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CONTENTS

PAPERS

MATHEMATICAL NOTES

CLASSROOM NOTES

BOOK REVIEWS

REPORT OF THE MEETING

GENERALIZED FEUERBACH'S THEOREM

By K. SITARAM

[Received May 17, 1959]

INTRODUCTION. In this paper a generalization of Feuerbach's theorem is obtained using mainly the Theorem 1 on desmic tetrahedra. The generalized theorem runs as follows : Each of the circles with the in- and ex-centres of a triangle as centres and their radii expanded from the corresponding in- and ex-radii in the same ratio is cut at equal angles by a circle belonging to the coaxal system of the circum-circle and the ninepoints circle.

It is known that the totality of circles in a plane can be represented as points in a 3-space, the point circles in the plane going over to points on a quadric, called the absolute quadric. Orthogonal circles go over in this correspondence to points conjugate with respect to the absolute, and circles belonging to a coaxal system to points on a straight line. We shall denote the points in space and the circles which they represent in the plane by the same symbol.

We shall start by proving the crucial theorem :

THEOREM 1. *If* $A_0A_1A_2A_3$, $B_0B_1B_2B_3$ *be any two desmic tetrahedra, then corresponding to a given tetrahedron* $B'_0B'_1B'_2B'_3$ *inscribed in the pencil* A_0B_0, A_0B_1, A_0B_2, A_0B_3, *there exist uniquely points* A'_1, A'_2, A'_3 *on* A_0A_1, A_0A_2 *and* A_0A_3 *respectively, such that the tetrahedron* $A_0A'_1A'_2A'_3$ *is desmic to it.*

Further, if $B''_0B''_1B''_2B''_3$ *be a variable tetrahedron inscribed in the pencil (but not desmic to* $B_0B_1B_2B_3$*) such that its vertices belong to the homography determined by the points* $A_0B_0B'_0$, $A_0B_1B'_1$, ... *on the corresponding lines* [*i.e.* $(A_0B_0B'_0B''_0) = (A_0B_1B'_1B''_1) = \dots$], *then the plane* $A''_1A''_2A''_3$ *that corresponds to* $B''_0B''_1B''_2B''_3$ *in the desmic system passes through the line of intersection of planes* $A_1A_2A_3$ *and* $A'_1A'_2A'_3$.

Without loss of generality we can take

$$
\begin{array}{llll}
A_0 : & (1, 0, 0, 0) & B_0 : & (1, 1, 1, 1) \\
A_1 : & (0, 1, 0, 0) & B_1 : & (1, 1, -1, -1) \\
A_2 : & (0, 0, 1, 0) & B_2 : & (1, -1, 1, -1) \\
A_3 : & (0, 0, 0, 1) & B_3 : & (1, -1, -1, 1).
\end{array}
$$

B'_0, B'_1, B'_2 and B'_3 being points on A_0B_0, A_0B_1, A_0B_2 and A_0B_3 can be taken as,

$$
\begin{array}{ll}
B'_0 : & (1+\lambda', 1, 1, 1) \\
B'_1 : & (1+\mu', 1, -1, -1) \\
B'_2 : & (1+\nu', -1, 1, -1) \\
B'_3 : & (1+\delta', -1, -1, 1).
\end{array}
$$

A'_1, A'_2 and A'_3 can be taken as

$$
\begin{array}{ll}
A'_1 : & (1, \alpha', 0, 0) \\
A'_2 : & (1, 0, \beta', 0) \\
A'_3 : & (1, 0, 0, \gamma').
\end{array}
$$

It can be verified that the condition for $A_0A'_1A'_2A'_3$ and $B'_0B'_1B'_2B'_3$ to be desmic is.

$$
\begin{aligned}
\lambda' + \mu' - \nu' - \delta' &= 4/\alpha' \\
\lambda' + \nu' - \mu' - \delta' &= 4/\beta' \\
\lambda' + \delta' - \mu' - \nu' &= 4/\gamma'.
\end{aligned}
$$

Thus given λ', μ', ν' and δ' ; α', β' and γ' can be uniquely determined, which proves the first part of our theroem.

We see that given α', β' and γ' alone there are ∞^1 tetrahedra inscribed in the pencil desmic to $A_0A'_1A'_2A'_3$.

If

$$
\begin{array}{ll}
B''_0 : & (1+\lambda'', 1, 1, 1) \\
B''_1 : & (1+\mu'', 1, -1, -1) \\
B''_2 : & (1+\nu'', -1, 1, -1) \\
B''_3 : & (1+\delta'', -1, -1, 1)
\end{array}
$$

be the vertices of a second tetrahedron inscribed in the pencil, not desmic to $B'_0B'_1B'_2B'_3$ and such that $(A_0B_0B'_0B''_0)=(A_0B_1B'_1B''_1) = \ldots$, then if B''_0 (or λ'') alone be given, we should have in terms of λ', (the parameter of B'_0)

$$\mu'' = \frac{\mu'}{\lambda'}\lambda'' ; \nu'' = \frac{\nu'}{\lambda'}\lambda'' ; \delta'' = \frac{\delta'}{\lambda'}\lambda''.$$

If $B'_0B'_1B'_2B'_3$ and $B''_0B''_1B''_2B''_3$ be desmic, we can verify that $\lambda''-\lambda' = \mu''-\mu' = \nu''-\nu' = \delta''-\delta'$; and $\lambda''+\mu''+\nu'+\delta' = -4$, so that $\lambda''+\mu''-\nu''-\delta'' = \frac{4}{\alpha'}$, etc. which means that the same plane $A'_1A'_2A'_3$ would suffice to complete the desmic system.

Otherwise, the plane $A''_1A''_2A''_3$ that completes the desmic system is $x_0 - \frac{\lambda''}{\lambda'}\left(\frac{x_1}{\alpha}+\frac{x_2}{\beta}+\frac{x_3}{\gamma}\right) = 0$, which passes through the line $x_0 = 0$ and $x_0 - \left(\frac{x_1}{\alpha}+\frac{x_2}{\beta}+\frac{x_3}{\gamma}\right) = 0$.

Hence the theorem.

Desmic tetrahedra arise in circle space in connection with circles cutting at equal angles any four given circles. The theorem proved below illustrates this point.

THEOREM 2. *There are eight circles cutting any given 4 circles at equal angles, which can be divided into two sets of 4 circles desmic to each other, the third set of 4 circles that complete the desmic system being circles orthogonal to 3 of the given circles.*

Choosing the given circles $K_0K_1K_2K_3$ as vertices of the tetrahedron of reference, let the equation of the absolute quadric be

$$a_{00}\,x^2_0 + a_{11}\,x^2_1 + \ldots + 2\,a_{01}\,x_0\,x_1 + \ldots = 0.$$

It is known [1] that a circle cutting two given circles at equal angles must be orthogonal to one of the two circles of antisimilitude of the circles, so that the circles we are seeking are orthogonal to one of each of the 6 pairs of circles of antisimilitude between pairs of given circles.

The edge K_0K_1 of the tetrahedron cuts the absolute in points given by

$$a_{00}\, x^2_0 + a_{11}\, x^2_1 + 2\, a_{01}\, x_0\, x_1 = 0.$$

The common harmonic pair of K_0, K_1 and the points where K_0K_1 cuts the absolute is the pair of circles w.r.t. to which they are mutually inverse, i.e. they correspond to the circles of antisimilitude between K_0, K_1.

They are given by $a_{00}x^2_1 - a_{11}x^2_0 = 0$. Their coordinates are

$$(\sqrt{a_{00}},\ \sqrt{a_{11}},\ 0,\ 0) \text{ and } (\sqrt{a_{00}},\ -\sqrt{a_{11}},\ 0,\ 0).$$

Similarly, the circles of antisimilitude on A_0A_2 are

$$(\sqrt{a_{00}},\ 0,\ \sqrt{a_{22}},\ 0) \text{ and } (\sqrt{a_{00}},\ 0,\ -\sqrt{a_{22}},\ 0) \text{ and so on.}$$

We can thus find the 6 pairs of points on the 6 edges, which can be seen to lie six-by-six on the eight planes

$$\frac{x_0}{\sqrt{a_{00}}} + \frac{x_1}{\sqrt{a_{11}}} + \frac{x_2}{\sqrt{a_{22}}} + \frac{x_3}{\sqrt{a_{33}}} = 0.$$

$$\frac{x_0}{\sqrt{a_{00}}} + \frac{x_1}{\sqrt{a_{11}}} - \frac{x_2}{\sqrt{a_{22}}} - \frac{x_3}{\sqrt{a_{33}}} = 0.$$

$$\frac{x_0}{\sqrt{a_{00}}} - \frac{x_1}{\sqrt{a_{11}}} + \frac{x_2}{\sqrt{a_{22}}} - \frac{x_3}{\sqrt{a_{33}}} = 0.$$

$$\frac{x_0}{\sqrt{a_{00}}} - \frac{x_1}{\sqrt{a_{11}}} - \frac{x_2}{\sqrt{a_{22}}} + \frac{x_3}{\sqrt{a_{33}}} = 0.$$

$$-\frac{x_0}{\sqrt{a_{00}}} + \frac{x_1}{\sqrt{a_{11}}} + \frac{x_2}{\sqrt{a_{22}}} + \frac{x_3}{\sqrt{a_{33}}} = 0.$$

$$\frac{x_0}{\sqrt{a_{00}}} - \frac{x_1}{\sqrt{a_{11}}} + \frac{x_2}{\sqrt{a_{22}}} + \frac{x_3}{\sqrt{a_{33}}} = 0.$$

$$\frac{x_0}{\sqrt{a_{00}}} + \frac{x_1}{\sqrt{a_{11}}} - \frac{x_2}{\sqrt{a_{22}}} + \frac{x_3}{\sqrt{a_{33}}} = 0.$$

$$\frac{x_0}{\sqrt{a_{00}}} + \frac{x_1}{\sqrt{a_{11}}} + \frac{x_2}{\sqrt{a_{22}}} - \frac{x_3}{\sqrt{a_{33}}} = 0.$$

These planes (or rather, their poles w.r.t. the absolute) correspond to the eight circles cutting the given circles at equal angles. The third tetrahedron completing the desmic system is the coordinate tetrahedron $x_0 = 0$, $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

Hence the theorem follows.

We will now establish the generalized Feuerbach's theorem.

THEOREM 3. *A circle belonging to the coaxal system of the circumcircle and the nine points circle of a triangle cuts at equal angles the four equally expanded in- and ex-circles.*

In Theorem 1, if a quadric through $A_0A_1A_2A_3$, $B_0B_1B_2B_3$ be taken as the absolute quadric, it is known[2] that by taking A_0 to be the point at infinity in the plane, B_0, B_1, B_2, B_3 go over into the in- and ex-centres of $\Delta\ A_1A_2A_3$. The plane $A_1A_2A_3$ in circle space corresponds to the circumcircle of $\Delta\ A_1A_2A_3$. Points B'_0, B'_1, B'_2, B'_3 can be taken to be the in- and ex-circles of the triangle, as this tetrahedron is not desmic to $B_0B_1B_2B_3$. For, if it be desmic[3] they must correspond to mutually orthogonal circles with the in- and ex-centres as centres. But the in- and ex-circles are known to be not mutually orthogonal.

Assuming Feuerbach's theorem, since the sides of the triangle and the nine points circle touch the in- and ex-circles of the triangle, we see that the plane $A'_1A'_2A'_3$ corresponds to the nine points circle. Points B''_0, B''_1, B''_2, B''_3 correspond to equally expanded in- and ex-circles. The plane $A''_1A''_2A''_3$ that completes the desmic system, we have seen, passes through the line of intersection of planes $A_1A_2A_3$ and $A'_1A'_2A'_3$. Hence the circle that cuts at equal angles the expanded in-and ex-circles belongs to the coaxal system of the circumcircle and the nine points circle.

By taking the point (x_0, x_1, x_2, x_3) in 3-space to represent a circle whose equation in plane areal coordinates can be written as

$-abc\,x_0\,(\text{line at infinity})^2 + ax_1\,(\text{point circle } A_1) + bx_2\,(\text{point circle } A_2) + cx_3\,(\text{point circle } A_3) = 0$, ($a, b, c$ being the sides of $\Delta\ A_1A_2A_3$), the absolute quadric has an equation of the form,

$$a(x_0 x_1 - x_2 x_3) + b(x_0 x_2 - x_3 x_1) + c(x_0 x_3 - x_1 x_2) = 0.$$

Taking the equal expansion as ρ, it can be verified that the expanded in- and ex-circles are cut at equal angles by the circle whose equation in areals is

$$\rho^2[(b^2+c^2-a^2)x+(c^2+a^2-b^2)y+(a^2+b^2-c^2)z]\ (x+y+z) - \\ - 4\,(a^2\,yz + b^2\,zx + c^2\,xy) = 0,$$

which belongs to the said coaxal system.

It is of interest to observe that $\rho = \sqrt{(2)}$ corresponds to the polar circle and $\rho = \infty$ corresponds to the radical axis of the coaxal system.

The author wishes to express his thanks to late Dr. R. Vaidyanathaswamy for his encouragement and guidance in preparing this paper.

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A NOTE ON MERSENNE'S NUMBERS

By U. V. SATYANARAYANA

[Received January 4, 1960]

M. SATYANARAYANA [1] proved that the Mersenne's numbers M_m defined by

$$M_m = 2^m - 1$$

are not triangular numbers if m is odd. In this note I consider the case of even m and prove

THEOREM 1. *There are infinitely many n for which M_{2n} is not a triangular number.*

PROOF. Suppose that p is a prime > 7 and is such that

(i) 2 is a primitive root of p^2 (and therefore of p also) and

(ii) 7 is a quadratic non-residue of p.

Firstly, we note that primes with properties (i) and (ii) do exist, for 11 is one such.

We make use of the well-known result that a necessary and sufficient condition for t to be triangular is that $1 + 8t$ should be a perfect square.

We prove the theorem by showing that there are infinitely many n for which

$$1 + 8M_{2n} = 2^{2n+3} - 7$$

is divisible by p but not by p^2. Here, 2 being a primitive root of p^2, the numbers

$$1, 2, 2^2, \ldots, 2^{(\phi(p^2)-1)}$$

form a reduced residue system mod p^2.

