters, the requirement for suiting alone will be $3x_1$ units. Similarly, x_2 meters of shirting and x_3 meters of woolen will require $4x_2$ and $3x_3$ minutes respectively. Thus the total requirement of weaving will be $3x_1 + 4x_2 + 3x_3$, which should not exceed the available 3600 minutes. So, the labour constraint becomes $3x_1 + 4x_2 + 3x_3 \leq 3600$.

Similarly the constraints for the processing department and packing departments are $2x_1 + x_2 + 3x_3 \leq 2400$ and $x_1 + 3x_2 + 3x_3 \leq 4800$ respectively.

Step: 5

The objective is to maximize the total profit from sales. Assuming that whatever is produced is sold in the market, the total profit is given by the linear relation $z = 2x_1 + 4x_2 + 3x_3$.

The linear programming problem can be put in the following mathematical format:

Maximize $Z = 2x_1 + 4x_2 + 3x_3$

Subject to the constraints

$$3x_1 + 4x_2 + 3x_3 \leq 3600$$

$$2x_1 + x_2 + 3x_3 \leq 2400$$

$$x_1 + 3x_2 + 3x_3 \leq 4800$$

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$x_1 \geq 0,\ x_2 \geq 0$ and $x_3 \geq 0$.

Example: 2.1.2

The manager of an oil refinery must decide on the optimum mix of two possible blending processes of which the input and output production runs are as follows.

Process	Input		Output	
	Crude A	Crude B	Gasoline X	Gasoline Y
1	6	4	6	9
2	5	6	5	5

The maximum amounts available of crudes A and B are 250 units and 200 units respectively. Market demand shows that at least 150 units of gasoline X and 130 units of gasoline Y must be produced. The profits per production run from process 1 and process 2 are Rs. 4 and Rs. 5 respectively. Formulate the problem for maximising the profit.

Solution:

Decision Variables: x_1 = number of units of Gasoline from process 1.

x_2 = number of units of Gasoline from process 2.

Objective function : Maximize $Z = 4x_1 + 5x_2$

Constraints : $6x_1 + 5x_2 \leq 250,\ 4x_1 + 6x_2 \leq 200$ (crude A)

$6x_1 + 5x_2 \geq 150,\ 9x_1 + 5x_2 \geq 130$ (crude B)

$x_1 \geq 0$ and $x_2 \geq 0$.

Example: 2.1.3

The owner of metro sports wishes to determine how many advertisements to place in the selected three monthly maganizes A, B and C. His objective is to advertise in such a way that the total exposure to principal buyers of expensive sports good is maximized. Percentages of readers for each magazine are known. Exposure in any particular magazine is . the number of advertisements placed multiplied by the number of principal buyers. The following data may be used.

	Magazine		
	A	B	C
Readers	1 Lakh	0.6 Lakh	0.4 Lakh
Principal Buyers	20 %	15 %	8 %
Cost per advertisement (Rs)	8000	6000	5000

The budgeted amount is at most Rs. 1 Lakh for the advertisement. The owner has already decided that magazine A should have no more than 15 advertisements and that B and C each have at least 80 advertisements. Formulate on LP model for the problem.

Solution:

Decision variables x_1 = number of insertions in Magazine A

x_2 = number of insertions in Magazine B and

x_3 = number of insertions in Magazine C.

Objective function:

Maximize (total exposure)

$$Z = (20\% \text{ of } 1{,}00{,}000)x_1 + (15\% \text{ of } 60{,}000)x_2 + (8\% \text{ of } 40{,}000)x_3$$
$$= 20{,}000x_1 + 9{,}000x_2 + 3{,}200x_3$$

Constraints

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$$8{,}000x_1 + 6{,}000x_2 + 5{,}000x_3 \le 1{,}00{,}000$$

$$x_1 \le 15, \qquad x_2 \ge 8, \qquad x_3 \ge 8$$

$$x_1 \ge 0, \qquad x_2 \ge 0 \text{ and } x_3 \ge 0$$

Example: 2.1.4

A complete unit of a certain product consists of four units of component A and three units of component B. The two components (A and B) are manufactured from two different raw materials of which 100 units and 200 units, respectively are available. Three departments are engaged in the production process with each department using a different method for manufacturing the components per production run and the recounting units of each component are given below.

Department	Input per run (units)		Output per run (units)	
	Raw Material I	Raw Material II	Component A	Component B
1	7	5	6	4
2	4	8	5	8
3	2	7	7	3

Formulate this problem as a linear programming model. So as to determine the number of production runs for each department which will maximize the total number of complete units of the final product.

Solution:

Decision Variables:

x_1 = number of production runs for department 1

x_2 = number of production runs for department 2 and

x_3 = number of production runs for department 3.

Objective function:

Since each unit of the final product requires 4 units of component A and 3 units of component B, therefore, maximum number of units of the final product cannot exceed the smaller value of

$$\left\{\frac{\text{Total number of units A produced}}{4}, \frac{\text{Total number of units B produced}}{3}\right\}$$

(i.e.), Minimum of $\left\{\frac{6x_1 + 5x_2 + 7x_3}{4}, \frac{4x_1 + 8x_2 + 3x_3}{3}\right\}$

Constraints: (i)

If y is the number of component units of final product, then obviously, we have $\frac{6x_1 + 5x_2 + 7x_3}{4} \geq y$ and $\frac{4x_1 + 8x_2 + 3x_3}{3} \geq y$.

(i.e.) $6x_1 + 5x_2 7x_3 - 4y \geq 0$

$4x_1 + 8x_2 + 3x_3 - 3y \geq 0$

(ii) $7x_1 + 4x_2 + 2x_3 \leq 100$

$5x_1 + 8x_2 + 7x_3 \leq 200$

$x_1 \geq 0,\ x_2 \geq 0$ and $x_3 \geq 0$

Example: 2.1.5

Old hens can be bought at Rs. 2 each and young ones at Rs. 5 each. The old hens lay 3 eggs per week and the young ones lay 5 eggs per week, each egg being worth 30 paise. A hen costs Rs. 1 per week to feed. A person has only Rs. 80 to spend for hens. How many of each kind should he buy to give a profit of more than Rs. 6 per week assuming that he cannot house more than 20 hens. Formulate this as a L.P.P.

Solution:

The person decides to by x_1 old hens and x_2 young hens to maximize his profit.

Since he has only Rs. 80 to spend for hens and old hen costs Rs. 2 and young hen costs Rs. 5 each,

$2x_1 + 5x_2 \leq 80$.

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Also, since he cannot house more than 20 hens,

$$x_1 + x_2 \le 20.$$

The total sale of eggs will be $= \text{Rs. } 0.3\ (3x_1 + 5x_2)$

Expenditure on feeding will be $= \text{Rs. } 1\ (x_1 + x_2)$

$\therefore$ The net profit is $= \text{Rs. } [0.3(3x_1 + 5x_2) - 1(x_1 + x_2)]$

$= \text{Rs. } (0.5x_2 - 0.1x_1)$

$$\therefore\ 0.5x_2 - 0.1x_1 \ge 6.$$

The constraints are

$$2x_1 + 5x_2 \le 80$$

$$x_1 + x_2 \le 20$$

$$0.5x_2 - 0.1x_1 \ge 6$$

and $x_1, x_2 \ge 0$.

$\therefore$ The complete formulation of the L.P.P is maximize $Z = 0.5x_2 - 0.1x_1$,

subject to the constraints

$$2x_1 + 5x_2 \le 80$$

$$x_1 + x_2 \le 20$$

$0.5x_2 - 0.1x_1 \ge 6$ and $x_1, x_2 \ge 0$.

2.2 Slack and Surplus Variables

Definition: Slack Variables

Let the constraints of a general L.P.P be $\sum_{j=1}^{n} a_{ij} x_j \le b_i,\ i = 1,2,\ldots\ldots,k$.

Then, the non – negative variables x_{n+i} which satisfy $\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i,\ i = 1,2,\ldots\ldots,k$ are called slack variables.

Definition: Surplus Variables

Let the constraints of a general L.P.P be

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$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i, i = k+1, k+2, \ldots\ldots, \ell.$$

Then the non – negative variables x_{n+i} which satisfy $\sum_{j=1}^{n} a_{ij}x_j - x_{n+i} = b_i,\ i = k+1, k+2, \ldots\ldots\ldots, \ell$ are called surplus variables.

2.3 Graphical Solution of a L.P.P

Linear programming problems involving only two variables can be effectively solved by a graphical method which provides a pictorial representation of the problems and its solutions and which gives the basic concepts used in solving general L.P.P which may involve any finite number of variables. This method is simple to understand and easy to use. Graphical method is not a powerful tool of linear programming as most of the practical situations do involve more than two variables. But the method is really useful to explain the basic concepts of L.P.P. to the persons who are not familiar with this. Though graphical method can deal with any number of constraints but since each constraint is shown as a line on a graph, a large number of lines make the graph difficult to read.

Working procedure for Graphical Method

Step: 1

Identify the problem – the decision variables, the objective and the restrictions.

Step: 2

Set up the mathematical formulation of the problem.

Step: 3

Plot a graph representing all the constraints of the problem and identify the feasible region (solution space). The feasible region is the intersection of all the regions represented by the

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constraints of the problem and is restricted to the first quadrant only.

Step: 4

The feasible region obtained in step 3 may be bounded or unbounded. Compute the coordinates of all the corner points of the feasible region.

Step: 5

Find out the value of the objective function at each corner (solution) point determined in step 4.

Step: 6

Select the corner point that optimize (maximizes or minimizes) the value of the objective function. It gives the optimum feasible solution.

Example: 2.3.1

A company makes two kinds of leather belts. Belt A is a high quality belt, and belt B is at low quantity. The respective profits are Rs. 4.00 and Rs. 3.00 per belt. Each belt of type A requires twice as much time as a belt of type B, and if all belts were of type B, the company could make 1000 per day. The supply of leather is sufficient for only 800 belts per day (both A and B combined). Belt A requires a fancy buckles and only 400 per day are available. There are only 700 buckles a day available for belt B. Determine the optimal product mix.

Solution:

Step: 1

The mathematical formulation of the given linear programming problem is

Maximize $Z = 4x_1 + 3x_2$

Subject to the constraints:

$$2x_1 + x_2 \leq 1{,}000$$

$$x_1 + x_2 \leq 800$$

$x_1 \leq 400$ and $x_2 \leq 700$

$x_1 \geq 0$ and $x_2 \geq 0$

Where x_1 = number of belts of type A.

x_2 = number of belts of type B.

Step: 2

Next we construct the graph by considering the Cartesian rectangular axis OX_1X_2 in the plane. As each point has the coordinates of the type (x_1, x_2); any point satisfying the conditions $x_1 \geq 0$ and $x_2 \geq 0$ lies in the first quadrant.

Now, the inequalities are graphed taking, them as equations, e.g., the first constraint $2x_1 + x_2 \leq 1000$. The equation is re – written as $\frac{x_1}{500} + \frac{x_2}{1000} = 1$.

This equation indicates that when it is plotted on the graph, it cuts an x_1- intercept of 500 and x_2- intercept of 1000. These two points are then connected by a straight line which is shown in fig. (i) as line AB. Any point (representing a combination of x_1 and x_2) that fall on this line or in the area below it, is acceptable in so far as this constraint is concerned. The region OAB formed by two axes and the line representing the equation $2x_1 + x_2 = 1000$ is the region containing accepted values of x_1 and x_2 in respect of this constraint.

Similarly, the constraint $x_1 + x_2 \leq 800$ can be plotted. The line CD in fig (ii) represents the equation $x_1 + x_2 = 800$. The region OCD, formed by the two axes and this line represents the area in which any point would satisfy this constraint of leather availability. Further, the constraints $x_1 \leq 400$ and $x_2 \leq 700$ are also plotted on the graph which represents the area between the two axes and the lines $x_1 = 400$ and $x_2 = 700$ as shown in fig (ii).

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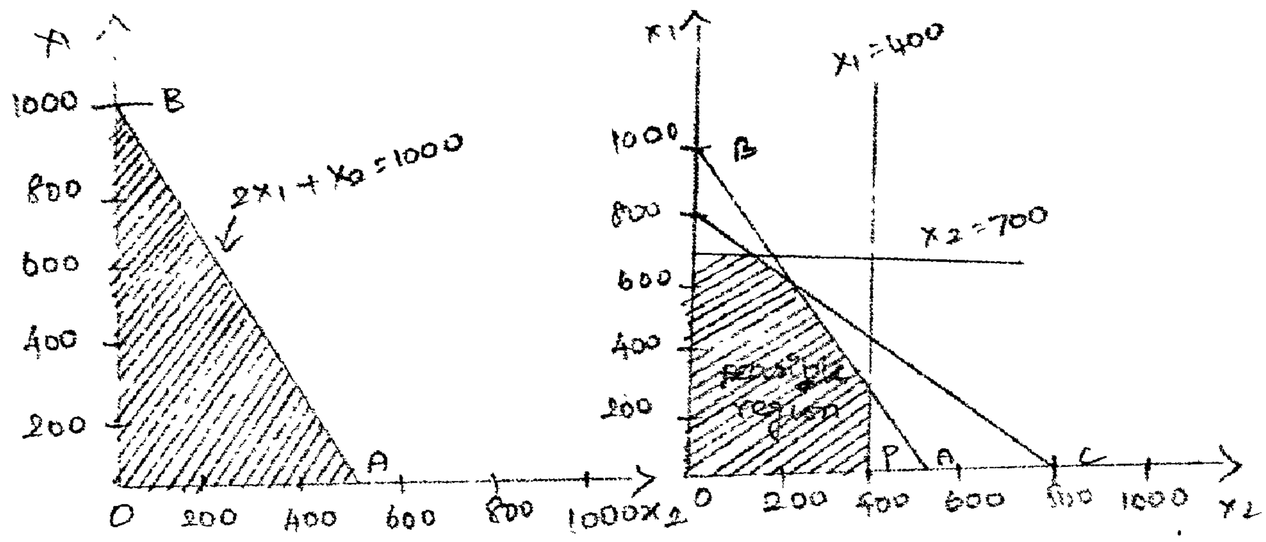

Now all the constraints have been graphed. The area bounded by all these constraints, called feasible region (or) solution space, as shown in fig (ii) by the shaded area OPQRST.

Step: 3

The optimum value of objective function occurs at one of the extreme (corner) points of the feasible region. The coordinates of the extreme points are

I = (0, 0), P = (400, 0), Q = (400, 200), R = (200, 600), S = (100, 700) and T = (0, 700).

Step: 4

We now compute the z – values corresponding to the extreme points

Extreme point	(x_1, x_2)	$z = 4x_1 + 3x_2$	
O	(0, 0)	0	
P	(400, 0)	1600	
Q	(400, 200)	2200	
R	(200, 600)	2600	← Maximum
S	(100, 700)	2500	
T	(0, 700)	2100	

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Step: 5

The optimum solution is that extreme point for which the objective function has the largest value. Thus the optimum solution occurs at the point R, i.e., $x_1 = 200$ and $x_2 = 600$ with the objective function value of Rs. 2600.

Hence, to maximize profit, the company should produce 200 belts of type A and 600 belts of type B per day.

Example: 2.3.2

Let us assume that you have inherited Rs. 1,00,000 from your father – in – law that can be invested in a combination of only two stock portfolios with the maximum investment allowed in either portfolio set at Rs. 75,000. The first portfolio has an average rate of return of 10%, whereas the second has 20%. In terms of risk factors associated with these portfolios, the first has a risk rating of 4 (on a scale from 0 to 10), and the second has 9. Since you wish to maximize your return, you will not accept an average rate of return below 12% or a risk factor above 6. Hence, you then face the important question. How much should you invest in each portfolio?

Solution:

Step: 1

The mathematical formulation of the linear programming problem is

Maximize $Z = 0.10x_1 + 0.20x_2$

Subject to the constraints

$x_1 + x_2 \leq 1{,}00{,}000, \quad x_1 \leq 75{,}000, \; x_2 \leq 75{,}000$

$0.10x_1 + 0.20x_2 \geq 0.12(x_1 + x_2)$ (or) $-0.02x_1 + 0.08x_2 \geq 0$

$4x_1 + 9x_2 \leq 6(x_1 + x_2)$ (or) $-2x_1 + 3x_2 \leq 0$

$x_1 \geq 0$ and $x_2 \geq 0$

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Where x_1 = amount invested in portfolio 1, and

x_2 = amount invested in portfolio 2.

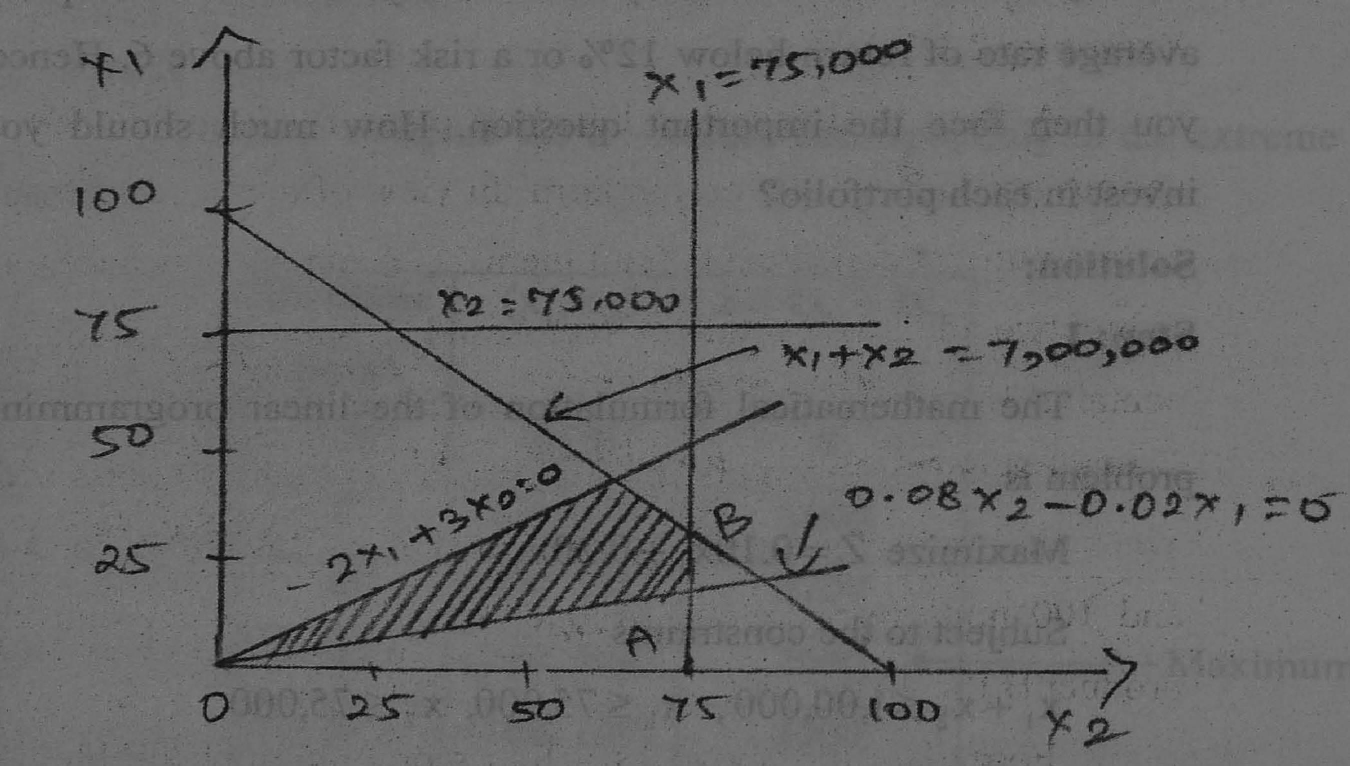
Step: 2

The first constraint $x_1 + x_2 \le 1{,}00{,}000$ can be graphed by plotting the straight line $\dfrac{x_1}{1{,}00{,}000} + \dfrac{x_2}{1{,}00{,}000} = 1$.

This cuts a x_1 - intercept and x_2 - intercept of 1,00,000 each. The area below this line represents the feasible area in respect of this constraint.

Similarly, the other constraints are depicted by plotting the straight lines corresponding to the equations $x_1 = 75{,}000$, $x_2 = 75{,}000$, $-2x_1 + 3x_2 = 0$ and $-0.02x_1 + 0.08x_2 = 0$. Here, the area below the first three lines and beyond the fourth line gives the feasible region in respect of these four constraints.

Thus the feasible region in respect of the given problem is as shown in figure.



Step: 3

The coordinates of the extreme points are:

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O (0,0), A = (75,000, 18,750) B = (75,000, 25,000) and

C = (60,000, 40,000)

Step: 4

The z – values corresponding to the extreme points are:

Extreme Point	(x_1, x_2)	$z = 0.10x_1 + 0.20x_2$
O	(0, 0)	0
A	(75,000, 18,750)	11,250
B	(75,000, 25,000)	12,500
C	(60,000, 40,000)	14,000 ← Maximum

Hence the optimal solution is $x_1 = 60{,}000$, $x_2 = 40{,}000$ and maximum return = Rs. 14,000.

Example: 2.3.3

A farm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. In view of the need to ensure certain nutrient constituents (call them X, Y and Z), it is necessary to buy two additional products say, A and B, one unit of product A contains 36 units of X, 3 units of Y and 20 units of Z. One unit of product B contains 6 units of X, 12 units of Y and 10 units of Z. The minimum requirement of X, Y and Z is 108 units, 36 units and 100 units respectively. Product A cost Rs. 20 per unit and product B Rs. 40 per unit.

Formulate the above as a linear programming problem to minimize the total cost, and solve the problem by using graphic method.

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Solution:

The data of the given problem can be summarized as follows:

Nutrient Constituents	Nutrient content in product		Minimum amount of nutrient
	A	B	
X	36	06	108
Y	03	12	36
Z	20	10	100
Cost of Product	Rs. 20	Rs. 40	

Making use of above information, the appropriate mathematical formulation of the linear programming problem is

Minimize $Z = 20x_1 + 40x_2$

Subject to the constraints

$$36x_1 + 6x_2 \geq 108$$

$$3x_1 + 12x_2 \geq 36$$

$$20x_1 + 10x_2 \geq 100 \text{ and } x_1, x_2 \geq 0.$$

Consider now a set of Cartesian rectangular axis OX_1X_2 in the plane. As each point has the coordinates of the type (x_1, x_2) any point satisfying the conditions $x_1 \geq 0$ and $x_2 \geq 0$ lies in the first quadrant only.

The constraints of the given problem are plotted as described earlier by treating them as equations:

$$36x_1 + 6x_2 = 108$$

$$3x_1 + 12x_2 = 36 \text{ and}$$

$$20x_1 + 10x_2 = 100$$

(or) $\dfrac{x_1}{3} + \dfrac{x_2}{18} = 1, \ \dfrac{x_1}{12} + \dfrac{x_2}{3} = 1$ and $\dfrac{x_1}{5} + \dfrac{x_2}{10} = 1.$

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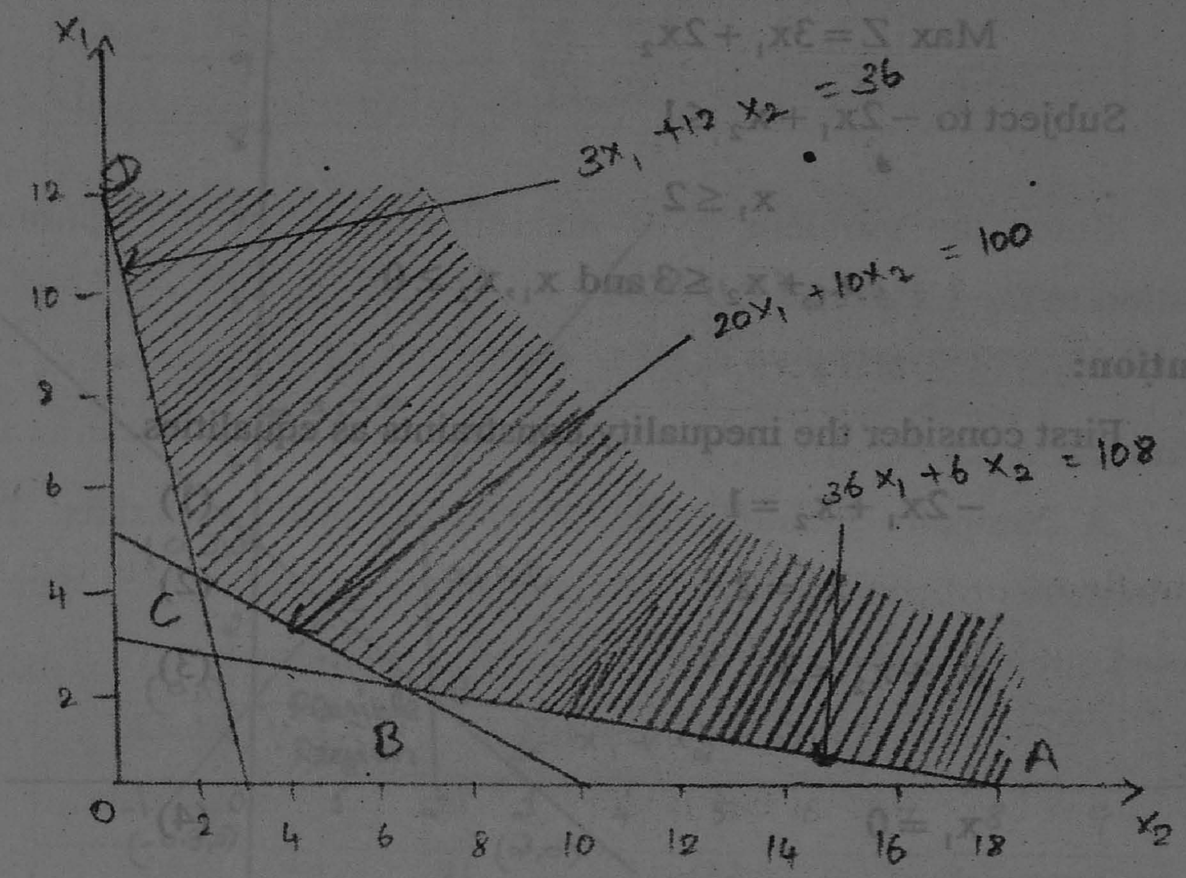
The area beyond these lines represents the feasible region in respect of these constraints; any point on the straight lines or in the region above these lines would satisfy the constraints. The feasible region of the problem is as shown in figure.



The coordinates of the extreme points of the feasible region are: $A = (0,18)$, $B = (2,6)$, $C = (4,2)$ and $D = (12,0)$.

The value of the objective function at each of the extreme points can be evaluated as follows:

Extreme Point	(x_1, x_2)	$z = 20x_1 + 40x_2$
A	(0,18)	720
B	(2,6)	280
C	(4,2)	160 ← Minimum
D	(12,0)	240

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Hence the optimum solution is to purchase 4 units of product A and 2 units of product B in order to maintain a minimum cost of Rs. 160.

Example: 2.3.4

Solve the following L.P.P by the graphical method

$$\text{Max } Z = 3x_1 + 2x_2$$

$$\text{Subject to } -2x_1 + x_2 \le 1$$

$$x_1 \le 2$$

$$x_1 + x_2 \le 3 \text{ and } x_1, x_2 \ge 0$$

Solution:

First consider the inequality constraints as equalities.

$$-2x_1 + x_2 = 1 \qquad (1)$$

$$x_1 = 2 \qquad (2)$$

$$x_1 + x_2 = 3 \qquad (3)$$

and

$$x_1 = 0 \qquad (4)$$

$$x_2 = 0 \qquad (5)$$

For the line,

$$-2x_1 + x_2 = 1,$$

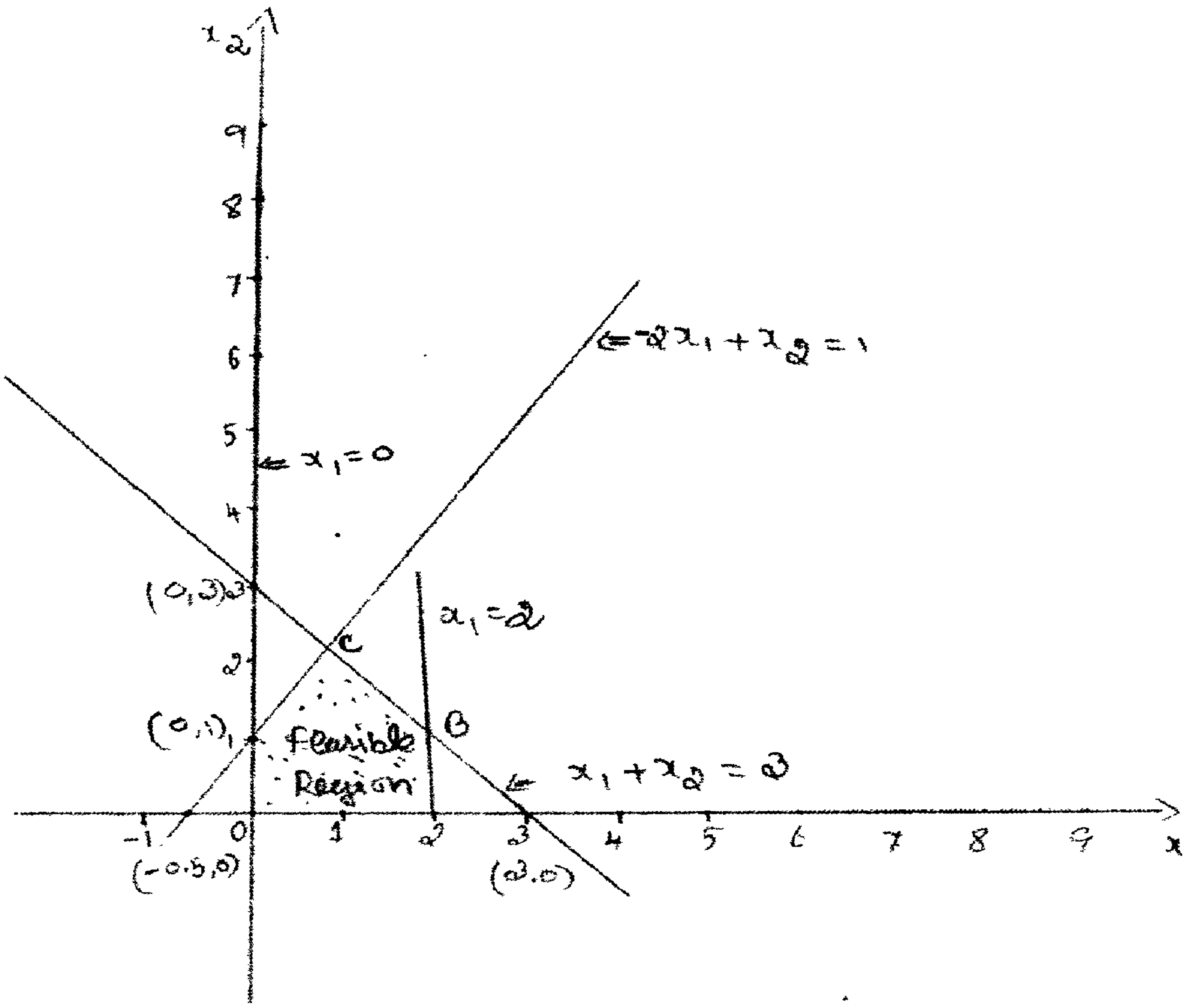
Put $x_1 = 0 \Rightarrow x_2 = 1 \Rightarrow (0,1)$

Put $x_2 = 0 \Rightarrow -2x_1 = 1 \Rightarrow x_1 = -0.5 \Rightarrow (-0.5,0)$

So, the line (1) passes through the points (0, 1) and (-.05, 0). The points on this line satisfy the equation $-2x_1 + x_2 = 1$. Now orgin (0, 0), on substitution, gives $-0 + 0 = 0 < 1$; hence it also satisfies the inequality $2x_1 + x_2 \le 1$.

Thus all points on the origin side and on this line satisfy the inequality $-2x_1 + x_2 \le 1$.

Similarly interpreting the other constraints we get the feasible region OABCD. The feasible region is also known as solution space of the L.P.P. Every point within this area satisfies all the constraints.



Now our aim is to find the vertices of the solution space. B is the point of intersection of $x_1 = 2$ and $x_1 + x_2 = 3$. Solving these two equations, we have $x_1 = 2, x_2 = 1$.

Therefore we have the vertex B (2, 1). Similarly, C is the intersection of $-2x_1 + x_2 = 1$ and $x_1 + x_2 = 3$. Solving these we have $C = \left(\frac{2}{3}, \frac{7}{3}\right)$.

$\therefore$ The vertices of the solution space are $0(0,0)$, $A(2,0)$, $B(2,1)$, $C\left(\frac{2}{3}, \frac{7}{3}\right)$ and $D(0,1)$.

The values of Z at these vertices are given by

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Extreme Point	Vertex (x_1, x_2)	$Z = 3x_1 + 2x_2$
O	(0, 0)	0
A	(2, 0)	6
B	(2, 1)	8
C	$\left(\frac{2}{3}, \frac{7}{3}\right)$	$\frac{20}{3}$
D	(0, 1)	2

Since the problem is of maximization type, the optimum solution to the L.P.P is

Maximize $Z = 8,\ x_1 = 2,\ x_2 = 1$.

Example: 2.3.5

A company manufactures 2 types of printed circuits. The requirements of transistors, resistors and capacitors for each type of printed circuits along with other data are given below:

	Circuit		Stock available
	A	B	
Transistor	15	10	180
Resistor	10	20	200
Capacitor	15	20	210
Profit	Rs. 5	Rs. 8	

How many circuits of each type should the company produce from the stock to earn maximum profit.

Solution:

Let x_1 be the number of type A circuits and x_2 be the number of type B circuits to be produced.

To produce these units of type A and type B circuits, the company requires

Transistors = $15x_1 + 10x_2$

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Resistor $= 10x_1 + 20x_2$

Capacitors $= 15x_1 + 20x_2$

Since the availability of these transistors, resistors and capacitors are 180, 200 and 210 respectively, the constraints are

$15x_1 + 10x_2 \leq 180$

$10x_1 + 20x_2 \leq 200$

$15x_1 + 20x_2 \leq 210$

and $x_1 \geq 0, x_2 \geq 0$.

Since the profit from type A is Rs. 5 and from type B is Rs. 8 per units the total profit is $5x_1 + 8x_2$.

$\therefore$ The complete formulation of the L.P.P is

$$\text{Maximize } Z = 5x_1 + 8x_2$$

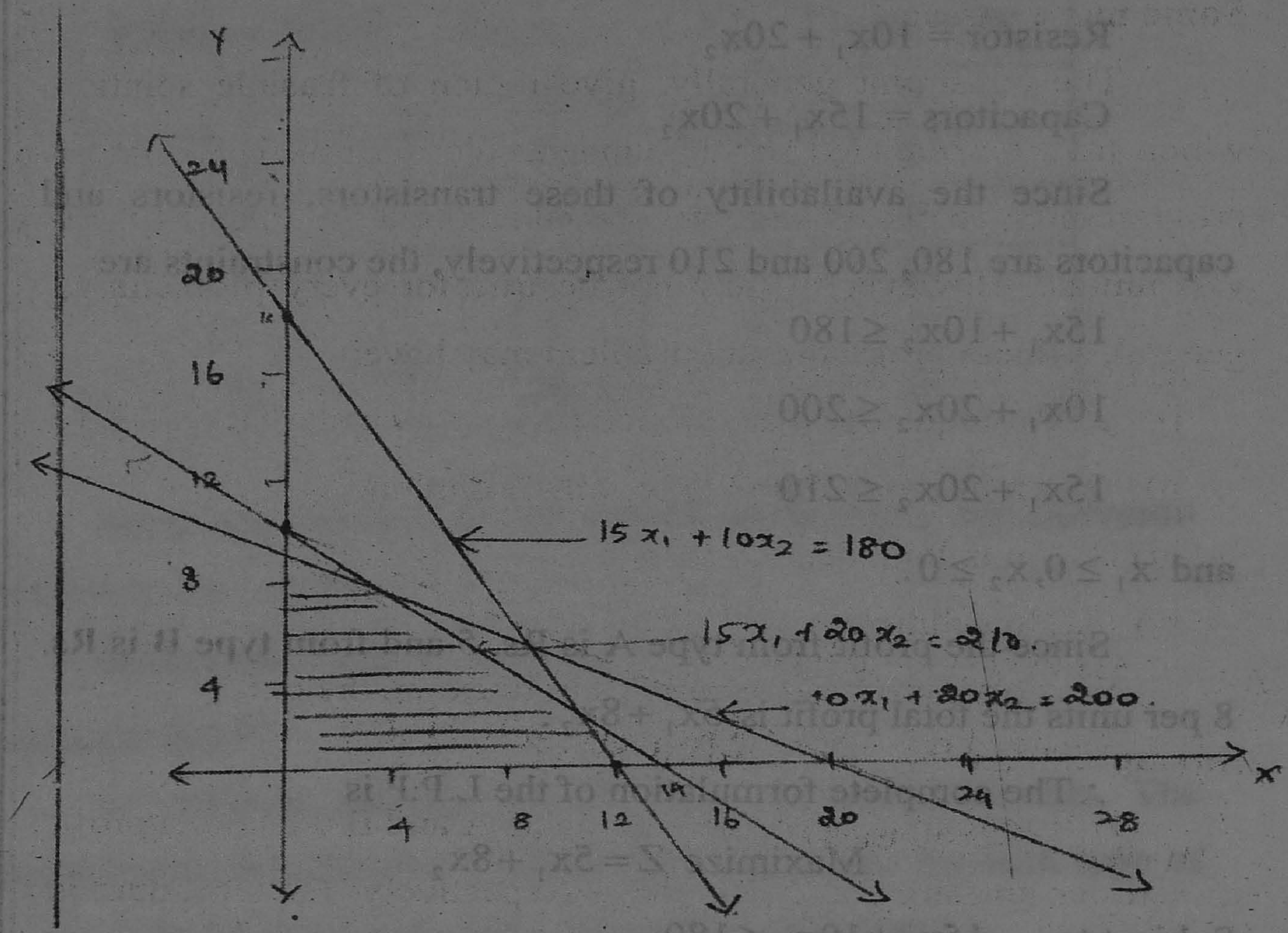
Subject to $15x_1 + 10x_2 \leq 180$

$10x_1 + 20x_2 \leq 200$

$15x_1 + 20x_2 \leq 210$

and $x_1, x_2 \geq 0$

By using graphical method, the solution space is given below with shaded area OABCD with vertices $O(0,0)$, $A(12,0)$, $B(10,3)$, $C(2,9)$ and $D(0,10)$.



The value of Z of these vertices are given by

Extreme point	(x_1, x_2)	$Z = 5x_1 + 8x_2$
O	(0, 0)	0
A	(12, 0)	60
B	(10, 3)	74
C	(2, 9)	82
D	(0, 10)	80

Since the problem is manimization type, the optimum solution is maximum $Z = 82,\ x_1 = 2,\ x_2 = 9$.

Note:

From the above examples, for problems involving two variables and having a finite solution, we observed that the optimal solution existed at a vertex of the feasible region that is "if there exists an optimal solution of an L.P.P, it will be at one of the vertices of the feasible region".

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Some more cases:

The constraint generally, give region of feasible solution which may be bounded or unbounded. We discussed five linear programming problems and the optimal solution for either of them was unique. However, it may not be true for every problem. In general, a linear programming problem may have:

i) A unique optimal solution

ii) An infinite number of optimal solution

iii) An unbounded solution

iv) No solution

We now give a few examples to illustrate the cases.

Example: 2.3.6

A firm manufactures two products A and B which the profit earned per unit are Rs. 3 and Rs. 4 respectively. Each product is processed on two machines M_1 and M_2. Product A requires one minute of processing time on M_1 and two minutes on M_2. While B requires one minute on M_1 and one minute on M_2. Machine M_1 is available for not more than 7 hours 30 minutes while machine M_2 is available for 10 hours during any working day. Find the number of units products A and B to be manufactured to get maximum profit. Formulate the above as a L.P.P and solve by graphical method.

Solution:

Let the firm decide to manufacture x_1 units of product A and x_2 units of product B.

To produce these units of products A and B, it requires

$x_1 + x_2$ hours of processing times on M_1.

$2x_1 + x_2$ hours of processing times on M_2.

But the availability of these two machines M_1 and M_2 are

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450 minutes and 600 minutes respectively, the constraints are

$$x_1 + x_2 \le 450$$

$$2x_1 + x_2 \le 600$$

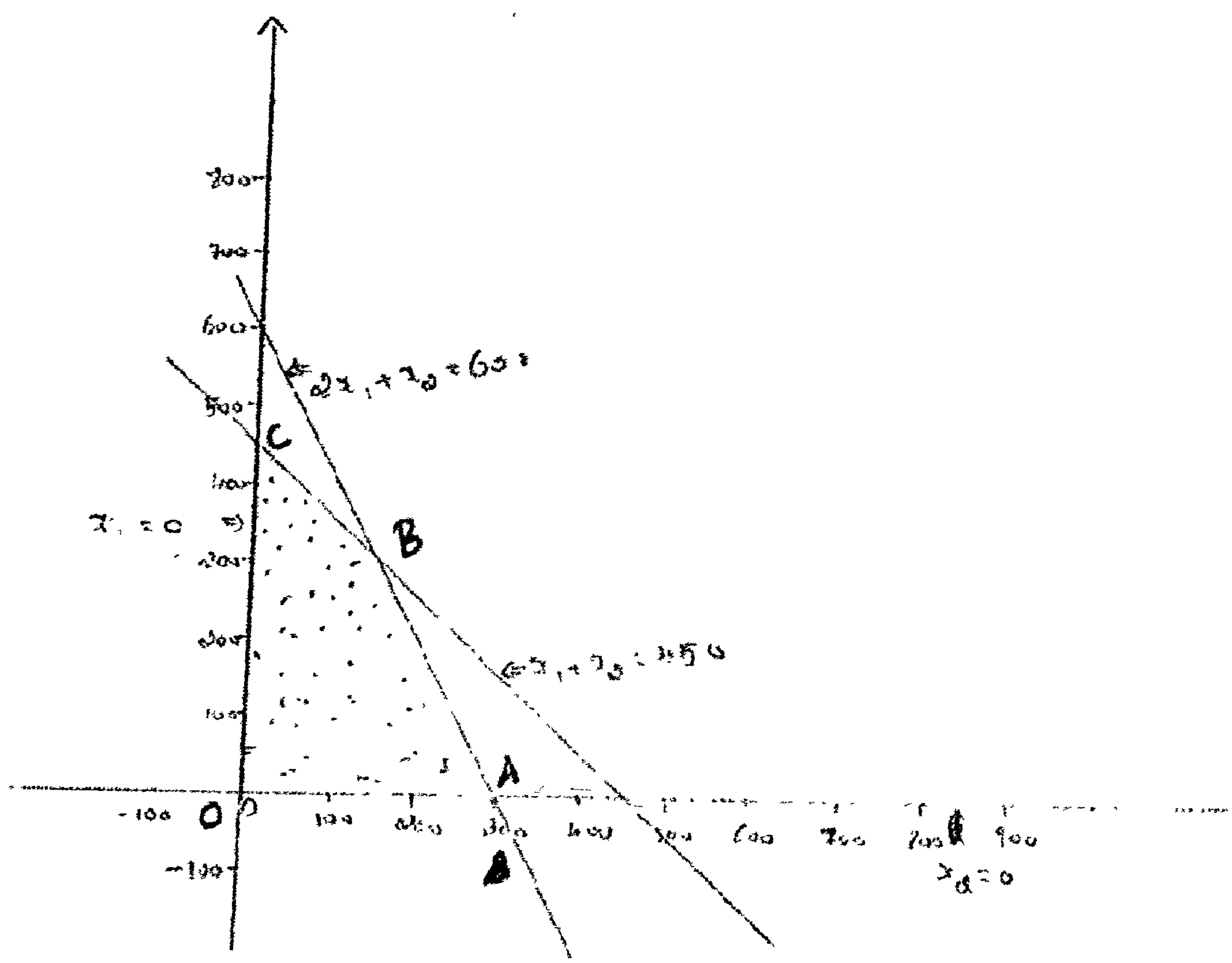
$$\text{and } x_1, x_2 \ge 0.$$

Since the profit from product A is Rs. 3 per unit and from product B is Rs. 4 per unit, the total profit is Rs. $3x_1 + 4x_2$ and our objective is to maximize the profit.