Hence there is an integer α with $0 \leqslant \alpha \leqslant (\phi(p^2) - 1)$, such that

$$2^\alpha \equiv 7 (\text{mod } p^2),\ p \text{ being different from } 7. \tag{1.1}$$

Also, obviously this integer α is odd by virtue of (ii) above.

Now, taking $n = (p-1)k/2 + (\alpha - 3)/2$, we have

$$(2n+3) \equiv \alpha \pmod{p-1},$$

so that we have, by Fermat's theorem and (1.1), the relation

$$2^{2n+3} - 7 \equiv 0 \pmod{p}. \tag{1.2}$$

Also, reducing $2^{2n+3} - 7$ mod p^2, we have

$$2^{2n+3} - 7 \doteq 2^{(p-1)k}.\,2^{\alpha} - 7 \equiv 7\{2^{(p-1)k} - 1\} \pmod{p^2}.$$

Now, since 2 is a primitive root of p^2, 2 belongs to the exp $\phi(p^2) = p^2 - p$ mod p^2. Hence, if we choose the k such that $p\ k, \nmid$ we would have

$$\phi(p^2) \nmid (p-1)k,$$

which shows that $2^{(p-1)k} - 1 \not\equiv 0 \pmod{p^2}$.

Hence the theorem.

THEOREM 2.† *If $n = 5k + 3$ or $5k + 4$, then M_{2n} is not triangular.*

PROOF : If $n = 5k + 3$, we can easily verify that

$$(1 + 8M_{2n}) \equiv 10 \pmod{11};$$

also if $n = 5k + 4$, $(1 + 8M_{2n}) \equiv 6 \pmod{11}$.

But the square of an integer is not congruent to 10 or 6 (mod 11).

This shows that

$(1 + 8M_{2n})$ is not a perfect square if $n \equiv 3$ or 4 (mod 5).

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† If $n = \frac{p-1}{2}K + l$, $l = 3$ or 4, then M_{2n} is not triangular p being a prime such that $\left(\frac{505}{p}\right) = -1$, for $l = 3$, and $\left(\frac{2041}{p}\right) = -1$, for $l = 4$. [Referee]

ASYMPTOTIC EXPRESSIONS FOR CERTAIN TYPE OF SUMS INVOLVING THE ARITHMETIC FUNCTIONS IN THE THEORY OF NUMBERS(†)

By S. SWETHARANYAM

1. Introduction. In this paper, we establish the asymptotic expressions with the best possible error terms for the sums $\sum_{n \leqslant x} f(n)\, n^{-a}$ for all values of a (positive, negative and zero), where $f(n)$ is one of the arithmetic functions $\phi(n)$, $d(n)$, $\sigma(n)$ and $\sigma_k(n)$, defined as below: $\phi(n)$ is the Euler function defined as the number of numbers $\nu \leqslant n$ and prime to it; $d(n)$ is the divisor function defined as the number of positive divisors of n including 1 and n; $\sigma(n)$ is the sum of the divisors of n and $\sigma_k(n)$ is the sum of the k-th powers of the divisors of n. The asymptotic expressions enable us to find certain average orders (in the sense of Hardy and Wright ([5], p.263)) connected with the above mentioned functions. These can be called the *weighted average orders* or the *average orders of the weighted functions.* In section 2, we list the lemmas and certain other results that we require in the other sections and sections 3-6 are concerned with the functions $\phi(n)$, $d(n)$, $\sigma(n)$ and $\sigma_k(n)$ respectively. In the last section we summarise our results.

REMARK 1. Throughout this paper, $[x]$ denotes the integral part of x and $\{x\}$ the fractional part of x, so that $x = [x] + \{x\}$ and unless otherwise stated sums of the form $\Sigma f(n)$ and Σa_n without any indication of the range of summation imply summation over the integers $n \leqslant x$.

2. Lemmas and approximations. The following lemmas and approximations or slightly varied versions of these will be made use of in the remaining sections.

LEMMA 1. (*vide* Hardy and Wright [5], Theorem 421). *Let* c_1, c_2, ... *be a sequence of numbers and* $C(x) = \Sigma c_n$. *If* $f(x)$ *is any function of* x

(†) Presented at the XXII Conference of the Indian Mathematical Society a Baroda in December 1956; abstract appeared in *Math. Student* 25 (1957), 81.

and $f(x)$ has a continuous derivative for $x \geqslant 1$, then

$$\sum c_n f(n) = C(x) f(x) - \int_1^x C(t) f'(t)\, dt.$$

Corollary 1. *In the particular case when $f(x) = x^{-\lambda}$, $\lambda \neq 0$, if $C(x)$ has an approximation of the type $C(x) = F(x) + E(x)$, where $F(x)$ is the order (or the main) term and $E(x)$, the error term, we can deduce from* Lemma 1 *that*

$$\sum c_n n^{-\lambda} = F(x) x^{-\lambda} + \lambda \int_1^x F(t) t^{-(\lambda+1)}\, dt + E(x) x^{-\lambda} + \\ + \lambda \int_1^x E(t) t^{-(\lambda+1)}\, dt. \quad (1)$$

Corollary 2. *In particular, if $E(x)$ is such that $E(x) x^{-\lambda}$ tends to zero so fast that $\int_1^\infty E(x) x^{-(\lambda+1)} dx$ converges, then* (1) *can be written as*

$$\sum c_n n^{-\lambda} = F(x) x^{-\lambda} + \lambda \int_1^x F(t) t^{-(\lambda+1)}\, dt + \lambda \int_1^\infty E(t) t^{-(\lambda+1)}\, dt + \\ + E(x) x^{-\lambda} - \lambda \int_x^\infty E(t) t^{-(\lambda+1)}\, dt. \quad (2)$$

Remark 2. So, for any particular case if (1) is applicable, then

$$\text{the order term of } \sum c_n n^{-\lambda} = F(x) x^{-\lambda} + \lambda \int_1^x F(t) t^{-(\lambda+1)}\, dt, \quad (1')$$

and

$$\text{the error term of } \sum c_n n^{-\lambda} = E(x) x^{-\lambda} + \lambda \int_1^x E(t) t^{-(\lambda+1)}\, dt, \quad (1'')$$

whereas if (2) is applicable, then

$$\text{the order term of } \sum c_n n^{-\lambda} = F(x) x^{-\lambda} + \lambda \int_1^x F(t) t^{-(\lambda+1)}\, dt + \\ + \lambda \int_1^\infty E(t) t^{-(\lambda+1)}\, dt, \quad (2')$$

and

$$\text{the error term of } \sum c_n\, n^{-\lambda} = E(x)\, x^{-\lambda} - \lambda \int_x^\infty E(t)\, t^{-(\lambda+1)}\, dt. \quad (2'')$$

REMARK 3.† In the second case, since only the order of $E(x)$ is known we cannot precisely estimate the order term of $\Sigma c_n\, n^{-\lambda}$ by (2′). In such cases, we usually employ a different method for the estimation of the order term and use (2″) for the estimation of the error term. However, in those cases, the method that we employ to estimate the order terms can also be made use of to get the error terms; but it is seen that by using (2″) we are, in most cases, able to get better error terms than those that are otherwise obtained, since using (2″) any refinement in the order of $E(x)$ can be immediately carried over to the case concerned. In this paper, the other method that we make use of to estimate the order terms is the one of expressing the function $f(n)$ as a sum $\sum_{\alpha|n} g(\alpha)$ over its divisors and then summing up $\sum_{n\leqslant x} f(n) = \sum_{n\leqslant x} \sum_{\alpha|n} g(\alpha)$ on the lines of Hardy and Wright ([5], Theorems 324 and 330). We find that though in most cases, this yields the best order terms possible, the following function-theoretic interpretation of the significance of the order terms is capable of a wider application. The function-theoretic interpretation is: if the sequence $a_1, a_2, \ldots,$ is such that $\Sigma a_n = F(x) + E(x)$, $E(x) = O(x^\theta)$ and $F(x)$ is a sum of a finite number of terms of the form $x^\alpha \log^\beta x$, where $\mathscr{R}(\alpha) > \theta$ and β is a non-negative integer, then $F(x)$ is precisely the sum of the residues of the function $x^s f(s)/s$, in the half plane $\mathscr{R}(s) > \theta$, where $f(s)$ is the continuation of $\Sigma a_n\, n^{-s}$. This is simply Lemma 3.12 of Titchmarsh [11] in a different form. An application of this lemma to sums of the type that we deal with in this paper, has been treated by Titchmarsh ([11], Chapter XII). Buschman [3] has made use of Laplace transforms and a tauberian theorem to estimate the average orders of a good number of

† The substance of this remark as also the application of Lemma 1 to get improved error terms in the various theorems has been pointed out to me by the referee and the paper has consequently undergone a drastic revision in the methods used to get the theorems.

functions of the type $n^a f(n) \log^r n$, involving almost all the arithmetical functions. Recently Briggs and Buschman [2] have improved the work of Buschman [3] to obtain the order terms as well as the error terms and their results for the functions treated in this paper are the same as those obtained in this paper except for some cases for the function $d(n)$. This is so because we have not made use of the Lemma 3.12 of Titchmarsh [11]. But the error terms obtained in this paper for many of the sums are the best possible at the present moment and are better than or are as good as those of Briggs and Buschman [2].

Lemma 2. (*vide* Titchmarsh [11], 2.1.4). *If* $0 < \mathscr{R}(s) \leqslant 1$, $s \neq 1$, (s) *can be defined by*

$$\sum_{n \leqslant \infty} n^{-s} = \zeta(s) = \frac{1}{s-1} + \tfrac{1}{2} - s \int_1^{\infty} (x - [x] - \tfrac{1}{2})\, x^{-(s+1)}\, dx, \tag{3}$$

and making an approximation as in Titchmarsh ([11], *p.* 16) *very near* $s = 1$, *we get*

$$\gamma = \tfrac{1}{2} - \int_1^{\infty} (x - [x] - \tfrac{1}{2})\, x^{-2}\, dx, \tag{4}$$

γ being the Euler's constant.

Lemma 3. (*vide* Hardy and Wright [5], Theorem 287).

$$\sum_{n \leqslant \infty} \mu(n)\, n^{-s} = 1/\zeta(s),\ s > 1. \tag{5}$$

Lemma 4. $\sum_{n \leqslant \infty} \mu(n) \log n . n^{-s} = \zeta'(s)/(\zeta(s))^2,\ s > 1.$ (6)

This is obtained by differentiation of both sides of (5), it being term by term on the left side.

Lemma 5.

$$\sum n^{-s} = x^{1-s}/(1-s) + \zeta(s) + O(x^{-s}), \quad 0 < s < 1, \tag{7}$$

where $\zeta(s)$ *is defined by* (3) *and*

$$\sum n^{-s} = x^{1-s}/(1-s) + O(x^{-s}), \quad \textit{for all } s < 0. \tag{8}$$

REMARK 4. For $s > 1$, the sum $\sum_{1}^{\infty} n^{-s} = \zeta(s)$ and the case $s = 1$ is the very familiar one, viz.

$$\sum n^{-1} = \log x + \gamma + O(x^{-1}).$$

The proof of Lemma 5 follows directly from the following:

LEMMA 6. (Euler-Mclaurin sum formula, *vide* Knopp [6], p. 521).

$$\sum f(n) = \int_{1}^{x} f(t)\, dt + \tfrac{1}{2}(f(1) + f(x)) + \int_{1}^{x} (t - [t] - \tfrac{1}{2})\, f'(t)\, dt.$$

APPROXIMATION 1. When $1 \leqslant d \leqslant x/2$,

$$\log\, [x/d] = \log x - \log d + O(d/x). \tag{9}$$

For, let ϵ be the fractional part of x/d so that

$$[x/d] = x/d - \epsilon = \frac{x}{d}\left(1 - \frac{\epsilon d}{x}\right), \quad 0 \leqslant \epsilon < 1,$$

and

$$\log\, [x/d] = \log\, (x/d) + \log\left(1 - \frac{\epsilon d}{x}\right),$$

and in $1 \leqslant d \leqslant x/2$, $d/x \leqslant \frac{1}{2}$ and so $\epsilon d/x < \frac{1}{2}$ uniformly and so we have $\log\left(1 - \dfrac{\epsilon d}{x}\right) = O(d/x)$.

APPROXIMATION 2. When $x/2 \leqslant d \leqslant x$,

$$\sum_{d' \leqslant [x/d]} d'^{k} = O(1) \text{ for any } k. \tag{10}$$

For, in the interval $x/2 \leqslant d \leqslant x$, $x/d \leqslant 2$ and hence in the sum $\sum_{d' \leqslant [x/d]} d'^{k}$ there are at most two terms and so we have (10).

APPROXIMATION 3. When $1 \leqslant d \leqslant x/2$,

$$[x/d]^{k} = (x/d)^{k}\big(1 + O(d/x)\big) \text{ for any } k. \tag{11}$$

To see this, let $[x/d] = x/d - \epsilon$, $0 \leqslant \epsilon < 1$; then

$$[x/d]^{k} = (x/d)^{k}\,(1 - \epsilon d/x)^{k},$$

and since $\epsilon d/x < \frac{1}{2}$ uniformly in $1 \leqslant d \leqslant x/2, \quad 0 \leqslant \epsilon < 1$,

$$(1 - \epsilon d/x)^k = 1 + O(d/x).$$

3. The Euler function $\phi(n)$. Regarding the average order of $\phi(n)$, the best result that is known yet is

THEOREM 3.A. $\Sigma \phi(n) = x^2/2\,\zeta(2) + O\big(x \log^{3/4} x\,(\log\log x)^2\big)$.

This is due to Walfisz ([14], [15], [16]). This is an improvement on the well-known result

$$\Sigma \phi(n) = x^2/2\,\zeta(2) + O(x \log x)$$

of Mertens [8] (*vide* Hardy and Wright [5], Theorem 330).

We prove the following theorems :

THEOREM 3.1.

$$\Sigma \phi(n)/n = x/\zeta(2) + O\big(\log^{3/4} x\,(\log\log x)^2\big).$$

PROOF. This is obtained easily by combining Walfisz's work and some results of Pillai and Chowla [9], as indicated below. Let

$$\Sigma \phi(n) = x^2/2\,\zeta(2) + R(x), \tag{12}$$

where

$$R(x) = O\big(x \log^{3/4} x\,(\log\log x)^2\big)$$

by Theorem 3.A.

Applying (1′) to the case $c_n = \phi(n)$, $\lambda = 1$, $F(x) = x^2/2\,\zeta(2)$, the order term of $\Sigma \phi(n)/n = x/\zeta(2) - 1/2\,\zeta(2)$. (13)

A direct estimation of the error term by (1″) with $E(x) = R(x)$ yields the error term $O\big(\log^{7/4} x\,(\log\log x)^2\big)$. But a refinement is possible if we adopt the following procedure. Let

$$\Sigma \phi(n)/n = x/\zeta(2) + H(x), \tag{14}$$

where, by Pillai and Chowla ([9], 3.4),

$$H(x) = -\sum \frac{\mu(n)}{n}\left\{\frac{x}{n}\right\} + o(1). \tag{15}$$

But we have from Walfisz ([14], (2); [15], (5); [16], (5)) that

$$\chi(x) = \sum \frac{\mu(n)}{n}\,\psi(x/n) = O\big(\log^{3/4} x\,(\log\log x)^2\big), \tag{16}$$

where

$$\psi(x) = x - [x] - \tfrac{1}{2} = \{x\} - \tfrac{1}{2}. \tag{17}$$

Hence

$$\chi(x) = \sum \frac{\mu(n)}{n} (\{x/n\} - \tfrac{1}{2})$$

$$= \sum \frac{\mu(n)}{n} \{x/n\} - \frac{1}{2} \sum \frac{\mu(n)}{n}$$

$$= - H(x) + o(1) - \frac{1}{2} \sum \frac{\mu(n)}{n} \qquad \text{by (15).}$$

But $\Sigma\, \mu(n)/n = O(1)$ (*vide* Landau [7], p. 582) and hence

$$H(x) = - \chi(x) + o(1) + O(1) = O(\log^{3/4} x\, (\log \log x)^2) \tag{18}$$

in view of (16). Combining (14) and (18) we have the theorem.

THEOREM 3.2.

$$\sum \phi(n)\, n^{-a} = \zeta\,(a-1)/\zeta(a) + O(x^{2-a}),\ a > 2.$$

PROOF. We have

$$\sum \phi(n)\, n^{-a} = \sum_{1}^{\infty} \phi(n)\, n^{-a} - \sum_{x}^{\infty} \phi(n)\, n^{-a}$$

and since $a > 2$, $\sum_{1}^{\infty} \phi(n)\ n^{-a} = \zeta(a - 1)/\zeta(a)$ (*vide* Hardy and Wright [5], Theorem 288) and since $\phi(n) = O(n)$, we have

$$\sum_{x}^{\infty} \phi(n)\, n^{-a} = O\left(\sum_{x}^{\infty} n.\, n^{-a}\right) = O(x^{2-a})$$

and hence the theorem.

THEOREM 3.3.

$$\sum \phi(n)\, n^{-2} = \log x/\zeta(2) + (1/\zeta(2))\, (\gamma - \zeta'(2)/\zeta(2)) + $$
$$+ O(\log^{3/4} x\, (\log \log x)^2/x).$$

PROOF. Using (12) and putting $c_n = \phi(n)$, $\lambda = 2$, and $E(x) = R(x)$ in (2″) we get

$$\text{the error term of} \sum \phi(n)\, n^{-2} = R(x)\, x^{-2} - 2 \int_x^\infty R(t)\, t^{-3}\, dt$$

$$= O(\log^{3/4} x\, (\log \log x)^2/x). \quad (19)$$

Putting $F(x) = x^2/2\ \zeta(2)$ in (2′), the main term of $\Sigma\phi(n)\, n^{-2}$

$$= \frac{1}{2\,\zeta(2)} + 2\int_1^x \frac{dx}{2\,\zeta(2)\, x} + 2\int_1^\infty \frac{R(t)}{t^3}\, dt$$

$$= \frac{1}{2\,\zeta(2)} + \frac{\log x}{\zeta(2)} + 2\int_1^\infty \frac{R(t)}{t^3}\, dt. \quad (20)$$

The last integral in (20) cannot be evaluated since we know only the order of $R(x)$; and all that we can say about this integral is that it is $O(1)$, in which case the order term (20) becomes $\log x/\,\zeta(2) + O(1)$ which is however a weak one. But the order term can be calculated more precisely, if we adopt the method mentioned in Remark 3.

By definition (*vide* [5], 16.3.1),

$$\phi(n) = n \sum_{d|n} \mu(d)/d.$$

Hence

$$S = \sum \phi(n)\, n^{-2} = \sum n^{-1} \sum_{d|n} \mu(d)/d$$

$$= \sum_{dd' \leqslant x} \mu(d)/d^2\, d'$$

which we rewrite as

$$S = \sum_{d \leqslant x|2} \mu(d)/d^2 \sum_{d' \leqslant [x|d]} 1/d' + \sum_{d=x|2}^{x} \mu(d)/d^2 \sum_{d' \leqslant [x|d]} 1/d'$$

$$= S_1 + S_2.$$

$$S_1 = \sum_{d \leqslant x|2} (\mu(d)/d^2)\,(\log [x/d] + \gamma + O(1/[x/d]\,))\ \text{by Remark 4};$$

$$= \sum_{d \leqslant x|2} (\mu(d)/d^2)\,(\log x - \log d + \gamma + O(d/x)) \quad \text{by (9)}.$$

Making use of Lemmas 3 and 4 in the right side of the above, we get

$$S_1 = (\log x + \gamma)\big(1/\zeta(2) + O(2/x)\big) - \big(\zeta'(2)/(\zeta(2))^2 + O((\log x)/x)\big) + O((\log x)/x).$$

Hence the order term of $S_1 = (\log x + \gamma)/\zeta(2) - \zeta'(2)/(\zeta(2))^2$. (21)

From (10) we have

$$S_2 = O\Big(\sum_{x|2}^{x} \mu(d)/d^2\Big) = O\Big(\sum_{x|2}^{x} 1/d^2\Big) = O(2/x). \qquad (22)$$

Hence from (21) and (22)

$$\text{the order term of } S = (\log x + \gamma)/\zeta(2) - \zeta'(2)/(\zeta(2))^2 \qquad (23)$$

and this with (19) gives Theorem 3.3.

REMARK 5. From (20) and (23) we have

$$\frac{1}{2\,\zeta(2)} + 2\int_1^\infty R(t)\,t^{-3}\,dt = \frac{1}{\zeta(2)}\left(\gamma - \frac{\zeta'(2)}{(\zeta(2))^2}\right)$$

or

$$\int_1^\infty R(t)\,t^{-3}\,dt = \frac{1}{2\,\zeta(2)}\left(\gamma - \frac{1}{2} - \frac{\zeta'(2)}{(\zeta(2))^2}\right),$$

where $R(t)$ is defined by (12)

THEOREM 3.4.

$$\sum \phi(n)\,n^{-a} = x^{2-a}/(2-a)\,\zeta(2) + O\big(x^{1-a}\log^{3/4} x\,(\log\log x)^2\big)$$

for all $a < 1$, $a \neq 0$.

PROOF. It is easily obtained by a direct application of (1) to the case $c_n = \phi(n)$, $\lambda = a$, $F(x) = x^2/2\,\zeta(2)$ and

$$E(x) = R(x) = O\big(x\log^{3/4} x\,(\log\log x)^2\big).$$

REMARK 6. It is seen that Theorems 3.A and 3.4 can be combined to give

$$\sum \phi(n)\,n^{-a} = x^{2-a}/(2-a)\,\zeta(2) + O\big(x^{1-a}\log^{3/4} x\,(\log\log x)^2\big)$$

for all $a < 1$.

THEOREM 3.5.

$$\sum \phi(n)\, n^{-a} = \frac{x^{2-a}}{(2-a)\,\zeta(2)} + \frac{\zeta(a-1)}{\zeta(a)} + O(x^{1-a}\log^{3/4} x\,(\log\log x)^2),$$

for all a, $1 < a < 2$.

REMARK 7. In this case since $1<a<2$, $\zeta(a-1)$ is defined by (3).

PROOF. We adopt a procedure similar to that in Theorem 3.3. We estimate the order term by the method of Hardy and Wright ([5], p. 265) and for the error term we use (2″). So

$$\text{the error term of } \Sigma\phi(n)\, n^{-a} = O\big(x^{1-a}\log^{3/4} x\,(\log\log x)^2\big) \quad (24)$$

for $1 < a < 2$.

Let $S_a = \Sigma\, \phi(n)\, n^{-a}$, $1 < a < 2$. Then it is easily obtained

$$S_a = \sum_{d\leqslant x|2} \mu(d)/d^a \sum_{d'\leqslant [x|d]} 1/d'^{(a-1)} + \sum_{x|2}^{x} \mu(d)/d^a \sum_{d'\leqslant [x|d]} 1/d'^{(a-1)}$$

$$= S_{a,1} + S_{a,2}.$$

Since $0 < a-1 < 1$, we apply (7) to the sum $S_{a,1}$ and get

$$S_{a,1} = \sum_{d\leqslant x|2} \frac{\mu(d)}{d^a}\left(\frac{[x/d]^{2-a}}{2-a} + \zeta(a-1) + O([x/d]^{1-a})\right).$$

Using (11) in this, we obtain

$$S_{a,1} = \sum_{d\leqslant x|2} \frac{\mu(d)}{d^a}\left(\frac{(x/d)^{2-a}}{2-a} + \zeta(a-1) + O((x/d)^{1-a})\right).$$

And by (5) we have

$$\text{the order term of } S_{a,1} = x^{2-a}/(2-a)\,\zeta(2) + \zeta(a-1)/\zeta(a).$$

$S_{a,2}$ gives only an error term of $O(x^{1-a})$ and so

$$\text{the order term of } S_a = x^{2-a}/(2-a)\,\zeta(2) + \zeta(a-1)/\zeta(a) \quad (25)$$

and this together with (24) proves the theorem.

REMARK 8. An application of (2′) to this case gives the order term of $\Sigma\phi(n)\, n^{-a}$, $1 < a < 2$ as

$$\frac{x^{2-a}}{2\,\zeta(2)} + a\int_1^x \frac{t^2}{2\zeta(2)t^{a+1}}\,dt + a\int_1^\infty \frac{R(t)}{t^{a+1}}\,dt;$$

$$= \frac{x^{2-a}}{2\,\zeta(2)} + \frac{a}{2\,\zeta(2)}\left(\frac{x^{2-a}}{2-a} - \frac{1}{2-a}\right) + a\int_1^\infty \frac{R(t)}{t^{a+1}}\,dt;$$

$$= \frac{x^{2-a}}{(2-a)\,\zeta(2)} - \frac{a}{2\,\zeta(2)\,(2-a)} + a\int_1^\infty \frac{R(t)}{t^{a+1}}\,dt;$$

$$= x^{2-a}/(2-a)\,\zeta(2) + \zeta(a-1)/\zeta(a) \qquad \text{by (25)}$$

Hence

$$\int_1^\infty \frac{R(t)}{t^{a+1}}\,dt = \frac{\zeta(a-1)}{a\,\zeta(a)} + \frac{1}{2\,\zeta(2)\,(2-a)}.$$

4. Dirichlet's divisor function $d(n)$. The function $d(n)$ and the precise estimation of the asymptotic value of $d(n)$ has been the matter of much study since the days of Dirichlet. Dirichlet [4] established that

$$\Sigma\, d(n) = x \log x + (2\gamma - 1)\, x + O(x^{1/2})$$

(*vide* Hardy and Wright [5], Theorem 330). However as a result of the works of later mathematicians, the error term has been refined to $O(x^\delta)$, $\frac{1}{4} \leqslant \delta < \frac{1}{3}$. Recently Richert ([10], p. 205, (3)) showed that

if the error term is $\Delta(x)$, then $\Delta(x) = O(x^{15/46} \log^{30/23} x)$.

However for our purpose we shall use the following

Theorem 4.A.

$$\sum d(n) = x \log x + (2\gamma - 1)\, x + \Delta(x),$$

$$\Delta(x) = O(x^\delta),\ \tfrac{1}{4} \leqslant \delta < \tfrac{1}{3}.$$

The following theorems are proved.

Theorem 4.1.

$$\sum d(n)\, n^{-a} = (\zeta(a))^2 + O(x^{1-a} \log x),\ a > 1.$$

PROOF.

$$\sum d(n)\, n^{-a} = \sum_{1}^{\infty} d(n)\, n^{-a} - \sum_{x}^{\infty} d(n)\, n^{-a}$$

$$= (\zeta(a))^2 + O\left(\sum_{x}^{\infty} \log n.\, n^{-a}\right),$$

since $\sum\limits_{1}^{\infty} d(n)n^{-a} = (\zeta(a))^2$, $a > 1$ (*vide* Hardy and Wright [5], Theorem 289) and the order of $d(n)$ is $\log n$. Hence

$$\sum d(n)\, n^{-a} = (\zeta(a))^2 + O(x^{1-a} \log x).$$

THEOREM 4.2.

$$\sum d(n)/n = \log^2 x/2 + 2\gamma \log x + K + O(x^{\delta-1}),$$

where K is the constant term.

PROOF. Applying Corollary 2 to the case $c_n = d(n), \lambda = 1$, $F(x) = x(\log x + 2\gamma - 1)$ and $E(x) = \Delta(x) = O(x^{\delta})$, we have

$$\Sigma\, d(n)/n = \log^2 x/2 + 2\gamma \log x + (2\gamma - 1) + \int_1^{\infty} \Delta(x)\, x^{-2}\, dx + O(x^{\delta-1}),$$

where since $\Delta(x) = O(x^{\delta})$, $\delta < 1$, $\int_1^{\infty} \Delta(x)\, x^{-2} dx$ is convergent. If we put $K = 2\gamma - 1 + \int_1^{\infty} \Delta(x)\, x^{-2}\, dx$, then we have the theorem.

THEOREM 4.3.

$$\sum d(n)\, n^{-a} = \frac{x^{1-a} \log x}{1-a} + x^{1-a}\left(\frac{2\gamma}{1-a} - \frac{1}{(1-a)^2}\right) + K' + O(x^{\delta-a})$$

in the interval $\delta < a < 1$, K' being the constant term.