$\therefore$ The complete formulation of the L.P.P is max $Z = 3x_1 + 4x_2$

Subject to	$x_1 + x_2 \le 450$	(i)
	$2x_1 + x_2 \le 600$	(ii)
	and $x_1, x_2 \ge 0$	(iii)

By graphical method, the solution space satisfying the constrains (i), (ii) and meeting the non – negativity restriction (iii) is shown shaded in the following figure.



The solution space is the region OABC. The vertices of this solution spaces are O(0,0), A(300,0), B(150,300) and C(0,450).

The values of Z at these vertices are given by

Extreme Point	(x_1, x_2)	$Z = 3x_1 + 4x_2$
O	(0, 0)	0
A	(300, 0)	900
B	(150, 300)	1650
C	(0, 450)	1800

Since the problems is of maximization type and the minimum value of Z is attained at a single vertex, this problem has a unique optimal solution.

$\therefore$ The optimal solution is

$$\text{Maximum } Z = 1800, \ x_1 = 0, \ x_2 = 450.$$

Example: 2.3.7

Solve the following L.P.P graphically

Maximize $Z = 100x_1 + 40x_2$

Subject to $5x_1 + 2x_2 \leq 1000$

$3x_1 + 2x_2 \leq 900$

$x_1 + 2x_2 \leq 500$

and $x_1, x_2 \geq 0$.

Solution:

By using graphical method, the solution space OABC shown shaded in the following figure.

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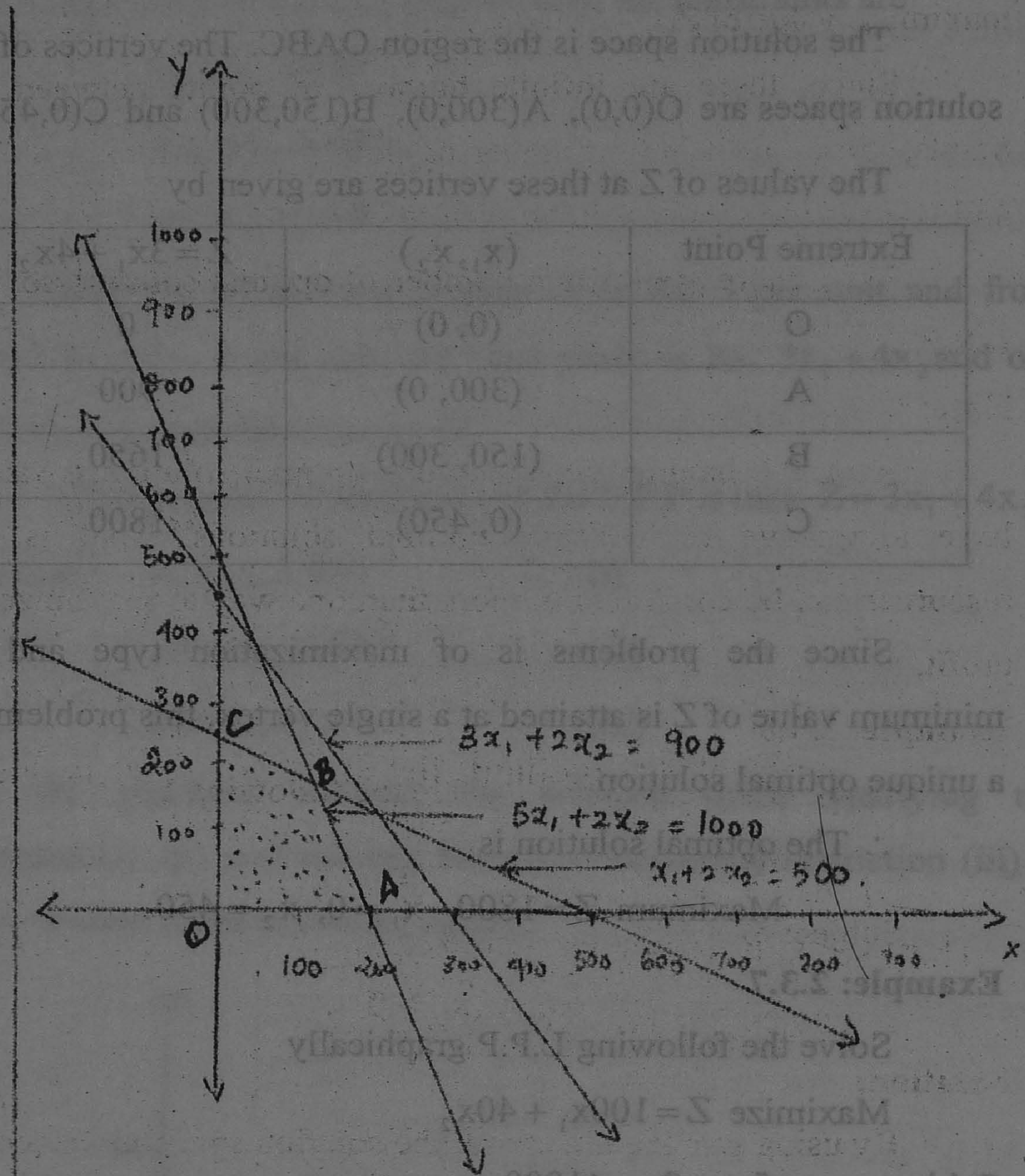

The vertices of this convex region are O(0,0), A(200,0), B(125,187.5) and C(0,250).

The values of Z at these vertices are given by

Extreme Values	(x_1, x_2)	$Z = 100x_1 + 40x_2$
O	(0, 0)	0
A	(200, 0)	20,000
B	(125, 187.5)	20,000
C	(0, 250)	10,000

Here the minimum value of Z occurs at two vertices A and B.

Any point on the line joining A and B will also give the same

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maximum value of Z.

Since, there are infinite number of points between any points, there are infinite number of points on the line joining A and B gives the same maximum value of Z.

Thus, there are infinite number of optimal solution for this L.P.P.

Note:

An L.P.P having more than one optimal solution is said to have alternative or multiple optimal solutions. That is, the resources can be combined in more than one way to maximize the profit.

Example: 2.3.8

Using graphical method, solve the following L.P.P

$$\text{Maximize } Z = 2x_1 + 3x_2$$

Subject to $x_1 - x_2 \le 2$

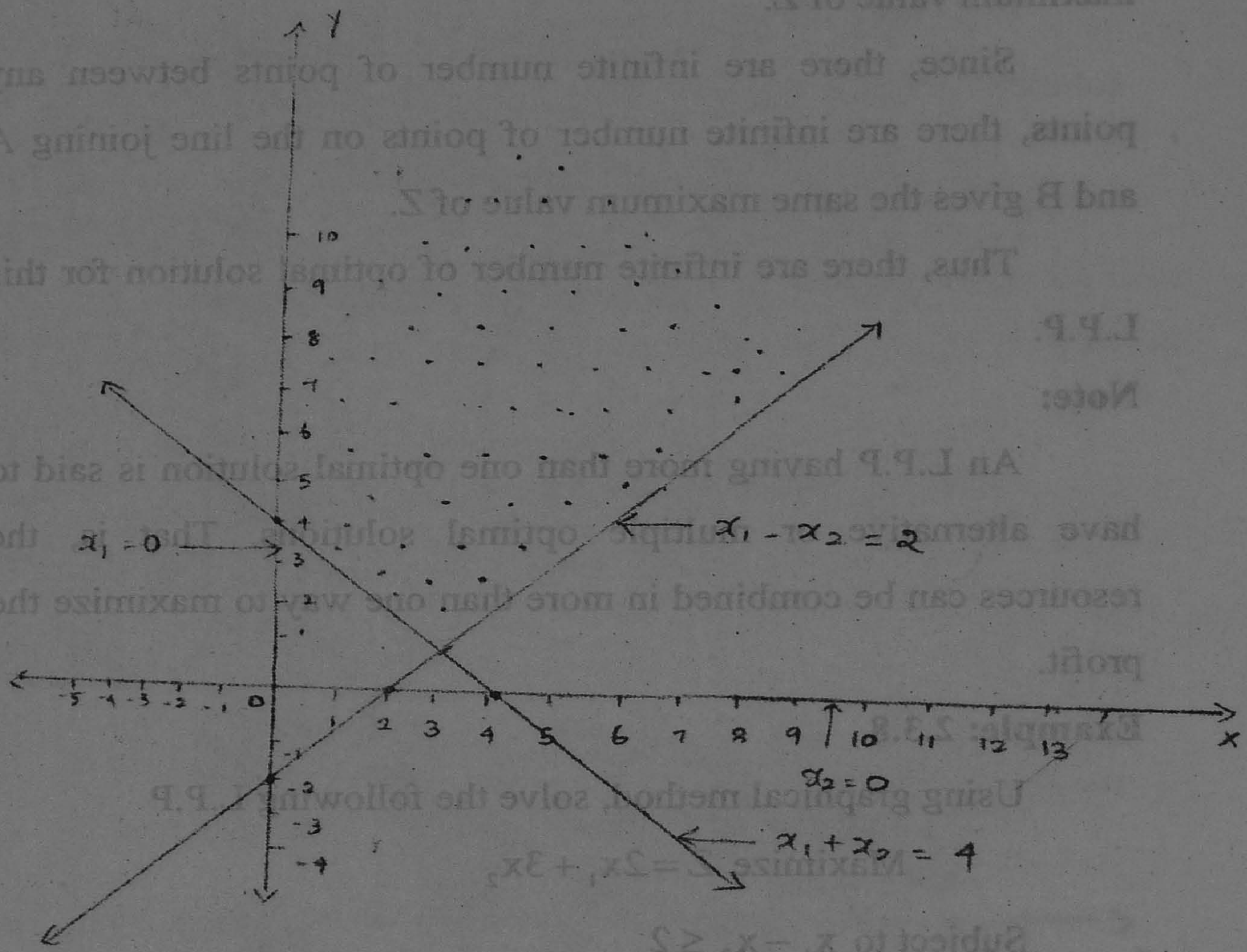
$$x_1 + x_2 \ge 4 \text{ and } x_1, x_2 \ge 0$$

Solution:

By using graphical method, the solution space is shaded the following figure.

Here the solution space is unbounded. The vertices of the feasible region (in the finite plane) are A(3, 1) and B(0, 4).

Value of the objective function $Z = 2x_1 + 3x_2$ at these vertices are Z(A) = 9 and Z(B) = 12.



But these are points in this convex region for which Z will have much higher value. In fact, the maximum values of Z occurs at infinity. Here this problem has an unbounded solution.

Example: 2.3.9

Solve graphically the following L.P.P:

Maximize $Z = 4x_1 + 3x_2$

Subject to $x_1 - x_2 \leq -1$ (1)

$-x_1 + x_2 \leq 0$ (2)

and $x_1, x_2 \geq 0$ (3)

Solution:

Any point satisfying the non – negativity restrictions (iii) lies in the first quadrant only. The two solution spaces, one satisfying (i), and other satisfying (ii) are shown shaded in the following figure.

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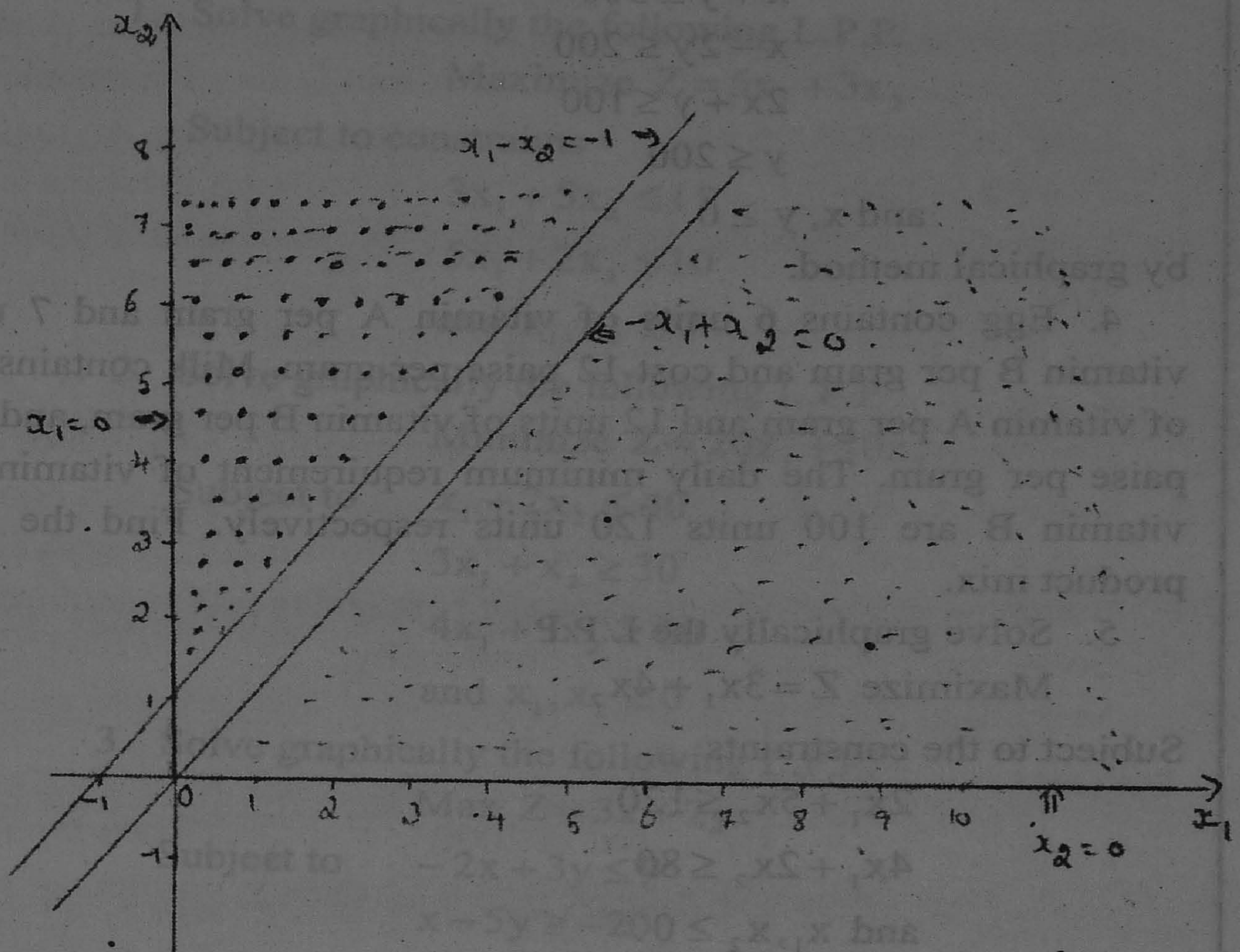

These being no point (x_1, x_2) common to both the shaded regions. That is, we can not find a convex region for this problem. So the problem cannot be solved. Hence the problem have no feasible solution.

Check your progress: 2.1

1. Solve the following problem graphically.

$$\text{Maximize} \quad Z = 2x_1 + x_2$$

$$\text{Subject to} \quad 3x_1 + 2x_2 \le 12.0$$

$$x_1 + 2x_2 \le 7$$

$$x_1 + x_2 \le 5$$

$$\text{and} \quad x_1, x_2 \ge 0$$

2. Use graphical method to Maximize $Z = 6x_1 + 4x_2$

$$\text{Subject to} \quad -2x_1 + x_2 \le 2$$

$$x_1 - x_2 \le 2$$

$$3x_1 + 2x_2 \le 9$$

$$x_1 \ge 0, x_2 \ge 0$$

3. Maximize $Z = x - 3y$

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Subject to the constrains

$$x + y \le 300$$
$$x - 2y \le 200$$
$$2x + y \le 100$$
$$y \le 200$$
$$\text{and } x, y \ge 0$$

by graphical method.

4. Egg contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and cost 12 paise per gram. Milk contains 8 units of vitamin A per gram and 12 units of vitamin B per gram, and cost 20 paise per gram. The daily minimum requirement of vitamin A and vitamin B are 100 units 120 units respectively. Find the optimal product mix.

5. Solve graphically the L.P.P

   Maximize $Z = 3x_1 + 4x_2$

Subject to the constraints

$$2x_1 + 5x_2 \le 120$$
$$4x_1 + 2x_2 \le 80$$
$$\text{and } x_1, x_2 \ge 0.$$

2.4 KEY WORDS

Linear Programming, Mathematical Formulation, Graphical Solution.

2.5 ANSWER TO CHECK YOUR PROGRES QUESTIONS

Check your progress: 2.1

1. Max $Z = 8$, $x_1 = 4$, $x_2 = 0$
2. An infinite number of solution with $Z = 18$.

   (i) $x_1 = \frac{13}{5}$, $x_2 = \frac{3}{5}$

   (ii) $x_1 = \frac{5}{7}$, $x_2 = \frac{24}{7}$ etc.
3. Min $Z = -600$, $x = 0$, $y = 200$.
4. Min $Z = 12x_1 + 20x_2$

   Subject to $6x_1 + 8x_2 \ge 100$

   $7x_1 + 12x_2 \ge 120$

   and $x_1\ x_2 \ge 0$

Also min $Z = 205$, $x_1 = 15, x_2 = 1.25$.

5. Max $Z = 110$, $x_1 = 10$, $x_2 = 20$.

Space for Hints

2.6 MODEL QUESTIONS

1. Solve graphically the following L.P.P:

$$\text{Maximize } Z = 5x_1 + 3x_2$$

Subject to constraints

$$3x_1 + 5x_2 \le 15$$
$$5x_1 + 2x_2 \le 10$$
$$\text{and } x_1, x_2 \ge 0$$

2. Solve graphically the following L.P.P

$$\text{Minimize } Z = 20x_1 + 10x_2$$

Subject to $x_1 + 2x_2 \le 40$

$$3x_1 + x_2 \ge 30$$
$$4x_1 + 3x_2 \ge 60$$
$$\text{and } x_1, x_2 \ge 0$$

3. Solve graphically the following L.P.P.

$$\text{Max } Z = 3x + 2y$$

Subjeet to $-2x + 3y \le 9$

$$x - 5y \ge -20$$
$$\text{and } x, y \ge 0$$

4. Solve graphically the following L.P.P:

$$\text{Minimize } Z = -6x_1 - 4x_2$$

Subject to $2x_1 + 3x_2 \ge 30$

$$3x_1 + 2x_2 \le 24$$
$$x_1 + x_2 \ge 3$$
$$\text{and } x_1, x_2 \ge 0$$

5. Solve graphically the following L.P.P.

$$\text{Maximize } Z = 3x_1 - 2x_2$$

Subject to $x_1 + x_2 \le 1$

$$2x_1 + 2x_2 \ge 4$$
$$\text{and } x_1, x_2 \ge 0$$

6. Solve graphically the following L.P.P.

$$\text{Maximize } Z = x_1 + x_2$$

Subject to $x_1 - x_2 \ge 0$

$$-3x_1 + x_2 \ge 3$$
$$\text{and } x_1, x_2 \ge 0$$

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7. A manufacture of furniture makes two products, chairs and tables. Processing of these products is done on two machines A and B. A chair requires 2 hours on machine A and 6 hours on machine B. A table requires 5 hours on machine A and 6 hours on machine B. There are 16 hours of time per day available on machine A and 30 hours on machine B. Profit gained by the manufacturer from a chair and a table is Rs. 2 and Rs. 10 respectively. What should be the daily production of each of the products.
8. A person requires 10, 12 and 12 units of chemicals A, B and C respectively for his garden. A liquid product contains 5, 2 and 1 units of A, B and C respectively per Jar. A dry product contain 1, 2 and 4 units of A, B and C per carton. How many of each should be purchase in order to minimize the cost and meet the requirement.
9. Solve the following problem graphically,

   Maximize $Z = 40x_1 + 100x_2$

Subject to $12x_1 + 6x_2 \leq 3000$

$$4x_1 + 10x_2 \leq 2000$$

$$2x_1 + 3x_2 \leq 900$$

$$\text{and } x_1, x_2 \geq 0$$

10. A garment manufacturing company can make two products, Prima and Seconda. Each of the products requires time on a cutting machine and a finishing Machine. Relevant data are

	Product	
	Prima	Seconda
Cutting hours (per unit)	2	1
Finishing hours (per unit)	3	3
Unit cost Rs.	128	120
Selling price Rs.	134	129
Maximum sales (units per week)	200	200

The number of cutting hours available per week is 390 and the number of finishing hours available per week is 810. How much should each product be produced in order to maximize the profit?

Space for Hints

UNIT – 3

SIMPLEX METHODS

Introduction:

In the present chapter, we shall introduce and explain the computational procedure of the simplex method. The theory behind the method will be first developed and then the computational techniques explained and illustrated.

Objectives:

After working through examples and exercise of this unit you will be able to

1. Solve the L.P.P using simple method, Big – M method and two phase method.
2. Solve the industry problems using simplex method.

Structure:

3.1 The Simplex Method

3.2 Big – M Method

3.3 Two Phase Method

3.4 Degeneracy and Cycling in L.P.P.

3.5 Application of Simplex Method

3.6 Key words

3.7 Answer to check your progress questions

3.8 Model Questions

3.1 The Simplex Method

The simplex method also called the simplex technique or the simplex algorithm is an iterative procedure for solving a linear programming problem in a finite number of steps. The method provides an algorithm which consists in moving from one vertex of the region of feasible solutions to another in such a manner that the value of the objective function at the succeeding vertex is less (or more, as the case may be) than at the preceding vertex. This procedure of jumbing from one vertex to another is then repeated.

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Since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps or indicates the existence of an unbounded solution.

Definition: 1

Given a system of m linear equations with n variables (m < n). The solution obtained by setting (n – m) variables equal to zero and solving for the remaining m variables is called a basic solution. The m variables are called basic variables and they form the basic solution. The (n – m) variables which are put to zero are called as non – basic variables.

Definition: 2

A basic solution is said to be a non – degenerate basic solution if none of the basic variables is zero.

Definition: 3

A basic solution is said to be a degenerate basic solution if one or more of the basic variables are zero.

Definition: 4

A feasible solution which is also basic is called a basic feasible solution.

Definition: 5

Let X_B be a basic feasible solution to the L.P.P.

Maximize $Z = cX$

Subject to $AX = b$

and $X \geq 0$.

Then the vector $C_B = (C_{B1}, C_{B2}, \ldots\ldots, C_{Bm})$ where C_{Bi} are components of C associated with the basic variables, is called the cost vector associated with the basic feasible solution X_B.

Note: 1

1. If a LPP has a feasible solution, then it also has a basic feasible solution.

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2. There exists only finite number of basic feasible solutions to a L.P.P.
3. Let a L.P.P have a feasible solution. If we drop one of the basic variables and introduce another variable in the basis set, then the new solution obtained is also a basic feasible solution.

Definition: 6

Let $X_B = B^{-1}b$ be a basic feasible solution to the L.P.P.

Maximize $Z = CX$, where $AX = b$ and $X \geq 0$.

Let C_B be the cost vector corresponding to X_B. For each column vector a_j in A which is not a column vector of B, let

$$a_j = \sum_{i=1}^{m} a_{ij} b_i$$

Then the number $Z_j = \sum_{i=1}^{m} C_{Bi} a_{ij}$ is called the evaluation corresponding to a_j and the number $(Z_j - C_j)$ is called the net evaluation corresponding to a_j.

Note: 1

If $(Z_j - C_j) = 0$ for atleast one j for which $a_{ij} > 0,\ i = 1,2,\ldots\ldots,m$; then another basic feasible solution is obtained which gives an unchanged value of the objective function.

Note: 2 (Unbounded solution)

Let there exist a basic feasible solution to a given L.P.P. If for atleast one j, for which $a_{ij} \leq 0 (i = 1,2,3,\ldots\ldots m)$ and $(Z_j - C_j)$ is negative, then there does not exist any optimum solution to this LPP.

Note: 3

A necessary and sufficient condition for a basic feasible

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solution to a LPP to bc an optimum (maximum) is that $(Z_j - C_j) \geq 0$ for all j, for which $a_j \notin B$.

Note: 4

The two fundamental conditions on which the simplex method is based are (i) **Feasibility Condition:** It ensures that if the initial (starting) solution is basic feasible then during computation only basic feasible solution will be obtained. (ii) **Optimality Condition:** It guarantees that only improved solutions will be obtained.

The simplex Algorithm:

For the solution of any LPP by simplex algorithm, the existence of an initial basic feasible solution is always assumed. The steps for the computation of an optimum solution are as follows.

Step: 1

Check whether the objective function is to be maximized or minimized. If it is to be minimized, then convert it into a problem of maximization, by

$$\text{Minimize } Z = -\text{ Maximize } (-Z)$$

Step: 2

Check whether all b_i's are positive. If any of the b_i's are negative, multiply both sides of that constraint by -1 so as to make its right hand side positive.

Step: 3

By introducing slack and/ or surplus variables, convert the inequality constraints into equations and express the given L.P.P into its standard form.

Step: 4

Find an initial basic feasible solution and express the above information conveniently in the following simplex table.

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C_B	Y_B	X_B	x_1	x_2	x_3	.........S_1	S_2	S_3	....
C_{B1}	S_1	b_1	a_{11}	a_{12}	a_{13}	 1	0	0	....
C_{B2}	S_2	b_2	a_{21}	a_{22}	a_{23}	 0	1	0	....
C_{B3}	S_3	b_3	a_{31}	a_{32}	a_{33}	 0	0	1	....
.	.	.	.	.	.	....			....
.	.	.	.	.	.	....			....
.	.	.	.	.	.	....			....
.	.	.		bodymatrix		....	unitmatrix		....
$(Z_j - C_j)$		Z_0	$Z_1 - C_1$	....	....				....

(where C_j- row denotes the coefficients of the variables in the objective function. C_B- column denotes the coefficients of the basic variables in the objective function. Y_B- column denotes the basic variables. X_B- column denotes the values of the basic variables. The coefficients of the non – basic variables in the constraint equations constitute the body matrix which the coefficients of the basic variables constitute the unit matrix. The row $(Z_j - C_j)$ denotes the net evaluations (or) index for each column).

Step: 5

Compute the net evaluations $(Z_j - C_j)$ $(j = 1,2,.......,n)$ by using the relation $Z_j - C_j = C_B \ a_j - c_j$.

Examine the sign of $Z_j - C_j$.

(i) If all $(Z_j - C_j) \geq 0$ then the current basic feasible solution X_B is optimal.

(ii) If atleast one $(Z_j - C_j) < 0$ then the current basic feasible solution is not optimal, go to the next step.

Step: 6 (To find the entering variable)

The entering variable is the non – basic variable corresponding to the most negative value of $(Z_j - C_j)$. Let it be

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x_r for some $j = r$. The entering variable column is known as the key column (or) pivot column which is shown marked with an arrow at the bottom. If more than one variable has the same most negative $(Z_j - C_j)$, any of these variables may be selected arbitrarily as the entering variable.

Step: 7 (To find the leaving variable)

Compute the ratio $\theta = \text{Min}\left\{\frac{X_{Bi}}{a_{ir}},\ a_{ir} > 0\right\}$ (i.e., the ratio between the solution column and the entering variable column by considering only the positive denominators).

(i) If all $a_{ir} \le 0$, then there is an unbounded solution to the given L.P.P.

(ii) If atleast one $a_{ir} > 0$, then the leaving variable is the basic variable corresponding to the minimum ratio θ. If $\theta = \frac{X_{BK}}{a_{kr}}$, then the basic variable x_k leaves the basis. The leaving variable row is called the key row (or) pivot equation and the element at the intersection of the pivot column and pivot row is called the pivot element or key element (or) leading element.

Step: 8

Drop the leaving variable and introduce the entering variable along with its associated value under C_B column. Convert the pivot element to unity by dividing the pivot equation by the pivot element and all other elements in its column to zero by making use of

(i) New pivot equation = Old pivot equation $\div$ Pivot element

(ii) New equation(all other rows including $(Z_j - C_j)$ = Old equation $-$

$$\begin{pmatrix}\text{Corresponding}\\ \text{Column}\\ \text{coefficient}\end{pmatrix} - \begin{pmatrix}\text{New pivot}\\ \text{equation}\end{pmatrix}$$

Step: 9

Go to step (5) and repeat the procedure until either an optimum solution is obtained or there is an indication of an unbounded solution.

Note: 1

For maximization problems:

(i) If all $(Z_j - C_j) \leq 0$, then the current basic feasible solution is optimal.

(ii) If atleast one $(Z_j - C_j) < 0$, then the current basic feasible solution is not optimal.

(iii) The entering variable is the non – basic variable corresponding to the most negative value of $(Z_j - C_j)$.

Note: 2

For minimization problems:

(i) If all $(Z_j - C_j) > 0$, then the current basic feasible solution is not optimal.

(ii) If atleast one $(Z_j - C_j) > 0$, then the current basic feasible solution is not optimal.

(iii) The entering variable is the non – basic variable corresponding to the most positive value of $(Z_j - C_j)$.

Note: 3

For both maximization and minimization problems, the leaving variable is the basic variable corresponding to the minimum ratio θ.

Example: 3.1.1

Use simplex method to solve the L.P.P

Maximize $Z = 4x_1 + 10x_2$

Subject to

$$2x_1 + x_2 \leq 50$$

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$$2x_1 + 5x_2 \le 100$$

$$2x_1 + 3x_2 \le 90 \text{ and } x_1, x_2 \ge 0.$$

Solution:

By introducing the slack variables S_1, S_2 and S_3 the problem in standard form becomes

Maximize $Z = 4x_1 + 10x_2 + 0.S_1 + 0.S_2 + 0.S_3$

Subject to

$$2x_1 + x_2 + S_1 + 0.S_2 + 0.S_3 = 50$$

$$2x_1 + 5x_2 + 0.S_1 + S_2 + 0.S_3 = 100$$

$$2x_1 + 3x_2 + 0.S_1 + 0.S_2 + S_3 = 90$$

and $x_1, x_2, S_1, S_2, S_3 \ge 0$

Since there are 3 equations with 5 variables, the initial basic feasible solution is obtained by equating $(5-3) = 2$ variables to zero.

$\therefore$ The initial basic feasible solution is $S_1 = 50,\ S_2 = 100,\ S_3 = 90$.

$(x_1 = 0,\ x_2 = 0$, non – basic),

The initial simplex table is given by

C_B	Y_B	X_B	X_1	X_2	S_1	S_2	S_3	$\theta = \min\left(\frac{X_{Bi}}{a_{ir}}\right)$
0	S_1	50	2	1	1	0	0	50
0	S_2	100	2	(5)	0	1	0	20*
0	S_3	90	2	3	0	0	1	30
Z_j	$-C_j$	0	-4	-10	0	0	0	

Here the net evaluations are calculated as

$$Z_j - C_j = C_B a_j - C_j,$$

$Z_1 - C_1 = C_B\, a_1 - C_1 = (0 \quad 0 \quad 0)[2 \quad 2 \quad 2]^T - 4$ [where T denotes transpose]

$$= (0 \quad 0 \quad 0)\begin{pmatrix}2\\2\\2\end{pmatrix} - 4 = -4.$$

$$Z_2 - C_2 = C_B\, a_2 - C_2 = (0 \quad 0 \quad 0)[1 \quad 5 \quad 3]^T - 10 = -10$$

$$Z_3 - C_3 = C_B\, a_3 - C_3 = (0 \quad 0 \quad 0)[1 \quad 0 \quad 0]^T - 10 = 0$$

$$Z_4 - C_4 = C_B\, a_4 - C_4 = (0 \quad 0 \quad 0)[0 \quad 1 \quad 0]^T - 0 = 0$$

$$Z_5 - C_5 = C_B\, a_5 - C_5 = (0 \quad 0 \quad 0)[0 \quad 0 \quad 1]^T - 0 = 0$$

Since these are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

To find the entering variable

Since $(Z_2 - C_2) = -10$ is the most negative, the corresponding non – basic variable x_2 enters the basis. The column corresponding to this x_2 is called the key column or pivot column.

To find the leaving variable:

Find the ratio $\theta = \min\left\{\dfrac{X_{Bi}}{a_{ir}},\ a_{ir} > 0\right\}$

$$= \min\left\{\frac{X_{Bi}}{a_{i2}},\ a_{i2} > 0\right\}$$

$$\theta = \min\left\{\frac{50}{1},\ \frac{100}{5},\ \frac{90}{3}\right\}$$

$= \min\{50,\ 20,\ 30\} = 20$ which corresponds to S_2 the minimum ratio $\theta = 20$. The leaving variable row is called the pivot – row or key row or pivot equation and 5 is the pivot element. Now pivot equation = old pivot equation $\div$ pivot element.

$$= (100 \quad 2 \quad 5 \quad 0 \quad 1 \quad 0) \div 5$$

$$= 20 \quad \frac{2}{5} \quad 1 \quad 0 \quad \frac{1}{5} \quad 0$$

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Now S_1 equation = old S_1 equation - $\begin{pmatrix}\text{Corresponding}\\ \text{column}\\ \text{coefficient}\end{pmatrix} - \begin{pmatrix}\text{New}\\ \text{pivot}\\ \text{equation}\end{pmatrix}$

$$\begin{array}{rrrrrrr} = & 50 & 2 & 1 & 1 & 0 & 0 \\ (-) & 20 & \frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 \\ \hline = & 30 & \frac{8}{5} & 0 & 1 & -\frac{1}{5} & 0 \\ \hline \end{array}$$

Now S_3 equation

$$\begin{array}{rrrrrrr} = & 90 & \cdot 2 & 3 & 0 & 0 & 1 \\ (-) & 60 & \frac{6}{5} & 3 & 0 & \frac{3}{5} & 0 \\ \hline = & 30 & \frac{4}{5} & 0 & 0 & -\frac{3}{5} & 1 \\ \hline \end{array}$$

Now $(Z_j - C_j)$ eqn.

$$\begin{array}{rrrrrrr} = & 0 & -4 & -10 & 0 & 0 & 0 \\ (-) & -200 & \frac{-20}{5} & -10 & 0 & \frac{-10}{5} & 0 \\ \hline = & 200 & 0 & 0 & 0 & 2 & 0 \\ \hline \end{array}$$

$\therefore$ The improved basic feasible solution is given in the following simplex table.

First Iteration:

	C_j		4	10	0	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3
0	S_1	30	$\frac{8}{5}$	0	1	$\frac{-1}{5}$	0
10	x_2	20	$\frac{2}{5}$	1	0	$\frac{1}{5}$	0
0	S_3	30	$\frac{4}{5}$	0	0	$\frac{-3}{5}$	1
Z_j	$-C_j$	200	0	0	0	2	0

Since all $Z_j - C_j \geq 0$ the current basic feasible solution is optimal.

$\therefore$ The optimal solution is max $Z = 200,\ x_1 = 0,\ x_2 = 20$

Example: 3.1.2

Solve the following

Maximize $\quad Z = 15x_1 + 6x_2 + 9x_3 + 2x_4$

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Subject to $2x_1 + x_2 + 5x_3 + 6x_4 \leq 20$

$3x_1 + x_2 + 3x_3 + 25x_4 \leq 24$

$7x_1 + x_4 \leq 70$

$x_1, x_2, x_3, x_4 \geq 0.$

Solution:

By introducing non – negative slack variables S_1, S_2 and S_3, the standard form of L.P.P becomes

Maximize $Z = 15x_1 + 6x_2 + 9x_3 + 2x_4 + 0.S_1 + 0.S_2 + 0.S_3$

Subject to $2x_1 + x_2 + 5x_3 + 6x_4 + S_1 + 0.S_2 + 0.S_3 = 20$

$3x_1 + x_2 + 3x_3 + 25x_4 + 0.S_1 + S_2 + 0.S_3 = 24$

$7x_1 + 0x_2 + 0x_3 + x_4 + 0.S_1 + 0.S_2 + S_3 = 70$

$x_1, x_2, x_3, x_4, S_1, S_2, S_3 \geq 0.$

$\therefore$ The initial basic feasible solution is $S_1 = 20$, $S_2 = 24$, $S_3 = 70$. ($x_1 = x_2 = x_3 = x_4 = 0$ non basic). The initial simplex table is given by

Initial Iteration:

	C_j		15	6	9	2	0	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	S_1	S_2	S_3	θ
0	S_1	20	2	1	5	6	1	0	0	$\frac{20}{2} = 10$
0	S_2	24	(3)	1	3	25	0	1	0	$\frac{24}{3} = 8*$
0	S_3	70	7	0	0	1	0	0	1	$\frac{70}{7} = 10$
Z_j	$-C_j$	0	−15	−6	−9	−2	0	0	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non – basic variable x_1 enters into the basis and the basic variable S_2 leaves the basis.

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First Iteration:

		C_j	15	6	9	2	0	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	S_1	S_2	S_3	θ
0	S_1	4	0	$\frac{1}{3}$	3	$\frac{-32}{3}$	1	$\frac{-2}{3}$	0	12*
15	x_1	8	1	$\frac{1}{3}$	1	$\frac{25}{3}$	0	$\frac{1}{3}$	0	24
0	S_3	14	0	$\frac{-7}{3}$	-7	$\frac{-172}{3}$	0	$\frac{-7}{3}$	1	–
Z_j	$-C_j$	120	0	-1	6	123	0	5	0	

Since $(Z_2 - C_2) = -1 < 0$, the current basic feasible solution is not optimal. The non – basic variable x_2 enters into the basis and the basic variable S_1 leaves the basis.

Second Iteration:

		C_j	(15	6	9	2	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	S_1	S_2	S_3
6	x_2	12	0	1	9	-32	3	-2	0
15	x_1	4	1	0	-2	$\frac{57}{3}$	-1	1	0
0	S_3	42	0	0	14	-132	7	-7	1
Z_j	$-C_j$	132	0	0	15	91	3	3	0

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is given by

$$\text{Max } Z = 132,\ x_1 = 4,\ x_2 = 12,\ x_3 = 0 \text{ and } x_4 = 0.$$

Example: 3.1.3

Solve the following L.P.P by simplex method

$$\text{Minimize } Z = 8x_1 - 2x_2$$

Subject to $-4x_1 + 2x_2 \leq 1$

$$5x_1 - 4x_2 \leq 3$$

and $x_1, x_2 \geq 0$.

Space for Hints

Solution:

Since the given objective function is of minimization type, we shall convert it into a maximization type as follows.

Maximize (-Z) = Maximize $Z^* = -8x_1 + 2x_2$

Subject to

$$-4x_1 + 2x_2 \le 1$$

$$5x_1 - 4x_2 \le 3$$

$$x_1,\ x_2 \ge 0.$$

By introducing non – negative slack variable S_1, S_2, the standard form of L.P.P becomes

$$\text{Maximize } Z^* = -8x_1 + 2x_2 + 0.S_1 + 0.S_2$$

Subject to the constraints

$$-4x_1 + 2x_2 + S_1 + 0.S_2 = 1$$

$$5x_1 - 4x_2 + 0.S_1 + S_2 = 3$$

$$\text{and } x_1,\ x_2,\ S_1,\ S_2 \ge 0.$$

$\therefore$ The initial basic feasible solution is given by $S_1 = 1,\ S_2 = 3,\ (x_1 = x_2 = 0,\ \text{non-basic})$

Initial Iteration:

		C_j	-8	2	0	0	
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	θ
0	S_1	1	-4	2	1	0	$\frac{1}{2}$*
0	S_2	3	5	-4	0	1	–
Z_j^*	$-C_j$	0	8	-2	0	0	

Since $(Z_2^* - C_2) = -2 < 0$, the current basic feasible solution is not optimal.

The non – basic variable x_2 enters the basis and the basic variable S_1 leaves the basis.

Space for Hints

First Iteration:

		C_j	-8	2	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2
2	x_2	$\frac{1}{2}$	-2	1	$\frac{1}{2}$	0
0	S_2	5	-3	0	2	1
Z_j^*	$-C_j$	1	4	0	1	0

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is given by

Maximize $Z^* = 1, \; x_1 = 0, \; x_2 = \frac{1}{2}$.

But Minimize Z = - Maximize (-Z) = - Maximize Z*

= -1

$\therefore$ Min Z = -1, $x_1 = 0, \; x_2 = \frac{1}{2}$

Aliter:

The above problem can be solved without converting the objective function in the maximization type.

Given Minimize $Z = 8x_1 - 2x_2$

Subject to the constraints

$$-4x_1 + 2x_2 \leq 1$$

$$5x_1 - 4x_2 \leq 3$$

$$x_1, x_2 \geq 0.$$

$\therefore$ By introducing the non – negative slack variable S_1, S_2 the L.P.P becomes

Minimize $Z = 8x_1 - 2x_2 + 0.S_1 + 0.S_2$

Subject to the constraints

$$-4x_1 + 2x_2 + S_1 + 0.S_2 = 1$$

$$5x_1 - 4x_2 + 0.S_1 + S_2 = 3$$

$x_1, x_2, S_1, S_2 \geq 0.$

The initial basic feasible solution is given by $S_1 = 1,\ S_2 = 3$(basic)

($x_1 = x_2 = 0$, non - basic).

Initial Iteration:

C_B	Y_B	C_j / X_B	x_1 (8)	x_2 (−2)	S_1 (0)	S_2 (0)	θ
0	S_1	1	−4	2	1	0	$\frac{1}{2}$*
0	S_2	3	5	−4	0	1	−
Z_j	$-C_j$	0	−8	2	0	0	

Since $(Z_2 - C_2) = 2 > 0$, the current basic feasible solution is not optimal.

To find the entering variable:

Since $(Z_2 - C_2) = 2$ is most positive, the corresponding non – basic variable x_2 enters into the basis.

To find the leaving variable:

The leaving variable is the basic variable S_1 corresponding to the minimum ratio $\theta = \frac{1}{2}$.

First Iteration:

C_B	Y_B	C_j / X_B	x_1 (8)	x_2 (−2)	S_1 (0)	S_2 (0)
−2	x_2	1/2	−2	1	1/2	0
0	S_2	5	−3	0	2	1
Z_j	$-C_j$	−1	−4	0	−1	0

Since all $(Z_j - C_j) \leq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Min Z = -1, $x_1 = 0, x_2 = \frac{1}{2}$.

Space for Hints

Example: 3.1.4

Use simplex method to

$$\text{Min } Z = x_2 - 3x_3 + 2x_5$$

Subject to

$$3x_2 - x_3 + 2x_5 \le 7$$

$$-2x_2 + 4x_3 \le 12$$

$$-4x_2 + 3x_3 + 8x_5 \le 10$$

and $x_2, x_3, x_5 \ge 0$

Solution:

Since the given objective function is minimization type. We shall convert it into a maximization type as follows.

Maximize (-Z) = Maximize $Z^* = -x_2 + 3x_3 - 2x_5$

Subject to $3x_2 - x_3 + 2x_5 \le 7$

$$-2x_2 + 4x_3 \le 12$$

$$-4x_2 + 3x_3 + 8x_5 \le 10$$

$$x_2, x_3, x_5 \ge 0.$$

By introducing non – negative slack variables S_1, S_2 and S_3 the standard form of the L.P.P becomes

Maximize $Z^* = -x_2 + 3x_3 - 2x_5 + 0.S_1 + 0.S_2 + 0.S_3$

Subject to the constraints

$$3x_2 - x_3 + 2x_5 + S_1 + 0.S_2 + 0.S_3 = 7$$

$$-2x_2 + 4x_3 + 0.x_5 + 0.S_1 + S_2 + 0.S_3 = 12$$

$$-4x_2 + 3x_3 + 8x_5 + 0.S_1 + 0.S_2 + S_3 = 10$$

and $x_2, x_3, x_5, S_1, S_2, S_3 \ge 0$.

$\therefore$ The initial basic feasible solution is given by

$S_1 = 7,\ S_2 = 12,\ S_3 = 10 (x_2 = x_3 = x_5 = 0, \text{non – basic})$

Space for Hints

Initial Iteration:

		C_j	(−1	3	−2	0	0	0)	
C_B	Y_B	X_B	x_2	x_3	x_5	S_1	S_2	S_3	θ
0	S_1	7	3	−1	2	1	0	0	−
0	S_2	12	−2	(4)	0	0	1	0	$\frac{12}{4}=3^*$
0	S_3	10	−4	3	8	0	0	1	$\frac{10}{3}=3.33$
Z_j^*	$-C_j$	0	1	−3	2	0	0	1	

Since $(Z_2^* - C_2) < 0$, the current basic feasible solution is not optimal.

The non – basic variable x_3 enters into the basis and the basic variable S_2 leaves the basis.

First Iteration:

		C_j	(−1	3	−2	0	0	0)	
C_B	Y_B	X_B	x_2	x_3	x_5	S_1	S_2	S_3	θ
0	S_1	10	5/2	0	2	1	1/4	0	$\frac{20}{5}=4^*$
3	x_3	3	−1/2	1	0	0	1/4	0	−
0	S_3	1	−5/2	0	8	0	−3/4	1	−
Z_j^*	$-C_j$	9	−1/2	0	2	0	3/4	0	

Since $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

The non – basic variable x_2 enters into the basis and the basic variable S_1 leaves the basis.

Second Iteration:

		C_j	(−1	3	2	0	0)	
C_B	Y_B	X_B	x_2	x_3	x_5	S_1	S_2	S_3
−1	x_2	4	1	0	4/5	2/5	1/10	0
3	x_3	5	0	1	2/5	1/5	3/10	0
0	S_3	11	0	0	10	1	−1/2	1
Z_j^*	$-C_j$	11	0	0	12/5	1/5	4/5	0

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Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is given by

Maximize $Z^* = 11,\ x_2 = 4,\ x_3 = 5, x_5 = 0$.

But

Minimize Z = - Maximize $Z^* = -11$

$\therefore$ Min Z = -11, $x_2 = 4,\ x_3 = 5,\ x_5 = 0$.

Check your progress 3.1

1. Using simplex method, find non – negative values of x_1, x_2, x_3 which maximize $Z = x_1 + 4x_2 + 5x_3$ subject to the constraints $3x_1 + 6x_2 + 3x_3 \leq 22,\ x_1 + 2x_2 + 3x_3 \leq 14$ and $3x_1 + 2x_2 \leq 14$.
2. Apply the simplex method to solve the problem.

   Maximize $Z = 100x_1 + 200x_2 + 50x_3$

   Subject to $5x_1 + 5x_2 + 10x_3 \leq 1000$

   $10x_1 + 8x_2 + 5x_3 \leq 2000$

   $10x_1 + 5x_2 \leq 500$

   $x_1, x_2, x_3 \geq 0$.
3. Using simplex method, find the non – negative values of x_1 and x_2 which maximize the objective function $Z = 2x_1 + x_2$ subject to the constraints

   $x_1 - 2x_2 \leq 1$

   $x_1 + x_2 \leq 6$

   $x_1 + 2x_2 \leq 10$

   and $x_1 - x_2 \leq 2$
4. Solve the following LPP using simplex method:

   Maximize $Z = x_1 + x_2 + 3x_3$

   Subject to $3x_1 + 2x_2 + x_3 \leq 3$

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$$2x_1 + x_2 + 2x_3 \leq 2$$

$$x_1, x_2, x_3 \geq 0.$$

3.2 BIG – M Method

To solve a LPP by simplex method, we have to start with the initial basic feasible solution and construct the initial simplex table. In the previous problems, we see that the slack variables readily provided the initial basic feasible solution. However, in some problems, the slack variables cannot provide the initial basic feasible solution. In these problems atleast one of the constraints is of < or ≥ type. To solve such linear programming problems, there are two (closely related) methods available.

(i) The "**Big M – method**" or "**M – technique**" or the "**Method of penalties**".

(ii) The "**Two phase**" method

The Big M – Method:

Step: 1

Express the linear programming problem in the standard form by introducing slack and/or surplus variables, if necessary.

Step: 2

Introduce the non – negative artificial variables $R_1, R_2, \ldots$ to the left hand side of all the constraints of ≥ or = type. The purpose of introducing artificial variables is just to obtain an initial basic feasible solution. However, addition of these artificial variables causes violation of the corresponding constraints. Therefore we would like to get rid of these variables and would not allow them to appear in the final solution. To achieve this we assign a very large penalty (-M for maximization problems and +M for minimization problems) as the coefficients of the artificial variables in the objective function.

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Step: 3

Solve the modified linear programming problem by simplex method.

While making iterations, using simplex method, one of the following three cases may arise:

(i) If no artificial variable remains in the basis and the optimality condition is satisfied, then the current solution is an optimal basic feasible solution.

(ii) If atleast one artificial variable appears in the basis at zero level (with zero value of X_B column) and the optimality condition is satisfied, then the current solution is an optimal basic feasible (though degenerated) solution.

(iii) If atleast one artificial variable appears in the basis at non – zero level (with positive value in X_B column) and the optimality condition is satisfied, then the original problem has no feasible solution. The solution satisfies the constraints but does not optimize the objective function since it contains a very large penalty M and is called **pseudo – optimal solution.**

Note:

While applying simplex method, whenever an artificial variable happens to leave the basis, we drop that artificial variable and omit all the entries corresponding to its column from the simplex table.

Example: 3.2.1

Solve the following LPP by simplex method:

Maximize $Z = 3x_1 + 2x_2$

Subject to $2x_1 + x_2 \leq 2$

$3x_1 + 4x_2 \geq 12$

and $x_1, x_2 \geq 0$.

Solution:

By introducing the non – negative slack variable S_1 and surplus variable S_2, the standard form of the LPP becomes

Maximize $Z = 3x_1 + 2x_2 + 0.S_1 + 0.S_2$

Subject to $\quad 2x_1 + x_2 + S_1 + 0.S_2 = 2$

$3x_1 + 4x_2 + 0.S_1 - S_2 = 12$

$x_1, x_2, S_1, S_2 \geq 0.$

But this will not yield a basic feasible solution. To get the basic feasible solution add the artificial variable R, to the left hand side of the constraint equation which does not possess the slack variable and assign – M to the artificial variable in the objective function. The LPP becomes

Max $\quad Z = 3x_1 + 2x_2 + 0.S_1 + 0.S_2 - MR_1$

Subject to $\quad 2x_1 + x_2 + S_1 + 0.S_2 = 2$

$3x_1 + 4x_2 + 0.S_1 - S_2 + R_1 = 12$

$x_1, x_2, S_1, S_2, R_1 \geq 0$

The initial basic solution is given by

$S_1 = 2,\ R_1 = 12$(basic) $(x_1 = x_2 = S_2 = 0,$ non – basic).

Initial Iteration:

		C_j	(3	2	0	0	−M)	
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	R_1	θ
0	S_1	2	2	(1)	1	0	0	2/1=2
−M	R_1	12	3	4	0	−1	1	12/4=3
Z_j	$-C_j$	−12M	−M−3	−4M−2	0	M	0	

Since there are some $(Z_j - C_j) < 0$, the current basis feasible solution is not optimal.

Space for Hints

Space for Hints

The non – basic variable x_2 enters into the basis and the basic variable S_1 leaves the basis.

First Iteration:

		C_j	(3	2	0	0	$-M$)
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	R_1
2	x_2	2	2	1	1	0	0
$-M$	R_1	4	-5	0	-4	-1	1
Z_j	$-C_j$	$-4M+4$	$5M+1$	0	$4M+2$	M	0

Since all $(Z_j - C_j) \geq 0$ and an artificial variable R_1 appears in the basis at non – zero level, the given LPP does not possess any feasible solution. But the LPP possess a pseudo optimal solution.

Example: 3.2.2

Use Big – M method to solve

Minimize $\quad Z = 4x_1 + 3x_2$

Subject to $\quad -3x_1 + 2x_2 \geq 10$

$x_1 + x_2 \geq 6$

and $x_1, x_2 \geq 0$

Solution:

Given Min $\quad Z = 4x_1 + 3x_2$

Subject to $\quad 2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6$

$x_1, x_2 \geq 0$

That is Max $\quad Z^* = -4x_1 - 3x_2$

Subject to $\quad 2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6$

$x_1, x_2 \geq 0$

By introducing the non – negative slack, surplus and artificial variables, the standard form of LPP becomes

$$\text{Max } Z^* = -4x_1 - 3x_2 + 0.S_1 + 0.S_2 + 0.S_3 - MR_1 - MR_2$$

Subject to $\quad 2x_1 + x_2 - S_1 + 0.S_2 + 0.S_3 + R_1 = 10$

$$-3x_1 + 2x_2 + 0.S_1 + S_2 + 0.S_3 = 6$$

$$x_1 + x_2 + 0.S_1 + 0.S_2 - S_3 + R_2 = 6$$

and $\quad x_1, x_2, x_3, S_1, S_2, S_3, R_1, R_2 \geq 0$

The initial basic feasible solution is given by $R_1 = 10,\ S_2 = 6,\ R_2 = 6$ (basic) $(x_1 = x_2 = S_1 = S_3 = 0,$ non – basic)

Initial Iteration:

	C_j		(–4	–3	0	0	0	–M	–M)	
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	R_1	R_2	θ
–M	R_1	10	(2)	1	–1	0	0	1	0	10/2=5
0	S_2	6	–3	2	0	1	0	0	0	–
–M	R_2	6	1	1	0	0	–1	0	1	6/1=6
Z_j^*	$-C_j$	–16M	–3M+4	–2M+3	M	0	M	0	0	

Since these are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

The non – basic variable x_1 enters into the basis and the basic variable R_1 leaves the basis.

First Iteration:

	C_j		(–4	–3	0	0	0	–M)	
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	R_2	θ
–4	x_1	5	1	1/2	–1/2	0	0	0	10
0	S_2	21	0	7/2	–3/2	1	0	0	42/7
–M	R_2	1	0	(1/2)	1/2	0	–1	1	2
Z_j^*	$-C_j$	–M–20	0	$\frac{-M+2}{2}$	$\frac{-M+4}{2}$	0	M	0	

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Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

The non – basic variable x_2 enters into the basis and the basic variable R_2 leaves the basis.

Second Iteration:

C_j (−4 −3 0 0 0)

C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3
−4	x_1	4	1	0	−1	0	1
0	S_2	14	0	0	−5	1	7
−3	x_2	2	0	1	1	0	−2
Z_j^*	$-C_j$	−22	0	0	1	0	2

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal

$$\therefore \text{Max } Z^* = -22,\ x_1 = 4,\ x_2 = 2$$

But Min $Z = -\text{Max}(-Z) = -\text{Max}Z^* = -(-22) = 22$

$\therefore$ The optimal solution is $\text{Min } Z = 22,\ x_1 = 4,\ x_2 = 2$

Example: 3.2.3

Use penalty method to

Maximize $z = 2x_1 + x_2 + x_3$

Subject to $4x_1 + 6x_2 + 3x_3 \leq 8$

$3x_1 - 6x_2 - 4x_3 \leq 1$

$2x_1 + 3x_2 - 5x_3 \geq 4$

and $x_1, x_2, x_3 \geq 0$

Solution:

By introducing the non-negative slack, surplus and artificial variables, the standard form of the LPP becomes.

Max $z = 2x_1 + x_2 + x_3 + 0.S_1 + 0.S_2 + 0.S_3 - MR_1,$

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Subject to $4x_1 + 6x_2 + 3x_3 + S_1 + 0.S_2 + 0.S_3 = 8$

$3x_1 - 6x_2 - 4x_3 + 0.S_1 + 0.S_2 + 0.S_3 = 1$

$2x_1 + 3x_2 - 5x_3 + 0.S_1 + 0.S_2 - S_3 + R_1 = 4$

and $x_1, x_2, x_3, s_1, s_2, s_3, R_1 \geq 0.$

(Here: S_1, S_2 – slack, S_3 – surplus, R_1,-artificial)

The initial basic feasible solution is given by

$S_1 = 8, S_2 = 1, R_1 = 4$ (basic) ($x_1 = x_2 = S_3 = 0$, non–basic)

Initial Iteration:

		C_j	(2	1	1	0	0	0	–M)	
C_B	Y_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	R_1	Q
0	S_1	8	4	6	3	1	0	0	0	$\frac{8}{6}=1.33$
0	S_2	1	3	–6	–4	0	1	0	0	–
–M	R_1	4	2	(3)	–5	0	0	–1	1	$\frac{4}{3}=1.33$
Z_j	$-C_j$	–4M	–2M–2	–3M–1	5M–1	0	0	M	0	

Since there are some $(z_j - c_j) < 0$, the current base feasible solution is not optimal.

The non-basic variable x_2 enters into the basis and the basic variable R_1 leaves the basis.

First iteration:

		C_j	C_2	1	1	0	0	0	
C_B	Y_B	Y_B	x_1	x_2	x_3	S_1	S_2	S_3	θ
0	S_1	0	0	0	(13)	1	0	2	0
0	S_1	9	7	0	–14	0	1	–2	–
1	x_2	$\frac{4}{3}$	$\frac{2}{3}$	1	$\frac{-5}{3}$	0	0	$\frac{-1}{3}$	–
$Z_j -$	C_j	$\frac{4}{3}$	$\frac{-4}{3}$	0	$\frac{-8}{3}$	0	0	$\frac{-1}{13}$	

Since there are some $(Z_j - C_j) < 0$, The current basic

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feasible solution is not optimal. The non – basic variable x_3 enters into the basis and the basic variable S_1 leaves the basis.

Second Iteration:

C_B	Y_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	θ
		C_j	2	1	1	0	0	0	
1	x_3	0	0	0	1	$\frac{1}{13}$	0	$\frac{2}{13}$	–
0	S_2	9	(7)	0	0	$\frac{14}{13}$	1	$\frac{2}{13}$	$\frac{9}{7}$
1	x_2	$\frac{4}{3}$	$\frac{2}{3}$	1	0	$\frac{5}{39}$	0	$\frac{-1}{13}$	2
$Z_j -$	C_j	$\frac{4}{3}$	$\frac{-4}{3}$	0	0	$\frac{8}{39}$	0	$\frac{1}{13}$	

Since there are some $(Z_j - C_j) < 0,$ the current basic feasible solution is not optimal

The non-basic variable X_3 enters into the basis and the basic variable S_2 leaves the basis.

Third Iteration:

C_B	Y_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
		C_j	2	1	1	0	0	
1	x_3	0	0	0	1	$\frac{1}{13}$	0	$\frac{2}{13}$
2	x_1	$\frac{9}{7}$	1	0	0	$\frac{1}{13}$	$\frac{1}{7}$	$\frac{2}{91}$
1	x_2	$\frac{10}{21}$	0	1	0	$\frac{1}{39}$	$\frac{-2}{21}$	$\frac{-25}{273}$
$Z_j -$	C_j	$\frac{64}{21}$	0	0	0	$\frac{16}{39}$	$\frac{4}{21}$	$\frac{29}{273}$

Since all $(Z_j - C_j) \geq 0,$ the current basic feasible solution is optimal.

The optimal solution is

$$\text{Max } Z = \frac{64}{21}$$

$$x_1 = \frac{9}{7},\ x_2 = \frac{10}{21},\ x_3 = 0$$

Check your progress 3.2

1) Solve the following LPP Minimize $Z = 4x_1 + 2x_2$

   Subject to $3x_1 + x_2 \geq 27$

   $x_1 + x_2 \geq 21$

   $x_1 + 2x_2 \geq 30$

   $x_1, x_2 \geq 0.$

2) Solve the following LPP: Minimize $Z = 2x_1 + 3x_2$

   Subject to $x_1 + x_2 \geq 5$

   $x_1 + 2x_2 \geq 6$

   $x_1, x_2 \geq 0.$

3.3. TWO PHASE METHOD

The two phase method is another method to solved a given problem in which some artificial variables are involved. The solution is obtained in two phases as follows:

Phase: I

In this phase, the simplex method is applied to a specially constructed auxillary linear programming problem leading to a final simplex table containing a basic feasible solution to the original problem.

Step: 1

Assign a cost -1 to each artificial variable and a cost 0 to all other variables (in place of their original cost) in the objective function. Thus the new objective function is $Z^* = -R_1 - R_2 - R_3 - \ldots - R_n$

where R_i's are the artificial variables.

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Step: 2

Construct the auxiliary LPP in which the new objective function Z^* is to be maximized subject to the given set of constraints.

Step: 3

Solve the auxiliary LPP by simplex method until either of the following three possibilities arises.

(i) Max $Z^* < 0$ and atleast one artificial variable appears in the optimum basis at a non – zero level. In this case the given LPP does not possess any feasible solution, stop the procedure.

(ii) Max $Z^* = 0$ and atleast one artificial variable appears in the optimum basis at zero level. In this case proceed to phase – II.

(iii) Max $Z^* = 0$ and no artificial variable appears in the optimum basis. In this case proceed to phase – II.

Phase: II

Use the optimum basic feasible solution of Phase – I as a starting solution for the original LPP. Assign the actual costs to the variables in the objective function and a 0 cost to every artificial variable that appears in the basis at the zero level. Use simplex method to the modified simplex table obtained at the end of Phase – I, till an optimum basic feasible solution (if any) is obtained.

Note: 1

In Phase – I, the iterations are stopped as soon as the value of the new objective function becomes zero because this is its maximum value. There is no need to continue till the optimality is reached if the value becomes zero earlier than that.

Note: 2

The new objective function is always of maximization type regardless of whether the original problem is of maximization or minimization type.

Note: 3

Before starting phase – II, remove all artificial variables from the table which were non – basic at the end of phase – I.

Example: 3.3.1

Use Two – Phase simplex method to solve

Maximize $Z = 5x_1 + 8x_2$

Subject to the constraints

$$3x_1 + 2x_2 \geq 3$$

$$x_1 + 4x_2 \geq 4$$

$$x_1 + x_2 \leq 5$$

$$x_1, x_2 \geq 0$$

Solution:

By introducing the non – negative slack, surplus and artificial variables, the standard form of the LPP becomes

$$\text{Max } Z = 5x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3$$

Subject to $\quad 3x_1 + 2x_2 - s_1 + 0s_2 + 0s_3 + R_1 = 3$

$$x_1 + 4x_2 + 0s_1 - s_2 + R_2 = 4$$

$$x_1 + x_2 + 0s_1 + 0s_2 + s_3 = 5$$

$$x_1, x_2, s_1, s_2, s_3, R_1, R_2 \geq 0$$

(**Here:** s_1, s_2 - surplus, s_3 - slack, R_1, R_2 - artificial)

The initial basic feasible solution is given by

$R_1 = 3, R_2 = 4, s_3 = 5$ (basic) ($x_1 = x_2 = s_1 = s_2 = 0$, non – basic)

Phase: I

Assigning a cost – 1 to the artificial variables and costs 0 to all other variables, the objective function of the auxiliary LPP becomes

$$\text{Max } Z^* = -R_1 - R_2$$

Subject to the given constraints

The iterative simplex tables for the auxiliary LPP are:

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Initial Iteration:

		C_j	(0	0	0	0	0	−1	−1)	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	R_1	R_2	θ
−1	R_1	3	3	2	−1	0	0	1	0	$\frac{3}{2}$
−1	R_2	4	1	(4)	0	−1	0	0	1	$\frac{4}{4}$
0	s_3	5	1	1	0	0	1	0	0	$\frac{5}{1}$
Z_j^*	$-C_j$	−7	−4	−6	1	1	0	0	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

First Iteration:

Introduce x_2 and drop R_2

		C_j	(0	0	0	0	0	−1	−1)	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	R_1	R_2	θ
−1	R_1	1	$\left(\frac{5}{2}\right)$	0	−1	$\frac{1}{2}$	0	1	$\frac{-1}{2}$	$\frac{2}{5}$
0	x_2	1	$\frac{1}{4}$	1	0	$\frac{-1}{4}$	0	0	$\frac{1}{4}$	4
0	s_3	4	$\frac{3}{4}$	0	0	$\frac{1}{4}$	1	0	$\frac{-1}{4}$	$\frac{16}{3}$
Z_j^*	$-C_j$	−1	$\frac{-5}{2}$	0	1	$\frac{-1}{2}$	0	0	$\frac{3}{2}$	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

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Second Iteration:

Introduce x_1 and drop R_1

C_J (0 0 0 0 0 −1 −1)

C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	R_1	R_2
0	x_1	$\frac{2}{5}$	1	0	$\frac{-2}{5}$	$\frac{1}{5}$	0	$\frac{2}{5}$	$\frac{-1}{5}$
0	x_2	$\frac{9}{10}$	0	1	$\frac{1}{10}$	$\frac{-3}{10}$	0	$\frac{-1}{10}$	$\frac{3}{10}$
0	s_3	$\frac{37}{10}$	0	0	$\frac{3}{10}$	$\frac{1}{10}$	1	$\frac{-3}{10}$	$\frac{-1}{10}$
Z_J^*	$-C_J$	0	0	0	0	0	0	1	1

Since all $(Z_J^* - C_J) \geq 0$, the current basic feasible solution is optimum. Furthermore, no artificial variable appears in the optimum basis so proceed to phase – II.

Phase: II

Here, we consider the actual costs associated with the original variables. The new objective function then becomes

$$\text{Max } Z = 5x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3$$

The initial basic feasible solution for this phase is the one obtained the end of Phase – I.

The iterative simplex tables for this phase are:

Initial Iteration:

C_J (5 8 0 0 0)

C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	θ
5	x_1	$\frac{2}{5}$	1	0	$\frac{-2}{5}$	$\left(\frac{1}{5}\right)$	0	2
8	x_2	$\frac{9}{10}$	0	1	$\frac{1}{10}$	$\frac{-3}{10}$	0	–
0	s_3	$\frac{37}{10}$	0	0	$\frac{3}{10}$	$\frac{1}{10}$	1	37
$(Z_J$	$-C_J)$	$\frac{46}{5}$	0	0	$\frac{-6}{5}$	$\frac{-7}{5}$	0	

Space for Hints

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

First Iteration:

Introduce s_2 and drop x_1

	C_j		(5	8	0	0	0)	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	θ
0	s_2	2	5	0	−2	1	0	–
8	x_2	$\frac{3}{2}$	$\frac{3}{2}$	1	$\frac{-1}{2}$	0	0	–
0	s_3	$\frac{7}{2}$	$\frac{-1}{2}$	0	$\left(\frac{1}{2}\right)$	0	1	7
$(Z_j$	$-C_j)$	12	7	0	−4	0	0	

Since there are some $(Z_j - C_j) < 0$, current basic feasible solution is not optimal.

Second Iteration:

Introduce s_1 and drop s_3

	C_j		(5	8	0	0	0)
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3
0	s_2	16	3	0	0	1	4
8	x_2	5	1	1	0	0	1
0	s_1	7	−1	0	1	0	2
$(Z_J$	$-C_J)$	40	3	0	0	0	8

Since all $(Z_J - C_J) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $z = 40, x_1 = 0, x_2 = 5$.

Example: 3.3.2

Solve by two phase simple method

Maximize $X_0 = -4x_1 - 3x_2 - 9x_3$

Subject to

$2x_1 + 4x_2 + 6x_3 - s_1 + R_1 = 15$

Space for Hints

$6x_1 + x_2 + 6x_3 - s_2 + R_2 = 12$

$x_1, x_2, x_3, s_1, s_2, R_1, R_2 \geq 0$.

Solution:

The initial basic feasible solution is given by

$R_1 = 15, R_2 = 12$(basic)$(x_1 = x_2 = x_3 = s_1 = s_2 = 0$, non – basic)

Phase: I

Assigning a cost – 1 to the artificial variables and costs 0 to all other variables, the objective function of the auxiliary LPP becomes

$$\text{Max}Z^* = -R_1 - R_2$$

The iterative simplex tables for the auxiliary LPP are:

Initial Iteration:

C_j (0 0 0 0 0 −1 −1)

C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	R_1	R_2	θ
−1	R_1	15	2	4	6	−1	0	1	0	$\frac{15}{6}$
−1	R_2	12	6	1	(6)	0	−1	0	1	$\frac{12}{6}$
Z_j^*	$-C_j$	−27	−8	−5	−12	1	1	0	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

First Iteration:

Introduce x_3 and drop R_2.

C_j (0 0 0 0 0 −1 −1)

C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	R_1	R_2	θ
−1	R_1	3	−4	(3)	0	−1	1	1	−1	$\frac{3}{3}$
0	x_3	2	1	$\frac{1}{6}$	1	0	$\frac{-1}{6}$	0	$\frac{1}{6}$	12
Z_j^*	$-C_j$	−3	4	−3	0	1	−1	0	2	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

Second iteration:

Introduce x_2 and drop R_1.

	C_j		(0	0	0	0	0	−1	−1
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	R_1	R_2
0	x_2	1	$\frac{-4}{3}$	1	0	$\frac{-1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{-1}{3}$
0	x_3	$\frac{11}{6}$	$\frac{22}{18}$	0	1	$\frac{1}{18}$	$\frac{-4}{18}$	$\frac{-1}{18}$	$\frac{4}{18}$
Z_J^*	$-C_J$	0	0	0	0	0	0	1	1

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimum.

Further, no artificial variable appears in the basis, so we proceed phase – II.

Phase: II

Here, we consider the actual costs associated with the original variables. The new objective function then becomes

$$\text{Max } X_0 = -4x_1 - 3x_2 - 9x_3 + 0s_1 + 0s_2$$

The initial basic feasible solution for this phase is the one obtained at the end of Phase – I.

The iterative simplex tables for this phase are:

Initial Iteration:

	C_j		(−4	−3	−9	0	0)	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	θ
−3	x_2	1	$\frac{-4}{3}$	1	0	$\frac{-1}{3}$	$\frac{1}{3}$	–
−9	x_3	$\frac{11}{6}$	$\left(\frac{22}{18}\right)$	0	1	$\frac{1}{18}$	$\frac{-4}{18}$	$\frac{3}{2}$
$(X_0$	$-C_j)$	$\frac{-39}{2}$	−3	0	0	$\frac{1}{2}$	1	

Space for Hints

Since there are some $(X_0 - C_j) < 0$, the current basic feasible solution is not optimal.

First Iteration:

Introduce x_1 and drop x_3

C_j $(-4 \quad -3 \quad -9 \quad 0 \quad 0)$

C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2
-3	x_2	3	0	1	$\frac{12}{11}$	$\frac{-3}{11}$	$\frac{1}{11}$
-4	x_1	$\frac{3}{2}$	1	0	$\frac{18}{22}$	$\frac{1}{22}$	$\frac{-4}{22}$
$(X_0$	$-C_j)$	-15	0	0	$\frac{27}{11}$	$\frac{7}{11}$	$\frac{5}{11}$

Since all $(X_0 - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max

$$X_0 = -15, x_1 = \frac{3}{2}, x_2 = 3, x_3 = 0$$

Example: 3.3.3

Use two – phase method to

$$\text{Maximize} \quad Z = 2x_1 + x_2 + \frac{1}{4}x_3$$

Subject to constraints

$$4x_1 + 6x_2 + 3x_3 \leq 8$$

$$3x_1 - 6x_2 - 4x_3 \leq 1$$

$$2x_1 + 3x_2 - 5x_3 \geq 4$$

$$x_1, x_2, x_3 \geq 0.$$

Solution:

By introducing slack, surplus and artificial variables, the standard form of the LPP becomes

$$\text{Max} \quad Z = 2x_1 + x_2 + \frac{1}{4}x_3$$

Space for Hints

Subject to $4x_1 + 6x_2 + 3x_3 + s_1 + 0s_2 + 0s_3 = 8$

$$3x_1 - 6x_2 - 4x_3 + 0s_1 + s_2 + 0s_3 = 1$$

$$2x_1 + 3x_2 - 5x_3 + 0s_1 + 0s_2 - s_3 + R_1 = 4$$

and $x_1, x_2, x_3, s_1, s_2, s_3, R_1 \geq 0$

(Here: s_1, s_2 - slack, s_3 - surplus, R_1 - artificial)

The initial basic feasible solution is given by

$$s_1 = 8, s_2 = 1, R_1 = 4 \text{(basic)} (x_1 = x_2 = x_3 = s_3 = 0, \text{non-basic})$$

Phase: I

The objective function of the auxiliary LPP is

Max $Z^* = -R_1$

The iterative simplex tables for the auxiliary LPP are:

Initial Iteration:

		C_j	(0	0	0	0	0	0	−1)	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	R_1	θ
0	s_1	8	4	(6)	3	1	0	0	0	$\frac{8}{6}$
0	s_2	1	3	−6	−4	0	1	0	0	−
−1	R_1	4	2	3	−5	0	0	−1	1	$\frac{4}{3}$
Z_j^*	$-C_j$	−4	−2	−3	5	0	0	1	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

First Iteration:

Introduce x_2 and drop s_1.

		C_j	(0	0	0	0	0	0	−1)
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	R_1
0	x_2	$\frac{4}{3}$	$\frac{2}{3}$	1	$\frac{1}{2}$	$\frac{1}{6}$	0	0	0
0	s_2	9	7	0	−1	1	1	0	0
−1	R_1	0	0	0	$\frac{-13}{2}$	$\frac{-1}{2}$	0	−1	1
Z_j^*	$-C_j$	0	0	0	$\frac{13}{2}$	$\frac{1}{2}$	0	1	0

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal for the auxiliary L.P.P.

But at the same time the artificial variable R_1 appears in the optimum basis at the zero level. This optimal solution may or may not be optimal to the given (original) L.P.P. So we proceed to Phase – II.

Phase – II:

Here, we consider the actual costs associated with the original variables and assign a cost 0 to the artificial variable R_1, which appeared at zero level in phase – I, in the objective function. The new objective function then becomes

$$\text{Max } Z = 2x_1 + x_2 + \frac{1}{4}x_3 + 0s_1 + 0s_2 + 0s_3$$

The initial basic feasible solution for this phase is the one obtained at the end of Phase – I.

The iterative simplex tables for this phase are:

Initial Iteration:

$C_j \quad (2 \quad 1 \quad \frac{1}{4} \quad 0 \quad 0 \quad 0 \quad 0)$

C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	R_1	θ
1	x_2	$\frac{4}{3}$	$\frac{2}{3}$	1	$\frac{1}{2}$	$\frac{1}{6}$	0	0	0	$\frac{4}{2}$
0	s_2	9	(7)	0	-1	1	1	0	0	$\frac{9}{7}$
0	R_1	0	0	0	$\frac{-13}{2}$	$\frac{-1}{2}$	0	-1	1	–
$(Z_j - C_j)$		$\frac{4}{3}$	$\frac{-4}{3}$	0	$\frac{1}{4}$	$\frac{1}{6}$	0	0	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

First Iteration:

Introduce x_1 and drop s_2

Space for Hints

	C_j	$(2$	1	$\frac{1}{4}$	0	0	0	$0)$
C_B Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	R_1
1 x_2	$\frac{10}{21}$	0	1	$\frac{25}{42}$	$\frac{1}{14}$	$\frac{-2}{21}$	0	0
2 x_1	$\frac{9}{7}$	1	0	$\frac{-1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	0	0
0 R_1	0	0	0	$\frac{-13}{2}$	$\frac{-1}{2}$	0	-1	1
$(Z_j - C_j)$	$\frac{64}{21}$	0	0	$\frac{5}{84}$	$\frac{5}{14}$	$\frac{4}{21}$	0	0

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $Z = \frac{64}{21}, x_1 = \frac{9}{7}, x_2 = \frac{10}{21}, x_3 = 0$.

3.4 DEGENERACY AND CYCLING

The concept of obtaining a degeneracy basic feasible solution in a LPP is known as **Degeneracy**. Degeneracy in LPP may arise

(i) At the initial stage when atleast one basic variable is zero in the initial basic feasible solution.

(ii) At any subsequent iteration when more than one basic variable is eligible to leave the basis and hence one or more variable becoming zero in the next iteration and the problem is said to degenerate. There is no assurance that the value of the objective function will improve, since the new solutions may return degenerate. As a result, it is possible to repeat the same sequence of simplex iterations endlessly without improving the solution. This concept is known as **cycling** or **circling.**

Perturbation rule to avoid cycling:

(a) Divide each element in tied rows by the positive co – efficient of the key column in that row.

(b) Compare the resulting ratios, column by column, first in the **matrix** and then in the **body matrix** from left to right.

(c) The row which first contains the smallest algebraic ratio the leaving variable.

Example: 3.4.1

Solve the following LPP by a simplex method:

Maximize F = $x_1 + 2x_2 + x_3$

Subject to $2x_1 + x_2 - x_3 \le 2$

$-2x_1 + x_2 - 5x_3 \le -6$

$4x_1 + x_2 + x_3 \le 6$

$x_1, x_2, x_3 \ge 0.$

Space for Hints

Solution:

Given

$$\text{Maximize } F = x_1 + 2x_2 + x_3$$

Subject to $2x_1 + x_2 - x_3 \le 2$

$2x_1 - x_2 + 5x_3 \le 6$

$4x_1 + x_2 + x_3 \le 6$

$x_1, x_2, x_3 \ge 0.$

By introducing the non – negative slack variables s_1, s_2, s_3 the standard form of LPP becomes

$$\text{Maximize } F = x_1 + 2x_2 + x_3$$

Subject to $2x_1 + x_2 - x_3 + s_1 + 0.s_2 + 0.s_3 = 2$

$2x_1 - x_2 + 5x_3 + 0.s_1 + s_2 + 0.s_3 = 6$

$4x_1 + x_2 + x_3 + 0.s_1 + 0.s_2 + s_3 = 6$

and $x_1, x_2, x_3, s_1, s_2, s_3 \ge 0.$

The initial basic feasible solution is given by

$s_1 = 2,\ s_2 = 6,\ s_3 = 6$(basic) $(x_1 = x_2 = x_3 = 0$, non basic)

Initial Iteration:

C_j (1 2 1 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
0	s_1	2	2	(1)	-1	1	0	0	$\frac{2}{1} = 1^*$
0	s_2	6	2	-1	5	0	1	0	–
0	s_3	6	4	1	1	0	0	1	$\frac{6}{1} = 6$
F_j	$-C_j$	0	-1	-2	-1	0	0	0	

Since there are some $(F_j - C_j) < 0$, the current basic feasible solution is not optimal.

First Iteration:

Introduce x_2 and drop s_1.

Space for Hints

		C_j	(1	2	1	0	0	0)	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
2	x_2	2	2	1	-1	1	0	0	–
0	s_2	8	4	0	4	1	1	0	$\frac{8}{4}=2$
0	s_3	4	2	0	(2)	-1	0	1	$\frac{4}{2}=2*$
F_j	$-C_j$	4	3	0	-3	2	0	0	

Since there are some $(F_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non – basic variable x_3 enters in to the basis. Since both the basic variables s_2 and s_3 having the same minimum ratio 2, there is a tie in selecting the leaving variable. To resolve this degeneracy, we divide each entry corresponding to basic variables s_2 and s_3 and then corresponding to non – basic variables x_1, x_2, x_3 and s_1. We get the following quotients.

	s_2	s_3	x_1	x_2	x_3	s_1
Row: 2	$\frac{1}{4}$	$\frac{0}{4}$	$\frac{4}{4}$	$\frac{0}{4}$	$\frac{4}{4}$	$\frac{1}{4}$
Row: 3	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{0}{2}$	$\frac{2}{2}$	$\frac{-1}{2}$

The columnwise comparison of quotients starting with basic variables s_2 and s_3. We find that column s_2 gives algebraically smaller ratio: for Row 3 and as such Row 3 is selected as key row. So the basic variable x_3 leaves the basis.

Second Iteration:

Introduce x_3 and drop s_3.

	C_j	$(1$	2	1	0	0	$0)$

C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
2	x_2	4	3	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$
0	s_2	0	0	0	0	3	1	-2
1	x_3	2	1	0	1	$\frac{-1}{2}$	0	$\frac{1}{2}$
F_j	$-C_j$	10	6	0	0	$\frac{1}{2}$	0	$\frac{3}{2}$

Since all $(F_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $F = 10$, $x_1 = 0$, $x_2 = 4$, $x_3 = 2$.

Example: 3.4.2

Solve the LPP Maximize $Z = 5x_1 - 2x_2 + 3x_3$

Subject to
$$2x_1 + 2x_2 - x_3 \geq 2$$
$$3x_1 - 4x_2 \leq 3$$
$$x_2 + 3x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

Solution:

By introducing the non – negative surplus variable s_1, slack variables s_2, s_3 and an artificial variable R_1, the standard form of the LPP becomes

Maximize $Z = 5x_1 - 2x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3 - MR_1$

Subject to
$$2x_1 + 2x_2 - x_3 - s_1 + 0.s_2 + 0.s_3 + R_1 = 2$$
$$3x_1 - 4x_2 + 0.x_3 + 0.s_1 + s_2 + 0.s_3 = 3$$
$$0.x_1 + x_2 + 3x_3 + 0.s_1 + 0.s_2 + s_3 = 5$$
and $x_1, x_2, x_3, s_1, s_2, s_3, R_1 \geq 0$.

The initial basic feasible solution is given by

Space for Hints

$R_1 = 2, \ s_2 = 3, \ s_3 = 5$ (basic) $(x_1 = x_2 = x_3 = s_1 = 0$, non – basic)

Initial Iteration:

C_B	Y_B	C_j / X_B	(5) x_1	−2 x_2	3 x_3	0 s_1	−M R_1	0 s_2	0) s_3	θ
M	R_1	2	(2)	2	−1	−1	1	0	0	$\frac{2}{2}$*
0	s_2	3	3	−4	0	0	0	1	0	$\frac{3}{3}$
0	s_3	5	0	1	3	0	0	0	1	—
Z_j	$-C_j$	2M	−2M−5	−2M+2	M−3	M	0	0	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal. The non – basic variable x_1 enters into the basis.

Since there is a tie in selecting the leaving variable among R_1 and s_2, is an indication of the existence of degeneracy. But since R_1 is an artificial basic variable, we select R_1 as the leaving variable.

First Iteration:

Introduce x_1 and drop R_1.

C_B	Y_B	C_j / X_B	5 x_1	−2 x_2	3 x_3	0 s_1	0 s_2	0 s_3	θ
5	x_1	1	1	1	$\frac{-1}{2}$	$\frac{-1}{2}$	0	0	—
0	s_2	0	0	−7	$\left(\frac{3}{2}\right)$	$\frac{3}{2}$	1	0	0*
0	s_3	5	0	1	3	0	0	1	$\frac{5}{3}$
Z_j	$-C_j$	5	0	7	$\frac{-11}{2}$	$\frac{-5}{2}$	0	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

Space for Hints

Second Iteration:

Introduce x_3 and drop s_2.

C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
	C_j		(5	-2	3	0	0	0)	
5	x_1	1	1	$\frac{-4}{3}$	0	0	$\frac{1}{3}$	0	–
3	x_3	0	0	$\frac{-14}{3}$	1	1	$\frac{2}{3}$	0	–
0	s_3	5	0	(15)	0	-3	-2	1	$\frac{5}{15}$
Z_j	$-C_j$	5	0	$\frac{-56}{3}$	0	3	$\frac{11}{3}$	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

Third Iteration:

Introduce x_2 and drop s_3.

C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
	C_j		(5	-2	3	0	0	0)	
5	x_1	$\frac{13}{9}$	1	0	0	$\frac{-4}{15}$	$\frac{7}{45}$	$\frac{4}{45}$	–
3	x_3	$\frac{14}{9}$	0	0	1	$\left(\frac{1}{15}\right)$	$\frac{2}{45}$	$\frac{14}{45}$	$\frac{14}{9} \times \frac{15}{1}^*$
-2	x_2	$\frac{1}{3}$	0	1	0	$\frac{-1}{5}$	$\frac{-2}{15}$	$\frac{1}{15}$	–
Z_j	$-C_j$	$\frac{101}{9}$	0	0	0	$\frac{-11}{15}$	$\frac{53}{45}$	$\frac{56}{45}$	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

Fourth Iteration:

Introduce s_1 and drop x_3.

Space for Hints

		C_j	5	−2	3	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
5	x_1	$\frac{23}{3}$	1	0	4	0	$\frac{1}{3}$	$\frac{4}{3}$
0	s_1	$\frac{70}{3}$	0	0	15	1	$\frac{2}{3}$	$\frac{14}{3}$
−2	x_2	5	0	1	3	0	0	1
Z_j	$-C_j$	$\frac{85}{3}$	0	0	11	0	$\frac{5}{3}$	$\frac{14}{3}$

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal..

$\therefore$ The optimal solution is Max $Z = \frac{85}{3}$, $x_1 = \frac{23}{3}$, $x_2 = 5$, $x_3 = 0$.

Check your progress 3.3

1. Solve the LPP: Maximum $Z = 3x_1 + 9x_2$

   Subject to $x_1 + 4x_2 \leq 8$

   $x_1 + 2x_2 \leq 4$

   and $x_1, x_2 \geq 0$.

2. Solve the LPP:

   Max $Z = 2x_1 + x_2$

   Subject to $4x_1 + 3x_2 \leq 12$

   $4x_1 + x_2 \leq 8$

   $4x_1 - x_2 \leq 8$

   and $x_1, x_2 \geq 0$.

3. Solve the LPP:

   $$\text{Min } Z = \frac{3}{4}x_1 + 20x_2 - \frac{1}{2}x_3 + 6x_4$$

   Subject to $\frac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 \leq 0$

   $\frac{1}{2}x_1 - 12x_2 - \frac{1}{2}x_3 + 3x_4 \leq 0$

   $x_3 \leq 1$

   and $x_1, x_2, x_3, x_4 \geq 0$.

Space for Hints

3.5 APPLICATION OF SIMPLEX METHOD

Example: 3.5.1

An automobile manufacturer makes auto-mobiles and trucks in a factory that is divided into two shops. Shop A, which performs the basic assembly operation must work 5 man – days on each truck but only 2 man – days on each automobile. Shop B, which performs finishing operation must work 3 man – days for each truck or automobile that is produces. Because of men and machine limitations shop A has 180 man – days per week available while shop B has 135 man – days per week. If the manufacturer makes a profit of Rs. 300 on each truck and Rs. 200 on each automobile, how many of each should he produce to maximize his profit?