PROOF. This is also obtained by an application of Corollary 2 with $c_n = d(n)$, $\lambda = a$, $F(x) = x\,(\log x + 2\gamma - 1)$ and $E(x) = \Delta(x)$ so that

$$\sum d(n)\, n^{-a} = \frac{x^{1-a} \log x}{1-a} + \frac{x^{1-a}}{(1-a)^2}\left(\frac{2\gamma}{1-a} - \frac{1}{(1-a)^2}\right) +$$

$$+ \frac{a(2-a)}{(1-a)^2} - \frac{2\gamma a}{1-a} + a \int_1^{\infty} \Delta(x)\, x^{-(a+1)}\, dx + O(x^{\delta-a}),$$

and we put

$$K' = \frac{a(2-a)}{(1-a)^2} - \frac{2\gamma a}{1-a} + a \int_1^\infty \Delta(x)\, x^{-(a+1)}\, dx.$$

REMARK 9. Briggs and Buschman ([2], § 2, Corollary 6) have obtained the values $K = \gamma^2 - 2\gamma'$, $K' = (\zeta(a))^2$ and the referee has also pointed out the same in his remarks. These are obtained if we use the function-theoretic interpretation mentioned in Remark 3.

THEOREM 4.4.

$$\sum d(n)\, n^{-\delta} = \frac{x^{1-\delta}}{1-\delta} \log x + x^{1-\delta} \left(\frac{2\gamma}{1-\delta} - \frac{1}{(1-\delta)^2} \right) + O(\log x).$$

PROOF. Here $\Delta(x)/n^\delta = O(x^\delta/x^\delta) = O(1)$ and hence we apply Corollary 1 with $c_n = d(n)$, $\lambda = \delta$, $F(x) = x\,(\log x + 2\gamma - 1)$ and $E(x) = O(x^\delta)$, to get the theorem.

THEOREM 4.5.

$$\sum d(n)\, n^{-a} = \frac{x^{1-a} \log x}{1-a} + x^{1-a} \left(\frac{2\gamma}{1-a} - \frac{1}{(1-a)^2} \right) + O(x^{\delta-a}), \quad a < \delta,\ a \neq 0.$$

PROOF. The proof is the same as in Theorem 4.4, except that $a \neq \delta$ and hence the error terms become

$$\Delta(x)\, x^{-a} + O\left(\int_1^x \Delta(t)\, t^{-(a+1)}\, dt \right) = O(x^{\delta-a}), \quad a < \delta, a \neq 0.$$

REMARK 10. Theorem 4.5 can be combined with Theorem 4.A to give

$$\sum d(n)\, n^{-a} = \frac{x^{1-a}}{1-a} \left(\log x + 2\gamma - \frac{1}{1-a} \right) + O(x^{\delta-a}), a < \delta.$$

5. The function $\sigma(n)$. This function is similar to $\phi(n)$ in behaviour and the results concerning this can be got in a manner similar to those in section 3. The average order of $\sigma(n)$ is $\zeta(2)\, x$ and the more precise result on the average order of $\sigma(n)$ is the following

THEOREM 5.A.

$$\sum \sigma(n) = \zeta(2)\, x^2/2 + O(x \log^{4/5} x \log\log x).$$

A slightly weaker form of the above with the error term $O(x \log x)$ (*vide* Hardy and Wright [5], Theorem 324 and Bachmann [1], p. 402) is more well known. Walfisz ([13], p. 182, (3) ; [12]) showed that this can be refined to $O(x \log x/\log\log x)$. But the result in the form of Theorem 5.A is also due to Walfisz ([13], p. 182, II).

Wigert [17] proved that

$$\sum \sigma(n)/n = \zeta(2)\, x + O(\log x)$$

and gave the bounds within which the error term lies. He has also discussed therein the relationship between the error term of $\Sigma\sigma(n)$ and that of $\Sigma\sigma(n)/n$. But Walfisz ([13], p. 182, II) has proved the following

THEOREM 5.B.

$$\sum \sigma(n)/n = \zeta(2)\, x - \tfrac{1}{2} \log x + O(\log^{4/5} x \log\log x).$$

Applying Lemma 1 to Theorem 5.A (or to Theorem 5.B) and using (wherever necessary) the methods mentioned in Remark 3 (as has been done in section 3) we have the following theorems.

THEOREM 5.1.

$$\sum \sigma(n)\, n^{-2} = \zeta(2) \log x + \gamma\zeta(2) + \zeta'(2) + O(\log^{4/5} x \log\log x/x).$$

THEOREM 5.2.

$$\sum \sigma(n)\, n^{-a} = \zeta(2)\, x^{2-a}/(2-a) + O(x^{1-a} \log^{4/5} x \log\log x), \quad a < 1,\ a \neq 0.$$

THEOREM 5.3.

$$\sum \sigma(n)\, n^{-a} = \zeta(2)\, x^{2-a}/(2-a) + \zeta(a-1)\, \zeta(a) + O(x^{1-a} \log^{4/5} x \log\log x), \quad 1 < a < 2.$$

Finally we have

THEOREM 5.4.

$$\sum \sigma(n)\, n^{-a} = \zeta(a)\ \zeta(a-1) + O(x^{2-a}), \quad a > 2.$$

PROOF.

$$\sum \sigma(n)\, n^{-a} = \sum_{1}^{\infty} \sigma(n)\, n^{-a} - \sum_{x}^{\infty} \sigma(n)\, n^{-a}$$

$$= \zeta(a)\ \zeta(a-1) + O\left(\sum_{x}^{\infty} n . n^{-a}\right)$$

(*vide* Hardy and Wright [5], Theorem 290) ;

$$= \zeta(a)\ \zeta(a-1) + O(x^{2-a}).$$

REMARK 11. Here also Theorem 5.A can be combined with Theorem 5.2 and we can write

$$\sum \sigma(n)\, n^{-a} = \zeta(2)\, x^{2-a}/(2-a) + O(x^{1-a} \log^{4/5} x \ \log\log x)$$

for all $a < 1$. Theorems 5.A, 5.1, 5.2 and 5.3 can be compared with Theorems 3.A, 3.3, 3.4 and 3.5, respectively, to observe the similarity between the behaviour of the functions $\sigma(n)$ and $\phi(n)$.

6. The function $\sigma_k(n)$. In this paper, in as much as this function is concerned, we take $k \geqslant 2$ and $k = 1$ is the function $\sigma(n)$ and $k = 0$ is the function $d(n)$ which are treated in sections 5 and 4 respectively. The average order of $\sigma_k(n)$ is $\zeta(k+1)\ x^k$ and in this connection we have

THEOREM 6.1.

$$\sum \sigma_k(n) = \zeta(k+1)\, x^{k+1} / (k+1) + O(x^k).$$

PROOF.

$$\sum \sigma_k(n) = \sum \sum_{d|n} d^k$$

$$= \sum_{dd' \leqslant x} d^k$$

$$= \sum_{d' \leqslant x/2} \ \sum_{d \leqslant [x/d']} d^k + \sum_{d' = x/2}^{x} \ \sum_{d \leqslant [x/d']} d^k$$

$$= S_1 + S_2.$$

$$\begin{aligned}
S_1 &= \sum_{d' \leqslant x|2} \left(\frac{[x/d']^{k+1}}{k+1} + O([x/d']^k) \right) && \text{by (8)} \\
&= \sum_{d' \leqslant x|2} \left(\frac{(x/d')^{k+1}}{k+1} + O((x/d')^k) \right) && \text{by (11)} \\
&= \frac{x^{k+1}}{k+1} \sum_{d' \leqslant x|2} \frac{1}{d'^{k+1}} + O\left(x^k \sum_{d' \leqslant x|2} \frac{1}{d'^k} \right) \\
&= \frac{x^{k+1}}{k+1} \left(\sum_1^\infty \frac{1}{d'^{k+1}} - \sum_{x/2}^\infty \frac{1}{d'^{k+1}} \right) + O(x^k) \\
&= \zeta(k+1)\, x^{k+1}/(k+1) + O(x^{k+1}/x^k) + O(x^k) \\
&= \zeta(k+1)\, x^{k+1}/(k+1) + O(x^k) && \text{since } k \geqslant 2.
\end{aligned}$$

Using the same method we have the following theorems.

THEOREM 6.2. $\sum \sigma_k(n)/n^k = \zeta(k+1)\, x + O(1)$.

THEOREM 6.3.

$$\sum \sigma_k(n)\, n^{-(k+1)} = \zeta(k+1) \log x + \gamma \zeta(k+1) + \zeta'(k+1) + O(1/x).$$

THEOREM 6.4.

$$\sum \sigma_k(n)\, n^{-a} = \zeta(k+1)\, x^{k+1-a}/(k+1-a) + O(x^{k-a}),\ a < k,\ a \neq 0.$$

THEOREM 6.5.

$$\sum \sigma_k(n)\, n^{-a} = \zeta(k+1)\, x^{k+1-a}/(k+1-a) + \zeta(a-k)\, \zeta(a) + O(x^{k-a}), \quad k < a < (k+1).$$

THEOREM 6.6.

$$\sum \sigma_k(n)\, n^{-a} = \zeta(a)\, \zeta(a-k) + O(x^{k+1-a}), \quad a > (k+1).$$

However Theorems 6.2–6.5 can be deduced from Theorem 6.1 by an application of Corollary 1 or 2, whichever is applicable; but the order terms of Theorems 6.3 and 6.4 cannot be so precisely estimated except by the method adopted in the proof of Theorem 6.1 or the function-theoretic interpretation pointed out in Remark 3. Here, Theorems 6.1 and 6.4 can be combined to give

$$\sum \sigma_k(n)\, n^{-a} = \zeta(k+1)\, x^{k+1-a}/(k+1-a) + O(x^{k-a}), \quad a < k.$$

7. Summary. The results of sections 3 to 6 can be put in compac tabular forms as shown at the end of this section. This would afford a ready reference to the various cases as also a comparative study of the corresponding cases for two or more functions. For example, in the case of the functions $\sigma(n)$ and $\sigma_k(n)$, the order terms of the former can be obtained (except when $a=1$) by putting $k=1$ in the result for the corresponding case for the latter function. But the error terms show variation and this is accounted for by the fact that the order of $\Sigma\ 1/n$ is $\log x$ whereas that of $\Sigma\ 1/n^a$ for any value of $a<1$ is x^{1-a}. To find the average order of any function $f(n)$, we simply find that function $g(n)$ for which $\int_1^x g(n)\,dn \sim \Sigma f(n)$.

In the tables, γ denotes Euler's constant $(=\zeta(1)$ in the limiting sense) and $\zeta(a)$ for $0<a<1$ is defined as in (3).

TABLES OF ASYMPTOTIC VALUES OF $\Sigma\ F(n)=\Sigma f(n)\,n^{-a}$.

Value of (a)	Average order of $F(n)$	Asymptotic value of $\Sigma\ F(n)$
TABLE I: $f(n)=\phi(n)$.		
$a\leqslant 1$	$x^{1-a}/\zeta(2)$	$\dfrac{x^{2-a}}{\zeta(2)\,(2-a)}+O(x^{1-a}\log^{3/4}x\,(\log\log x)^2)$
$1<a<2$	$x^{1-a}/\zeta(2)$	$\dfrac{x^{2-a}}{\zeta(2)\,(2-a)}+\zeta(a-1)/\zeta(a)+$ $+O(x^{1-a}\log^{3/4}x\,(\log\log x)^2)$
$a=2$	$1/x\,\zeta(2)$	$\log x/\zeta(2)+\gamma/\zeta(2)-\zeta'(2)/(\zeta(2))^2+$ $+O(\log^{3/4}x\,(\log\log x)^2/x)$
$a>2$	...	$\zeta(a-1)/\zeta(a)+O(x^{2-a})$.

TABLE II : $f(n) = d(n)$.

$a < \delta$	$x^{-a} \log x$	$\dfrac{x^{1-a}\log x}{1-a} + \dfrac{x^{1-a}}{1-a}\left(2\gamma - \dfrac{1}{1-a}\right) + O(x^{\delta-a})$
$a = \delta$	$x^{-a} \log x$	$\dfrac{x^{1-a}\log x}{1-a} + \dfrac{x^{1-a}}{1-a}\left(2\gamma - \dfrac{1}{1-a}\right) + O(\log x)$
$\delta < a < 1$	$x^{-a} \log x$	$\dfrac{x^{1-a}\log x}{1-a} + \dfrac{x^{1-a}}{1-a}\left(2\gamma - \dfrac{1}{1-a}\right) + (\zeta(a))^2 + O(x^{\delta-a})$
$a = 1$	$\log x/x$	$\frac{1}{2}\log^2 x + 2\gamma \log x + \gamma^2 - 2\gamma' + O(x^{\delta-1})$
$a > 1$	...	$(\zeta(a))^2 + O(x^{1-a}\log x)$.

Here δ satisfies the inequality $\frac{1}{4} \leqslant \delta < \frac{1}{3}$.

TABLE III : $f(n) = \sigma(n) = \sigma_1(n)$.

$a < 1$	$\zeta(2)\, x^{1-a}$	$\dfrac{\zeta(2)\, x^{2-a}}{2-a} + O(x^{1-a} \log^{4/5} x \log\log x)$
$1 < a < 2$	$\zeta(2)\, x^{1-a}$	$\dfrac{\zeta(2)\, x^{2-a}}{2-a} + \zeta(a-1)\,\zeta(a) + O(x^{1-a} \log^{4/5} x \log\log x)$
$a = 1$	$\zeta(2)$	$\zeta(2)\, x - \frac{1}{2}\log x + O(\log^{4/5} x \log\log x)$
$a = 2$	$\zeta(2)/x$	$\zeta(2)\log x + \gamma\,\zeta(2) + \zeta'(2) + O(\log^{4/5} x \log\log x/x)$
$a > 2$	...	$\zeta(a)\,\zeta(a-1) + O(x^{2-a})$

TABLE IV : $f(n) = \sigma_k(n)$, $k \geqslant 2$.