Solution:

Let. x_1 units of trucks and x_2 units of automobiles be manufactured per week to maximize his profit.

Then the mathematical form of the given problem will be:

$$\text{Maximize } Z = 300x_1 + 200x_2$$

Subject to
$$5x_1 + 2x_2 \le 180$$
$$3x_1 + 3x_2 \le 135$$
$$\text{and } x_1, x_2 \ge 0.$$

By introducing non – negative slack variables s_1 and s_2, the standard form of the LPP becomes

$$\text{Maximize } Z = 300x_1 + 200x_2 + 0.s_1 + 0.s_2$$

Subject to
$$5x_1 + 2x_2 + s_1 + 0.s_2 = 180$$
$$3x_1 + 3x_2 + 0.s_1 + s_2 = 135$$
$$\text{and } x_1, x_2, s_1, s_2 \ge 0.$$

$\therefore$ The initial basic feasible solution is given by $s_1 = 180,\ s_2 = 135,\ (x_1 = x_2 = 0, \text{non} - \text{basic})$

Space for Hints

Initial Iteration:

C_j (300 200 0 0)

C_B	Y_B	X_B	x_1	x_2	s_1	s_2	θ
0	s_1	180	(5)	2	1	0	$\frac{180}{5}=36^*$
0	s_2	135	3	3	0	1	$\frac{135}{3}=45$
Z_j	$-C_j$	0	-300	-200	0	0	

First Iteration:

Introduce x_1 and drop s_1.

C_j (300 200 0 0)

C_B	Y_B	X_B	x_1	x_2	s_1	s_2	θ
300	x_1	36	1	$\frac{2}{5}$	$\frac{1}{5}$	0	$36x\frac{5}{2}=90$
0	s_2	27	0	$\left(\frac{9}{5}\right)$	$\frac{-3}{5}$	1	$27x\frac{5}{9}=15^*$
Z_j	$-C_j$	10,800	0	-80	60	0	

Second Iteration:

Introduce x_2 and drop s_2

C_j (300 200 0 0)

C_B	Y_B	X_B	x_1	x_2	s_1	s_2
300	x_1	30	1	0	$\frac{1}{3}$	$\frac{-2}{9}$
200	x_2	15	0	1	$\frac{-1}{3}$	$\frac{5}{9}$
Z_j	$-C_j$	12,000	0	0	$\frac{100}{3}$	$\frac{400}{9}$

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $Z = 12{,}000,\ x_1 = 30,\ x_2 = 15.$

Space for Hints

The manufacturer should manufacture 30 trucks and 15 automobiles per week in orders to get a maximum profit of Rs. 12,000.

Example: 3.5.2

A gear manufacturing company received on order for three specific types of gears for regular supply. The management is considering to devote the available excess capacity to one or more of the three types, say A, B and C. The available capacity on the machines which might limit output and the number of machine hours required for each unit of the respective gear is also given below:

Machine Type	**Available machine Hours/Week**	**Productivity in Machine Hours/ Unit**		
		Gear A	**Gear B**	**Gear C**
Gear Hobbing m/c	250	8	2	3
Gear Shaping m/c	150	4	3	0
Gear Griding m/c	50	2	-	1

The unit profit would be Rs. 20, Rs. 6 and Rs. 8 respectively for the gears A, B and C. Find how much of each gear the company should produce in order to maximize profit?

Solution:

Let x_1, x_2 and x_3 be the number of units of gears A, B and C produced respectively to maximize the profit. The mathematical formulation of the LPP is given by

Maximize $Z = 20x_1 + 6x_2 + 8x_3$

Subject to $8x_1 + 2x_2 + 3x_3 \leq 250$

$4x_1 + 3x_2 \leq 150$

$2x_1 + x_3 \leq 50$

and $x_1, x_2, x_3 \geq 0$.

By introducing non – negative slack variables s_1, s_2 and s_3, the standard form of the LPP becomes

Maximize $Z^* = 20x_1 + 6x_2 + 8x_3 + 0.s_1 + 0.s_2 + 0.s_3$

Subject to the constraints

$8x_1 + 2x_2 + 3x_3 + s_1 + 0.s_2 + 0.s_3 = 250$

$4x_1 + 3x_2 + 0.s_1 + s_2 + 0.s_3 = 150$

$2x_1 + 0.x_2 + x_3 + 0.s_1 + 0.s_2 + s_3 = 50$

and $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$.

The initial basic feasible solution is given by $s_1 = 250,\ s_2 = 150,\ s_3 = 50.$ ($x_1 = x_2 = x_3 = 0$, non – basic).

Initial Iteration:

		C_j	20	6	8	0	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
0	s_1	250	8	2	3	1	0	0	250/8
0	s_2	150	4	3	0	0	1	0	150/4
0	s_3	50	(2)	0	1	0	0	1	50/2*
Z_j	$-C_j$	0	−20	−6	−8	0	0	0	

First Iteration:

Introduce x_1 and drop s_3

		C_j	20	6	8	0	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
0	s_1	50	0	2	−1	1	0	−4	$\frac{50}{2}$
0	s_2	50	0	(3)	−2	0	1	−2	$\frac{50}{3}$*
20	x_1	25	1	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	–
Z_j	$-C_j$	500	0	−6	2	0	0	10	

Space for Hints

Second Iteration:

Introduce x_2 and drop s_2

		C_j	20	6	8	0	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
0	s_1	$\frac{50}{3}$	0	0	$\frac{1}{3}$	1	$\frac{-2}{3}$	$\frac{-8}{3}$	50
6	x_2	$\frac{50}{3}$	0	1	$\frac{-2}{3}$	0	$\frac{1}{3}$	$\frac{-2}{3}$	–
20	x_1	25	1	0	$\left(\frac{1}{2}\right)$	0	0	$\frac{1}{2}$	50*
Z_j	$-C_j$	600	0	0	-2	0	2	6	

Third Iteration:

Introduce x_3 and drop x_1

		C_j	20	6	8	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
0	s_1	0	$\frac{-2}{3}$	0	0	1	$\frac{-2}{3}$	-3
6	x_2	50	$\frac{4}{3}$	1	0	0	$\frac{1}{3}$	0
8	x_3	50	2	0	1	0	0	1
Z_j	$-C_j$	700	4	0	0	0	2	8

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is

Max $Z = 700,\ x_1 = 0,\ x_2 = 50,\ x_3 = 50$.

$\therefore$ The company should produce 50 units of gear B, 50 units of gear of C and none of gear A in order to have a maximum profit Rs. 700.

3.6 KEYWORDS

Simplex Method, Big – M Method, Two – Phase Method, Degeneracy, Method of Penalties.

3.7 ANSWERS TO CHECH YOUR PROGRESS

Space for Hints

Check your progress 3.1

1. Max $Z = \frac{74}{3}$, $x_1 = 0$, $x_2 = 2$, $x_3 = \frac{10}{3}$
2. Max $Z = 22500$, $x_1 = 0$, $x_2 = 100$, $x_3 = 50$
3. Max $Z = 10$, $x_1 = 4$, $x_2 = 2$
4. Max $Z = 3$, $x_1 = 0$, $x_2 = 0$, $x_3 = 1$.

Check your progress 3.2

1. Min $Z = 48$, $x_1 = 3$, $x_2 = 18$
2. Min $Z = 11$, $x_1 = 4$, $x_2 = 1$

Check your progress 3.3

1. Max $Z = 18$, $x_1 = 0$, $x_2 = 2$
2. Max $Z = 5$, $x_1 = \frac{3}{2}$, $x_2 = 2$
3. Min $Z = -0.5$, $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$

3.8 Model Questions

1. Solve by simplex method:

$$\text{Maximize } Z = 3x_1 + 5x_2 + 4x_3$$

Subject to

$$2x_1 + 3x_2 \leq 8$$
$$2x_2 + 5x_3 \leq 10$$
$$3x_1 + 2x_2 + 4x_3 \leq 15$$
$$x_1, x_2, x_3 \geq 0.$$

2. Solve by simplex method:

$$\text{Maximize } Z = 2x_1 - 4x_2 + 5x_3 - 6x_4$$

Subject to the constraints

$$x_1 + 4x_2 - 2x_3 + 8x_4 \leq 2$$
$$-x_1 + 2x_2 + 3x_3 + 4x_4 \leq 1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Space for Hints

3. Food X contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and costs 12 paise per gram. Food Y contains 8 units of vitamin A per gram and 12 units of vitamin B per gram and costs 20 paise per gram. The daily minimum requirements of vitamin A and vitamin B are 100 units and 120 units respectively. Find the minimum cost of product mix by simplex method.

4. Solve the following LPP:

Maximize $Z = x_1 + 1.5x_2 + 2x_3 + 5x_4$

Subject to
$$3x_1 + 2x_2 + 4x_3 + x_4 \le 6$$
$$2x_1 + x_2 + x_3 + 5x_4 \le 4$$
$$2x_1 + 6x_2 - 8x_3 + 4x_4 = 0$$
$$x_1 + 3x_2 - 4x_3 + 3x_4 = 0$$
$$x_1, x_2, x_3, x_4 \ge 0.$$

5. A company possesses two manufacturing plants, each of which can produce three products A, B, C from common raw material. However, the proportions in which the products are produced are different in each plant and so are the plant's operating cost per hour. Data on production per hour and costs are given below, together with current orders in hand for each product.

	Product			**Operating cost per hour (Rs)**
	X	**Y**	**Z**	
Plant I	2	4	3	9
Plant II	4	3	2	10
Order on hand	50	24	60	

You are required to use simplex method to find the number of production hours needed to fulfill the orders on hand at a minimum cost.

UNIT – 4

DUALITY

Introduction:

For every linear programming problem there is a unique linear programming problem associated with it, involving the same data and closely related optimal solutions. The original (given) problem is then called the **Primal** problem while the other is called its **dual** problem. But in general, the two problems are said to be **duals** of each other.

The importance of the duality concept is due to two main reasons. Firstly, if the primal contains a large number of constraints and a smaller number of variables, the labour of computation can be considerably reduced by converting in to the dual variables from the cost of economic point of view proves extremely useful in making future decisions in the activities being programmed.

A linear programming problem in which all or some of the decision variable are constrained to assume non – negative integer values is called an Integer Programming Problem. This type of problem is of particular importance in business and industry, where quite often, the fractional solution are unrealistic because the units are not divisible. For example, it is absurd to speak of 2.3 men working on a project or 8.7 machines in a workshop. The integer solution to a problem can, however, be obtained by rounding off the optimum values of the variables to the nearest integer values. But, it is generally inaccurate to obtain an integer solution by rounding off in this manner, for there is no guarantee that the deviation from the 'exact' integer solution will not be too large to retain the feasibility.

The linear programming problem with the additional requirements that the variables can take on only, integer values

Space for Hints

may have the following mathematical form

Objectives:

After working through examples and exercise of this unit you will be able to

1. Formulate the dual problem from primal.
2. Solve the L.P.P using dual simplex method.
3. Solve the integer programming problem using culting plan method.

Structure:

4.1 Formulation of dual problems

4.2 Duality Theorems

4.3 Duality and Simplex Methods

4.4 Dual Simplex Method

4.5 Integer programming

4.6 Keywords

4.7 Answers to check your progress

4.8 Model Questions

4.1 Formulations of Dual Problems

Formulation of dual problems

There are two important forms of primal – dual pairs, namely **symmetric form** and **unsymmetric form.**

Symmetric Form:

Consider the following LPP.

Maximize $Z = c_1x_1 + c_2x_2 + c_3x_3 + \dots\dots + c_nx_n$

Subject to the constraints

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n \le b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n \le b_m \end{array}\right. \qquad (1)$$

Space for Hints

and $x_1, x_2, \ldots\ldots\ldots, x_n \geq 0$

i.e., $\quad$ Max $Z = CX$

subject to $\quad AX \leq b$

and $\quad X \geq 0$

where $C = (c_1 c_2 \ldots\ldots\ldots c_n)$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots\cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots\cdots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ . \\ b_m \end{pmatrix}$$

To construct the dual problem, we adopt the following guidelines:

(i) The maximization problem in the primal becomes the minimization problem in the dual and vice versa.

(ii) The maximization problem has $(\leq)$ constraints while the minimization problems has $(\geq)$ constraints.

(iii) If the primal contains m constraints and n variables, then the dual will contain n constraints and m variables. i.e., the transpose of the body matrix of the primal problem gives the body matrix of the dual and vice versa.

(iv) The constants $c_1, c_2, c_3, \ldots\ldots\ldots, c_n$ in the objective function of the primal appear in the constraints of the dual.

(v) The constants $b_1, b_2, \ldots\ldots\ldots, b_m$ in the constraints of the primal appear in the objective function of the dual.

(vi) The variables in both problems are non – negative.

The primal – dual relationships can be conveniently displayed as below:

Primal variables

$$\begin{array}{cc} & \begin{array}{cccccc} x_1 & x_2 & x_3 & \cdots & \cdots & x_n \end{array} \\ \text{Dual Variables}\ \begin{array}{c} y_1 \\ y_2 \\ \cdots \\ \cdots \\ y_m \end{array} & \left[\begin{array}{cccccc} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mn} \end{array}\right] \begin{array}{l} \le b_1 \\ \le b_2 \ \text{R.H.S. of Primal constraints} \\ \cdots \\ \cdots \\ \le b_m \end{array} \\ & \begin{array}{cccccc} \ge & \ge & \ge & \cdots & \cdots & \ge \\ c_1 & c_2 & c_3 & \cdots & \cdots & c_n \end{array} \end{array}$$

R.H.S of dual constraints

Then the dual problem of (1) will be $\left(\text{where } Y = \begin{pmatrix} y_1 \\ y \\ . \\ y_m \end{pmatrix} \right)$

Minimize $W = b^T Y$

Subject to the constraints $\quad A^T Y \ge C^T$

and $\quad Y \ge 0$

i.e., Minimize $W = b_1 y_1 + b_2 y_2 + \dots\dots\dots + b_m y_m$

subject to the constraints

$$\left.\begin{array}{l} a_{11} y_1 + a_{21} y_2 + \dots\dots\dots\dots + a_{m1} y_m \ge c_1 \\ a_{12} y_1 + a_{22} y_2 + \dots\dots\dots\dots + a_{m2} y_m \ge c_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{1n} y_1 + a_{2n} y_2 + \dots\dots\dots\dots + a_{mn} y_m \ge c_n \\ \quad \text{and } y_1, y_2, \dots\dots, y_m \ge 0 \end{array}\right\} \quad (2)$$

Equations (1) and (2) are called **symmetric primal – dual pairs.**

Example: 4.1.1

Write the dual of the following primal LPP.

Maximize $F = x_1 + 2x_2 + x_3$

Subject to $\quad 2x_1 + x_2 - x_3 \le 2$

$\quad -2x_1 + x_2 - 5x_3 \ge -6$

$$4x_1 + x_2 + x_3 \le 6$$

$$x_1, x_2, x_3 \ge 0$$

Solution:

Given primal LPP Maximize $F = x_1 + 2x_2 + x_3$

Subject to
$$2x_1 + x_2 - x_3 \le 2$$
$$-2x_1 + x_2 - 5x_3 \ge -6$$
$$4x_1 + x_2 + x_3 \le 6$$
and $x_1, x_2, x_3 \ge 0$.

That is, Maximize $F = x_1 + 2x_2 + x_3$

Subject to
$$2x_1 + x_2 - x_3 \le 2$$
$$2x_1 - x_2 + 5x_3 \le 6$$
$$4x_1 + x_2 + x_3 \le 6$$
and $x_1, x_2, x_3 \ge 0$.

That is Max $F = CX$

Subject to $AX \le b$

and $X \ge 0$

Where $C = (1\ 2\ 1)$, $A = \begin{pmatrix} 2 & 1 & -1 \\ 2 & -1 & 5 \\ 4 & 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Since the primal problems is maximization type with $(\le)$ type constraints, with 3 constraints and 3 variables, the dual problem is minimization type with $(\ge)$ type constraints, with 3 constraints and 3 dual variables y_1, y_2, y_3.

The dual problem is Minimize $W = b^T Y$

subject to the constraints $A^T Y \ge C^T$

and $Y \ge 0$

Space for Hints

$$\text{i.e., Min } W = \begin{pmatrix} 2 & 6 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\text{subject to } \begin{pmatrix} 2 & 2 & 4 \\ 1 & -1 & 1 \\ -1 & 5 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq 0$$

i.e., Min $W = 2y_1 + 6y_2 + 6y_3$

subject to

$$2y_1 + 2y_2 + 4y_3 \geq 1$$
$$y_1 - y_2 + y_3 \geq 2$$
$$-y_1 + 5y_2 + y_3 \geq 1$$
$$y_1, y_2, y_3 \geq 0$$

Example: 4.1.2

Find the dual of the LPP

$$\text{Max } Z = 3x_1 - x_2 + x_3$$

Subject to

$$4x_1 - x_2 \leq 8$$
$$8x_1 + x_2 + 3x_3 \geq 12$$
$$5x_1 - 6x_3 \leq 13$$
$$x_1, x_2, x_3 \geq 0$$

Solution:

Given primal LPP is

Max $Z = 3x_1 - x_2 + x_3$

Subject to

$$4x_1 - x_2 + 0x_3 \leq 8$$
$$-8x_1 - x_2 - 3x_3 \leq -12$$
$$5x_1 + 0x_2 - 6x_3 \leq 13$$

and $x_1, x_2, x_3 \geq 0$.

That is Max $Z = CX$

Subject to $AX \leq b$

and $X \geq 0$

Space for Hints

Where

$$C=(3 \quad -1 \quad 1),\ A=\begin{pmatrix} 4 & -1 & 0 \\ -8 & -1 & -3 \\ 5 & 0 & 6 \end{pmatrix},\quad X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},\quad b=\begin{pmatrix} 8 \\ -12 \\ 13 \end{pmatrix}$$

The dual problem is Minimize $\quad W = b^T Y$

Subject to the constraints $\quad A^T Y \geq C^T$

and $Y \geq 0$

i.e., Min $W=(8 \quad -12 \quad 13)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

subject to $\begin{pmatrix} 4 & -8 & 5 \\ -1 & -1 & 0 \\ 0 & -3 & -6 \end{pmatrix} \geq \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq 0$

i.e., Min $W=8y_1-12y_2+13y_3$

subject to $\quad 4y_1-8y_2+5y_3 \geq 3$

$-y_1-y_2+0y_3 \geq -1$

$0y_1-3y_2-6y_3 \geq 1$

$y_1, y_2, y_3 \geq 0$

i.e., The dual problem is

Min $W=8y_1-12y_2+13y_3$

Subject to $\quad 4y_1-8y_2+5y_3 \geq 3$

$y_1+y_2 \leq 1$

$-3y_2-6y_3 \geq 1$

and $y_1, y_2, y_3 \geq 0$

Example: 4.1.3

Construct the dual of the LPP

Min $Z=4x_1+6x_2+18x_3$

Subject to $\quad x_1+3x_2 \geq 3$

Space for Hints

$$x_2 + 2x_3 \geq 5$$

$$\text{and} \quad x_1, x_2, x_3 \geq 0$$

Solution:

Given primal LPP is

$$\text{Min } Z = 4x_1 + 6x_2 + 18x_3$$

Subject to
$$x_1 + 3x_2 + 0x_3 \geq 3$$
$$0x_1 + x_2 + 2x_3 \geq 5$$
$$\text{and } x_1, x_2, x_3 \geq 0.$$

That is $\quad \text{Min } Z = CX$

Subject to $AX \geq b$

and $X \geq 0$.

Where $C = (4 \quad 6 \quad 18)$, $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Since the primal contains 2 constraints and 3 variables, the dual will contain 3 constraints and 2 variables y_1, y_2

The dual LPP is Max $W = b^T Y$

Subject to the constraints $A^T Y \leq C^T$

and $Y \geq 0$.

i.e., Max $W = (3 \quad 5)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ subject to $\begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \leq \begin{pmatrix} 4 \\ 6 \\ 18 \end{pmatrix}$

and $y_1, y_2, y_3 \geq 0$

i.e., The dual problem is Max $W = 3y_1 + 5y_2$

subject to $\quad y_1 \leq 4$

$$3y_1 + y_2 \leq 6$$

$$2y_2 \leq 18$$

and $\quad y_1, y_2 \geq 0$.

Example: 4.1.4

Write the dual of the LPP

$$\text{Min } Z = 3x_1 - 2x_2 + 4x_3$$

Subject to
$$3x_1 + 5x_2 + 4x_3 \geq 7$$
$$6x_1 + x_2 + 3x_3 \geq 4$$
$$7x_1 - 2x_2 - x_3 \leq 10$$
$$x_1 - 2x_2 + 5x_3 \geq 3$$
$$4x_1 + 7x_2 - 2x_3 \geq 2$$
and $x_1, x_2, x_3 \geq 0$.

Solution:

Given primal LPP is

$$\text{Min } Z = 3x_1 - 2x_2 + 4x_3$$

Subject to
$$3x_1 + 5x_2 + 4x_3 \geq 7$$
$$6x_1 + x_2 + 3x_3 \geq 4$$
$$-7x_1 + 2x_2 + x_3 \geq -10$$
$$x_1 - 2x_2 + 5x_3 \geq 3$$
$$4x_1 + 7x_2 - 2x_3 \geq 2$$
and $x_1, x_2, x_3 \geq 0$.

i.e.,
$$\text{Min } Z = CX$$
subject to $AX \geq b$

and $X \geq 0$.

Where

$$C = (3 \quad -2 \quad 4),\ A = \begin{pmatrix} 3 & 5 & 4 \\ 6 & 1 & 3 \\ -7 & 2 & 1 \\ 1 & -2 & 5 \\ 4 & 7 & -2 \end{pmatrix},\ b = \begin{pmatrix} 7 \\ 4 \\ -10 \\ 3 \\ 2 \end{pmatrix},\ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

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Since the primal contains 5 constraints and 3 variables, the dual will contain 3 constraints and 5 variables y_1, y_2, y_3, y_4 and y_5.

The dual LPP is Max $\quad W = b^T Y$

Subject to the constraints $\quad A^T Y = C^T$

and $\quad Y \geq 0$.

$$\text{Max } W = \begin{pmatrix} 7 & 4 & -10 & 3 & 2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}$$

$$\text{subject to } \begin{pmatrix} 3 & 6 & -7 & 1 & 4 \\ 5 & 1 & 2 & -2 & 7 \\ 4 & 3 & 1 & 5 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} \leq \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \text{ and } \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} \geq 0.$$

i.e., Max $W = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$

subject to $\quad 3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \leq 3$

$5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2$

$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4$

$y_1, y_2, y_3, y_4, y_5 \geq 0$.

Unsymmetric Form

S. No	**Primal / Dual**	**Dual / Primal**
1	Max $Z = CX$ Subject to $AX = b$ $X \geq 0$	Min $W = b^T Y$ Subject to $A^T Y \geq C^T$ Variables are unrestricted
2	Min $Z = CX$ Subject to $AX = b$ $X \geq 0$	Max $W = b^T Y$ Subject to $A^T Y \leq C^T$ Variables are unrestricted

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Remark: 1

If the k^{th} constraints of the primal problem is an equality, then the corresponding dual variable y_k is unrestricted in sign and vice versa.

Remark: 2

If any variable of the primal problem is unrestricted in sign, the corresponding constraint in the dual problem will be an equality and vice versa.

Example: 4.1.5

Write the dual of the LPP

$$\text{Max} \quad Z = 3x_1 + 10x_2 + 2x_3$$

Subject to
$$2x_1 + 3x_2 + 2x_3 \le 7$$
$$3x_1 - 2x_2 + 4x_3 = 3$$
$$\text{and } x_1, x_2, x_3 \ge 0.$$

Solution:

Given primal LPP is

$$\text{Max } Z = 3x_1 + 10x_2 + 2x_3$$

Subject to
$$2x_1 + 3x_2 + 2x_3 \le 7$$
$$3x_1 - 2x_2 + 4x_3 = 3$$
$$\text{and } x_1, x_2, x_3 \ge 0.$$

Since the primal problem contains 2 constraints and 3 variables, the dual problem will contain 3 constraints and 2 dual variables y_1, y_2.

Also, since the second constraint in the primal problem is an equality, the corresponding second dual variable y_2 is unrestricted in sign.

∴ The dual problem is

$$\text{Min} \quad W = 7y_1 + 3y_2$$

Subject to
$$2y_1 + 3y_2 \ge 3$$

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$$3y_1 - 2y_2 \geq 10$$

$$2y_1 + 4y_2 \geq 2$$

and $y_1 \geq 0$, y_2 is unrestricted.

Example: 4.1.6

Write the dual of the LPP

$$\text{Min} \quad Z = x_2 + 3x_3$$

Subject to

$$2x_1 + x_2 \leq 3$$

$$x_1 + 2x_2 + 6x_3 \geq 5$$

$$-x_1 + x_2 + x_3 = 2$$

$$x_1, x_2, x_3 \geq 0.$$

Solution:

Given primal LPP is

$$\text{Min} \quad Z = 0x_1 + x_2 + 3x_3$$

Subject to

$$-2x_1 - x_2 + 0x_3 \geq -3$$

$$x_1 + 2x_2 + 6x_3 \geq 5$$

$$-x_1 + x_2 + x_3 = 2$$

and $x_1, x_2, x_3 \geq 0$.

Since the primal problem contains 3 constraints and 3 variables, the dual problem will contain 3 constraints and 3 dual variables y_1, y_2, y_3.

Also, since the third constraint in the primal problem is an equality the corresponding third dual variable y_3 is unrestricted in sign.

The dual LPP is $\quad \text{Max} \quad W = -3y_1 + 5y_2 + 2y_3$

Subject to

$$-2y_1 + y_2 - y_3 \leq 0$$

$$-y_1 + 2y_2 + y_3 \leq 1$$

$$6y_2 + y_3 \leq 3$$

and $y_1, y_2 \geq 0$, y_3 is unrestricted.

Example: 4.1.7

Write the dual of the following primal LPP

$$\text{Min } Z = 4x_1 + 5x_2 - 3x_3$$

Subject to $\quad x_1 + x_2 + x_3 = 22$

$$3x_1 + 5x_2 - 2x_3 \le 65$$

$$x_1 + 7x_2 + 4x_3 \ge 120$$

$$x_1 \ge 0,\ x_2 \ge 0 \text{ and } x_3 \text{ unrestricted.}$$

Solution:

Given primal LPP is

$$\text{Min } \quad Z = 4x_1 + 5x_2 - 3x_3$$

Subject to $\quad x_1 + x_2 + x_3 = 22$

$$-3x_1 - 5x_2 + 2x_3 \ge -65$$

$$x_1 + 7x_2 + 4x_3 \ge 120$$

$x_1 \ge 0, x_2 \ge 0$ and x_3 unrestricted.

Since the primal problem contains 3 constraints and 3 variables, the dual problem will contain 3 constraints and 3 dual variables y_1, y_2, y_3. Since the first primal constraint is an equality, the corresponding first dual variable y_1 is unrestricted in sign. Also, since the third primal variable x_3 is unrestricted in sign, the corresponding third dual constraint will be an equality.

The dual LPP is $\quad \text{Max } W = 22y_1 - 65y_2 + 120y_3$

Subject to $\quad y_1 - 3y_2 + y_3 \le 4$

$$y_1 - 5y_2 + 7y_3 \le 5$$

$$y_1 + 2y_2 + 4y_3 = -3$$

and $y_2, y_3 \ge 0$, y_1 is unrestricted.

Example: 4.1.8

Write the dual of the primal:

$$\text{Max } \quad Z = 6x_1 + 6x_2 + x_3 + 7x_4 + 5x_5$$

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Subject to $3x_1 + 7x_2 + 8x_3 + 5x_4 + x_5 = 2$

$2x_1 + x_2 + 3x_4 + 9x_5 = 6$

$x_1, x_2, x_3, x_4 \geq 0$ and x_5 unrestricted.

Solution:

Given primal LPP is

Max $Z = 6x_1 + 6x_2 + x_3 + 7x_4 + 5x_5$

Subject to $3x_1 + 7x_2 + 8x_3 + 5x_4 + x_5 = 2$

$2x_1 + x_2 + 0x_3 + 3x_4 + 9x_5 = 6$

$x_1, x_2, x_3, x_4 \geq 0$ and x_5 unrestricted.

Since the primal problem contains 2 constraints and 5 variables, the dual problem will contain 5 constraints and 2 dual variables y_1, y_2. Since all the constraints in the primal are equality, all the dual variable are unrestricted in sign. Also, since the primal variable x_5 is unrestricted in sign, the corresponding fifth dual constraint is an equality.

The dual LPP is Min $W = 2y_1 + 6y_2$

Subject to $3y_1 + 2y_2 \geq 6$

$7y_1 + y_2 \geq 6$

$8y_1 \geq 1$

$5y_1 + 3y_2 \geq 7$

$y_1 + 9y_2 = 5$

and y_1, y_2 are unrestricted.

Check your progress 4.1

1. Obtain the dual problem of the following L.P.P:

   Maximize $f(x) = 2x_1 + 5x_2 + 6x_3$ subject to the constraints:

   $5x_1 + 6x_2 - x_3 \leq 3, -2x_1 + x_2 + 4x_3 \leq 4,\ x_1 - 5x_2 + 3x_3 \leq 1.$

   $-3x_1 - 3x_2 + 7x_3 \leq 6,\ x_1, x_2, x_3 \geq 0.$

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2. Find the dual of the following L.P.P:

   Maximize $z = 2x_1 + x_2$ subject to the constraints:

   $x_1 + 5x_2 \leq 10,\ x_1 + 3x_2 \geq 6,\ 2x_1 + 2x_2 \leq 8;\ x_2 \geq 0$ and x_1 unrestricted.

3. Obtain the dual of the following linear programming problem:

   Maximize $z = 2x_1 + 3x_2 + x_3$ subject to the constraints:

   $4x_1 + 3x_2 + x_3 = 6,\ x_1 + 2x_2 + 5x_3 = 4,\ x_1, x_2, x_3 \geq 0$.

4. Obtain the dual problem of the following L.P.P:

   Maximize $z = x_1 - 2x_2 + 3x_3$ subject to the constraints:

   $-2x_1 + x_2 + 3x_3 = 2,\ 2x_1 + 3x_2 + 4x_3 = 1;\ x_1, x_2, x_3 \geq 0$.

5. Obtain the dual of the linear programming problem:

   Minimize $z = x_3 + x_4 + x_3$ subject to the constraints:

   $x_1 - x_3 + x_4 - x_5 = -2;\ x_2 - x_3 - x_4 + x_5 = 1,\ x_j \geq 0$ for $j = 1,2,3,4,5$.

6. Write the dual of the following linear programming problem:

   Minimize $z = 3x_1 - 2x_2 + 4x_3$ subject to the constraints:

   $$3x_1 + 5x_2 + 4x_3 \geq 7,\ 6x_1 + x_2 + 3x_3 \geq 4$$

   $$7x_1 - 2x_2 - x_3 \leq 10,\ x_1 - 2x_2 + 5x_3 \geq 3$$

   $$4x_1 + 7x_2 - 2x_3 \geq 2,\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$$

4.2 Duality Theorems:

Theorem: 4.2.1

The dual of the dual is the primal.

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Proof:

Let the primal L.P.P be to determine $x^T \in R^n$ so as to

Maximize $f(x) = cx,\ c \in R^n$ subject to the constraints:

$$Ax = b \text{ and } x \geq 0,\ b^T \in R^m,$$

where A is an m x n real matrix.

The dual of this primal is the L.P.P of determining $w^T \in R^m$ so as to

Minimize $f(w) = b^T w,\ b^T \in R^m$ subject to the constraints:

$$A^T w \geq c^T, \text{ w unrestricted, } c \in R^n.$$

Now, introduce surplus variable $s \geq 0$ in the constraints of the dual and write $w = w_1 - w_2$, where $w_1 \geq 0$ and $w_2 \geq 0$.

The standard form of dual then is to

Minimize $g(w) = b^T(w_1 - w_2)$ $b^T \in R^m$ subject to the constraints:

$$A^T(w_1 - w_2) - I_n s = c^T \quad c \in R^n$$

$$w_1, w_2 \text{ and } s \geq 0.$$

Considering this linear programming problem as our standard primal, the associated dual problem will be to

Maximize $h(y) = cy,\ c \in R^n$ subject to the constraints:

$$(A^T)^T y \leq (b^T)^T,\ -(A^T)^T y \leq -(b^T)^T,$$

$$-y \leq 0\ (\Rightarrow y \geq 0) \text{ and y unrestricted.}$$

Eliminating redundancy, the dual problem may be re – written as:

Maximize $h(y) = cy,\ c \in R^n$ subject to the constraints:

$$\left.\begin{array}{l} Ay \leq b \\ Ay \geq b \end{array}\right\} \Rightarrow Ay = b,\ b^T \in R^m$$

$$y \geq 0.$$

This problem, which is the dual of the dual problem, is just the primal problem we had started with.

This completes the proof.

Theorem: 4.2.2 (Weak Duality Theorem)

Let $x_\circ$ be a feasible solution to the primal problem

Maximize $f(x) = cx$ subject to: $Ax \le b,\ x \ge 0$

where x^T and $c \in R^n$, $b^T \in R^m$ and A is an m x n real matrix. If $w_\circ$ be a feasible solution to the dual of the primal, namely

Minimize $g(w) = b^T w$ subject to: $A^T w \ge c^T,\ w \ge 0$

where $w^T \in R^m$, then $cx_\circ \le b^T w_\circ$.

Proof:

Since x_0 and w_0 are the feasible solution to the primal and its dual respectively, we must have

$$Ax_0 \le b,\ x_0 \ge 0 \text{ and } A^T w_0 \ge c^T,\ w_0 \ge 0.$$

Thus $\quad c \le w_\circ^T A$ or $cx_\circ \le w_\circ^T A x_\circ \le w_\circ^T b$

or $\quad cx_\circ \le b^T w_\circ \quad$ (since $w_\circ^T b = b^T w_\circ$)

Theorem: 4.2.3

Let $x_\circ$ be a feasible solution to the primal problem

Maximize $f(x) = cx$ subject to: $Ax \le b,\ x \ge 0$

and $w_\circ$ be a feasible solution to its dual:

Minimize $g(w) = b^T w$ subject to: $A^T w \ge c^T,\ w \ge 0$

Where x^T and $c \in R^n$, w^T and $b^T \in R^m$ and A is an m x n real matrix.

If $cx_\circ = b^T w_\circ$, then both $x_\circ$ and $w_\circ$ are optimum solutions to the primal and dual respectively.

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Proof:

Let $x_\circ^*$ be any other feasible solution to the primal problem then theorem 4.2.1 gives $cx_\circ^* \le b^T w_\circ$.

Thus $\quad cx_\circ^* \le cx_\circ \quad$ (since $cx_\circ = b^T w_\circ$)

and hence $x_\circ$ is an optimum solution to the primal problem, because primal is a maximization problem.

Similarly, if $w_\circ^*$ is any other feasible solution to the dual problem, then $b^T w_\circ \le b^T w_\circ^*$ and thus $w_\circ$ is an optimum solution to the dual problem, because dual is a minimization problem.

Theorem: 4.2.4 (Basic Duality Theorem) Let a primal problem be

Maximize $f(x) = cx$ $\quad$ subject to: $Ax \le b,\ x \ge 0\ \ x^T, c \in R^n$

and the associated dual be

Minimize $g(w) = b^T w$ subject to: $A^T w \ge c^T,\ w \ge 0\ \ w^T, b^T \in R^m$.

If $x_\circ (w_\circ)$ is an optimum solution to the primal (dual), then there exists a feasible solution $w_\circ (x_\circ)$ to the dual (primal) such that

$$cx_\circ = b^T w_\circ.$$

Proof:

The standard primal can be written as

$$\text{Maximize } z = cx \quad \text{subject to: } Ax + Ix_s = b$$

where $x_s^T \in R^m$ is the slack vector and I is the associated identity matrix.

Let $x_\circ = [x_B, 0]$ be an optimum solution to the primal, where $x_B^T \in R^m$ is the optimal basic feasible solution given by $x_B = B^{-1}b$, B being the optimal basis of A. Then, the optimum primal objective function is

$$z = cx_\circ = c_B x_B$$

where $x_s^T \in R^m$ is a slack vector and I is associated identity matrix.

Let $x_\circ = [x_B, 0]$ be an optimum solution to the primal, where $x_B^T \in R^m$ is the optimal basic feasible solution given by $x_B = B^{-1}b$, B being the optimal basis of A. Then, the optimum primal objective function is

$$z = cx_\circ = c_B x_B$$

where c_B is the cost vector associated with x_B.

Now, the net evaluations in the optimal simplex table are given by

$$z_j - c_j = c_B y_j - c_j = \begin{cases} c_B\ B^{-1}a_j - c_j & \text{for all } a_j \in A \\ c_B\ B^{-1}e_j - 0 & \text{for all } e_j \in I \end{cases}$$

Since x_B is optimal, we must have $z_j - c_j \geq 0$ for all j. This gives

$$c_B B^{-1} a_j \geq c_j \quad \text{and} \quad c_B B^{-1} e_j \geq 0 \quad \text{for all } j$$

or $$c_B B^{-1} A \geq c \quad \text{and} \quad c_B B^{-1} \geq 0 \quad \text{(in matrix form)}$$

or $$A^T B^{-1} c_B^T = c^T \quad \text{and} \quad B^{-1} c_B^T \geq 0$$

Now, if we let $B^{-1}c_B^T = w_\circ$, the above become

$$A^T w_\circ \geq c^T \text{ and } w_\circ \geq 0,\ w_\circ^T \in R^m.$$

This means that $w_\circ$ is a feasible solution to the dual problem. Moreover, the corresponding dual objective function value is,

$$b^T w_\circ = w_\circ^T b = c_B B^{-1} b = c_B x_B = cx_\circ.$$

Thus given an optimal solution $x_\circ$ to the primal, there exists a feasible solution $w_\circ$ to the dual such the $cx_\circ = b^T w_\circ$.

Similarly, starting with $w_\circ$, the existence of $x_\circ$ can be proved.

Corollary:

If $x_\circ$ is an optimal solution to the primal, an optimal

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solution to the dual is given by, where B is the primal optimal basis.

Note:

Observe that $B^{-1}c_B^T$ represents optimum $z_i - c_i$ under primal slack columns.

Theorem: 4.2.5

(Fundamental Theorem of Duality). If the primal or the dual has a finite optimum solution, then the other problem also possesses a finite optimum solution and the optimum values of the objective functions of the two problems are equal.

Proof:

Consider the primal – dual pair:

Primal. Maximize $f(x) = cx$ subject to: $Ax \le b$ and $x \ge 0$.

Dual. Minimize $g(w) = b^T w$ subject to: $A^T w \ge c^T$ and $w \ge 0$.

(Necessary condition). Let a feasible solution $x_\circ(w_\circ)$ be an optimum solution to the primal (dual) problem. It then follows from Theorem 4.2.4 that there exists a feasible solution $w_0(x_0)$ to the dual (primal) problem, such that $cx_\circ = b^T w_\circ$.

It now follows from Theorem 4.2.3 that $w_\circ$ must be optimal.

This proves the necessity of the condition.

(Sufficiency). It follows from Theorem 4.2.3 .

Theorem: 4.2.6

(Existence Theorem). If either the primal or the dual problem has an unbounded objective function value, then the other problem has no feasible solution.

Proof:

Let the given primal problem have unbounded solution. Then for any value of the objective function, say $+\infty$, there exists a feasible solution say x yielding this solution. i.e., $cx \to \infty$. If possible, let the dual problem have a feasible solution. Then from Theorem 4.2.1, for

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each feasible solution $w_\circ$ of the dual, there exists a feasible solution $x_\circ$ to the primal such that $cx_\circ \le b^T w_\circ$. That is, $b^T w_\circ \to \infty$. Now as b is constant and $w_\circ$ has to satisfy the constraint $A^T w_\circ \ge c^T$, therefore the dual objective function $b^T w_\circ$ must be finite. This contradicts the result $b^T w_\circ \to \infty$. Hence the dual problem has no feasible solution.

By similar argument it can be shown that when the dual has an unbounded solution, the primal has no solution.

Standard results on duality can be summarized as follows:

Dual Problem	Primal Problem	
	Feasible Solution	No Feasible Solution
Feasible Solution	Optimum	Dual Unbounded
No Feasible Solution	Primal Unbounded	Unbounded or Infeasible

In the following section we shall see how the above existence theorem helps us in understanding the relationship between the optimum values of the primal and dual variables.

COMPLEMENTARY SLACKNESS THEOREM

Theorem: 4.2.7

(Complementary Slackness). Let $x_\circ$ and $w_\circ$ be the feasible solutions to the primal $\{\text{max. } c^T x \mid Ax \le b,\ x \ge 0\}$ and its dual $\{\text{min. } b^T w \mid A^T w \ge c^T,\ w \ge 0\}$ respectively. Then, a necessary and sufficient condition for $x_\circ$ and $w_\circ$ to be optimal to their respective problems is that

$$w_\circ^T (b - Ax_\circ) = 0 \quad \text{and} \quad x_\circ^T (A^T w_\circ - c^T) = 0.$$

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Proof:

Necessity. Let $\alpha = w_\circ^T(b - Ax_\circ)$ and $\beta = x_\circ^T(A^T w_\circ - c^T)$. Since $x_\circ, w_\circ$ are feasible solutions to the primal and dual respectively, we have

$$\alpha \geq 0,\ \beta \geq 0 \text{ and } \alpha + \beta = w_\circ^T b - x_\circ^T c^T.$$

Now if $x_\circ, w_\circ$ are optimal, then $cx_\circ = b^T w_\circ$ so that $\alpha + \beta = 0$. But since $\alpha \geq 0$ and $\beta \geq 0$, this gives $\alpha = 0$ and $\beta = 0$.

Thus the conditions are necessary.

Sufficiency. Let the given conditions hold for the feasible solutions $x_\circ$ and $w_\circ$. That is, $\alpha = 0$ and $\beta = 0$.

Then $\quad 0 = \alpha + \beta = w_\circ^T b - x_\circ^T c^T$

$\Rightarrow \quad cx_\circ = b^T w_\circ$

$\Rightarrow \quad x_\circ$ and $w_\circ$ are optimal.

Thus the conditions are sufficient.

Corollary: 1

If x° and w° be feasible solutions to the primal and dual problems respectively, then they will be optimal if and only if

$$w_i^\circ(b_i - \sum_{j=1}^{n} a_{ij} x_i^\circ) = 0. \qquad i = 1,2,\ldots\ldots,m$$

and $$x_j^\circ(\sum_{i=1}^{m} a_{ji} w_i^\circ - c_j) = 0. \qquad j = 1,2,\ldots\ldots,n.$$

Proof:

From the above theorem, x° and w° will be optimal if and only if

$$w_\circ^T(b - Ax_\circ) = 0 \text{ and } x_\circ^T(c^T - A^T w_\circ) = 0.$$

Consider the first set of conditions. Since each term in the summation $w_\circ^T(b - Ax_\circ)$ is non – negative, it follows that

$$w_i^\circ(b_i - \sum_{i=1}^{n} a_{ij} x_j^\circ) = 0, \qquad j = 1,2,\ldots\ldots,m.$$

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Similarly, the second set of conditions is equivalent to

$$x_j^\circ\left(\sum_{i=1}^{m} a_{ji} w_i^\circ - c_j\right) = 0, \qquad j = 1,2,\ldots\ldots,n.$$

Corollary: 2

For optimal feasible solutions of the primal and dual systems, whenever the ith variable is strictly positive in either system, the ith relation of its dual is an equality.