$a \leqslant k$	$\zeta(k+1)\, x^{k-a}$	$\dfrac{\zeta(k+1)\, x^{k+1-a}}{k+1-a} + O(x^{k-a})$
$k < a < k+1$	$\zeta(k+1)\, x^{k-a}$	$\dfrac{\zeta(k+1)\, x^{k+1-a}}{k+1-a} + \zeta(a-k)\,\zeta(a) + O(x^{k-a})$
$a = k+1$	$\zeta(k+1)/x$	$\zeta(k+1)\log x + \gamma\,\zeta(k+1) + \zeta'(k+1) + O(1/x)$
$a > k+1$	...	$\zeta(a)\,\zeta(a-k) + O(x^{k+1-a})$

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A TRIGONOMETRIC SUM

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1. Introduction. Let n and r denote fixed integers, r positive. The following interesting identity was stated by H. Rademacher and proved by A. Brauer [1]:

$$\phi(r) \sum_{\substack{d \mid r \\ (d,n)=1}} \frac{d}{\phi(d)} \mu\left(\frac{r}{d}\right) = \mu(r) \sum_{d \mid (n,r)} d\mu\left(\frac{r}{d}\right), \tag{1.1}$$

where $\phi(r)$ and $\mu(r)$ denote the familiar functions of Euler and Möbius. In ([3], Corollary 3.5) the author proved the Brauer-Rademacher identity by showing that each member of (1.1) was an evaluation of the sum,

$$A_r(n) = \sum_{(a,r)=1} c_r(n-a), \tag{1.2}$$

where a ranges over a reduced residue system (mod r) and $c_r(n)$ denotes the trigonometric sum of Ramanujan. That proof was partly indirect in nature. In this note we deduce direct evaluations of a k-dimensional analogue (3.1) of $A_r(n)$, obtaining, as a by-product, an identity generalizing (1.1). The results are contained in (3.2), (3.3), and (3.4). For applications of the sum (3.1), denoted $A_r(n_1, \ldots, n_k)$, we refer the reader to ([3], § 5) and ([4], § 8), in the cases $k = 1$ and $k = 2$, respectively.

2. Definitions and preliminary lemmas. Denote by $\{n_i\}$ the k-dimensional, integral vector $\{n_1, \ldots, n_k\}$, $k \geqslant 1$, and place $[n_i] = [n_1, \ldots, n_k]$, $(n_i) = (n_i, \ldots, n_k)$, $(0) = 0$. If $A = \{a_i\}$, $B = \{b_i\}$, we write $A \equiv B \pmod{k, r}$ provided $a_i \equiv b_i \pmod r$, $i = 1, \ldots, k$, and say that A, B are congruent (mod k, r). Any maximal set V_r of vectors $\{a_i\}$, mutually incongruent (mod k, r), contains r^k vectors and is called a complete residue system (mod k, r). The subset T_r of all vectors $\{b_i\}$ of such a system V_r, satisfying $((b_i), r) = 1$, is called a reduced residue system (mod k, r). The number of elements of

T_r is denoted $J_k(r)$. The function $J_k(r)$ is the Jordan totient function and is independent of the choice of V_r.

Place $e_r(n) = \exp(2\pi i n/r)$ and define for integers $n_1, \ldots, n_k$,

$$c_r(n_1, \ldots, n_k) = \sum_{((x_i), r) = 1} e_r(n_1 x_1 + \ldots + n_k x_k), \tag{2.1}$$

where the summation is over a reduced residue system (mod k, r). Also place $c_r^{(k)}(n) = c_r(n_1, \ldots, n_k)$ in case $n = n_i (i = 1, \ldots, k)$; in particular $c_r^{(1)}(n) = c_r(n)$. We now state some known results that will be required in §3.

LEMMA 1. [2, Theorem 1].

$$c_r^{(k)}(n) = \sum_{d \mid (n,r)} d^k \mu\left(\frac{r}{d}\right). \tag{2.2}$$

For simplicity we write $n = (n_i)$; *specifically in the following* n *will represent the greatest common divisor of* $n_1, \ldots, n_k$.

LEMMA 2. [4, Lemma 2].

$$c_r(n_1, \ldots, n_k) = c_r^{(k)}((n, r)) = c_r^{(k)}(n). \tag{2.3}$$

LEMMA 3.

$$c_r(n_1, \ldots, n_k) = \sum_{[d_i] = r} c_{d_1}(n_1) \ldots c_{d_k}(n_k), \tag{2.4}$$

where the summation is over divisors d_i *of* r *such that* $[d_i] = r$.

The proof of Lemma 3 is based on the fact that $[d_i]\ (\delta_i) = r$, provided $d_i \delta_i = r$ $(i = 1, \ldots, k)$. For the details in case $k = 2$, see [4, Lemma 4].

LEMMA 4. [2, Lemma 7]. *If* $d \mid r$ *and* $\{n_i\}$ *is a vector such that* $(n, d) = 1$, *then there exist exactly* $J_k(r)/J_k(d)$ *vectors* (mod k, r), *congruent* (mod k, d) *to* $\{n_i\}$.

REMARK. Lemma 4 implies that a reduced residue (mod k, d) can be selected from such a system (mod k, r).

LEMMA 5. *If* $d_1, \ldots, d_k$ *are* k *integers and* $(d_i)_{k-1}$ *denotes the greatest common divisor of their products taken* $k - 1$ *at a time* $((d_i)_0 = 1)$ *then* $[d_i]\,(d_i)_{k-1} = d_1 \ldots d_k$.

This lemma is a simple extension of the familiar result ($k = 2$), $(d_1, d_2)\,[d_1, d_2] = d_1 d_2$.

3. Evaluations of $A_r(n_1, \dots, n_k)$. Define

$$A_r(n_1, \dots, n_k) = \sum_{((a_i), r) = 1} c_r(n_1 - a_1, \dots, n_k - a_k). \tag{3.1}$$

We first prove the evaluation,

$$A_r(n_1, \dots, n_k) = \mu(r) \sum_{d \mid (n, r)} d^k \mu\left(\frac{r}{d}\right). \tag{3.2}$$

PROOF OF (3.2). By Lemma 3, it follows that

$$\begin{aligned} A_r(n_1, \dots, n_k) &= \sum_{((a_i), r) = 1} \sum_{[d_i] = r} c_{d_1}(n_1 - a_1) \dots c_{d_k}(n_k - a_k) \\ &= \sum_{[d_i] = r} \sum_{\substack{(x_i, d_i) = 1 \\ i = 1, \dots, k}} e_{d_1}(x_1 n_1), \dots, e_{d_k}(x_k n_k) \times \\ &\qquad \times \sum_{((a_i), r) = 1} e_{d_1}(-a_1 x_1), \dots, e_{d_k}(-a_k x_k) \end{aligned}$$

Place $D = d_1 \dots d_k$ and $\Delta_i = D/d_i (i = 1, \dots, k)$, so that

$$\begin{aligned} A_r(n_1, \dots, n_k) &= \sum_{[d_i] = r} \sum_{\substack{(x_i, d_i) = 1 \\ i = 1, \dots, k}} e_{d_1}(x_1 n_1) \dots e_{d_k}(x_k n_k) \times \\ &\qquad \times \sum_{((a_i), r) = 1} e_D(-a_1 x_1 \Delta_1 - \dots - a_k x_k \Delta_k). \end{aligned}$$

Since in the summation, $D = (\Delta_i) r$ (Lemma 5), one obtains

$$\begin{aligned} A_r(n_1, \dots, n_k) &= \sum_{[d_i] = r} \sum_{\substack{(x_i, d_i) = 1 \\ i = 1, \dots, k}} e_{d_1}(x_1 n_1) \dots e_{d_k}(x_k n_k) \times \\ &\qquad \times c_r\left(-\frac{x_1 \Delta_1}{(\Delta_i)}, \dots, -\frac{x_x \Delta_k}{(\Delta_i)}\right). \end{aligned}$$

By the Remark of §2 one may suppose that $(x_i, r) = 1$, in the inner sum ; consequently, by Lemma 2,

$$A_r(n_1, \dots, n_k) = c_r^{(k)}(1) \sum_{[d_i] = r} c_{d_1}(n_1) \dots c_{d_k}(n_k).$$

Applying Lemmas 1 and 3, it follows that

$$A_r(n_1, \dots, n_k) = \mu(r)\, c_r(n_1, \dots, n_k),$$

and (3.2) results on applying Lemmas 2 and 1.

We now deduce a second evaluation

$$A_r(n_1, \ldots, n_k) = J_k(r) \sum_{\substack{d|r \\ (d,n)=1}} \frac{d^k}{J_k(d)} \mu\left(\frac{r}{d}\right) \tag{3.3}$$

PROOF OF (3.3). By Lemma 2, we have

$$A_r(n_1, \ldots, n_k) = \sum_{((a_i),r)=1} c_r^{(k)}((n_i - a_i)),$$

where $(n_i - a_i) = (n_1 - a_1, \ldots, n_k - a_k)$; hence by Lemma 1,

$$A_r(n_1, \ldots, n_k) = \sum_{((a_i),r)=1} \sum_{\substack{d|r \\ \{a_i\} \equiv \{n_i\} \pmod{k,d}}} d^k \mu\left(\frac{r}{d}\right)$$

$$= \sum_{\substack{d|r \\ (d,n)=1}} d^k \mu\left(\frac{r}{d}\right) \sum_{\substack{((a_i),r)=1 \\ \{a_i\} \equiv \{n_i\} \pmod{k,d}}} 1.$$

Therefore, on applying Lemma 4, the evaluation (3.3) results.

Finally, comparison of (3.2) and (3.3) yields the relation,

$$J_k(r) \sum_{\substack{d|r \\ (d,n)=1}} \frac{d^k}{J_k(d)} \mu\left(\frac{r}{d}\right) = \mu(r) \sum_{d|(n,r)} d^k \mu\left(\frac{r}{d}\right). \tag{3.4}$$

In case $k = 1$, (3.4) reduces to the Brauer-Rademacher identity (1.1).

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ON A FUNCTIONAL EQUATION

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RICHARD BELLMAN proposed the following research problem in the *Bulletin of the American Mathematical Society* (Vol. 60, p. 501)

" *Solve the functional equation*

$f(x) = \max \{g(x) + f(ax), h(x) + f(bx)\}$ *given that*

(i) $0 < a, b < 1$,

(ii) $h(x), g(x) > 0$,

(iii) $h(0) = g(0) = 0$,

(iv) $h'(x), g'(x) > 0$, *and*

(v) $h''(x), g''(x) > 0$."

Although, as indicated in the *Bulletin* (l.c.) the problem seems to have arisen in Dynamic programming, M. K. Fort, Jr. [1] treated it as a pure Mathematical problem, and established the existence and uniquencess of a solution $f(x)$ defined for $x \geqslant 0$ and satisfying

$$\left.\begin{array}{l} (1)\ f(0) = 0, \text{ and} \\ (2)\ f(x) \text{ is continuous at } x = 0. \end{array}\right\} \qquad \textbf{(B)}$$

He further proved that the $f(x)$ is continuous for $x > 0$.

His proof explicitly makes use of conditions (ii) to (v) for $x \geqslant 0$. But in the special case when $0 < b \leqslant a < 1$, $g(0) = 0$, $|g(x)/x|$ is bounded in any interval containing 0, $g(x) \geqslant h(x)$, and $g(x)$ is monotonic increasing, then, irrespective of (iv) and (v), an obvious solution for all x is :

$$f(x) = \sum_{n=0}^{\infty} g(a^n x),$$

which further is unique under conditions (B).

This example suggests the possibility of an existence-solution (like Fort's) of Bellman's problem with his conditions (ii) to (v) considerably relaxed.

The following theorem of this paper, which provides such a solution for all x (in contrast with Fort's solution for $x > 0$ alone) and at the same time not only replaces Bellman's conditions (ii) to (v) by the single much wider condition (0.2) below, but in addition concerns itself with a class of functions $g(x, y)$ instead of just two functions $g(x)$ and $h(x)$, should be of interest.

THEOREM. *Let $g(x, y)$ be a function of two real variables, defined for all x and for all y belonging to a set A contained in a fixed closed interval $(-c, c)$ where $0 < c < 1$,* (0.1)

and such that to every positive ρ corresponds a positive $K(\rho)$ satisfying

$$|g(x, y)| \leqslant K(\rho)\,|x| \text{ for } |x| \leqslant \rho \text{ and for all } y \in A. \qquad (0.2)$$

Then

(a) *the functional equation*

$$f(x) = \sup_{y \in A} \{g(x, y) + f(yx)\} \qquad \text{(E)}$$

admits, subject to conditions (B) above, a unique solution defined for all x.

If, in addition,

for any a, $g(x, y) \to g(x', y)$ as $x \to x'$, the convergence being uniform w.r.t. $y \in A$, for $x' = a$ and for every x' with $|x'| \leqslant |ac|$, (0.3)
then

(b) *the $f(x)$ is continuous at $x = a$.*

We shall first prove some lemmas.

LEMMA 1. *If $\phi(x)$ and $\chi(x)$ are functions, each defined and bounded on a set X of real numbers, then*

$$|\sup_{x \in X} \phi(x) - \sup_{x \in X} \chi(x)| \leqslant \sup_{x \in X} |\phi(x) - \chi(x)|.$$

PROOF. Obvious.

LEMMA 2. *Subject to* (0.1) *and* (0.2) *the functional equation* (E) *admits at most one solution* $f(x)$, *defined for all* x, *and satisfying* (B).

PROOF. Suppose $f_1(x)$ and $f_2(x)$ are two such solutions. Let δ be a positive number. Then by Lemma 1 it follows that, for a fixed x,

$$|f_1(x) - f_2(x)| \leqslant \sup_{y \in A} |f_1(yx) - f_2(yx)|$$
$$< |f_1(y_1 x) - f_2(y_1 x)| + \delta/2, \qquad \text{for some } y_1 \in A.$$

Arguing similarly we get

$$|f_1(y_1 x) - f_2(y_1 x)| < |f_1(y_1 y_2 x) - f_2(y_1 y_2 x)| + \delta/2^2,$$
$$\text{for some } y_2 \in A.$$

Hence by induction follows the existence of a sequence of numbers $\{\alpha_n\}$ tending to zero (since $|\alpha_n| = |y_1 y_2 \cdots y_n| \leqslant c^n$), and such that

$$|f_1(\alpha_n x) - f_2(\alpha_n x)| < |f_1(\alpha_{n+1} x) - f_2(\alpha_{n+1} x)| + \delta/2^{n+1}.$$

Thus

$$|f_1(x) - f_2(x)| < |f_1(\alpha_n x) - f_2(\alpha_n x)| + (\delta/2) + (\delta/2^2) + \dots + (\delta/2^n).$$

Now making $n \to \infty$ and $\delta \to +0$ in this, it follows, by virtue of (B) that

$$f_1(x) \equiv f_2(x).$$

Hence the lemma.