Proof:

It follows from corollary 1, that

$$w_i^\circ > 0 \Rightarrow \sum_{j=1}^{n} a_{ij} x_j^\circ = b_i \qquad \text{(ith primal relation)}$$

and $$x_j^\circ > 0 \Rightarrow \sum_{i=1}^{m} a_{ji} w_i^\circ = c_j \qquad \text{(jth dual relation)}$$

Corollary: 3

For optimal feasible solutions of the primal and dual systems, whenever ith relation of either system is satisfied as a strict inequality, then the ith variable of its dual vanishes.

Proof:

If follows from corollary 1 that

$$\sum_{j=1}^{n} a_{ij} x_j^\circ < b_i \Rightarrow w_i^\circ = 0$$

and $$\sum_{i=1}^{m} a_{ji} w_i^\circ > c_j \Rightarrow x_j^\circ = 0.$$

Remarks:

The conditions of corollary 1 can also be written as

$$w_i^\circ . x_{n+i} = 0$$

$i = 1,2,\ldots\ldots\ldots,m$

and $$x_j^\circ . w_{m+j} = 0$$

$j = 1,2,\ldots\ldots\ldots\ldots,n$

where x_{n+1} is the ith slack variable in the primal problem and w_{m+j}

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is the jth surplus variable in the dual.

Thus the theorem relates the variables of one problem to the slack or surplus variables of the other.

The above relations are called 'complementary slackness' because they imply that whenever a constraint in one of the problems holds with strict inequality (so that there is slack in the constraint), the complementary dual variable vanishes.

4.3 DUALITY AND SIMPLEX METHOD

Since any L.P.P can be solved by using simplex method, the method is applicable to both the primal as well as to its dual. The fundamental theorem of duality suggests that an optimum solution to the associated dual can be obtained from that of its primal and vice versa.

If primal is a maximization problem, then following are the set of rules that govern the derivation of the optimum solution:

Rule: 1

Corresponding net evaluations of the starting primal variables

= Difference between the left and right sides of the dual constraints associated with the starting primal variables.

Rule: 2

Negative of the corresponding net evaluations of the starting dual variables

= Difference between the left and right side of the primal constraints associated with dual starting variables.

Rule: 3

If the primal (dual) problem is unbounded, then the dual (primal) problem does not have any feasible solution.

Note:

In rule 2, dual problem is to be solved by changing the objective from minimization to the maximization.

Example: 4.3.1

Write down the dual of the following LPP and solve it.

$$\text{Max } Z = 4x_1 + 2x_2$$

Subject to the constraints $-x_1 - x_2 \le -3$

$$-x_1 + x_2 \ge -2$$

$$x_1, x_2 \ge 0$$

Hence or otherwise write down the solution of the primal.

Solution:

Given primal LPP is Max $Z = 4x_1 + 2x_2$

Subject to $-x_1 - x_2 \le -3$

$$x_1 - x_2 \le 2$$

$$x_1, x_2 \ge 0$$

Its dual problem is Min $W = -3y_1 + 2y_2$

Subject to $-y_1 + y_2 \ge 4$

$$-y_1 - y_2 \ge 2$$

$$y_1, y_2 \ge 0$$

$$\therefore \text{Max } W^* = 3y_1 - 2y_2$$

Subject to $-y_1 + y_2 \ge 4$

$$-y_1 - y_2 \ge 2$$

$$y_1, y_2 \ge 0$$

By introducing the surplus variables s_1, s_2 and the artificial variables R_1, R_2, we have

$$\text{Max } W^* = 3y_1 - 2y_2 + 0s_1 + 0s_2 - MR_1 - MR_2$$

Subject to $-y_1 + y_2 - s_1 + 0s_2 + R_1 + 0R_2 = 4$

$$-y_1 - y_2 + 0s_1 - s_2 + 0R_1 + R_2 \ge 0$$

The initial basic feasible solution is given by $R_1 = 4$, $R_2 = 2$ (basic)

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$(y_1 = y_2 = s_1 = s_2 = 0$, non – basic)

Initial Iteration:

		b_j	(3	-2	0	0	$-M$	$-M$)
C_B	Y_B	X_B	y_1	y_2	s_1	s_2	R_1	R_2
M	R_1	4	-1	(1)	-1	0	1	0
M	R_2	2	-1	-1	0	-1	0	1
$(W_j^*$	$-b_j)$	$-6M$	$2M-3$	2	M	M	0	0

Since all $(W_j^* - b_j) \geq 0$ and the artificial variables R_1 and R_2 appears in the basis at non – zero level, the dual problem does not possess any optimum basic feasible solution.

$\therefore$ There exists no finite optimum solution to the given primal LPP.

Example: 4.3.2

Use duality to solve the following LPP

Minimize $Z = 2x_1 + 2x_2$

Subject to $2x_1 + 4x_2 \geq 1$

$-x_1 - 2x_2 \leq -1$

$2x_1 + x_2 \geq 1$

and $x_1, x_2 \geq 0$.

Solution:

Given primal LPP is Minimize $Z = 2x_1 + 2x_2$

Subject to $2x_1 + 4x_2 \geq 1$

$x_1 + 2x_2 \geq 1$

$2x_1 + x_2 \geq 1$

and $x_1, x_2 \geq 0$.

Its dual problem is Max $W = y_1 + y_2 + y_3$

Subject to $2y_1 + y_2 + 2y_3 \leq 2$

$4y_1 + 2y_2 + y_3 \leq 2$

and $y_1, y_2, y_3 \geq 0$.

By introducing the non – negative slack variables s_1 and s_2 the standard form of the dual LPP becomes

Max $\quad W = y_1 + y_2 + y_3 + 0s_1 + 0s_2$

Subject to $\quad 2y_1 + y_2 + 2y_3 + s_1 + 0s_2 = 2$

$4y_1 + 2y_2 + y_3 + 0s_1 + s_2 = 2$

and $y_1, y_2, y_3, s_1, s_2 \geq 0$.

The initial basic feasible solution is $s_1 = 2,\ s_2 = 2$.

Initial Iteration:

	b_j	(1	1	1	0	0)	
C_B $\quad Y_B$	X_B	y_1	y_2	y_3	s_1	s_2	θ
0 $\quad s_1$	2	2	1	2	1	0	$\frac{2}{2}$
0 $\quad s_2$	2	(4)	2	1	0	1	$\frac{2}{4}$
$(W_j \ -b_j)$	0	-1	-1	-1	0	0	

First Iteration:

Introduce y_1 and drop s_2.

	b_j	(1	1	1	0	0)	
C_B $\quad Y_B$	X_B	y_1	y_2	y_3	s_1	s_2	θ
0 $\quad s_1$	1	0	0	$\left(\frac{3}{2}\right)$	1	$\frac{-1}{2}$	$\frac{2}{3}$
1 $\quad y_1$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$	2
$(W_j \ -b_j)$	$\frac{1}{2}$	0	$\frac{-1}{2}$	$\frac{-3}{4}$	0	$\frac{1}{4}$	

Second Iteration:

Introduce y_3 and drop s_1.

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		b_j	(1	1	1	0	0)	
C_B	Y_B	X_B	y_1	y_2	y_3	s_1	s_2	θ
1	y_3	$\frac{2}{3}$	0	0	1	$\frac{2}{3}$	$\frac{-1}{3}$	–
1	y_1	$\frac{1}{3}$	1	$\left(\frac{1}{2}\right)$	0	$\frac{-1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$*
$(W_j$	$-b_j)$	1	0	$\frac{-1}{2}$	0	$\frac{1}{2}$	0	

Third Iteration:

Introduce y_2 and drop y_1.

		b_j	(1	1	1	0	0)
C_B	Y_B	X_B	y_1	y_2	y_3	s_1	s_2
1	y_3	$\frac{2}{3}$	0	0	1	$\frac{2}{3}$	$\frac{-1}{3}$
1	y_2	$\frac{2}{3}$	2	1	0	$\frac{-1}{3}$	$\frac{2}{3}$
$(W_j$	$-b_j)$	$\frac{4}{3}$	1	0	0	$\frac{1}{3}$	$\frac{1}{3}$

Since all $(W_j - b_j) \geq 0$, the current basic feasible solution is optimal.

The optimal solution to the dual LPP is Max $w = \frac{4}{3}$, $y_1 = 0$, $y_2 = \frac{2}{3}$, $y_3 = \frac{2}{3}$. Here it is observed that the primal variable x_1 and x_2 respectively.

$\therefore$ The optimum solution to the original primal LPP is Min $Z = \frac{4}{3}$, $x_1 = \frac{1}{3}$ and $x_2 = \frac{1}{3}$.

Example: 4.3.3

Prove using duality theory that the following linear program is feasible but has no optimal solution.

$$\text{Minimize } Z = x_1 - x_2 + x_3$$

Subject to $\quad x_1 - x_3 \geq 4$

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$$x_1 - x_2 + 2x_3 \geq 3$$

$$\text{and } x_1, x_2, x_3 \geq 0.$$

Solution:

Given primal LPP is

$$\text{Min } Z = x_1 - x_2 + x_3$$

Subject to $x_1 + 0x_2 - x_3 \geq 4$

$$x_1 - x_2 + 2x_3 \geq 3$$

$$\text{and } x_1, x_2, x_3 \geq 0.$$

Its dual problem is

$$\text{Max } W = 4y_1 + 3y_2$$

Subject to $y_1 + y_2 \leq 1$

$$0y_1 + y_2 \geq 1$$

$$-y_1 + 2y_2 \leq 1$$

$$\text{and } y_1, y_2 \geq 0.$$

By introducing the slack variables s_1, s_3 and surplus variable s_2 and an artificial variable R_1, the standard form of the dual LPP is

$$\text{Max } W = 4y_1 + 3y_2 + 0s_1 + 0s_2 + 0s_3 - MR_1$$

Subject to $y_1 + y_2 + s_1 + 0s_2 + 0s_3 = 1$

$$0y_1 + y_2 + 0s_1 - s_2 + 0s_3 + R_1 = 1$$

$$-y_1 + 2y_2 + 0s_1 + 0s_2 + s_3 = 1$$

$$\text{and } y_1, y_2, s_1, s_2, s_3, R_1 \geq 0.$$

The initial basic feasible solution is given by

$s_1 = 1$, $R_1 = 1$, $s_3 = 1$ (basic) ($y_1 = y_2 = s_2 = 0$, non-basic)

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Initial Iteration:

		b_j	(4	3	0	0	−M	0)	
C_B	Y_B	X_B	y_1	y_2	s_2	s_1	R_1	s_3	θ
0	s_1	1	1	1	0	1	0	0	1
−M	R_1	1	0	1	−1	0	1	0	1
0	s_3	1	−1	(2)	0	0	0	1	$\frac{1}{2}$*
$(W_j$	$-b_j)$	−M	−4	−M−3	M	0	0	0	

First iteration: Introduce y_2 and drop s_3.

		b_j	(4	3	0	0	−M	0)	
C_B	Y_B	X_B	y_1	y_2	s_2	s_1	R_1	s_3	θ
0	s_1	$\frac{1}{2}$	$\left(\frac{3}{2}\right)$	0	0	1	0	$-\frac{1}{2}$	$\frac{1}{3}$
−M	R_1	$\frac{1}{2}$	$\frac{1}{2}$	0	−1	0	1	$-\frac{1}{2}$	1
3	y_2	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	0	0	$\frac{1}{2}$	−
$(W_j$	$-b_j)$	$\frac{-M+3}{2}$	$\frac{-M-11}{2}$	0	M	0	0	$\frac{M+3}{2}$	

Second Iteration:

Introduce y_1 and drop s_1.

		b_j	(4	3	0	0	−M	0)
C_B	Y_B	X_B	y_1	y_2	s_2	s_1	R_1	s_3
4	y_1	$\left(\frac{1}{3}\right)$	1	0	0	$\frac{2}{3}$	0	$-\frac{1}{3}$
−M	R_1	$\frac{1}{3}$	0	0	−1	$-\frac{1}{3}$	1	$-\frac{1}{3}$
3	y_2	$\frac{2}{3}$	0	1	0	$\frac{1}{3}$	0	$\frac{1}{3}$
W_j	$-b_j$	$\frac{10-M}{3}$	0	0	M	$\frac{11+M}{3}$	0	$\frac{M-1}{3}$

Since all $(W_j - b_j) \leq 0$, and an artificial variables R_1 appears in the basis at non – zero level, the dual problem has no optimal basic feasible solution.

The exists no finite optimum solution to the given primal LPP.

Check your progress 4.2

1. Use duality to solve the following L.P.P:

   Maximize $z = x_1 - x_2 + 3x_3 + 2x_4$ subject to

$x_1 + x_2 \geq -1,\ x_1 - 3x_2 - x_3 \leq 7,\ x_1 + x_3 - 3x_4 = -2;\ \ x_j \geq 0 \ \ (j = 1,2,3,4)$

2. Use duality in solving the L.P.P:

   Minimize $z = 2x_1 - x_2 + x_3 + 5x_4 - 3x_5$ subject to the constraints:

$4x_1 + 2x_2 + x_3 + x_4 = 3,\ 2x_1 + 2x_2 + x_3 + x_5 = 2;\ x_j \geq 0 \ \ (j = 1,2,3,4,5)$

3. Consider the problem:

   Maximize $z = 2x_2 - 5x_3$ subject to the constraints:

$$x_1 + x_3 \geq 2,\ 2x_1 + x_2 + 6x_3 \leq 6,\ x_1 - x_2 + 3x_3 = 0.$$

$$x_1, x_2, x_3 \geq 0.$$

4.4 DUAL SIMPLEX METHOD

The dual simplex method is used to solve problems which start dual feasible i.e., whose primal is optimal but infeasible. In this method the solution starts optimum but infeasible and remains infeasible until the true optimum is reached at which the solution becomes feasible.

The regular simplex method starts with a basic feasible but non – optimal solution and works toward optimality, whereas the dual simplex method starts with a basic infeasible but optimal solution and works towards feasibility. Also in regular simplex method we first determine the entering variable and then the leaving variable while in the case of dual simplex method we first determine the leaving variable and then the entering variable.

Working procedure for dual simplex method:

Step: 1

Convert the problem to maximization form if it is initially in the minimization form.

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Step: 2

Convert $(\geq)$ type constraints; if any to $(\leq)$ type by multiplying both side by -1.

Step: 3

Convert the inequality constraints into equalities by introducing slack variables. Obtain the initial basic solution and express this information in the simplex table.

Step: 4

(Optimal condition) Test the nature of $(Z_j - C_j)$ and X_{Bi}.

Case: (i)

If all $(Z_j - C_j) \geq 0$ and all $X_{Bi} \geq 0$, then the current solution is an optimum feasible solution.

Case: (ii)

If all $(Z_j - C_j) \geq 0$ and atleast one $X_{Bi} < 0$, then the current solution is not an optimum basic feasible solution and go to the next step.

Case: (iii)

If any $(Z_j - C_j) < 0$, then this method fails.

Step: 5 (Feasibility condition)

(i) **(Leaving variable):** The leaving variable is the basic variable corresponding to the most negative value of X_{Bi}. Let x_k be the leaving variable.

(ii) **(Entering variable):** Compute the ratio between $(Z_j - C_j)$ row and the key row. i.e., compute $\theta = \max\left\{\frac{(Z_j - C_j}{a_{ik}},\ a_{ik} < 0\right\}$. Consider the ratios with –ve denominators alone). The entering vairbale is the non – basic variable corresponding to the maximum ratio θ. If there is no such ratio with –ve denominator, then the problem does not have a feasible solution.

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Example: 4.4.1

Using dual simplex method solve the LPP

$$\text{Minimize } Z = 2x_1 + x_2$$

Subject to
$$3x_1 + x_2 \geq 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \geq 3$$
$$\text{and } x_1, x_2 \geq 0.$$

Solution:

After converting the object function in to maximization type and all the constraints in $\leq$ type, the given LPP becomes

$$\text{Max } Z^* = -2x_1 - x_2$$

Subject to
$$-3x_1 - x_2 \leq -3$$
$$-4x_1 - 3x_2 \leq -6$$
$$-x_1 - 2x_2 \leq -3$$
$$\text{and } x_1, x_2 \geq 0.$$

By introducing the non – negative slack variables s_1, s_2 and s_3, the LPP becomes

$$\text{Max } Z^* = -2x_1 - x_2$$

Subject to
$$-3x_1 - x_2 \leq -3$$
$$-4x_1 - 3x_2 \leq -6$$
$$-x_1 - 2x_2 \leq -3$$
$$\text{and } x_1, x_2 \geq 0.$$

By introducing the non – negative slack variables s_1, s_2 and s_3, the LPP becomes

$$\text{Max } Z^* = -2x_1 - x_2 + 0s_1 + 0s_2 + 0s_3$$

Subject to
$$-3x_1 - x_2 + s_1 = -3$$
$$-4x_1 - 3x_2 + s_2 = -6$$

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$$-x_1 - 2x_2 + s_3 = -3$$

$$\text{and } x_1, x_2, s_1, s_2, s_3 \geq 0.$$

The initial basic solution is given by

$$s_1 = -3,\ s_2 = -6,\ s_3 = -3 \text{ (basic) } (x_1 = x_2 = 0, \text{ non-basic})$$

Initial Iteration:

C_j $(-2 \quad -1 \quad 0 \quad 0 \quad 0)$

C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3
0	s_1	-3	-3	-1	1	0	0
0	s_2	-6	-4	(-3)	0	1	0
0	s_3	-3	-1	-2	0	0	1
$(Z_j^*$	$-C_j)$	0	2	1	0	0	0

Since all $(Z_j^* - C_j) \geq 0$ and all $X_{Bi} < 0$, the current solution is not an optimum basic feasible solution.

Since $X_{B2} = -6$ is the most negative, the corresponding basic variable s_2 leaves the basis.

Now $\theta = \text{Max}\left\{\dfrac{(Z_j^* - C_j)}{a_{ik}},\ a_{ik} < 0\right\}$ where x_k is the leaving variable.

$$= \text{Max}\left\{\frac{2}{-4}, \frac{1}{-3},\right\} = \text{Max}\left\{\frac{-1}{2}, \frac{-1}{3}\right\} = \frac{-1}{3}$$

$\therefore$ The corresponding non – basic variable x_2 enters into the basis.

First Iteration:

Drop s_2 and introduce x_2.

		C_j	$(-2$	-1	0	0	$0)$
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3
0	s_1	-1	$\left(\frac{-5}{3}\right)$	0	1	$\frac{-1}{3}$	0
-1	x_2	2	$\frac{4}{3}$	1	0	$\frac{-1}{3}$	0
0	s_3	1	$\frac{5}{3}$	0	0	$\frac{-2}{3}$	1
$(Z_j^*$	$-C_j)$	-2	$\frac{2}{3}$	0	0	$\frac{1}{3}$	0

Since all $(Z_j^* - C_j) \geq 0$, and $X_{B1} = -1 < 0$, the current solution is not optimum basic feasible solution.

Since $X_{B1} = -1$ is negative, the corresponding basic variables s_1 leaves the basis.

$$\text{Now } \theta = \text{Max}\left\{\frac{(Z_j^* - C_j}{a_{ik}},\ a_{ik} < 0\right\}$$

$$= \text{Max}\left\{\frac{2/3}{-5/3}, \frac{1/3}{-1/3}\right\} = \text{Max}\left\{\frac{-2}{5}, -1\right\} = \frac{-2}{5}$$

$\therefore$ The corresponding non – basic variable x_1 enters the basis.

Second Iteration:

Drop s_1 and introduce x_1.

		C_j	$(-2$	-1	0	0	$0)$
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3
-2	x_1	$\frac{3}{5}$	1	0	$\frac{-3}{5}$	$\frac{1}{5}$	0
-1	x_2	$\frac{6}{5}$	0	1	$\frac{4}{5}$	$\frac{-3}{5}$	0
0	s_3	0	0	0	1	-1	1
$(Z_j^*$	$-C_j)$	$\frac{-12}{5}$	0	0	$\frac{2}{5}$	$\frac{1}{5}$	0

Since all $(Z_j^* - C_j) \geq 0$, and all $X_{Bi} \geq 0$, the current solution is an optimum basic feasible solution.

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The optimum solution is Max $Z^* = \frac{-12}{5}, x_1 = \frac{3}{5}, x_2 = \frac{6}{5}$.

But Min $Z = -\text{Max} \quad Z^* = -\left(\frac{12}{5}\right) = \frac{12}{5}$

$\therefore$ Min $\quad Z = \frac{12}{5}, x_1 = \frac{3}{5}, x_2 = \frac{6}{5}$.

Example: 4.4.2

Using dual simplex method solve the LPP

$$\text{Maximize} \quad Z = 6x_1 + 4x_2 + 4x_3$$

Subject to

$$3x_1 + x_2 + 2x_3 \geq 2$$
$$2x_1 + x_2 - x_3 \geq 1$$
$$-x_1 + x_2 + 2x_3 \geq 1$$
$$\text{and } x_1, x_2, x_3 \geq 0.$$

Solution:

The given LLP is

$$\text{Max } Z = 6x_1 + 4x_2 + 4x_3$$

Subject to

$$-3x_1 - x_2 - 2x_3 \leq -2$$
$$-2x_1 - x_2 + x_3 \leq -1$$
$$x_1 - x_2 - 2x_3 \leq -1$$
$$\text{and } x_1, x_2, x_3 \geq 0.$$

By introducing the non – negative slack variable s_1, s_2 and s_3, the LPP becomes

$$\text{Max } Z = 6x_1 + 4x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$$

Subject to

$$-3x_1 - x_2 - 2x_3 + s_1 = -2$$
$$-2x_1 - x_2 + x_3 + s_2 = -1$$
$$x_1 - x_2 - 2x_3 + s_3 = -1$$
$$\text{and } x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

The initial basic solution is given by

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$s_1 = -2,\ s_2 = -1,\ s_3 = -1$ (basic) ($x_1 = x_2 = x_3 = 0$, non – basic)

Initial Iteration:

C_B	Y_B	C_j: X_B	(6) x_1	(4) x_2	(4) x_3	(0) s_1	(0) s_2	(0) s_3
0	s_1	-2	-3	-1	-2	1	0	0
0	s_2	-1	-2	-1	1	0	1	0
0	s_3	-1	1	-1	-2	0	0	1
$(Z_j$	$-C_j)$	0	-6	-4	-4	0	0	0

Since there are some $(Z_j - C_j) < 0$, this method fails. i.e., we cannot solve this problem by this dual simplex method.

Example: 4.4.3

Using dual simplex method solve the LPP

Minimize $Z = x_1 + x_2$

Subject to $2x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq 1$

and $x_1,\ x_2 \geq 0$.

Solution:

The given LPP is Max $Z^* = -x_1 - x_2$

Subject to $-2x_1 - x_2 \leq -2$

$x_1 + x_2 \leq -1$

and $x_1, x_2 \geq 0$.

By introducing the non – negative slack variables s_1 and s_2, the LPP becomes

Max $Z^* = -x_1 - x_2 + 0s_1 + 0s_2$

Subject to $-2x_1 - x_2 + s_1 = -2$

$x_1 + x_2 + s_2 = -1$

and $x_1, x_2, s_1, s_2 \geq 0$.

The initial basic solution is given by

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$\therefore$ $s_1 = -2,\ s_2 = -1$, (basic) ($x_1 = x_2 = 0$, non – basic)

Initial Iteration:

C_j (−1 −1 0 0)

C_B	Y_B	X_B	x_1	x_2	s_1	s_2
0	s_1	−2	(−2)	−1	1	0
0	s_2	−1	1	1	0	1
$(Z_j^* - C_j)$		0	1	1	0	0

Since all $(Z_j^* - C_j) \geq 0$, and all $X_{Bi} < 0$, the current solution is not an optimum basic feasible solution.

Since $X_{B1} = -2$ is most negative, the corresponding basic variable s_1 leaves the basis.

$$\text{Now } \theta = \text{Max}\left\{\frac{(Z_j^* - C_j}{a_{ik}},\ a_{ik} < 0\right\}$$

$$= \text{Max}\left\{\frac{1}{-2}, \frac{1}{-1}\right\} = \text{Max}\left\{\frac{-1}{2}, -1\right\} = \frac{-1}{2}.$$

$\therefore$ The corresponding non – basic variable x_1 enters the basis.

First Iteration:

Drop s_1 and introduce x_1

C_j (−1 −1 0 0)

C_B	Y_B	X_B	x_1	x_2	s_1	s_2
−1	x_1	1	1	$\frac{1}{2}$	$\frac{-1}{2}$	0
0	s_2	−2	0	$\frac{1}{2}$	$\frac{1}{2}$	1
$(Z_j^* - C_j)$		−1	0	$\frac{1}{2}$	$\frac{1}{2}$	0

Since all $(Z_j^* - C_j) \geq 0$, and $X_{B2} = -2 < 0$, the current solution is not an optimum basic feasible solution.

Since $X_{B2} = -2$, corresponding basic variable s_2 leaves the basis.

Now $\theta = \text{Max}\left\{\frac{(Z_j - C_j)}{a_{ik}},\ a_{ik} < 0\right\}$ where x_k is the leaving variable.

Since all the entries in the key row are positive, we cannot find the ratio θ with negative denominators. So, there is no feasible solution to the given LPP.

Example: 4.4.4

Use dual simplex method to solve the LPP.

Maximize $Z = -3x_1 - 2x_2$

Subject to
$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \geq 10$$
$$x_2 \leq 3$$
and $x_1, x_2 \geq 0$.

Solution:

The given LPP is Maximize $Z = -3x_1 - 2x_2$

Subject to
$$-x_1 - x_2 \leq -1$$
$$x_1 + x_2 \leq 7$$
$$-x_1 - 2x_2 \leq -10$$
$$0x_1 + x_2 \leq 3$$
and $x_1, x_2 \geq 0$

By introducing the non – negative slack variables s_1, s_2, s_3 and s_4, the LPP becomes

Max $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

Subject to
$$-x_1 - x_2 + s_1 = -1$$
$$x_1 + x_2 + s_2 = 7$$
$$-x_1 - 2x_2 + s_3 = -10$$
$$0x_1 + x_2 + s_4 = 3$$

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and $x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$.

The initial basic solution is given by

$s_1 = -1,\ s_2 = 7,\ s_3 = -10,\ s_4 = 3$ (basic) $(x_1 = x_2 = 0$, non – basic)

Initial Iteration:

		C_j	$(-3$	-2	0	0	0	$0)$
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	s_4
0	s_1	-1	-1	-1	1	0	0	0
0	s_2	7	1	1	0	1	0	0
0	s_3	-10	-1	(-2)	0	0	1	0
0	s_4	3	0	1	0	0	0	1
$(Z_j$	$-C_j)$	0	3	2	0	0	0	0

First Iteration:

Drop s_3 and introduce x_2

		c_j	$(-3$	-2	0	0	0	$0)$
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	s_4
0	s_1	4	$-\frac{1}{2}$	0	1	0	$-\frac{1}{2}$	0
0	s_2	2	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	0
-2	x_2	5	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	0
0	s_4	-2	$(-\frac{1}{2})$	0	0	0	$\frac{1}{2}$	1
$(Z_j$	$-C_j)$	-10	2	0	0	0	1	0

Second Iteration:

Drop s_4 and introduce x_1

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		C_j	(−3	−2	0	0	0	0)
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	s_4
0	s_1	2	0	0	1	0	−1	−1
0	s_2	0	0	0	0	1	1	1
−2	x_2	3	0	1	0	0	0	1
−3	x_1	4	1	0	0	0	−1	−2
$(Z_j$	$-C_j)$	−18	0	0	0	0	3	4

Since all $(Z_j - C_j) \geq 0$ and all $X_{Bi} \geq 0$, the current solution is an optimum basic feasible solution.

$\therefore$ The optimum solution is Max $Z = -18,\ x_1 = 4,\ x_2 = 3$.

Check your progress 4.3

Use dual simplex method to solve the following linear programming problem:

1. Minimize $z = x_1 + x_2$ subject to:

   $2x_1 + x_2 \geq 4,\ x_1 + 7x_2 \geq 7;\ x_1, x_2 \geq 0$.

2. Maximize $z = -2x_1 - x_2$ subject to:

   $3x_1 + x_2 \geq 3$, $4x_1 + 3x_2 \geq 6$,

   $x_1 + 2x_2 \geq 3;\ x_1, x_2 \geq 0$

3. Minimize $z = 10x_1 + 6x_2 + 2x_3$ subject to the constraints:

   $-x_1 + x_2 + x_3 \geq 1, 3x_1 + x_2 - x_3 \geq 2, x_1, x_2, x_3 \geq 0$.

4. Minimize $z = 6x_1 + 7x_2 + 3x_3 + 5x_4$ subject to the constraints:

$5x_1 + 6x_2 - 3x_3 + 4x_4 \geq 12x_2 + 5x_3 - 6x_4 \geq 10$

$2x_1 + 5x_2 + x_3 + x_4 \geq 8, x_1, x_2, x_3, x_4 \geq 0$

4.5 INTEGER PROGRAMMING

A linear programming problem in which some or all of the variables in the optimal solution are restricted to assume non – negative integer values is called an integer programming problem [or I.P.P or integer linear programming].

Space for Hints

In a linear programming problem, if all the variables in the optimal solution are restricted to assume non – negative integer values, then it is called the Pure (all) integer programming problem [Pure I.P.P].

In a linear programming problem, if only some of the variables in the optimal solution are restricted to assume non – negative integer values, while the remaining variables are free to take any non – negative values, then it is called a Mixed integer programming problem [Mixed I.P.P]

Further, if all the variables in the optimum solution are allowed to take values either 0 or 1 as in 'do' or 'not to do' type decisions, then the problem is called the Zero – one programming problem (or) standard discrete programming problem.

The general integer programming problem is given by

$$\text{Maximize } Z = CX$$

Subject to the constraints

$$AX \leq b,$$

$$X \geq 0 \text{ and some or all variables are integers.}$$

Importance of Integer Programming:

In linear programming problem, all the decision variables were allowed to take any non – negative real (continuous or fractional) values, as it is quit possible and appropriate to have fractional values in many situations. For example, it is quit possible to use 6.38 kg of raw material, or 5.62 machine hours etc. However in many situations, especially in business and industry, these decision variables make sense only if they business and industry, these decision variables make sense only if they have integer values in the optimal solution. For example, it is meaningless to produce 8.13 chairs or 6.85 tables, or to open 3.83 branches of a bank or to run 9.6 cars etc. Hence a new procedure has been developed in this direction for the case of LPP subjected to the additional restriction that the decision variables must have integer values.

Space for Hints

Applications of Integer Programming

Integer programming problems occur quit frequently in business and industry.

All transportation, assignment and travelling salesman problems are integer programming problems, since the decision variables are either zero or one. i.e., $x_{ij} = 0$ or 1.

All sequencing and routing decisions and integer programming problems as it requires the integer values of the decision variables.

Capital budgeting and production scheduling problems are integer programming problems. In fact, any situation involving decisions of the type "either to do a job or not to do" (either – or) can be treated as an integer programming problem. In these situations,

$$x_j = \begin{cases} 1, \text{ if } j^{th} \text{ job is performed} \\ 0, \text{ if } j^{th} \text{ job is not performed} \end{cases}$$

All allocation problems involving the allocation of goods, men, machines, give rise to integer programming problems, since such commodities can be assigned only integer and not fractional values.

Pitfalls in rounding the optimum solution of an I.P.P.

We may think to solve such problems by the usual simplex method (ignoring the integrality restriction) and then rounding off the non – integer values to integers in the optimal solution obtained by the simplex method. But there is no guarantee that the integer values solution thus obtained will satisfy all the constraints i.e., it may not satisfy one or more constraints and as such the new solution may not be feasible. For example, consider the problem

Space for Hints

$$\text{Max } Z = 20x_1 + 8x_2$$

Subject to $3x_1 + 4x_2 \le 8$

$x_1, x_2 \ge 0$ and integers

By using graphical method (or simplex method), the optimal non integer solution (ignoring the integrality condition) of this problem is given by Max $Z = 53.40$, $x_1 = 2.67$, $x_2 = 0$.

Now rounding the solution to $x_1 = 3$, $x_2 = 0$, it does not satisfy the constraint $3x_1 + 4x_2 \le 8$. Hence the solution $x_1 = 3$, $x_2 = 0$ is not feasible. Thus a new rounded solution may not be feasible.

Further if we rounding the solution to $x_1 = 2$, $x_2 = 0$, then the solution is feasible but gives Max Z = 40, which is far away from the optimum solution Max Z = 53.4. There is no guarantee that the rounded down solution will be optimum also.

Due to these difficulties, there is a need for developing a systematic and efficient algorithm for obtaining the exact optimum integer solution to an integer programming problem.

Methods of Integer Programming

Integer programming methods can be categorized as (1) cutting methods and (2) search methods.

Cutting Methods:

A systematic procedure for solving pure integer programming problem was first developed by R.E. Gomory in 1958. Later on, he extended the procedure to solve mixed I.P.P named as Cutting plane algorithm, the method consists in first solving the I.P.P as ordinary L.P.P. by ignoring the integrality restriction and then introducing additional constraints one after the other to cut (eliminate) certain part of the solution space until an integral solution is obtained.

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Search Method:

It is an enumeration method in which all feasible integer points are enumerated. The widely used search method is the branch and bond technique. It also starts with the continuous optimum, but systematically partitions the solution space into sub problems that eliminate parts that contain no feasible integer solution. It was originally developed by A.H Land and A.G. Doig.

Gomory's Fractional Cut algorithm (or) Cutting Plane Method for pure (all) I.P.P:

Step: 1

Convert the minimization I.P.P. in to an equivalent maximization I.P.P. and all the coefficients and constants should be integers. Ignore the integerality condition.

Step: 2

Find the optimum solution of the resulting maximization L.P.P by using simplex method.

Step: 3

Test the integerality of the optimum solution.

(i) If all $X_{Bi} \geq 0$ and the integers, an optimum integer solution is obtained.

(ii) If all $X_{Bi} \geq 0$ and atleast one X_{Bi} is not an integer, then go to the next step.

Step: 4

Rewrite each X_{Bi} as $X_{Bi} = [X_{Bi}] + f_i$, where $[X_{Bi}]$ is the integral part of X_{Bi} and f_i is the positive fractional part of X_{Bi}, $0 \leq f_i < 1$. Choose the largest fraction of X_{Bi},s, i.e., choose Max $\{f_i\}$. In case of a tie, select arbitrarily. Let Max $\{f_i\} = f_k$ corresponding to X_{Bk} (the k^{th} row corresponding to this f_k is called source row).

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Step: 5

Express each of the negative fractions if any, in the k^{th} row (source row) of the optimum simplex table as the sum of a negative integer and a non – negative fraction.

Step: 6

Find the fractional cut constraint (Gomorian constraint or secondary constraint)

From the source row $\sum_{j=1}^{n} a_{kj}x_j = X_{Bk}$

i.e., $\sum_{j=1}^{n} ([a_{kj}] + f_{kj})x_j = [X_{Bk}] + f_k$

in the form $\sum_{j=1}^{n} f_{kj}x_j \geq f_k$

(or) $-\sum_{j=1}^{n} f_{kj}x_j \leq -f_k$ or $-\sum_{j=1}^{n} f_{kj} + s_1 = -f_k$

Where s_1 is the Gomorian slack.

Step: 7

Add the fractional cut constraint obtained in step 6 at the bottom of the optimum simplex table obtained in step 2. Find the new feasible optimum solution using dual simplex method.

Step: 8

Go to step 3 and repeat the procedure until an optimum integer solution in obtained.

Note:

In this cutting plane method, the fractional cut constraints cut the unuseful area of the feasible region in the graphical solution of the problem. i.e., cut that area which has no integer – valued feasible solution. Thus these Gomorian constraints eliminate all the non – integral solutions without loosing any integer – valued solution.

Example: 4.5.1

Find the optimum integer solution to the following L.P.P.

$$\text{Max } Z = x_1 + x_2$$

Subject to constraints

$$3x_1 + 2x_2 \le 5$$

$$x_2 \le 2$$

and $x_1 > 0,\ x_2 \ge 0$ and are integers.

Solution:

Ignoring the integrality condition and introducing the non – negative slack variables x_3 and x_4, the standard form of the continuous LPP becomes

$$\text{Max } Z = x_1 + x_2 + 0x_3 + 0x_4$$

Subject to $3x_1 + 2x_2 + x_3 + 0x_4 = 5$

$$0x_1 + x_2 + 0x_3 + x_4 = 2$$

and $x_1, x_2, x_3, x_4 \ge 0$

The initial basic feasible solution is given by $x_3 = 5,\ x_4 = 2,\ (x_1 = x_2 = 0$, non – basic).

Initial Iteration:

C_j (1 1 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	5	(3)	2	1	0	$\frac{5}{3}$
0	x_4	2	0	1	0	1	–
$(Z_j$	$-C_j)$	0	–1	–1	0	0	

Since some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

First Iteration:

Introduce x_1 and drop x_3.

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C_j (1 1 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
1	x_1	$\frac{5}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{5}{2}$
0	x_4	2	0	1	0	1	2*
$(Z_j$	$-C_j)$	$\frac{5}{3}$	0	$\frac{-1}{3}$	$\frac{1}{3}$	0	

Since some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

Second Iteration:

Introduce x_2 and drop x_4.

C_j (1 1 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4
1	x_1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	$\frac{-2}{3}$
1	x_2	2	0	1	0	1
$(Z_j$	$-C_j)$	$\frac{7}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal and non – integer, i.e., Max $Z = \frac{7}{3}$, $x_1 = \frac{1}{3}$, $x_2 = 2$

To obtain the optimum integer solution, we have to add a fractional cut constraint in the optimum simplex table.

Since $x_1 = \frac{1}{3}$, from the source row (first row)

We have $\frac{1}{3} = x_1 + \frac{1}{3}x_3 - \frac{2}{3}x_4$

Expressing the negative fraction as a sum of a negative integer and non – negative fraction, we have

$$\frac{1}{3} = x_1 + \frac{1}{3}x_3 + \left(-1 + \frac{1}{3}\right)x_4$$

The fractional cut (Gomorian) constraint is given by

$$\frac{1}{3}x_3 + \frac{1}{3}x_4 \geq \frac{1}{3} \Rightarrow \frac{-1}{3}x_3 - \frac{1}{3}x_4 \leq \frac{-1}{3}$$

$$\Rightarrow -\frac{1}{3}x_3 - \frac{1}{3}x_4 + x_1 = -\frac{1}{3}$$

where s_1 is Gomorian slack.

Add this fractional cut constraint at the bottom of the above optimum simplex table, we have the new simplex table.

	C_j	(1	1	0	0	0)
C_B Y_B	X_B	x_1	x_2	x_3	x_4	s_1
1 x_1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	$\frac{-2}{3}$	0
1 x_2	2	0	1	0	1	0
0 s_1	$-\frac{1}{3}$	0	0	$\frac{-1}{3}$	$\frac{-1}{3}$	1
$(Z_j - C_j)$	$\frac{7}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	0

Here the solution is optimal but infeasible.

To obtain the feasible optimal solution, we have to use the dual simplex method.

Since $x_1 = -\frac{1}{3}$, s_1 leaves the basis.

To find the entering variable: Let Max $\left\{\frac{Z_j - C_j}{a_{ik}},\ a_{ik} < 0\right\}$

$= \text{Max}\left\{\frac{\frac{1}{3}}{-\frac{1}{3}}, \frac{\frac{1}{3}}{-\frac{1}{3}}\right\} = \text{Max}\{-1,-1\} = -1$ which corresponds to both x_3 and x_4. We choose x_3 as the entering variable arbitrarily.

Third Iteration:

Drop s_1 and introduce x_3.

	C_j	(1	1	0	0	0)
C_B Y_B	X_B	x_1	x_2	x_3	x_4	s_1
1 x_1	0	1	0	0	-1	1
1 x_2	2	0	1	0	1	1
0 x_3	1	0	0	1	1	-3
$(Z_j - C_j)$	2	0	0	0	0	1

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Space for Hints

Since all $(Z_j - C_j) \geq 0$ and all $X_{Bi} \geq 0$, the current solution is feasible and optimal and integer.

$\therefore$ The optimum integer solution is

$$\text{Max } Z = 2,\ x_1 = 0,\ x_2 = 2.$$

Geometrical Interpretation of cutting plane method:

For the above I.P.P, the feasible regions OABC, is shown shaded in the following figure.

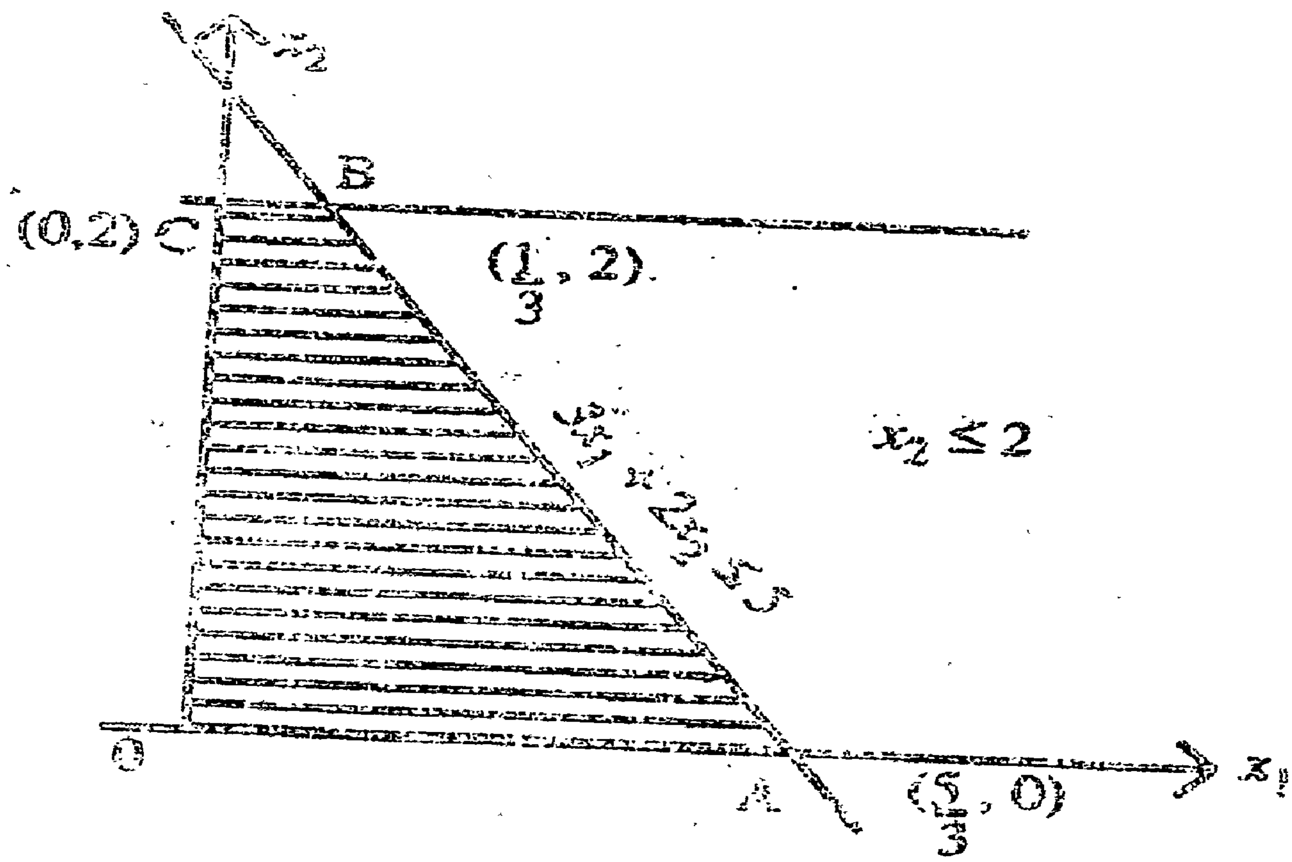

Figure 9.2

The optimum solution is

$$\text{Max } Z = \frac{7}{3},\ x_1 = \frac{1}{3},\ x_2 = 2$$

Since this solution is not an integer optimum solution, we introduce the secondary (Gomorian) Constraint

$$\frac{x_3}{3} + \frac{x_4}{3} \geq \frac{1}{3}$$

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The express this in terms of x_1 and x_2, we know that

$$3x_1 + 2x_2 + x_3 = 5 \Rightarrow x_3 = 5 - 3x_1 - 2x_2$$

$$\text{and } x_2 + x_4 = 2 \Rightarrow x_4 = 2 - x_2$$

Substituting in the Gomory Constraints, we have

$$\frac{1}{3}(5 - 3x_1 - 2x_2) + \frac{1}{3}(2 - x_2) \geq \frac{1}{3}$$

$$\Rightarrow 5 - 3x_1 - 2x_2 + 2 - x_2 \geq 1$$

$$\Rightarrow -3x_1 - 3x_2 + 7 \geq 1 \Rightarrow -3x_1 - 3x_2 \geq -6$$

$$\Rightarrow 3x_1 + 3x_2 \leq 6 \Rightarrow x_1 + x_2 \leq 2$$

Drawing the line $x_1 + x_2 = 2$, the above feasible region is cut off to the shaded region shown below:

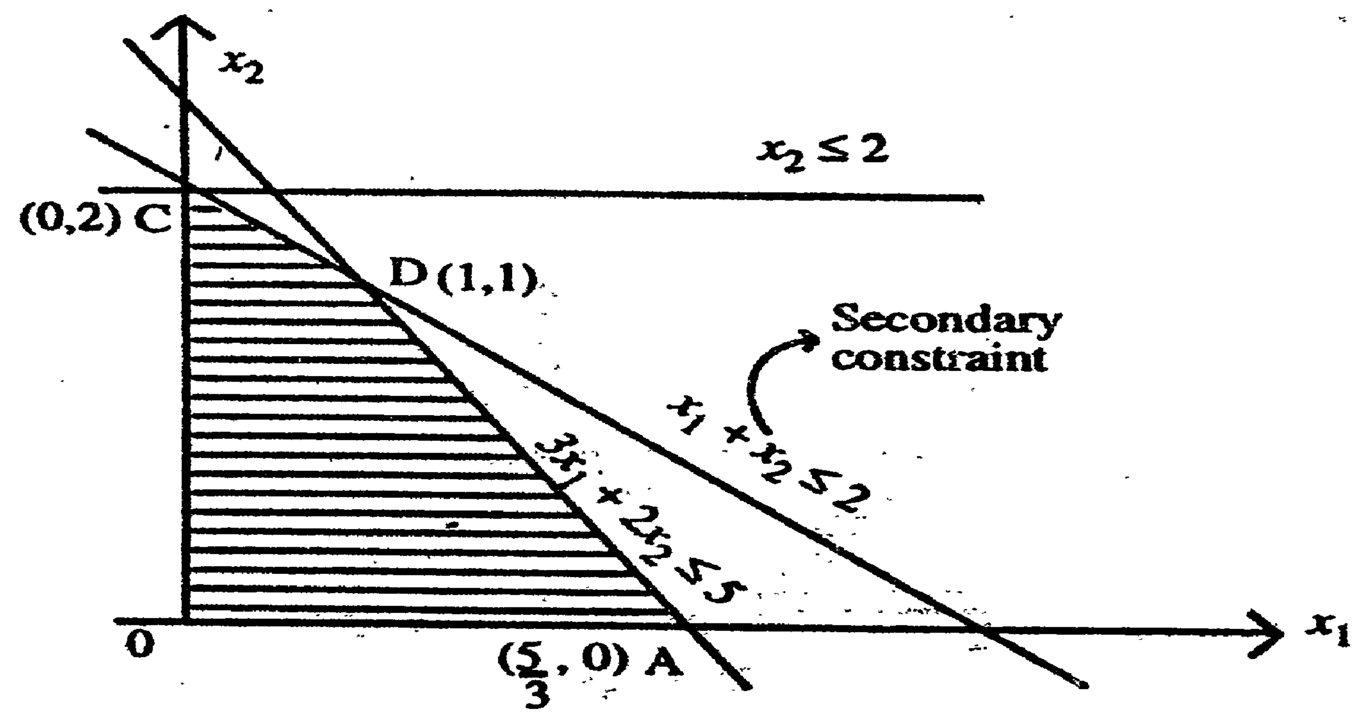

Figure 9.2

Thus the required optimal integer valued solution is

$$\text{Max } Z = 2,\ x_1 = 0,\ x_2 = 2 \text{ (or)}$$

$$\text{Max } Z = 2,\ x_1 = 1,\ x_2 = 1$$

Example: 4.5.2

Solve the following IPP

$$\text{Minimize } Z = -2x_1 - 3x_2$$

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Subject to $2x_1 + 2x_2 \leq 7$

$x_1 \leq 2$

$x_2 \leq 2$

$x_1, x_2 \geq 0$ and integers.

Solution:

Given I.P.P is

Minimize $Z = -2x_1 - 3x_2$

Subject to $2x_1 + 2x_2 \leq 7$

$x_1 \leq 2$

$x_2 \leq 2$

$x_1, x_2 \geq 0$ and integers.

i.e., Maximize $Z^* = 2x_1 + 3x_2$

subject to $2x_1 + 2x_2 \leq 7$

$x_1 \leq 2$

$x_2 \leq 2$

$x_1, x_2 \geq 0$ and integers.

Ignoring the integrality condition and introducing the non – negative slack variables x_3, x_4 and x_5, the standard form of the continuous L.P.P becomes

Maximize $Z^* = 2x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5$

Subject to $2x_1 + 2x_2 + x_3 + 0x_4 + 0x_5 = 7$

$x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 2$

$0x_1 + x_2 + 0x_3 + 0x_4 + x_5 = 2$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

The initial basic feasible solution is given by

$x_3 = 7,\ x_4 = 2,\ x_5 = 2$ (basic) ($x_1 = x_2 = 0$, non basic)

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Initial Iteration:

C_B	Y_B	C_j / X_B	2 / x_1	3 / x_2	0 / x_3	0 / x_4	0 / x_5	θ
0	x_3	7	2	2	1	0	0	$\frac{7}{2}$
0	x_4	2	1	0	0	1	0	–
0	x_5	2	0	1	0	0	1	2*
$(Z_j^*$	$-C_j)$	0	−2	−3	0	0	0	

First Iteration:

Introduce x_2 and drop x_5.

C_B	Y_B	C_j / X_B	2 / x_1	3 / x_2	0 / x_3	0 / x_4	0 / x_5	θ
0	x_3	3	2	0	1	0	−2	$\frac{3}{2}$*
0	x_4	2	1	0	0	1	0	2
3	x_2	2	0	1	0	0	1	–
$(Z_j^*$	$-C_j)$	6	−2	0	0	0	3	

Second Iteration:

Introduce x_1 and drop x_3.

C_B	Y_B	C_j / X_B	2 / x_1	3 / x_2	0 / x_3	0 / x_4	0 / x_5
2	x_1	$\frac{3}{2}$	1	0	$\frac{1}{2}$	0	−1
0	x_4	$\frac{1}{2}$	0	0	$\frac{-1}{2}$	1	1
3	x_2	2	0	1	0	0	1
$(Z_j^*$	$-C_j)$	9	0	0	1	0	1

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal but non-integer.

To obtain the optimum integer solution, we have to construct a fractional cut constraint.

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$$\text{Now } x_1 = \frac{3}{2} = 1 + \frac{1}{2} = [X_{B1}] + f_1$$

$$x_2 = \frac{1}{2} = 0 + \frac{1}{2} = [X_{B2}] + f_2$$

$\therefore$ Max $\{f_1, f_2\}$ = Max $\left\{\frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2}$ which corresponds to both first and second rows. We select the first row arbitrarily as the source row.

Now from this source row we have,

$$\frac{3}{2} = x_1 + \frac{1}{2}x_3 - x_5$$

$$1 + \frac{1}{2} = x_1 + \frac{1}{2}x_3 - x_5$$

$\therefore$ The fractional cut (Gomorian) constraint is given by

$$\frac{1}{2}x_3 \geq \frac{1}{2} \Rightarrow \frac{-1}{2}x_3 \leq \frac{-1}{2}$$

$$\Rightarrow \frac{-1}{2}x_3 + x_1 = -\frac{1}{2}$$

Where s_1 is the Gomorian slack.

Add this fractional cut constraint at the bottom of the above optimum simplex table and using the dual simplex method, we have

		C_j	(2	3	0	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	s_1
2	x_1	$\frac{3}{2}$	1	0	$\frac{1}{2}$	0	−1	0
0	x_4	$\frac{1}{2}$	0	0	$\frac{-1}{2}$	1	1	0
3	x_2	2	0	1	0	0	1	0
0	s_1	$\frac{-1}{2}$	0	0	$(-\frac{1}{2})$	0	0	1
$(Z_j^* - C_j)$		9	0	0	1	0	1	0

Since $s_1 = -\frac{1}{2}$, s_1 leaves the basis. Further.

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$$\text{Max} \quad \left\{\frac{Z_j^* - C_j}{a_{ik}},\ a_{ik} < 0\right\} = \text{Max}\left\{\frac{1}{-\frac{1}{2}}\right\} = -2 \quad \text{which}$$

corresponds to x_3. So x_3 enters the basis.

Third Iteration:

Drop s_1 and introduce x_3.

C_B	Y_B	C_j → X_B	x_1 (2)	x_2 (3)	x_3 (0)	x_4 (0)	x_5 (0)	s_1 (0)
2	x_1	1	1	0	0	0	−1	1
0	x_4	1	0	0	0	1	1	−1
3	x_2	2	0	1	0	0	1	0
0	x_3	1	0	0	1	0	0	−2
$(Z_j^*$	$-C_j)$	8	0	0	0	0	1	2

Since all $(Z_j^* - C_j) \geq 0$ and all $X_{Bi} \geq 0$, the current solution is feasible and integer optimal.

$\therefore$ The optimal integer solution is

$$\text{Max } Z^* = 8, x_1 = 1, x_2 = 2.$$

But Min $Z = -\text{Max}(-Z) = -\text{Max } Z^* = -8$

$$\therefore \text{Max } Z = -8, x_1 = 1, x_2 = 2.$$

Example: 4.5.3

A manufacturer of baby – dolls makes two types of dolls, doll X and doll Y. Processing of these two dolls is done on two machines, A and B. Doll X requires two hours on machine A and six hours on machine B. Doll Y requires five hours on machine A and also five hours on machine B. There are sixteen hours of time per day available on machine A and thirty hours on machine B. The profit gained on both the dolls is same, it., one rupee per doll. What should be the daily production of each of the two dolls?

(a) Set up and solve the I.P.P.

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(b) If the optimal solution is not integer valued, use the Gomory technique to derive the optimal solution.

Solution:

Let the manufacture decide to manufacture x_1 number of Doll X and x_2 number of Doll Y so as to maximize his profit. Then the complete formulation of the I.P.P. is given by

$$\text{Maximize } Z = x_1 + x_2$$

Subject to $\quad 2x_1 + 5x_2 \leq 16$

$$6x_1 + 5x_2 \leq 30$$

and $x_1, x_2 \geq 0$ and are integers.

Ignoring the integrality condition and introducing the non – negative slack variables x_3 and x_4, the standard form of the continuous L.P.P. becomes

$$\text{Maximize } Z = x_1 + x_2 + 0x_3 + 0x_4$$

Subject to

$$2x_1 + 5x_2 + 0x_3 + 0x_4 = 16$$

$$6x_1 + 5x_2 + 0x_3 + 0x_4 = 30$$

and $x_1, x_2, x_3, x_4 \geq 0$

The initial basic feasible solution is given by

$$x_3 = 16,\ x_4 = 30 \text{ (basic)}$$

$$(x_1 = x_2 = 0, \text{ non basic})$$

Initial Iteration:

		C_j	(1	1	0	0)	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	16	2	5	1	0	8
0	x_4	30	(6)	5	0	1	5*
$(Z_j$	$-C_j)$	0	−1	−1	0	0	

First Iteration:

Introduce x_1 and drop x_4.

C_j (1 1 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	6	0	$\frac{10}{3}$	1	$\frac{-1}{3}$	$\frac{18}{10}=\frac{9}{5}$*
1	x_1	5	1	$\frac{5}{6}$	0	$\frac{1}{6}$	$\frac{30}{5}=6$
$(Z_j$	$-C_j)$	5	0	$-\frac{1}{6}$	0	$\frac{1}{6}$	

Second Iteration:

Introduce x_2 and drop x_3

C_j (1 1 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4
1	x_2	$\frac{9}{5}$	0	1	$\frac{3}{10}$	$\frac{-1}{10}$
1	x_1	$\frac{7}{2}$	1	0	$\frac{-1}{4}$	$\frac{1}{4}$
$(Z_j$	$-C_j)$	$\frac{53}{10}$	0	0	$\frac{1}{20}$	$\frac{3}{20}$

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal and non – integer.

i.e., $\quad$ Max $Z = \frac{53}{10}$, $x_1 = \frac{7}{2}$, $x_2 = \frac{9}{5}$.

In order to obtain the optimum integer solution, we have to construct a fractional cut constraint.

Now $x_1 = \frac{7}{2} = 3 + \frac{1}{2} = [X_{B1}] + f_1$

$x_2 = \frac{9}{5} = 1 + \frac{4}{5} = [X_{B2}] + f_2$

$\therefore$ Max $\{f_1, f_2\}$ = Max$\left\{\frac{1}{2}, \frac{4}{5}\right\} = \frac{4}{5}$ which corresponds to the first row (called the source row). Then from this source row,

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we have

$$1+\frac{4}{5} = 0x_1 + x_2 + \frac{3}{10}x_3 - \frac{1}{10}x_4$$

Expressing the negative fraction as a sum of a negative integer and a non – negative fraction, we have

$$1+\frac{4}{5} = x_2 + \frac{3}{10}x_3 + \left(-1+\frac{9}{10}\right) x_4$$

∴ The fractional cut (Gomorian) construct is given by

$$\frac{3}{10}x_3 + \frac{9}{10}x_4 \geq \frac{4}{5}$$

$$\Rightarrow \frac{-3}{10}x_3 - \frac{9}{10}x_4 \leq -\frac{4}{5}$$

$$\Rightarrow -\frac{3}{10}x_3 - \frac{9}{10}x_4 + s_1 = -\frac{4}{5}$$

Where s_1 is the Gomorian slack.

Add this fractional cut constraint at the bottom of the above optimum simplex table, we have

		C_j	(1	1	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1
1	x_2	$\frac{9}{5}$	0	1	$\frac{3}{10}$	$\frac{-1}{10}$	0
1	x_1	$\frac{7}{2}$	1	0	$\frac{-1}{4}$	$\frac{1}{4}$	0
0	s_1	$\frac{-4}{5}$	0	0	$\left(\frac{-3}{10}\right)$	$\frac{-9}{10}$	1
$(Z_j$	$-C_j)$	$\frac{53}{10}$	0	0	$\frac{1}{20}$	$\frac{3}{20}$	0

Here the solution is optimal but infeasible

To find the feasible optimal solution, we have to use the **dual** simplex method.

Since $s_1 = -\frac{4}{5}$, s_1 leaves the basis.

To find the entering variable:

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$$\text{Also Max}\left\{\frac{Z_j - C_j}{a_{ik}},\ a_{ik} < 0\right\} = \text{Max}\left\{\frac{\frac{1}{20}}{-\frac{-3}{10}},\ \frac{\frac{3}{20}}{-\frac{-9}{10}}\right\}$$

$$= \text{Max}\left\{\frac{-1}{6}, \frac{-1}{6}\right\} = \frac{-1}{6} \text{ which corresponds to } x_3 \text{ and } x_4.$$

We shall choose x_3 arbitrarily as the entering variable,

Third iteration:

Drop s_1 and introduce x_3.

	C_j		(1	1	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1
1	x_2	1	0	1	0	−1	1
1	x_1	$\frac{25}{6}$	1	0	0	1	$\frac{-5}{6}$
0	x_3	$\frac{8}{3}$	0	0	1	3	$\frac{-10}{3}$
$(Z_j$	$-C_j)$	$\frac{31}{6}$	0	0	0	0	$\frac{1}{6}$

Since all $(Z_j - C_j) \geq 0$ and all $X_{Bi} \geq 0$, the current solution is feasible and optimal but non – integer.

∴ The obtain the optimum integer solution, we have to construct a fractional cut constraint.

$$\text{Now } x_1 = \frac{25}{6} = 4 + \frac{1}{6} = [X_{B1}] + f_1$$

$$x_2 = \frac{8}{3} = 2 + \frac{2}{3} = [X_{B2}] + f_3$$

∴ $\text{Max}\{f_1, f_3\} = \text{Max}\left[\frac{1}{6}, \frac{2}{3}\right] = \frac{2}{3}$ which corresponds to the third row.

From this source row, we have

$$2 + \frac{2}{3} = x_3 + 3x_4 - \frac{10}{3}s_1$$

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$$= x_3 + 3x_4 + \left(-4 + \frac{2}{3}\right)s_1$$

$\therefore$ The fractional cut (Gomorian) construct is given by

$$\frac{2}{3}s_1 \geq \frac{2}{3} \Rightarrow \frac{-2}{3}s_1 \leq \frac{-2}{3}$$

$$\Rightarrow \frac{-2}{3}s_1 + s_2 = \frac{-2}{3}$$

Where s_2 is the Gomorian slack.

Add this fractional cut constraint at the bottom of the above optimum simplex table and using the dual simplex method, we have

		C_j	(1	1	0	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2
1	x_2	1	0	1	0	−1	1	0
1	x_1	$\frac{25}{6}$	1	0	0	1	$\frac{-5}{6}$	0
0	x_3	$\frac{8}{3}$	0	0	1	3	$\frac{-10}{3}$	0
0	s_2	$\frac{-2}{3}$	0	0	0	0	$\left(\frac{-2}{3}\right)$	1
$(Z_j - C_j)$		$\frac{31}{6}$	0	0	0	3	$\frac{1}{6}$	0

Since $s_2 = \frac{-2}{3}$, s_2 leaves the basis.

Now Max $\left\{\frac{Z_j - C_j}{a_{ik}}, a_{ik} < 0\right\} = \text{Max}\left\{\frac{\frac{1}{6}}{\frac{-2}{3}}\right\} = \frac{-1}{4}$ which corresponds to s_1. So, s_1 enters the basis.

Fourth Iteration: Introduce s_1 and drop s_2.

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	C_j	$(1$	1	0	0	0	$0)$

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2
1	x_2	0	0	1	0	-1	0	$\frac{3}{2}$
1	x_1	5	1	0	0	1	0	$\frac{-5}{4}$
0	x_3	6	0	0	1	3	0	-5
0	s_1	1	0	0	0	0	1	$\frac{-3}{2}$
$(Z_j$	$-C_j)$	5	0	0	0	0	0	$\frac{1}{4}$

Since all $(Z_j - C_j) \geq 0$ and all $X_{Bi} \geq 0$, and integers, the current solution is feasible and integer optimal.

$\therefore$ The optimal solution to the new problem is

Max $Z = 5$, $x_1 = 5$, $x_2 = 0$.

i.e., the manufacturer should produce 5 numbers of doll X alone in order to get the maximum profit Rs. 5.

Note:

For this problem, since $(Z_4 - C_4) = 0$ corresponding to the non- basic variable x_4, there exists alternative optimal solutions. Such solutions are

$x_1 = 3$, $x_2 = 2$, and $x_1 = 4$, $x_2 = 1$ with same Max $Z = 5$.

Example 4.5.4

Find the optimum integer solution to the following linear programming problem:

$$\text{Maximize } Z = x_1 + 2x_2$$
$$2x_2 \leq 7$$
$$x_1 + x_2 \leq 7$$
$$2x_1 \leq 11$$
$$\text{and } x_1, x_2 \geq 0 \text{ and are integers}$$

Space for Hints

Solution:

Introducing the non-negative slack variables x_3, x_4, and x_5, the standard form of the continuous L.P.P. becomes.

Maximize $Z = x_1 + 2x_2 + 0x_3 + 0x_5$

Subject to $0x_1 + 2x_2 + x_3 + 0x_4 + 0x_5 = 7$

$x_1 + x_2 + 0x_3 + x_4 + 0x_5 = 7$

$2x_1 + 0x_2 + 0x_3 + 0x_4 + x_5 = 11$

and $x_i \geq 0,\ i = 1,2,3,4,5.$

The initial basic feasible solution is given by

$x_3 = 7,\ x_4 = 7,\ x_5 = 11$ (basic)

$(x_1 = x_2 = 0,$ non basic$)$

Initial Iteration:

C_B	Y_B	C_j / X_B	x_1 (1)	x_2 (2)	x_3 (0)	x_4 (0)	x_5 (0)	θ
0	x_3	7	0	(2)	1	0	0	$\frac{7}{2}$
0	x_4	7	1	1	0	1	0	7
0	x_5	11	2	0	0	0	1	–
$(Z_j$	$-C_j)$	0	-1	-2	0	0	0	

First Iteration:

Introduce x_2 and drop x_3.

C_B	Y_B	C_j / X_B	x_1 (1)	x_2 (2)	x_3 (0)	x_4 (0)	x_5 (0)	θ
2	x_2	$\frac{7}{2}$	0	1	$\frac{1}{2}$	0	0	–
0	x_4	$\frac{7}{2}$	(1)	0	$\frac{-1}{2}$	1	0	$\frac{7}{2}$
0	x_5	11	2	0	0	0	1	$\frac{11}{2}$
$(Z_j$	$-C_j)$	7	-1	0	1	0	0	

Second Iteration:

Introduce x_1 and drop x_4.

Space for Hints

C_j (1 2 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_3	x_4
2	x_2	$\frac{7}{2}$	0	1	$\frac{1}{2}$	0	0
1	x_1	$\frac{7}{2}$	1	0	$\frac{-1}{2}$	1	0
0	x_5	4	0	0	1	-2	1
$(Z_j$	$-C_j)$	$\frac{21}{2}$	0	0	$\frac{1}{2}$	1	0

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal but non – integer.

i.e., Max $Z = \frac{21}{2}$, $x_1 = \frac{7}{2}$, $x_2 = \frac{7}{2}$

To obtain the integer optimal solution, we have to construct a fractional cut constraint.

$$\text{Now} \quad x_1 = \frac{7}{2} = 3 + \frac{1}{2} = [X_{B1}] + f_1$$

$$x_2 = \frac{7}{2} = 3 + \frac{1}{2} = [X_{B2}] + f_2$$

$\therefore$ Max $\{f_1, f_2\}$ = Max$\left\{\frac{1}{2}, \frac{1}{2}\right\}$ = $\frac{1}{2}$ which corresponds to both first and second rows. We choose the first row arbitrarily as the first row arbitrarily as the source row.

From this source row, we have

$$\frac{7}{2} = x_2 + \frac{1}{2}x_3$$

$$\text{i.e., } 3 + \frac{1}{2} = x_2 + \frac{1}{2}x_3$$

$\therefore$ The fractional cut (Gomorian) construct is given by

$$\frac{1}{2}x_3 \geq \frac{1}{2}$$

$$\Rightarrow \frac{-1}{2}x_3 \leq -\frac{1}{2}x_3 + s_1 = -\frac{1}{2}$$

Space for Hints

Where s_1 is the Gomorian slack.

Add this fractional cut constraint at the bottom of the above optimum simplex table, we have

		C_j	(1	2	0	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	s_1
2	x_2	$\frac{7}{2}$	0	1	$\frac{1}{2}$	0	0	0
1	x_1	$\frac{7}{2}$	1	0	$\frac{-1}{2}$	1	0	0
0	x_5	4	0	0	1	-2	1	0
0	s_1	$\frac{-1}{2}$	0	0	$(\frac{-1}{2})$	0	0	1
$(Z_j$	$-C_j)$	$\frac{21}{2}$	0	0	$\frac{1}{2}$	1	0	0

Since $s_1 = -\frac{1}{2}$, the solution is infeasible. To find the feasible optimal solution, we have to use the dual simplex method.

Since $s_1 = -\frac{1}{2}$, s_1 leaves the basis.

Also Max $\left\{\frac{Z_j - C_j}{a_{ik}}, a_{ik} < 0\right\}$ = Max$\left\{\frac{\frac{1}{2}}{-\frac{-1}{2}}\right\} = -1$ which corresponds to x_3. So x_3 enters the basis.

Third Iteration:

Introduce x_3 and drop s_1.

		C_j	(1	2	0	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	s_2
2	x_2	3	0	1	0	0	0	1
1	x_1	4	1	0	0	1	0	-1
0	x_5	3	0	0	0	-2	1	2
0	x_3	1	0	0	1	0	0	-2
$(Z_j$	$-C_j)$	10	0	0	0	1	0	1

Since all $(Z_j - C_j) \geq 0$ and all $X_{Bi} \geq 0$, the current solution is feasible integer optimal.

$\therefore$ The optimal integer solution is

Max $Z = 10,\ x_1 = 4,\ x_2 = 3$.

Gomory's Mixed Integer Method:

In mixed integer programming problem only some of the variables are integer constrained, while the other variables may take integer or other real values. Like the pure integer problem, the mixed integer or other real values. Like the pure integer problem, the mixed integer problem should be of the maximization type and all the coefficients and constants should be integers.

The problem is first solved as a continuous LPP by ignoring the integrality condition. If the values of the integer constrained variables are integers, then the current solution is an optimum solution to the given mixed IPP. Otherwise, select the source row which corresponds to the largest fractional part f_k among those basic variables which are constrained to be integers. Then construct the Gomorain constraint (secondary constraint) form the source row.

Form the source row $\sum_{j=1}^{n} a_{kj}x_j = X_{Bk}$

i.e., $$\sum_{j=1}^{n} ([a_{kj}] + f_{kj})x_j = [X_{Bk}] + f_k$$

in the form of $$\sum_{j=1}^{n} f_{kj}x_j \geq f_k$$

i.e., $$\sum_{j\in J^+} f_{kj}x_j + \left(\frac{f_k}{f_{k-1}}\right) \sum_{j\in J^-} f_{kj}x_j \geq f_k$$

i.e., $$-\sum_{j\in J^+} f_{kj}x_j - \left(\frac{f_k}{f_{k-1}}\right) \sum_{j\in J^-} f_{kj}x_j \geq -f_k$$

i.e., $$-\sum_{j\in J^+} f_{kj}x_j - \left(\frac{f_k}{f_{k-1}}\right) \sum_{j\in J^-} f_{kj}x_j + s_k = -f_k$$

where s_k : Gomorian slack

$J^+ = \{J / f_{kj} \geq 0\}$

$J^- = \{J / f_{kj} < 0\}$

Add this secondary constraint at the bottom of the optimum simplex table and use dual simplex method to obtain the new feasible optimal solution. Repeat the procedure until the values of the integer restricted variables are integers in the optimum solution obtained.

Example: 4.5.5

Solve the following mixed integer programming problem:

$$\text{Max } Z = x_1 + x_2$$

subject to constraints

$$2x_1 + 5x_2 \leq 16$$

$$6x_1 + 5x_2 \leq 30$$

$$x_2 \geq 0 \; x_1, \text{ non} - \text{negative integer.}$$

Solution:

Ignoring the integrality condition and introducing the non – negative slack variables x_3 and x_4, the standard form of the continuous LPP becomes

$$\text{Max } Z = x_1 + x_2 + 0x_3 + 0x_4$$

Subject to $2x_1 + 5x_2 + x_3 + 0x_4 = 16$

$$6x_1 + 5x_2 + 0x_3 + x_4 = 30$$

$$\text{and } x_1, x_2, x_3, x_4 \geq 0$$

The initial basic feasible solution is given by $x_3 = 16, x_4 = 30$, ($x_1 = x_2 = 0$, non – basic).

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Initial Iteration:

C_B	Y_B	X_B	x_1 (1)	x_2 (1)	x_3 (0)	x_4 (0)	θ
0	x_3	16	2	5	1	0	8
0	x_4	30	(6)	5	0	1	5
$(Z_j$	$-C_j)$	0	-1	-1	0	0	

First Iteration:

Introduce x_1 and drop x_4.

C_B	Y_B	X_B	x_1 (1)	x_2 (1)	x_3 (0)	x_4 (0)	θ
0	x_3	6	0	$(\frac{10}{3})$	1	$\frac{-1}{3}$	$\frac{9}{5}$
1	x_1	5	1	$\frac{5}{6}$	0	$\frac{1}{6}$	6
$(Z_j$	$-C_j)$	5	0	$\frac{-1}{6}$	0	$\frac{1}{6}$	

Second Iteration:

Introduce x_2 and drop x_3

C_B	Y_B	X_B	x_1 (1)	x_2 (1)	x_3 (0)	x_4 (0)
1	x_2	$\frac{18}{10}$	0	1	$\frac{3}{10}$	$\frac{-1}{10}$
1	x_1	$\frac{7}{2}$	1	0	$\frac{-1}{4}$	$\frac{1}{4}$
$(Z_j$	$-C_j)$	$\frac{53}{10}$	0	0	$\frac{1}{20}$	$\frac{3}{20}$

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

Since the integer constrained variable x_1 is non – integer, we have from the second (source) row

$$\frac{7}{2} = x_1 + 0x_2 - \frac{1}{4}x_3 + \frac{1}{4}x_4$$

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$$3+\frac{1}{2} = x_1 + 0x_2 - \frac{1}{4}x_3 + \frac{1}{4}x_4$$

The Gomorian constraint is given by

$$\left(\frac{\frac{1}{2}}{\frac{1}{2}-1}\right)\left(\frac{-1}{4}\right)x_3 + \frac{1}{4}x_4 \geq \frac{1}{2}$$

$$\Rightarrow \qquad \frac{1}{4}x_3 + \frac{1}{4}x_4 \geq \frac{1}{2}$$

$$\Rightarrow \qquad -\frac{1}{4}x_3 - \frac{1}{4}x_4 \leq -\frac{1}{2}$$

$$\Rightarrow \qquad -\frac{1}{4}x_3 - \frac{1}{4}x_4 + s_1 = -\frac{1}{2}$$

where s_1 is the Gomorian slack.

Add this Gomorian constraint at the bottom of the above optimum simplex table, we have

		C_j	(1	1	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1
1	x_2	$\frac{9}{5}$	0	1	$\frac{3}{10}$	$\frac{-1}{10}$	0
1	x_1	$\frac{7}{2}$	1	0	$\frac{-1}{4}$	$\frac{1}{4}$	0
0	s_1	$\frac{-1}{2}$	0	0	$\frac{-1}{4}$	$\frac{-1}{4}$	1
$(Z_j$	$-C_j)$	$\frac{53}{10}$	0	0	$\frac{1}{20}$	$\frac{3}{20}$	0

Here the solution is optimal but infeasible.

So, we have to use the dual simplex method.

Since $s_1 = -\frac{1}{2}$, s_1 leaves the basis.

$$\text{Now Max}\left\{\frac{Z_j - C_j}{a_{ik}},\ a_{ik} < 0\right\} = \text{Max}\left\{\frac{\frac{1}{20}}{\frac{-1}{4}}, \frac{\frac{3}{20}}{\frac{-1}{4}}\right\}$$

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$$= \text{Max}\left\{\frac{-4}{20}, \frac{-12}{20}\right\} = \text{Max}\left\{\frac{-1}{5}; \frac{-3}{5}\right\} = \frac{-1}{5}$$ which

corresponds to the variable x_3, so x_3 enters the basis.

Third Iteration:

Drop s_1 and introduce x_3.

	C_j	(1	1	0	0	0)
C_B Y_B	X_B	x_1	x_2	x_3	x_4	s_1
1 x_2	$\frac{6}{5}$	0	1	0	$\frac{-2}{5}$	$\frac{6}{5}$
1 x_1	4	1	0	0	$\frac{1}{2}$	-1
0 x_3	2	0	0	1	1	-4
$(Z_j - C_j)$	$\frac{26}{5}$	0	0	0	$\frac{1}{10}$	$\frac{1}{5}$

Since all $(Z_j - C_j) \geq 0$ and all $X_{Bi} \geq 0$, the current solution is feasible and optimal.

Also, since the integer constrained variable x_1 is integer, the required optimum solution is

$$\text{Max } Z = \frac{26}{5},\ x_1 = 4,\ x_2 = \frac{6}{5}$$

Example: 4.5.6

Solve the following mixed integer programming problem by Gomory's cutting plane algorithm:

$$\text{Maximize } Z = x_1 + x_2$$

Subject to $3x_1 + 2x_2 \leq 5$

$x_2 \leq 2$

and $x_1, x_2 \geq 0$ and x_1 an integer.

Solution:

Ignoring the integrality condition and introducing the non – negative slack variables x_3, x_4, the standard form of the continuous LPP becomes

Space for Hints

$$\text{Maximize } Z = x_1 + x_2 + 0x_3 + 0x_4$$

Subject to $\quad 3x_1 + 2x_2 + x_3 + 0x_4 = 5$

$$0x_1 + x_2 + 0x_3 + 0x_4 = 2$$

$$\text{and } x_1, x_2, x_3, x_4 \geq 0$$

The initial basic feasible solution is given by

$$x_3 = 5,\ x_4 = 2 \text{ (basic)}$$

$$(x_1 = x_2 = 0, \text{ non basic})$$

Initial Iteration:

C_j (1 1 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	5	(3)	2	1	0	$\frac{5}{3}$
0	x_4	2	0	1	0	1	–
$(Z_j$	$-C_j)$	0	-1	-1	0	0	

First Iteration:

Introduce x_1 and drop x_3

C_j (1 1 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
1	x_1	$\frac{5}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{5}{2}$
0	x_4	2	0	(1)	0	1	2
$(Z_j$	$-C_j)$	$\frac{5}{3}$	0	$\frac{-1}{3}$	$\frac{1}{3}$	0	

Second Iteration:

Introduce x_2 and drop x_4.

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C_B	Y_B	C_j / X_B	x_1 (1)	x_2 (1)	x_3 (0)	x_4 (0)
1	x_1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	$-\frac{2}{3}$
1	x_2	2	0	1	0	1
$(Z_j - C_j)$		$\frac{7}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal

But x_1 is non – integer.

From the source row (first row), we have

$$\frac{1}{3} = x_1 + 0x_2 + \frac{1}{3}x_3 - \frac{2}{3}x_4$$

$\therefore$ The Gomorian construct is given by

$$\frac{1}{3}x_3 + \left(\frac{\frac{1}{3}}{\frac{1}{3}-1}\right)\left(\frac{-2}{3}\right)x_4 \geq \frac{1}{3}$$

$$\Rightarrow \quad \frac{1}{3}x_3 + \frac{1}{3}x_4 \geq \frac{1}{3} \Rightarrow -\frac{1}{3}x_3 - \frac{1}{3}x_4 \leq -\frac{1}{3}$$

$$\Rightarrow \quad -\frac{1}{3}x_3 - \frac{1}{3}x_4 + s_1 = -\frac{1}{3}$$

Where s_1 is the Gomorian slack.

Add this Gomarian constraint at the bottom of the above optimum simplex table, we have

C_B	Y_B	C_j / X_B	x_1 (1)	x_2 (1)	x_3 (0)	x_4 (0)	S_1 (0)
1	x_1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	$-\frac{2}{3}$	0
1	x_2	2	0	1	0	1	0
0	S_1	$-\frac{1}{3}$	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	1
$(Z_j - C_j)$		$\frac{7}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	0

Space for Hints

Here the solution is optimal but infeasible so, we have to use the dual simplex method.

Since $S_1 = -\frac{1}{3}, S_1$ leaves the basis.

Also $\text{Max}\left\{\frac{Z_j - C_j}{a_{ik}}, a_{ik} < 0\right\} = \text{Max}\left\{\frac{1/3}{-1/3}, \frac{1/3}{-1/3}\right\}$

$= \text{Max}\{-1,-1\} = -1$ which corresponds to both x_3 and x_4. We choose x_3 particularly as the entering variables.

Third Iteration:

Drop S_1 and introduce x_3

		C_j	(1	1	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1
1	x_1	0	1	0	0	−1	1
1	x_2	2	0	1	0	1	0
0	x_3	1	0	0	1	1	−3
$(Z_j$	$-C_j)$	2	0	0	0	0	1

Since all $(Z_j - C_j) \geq 0$ and all $X_{Bi} \geq 0$, the current solution is feasible and optimal.

$\therefore$ The required solution is

$$\text{Max } Z = 2,\ x_1 = 0,\ x_2 = 2.$$

Example: 4.5.7

Solve the following mixed integer programming problem.

$$\text{Minimize } Z = x_1 - 3x_2$$

Subject to $\quad x_1 + x_2 \leq 5$

$$-2x_1 + 4x_2 \leq 11$$

$$x_1, x_2 \geq 0 \text{ and } x_2 \text{ is an integer.}$$

Solution:

Given mixed I.P.P be

$$\text{Min } Z = x_1 - 3x_2$$

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Subject to $x_1 + x_2 \le 5$

$-2x_1 + 4x_2 \le 5$

$x_1, x_2 \ge 0$ and x_2 is an integer.

i.e., Maximize $Z^* = -x_1 + 3x_2$

subject to $x_1 + x_2 \le 5$

$-2x_1 + 4x_2 \le 11$

$x_1, x_2 \ge 0$ and x_2 is an integer.

Ignoring the integrality condition and introducing the non – negative slack variables x_3 and x_4, the standard form of the continuous L.P.P becomes.

Maximize $Z^* = -x_1 + 3x_2 + 0x_3 + 0x_4$

Subject to $x_1 + x_2 + x_3 + 0x_4 = 5$

$-2x_1 + 4x_2 + 0x_3 + x_4 = 11$

$x_1, x_2, x_3, x_4 \ge 0$

The initial basic feasible solution is given by

$x_3 = 5,\ x_4 = 11$, (basic)

$(x_1 = x_2 = 0$, non – basic)

Initial Iteration:

$C_j \quad (-1 \quad 3 \quad 0 \quad 0)$

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	5	1	1	1	0	$\frac{5}{1}$
0	x_4	11	-2	(4)	0	1	$\frac{11}{4}$
$(Z_j^*$	$-C_j)$	0	1	-3	0	0	

First Iteration:

Introduce x_2 and drop x_4.

Space for Hints

		C_j	(−1	3	0	0)	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	$\frac{9}{4}$	$(\frac{3}{2})$	0	1	$\frac{-1}{4}$	$\frac{3}{2}$
3	x_2	$\frac{11}{4}$	$\frac{-1}{2}$	1	0	$\frac{1}{4}$	–
$(Z_j^*$	$-C_j)$	$\frac{33}{4}$	$\frac{-1}{2}$	0	0	$\frac{3}{4}$	

Second Iteration:

Introduce x_1 and drop x_3.

		C_j	(−1	3	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4
−1	x_1	$\frac{3}{2}$	1	0	$\frac{2}{3}$	$\frac{-1}{6}$
3	x_2	$\frac{7}{2}$	0	1	$\frac{1}{3}$	$\frac{1}{6}$
$(Z_j^*$	$-C_j)$	9	0	0	$\frac{1}{3}$	$\frac{2}{3}$

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal.

But the integer constrained variable $x_2 = \frac{7}{2}$ is non – integer.

To obtain the integer value for x_2, we have to construct the Gomorian constraint. Now from the source row, we have

$$\frac{7}{2} = x_2 + \frac{1}{3}x_3 + \frac{1}{6}x_4$$

$$3 + \frac{1}{2} = x_2 + \frac{1}{3}x_3 + \frac{1}{6}x_4$$

∴ The Gomorian constraint is given by

$$\frac{1}{3}x_3 + \frac{1}{6}x_4 \geq \frac{1}{2} \Rightarrow \frac{-1}{3}x_3 - \frac{1}{6}x_4 \leq -\frac{1}{2}$$

$$\Rightarrow \frac{-1}{3}x_3 - \frac{1}{6}x_4 + s_1 = -\frac{1}{2}$$

where s_1 is the Gomorian slack.

Add this secondary (Gomorian) constraint at the bottom of the above optimum simplex table, we have

	C_j	(−1	3	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1
−1	x_1	$\frac{3}{2}$	1	0	$\frac{2}{3}$	$\frac{-1}{6}$	0
3	x_2	$\frac{7}{2}$	0	1	$\frac{1}{3}$	$\frac{1}{6}$	0
0	s_1	$\frac{-1}{2}$	0	0	$(\frac{-1}{3})$	$\frac{-1}{6}$	1
$(Z_j^*$ | $-C_j)$	9	0	0	$\frac{1}{3}$	$\frac{2}{3}$	0

Here the solution is optimal but infeasible. So we have to use dual simplex method.

Since $s_1 = -\frac{1}{2}$, s_1 leaves the basis.

$$\text{Also, Max}\left\{\frac{Z_j^* - C_j}{a_{ik}},\ a_{ik} < 0\right\} = \text{Max}\left\{\frac{\frac{1}{3}}{\frac{-1}{3}}, \frac{\frac{2}{3}}{\frac{-1}{6}}\right\}$$

$= \text{Max } \{-1, -4\} = -1$ which corresponds to x_3. So x_3 enters the basis.

Third Iteration:

Drop s_1 and introduce x_3.

	C_j	(−1	3	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1
−1	x_1	$\frac{1}{2}$	1	0	0	$\frac{-1}{2}$	2
3	x_2	3	0	1	0	0	1
0	x_3	$\frac{3}{2}$	0	0	1	$\frac{1}{2}$	−3
$(Z_j^*$ | $-C_j)$	$\frac{17}{2}$	0	0	0	$\frac{1}{2}$	1

Space for Hints

Since all $(Z_j^* - C_j) \geq 0$ and all $X_{Bi} \geq 0$, the current solution is feasible optimal.