LEMMA 3. *The sequence of functions* $f_n(x)$ *defined by the recurrence relation*

$$f_{n+1}(x) = \sup_{y \in A} \{g(x, y) + f_n(yx)\}, \tag{S}$$

with $f_1(x) = \sup_{y \in A} \{g(x, y)\}$, *satisfies*

(1) $f_n(0) = 0$ *for every* n, *and*

(2) $\{f_n(x)\}$ *is equicontinuous at* $x = 0$.

PROOF OF (1). Follows from (0.2).

PROOF OF (2). Let ∂ and ρ be two positive numbers. Let x be a fixed number with $|x| \leqslant \rho$. Then, in virtue of (0.2) and the relation (S), it follows that

$$|f_n(x)| \leqslant K(\rho)\,|x| + \sup_{y\in A} |f_{n-1}(yx)|$$

$$< K(\rho)\,|x| + |f_{n-1}(y_1 x)| + (\partial/2), \qquad \text{for some } y_1 \in A.$$

Similar argument shows that

$$|f_{n-1}(y_1 x)| < K(\rho)\,|y_1 x| + |f_{n-2}(y_1 y_2 x)| + (\partial/2^2)$$

$$< K(\rho)\,c\,|x| + |f_{n-2}(y_1 y_2 x)| + (\partial/2^2), \qquad \text{for some } y_2 \in A.$$

Hence by induction follows the existence of numbers $y_1, y_2, \dots y_{n-1}$ of A such that

$$|f_{n-1}(y_1 y_2 \cdots y_i x)| < K(\rho)\,c^i\,|x| + |f_{n-i-1}(y_1 y_2 \cdots y_{i+1} x)| + (\partial/2^{i+1})$$

$$\text{for } 1 \leqslant i \leqslant n-1.$$

Hence we obtain

$$|f_n(x)| < K(\rho)\,|x|\,(1 + c + c^2 + \dots) + \left(\frac{\partial}{2} + \frac{\partial}{2^2} + \dots\right)$$

$$< \frac{K(\rho)\,|x|}{1-c} + \partial.$$

∂ being an arbitrary positive number, we have

$$|f_n(x)| \leqslant \frac{K(\rho)\,|x|}{1-c}\,.$$

This being true for every x in $(-\rho, \rho)$, (2) of the lemma follows.

Lemma 4. *For any positive ρ, the sequence of Lemma 3 is uniformly convergent with respect to x in $(-\rho, \rho)$.*

Proof. Let ∂ be a positive number. Then by Lemma 1, we have for $m > n$,

$$|f_m(x) - f_n(x)| \leqslant \sup_{y\in A} |f_{m-1}(yx) - f_{n-1}(yx)|$$

$$< |f_{m-1}(y_1 x) - f_{n-1}(y_1 x)| + \partial/2^2$$

for some y_1 of A depending on x, m and n.

Arguing similarly we obtain,

$$|f_{m-1}(y_1 x) - f_{n-1}(y_1 x)| < |f_{m-2}(y_1 y_2 x) - f_{n-2}(y_1 y_2 x)| + \partial/2^3$$

for some y_2 of A depending on y_1, x, m and n.

By induction follows the existence of numbers $y_1, y_2, \dots y_{n-1}$ depending on x, m and n such that

$$|f_m(x) - f_n(x)| < |f_{m-n+1}(x_{n-1}x) - f_1(x_{n-1}x)| + (\partial/2^2 + \ldots + \partial/2^n), \tag{4.1}$$

where $x_{n-1} = y_1 y_2 \cdots y_{n-1}$, which is numerically less than c^{n-1} and hence tends to zero as n tends to infinity. Hence by (2) of Lemma 3 there exists an integer $N(\partial)$ such that

$$|f_{m-n+1}(x_{n-1}x) - f_1(x_{n-1}x)| < \partial/2, \text{ for } m > n > N(\partial) \text{ and } |x| \leqslant \rho.$$

Now (4.1) yields

$$|f_m(x) - f_n(x)| < \partial \text{ for } m > n > N(\partial) \text{ and } |x| \leqslant \rho.$$

This establishes the lemma.

LEMMA 5. *The limit function $f(x)$ of the sequence of Lemma* 3 *is such that*

(1) $f(0) = 0$,

(2) *$f(x)$ is continuous at $x = 0$, and*

(3) *$f(x)$ satisfies the functional equation* (E).

PROOF OF (1). Immediate from (1) of Lemma 3.

PROOF OF (2). Follows from (2) of Lemma 3, and Lemma 4.

PROOF OF (3). Let ρ and ∂ be two positive numbers. Then by Lemma 4 there exists an $N(\partial)$ such that

$$f(x) - \partial < f_n(x) < f(x) + \partial, \text{ for } n > N(\partial) \text{ and } |x| \leqslant \rho.$$

Therefore for every $y \in A$, we have

$$f(yx) - \partial < f_n(yx) < f(yx) + \partial, \text{ for } n > N(\partial) \text{ and } |x| \leqslant \rho.$$

Adding $g(x, y)$ throughout and taking suprema w.r.t. $y \in A$, we obtain

$$\sup_{y \in A} \{g(x, y) + f(yx)\} - \partial \leqslant f_{n+1}(x) \leqslant \sup_{y \in A} \{g(x, y) + f(yx)\} + \partial,$$

for $n > N(\partial)$ and $|x| \leqslant \rho$.

Hence it follows that

$$f(x) = \sup_{y \in A} \{g(x, y) + f(yx)\}.$$

This proves the lemma.

PROOF OF THE THEOREM.

PROOF OF (a). Follows from Lemmas 2 and 5.

PROOF OF (b). Corresponding to any non-zero number a, let E_a denote the set of numbers of the closed $[-|ac|, |ac|]$ together with a.

We first prove that to each $\epsilon > 0$ corresponds an $\eta = \eta$ (ϵ such that

$$|g(x, y) - g(x', y)| < \epsilon \tag{1}$$

whenever one of x, x' belongs to E_a and $|x - x'| < \eta$ and y belong to A.

By (0.3) to each point x_1 of E_a corresponds an open interval I_{x_1} with centre at x_1, such that $|g(x_1, y) - g(x, y)| < \epsilon/4$ whenever x belongs to I_{x_1} and y belongs to A. Now E_a being a closed set, by Heine-Borel theorem, there exists a finite number of the intervals I_{x_1}, which include I_a, I_{ac}, I_{-ac}, covering E_a. Now taking η to be the minimum of the semi-lengths of these intervals, we see that (1) is satisfied.

We next show that to each $\partial > 0$ corresponds an $\eta_1 = \eta_1(\partial)$ such that

$$|f_1(x_1) - f_1(x_2)| < \partial \tag{2}$$

whenever one of $x_1, x_2 \in E_a$ and $|x_1 - x_2| < \eta_1$.

By Lemma 1 we have

$$f_1(x_1) - f_1(x_2)| \leqslant \sup_{y \in A} |g(x_1, y) - g(x_2, y)|$$
$$< |g(x_1, y_1) - g(x_2, y_1)| + \partial/2$$

for some $y_1 \in A$, depending on x_1, and x_2.

Now the result (2) follows from (1) by taking $\eta_1 = \eta(\partial/2)$.

We now prove, for each f_n, by induction, that to each $\partial > 0$ corresponds an $\eta_n = \eta_n(\partial)$, such that

$$|f_n(x_1) - f_n(x_2)| < \partial \tag{3}$$

wherever one of x_1, x_2 is in E_a, and $|x_1 - x_2| < \eta_n$.

We assume the result for $f_{n-1}(x)$. By virtue of Lemma 1 and relation (S) we have

$$|f_n(x_1) - f_n(x_2)| \leqslant \sup_{y \in A} |g(x_1, y) + f_{n-1}(yx_1) - g(x_2, y) - f_{n-1}(yx_2)|$$

$$\leqslant \sup_{y \in A} |g(x_1, y) - g(x_2, y)| + \sup_{y \in A} |f_{n-1}(yx_1) - f_{n-1}(yx_2)|$$

$$< |g(x_1, y_1) - g(x_2, y_1)| + \partial/4 + |f_{n-1}(y_2x_1) - f_{n-1}(y_2x_2)| + \partial/4,$$

for some y_1, y_2 of A depending on x_1 and x_2. Now observing that one of y_2x_1, y_2x_2 is in E_a if one of x_1, x_2 is in E_a and $|y_2x_1 - y_2x_2|$ is less than $c\,|x_1 - x_2| < |x_1 - x_2|$, and taking

$$\eta_n = \min\left\{\eta\left(\frac{\partial}{4}\right), \eta_{n-1}\left(\frac{\partial}{4}\right)\right\}$$

(3) follows. Since the result (3) is true when $n = 1$ by (2), the proof is complete by induction.

Thus we see that each $f_n(x)$ is continuous at each point of E_a. Hence by Lemma 4, $f(x)$ is continuous at each point of E_a and in particular at a. This proves (b) of the theorem.

Remark 1. In (0.2) of the theorem, $|x|$ can be replaced by $|x|^\sigma$, where σ is a fixed positive number.

Remark 2. Every solution $\phi(x)$, continuous at $x = 0$, of the equation (E), with an assigned value for $\phi(0)$, is easily seen to be $\phi(0) + f(x)$, where $f(x)$ is the function constructed in Lemma 5.

Remark 3. It may be noted that the following generalization of the problem to Hilbert's separable coordinate space (λ_0) admits of an analogous solution.

Theorem. *Let $g(\{x_n\}, \{y_n\})$ be a real valued function of ordered pairs of sequences, defined for every sequence $\{x_n\}$ belonging to λ_0 (i.e. all sequences $\{x_n\}$ for which $\sum_1^\infty |x_n|^2 < \infty$) and for every sequence $\{y_n\}$, for which $|y_n|$ belongs to a set A_n for all n, each A_n being contained in a fixed closed interval $(0, c)$, where $0 < c < 1$,* (0.1)
and such that to every positive ρ corresponds a $K(\rho)$ with the property

$$|g(\{x_n\}, \{y_n\})| \leqslant K(\rho) \left[\sum_{1}^{\infty} |x_n|^2 \right]^{1/2} \tag{0.2}$$

for every $\{x_n\}$ *of* λ_0, *with* $\left[\sum_{1}^{\infty} |x_n|^2 \right]^{1/2} \leqslant \rho$, *and* $\{y_n\}$ *such that* $|y_n|$ *is contained in* A_n *for all* n.

Then,

(a) *the functional equation*

$$f(\{x_n\}) = \sup_{\substack{y_n \in A_n \\ 1 \leqslant n < \infty}} \left[g(\{x_n\}, \{y_n\}) + f(\{y_n x_n\}) \right]$$

admits, subjects to the conditions

(1) $f(\{0\}) = 0$, *and*

(2) $f(\{x_n\})$ *is continuous at* $\{0\}$,

a unique solution $f(\{x_n\})$ *defined over* λ_0.

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REMARKS ON A SERIES OF RAMANUJAN

By K. NARASIMHA MURTHY RAO

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RAMANUJAN [1] has set the following question.

If n is a multiple of 4, excluding zero, show that

$$1^{n-1}.\ \text{sech}\,\frac{\pi}{2} - 3^{n-1}.\ \text{sech}\,\frac{3\pi}{2} + 5^{n-1}\ \text{sech}\,\frac{5\pi}{2} - \ldots + \ldots = 0.$$

This has been solved by M. Bhimasena Rao [2] by the method of real variables.

An attempt to prove this by making use of Jacobian elliptic functions, has given me a simple proof when $n = 4$.

We have the Fourier series [3], with $u = \dfrac{2Kx}{\pi}$,

$$sd\ u = \frac{2\pi}{K\kappa\kappa'} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n+\frac{1}{2}}}{1+q^{2n+1}} \sin\,(2n+1)x, \tag{1}$$

valid in the strip $|I\,(x)| < \dfrac{\pi}{2} I\,(\tau)$.

By Maclaurin's theorem, we have

$$sd\ u = u - \frac{1}{\lfloor 3}\,(1-2\,k^2)\,u^3 + \frac{1}{\lfloor 5}\,(\qquad)\,u^5 - \ldots . \tag{2}$$

Equating the coefficients of x^3 on the right side of the two equations (1) and (2), we get,

$$\frac{2\,\pi}{K\kappa\kappa'} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n+\frac{1}{2}}}{1+q^{2n+1}}\,(2n+1)^3 = \frac{8\,K^3}{\pi^3}\,(1-2k^2),$$

i.e.

$$\sum_{n=0}^{\infty} (-1)^n.\ \frac{q^{n+\frac{1}{2}}}{1+q^{2n+1}}\,(2n+1)^3 = \frac{4}{\pi^4}\,K^4\kappa\kappa'\,(1-2k^2).$$

We have $\dfrac{q^{n+\frac{1}{2}}}{1+q^{2n+1}} = \frac{1}{2}\sec\,(n+\frac{1}{2})\,\pi\,\tau.$

Put $k = 1/\sqrt{2}$ in the above result. Observing that $\tau = i$, when $k = 1/\sqrt{2}$ on simplification, we obtain

$$\sum_{n=0}^{\infty} (-1)^n (2n+1)^3 . \operatorname{sech} (n+\tfrac{1}{2})\pi = 0$$

which is the required result.