$\therefore$ The optimal integer solution is

$$\text{Max } Z^* = \frac{17}{2},\ x_1 = \frac{1}{2},\ x_2 = 3$$

$$\text{But Min } Z = -\text{Max } Z^* = -\frac{17}{2}$$

$$\therefore \text{Min } Z = \frac{-17}{2},\ x_1 = \frac{1}{2},\ x_2 = 3.$$

Example: 4.5.8

Solve the following mixed integer problem

$$\text{Minimize } Z = 10x_1 + 9x_2$$

Subject to
$$x_1 \leq 8$$
$$x_2 \leq 10$$
$$5x_1 + 3x_2 \geq 45$$
$$x_1, x_2 \geq 0,\ x_1 \text{ integer.}$$

i.e., $$\text{Max } Z^* = -10x_1 - 9x_2$$

subject to
$$x_1 \leq 8$$
$$x_2 \leq 10$$
$$5x_1 + 3x_2 \geq 45$$
$$x_1, x_2 \geq 0,\ x_1 \text{ integer.}$$

Ignoring the integrality condition and introducing the non – negative slack variable x_3, x_4, surplus variable x_5 and artificial variable R_1, the standard form of the continuous LPP becomes:

$$\text{Max } Z^* = -10x_1 - 9x_2 + 0x_3 + 0x_4 + 0x_5 - MR_1$$

Subject to
$$x_1 + 0x_2 + x_3 + 0x_4 + 0x_5 + 0R_1 = 8$$
$$0x_1 + x_2 + 0x_3 + x_4 + 0x_5 + 0R_1 = 10$$
$$5x_1 + 3x_2 + 0x_3 + 0x_4 - x_5 + R_1 = 45$$

Space for Hints

and $x_1, x_2, x_3, x_4, x_5, R_1 \geq 0$

The initial basic feasible solution is given by

$x_3 = 8,\ x_4 = 10,\ R_1 = 45$ (basic)

$(x_1 = x_2 = 0,\ x_5 = 0$, non basic)

Initial Iteration:

		C_j	(−10	−9	0	0	0	−M)	
C_B	Y_B	X_B	x_1	x_2	x_5	x_3	x_4	R_1	θ
0	x_3	8	(1)	0	0	1	0	0	8
0	x_4	10	0	1	0	0	1	0	—
−M	R_1	45	5	3	−1	0	0	1	9
$(Z_j^*$	$-C_j)$	−45M	−5M+10	−3M+9	M	0	0	0	

First Iteration:

Introduce x_1 and drop x_3.

		C_j	(−10	−9	0	0	0	−M)	
C_B	Y_B	X_B	x_1	x_2	x_5	x_3	x_4	R_1	θ
−10	x_1	8	1	0	0	1	0	0	—
0	x_4	10	0	1	0	0	1	0	10
−M	R_1	5	0	(3)	−1	−5	0	1	$\frac{5}{3}$
$(Z_j^*$	$-C_j)$	−5M−80	0	−3M+9	M	5M−10	0	0	

Second Iteration:

Introduce x_2 and drop R_1.

		C_j	(−10	−9	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_5	x_3	x_4
−10	x_1	8	1	0	0	1	0
0	x_4	$\frac{25}{3}$	0	0	$\frac{1}{3}$	$\frac{5}{3}$	1
−9	x_2	$\frac{5}{3}$	0	1	$-\frac{1}{3}$	$\frac{-5}{3}$	0
$(Z_j^*$	$-C_j)$	−95	0	0	3	5	0

Since all $(Z_j - C_j) \geq 0$, and the integer constrained variable x_1 is an integer, the current basic feasible solution is optimal.

Space for Hints

$$\text{Max } Z^* = -95,\ x_1 = 8,\ x_2 = \frac{5}{3}$$

$$\text{But Min } Z = -\text{Max}(-Z) = -\text{Max } Z^*$$

$$= -(-95) = 95$$

$\therefore$ The optimal solution is

$$\therefore \text{Min } Z = 95,\ x_1 = 8,\ x_2 = \frac{5}{3}$$

Check your progress 4:4

1) Solve the following mixed integer problem

Maximize $Z = -3x_1, +x_2 + 3x_3$

Subject to $-x_1 + 2x_2 + x_3 \le 4$

$$2x_2 - \frac{3}{2}x_3 \le 1$$

$$x_1 - 3x_2 + 2x_3 \le 3$$

And $x_1, x_2 \ge 0, x_3$ non negative integer.

2) Solve the following mixed integer programming problem

Maximize $Z = 4x_1 + 6x_2 + 2x_3$

Subject to $4x_1 - 4x_2 \le 5$

$$-x_1 + 6x_2 \le 5$$

$$-x_1 + x_2 + x_3 \le 5$$

$x_1, x_2, x_3, \ge 0$ and x_1, x_3 are integers

4.6 KEY WORDS

Primal problems, Dual problems, Dual simplex methods, Integer programming

Space for Hints

4.7 AWSWERS TO CHECK YOUR PROGRESS

Check your Progress 4.1

1) Minimize $g(w)=2w_1+5w_2+6w_3$ Subject to:

$5w,+6w_3-w_3\le 3,\ -2w,+w_2+4w_3\le 4,\ w,-5w_2+3w_3\le 1,\ -3w,$
$-3w_2+7w_3\le 6,\ w;\ge 0\ (j=1,2,3,4)$

2) Minimize $Z^*=\text{low},+6w_2+8w_3$ Subject to:

$w_1+w_2+2w_2=2,\ 5w,+3w_2+2w_3\ge 1,\ w;\ge 0\ (j=1,2,3)$

3) Minimize $Z^*=6w,+4w_2$ Subject to:

$4w,+2w_2\ge 2,\ \ 3w,+2w=\ge 3,\ w,+5w_2\ge 1,$ w, and w_2 are cinreslricted.

4) Minimize $Z^*=2w,\ w_2$ Subject to:

$-2w,+2w_2\ge 1,\ w,+3w_2\ge -2,\ 3w,+4w_2\ge 3$; w, and w_2 are cinreslricted.

5) Maximize $Z^*=2w,+w_2$ Subject to:

$w_1-w_2\le 1,\ w_1+w_2\le 1,\ w,\ w_2\ge 0$

6) Maximize $Z^*=7w,+4w_2-10w_3+3w_4+2w_5$ Subject to:

$3w,+6w_2-7w_3+w_4+4w_5\le 3,$

$5w,+w_2+2w_3-2w_4+7w_5\le -2,$

$4w,+3w_2+w_3+5w_4+2w_5\le 4,\ w;\ge 0\ (j=1,2,3,4,5)$

Check Your Progress 4.2

1) Unbounded solution.

2) Min $Z=0,\ x_1=\frac{3}{4},\ x_2=0,\ x_3=0,\ x_4=0,\ x_5=\frac{1}{2}$

Space for Hints

Check Your Progress 4.3

1) Minimum $Z = \frac{258}{11}$, $x_1 = 0$, $x_2 = \frac{30}{11}$, $x_3 = \frac{16}{11}$, $x_4 = 0$

2) Minimum $Z = \frac{215}{23}$, $x_1 \frac{65}{23}$, $x_2 = 0$, $x_3 = \frac{20}{23}$

3) Minimum $Z = 24$, $x_1 = 0$, $x_2 = 0$, $x_3 = \frac{1}{2}$, $x_4 = \frac{3}{2}$, $x_5 = 0$

4) Maximum $Z = \frac{4}{3}$, $x_1 = 0$, $x_2 = \frac{4}{3}$, $x_3 = 0$

Check Your Progress 4.4

1) Max $Z = 7$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$

2) Max $Z = 26$, $x_1 = 2$, $x_2 = 1$, $x_3 = 6$

4.8 Model Questions

Write down the dual of the following L.P.P. and solve:

1) Maximize $Z = 4x_1 + 2x_2$ Subject to:

   $x_1 + x_2 \geq 3$, $x_1 - x_2 \geq 2$, $x_1, x_2 \geq 0$

2) Minimize $Z = 15x_1 + 10x_2$ Subject to:

   $3x_1 + 5x_2 \geq 5$, $5x_1 + 2x_2 \geq 3$, $x_1, x_2 \geq 0$

3) Maximize $Z = 5x_1 + 2x_2$ Subject to:

   $6x_1 + x_2 \geq 6$, $4x_1 + 3x_2 \geq 12$ x_1, $x_1 + 2x_2 \geq 4$, $x_1, x \geq 0$

4) Minimize $Z = 2x_1 + 9x_2 + x_3$ Subject to:

   $x_1 + 4x_2 + 2x_3 \geq 5$, $3x_1 + x_2 + 2x_3 \geq 4$ and $x_1, x_2, x_3 \geq 0$

Space for Hints

5) Consider the following L.P.P.:

Maximize $Z = x_1 + 5x_2 + 3x_3$ Subject to:

$x_1 + 2x_2 + x_3 = 3,\ 2x_1 - x_2 = 4;\ x_1, x_2, x_3 \geq 0$

(a) Write the associated dual problem.

(b) Given the information that the optimal basic variables are x_1 and x_3, determine the associated optimal dual solution.

6) Given the L.P.P. :

Maximize $Z = 2x_1 + 4x_2 + 4x_3 - 3x_4$ Subject to:

$x_1 + x_2 + x_3 = 4,\ x_1 + 4x_2 + x_4 = 8;\ x_1, x_2, x_3, x_4 \geq 0$

Use the dual problem to verify that the basic solution (x_1, x_2) is not optimal.

7) Find the optimal value of the objective function for the following problem by only inspecting its dual :

Minimize $Z = 10x_1 + 4x_2 + 5x_3 + x_4$ Subject to:

$5x_1 - 7x_2 + 3x_3 + 0.5x_4 \geq 150;\ x_j \geq 0\ (j = 1,2,3,4)$

8) Find the optimum solution to the I.P.P. :

Maximize $Z = x_1 - 2x_2$ Subject to the constraints:

$4x_1 + 2x_2 \leq 15,\ x_1, x_2 \geq 0$ and are integers.

9) Find the optimum integer solution to the all-integer programming problem:

Maximize $Z = x_1 + x_2$ Subject to the constraints:

$3x_1 + 2x_2 \leq 5,\ x_2 \leq 2,\ x_1 \geq 0,\ x_2 \geq 0$ and are integers

Space for Hints

10) Find the optimal solution to the following integer programming problem:

Maximize $Z = x_1 + 2x_2 \leq 4,\ 6x_1 + 2x_2 \leq 9,\ x_1,\ x_2 \geq 0$ and x_1 and x_2 are integers.

11) Solve the following I.P.P. :

Maximize $Z = 2x_1 + 3x_2$ Subject to:

$-3x_1 + 7x_2 \leq 14,\ 7x_1 - 3x_2 \leq 14 : x_1,\ x_2 \geq 0$ and are integers.

12) Describe any method of solving an integer programming problem. Use it to solve the problem :

Maximize $Z = 2x_1 + 2x_2$ Subject to the constraints:

$5x_1 + 3x_2 \leq 8,\ x_1 = 2x_2 \leq 4$; $x_1,\ x_2$ non –negative integers.

13) Solve the following I.P.P. :

Minimize $Z = 9x_1 + 10x_2$ Subject to the constraints:

$x_1 \leq 9,\ x_2 \leq 8,\ 4x_1 + 3x_2 \geq 40;\ x_1,\ x_2 \geq 0$ and are integers.

14) Solve the following I.P.P:

Maximize $Z = 11x_1 + 4x_2$ Subject to the constraints:

$-x_1 + 2x_2 \leq 4,\ 5x_1 + 2x_2 \leq 16,\ 2x_1 - x_2 \leq 4,\ x_1$ and x_2 are non-negative integers

Solve the following mixed-integer programming problems, using Gomory's cutting plane method:

15) Maximize $Z = x_1 + x_2$ Subject to the constraints:

$3x_1 + 2x_2 \leq 5,\ x_2 \leq 2;\ x_1,\ x_2 \geq 0$ and x_1 an integer.

16) Minimize $Z = x_1 - 3x_2$ Subject to the constraints:

$x_1 + x_2 \leq 5,\ -2x_1 + 4x_2 \leq 11$; $x_1,\ x_2 \geq 0$ and x_2 is an integer.

17) Maximize $Z = 7x_1 + 9x_2$ Subject to:

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$-x_1 + 3x_2 \leq 6,\ 7x_1 + x_2 \leq 35$; $x_1, x_2 \geq 0$ and x_1 is an integer.

18) Maximize $Z = -3x_1 + x_2 + 3x_3$ Subject to the constraints :

$-x_1 + 2x_2 + x_3 \leq 4,\ 4x_2 - 3x_3 \leq 2,\ x_1 - 3x_2 + 2x_2 \leq 3$:

x_1 and x_3 are integers and $x_j \geq 0$ $\qquad (j = 1,2,3)$

19) Maximize $Z = 1.5x_1 + 3x_2 + 4x_3$ Subject to the constraints:

$2.5x_1 + 2x_2 + 4x_3 \leq 12,\ 2x_1 + 4x_2 - x_3 \leq 7$; $x_1, x_2, x_3 \geq 0$ and x_3 is an integer.

Space for Hints

UNIT – 5

ASSIGNMENT PROBLEMS

Introduction

The assignment problem is a particular case of the transportation problem in which the objective is to assign a number of tasks (Jobs or origins or sources) to an equal number of facilities (machines or persons or destinations) at a minimum cost (or maximum profit).

Suppose that we have 'n' jobs to be performed on 'm' machines (one Job to one machine) and our objective is to assign the jobs to the machines at the minimum cost (or maximum profit) under the assumption that each machine can perform each job but with varying degree of efficiencies.

The assignment problem can be stated in the form of $m \times n$ matrix (c_{ij}) called a **Cost matrix** (or) **Effectiveness matrix** where c_{ij} is the cost of assigning i^{th} machine to the j^{th} job.

Jobs

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & \cdots\cdots & n \end{array} \\ \text{Machines} \begin{array}{c} 1 \\ 2 \\ 3 \\ : \\ : \\ : \\ m \end{array} & \left[\begin{array}{ccccc} c_{11} & c_{12} & c_{13} & \cdots\cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots\cdots & c_{2n} \\ c_{31} & c_{32} & c_{33} & \cdots\cdots & c_{3n} \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ c_{m1} & c_{m2} & c_{m3} & \cdots\cdots & c_{mn} \end{array}\right] \end{array}$$

OBJECTIVES

After studying this unit you will be able to

1. Structure special L.P.P using assignment models
2. Solve assignment problems with the Hungarian method.

STRUCTURE

5.1 MATHEMATICAL FORMULATION OF AN ASSIGNMENT PROBLEM

Consider an assignment problem of assigning *n* jobs to *n* machines (one job to one machine). Let c_{ij} be the unit cost of assigning i^{th} machine to the j^{th} job and

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$$\text{Let } x_{ij} = \begin{cases} 1, \text{ if } j^{th} \text{ job is assigned to } i^{th} \text{ machine} \\ 0, \text{ if } j^{th} \text{ is not assigned to } i^{th} \text{ machine} \end{cases}$$

The assignment model is then given by the following LPP

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to the constraints

$$\sum_{i=1}^{n} x_{ij} = 1, \ j = 1, 2, \ldots n$$

$$\sum_{j=1}^{n} x_{ij} = 1, \ i = 1, 2, \ldots n$$

$$\text{and } x_{ij} = 0 \text{ (or) } 1.$$

Theorem 5.1.1 (Reduction Theorem). In an assignment problem, if we add or subtract a constant to every element of any row (or column) of the cost matrix $[c_{ij}]$, then an assignment that minimizes the total cost on one matrix also minimizes the total cost on the other matrix. In other words, if $x_{ij} = x_{ij}^*$ minimizes

$$z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad \sum_{i=1}^{n} x_{ij} = 1, \ \sum_{j=1}^{n} x_{ij} = 1; \ x_{ij} = 0 \text{ or } 1;$$

then x_{ij}^* also minimizes $z^* = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}^* x_{ij}$, where $c_{ij}^* = c_{ij} - u_i - v_j$ for all $i, j = 1, 2, \ldots, n$ and u_i, v_j are some real numbers.

Proof

We write

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$$z^* = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}^* x_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (c_{ij} - u_i - v_j)\, x_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - \sum_{i=1}^{n} u_i \sum_{j=1}^{n} x_{ij} - \sum_{i=1}^{n} x_{ij} \sum_{j=1}^{n} v_j$$

$$= z - \sum_{i=1}^{n} u_i - \sum_{j=1}^{n} v_j;$$

Since $\sum_{i=1}^{n} x_{ij} = \sum_{j=1}^{n} x_{ij} = 1.$

This shows that the minimization of the new objective function z^* yields the same solution as the minimization of original objective function z, because $\sum u_i$ and $\sum v_j$ are independent of x_{ij}.

Theorem 5.1.2

If $c_{ij} \geq 0$ such that minimum $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} = 0$, then x_{ij} provides an optimum solution (assignment)

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The above two theorems form the basis of *Assignment Algorithm*. By selecting suitable constants to be added to or subtracted from the elements of the cost matrix we can ensure that $c_{ij}^* \geq 0$ and can produce at least one $c_{ij}^* = 0$ in each row and each column and try assignments from among these 0 positions. The assignment schedule will be optimal if there is exactly one assignment (i.e., exactly one assigned 0) in each row and each column.

Remarks:

It may be noted that assignment problem is a variation of transportation problem with two characteristics (i) the cost matrix is a square matrix, and (ii) the optimum solution for the problem would always be such that there would be only one assignment in a given row or column of the cost matrix.

5.2 ASSINGEMENT ALGORITHM OR HUNGARIAN METHOD

An efficient method for solving an assignment problem as developed by the Hungarian mathematician D. Konig is summarized below.

Step: 1

Determine the cost table from the given problem

I. If the number of sources is equal to the number of destination, go to step 3.

II. If the number of sources is not equal to the number of destinations, go to step 2.

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Step: 2

Add a dummy source or dummy destination, so that the cost table becomes a square matrix. The cost entries of dummy source/destinations are always zero.

Step: 3

Locate the smallest element in each row of the given cost matrix and then subtract the same from each element of that row.

Step: 4

In the reduced matrix obtained in step 3, locate the smallest element of each column and then subtract the same from each element of that column. Each column and row now have at least one zero.

Step: 5

In the modified matrix obtained in step 4, search for an optimal assignment as follows:

a) Examine the rows successively until a row with a single zero is found. Enrectangle this zero (□) and cross off (X) all other zeros in its column. Continue in this manner until all the rows have been taken care of.

b) Repeat the procedure for each column of the reduced matrix.

c) If a row and / or column has two or more zeros and one cannot be chosen by inspection then assign arbitrary any one of the zeros and cross off all other zeros of that row/ column

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d) Repeat (a) through (c) above successfully until the chain of assigning (□) or cross (X) ends.

Step: 6

If the number of assignments (□) is equal to n (the order of the cost matrix), an optimal solution is reached.

If the numbers of assignments is less than n (the order of the matrix), go to the next step.

Step: 7

Draw the minimum number of horizontal and / or vertical lines to cover all the zeros of the reduced matrix. This can be conveniently alone by using a simple procedure:

a) Mark (✓) rows that do not have any assigned zero.

b) Mark (✓) columns that have zeros in the marked rows.

c) Mark (✓) rows that have assigned zeros in the marked columns.

d) Repeat (b) and (c) above until the chain of marking is completed.

e) Draw lines through all the unmarked rows and marked columns. This given us the desired minimum number of lines.

Step: 8

Develop the new revised cost matrices as follows:

a) Find the smallest element of the reduced matrix not covered by any of the lines.

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b) Subtract this element from all the uncovered elements and add the same to all the element lying at the intersection of any two lines.

Step: 9

Go to step 6 and respect the procedure until an optimum solution is attained.

Example: 5.1.1

A departmental head has four subordinates, and four tasks to be performed. The subordinates differ in efficiency, and the tasks differ in their intrinsic difficulty. His estimate, of the time each man would take to perform each task, is given in the matrix below:

Tasks	Men			
	E	F	G	H
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

How should the tasks be allocated, one to a man, so as to minimize the total man – hours?

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Solution:

Step: 3

Subtracting the smallest element of each row from every element of the corresponding row, we get the reduced matrix

7	15	6	0
0	15	1	13
23	4	3	0
9	16	14	0

Step: 4

Subtracting the smallest element of each column of the reduced matrix from every element of the corresponding column. We get the following reduced matrix:

7	11	5	0
0	11	0	13
23	0	2	0
9	12	13	0

Step: 5

Starting with row 1, we enrectangle (☐) (i.e., make assignment 1 a single zero, if any, and cross (X) all other zeros in the column so marked. Thus, we get

7	11	5	$\boxed{0}$
$\boxed{0}$	11	$\cancel{0}$	13
23	$\boxed{0}$	2	$\cancel{0}$
9	12	13	$\cancel{0}$

Space for Hints

In the above matrix, we arbitrarily enrectangled a zero in column 1, because row 2 had two zeros.

It may be noted that column 3 and row 4 do not have any assignment. So, we move on to the next step.

Step: 7

I) Since row 4 does not have any assignment, we mark this row (✓).

II) Now there is a zero in the fourth column of the marked row. So, mark four in column (✓).

III) Further there is an assignment in the first row of the ticked column. So we mark first row (✓).

IV) Draw straight lines through all unmarked rows and marked columns. Thus, we have

$$\begin{array}{cccc|c} 7 & 11 & 5 & \boxed{0} & \checkmark \\ \boxed{0} & 11 & \cancel{0} & 13 & \\ 23 & \boxed{0} & 2 & \cancel{0} & \\ 9 & 12 & 13 & \cancel{0} & \checkmark \\ & & & \checkmark & \end{array}$$

Step: 8

In step 7, we observe that the minimum numbers of lines so drawn in 3, which is less than the orders of the cost matrix, indicating that the current assignment is not optimum.

Space for Hints

To increase the minimum numbers of lines, we generate new zeros in the modified matrix.

The smallest element not covered by the lines is 5. Subtracting this element from all the obtain the following new reduced cost matrix:

2	6	0	0
0	11	0	18
23	0	2	5
4	7	8	0

Step: 9

Repeating step 5 on the reduced matrix, we get

2	6	$\boxed{0}$	$\cancel{0}$
$\boxed{0}$	11	$\cancel{0}$	18
23	$\boxed{0}$	2	5
4	7	8	$\boxed{0}$

Now, since each row and each column has one and only one assignment, an optimal solution is reached. The optimum assignment is:

$A \rightarrow G,\ B \rightarrow E,\ C \rightarrow F$ and $D \rightarrow H$.

The minimum total time for this assignment scheduled is 17+13+19+10 or 59 man – hours.

Example: 5.1.2

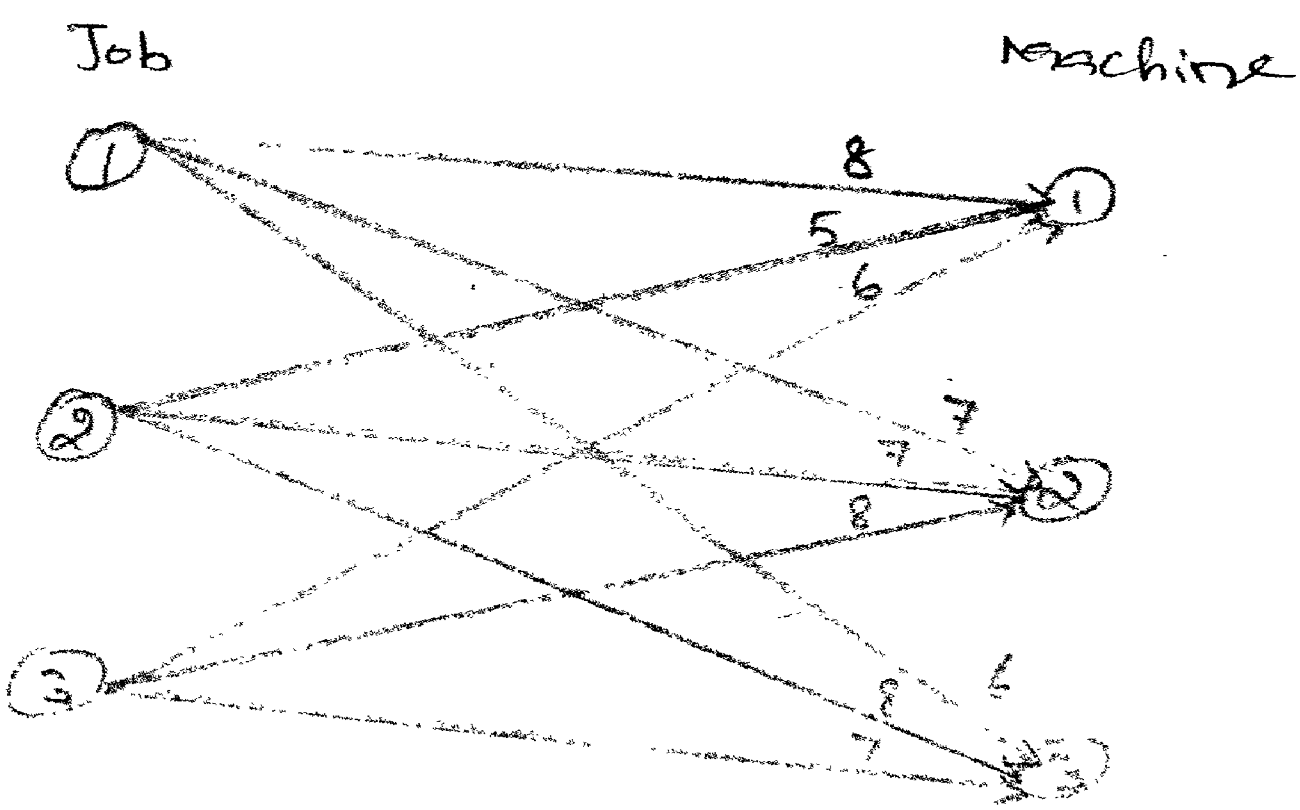
A Company wishes to assign 3 jobs to 3 machines in such a way that each job is assigned to some machine and no machine works on more than one job. The cost of assigning job i to machine j is given by the matrix below (ij^{th} entry):

$$\text{Cost Matrix: } \begin{bmatrix} 8 & 7 & 6 \\ 5 & 7 & 8 \\ 6 & 8 & 7 \end{bmatrix}$$

Draw the associated network. Formulate the network LPP and find the minimum cost of making the assignment.

Solution:

a) Network formulation of the given problem is given as under:



b) Linear programming formulation of the given problem is minimize the total cost involved, i.e., Minimize Z =

$$(8x_{11} + 7x_{12} + 6x_{13}) + (5x_{21} + 7x_{22} + 8x_{23}) + (6x_{31} + 8x_{32} + 7x_{33})$$

Subject to the constraints:

$$x_{i1} + x_{i2} + x_{i3} = 1; \qquad i = 1, 2, 3.$$

$$x_{1j} + x_{2j} + x_{3j} = 1; \qquad j = 1, 2, 3.$$

$$x_{ij} = 0 \text{ or } 1, \text{ for all } i \text{ and } j.$$

c) Reduce the cost matrix by subtracting smallest element of each row (column) from the corresponding row (column) elements. In the reduced matrix, make assignment in rows and columns having single zeros.

Initial Iteration:

Draw the minimum number of lines to cover all the zeros of the reduced matrix. See Table 1.

First Iteration:

Modify the reduced cost matrix by subtracting element 1 from all the elements not covered by the lines and adding the same at the intersection of two lines. See table 2.

Draw the minimum numbers of lines again.

2	1	$\boxed{0}$	
$\boxed{0}$	2	3	✓
$\cancel{0}$	2	1	✓
✓			

Table 1

3	1	$\boxed{0}$	✓
$\boxed{0}$	1	2	✓
$\cancel{0}$	1	$\cancel{0}$	✓
✓			

Table 1

Final Iteration:

Modify further the reduced table 2 by subtracting element 1 from all the elements not covered by the lines adding the same at the intersection of two lines. Thus, we get table 3

4	$\cancel{0}$	$\boxed{0}$
$\cancel{0}$	$\boxed{0}$	1
$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

Or

4	1	$\boxed{0}$
$\boxed{0}$	$\cancel{0}$	1
$\cancel{0}$	$\boxed{0}$	$\cancel{0}$

Or

4	$\boxed{0}$	$\cancel{0}$
$\boxed{0}$	$\cancel{0}$	1
$\cancel{0}$	$\cancel{0}$	$\boxed{0}$

Table 3

Since the number of assignments is equal to the order of the matrix, we have the optimum assignment schedule:

Job 1 →Machine 3, Job 2 →Machine 2, Job 3 →Machine 1;

or

Job 1 →Machine 3, Job 2 →Machine 1, Job 3 →Machine 2;

or

Job 1 →Machine 2, Job 2 →Machine 1, Job 3 →Machine 3

Total minimum cost is both the cases will be 19.

Space for Hints

Example 5.1.3

A department head has four tasks to be performed and three subordinates, the subordinates differ in efficiency. The estimates of the time, each subordinate would take to perform, is given below in the matrix.

How should he allocate the tasks one to each man, so as to minimize the total man-hours?

Task	Men		
	1	2	3
I	9	26	15
II	13	27	6
III	35	20	15
IV	18	30	20

Solution:

Since the problem is unbalanced, we add a dummy column with all the entries as zero and use assignment methods for optimum solution.

Now reduce the balanced cost matrix and make assignments in rows and columns having single zeros. Thus, we have:

$$\begin{array}{|cccc|}
\hline
\boxed{0} & 6 & 9 & \cancel{0} \\
4 & 7 & \boxed{0} & \cancel{0} \\
26 & \boxed{0} & 9 & \cancel{0} \\
9 & 10 & 14 & \boxed{0} \\
\hline
\end{array}$$

The optimum assignment is

$I \rightarrow 1$, $II \rightarrow 3$ and $III \rightarrow 2$; while task IV should be assigned to a dummy man, i.e., it remains to be done.

The minimum time is 35 hours.

Example 5.1.4

The assignment cost of assigning any one operator to any one machine is given in the following table

		Operators			
		I	II	III	IV
	A	10	5	13	15
Machine	B	3	9	18	3
	C	10	7	3	2
	D	5	11	9	7

Find the optimal assignment by Hungarian method.

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Solution: The cost matrix of the assignment problem is

$$\begin{pmatrix} 10 & 5 & 13 & 15 \\ 3 & 9 & 18 & 3 \\ 10 & 7 & 3 & 2 \\ 5 & 11 & 9 & 7 \end{pmatrix}$$

Since the number of rows is equal to the number of columns in the cost matrix, the given assignment problem is balanced.

Space for Hints

Select the smallest cost element in each row and subtract this from all the elements of the corresponding row, we get the reduced matrix.

$$\begin{pmatrix} 5 & 0 & 8 & 10 \\ 0 & 6 & 15 & 0 \\ 8 & 5 & 1 & 0 \\ 0 & 6 & 4 & 2 \end{pmatrix}$$

Select the smallest cost element in each column and subtract this from all the element of the corresponding column, we get the reduced matrix

$$\begin{pmatrix} 5 & 0 & 7 & 10 \\ 10 & 6 & 14 & 0 \\ 8 & 5 & 0 & 0 \\ 0 & 6 & 3 & 2 \end{pmatrix}$$

Since each row and each column contain exactly one zero, we shall make the assignment in rows and columns of this reduced cost matrix.

$$\begin{array}{|cccc|} \hline 5 & \boxed{0} & 7 & 10 \\ \cancel{0} & 6 & 14 & \boxed{0} \\ 8 & 5 & \boxed{0} & \cancel{0} \\ \boxed{0} & 6 & 3 & 2 \\ \hline \end{array}$$

Since each row and each column contain exactly one assignment (i.e., exactly one encircled zero), the current assignment is optimal.

The optimum assignment schedule is

$$A \rightarrow II,\ B \rightarrow IV,\ C \rightarrow III,\ D \rightarrow I$$

and the optimum (minimum) assignment cost

$$= Rs.(5+3+3+5) = Rs.16/-$$

Check your progress 5.1

1) Four professors are each capable of teaching any one of four different courses. Class preparation time in hours for different topics varies form professor to professor and is given in the table below. Each professor is assigned only one course. Determine an assignment schedule so as to minimize the total course preparation time for all' courses:

Professor	Linear programmes	Queueing theory	Dynamic programme	Regression analysis
A	2	10	9	7
B	15	4	14	8
C	13	14	16	11
D	4	15	13	9

2) An automobile dealer wishes to put four repairmen to four different jobs. The repairmen have somewhat different kinds of skills and they exhibit different levels of efficiency from one job to another. The dealer has estimated the number of man-hours that would be required for each job-man combination. This is given in the matrix form in the following table:

Space for Hints

Man	Job A	B	C	D
1	5	3	2	8
2	7	9	2	6
3	6	4	5	7
4	5	7	7	8

Find the optimum assignment that will result in minimum man-hours needed.

3) Solve the following assignment problems:

(a) $$\begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 4 & 6 & 3 \\ 9 & 7 & 10 & 9 \\ 4 & 5 & 11 & 7 \\ 8 & 7 & 8 & 5 \end{bmatrix} \end{array}$$

(b) $$\begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 10 & 25 & 15 & 20 \\ 15 & 10 & 5 & 15 \\ 35 & 20 & 12 & 24 \\ 17 & 25 & 24 & 20 \end{bmatrix} \end{array}$$

(c) $$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 10 & 12 & 19 & 11 \\ 5 & 10 & 7 & 8 \\ 12 & 14 & 13 & 11 \\ 8 & 15 & 11 & 9 \end{bmatrix} \end{array}$$

(d) $$\begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array} \begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} 5 & 8 & 3 & 2 \\ 10 & 7 & 5 & 8 \\ 4 & 10 & 12 & 10 \\ 8 & 6 & 9 & 4 \end{bmatrix} \end{array}$$

5.3 Maximization case in Assignment problem

In an assignment problem, we may have to deal with maximization of an objective function. For example, we may have to assign persons to jobs in such a way that the total profit is maximized. The maximization problem has to be converted into an equivalent minimization problem and then solved by the usual Hungarian Method.

The conversion of the maximization problem into an equivalent minimization problem can be done by any one of the following methods:

(i) Since max Z = - min (-Z), multiply all the cost elements c_{ij} of the cost matrix by -1.

(ii) Subtract all the cost elements c_{ij} of the cost matrix from the highest cost element in that cost matrix.

Example 5.3.1

A company has a team of four salesmen and there are four districts where the company wants to start its business. After taking into account the capabilities of salesman and the nature of districts, the company estimates that the profit per day in rupees for each salesman in each district is as below:

		Districts			
		1	2	3	4
	A	16	10	14	11
Salesmen	B	14	11	15	15
	C	15	15	13	12
	D	13	12	14	15

Space for Hints

Find the assignment of salesmen to various districts which will yield maximum profit.

Solution: The cost matrix of the given assignment problem is

$$\begin{pmatrix} 16 & \boxed{10} & 14 & 11 \\ 14 & 11 & 15 & 15 \\ 15 & 15 & 13 & 12 \\ 13 & 12 & 14 & 15 \end{pmatrix}$$

Since this is a maximization problem, it can be converted into an equivalent minimization problem by subtracting all the cost elements in the cost matrix from the highest cost element 16 of this cost matrix. Thus the cost matrix of the equivalent minimization problem is

$$\begin{pmatrix} 0 & 6 & 2 & 5 \\ 2 & 5 & 1 & 1 \\ 1 & 1 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

Select the smallest cost element in each row (column) and subtract this from all the cost elements of the corresponding row (column). We get the reduced cost matrix

$$\begin{pmatrix} 0 & 6 & 2 & 5 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 2 & 3 & 1 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} 0 & 6 & 2 & 5 \\ 1 & 4 & 0 & 0 \\ 0 & (0) & 2 & 3 \\ 2 & 3 & 1 & 0 \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by

$A \rightarrow 1$, $B \rightarrow 3$, $C \rightarrow 2$, $D \rightarrow 4$ and the optimum (maximum) profit

$$= \text{Rs. } (16+15+15+15)$$

$$= \text{Rs. } 61/-$$

Example 5.3.2

Solve the assignment problem for maximization given the profit matrix (Profit in rupees).

Machines

Job		P	Q	R	S
	A	51	53	54	50
	B	47	50	48	50
	C	49	50	60	61
	D	63	64	60	60

Solution: The profit matrix of the given assignment problem is

$$\begin{pmatrix} 51 & 53 & 54 & 50 \\ 47 & 50 & 48 & 50 \\ 49 & 50 & 60 & 61 \\ 63 & (64) & 60 & 60 \end{pmatrix}$$

Space for Hints

Since this is a maximization problem, it can be converted into an equivalent minimization problem by subtracting all the profit elements in the profit matrix from the highest profit element 64 of this profit matrix. Thus the cost matrix of the equivalent minimization problem is

$$\begin{pmatrix} 13 & 11 & 10 & 14 \\ 17 & 14 & 16 & 14 \\ 15 & 14 & 4 & 3 \\ 1 & 0 & 4 & 4 \end{pmatrix}$$

Select the smallest cost in each row and subtract this from all the cost elements of the corresponding row. We get

$$\begin{pmatrix} 3 & 1 & 0 & 4 \\ 3 & 0 & 2 & 0 \\ 12 & 11 & 1 & 0 \\ 1 & 0 & 4 & 4 \end{pmatrix}$$

Select the smallest cost element in each column and subtract this from all the cost elements of the corresponding column. We get

$$\begin{pmatrix} 2 & 1 & 0 & 4 \\ 2 & 0 & 2 & 0 \\ 11 & 11 & 1 & 0 \\ 0 & 0 & 4 & 4 \end{pmatrix}$$

Since eacn row and each column contains atleast one zero, we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} 2 & 1 & (0) & 4 \\ 2 & \boxed{0} & 2 & \cancel{0} \\ 11 & 11 & 1 & \boxed{0} \\ (0) & \cancel{0} & 4 & 4 \end{pmatrix}$$

Space for Hints :

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by $A \to R$, $B \to Q$, $C \to S$, $D \to P$ and the optimum (maximum) profit

$$= \text{Rs. } (54+50+61+63)$$

$$= \text{Rs. } 228/-$$

Example 5.3.3

A company is faced with the problem of assigning four different salesman to four territories for promoting its sales. Territories are not equally rich in their sales potential and the salesman also differ in their ability to promote sales. The following table gives the expected annual sales (in thousands of Rs) for each salesman if assigned to various territories. Find the assignment of salesman so as to maximize the annual sales.

		Territories			
		1	2	3	4
Salesmen	1	60	50	40	30
	2	40	30	20	15
	3	40	20	35	10
	4	30	30	25	20

Solution: The cost matrix of the given assignment problem is

$$\begin{pmatrix} (60) & 50 & 40 & 30 \\ 40 & 30 & 20 & 15 \\ 40 & 20 & 35 & 10 \\ 30 & 30 & 25 & 20 \end{pmatrix}$$

Since this is a maximization problem, it can be converted into an equivalent minimization problem by subtracting all the cost elements in the cost matrix from the highest cost element 60 of this cost matrix. Thus the cost matrix of the equivalent minimization problem is

$$\begin{pmatrix} 0 & 10 & 20 & 30 \\ 20 & 30 & 40 & 45 \\ 20 & 40 & 25 & 50 \\ 30 & 30 & 35 & 40 \end{pmatrix}$$

Select the smallest cost element in each row (column) and subtract this from all the cost elements of the corresponding row (column). We get the reduced cost matrix

$$\begin{pmatrix} 0 & 10 & 15 & 20 \\ 0 & 10 & 15 & 15 \\ 0 & 20 & 0 & 20 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we make the assignment in rows and columns of this reduced cost matrix

$$\begin{pmatrix} \boxed{0} & 10 & 15 & 20 \\ \cancel{0} & 10 & 15 & 15 \\ \cancel{0} & 20 & \boxed{0} & 20 \\ \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since there are some rows and columns without assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines

$$\begin{array}{c} \begin{pmatrix} 0 & 10 & 15 & 20 \\ 0 & \boxed{10} & 15 & 15 \\ 0 & 20 & 0 & 20 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \checkmark \\ \checkmark \\ \\ \\ \end{array}$$

Here 10 is the smallest cost element not covered by these straight lines. Subtract this 10 from all the uncovered elements, add this 10 to those elements which lie in the intersection of these straight lines and do not change the remaining elements which lie on these straight lines. We get

$$\begin{pmatrix} 0 & 0 & 5 & 10 \\ 0 & 0 & 5 & 5 \\ 10 & 20 & 0 & 20 \\ 10 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we make the assignment in rows and columns of this reduced cost matrix.

$$\begin{array}{|cccc|} \hline \boxed{0} & \cancel{0} & 5 & 10 \\ \cancel{0} & \boxed{0} & 5 & 5 \\ 10 & 20 & \boxed{0} & 20 \\ 10 & \cancel{0} & \cancel{0} & \boxed{0} \\ \hline \end{array}$$

Since each row and each column contains exactly one assignment (i.e., exactly one encircled zero), the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by Salesman 1 $\rightarrow$ Territory 1, Salesman 2 $\rightarrow$ Territory 2, Salesman 3 $\rightarrow$ Territory 3, Salesman 4 $\rightarrow$ Territory 4.

Space for Hints

The optimum (maximum) annual sales

= 60+30+35+20 (in thousand of rupees)

= 145 (in thousand of rupees)

= Rs. 1,45,000/-.

Note: For this problem, there exists alternative optimal assignment schedule with the same maximum sales Rs. 1,45,000/-.

Check your progress 5.2

1) MCS Inc is a software company that has three projects of Y2K with the department of health, education, and housing of Maharashtra Government. Based on the background and experiences of the project leaders, they differ in terms of their performance at various projects. The performance score matrix is given below:

Project Leaders	Projects		
	Health	Education	Housing
P_1	20	26	42
P_2	24	32	50
P_3	32	34	44

Help the management by determining the optimal assignment that maximises the total performance score.

2) (a) Suggest the optimal assignment schedule for the following assignment problem :

Sales (Rs. in lakh)

Salesmen	Markets			
	I	II	III	IV
A	80	70	75	72
B	75	75	80	85
C	78	78	82	78

What will be the total maximum sale?

3) (b) Solve the following assignment problem to find the maximum total expected sale

$$\begin{array}{ccc} & \text{Area} & \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ & \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 42 & 35 & 28 & 21 \\ 30 & 25 & 20 & 15 \\ 30 & 25 & 20 & 15 \\ 24 & 20 & 16 & 12 \end{bmatrix} \end{array}$$

Salesman

5.4 UNBALANCED ASSIGNMENT MODELS

If the number of rows is not equal to the number columns in the cost matrix of the given assignment problem, then the given assignment problem said to be unbalanced.

First convert the unbalanced assignment problem in to a balanced one by adding dummy rows or dummy columns with zero cost elements in the cost matrix depending upon whether $m < n$ or $m > n$ and then solve by the usual method.

Space for Hints

Example 5.4.1:

A company has four machines to do three jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table.

Machines

Jobs		1	2	3	4
	A	18	24	28	32
	B	8	13	17	19
	C	10	15	19	22

What are job assignments which will minimize the cost?

Solution:

The cost matrix of the given assignment problem is

$$\begin{pmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 19 \\ 10 & 15 & 19 & 22 \end{pmatrix}$$

Since the number of rows is less than the number of columns in the cost matrix, the given assignment problem is unbalanced.

To make it a balanced one, add a dummy job D (row) with zero cost elements. The balanced cost matrix is given by

$$\begin{pmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 19 \\ 10 & 15 & 19 & 22 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now select the smallest cost element in each row (column) and subtract this from all elements of the corresponding row (column), we get the reduced matrix

$$\begin{pmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & 5 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In this reduced matrix, we shall make the assignment in rows and columns having single zero. We have

$$\begin{array}{|cccc|} \hline \boxed{0} & 6 & 10 & 14 \\ \cancel{0} & 5 & 9 & 11 \\ \cancel{0} & 5 & 9 & 12 \\ \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} \\ \hline \end{array}$$

Since there are some rows and columns without assignment, the current assignment is not optimal.