By actual calculation, denoting the rth derivative of y w.r.t x by y_r we have, if

$$y = sd\, x$$

$$y_1 = \sqrt{(1+k^2y^2).(1-k'^2y^2)}$$

$$y_2 = (2k^2-1)\,y - 2k^2k'^2y^3$$

$$y_3 = [P].\ y_1 \qquad \text{where } [P] \equiv (2k^2-1) - 6k^2k'^2y^2$$

$$y_4 = [P].\ y_2 - 12\,k^2k'^2y^2{y_1}^2$$

$$y_5 = [P].\ y_3 - 12\,k^2k'^2.(3yy_2+{y_1}^2).y_1$$

$$y_6 = [P].\ y_4 - 36\,k^2k'^2.(yy_2+2{y_1}^2).y_2 - 48k^2k'^2yy_1y_3$$

$$y_7 = [P].\ y_5 - 120\,k^2k'^2.(yy_2+{y_1}^2).y_3 - 60k^2k'^2.(yy_4+3{y_2}^2).y_1$$

$$y_8 = [P].\ y_6 - 180\,k^2k'^2.(yy_2+{y_1}^2).y_4 -$$

$$-180k^2k'^2.(4y_1y_3+{y_2}^2).y_2 - 24k^2k'^2.(3y_1y_5+5{y_3}^2).y$$

$$y_9 = [P].\ y_7 - 252\,k^2k'^2.(yy_2+{y_1}^2).y_5 -$$

$$-420k^2k'^2.(yy_4+2y_1y_3-3{y_2}^2).y_3 - 84k^2k'^2.(yy_6+15y_2y_4).y$$

When $k = 1/\sqrt{2}$, it is easily seen that y_3 and y_7 vanish at $x = 0$. Hence in the Maclaurin expansion of $sd\,x$, the coefficients of x^3 and x^7 are zero when $k = 1/\sqrt{2}$.

Now considering the Fourier series expansion of $sd\,u$, $u = 2Kx/\pi$ and equating the coefficients of x^3 and x^7 to 0, as in the previous case, we obtain Ramanujan's result when $n = 4$ and 8.

In the similar manner by actual calculation we can show that y_{11} is also zero when $k = 1/\sqrt{2}$ and $x = 0$, which gives Ramanujan's result when $n = 12$ also.

The general case depends upon showing

$$y_{4m-1} = 0 \text{ when } x = 0,\ m \text{ an integer.}$$

This does not appear to be straight forward. But it has been thought worthwhile to record the above proof for the particular cases $n = 4$ and 8.

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A SUBSTITUTION RELATION FOR INTEGRAL TRANSFORMS

By R. G. BUSCHMAN

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EFROSS [3] has given a rather general substitution theorem for the Laplace transforms and McLachlan [5] has Later obtained a similar result. It is the purpose of this paper to point out a generalization to other types of integral transformations and to give a few examples. Symmetric kernels are not required nor is the same range of integration required as in the case of the substitution relation for original functions given in [1].

We use the notation as in Doetsch [2]

$$f(s) = \underset{s}{\overset{\mathscr{K}}{\bullet\!-\!\circ}} F(t)$$

to denote

$$f(s) = \int_a^b K(s, t)\, F(t)\, dt,$$

where the range of integration may be infinite and the script letter denotes the type of transformation.

Consider the expression

$$k(s)\, f[\,g(s)\,] = \int_a^b k(s)\, K_1\,[\,g(s),\, t\,]\, F(t)\, dt.$$

If there exists a function $\theta(u, t)$ such that with respect to a second kernel $K_2\,(s, u)$

$$k(s)\, K_1\,[g(s),\, t] = \int_c^d K_2(s, u)\, \theta\,(u, t)\, du,$$

then

$$k(s)\, f\,[\,g(s)\,] = \int_a^b \left(\int_c^d K_2(s,u)\, \theta(u,t)\, du \right) F(t)\, dt.$$

Now if the function $F(t)$ is such that the order of integration can be changed, we have

$$k(s)\, f[g(s)] = \int_c^d K_2(s,u) \left(\int_a^b \theta(u,t)\, F(t)\, dt \right) du,$$

which is in the form of a transformation with respect to the kernel $K_2(s, u)$. Finally then we have, if $F(t)$ satisfies the conditions and if

$$f(s) \underset{s\quad t}{\overset{\mathscr{K}_1}{\bullet\!-\!\circ}} F(t), \tag{1}$$

$$k(s)\, K_1[g(s), t] \underset{s\quad u}{\overset{\mathscr{K}_2}{\bullet\!-\!\circ}} \theta(u,t), \tag{2}$$

then

$$k(s)\, f[g(s)] \underset{s\quad u}{\overset{\mathscr{K}_2}{\bullet\!-\!\circ}} \int_a^b \theta(u,t)\, F(t)\, dt. \tag{3}$$

Examples which hold, only, of course, provided that $F(t)$ satisfies the rather stringent conditions for the existence of the various integrals involved and for the interchange of the order of integration, will next be given. Because of the variety of kernels involved, these conditions on $F(t)$ would have to be established for each example individually.

Example 1. As a very simple example we derive a known formula for Laplace transforms. Let $K_1(s,t) = K_2(s,t) = e^{-st}$, $k(s) = s^{-1}$, and $g(s) = s$, then $\theta(u,t) = U(u-t)$ (i.e. $= 1,\ u > t;\ = 0,\ u < t$), and

$$s^{-1} f(s) \underset{s\quad u}{\overset{\mathscr{L}}{\bullet\!-\!\circ}} \int_0^u F(t)\, dt.$$

Example 2. Again consider $K_1(s,t) = K_2(s,t) = e^{-st}$, but with $k(s) = (s^2 + b^2)^{-1/2}$, $g(s) = a \sinh^{-1}(s/b)$, so that for $a, b > 0$, we have from [4 : 5.4(23)], $\theta(u,t) = J_{at}(bu)$. Thus for

$$f(s) \overset{\mathscr{L}}{\underset{s\quad t}{\bullet\!-\!\circ}} F(t),$$

$$(s^2 + b^2)^{-1/2} f[a \sinh^{-1}(s/b)] \overset{\mathscr{L}}{\underset{s\quad u}{\bullet\!-\!\circ}} \int_0^\infty J_{at}(bu)\, F(t)\, dt.$$

EXAMPLE 3. As an example of mixed kernels, consider $K_1(s,t) = e^{-st}$, $K_2(s,t) = \cos st$, $k(s) = (s^2 + a^2)^{-1/2}$, and $g(s) = (s^2 + a^2)^{1/2}$, so that from [4 : 1.5(27)] $\theta(u,t) = 2\pi^{-1} K_0[a(t^2 + u^2)^{1/2}]$. Thus for

$$f(s) \overset{\mathscr{L}}{\underset{s\quad t}{\bullet\!-\!\circ}} F(t)$$

$$(s^2 + a^2)^{-1/2} f[(s^2 + a^2)^{1/2}] \overset{\mathscr{C}}{\underset{s\quad u}{\bullet\!-\!\circ}} 2\pi^{-1} \int_0^\infty K_0[a(t^2 + u^2)^{1/2}]\, F(t)\, dt.$$

EXAMPLE 4. Let $K_1(s, t) = J_\nu(st)\ (st)^{1/2}$, $K_2(s, t) = \cos st$, $g(s) = s^{1/2}$, and $k(s) = s^{\nu/2-1/4}$ with $-1 < \nu < 1/2$. Then using [4 : 1.13 (26)] $\theta(u, t) = 2^{1-\nu} \pi^{-1} t^{\nu+1/2} u^{-\nu-1} \sin(t^2/4u - \nu\pi/2)$ and for

$$f(s) \overset{\mathscr{H}_\nu}{\underset{s\quad t}{\bullet\!-\!\circ}} F(t),$$

$$s^{\nu/2-1/4} f(s^{1/2}) \overset{\mathscr{C}}{\underset{s\quad u}{\bullet\!-\!\circ}} \pi^{-1}\, 2^{1-\nu}\, u^{\nu+1} \int_0^\infty t^{\nu+1/2} \sin(t^2/4u - \nu\pi/2)\, F(t)\, dt.$$

EXAMPLE 5. As a final illustration consider one kernel non-symmetric $K_1(s,t) = e^{-st}$, $K_2(s,t) = t^{s-1}$, $g(s) = s^{1/2}$, and $k(s) = s^{-1/2}$, so that from [4: 7.2(9)] $\theta(u,t) = (-\pi \ln u)^{-1/2} e^{-t^2}/4 \ln u$. Thus for

$$f(s) \overset{\mathscr{L}}{\underset{s\quad u}{\bullet\!-\!\circ}} F(t)$$

$$s^{-1/2} f(s^{1/2}) \overset{\mathscr{M}}{\underset{s\quad u}{\bullet\!-\!\circ}} (-\pi \ln u)^{-1/2} \int_0^1 e^{-t^2}/(4 \ln u)\, F(t)\, dt.$$

Other examples are easily constructed in a similar manner.

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MEDIAL SIMPLEX

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1. Introduction. (a) Following Court [4] it is proposed to project his ideas into higher spaces and then deduce his special theorems. Let $A_0 \ldots A_n$ be a simplex S in an n-space and S' be formed of the $n+1$ centroids G_i of its prime faces a_i opposite its respective vertices A_i. A_iG_i is then called a *median* of S and is divided by its centroid G in the ratio $n:1$. S' is called the *complementary* or *medial* simplex of S which is then referred to as the *anti-complimentary*, *anti-medial*, or, briefly, *an-ry* simplex of S'. S, S' are evidently *homothetic* w.r.t. their common centroid G with homothetic ratio $-n:1$. We may also introduce here their *midway* simplex S'' too, formed of the $n+1$ mid-points M_i of the medians of S such that every pair of corresponding vertices A_i, G_i of S, S' is symmetric w.r.t. a vertex M_i of S'' and therefore every pair of their corresponding primes a_i, g_i are symmetric w.r.t. a prime m_i of S''. *The midway tetrahedron T'' of a given tetrahedron T and its medial T' is the twin of T'*[4].

(b) Projectively, S' is the *cevian* simplex [2] of a point G w.r.t. S which is then the *anti-cevian* of G w.r.t. S' such that S, S' are in the *homology* $(G, g, -n)$ with centre at G, the common polar [6] prime g of G w.r.t. both S, S' as the hyperplane of this homology and $-n$ as its constant anharmonic ratio. When G is their common centroid as above, g is the hyperplane at infinity. A_iG_i is said to be a *cevian* of S and G_i is its *foot* such that $(A_iM_iG_iN_i) = -1$, N_i being the trace of A_iG_i in g. The secant through G to an edge A_iA_j and the opposite $(n-2)$-space of S may be said to be a *bicevian* of S meeting them in its *feet* M_{ij}, G_{ij} thereat, respectively, such that *the n cevians A_jG_{ij} of the face a_i of S CONCUR at G_i and thus G_{ij} form the n vertices of the cevian $(n-1)$-simplex of G_i* w.r.t. a_i[*cf.*1, pp. 118-19, Arts. 346-50]. *The $\binom{n+1}{2}$ points M_{ij} and the $\binom{n+1}{2}$ traces N_{ij} of the edges of S in g*

form an S-configuration with S as its diagonal simplex, G being one of the 2^n *vertices of its reciprocal configuration* [5].

2. Pairs of isotomic points on the edges of S. Let G_i' be the centroid of the prime face g_i of S' [§ 1 (a)] opposite G_i. The lines G_iA_i, G_jG_i' are the traces of the plane $A_iA_jGG_iG_j$ in the two parallel primes a_i, g_i and are therefore parallel. G_iG_j, A_iA_j are parallel due to the homothecy. Hence the trace A_{ij} of A_iA_j in g_i, or more precisely, on G_jG_i' is the fourth vertex of the parallelogram $A_jG_iG_jA_{ij}$. Thus $A_jA_{ij} = G_iG_j = A_iA_j/n$. A similar argument may show that if $A_{ji}(\neq A_{ij})$ denotes the trace of A_iA_j on the median G_iG_j' of g_j, we have $A_iA_{ji} = G_iG_j$. Likewise for the other edges of S. Hence: *An edge of the an-ry simplex S of a given simplex S′ meets the two prime faces of S′ non-parallel to it in a pair of isotomic points* (i.e., *equidistant from its midpoint* M_{ij}) [3]. Projectively: *An edge* A_iA_j *of the anti-cevian simplex S of a point G w.r.t. S′ meets the two prime faces of S′ other than those through its corresponding edge* G_iG_j *in a pair of harmonic conjugates of the pair of vertices on it of the associated S-configuration* [§ 1 (b)].

3. The *an-ry* simplexes. (a) Now the medians G_iG_j', G_jG_i' of S' [§ 2] meet in the centroid G_{ij}', of the common $(n-2)$-face of its prime faces g_i and g_j, through which then passes the common bi-median $M'_{ij}GM_{ij}$ of S,S' (Fig. 1). $M_{ij}A_{ij} = M_{ij}A_j - A_{ij}A_j = (n-2)A_iA_j/(2n) = (n-2)G_jM'_{ij}$ [§ 2]. Hence G'_{ij} divides G_jA_{ij} in the

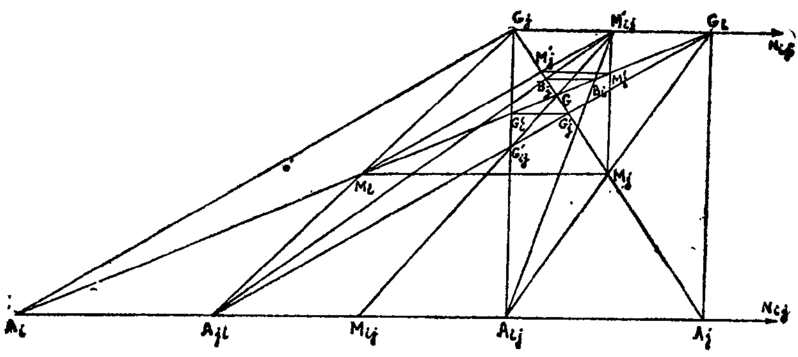

FIG. 1.

ratio $1:(n-2)$ and consequently G'_i in $1:(n-1)$ proving A_{ij} to be the vertex of the *an-ry* $(n-1)$-simplex of g_i corresponding to its vertex G_j. Similarly are related G_i, A_{ji}. Likewise for the other edges of S. Thus: *The $n(n+1)$ vertices of the $n+1$ an-ry $(n-1)$-simplexes of the $n+1$ prime faces of a simplex S in an n-space lie in pairs on the edges of its an-ry simplex S.*

(b) Now the prime face g_i of S' is homothetic, to the $(n-1)$-simplex $(M_{i\,j}')$ formed of the midpoints of the n edges of S' through G_i w.r.t. G_i with homothetic ratio $2:1$, and to its *an-ry* simplex (A_{ij}) w.r.t. their common centroid G_i' with homothetic ratio $-1:(n-1)$. Hence : (A_{ij}), $(M_{i\,j}')$, *are homothetic with ratio* $2(1-n):1$ *and centre B_i collinear with G_i, G'_i* such that $G_i\,B_i : B_i\,G_i' = n:(n-1)$.