Cover the all zeros by drawing a minimum number of straight lines. Choose the smallest cost element not covered by these straight lines.

$$\begin{pmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & 5 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Here 5 is the smallest cost element not covered by these straight lines. Subtract this 5 from all the uncovered elements, add this 5 to those elements which lie in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines. We get

$$\begin{pmatrix} 0 & 1 & 5 & 9 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 4 & 7 \\ 5 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make assignment in the rows and columns having single zero. We get

$$\begin{array}{|cccc|} \hline \boxed{0} & 1 & 5 & 9 \\ \cancel{0} & (0) & 4 & 6 \\ \cancel{0} & \cancel{0} & 4 & 7 \\ 5 & \cancel{0} & \boxed{0} & \cancel{0} \\ \hline \end{array}$$

Since there are some rows and columns without assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

$$\begin{pmatrix} 0 & 1 & 5 & 9 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & (4) & 7 \\ 5 & 0 & 0 & 0 \end{pmatrix}$$

Choose the smallest cost element not covered by these straight lines, subtract this from all the uncovered elements, add this to those elements which are in the intersection of the lines and do not change the remaining elements which lie on these straight lines. Thus we get

$$\begin{pmatrix} 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 9 & 4 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in the rows and columns having single zero. We get

$\boxed{0}$	1	1	5
$\cancel{0}$	(0)	$\cancel{0}$	2
$\cancel{0}$	$\cancel{0}$	(0)	3
9	4	$\cancel{0}$	$\boxed{0}$

Since each row and contains exactly one assignment (i.e., exactly one encircled zero) the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by $A \to 1,\ B \to 2,\ C \to 3,\ D \to 4$ and the optimum (minimum) assignment cost $= (18+17+15+0) = 50/-$, units of cost.

Note: 1

For this problem, the alternative optimum schedule is $A \to 1,\ B \to 3,\ C \to 2,\ D \to 4$ with the same optimum assignment cost $=$ Rs.$(18+17+15+0) = 50/-$units of cost.

Note: 2

Here the assignment $D \to 4$ means that the dummy Job D is assigned to the 4^{th} Machine. It means that machine 4 is left without any assignment.

Example 5.4.2

Assign four trucks 1, 2, 3 and 4 to vacant spaces A, B, C, D, E and F so that the distance travelled is minimized. The matrix below shows the distance.

	1	2	3	4
A	4	7	3	7
B	8	2	5	5
C	4	9	6	9
D	7	5	4	8
E	6	3	5	4
F	6	8	7	3

Solution

The matrix of the assignment problem is

$$\begin{pmatrix} 4 & 7 & 3 & 7 \\ 8 & 2 & 5 & 5 \\ 4 & 9 & 6 & 9 \\ 7 & 5 & 4 & 8 \\ 6 & 3 & 5 & 4 \\ 6 & 8 & 7 & 3 \end{pmatrix}.$$

Since the number of rows is more than the number of columns, the given assignment problem is unbalanced. To make it balanced, let us introduce two dummy trucks (columns) with zero costs. We get

$$\begin{pmatrix} 4 & 7 & 3 & 7 & 0 & 0 \\ 8 & 2 & 5 & 5 & 0 & 0 \\ 4 & 9 & 6 & 9 & 0 & 0 \\ 7 & 5 & 4 & 8 & 0 & 0 \\ 6 & 3 & 5 & 4 & 0 & 0 \\ 6 & 8 & 7 & 3 & 0 & 0 \end{pmatrix}$$

Select the smallest cost element in each row (column) and subtract this from all the elements of the corresponding row (column). We get

$$\begin{pmatrix} 0 & 5 & 0 & 4 & 0 & 0 \\ 4 & 0 & 2 & 2 & 0 & 0 \\ 0 & 7 & 3 & 6 & 0 & 0 \\ 3 & 3 & 1 & 5 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 0 \\ 2 & 6 & 4 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we make the assignment in rows and columns having single zero. We get

$$\begin{array}{|cccccc|} \boxed{0} & 5 & \boxed{0} & 4 & \boxed{0} & \boxed{0} \\ 4 & \boxed{0} & 2 & 2 & \cancel{0} & \cancel{0} \\ \boxed{0} & 7 & 3 & 6 & \cancel{0} & \cancel{0} \\ 3 & 3 & 1 & 5 & \boxed{0} & \cancel{0} \\ 2 & 1 & 2 & 1 & \cancel{0} & \boxed{0} \\ 2 & 6 & 4 & \boxed{0} & \cancel{0} & 0 \end{array}$$

Space for Hints

Since each row and each column contains exactly one assignment (i.e., exactly one encircled zero), the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by $A \to 3,\ B \to 2,\ C \to 1,\ D \to 5,\ E \to 6,\ F \to 4,$ and the optimum (minimum) distance.

$$=(3+2+4+0+0+3)$$

Units of distance =12/– units of distance.

Example: 5.4.3

A batch of 4 jobs can be assigned to 5 different machines. The set up time (in hours) for each job on various machines is given below:

Machine

Job		1	2	3	4	5
	1	10	11	4	2	8
	2	7	11	10	14	12
	3	5	6	9	12	14
	4	13	15	11	10	7

Find an optimal assignment of jobs to machines which will minimize the total set up time.

Solution:

The matrix of the given assignment problem is

$$\begin{pmatrix} 10 & 11 & 4 & 2 & 8 \\ 7 & 11 & 10 & 14 & 12 \\ 5 & 6 & 9 & 12 & 14 \\ 13 & 15 & 11 & 10 & 7 \end{pmatrix}$$

Space for Hints

Since the number of rows is less than the number of columns in the cost matrix, the given assignment problem is unbalanced.

To make it a balanced one, add a dummy job 5 (row) with zero cost elements. The balanced cost matrix is given by

$$\begin{pmatrix} 10 & 11 & 4 & 2 & 8 \\ 7 & 11 & 10 & 14 & 12 \\ 5 & 6 & 9 & 12 & 14 \\ 13 & 15 & 11 & 10 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now select the smallest cost element in each row (column) and subtract this from all the elements of the corresponding row (column). We get the reduced cost matrix.

$$\begin{pmatrix} 8 & 9 & 2 & 0 & 6 \\ 0 & 4 & 3 & 7 & 5 \\ 0 & 1 & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each columns contains atleast one zero, we shall make the assignment in rows and columns of this reduced cost matrix.

8	9	2	$\boxed{0}$	6
$\boxed{0}$	4	3	7	5
$\cancel{0}$	1	4	7	9
6	8	4	3	$\boxed{0}$
$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$

Space for Hints

Since there are some rows and columns with out assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

$$\begin{array}{c c}
\begin{pmatrix}
8 & 9 & 2 & 0 & 6 \\
0 & 4 & 3 & 7 & 5 \\
0 & (1) & 4 & 7 & 9 \\
6 & 8 & 4 & 3 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} &
\begin{array}{c}
\\ \checkmark \\ \checkmark \\ \\ \\
\end{array}
\end{array}$$

Here 1 is the smallest cost element not covered by these straight lines. Subtract this 1 from all the uncovered elements, and this 1 to those elements which lie in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines. We get

$$\begin{pmatrix}
9 & 9 & 2 & 0 & 6 \\
0 & 3 & 2 & 6 & 4 \\
0 & 0 & 3 & 6 & 8 \\
7 & 8 & 4 & 3 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns of this reduced cost matrix

9	9	2	$\boxed{0}$	6
$\boxed{0}$	3	2	6	4
$\cancel{0}$	$\boxed{0}$	3	6	8
7	8	4	3	$\boxed{0}$
1	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

Space for Hints

Since each row and each column contains exactly one assignment (i.e., exactly one encircled zero), the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by Job $1 \rightarrow M/c$ 4, Job $2 \rightarrow M/c$ 1, Job $3 \rightarrow M/c$ 2, Job $4 \rightarrow M/c3$ is left without any assignment.

The optimum (minimum) total set up time

$$= 2 + 7 + 6 + 7 \text{ hours}$$

$$= 22 \text{ hours.}$$

Check your progress 5.3

1) A department head has four tasks to be performed by three subordinates, the subordinates differing in efficiency. The estimates of the time, each subordinate would take to perform, is given below in the matrix. How should he allocate the tasks one to each to each man, so as to minimize the total man-hour?

Men

		1	2	3
	I	9	26	15
Tasks	II	13	27	6
	III	35	20	15
	IV	18	30	20

2) A company has four machines on which to do three jobs. Each job can be assigned to one and only machine. The cast of each job on each machine is given in the following table:

Space for Hints

Machine

		P	Q	R	S
Job	A	18	24	28	32
	B	8	13	17	19
	C	10	15	19	22

What are the job assignments which will minimize the cost?

3) Solve the following unbalanced problem of assigning four jobs to three different men (only one job to each man). The time to perform the job by different men is given in the following table:

Job

		J_1	J_2	J_3	J_4
Men	M_1	7	5	8	4
	M_2	5	6	7	4
	M_3	8	7	9	8

5.5 Routing problems:

Example 5.5.1

Kapil airlines that operates seven days a week has a time-table as shown below. Crews must have a minimum layover of 5 hours between flights. Obtain the pairing of fights that minimizes layover time away from home. For any given pairing, the crew will be based at the city that results in the smaller layover:

Delhi - Jaipur			Jaipur - Delhi		
Flight No.	Depart	Arrive	Flight No.	Depart	Arrive
1	7.00 A.M.	8.00 A.M.	101	8.00 A.M.	9.15 A.M.
2	8.00 A.M.	9.00 A.M.	102	8.30 A.M.	9.45 A.M.
3	1.30 P.M.	2.30 P.M.	103	12.00 Noon	1.15 P.M.
4	6.00 P.M.	7.30 A.M.	104	5.30 P.M.	6.45 P.M.

For each pair also mention the town where the crew should be based.

Solution:

It is assumed that a plane flying from Delhi for Jaipur must come back to Delhi at the immediate next opportunity. It is further assumed that each place will make only one forward and one return trip and thus there must be four planes for four forward and return flights. The problem is to determine the optimum pairing.

As the objective is to minimize the total layover time, both at first instance determine the lay-over time both at Delhi and at Jaipur for all possible pairings. For instance, for flight 1 and 101 pair, the plane leaves Delhi at 7 a.m. reaching Jaipur at 8 a.m. If it has to avail of 101 return flight, it can start at 8 a.m. next day, thereby making the layover of 24 hours at Jaipur at 7 a.m. next morning making the layover of 21.75 hours at Delhi. Likewise, we determine layover time at Jaipur and Delhi for all possible pairings. These are shown in the following tables:

Space for Hints

Layover Time in Hours

Crew based in Delhi

	101	102	103	104
1	24	24.5	28	9.5
2	23	23.5	27	8.5
3	17.5	18	21.5	27
4	12.5	13	16.5	22

Table 1

Crew based in Jaipur

	101	102	103	104
1	21.75	21.25	17.75	12.25
2	22.75	22.25	18.75	13.25
3	28.25	27.25	24.25	18.75
4	9.25	8.75	5.25	23.75

Table 2

Next construct the table for minimum layover time between flights with the help of above tables. The layover time marked '*' denotes that the crew is based in Jaipur. Thus we get the following table:

Minimum layover time

	101	102	103	104
1	21.75*	21.25*	17.75*	9.5
2	22.75*	22.25*	18.75*	8.5
3	17.5	18	21.5	18.75*
4	9.25*	8.75*	5.25*	22

Table 3

Now, subtract the minimum element of each row from all the elements of that row. Then subtract the minimum element of each column from all the elements of that column. In the reduced matrix make assignments in rows and columns that have single zeros. Thus, we have:

Initial Iteration

Drew the minimum number of lines to cover all the zeros of the reduced matrix. See table 4.

12.25	11.25	8.25	$\boxed{0}$
14.25	13.25	10.25	~~0~~
~~0~~	$\boxed{0}$	3.50	0.75
4.0	3.0	$\boxed{0}$	6.75

Table 4

First Iteration

Modify the reduced matrix by subtracting the element '8.25' from all the elements not covered by the lines and adding the same at the intersection of two lines. See table 5:

4	3	$\boxed{0}$	~~0~~
6	5	2	$\boxed{0}$
$\boxed{0}$	~~0~~	3.5	9
4	3	~~0~~	2.5

Table 5

Final Iteration

Modify further the reduced table 5 by subtracting element '3' from all the elements not covered by the lines and adding the same at the intersection of two lines. Thus we get table 6:

Space for Hints

	101	102	103	104
1	1	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$
2	3	2	2	$\boxed{0}$
3	$\boxed{0}$	$\cancel{0}$	6.5	12
4	1	$\cancel{0}$	$\boxed{0}$	25

Table 6

or

	101	102	103	104
1	1	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$
2	3	2	2	$\boxed{0}$
3	$\boxed{0}$	$\cancel{0}$	6.5	12
4	1	$\boxed{0}$	$\cancel{0}$	2.5

Table 7

Thus the optimum solution is obtained with the following assignment schedule:

Flights: $1 \to 102$ $2 \to 104$ $3 \to 101$ $4 \to 103$

or $1 \to 103$ or $4 \to 102$

Base Crew: Jaipur Delhi Delhi Jaipur

Total layover time = 52.5 hours.

Check your progress 5.4

1) XYZ Airline operation 7 days a week has given the following time-table. Crews must have a minimum layover of 5 hours between flights. Obtain the pairing flights that minimize layover time away from home. For any given pairing the crew will be based at the city results in the smaller layover:

Chennai – Mumbai			Mumbai – Chennai		
Flight No.	Depart	Arrive	Flight No.	Depart	Arrive
A_1	6.00 AM	8 AM	B_1	8.00 AM	10.00 AM
A_2	8.00 AM	10.00 A.M.	B_2	9.00 A.M.	11.00 A.M.
A_3	2.00 P.M.	4.00 P.M.	B_3	2.00 P.M.	4.00 P.M.
A_4	8.00 P.M.	10.00 P.M.	B_4	7 P.M.	9 P.M.

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2) A trip from Chennai to Bangalore takes six hours by bus. A typical time table of the bus service in both directions in both directions is given below:

Departure from chennai	Route number	Arrivalat Bangalore	Arrivalat chennai	Route number	Departure from Bangalore
06.00	$\xrightarrow{a}$	12.00	11.30	$\xleftarrow{1}$	05.30
07.30	$\xrightarrow{b}$	13.30	15.00	$\xleftarrow{2}$	09.00
11.30	$\xrightarrow{c}$	17.30	21.00	$\xleftarrow{3}$	15.00
19.00	$\xrightarrow{d}$	01.00	00.30	$\xleftarrow{4}$	18.30
00.30	$\xrightarrow{e}$	06.30	06.00	$\xleftarrow{5}$	00.00

The cost of providing this service by the transport company depends upon the time spent by the bus crew (driver and conductor) away from their places in addition to service time. There are five crews. There is a constraint that every crew should be provided with more than 4 hours of rest before residential facilities for the crew at Chennai as well as at Bangalore. Find which crew be assigned which line of service or which service line be connected with which other line, so as to reduce the waiting time to be the minimum.

5.6 Restrictions in Assignment Problems

The assignment technique assumes that the problem is free from practical restrictions and any task could be assigned to any facility. But in some cases, it may not be possible to assign a particular task to a particular facility due to space, size of the task, process capability of the facility. Technical difficulties or other restrictions. This can be overcome by assigning a very high processing time or cost element (it can be ∞) to the corresponding cell. This cell will be automatically excluded in the assignment because of the unused high time cost associated with it.

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Example: 5.6.1

A machine shop purchased a drilling machine and two lathes of different capacities. The positioning of the machines among 4 possible locations on the shop floor is important from the standard of materials handling. Given the cost estimate per unit time of materials below, determine the optimal location of the machines.

	Location 1	2	3	4
Lathe 1	12	9	12	9
Drill	15	Not suitable	13	20
Lathe 2	4	8	10	6

Solution

Since the drilling machine is not suitable for location 2, the corresponding cost element should be taken as ∞. Thus the cost matrix of the given assignment problem is

$$\begin{pmatrix} 12 & 9 & 12 & 9 \\ 15 & \infty & 13 & 20 \\ 4 & 8 & 10 & 6 \end{pmatrix}$$

Since the number of rows is less than the number of columns, we add a dummy row (a dummy drilling machine or a dummy lathe 3) with zero cost elements. The cost matrix for the balanced assignment problem is

$$\begin{pmatrix} 12 & 9 & 12 & 9 \\ 15 & \infty & 13 & 20 \\ 4 & 8 & 10 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Select the smallest cost in each row (column) and subtract this from all the cost elements of the corresponding row (column). We get the reduced matrix.

$$\begin{pmatrix} 3 & 0 & 3 & 0 \\ 2 & \infty & 0 & 7 \\ 0 & 4 & 6 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignments in rows and columns having single zero. We get

$$\begin{array}{|cccc|} \hline 3 & \boxed{0} & 3 & \cancel{0} \\ 2 & \infty & \boxed{0} & 7 \\ \boxed{0} & 4 & 6 & 2 \\ \cancel{0} & \cancel{0} & \cancel{0} & \boxed{0} \\ \hline \end{array}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by

lathe 1 $\rightarrow$ Location 2, Drill $\rightarrow$ Location 3,

Lathe 2 $\rightarrow$ Location 1, Dummy drill $\rightarrow$ Location 4

and the optimum (minimum) assignment cost

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$$=(9+13+4+0) \text{ unit of cost}$$

$$=26/- \text{ units of cost.}$$

Note:

For this the alternate optimum assignment is

Lathe 1	$\rightarrow$	Location 4,	Drill	$\rightarrow$	Location 3,
Lathe 2	$\rightarrow$	Location 1,	Dummy drill	$\rightarrow$	Location 2.

With the same optimum (minimum) assignment cost

$$=(9+13+4+0) \text{ units of cost}$$

$$=26/ \text{ units of cost.}$$

Example: 5.6.2

Five workers are available to work with the machine and the respective costs (in rupees) associated with each worker – machine assignment is given below. A sixth machine is available to replace one of the existing machines and the associated costs also given below:

Machines

Workers		M_1	M_2	M_3	M_4	M_5	M_6
	W_1	12	3	6	–	5	8
	W_2	4	11	–	5	–	3
	W_3	8	2	10	9	7	5
	W_4	–	7	8	6	12	10
	W_5	5	8	9	4	6	–

(i) Determine whether the new machine can be accepted?

(ii) Determine also optimal assignment and the associated saving in cost.

Solution

The cost matrix of the given assignment problem is

$$\begin{pmatrix} 12 & 3 & 6 & \infty & 5 & 8 \\ 4 & 11 & \infty & 5 & \infty & 3 \\ 8 & 2 & 10 & 9 & 7 & 5 \\ \infty & 7 & 8 & 6 & 12 & 10 \\ 5 & 8 & 9 & 4 & 6 & \infty \end{pmatrix}$$

Since the number of rows is less than the number of columns, the given assignment problem is unbalanced. Add a dummy worker W_6 (dummy row) with zero cost elements.

Thus the cost matrix of the balanced assignment problem is

$$\begin{pmatrix} 12 & 3 & 6 & \infty & 5 & 8 \\ 4 & 11 & \infty & 5 & \infty & 3 \\ 8 & 2 & 10 & 9 & 7 & 5 \\ \infty & 7 & 8 & 6 & 12 & 10 \\ 5 & 8 & 9 & 4 & 6 & \infty \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Select the smallest cost in each row and column and subtract this from all the elements of the corresponding row and column of the cost matrix. We get

Space for Hints

$$\begin{pmatrix} 9 & 0 & 3 & \infty & 2 & 5 \\ 1 & 8 & \infty & 2 & \infty & 0 \\ 6 & 0 & 8 & 7 & 5 & 3 \\ \infty & 1 & 2 & 0 & 6 & 4 \\ 1 & 4 & 5 & 0 & 2 & \infty \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} 9 & \boxed{0} & 3 & \infty & 2 & 5 \\ 1 & 8 & \infty & 2 & \infty & \boxed{0} \\ 6 & \times & 8 & 7 & 5 & 3 \\ \infty & 1 & 2 & \boxed{0} & 6 & 4 \\ 1 & 4 & 5 & \times & 2 & \infty \\ \boxed{0} & \times & \times & \times & \times & \times \end{pmatrix}$$

Since some rows and columns are without assignment, the current assignment is not optimal.

Cover all the zeros by drawing minimum number of straight lines:

$$\begin{array}{c} \begin{pmatrix} 9 & 0 & 3 & \infty & 2 & 5 \\ 1 & 8 & \infty & 2 & \infty & 0 \\ 6 & 0 & 8 & 7 & 5 & 3 \\ \infty & 1 & 2 & 0 & 6 & 4 \\ (1) & 4 & 5 & 0 & 2 & \infty \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \checkmark \\ \\ \checkmark \\ \checkmark \\ \checkmark \\ \\ \end{array}$$

Choose the smallest cost element not covered not covered by these straight lines. Here 1 is such element. Subtract this 1 from all the uncovered elements, add this 1 to those elements which are in the

Space for Hints

intersection of the straight lines and do not change the remaining elements which lie on these straight lines. We get

$$\begin{pmatrix} 8 & 0 & 2 & \infty & 1 & 4 \\ 1 & 9 & \infty & 3 & \infty & 0 \\ 5 & 0 & 7 & 7 & 4 & 2 \\ \infty & 1 & 1 & 0 & 5 & 3 \\ 0 & 4 & 4 & 0 & 1 & \infty \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

New we shall make the assignment in rows and columns having single zero.

$$\begin{pmatrix} 8 & \boxed{0} & 2 & \infty & 1 & 4 \\ 1 & 9 & \infty & 3 & \infty & \boxed{0} \\ 5 & \cancel{0} & 7 & 7 & 4 & 2 \\ \infty & 1 & 1 & \boxed{0} & 5 & 3 \\ \boxed{0} & 4 & 4 & \cancel{0} & 1 & \infty \\ 0 & 1 & \boxed{0} & 1 & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since some rows and columns are without assignment, the current assignment is not optimal.

Cover all the zeros by drawing minimum number of straight lines.

$$\begin{array}{cc} \begin{pmatrix} 8 & 0 & 2 & \infty & (1) & 4 \\ 1 & 9 & \infty & 3 & \infty & 0 \\ 5 & 0 & 7 & 7 & 4 & 2 \\ \infty & 1 & 1 & 0 & 5 & 3 \\ 0 & 4 & 4 & 0 & 1 & \infty \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} & \begin{matrix} \checkmark \\ \\ \checkmark \\ \\ \\ \\ \end{matrix} \\ \begin{matrix} \quad & \checkmark & \quad & \quad & \quad & \quad \end{matrix} & \end{array}$$

Subtract the smallest uncovered element 1 from all the uncovered elements, and this 1 to those elements which are in the

Space for Hints

intersection of these straight lines and do not change the remaining elements which lie on these straight lines. We get

$$\begin{pmatrix} 7 & 0 & 1 & \infty & 0 & 3 \\ 1 & 10 & \infty & 3 & \infty & 0 \\ 4 & 0 & 6 & 6 & 3 & 1 \\ \infty & 2 & 1 & 0 & 5 & 3 \\ 0 & 5 & 4 & 0 & 1 & \infty \\ 0 & 2 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Now we shall make the assignment in rows and columns having single zeros.

$$\begin{pmatrix} 7 & \cancel{0} & 1 & \infty & \boxed{0} & 3 \\ 1 & 10 & \infty & 3 & \infty & \boxed{0} \\ 4 & \boxed{0} & 6 & 6 & 3 & 1 \\ \infty & 2 & 1 & \boxed{0} & 5 & 3 \\ \boxed{0} & 5 & 4 & \cancel{0} & 1 & \infty \\ \cancel{0} & 2 & \boxed{0} & 1 & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by $W_1 \rightarrow M_5$, $W_2 \rightarrow M_6$, $W_3 \rightarrow M_2$, $W_4 \rightarrow M_4$, $W_5 \rightarrow M_1$, $W_6 \rightarrow M_3$,

and the optimum (minimum) assignment cost according to this schedule is

$$= \text{Rs.}(5+3+2+6+5+0)$$

$$= \text{Rs.}21/-$$

Now, if the sixth machine M_6 is not assigned to any of the workers, the given problem reduces to balanced one (deleting the sixth

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column). Applying the assignment algorithm to this balanced problem (reduced problem), the optimal assignment schedule is given by

$$W_1 \rightarrow M_5,\ W_2 \rightarrow M_1,\ W_3 \rightarrow M_2,\ W_4 \rightarrow M_3,\ W_5 \rightarrow M_4,$$

and the optimum (minimum) assignment cost assignment cost according to this schedule is

$$= \text{Rs.}(5+4+2+8+4)$$

$$= \text{Rs.}23/-$$

It is clear from the above that the minimum cost is more when there are only five machines. Hence, the sixth machine should be accepted. By accepting this sixth machine the associated saving cost will be Rs.$(23-21)$= Rs.2.

5.7 Travelling salesman problem

A salesman normally must visit a number of cities starting from his head quarters. The distance (or time or cost) between every pair of cities are assumed to be known. The problem of finding the shortest distance (or minimum time or minimum cost) if the salesman starts from his headquarters and passes through each city under his Jurisdiction exactly once and returns to the headquarters is called the **Travelling salesman problem** or **A Travelling salesperson problem.**

A travelling salesman problem is very similar to the assignment problem with the additional constraints.

(a) The salesman should go through every city exactly once except the starting city (headquarters).

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(b) The salesman starts from one city (head quarters) and comes back to that city (headquarters).

(c) Obviously going from any city to the same city directly is not allowed (i.e., no assignments should be made along the diagonal line).

Note: 1

Conditions (a) and (b) are usually called **route conditions.**

Note: 2

If a salesman has to visit n cities, then he will have a total of (n-1)! Possible round trips.

Therefore, the necessary basic steps to solve a travelling salesman problem are:

(i) Assigning an infinitely large element (∞) in each of the squares along the diagonal line in the cost matrix.

(ii) Solving the problem as a routine assignment problem.

(iii) Scrutinizing the solution obtained under (ii) to see if the "route" conditions are satisfied.

(iv) If not, making adjustments in assignments to satisfy the condition with minimum increase in total cost (i.e., to satisfy route condition "next best solution" may require to be considered).

Example: 5.7.1

Solve the following travelling salesman problem

To

Space for Hints

$$
\begin{array}{cc}
 & \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\
\begin{array}{r} \text{A} \\ \text{From B} \\ \text{C} \\ \text{D} \end{array} &
\begin{array}{cccc}
- & 46 & 16 & 40 \\
41 & - & 50 & 40 \\
82 & 32 & - & 60 \\
40 & 40 & 36 & -
\end{array}
\end{array}
$$

Solution

The cost matrix of the given travelling salesman problem is

$$
\begin{pmatrix}
\infty & 46 & 16 & 40 \\
41 & \infty & 50 & 40 \\
82 & 32 & \infty & 60 \\
40 & 40 & 36 & \infty
\end{pmatrix}
$$

Solve this as a routine assignment problem

Subtract the smallest cost element in each row from all the elements of the corresponding row. We get.

$$
\begin{pmatrix}
\infty & 30 & 0 & 24 \\
1 & \infty & 10 & 0 \\
50 & 0 & \infty & 28 \\
4 & 4 & 0 & \infty
\end{pmatrix}
$$

Subtract the smallest cost element in each column from all the elements of the corresponding column. We get

$$
\begin{pmatrix}
\infty & 30 & 0 & 24 \\
0 & \infty & 10 & 0 \\
49 & 0 & \infty & 28 \\
3 & 4 & 0 & \infty
\end{pmatrix}
$$

Now we shall make the assignment in rows and columns having single zero. We get

Space for Hints

$$\begin{pmatrix} \infty & 30 & \boxed{0} & 24 \\ \boxed{0} & \infty & 10 & \cancel{0} \\ 49 & \boxed{0} & \infty & 28 \\ 3 & 4 & \cancel{0} & \infty \end{pmatrix}$$

Since some rows and columns are without assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

$$\begin{array}{c} \begin{pmatrix} \infty & 30 & 0 & 24 \\ 0 & \infty & 10 & 0 \\ 49 & 0 & \infty & 28 \\ (3) & 4 & \cancel{0} & \infty \end{pmatrix} \begin{array}{c} \checkmark \\ \\ \\ \checkmark \end{array} \\ \qquad\qquad \checkmark \end{array}$$

Subtract the smallest uncovered cost element 3 from all uncovered elements, add this 3 to those elements which are in the intersection of these straight and do not change the remaining elements which lie on these straight lines. We have

$$\begin{pmatrix} \infty & 27 & 0 & 21 \\ 0 & \infty & 13 & 0 \\ 49 & 0 & \infty & 28 \\ 0 & 1 & 0 & \infty \end{pmatrix}$$

Now we shall make the assignment in rows and columns having single zeros. We get

$$\begin{pmatrix} \infty & 27 & \boxed{0} & 21 \\ \cancel{0} & \infty & 13 & \boxed{0} \\ 49 & \boxed{0} & \infty & 28 \\ \boxed{0} & 1 & \cancel{0} & \infty \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal for the assignment problem.

$\therefore$ The optimum assignment schedule is given by

$$A \to C,\ B \to D,\ C \to B,\ D \to A,$$

i.e., $A \to C,\ C \to B,\ B \to D,\ D \to A,$

i.e., $A \to C \to B \to D \to A$

Check whether the route conditions are satisfied.

$A \to C \to B \to D \to A$ satisfies the route condition.

$\therefore$ The required minimum costs.

$$= (16 + 32 + 40 + 40) \text{ units of cost.}$$

$$= 128/- \text{ units of cost.}$$

Example: 5.7.2

Solve the following travelling salesman problem so as to minimize the cost per cycle.

To

		A	B	C	D	E
From	A	–	3	6	2	3
	B	3	–	5	2	3
	C	6	5	–	6	4
	D	2	2	6	–	6
	E	3	3	4	6	–

Space for Hints

Space for Hints

Solution

The cost matrix of the given travelling salesman problem is

$$\begin{pmatrix} \infty & 3 & 6 & 2 & 3 \\ 3 & \infty & 5 & 2 & 3 \\ 6 & 5 & \infty & 6 & 4 \\ 2 & 2 & 6 & \infty & 6 \\ 3 & 3 & 4 & 6 & \infty \end{pmatrix}$$

Subtract the smallest cost elements in each row from all the elements of the corresponding row. We get

$$\begin{pmatrix} \infty & 1 & 4 & 0 & 1 \\ 1 & \infty & 3 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 4 & \infty & 4 \\ 0 & 0 & 1 & 3 & \infty \end{pmatrix}$$

Subtract the smallest cost element in each column from all the elements of the corresponding column. We get

$$\begin{pmatrix} \infty & 1 & 3 & 0 & 1 \\ 1 & \infty & 2 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 3 & \infty & 4 \\ 0 & 0 & 0 & 3 & \infty \end{pmatrix}$$

Now we shall make the assignment in rows and columns having single zeros we get

$$\begin{pmatrix} \infty & 1 & 3 & \boxed{0} & 1 \\ 1 & \infty & 2 & \cancel{0} & 1 \\ 2 & 1 & \infty & 2 & \boxed{0} \\ \cancel{0} & \boxed{0} & 3 & \infty & 4 \\ \cancel{0} & \cancel{0} & (0) & 3 & \infty \end{pmatrix}$$

Space for Hints

Since some rows and columns are without assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

$$
\begin{array}{c c}
\begin{pmatrix}
\infty & \boxed{1} & 3 & 0 & 1 \\
1 & \infty & 2 & 0 & 1 \\
2 & 1 & \infty & 2 & 0 \\
0 & 0 & 3 & \infty & 4 \\
0 & 0 & 0 & 3 & \infty
\end{pmatrix} & \begin{array}{c} \checkmark \\ \checkmark \\ \\ \\ \\ \end{array} \\
\begin{array}{ccccc} & & & \checkmark & \end{array} &
\end{array}
$$

Subtract the smallest uncovered cost element 1 from all uncovered elements, add this 1 to those elements which are in the intersection of these straight lines and do not change the remaining elements which lie on these straight lines. We have

$$
\begin{pmatrix}
\infty & 0 & 2 & 0 & 0 \\
0 & \infty & 1 & 0 & 0 \\
2 & 1 & \infty & 3 & 0 \\
0 & 0 & 3 & \infty & 4 \\
0 & 0 & 0 & 4 & \infty
\end{pmatrix}
$$

Now we shall make the assignment in rows and columns having single zero. We get

$$
\begin{pmatrix}
\infty & \cancel{0} & 2 & \boxed{0} & \cancel{0} \\
\boxed{0} & \infty & 1 & \cancel{0} & \cancel{0} \\
2 & 1 & \infty & 3 & \boxed{0} \\
\cancel{0} & \boxed{0} & 3 & \infty & 4 \\
\cancel{0} & \cancel{0} & \boxed{0} & 4 & \infty
\end{pmatrix}
$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

$\therefore$ The optimal assignment schedule is given by

Space for Hints

$A \rightarrow D, \quad B \rightarrow A, \quad C \rightarrow E, \quad D \rightarrow B, \quad E \rightarrow C$

i.e. $\quad A \rightarrow D \rightarrow B \rightarrow A, \quad C \rightarrow E \rightarrow C$

and the corresponding optimum (minimum) assignment cost

$= (2 + 3 + 4 + 2 + 4)$ units of cost

$= 15/-$ units of cost.

But this assignment schedule does not provide the solution this travelling salesman problem, because it does not satisfy the 'route' condition.

We try to find the next best solution which satisfies the route condition also. The next minimum (non- zero) cost element in the cost matrix is 1. So we try to bring 1 in to the solution. But the 1 occurs at two places. We shall consider all the cases separately until the acceptable solution is reached.

We start with making an assignment at (2, 3) instead of zero assignment at (2, 1). The resulting feasible solution will then be

$$\begin{pmatrix} \infty & \cancel{0} & 2 & \boxed{0} & \cancel{0} \\ \cancel{0} & \infty & \boxed{1} & \cancel{0} & \cancel{0} \\ 2 & 1 & \infty & 3 & (0) \\ \cancel{0} & \boxed{0} & 3 & \infty & 4 \\ \boxed{0} & \cancel{0} & \cancel{0} & 4 & \infty \end{pmatrix}$$

$\therefore$ The optimum assignment is given by

$A \rightarrow B, B \rightarrow C, C \rightarrow E, D \rightarrow B, E \rightarrow A,$

i.e., $\quad A \rightarrow D \rightarrow B \rightarrow C \rightarrow E \rightarrow A$

Also, when an assignment is made at (3, 2) instead of zero assignment (3, 5) at, the resulting feasible solution will be

$$\begin{pmatrix} \infty & \cancel{0} & 2 & \cancel{0} & \boxed{0} \\ \cancel{0} & \infty & 1 & 0 & \cancel{0} \\ 2 & \boxed{1} & \infty & 3 & \cancel{0} \\ \boxed{0} & \cancel{0} & 3 & \infty & 4 \\ \cancel{0} & \cancel{0} & \boxed{0} & 4 & \infty \end{pmatrix}$$

$\therefore$ The optimum assignment is given by

$$A \to E,\ B \to D,\ C \to B,\ D \to A,\ E \to C,$$

i.e., $A \to E \to C \to B \to D \to A$

$\therefore$ For the given travelling salesman problem, the optimum assignment schedule is given by

$A \to D \to B \to C \to E \to A$, (or)

$A \to E \to C \to B \to D \to A$

In both cases, the optimum (minimum) assignment cost is 16/– units of cost.

Check your progress 5.5

1) Given the following matrix of set-up costs, show how to sequence so as to minimize set-up cost per cycle:

From	To				
	A	**B**	**C**	**D**	**E**
A	∞	2	5	7	1
B	6	∞	3	8	2
C	8	7	∞	4	7
D	12	4	6	∞	5
E	1	3	2	8	∞

Space for Hints

2) Solve the following travelling salesman problem so as to minimize the cost per cycle.

From	To: A	B	C	D	E
A	-	3	6	2	3
B	3	-	5	2	3
C	6	5	-	6	4
D	2	2	6	-	6
E	3	3	4	6	-

3) Solve the travelling salesman problem given by the following data:

$$c_{12} = 20,\ c_{13} = 4,\ c_{14} = 10,\ c_{23} = 5,\ c_{34} = 6$$

$$c_{25} = 10,\ c_{35} = 6,\ c_{45} = 20,\ \text{where}\ c_{ij} = c_{ji},$$

and there is no route between cities i and j if a value for c_{ij} is not shown. [LAS 1991]

4) The ABC Ice Cream company has a distribution depot in Greater Kailash Part I for distributing ice cream in south Delhi. There are four vendors located in different parts of south Delhi (call them A, B, C and D) who have to be supplied ice cream every day. The following matrix displays the distances (in kilometres) between the depot and the four vendors:

Space for Hints

From	Depot	VendorA	VendorB	VendorC	VendorD
Depot	–	3.5	3	4	2
VendorA	3.5	–	4	2.5	3
VendorB	3	4	–	4.5	3.5
VendorC	4	2.5	4.5	–	4
VendorD	2	3	3.5	4	–

(To)

What route should the company van follow so that the total distance travelled is minimized?

5) A salesman must travel from city to city to maintain his accounts. This week he has to leave his home base and visit each other city and return home. The table shows the distances (in kilometres) between the various cities. The home city is city A. Use the assignment method to determine the tour that will minimize the total distances of visiting all cities and returning home.

From city	**To city**				
	A	**B**	**C**	**D**	**E**
A	-	375	600	150	190
B	375	-	300	350	175
C	600	300	-	350	500
D	160	350	350	-	300
E	190	175	500	300	-

Space for Hints

5.8 KEYWORDS

Assignment Algorithm, Hungarian method, Traveling salesmen problem

5.9 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

Check your progress 5.1

1) Prof A to Dynamic programme, Prof B to Queuing theory, prof C to Regression Analysis and proof D to linear programmer, Minimum total time will be 28 hours

2) $1 \rightarrow C,\ 2 \rightarrow D,\ 3 \rightarrow B,\ 4 \rightarrow A$; or $1 \rightarrow B,\ 2 \rightarrow C,\ 3 \rightarrow D,\ 4 \rightarrow A$

   Minimum total time $= 17$hours

3) a) $I \rightarrow A,\ II \rightarrow C,\ III \rightarrow B,\ IV \rightarrow D$, Minimum cost $= 21$

   b) Job 1 to A, Job 2 to 6, Job 3 to B and Job 4 to D total minimum time $= 17$hours

   c) $A \rightarrow 2,\ B \rightarrow 3,\ C \rightarrow 4,\ D \rightarrow 1$ Minimum cost$= 38$

   d) $J_1 \rightarrow M_4,\ J_2 \rightarrow M_3,\ J_3 \rightarrow M_1,\ J_2 \rightarrow M_2$, Minimum cost $= 17$.

Check your progress 5.2

1) $P \rightarrow A,\ Q \rightarrow C,\ R \rightarrow B$ and $S \rightarrow D$ maximum profit $= 918$

2) $A \rightarrow I,\ B \rightarrow II,\ C \rightarrow III,\ D \rightarrow IV$ (or) $A \rightarrow I,\ B \rightarrow III,\ C \rightarrow II,\ D \rightarrow IV$

   maximum sales $= 99$.

Check your progress 5.3

1) $I \rightarrow 1,\ II \rightarrow 3,\ III \rightarrow 2,\ IV \rightarrow 4$ minimum time $= 35$hours

Space for Hints

2) $A \to P,\ B \to Q,\ C \to R,\ D \to S$ minimum cost $=$ Rs.50.

3) $M \to J_4,\ M_2 \to J_1,\ M_3 \to J_2,\ M_4 \to J_3$

minimum cost $=$ Rs.16.

Check your progress 5.4

1) Flights: $I \to 103,\ 2 \to 104,\ 3 \to 101,\ 4 \to 102$

Base crew: Chennai, Chennai, Chennai, Mumbai, Total lay over time
= 4 hours

2)

crew	Residence at	Route nember	waiting time(hrs).
1	chennai	d – 1	4.5
2	Bangalore	e – 2	9.5
3	Bangalore	a – 3	9.0
4	chennai	b – 4	5.0
5	Bangalore	c – 5	5.5

Check your progress 5.5

1) $A \to B \to C \to D \to E \to A$, Corresponding cost $=15$

2) $A \to D \to B \to C \to E \to A$ or $A \to E \to C \to B \to D \to A$

Corresponding cost $=16$.

3) $1 \to 4 \to 5 \to 2 \to 3 \to 1$ corresponding cost $=49$.

4) Depot $\to$ vendor D $\to$ vendor A $\to$ vendor C $\to$ Vendor B $\to$ Depot.

Total distance covered in this sequence $=$ 15K.m.

5) $A \to E \to B \to C \to D \to A$

Minimum distance $= 1{,}165$ k.m.

Space for Hints

5.10 MODEL QUESTIONS

1) Solve the assignment problem

	A	B	C	D
I	1	4	6	3
II	9	7	10	9
III	4	5	11	7
IV	8	7	8	5

2) Consider the problem of assigning five jobs to five persons. The assignment costs are given as follows:

		Job				
		1	2	3	4	5
Person	A	8	5	2	6	1
	B	0	9	5	5	4
	C	3	8	9	2	6
	D	4	3	1	0	3
	E	9	5	8	9	5

Determine the optimum assignment schedule.

3) Solve the following assignment problem to find the maximum total expected sale.

Area

		I	II	III	IV
Salesman	A	42	35	28	21
	B	30	25	20	15
	C	30	25	20	15
	D	24	20	16	12

4) A Sales manager has to assign salesman to four territories. He has 4 candidates of varying experience and capabilities and assesses the possible profit for each salesman in each territory as given below. Find the assignment which maximizes the profit.

Territory

		A	B	C	D
Salesman	1	35	27	28	37
	2	28	34	29	40
	3	35	24	32	28
	4	24	32	25	28

5) Solve the following assignment problem

Task

Machine

	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	Not Suitable	5	4
M_3	Not suitable	6	9	6	2
M_4	9	3	7	2	3

Space for Hints

6) Solve the following assignment problem

Machine

Task		1	2	3	4	5
	A	7	7	∞	4	8
	B	9	6	4	5	6
	C	11	5	7	∞	5
	D	9	4	8	9	4
	E	8	7	9	11	3

7) Given the following matrix of setup costs show how to sequence production so as to minimize setup cost per cycle.

To

From		A	B	C	D	E
	A	–	2	5	7	1
	B	6	–	3	8	2
	C	8	7	–	4	7
	D	12	4	6	–	5
	E	1	3	2	8	–

8) Solve the following travelling salesman problem:

	A	B	C	D	E	F
A	∞	5	12	6	4	8
B	6	∞	10	5	4	3
C	8	7	∞	6	3	11
D	5	4	11	∞	5	8
E	5	2	7	8	∞	4
F	6	3	11	5	4	∞

Space for Hints

9) Solve the travelling salesman problem given by the following data.

$$c_{12} = 20,\ c_{13} = 4,\ c_{14} = 10,\ c_{23} = 6,$$

$$c_{25} = 10,\ c_{35} = 6,\ c_{45} = 20,\ \text{where} c_{ij} = c_{ji}$$

and there is not route between curies i and j if a value for c_{ij} is not shown above.

10) A company has five jobs to be done one five machines; any job can be done on any machine. The costs of doing the jobs in different machines are given below. Assign the jobs for different machines so as to minimize the total cost.

Machines

		A	B	C	D	E
Jobs	I	13	8	16	18	19
	II	9	15	24	9	12
	III	12	9	4	4	4
	IV	6	12	10	8	13
	V	15	17	18	12	20

11) The owner of a small machine shop has four machinists available to assign to jobs for the day. Five jobs are offered with expected profit for each machinist on each job as follows:

Space for Hints

Jobs

		A	B	C	D	E
Machinist	M_1	12	28	0	51	32
	M_2	12	34	11	23	9
	M_3	37	42	61	21	31
	M_4	0	14	37	27	31

Assign machinists to jobs which results in overall maximum profit. Which job should be declined?