(c) Projectively : *The $n(n+1)$ vertices of the $n+1$ anti-cevian $(n-1)$-simplexes (A_{ij}) of the $n+1$ traces G_i' of $n+1$ concurrent cevians G_iGG_i' of the simplex S' in its prime faces g_i* w.r.t. *them lie in pairs on the edges of the anti-cevian simplex S of G* w.r.t. S'. *An (A_{ij}) and the $(n-1)$-simplex (M_{ij}) formed of the n feet of the n bi-cevians $M'_{ij}\,GG'_{ij}$ of S' to its n edges through its vertex G_i thereat correspond in the homology $(B_i, g, 2-2n)$, with centre B_i on G_iG_i' such that $(B_iG_iB_i'G_i') = n/(1-n)$, where B_i' is the trace of this line in the hyperplane g, of this homology, which is the polar prime of G* w.r.t. S'.

4. Midway simplex S''. (a) M_i [§ 1 (a)] is evidently the centre of the parallelogram $A_iG_jG_iA_{ji}$ (§ 2) and therefore bisects its second diagonal G_jA_{ji}. Again $A_iA_{ij} : A_iA_j = (n-1):n$. Hence : *The an-ry $(n-1)$-simplex of a prime face g_i of the simplex S' is homothetic to the corresponding face a_i of the an-ry simplex S of S'* w.r.t. *its $(n+1)$th vertex A_i with ratio $(n-1):n$, and to the corresponding face m_i of their midway simplex S''* w.r.t. *the $(n+1)$th vertex G of S' with homothetic ratio* $2:1$.

(b) Let M_i' be the midpoint of the median G_iG_i' of S'. M_j [§ 4 (a)] is then related to M_{ij}' [§ 3 (a)] w.r.t. M_i' as is A_{ij} to G_j w.r.t. G_i' due to their homothecy w.r.t. G_i. Similar is the relation of M_i to M'_{ij} w.r.t. M_j' as of A_{ji} to G_i w.r.t. G_j' due to their homothecy w.r.t. G_j. Thus : *The midpoints of the n edges through a vertex G_i of S' form* *the medial* $(n-1)$*-simplex of the corresponding prime face* m_i *of* S''. Consequently: *The vertices of the* $n+1$ *medial* $(n-1)$*-simplexes of the prime faces of* S'' *lie at the midpoints of the edges of* S', *each occurring twice. The midway simplex* S''' *of* S' *and its medial coincides with the medial of* S''. For, M_i' are the centroids of the prime faces m_i of S''.

(c) Projectively: *An* $(n-1)$*-simplex* (A_{ij}) [§ 3 (c)] *is in the homology* $(A_i, g, (n-1)/n)$ *with the corresponding face* a_i *of* S, *and in* $(G_i, g, 2)$ *with the corresponding face* m_i, *of the midway simplex* $S''=(M_i)$ [§ 1 (b)]. *The cevian simplex* (M_i') *of* G w.r.t. S'' *coincides with the midway* S''' *of* S' *and the cevian* (G_i') *of* G w.r.t. S'. *The vertices of the* $n+1$ *cevian* $(n-1)$*-simplexes of the vertices of* S''' w.r.t. *the corresponding prime faces of* S'' *lie on the edges* G_iG_j *of* S' *as the harmonic conjugates* $M_i'{}_j$ *of their traces* N_{ij} [§ 1 (b)] *in* g w.r.t. *the respective pairs of its vertices thereat each occurring twice.*

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ON THREE TRICUSPS ASSOCIATED WITH A TRIANGLE

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1. It is known that the envelope of θ-pedal lines of a triangle is a three-cusped hypocycloid touching the line at infinity at the circular points. Such a hypocycloid is called a *tricusp*. In this paper we consider three tricusps associated with a triangle ABC, viz.: (i) the envelope of Simson lines ($\pi/2$-pedal lines) of its antimedial triangle, (ii) the one for its medial triangle, and (iii) the one for the given triangle ABC itself. These three tricusps are related to the given triangle ABC in another way also, as we will see in § 3.

2. THEOREM 1. *Any two perpendicular tangents to a parabola inscribed in a triangle ABC are the axes of*

(i) *a unique conic S_1 for which ABC is self-polar,*

(ii) *a unique conic S_2 circumscribing the antimedial triangle of ABC, and*

(iii) *a unique conic S_3 inscribed in the pedal triangle of ABC.*

PROOF. Let $P\alpha$, $P\beta$ be any two perpendicular tangents to a parabola π inscribed in ABC, α and β being the points at infinity on them.

(i) The six sides of the two triangles ABC and $P\alpha\beta$ touch π; hence there exists a unique conic S_1 for which the two triangles are self-polar. Thus ABC is self-polar for S_1 and $P\alpha$, $P\beta$, being perpendicular conjugate diameters of S_1, are the axes of S_1.

(ii) Let S_2 be the unique conic circumscribing the antimedial triangle of ABC and with centre at P. S_2 is out-polar to π and hence π is in-polar to S_2. Now, $P\alpha\beta$ circumscribes π and P and $\alpha\beta$ are pole and polar with respect to S_2; hence, $P\alpha$ and $P\beta$ are perpendicular conjugate diameters of S_2, that is, they are the axes of S_2.

(iii) Let S_3 be the unique conic inscribed in the pedal triangle of ABC and having P for its centre. Now, a unique rectangular hyperbola H passes through $ABCP\alpha\beta$. S_3 is in-polar to H, and hence, H is out-polar to S_3. $P\alpha\beta$ is inscribed in H, and P and $\alpha\beta$ are pole and polar with respect to S_3; hence, $P\alpha$, $P\beta$ are perpendicular conjugate diameters of S_3, i.e. they are the axes of S_3.

Incidentally we observe that

Given a triangle ABC, there exist sets of three co-axal conics, one for which ABC is self-polar, another circumscribing the antimedial triangle of ABC, and the third inscribed in the pedal triangle of ABC. (2.1)

3. If, in the above theorem, $P\alpha$ is taken to be the tangent at the vertex of π, then $P\beta$ becomes the line at infinity and the three conics S_1, S_2, S_3 become parabolas. Also, the tangents at the vertices of parabolas inscribed in a triangle are the Simson lines of the triangle. Hence, it follows from the theorem that

Simson lines of triangle ABC are axes of

(i) *Parabolas for which ABC is self-polar (i.e. parabolas inscribed in the medial triangle of ABC),*

(ii) *Parabolas circumscribed to the antimedial triangle of ABC, and*

(iii) *Parabolas inscribed in the pedal triangle of ABC.* (3.1)

Thus the tricusps mentioned in § 1 are, respectively, the envelopes of axes of parabolas inscribed in ABC, the one for parabolas circumscribing ABC, and the one for parabolas having ABC for a self-polar triangle. We will call them, respectively, the in-, the ex-, and the polar-tricusps of ABC.

4. We now get the areal tangential equations of the three tricusps, referred to ABC as the triangle of reference.

Since a tricusp is a class-cubic touching the line at infinity at the circular points, the areal tangential equation of any tricusp touching the sides of the triangle of reference is of the form

$$\sum mn \frac{\partial \Omega}{\partial l} + 2\lambda(m-n)(n-l)(l-m) = 0, \tag{4.1}$$

where $\Omega = 0$ is the tangential equation of the circular points, and λ is arbitrary. This shows that a tricusp is uniquely determined by any four of its tangents. Thus, by MacMahon's theorem [2], the above tricusp is the envelope of θ-pedal lines of the triangle of reference for a definite θ; and the value of θ in this case can by obtained by considering the third tangent through any one vertex of ABC, and is given by

$$2\Delta \cot \theta = \lambda, \tag{4.2}$$

where Δ is the area of triangle ABC. Thus, the polar-tricusp, viz. the envelope of Simson lines or $\pi/2$-pedal lines of ABC has the tangential equation

$$\sum mn \frac{\partial \Omega}{\partial l} = 0. \tag{4.3}$$

The tangential equation of the in-tricusp can be got from that of the polar-tricusp by the transformation

$$l \to m + n - l, \quad m \to n + l - m, \quad n \to l + m - n,$$

and is

$$\Sigma a^2 l (l - m)(l - n) = 0. \tag{4.4}$$

The tangential equation of the ex-tricusp can be obtained from that of the polar-tricusp by the transformation

$$l \to m + n, \quad m \to n + l, \quad n \to l + m,$$

and is

$$\sum l^2 \frac{\partial \Omega}{\partial l} = 0. \tag{4.5}$$

Also, since (4.4) may be written in the form

$$\sum (l^2 + mn) \frac{\partial \Omega}{\partial l} = 0,$$

we see that all the three tricusps belong to one tangential pencil.

5. It would also be advantageous to give the map equations of the three tricusps. For this we shall choose the conjugate coordinate system† referred to the circum-circle of ABC as the unit circle; then the vertices A,B,C correspond to three turns t_1, t_2, t_3 say, respectively. Let $s_1 = t_1 + t_2 + t_3$, $s_2 = \Sigma t_1 t_2$ and $s_3 = t_1 t_2 t_3$; the circum-centre, the ortho-centre and the ninepoint-centre of ABC are the points

$$x = 0, \quad x = s_1, \quad \text{and} \quad x = s_1/2, \quad \text{respectively.}$$

The polar-tricusp, being the envelope of Simson lines of triangle ABC, is found to have map-equation

$$x = \frac{s_1}{2} + \frac{T^2}{2}\left(2t + \frac{1}{t^2}\right), \; (T^6 = s_3) \tag{5.1}$$

$$= f(t), \text{ say.}$$

The cusps are given by $f'(t) = 0$; that is, the cusps of the polar-tricusp are given by

$$x = \frac{s_1}{2} + \frac{3}{2}T^2, \; x = \frac{s_1}{2} + \frac{3}{2}T^2\omega, \text{ and } x = \frac{s_1}{2} + \frac{3}{2}T^2\omega^2,$$

where ω is a cube root of unity. The cuspidal tangents concur at $x = s_1/2$, the ninepoint-centre of ABC. Also, the ninepoint-circle of ABC is inscribed in the tricusp; it touches the tricusp at the points where the cuspidal tangents meet the tricusp again. The point of concurrence of the cuspidal tangents (which is also the centre of the inscribed circle of the tricusp) is called the *centre of the tricusp*. The cusps are equidistant from the centre of the tricusp and form an equilateral triad, so that the cuspidal tangents are inclined to one another at an angle of 120°.

The ex- and the in-tricusps have respectively the map equations

$$x = \frac{s_1}{4} - \frac{T^2}{4}\left(2t + \frac{1}{t^2}\right), \tag{5.2}$$

and

$$x = -T^2\left(2t + \frac{1}{t^2}\right), \tag{5.3}$$

† For a detailed study of conjugate coordinate system as applied to triangle, circle, tricusp etc. reference may be made to Slaught Memorial Papers, No. 5, *American Math. Monthly*, Vol. 63 or *Inversive Geometry* by MORLEY and MORLEY.

It is clear from the map equations that the ex-tricusp can be got by contracting the in-tricusp about its centre (circum-centre of ABC) to one-fourth its size and then translating it through one-fourth the distance from the circum-centre to the ortho-centre of ABC in that direction, while the polar-tricusp can be obtained by contracting the in-tricusp about its centre to half its size, then rotating it about its centre through an angle π and finally translating it through half the distance from the circum-centre to the ortho-centre of ABC in that direction.

Since by (3.1) the in-tricusp is also the envelope of Simson lines of the anti-pedal triangle (i.e. triangle formed by the ex-centres) of ABC and its altitudes are the internal and external bisectors of the angles of ABC, we see that the in-tricusp touches the six bisectors of the angles of ABC.

Similarly, the ex-tricusp touches the perpendicular bisectors of the sides of ABC.

It may further be noticed that the centres of the three tricusps are collinear on the Euler-line of ABC; this can be easily seen from the fact that the Euler-lines of ABC, its medial and its anti-medial triangles are the same.

6. Theorem 2. *The asymptotes of rectangular hyperbolas circumscribing a triangle ABC are Simson lines of the triangle.*

Proof. We have only to prove that the asymptotes of the R.H.'s are the tangents at the vertices of parabolas inscribed in ABC.

Let $O\alpha$, $O\beta$ be the asymptotes of a R.H. Γ, through A, B, C, α and β being the points at infinity on Γ. Consider the parabola π inscribed in ABC and touching $O\alpha$. Now, any number of triangles can be inscribed in Γ so as to circumscribe π; $O\alpha$ touches π and meets Γ in the two coincident points α, α; the other tangents to π from α, α are the line at infinity repeated and their intersection should be taken as β as it should lie on Γ; but as it is the ultimate intersection of two coincident tangents to π, β should lie on π also. Thus, the point at infinity on the parabola π is also a point at infinity on Γ

and hence the asymptote $O\beta$ of Γ is parallel to the axis of π so that $O\alpha$ is the tangent to π perpendicular to its axis; hence, it is the tangent at the vertex of π and the theorem is proved.

It follows from the above theorem that *the polar-tricusp of ABC is also the envelope of asymptotes of R.H.'s through A, B, C.*

7. Since by MacMahon's theorem a given tricusp is the envelope of θ-pedal lines of a triangle circumscribed to it for a particular θ, it follows that if the sides of a triangle and the line through a vertex of the triangle making an angle θ with the opposite side touch a given tricusp, then the third tangents through the other two vertices also make the same angle θ with their opposite sides and the tricusp is the envelope of θ-pedal lines of the triangle. Such a triangle we will call a θ-tangent triangle. The case when $\theta = \pi/2$ is worth noticing; *any three concurrent tangents to a tricusp and the three tangents perpendicular to them form an ortho-centric quadrangle giving rise to four triangles, each a $\pi/2$-tangent triangle.* Hence, given a tricusp, we can choose a $\pi/2$-tangent triangle by taking any two tangents to it and the tangent perpendicular to the third tangent through their intersection. By a similar method we can choose a θ-tangent triangle of the tricusp. Thus there are ∞^2 θ-tangent triangles for a given tricusp and known θ. Incidentally we may observe that the circum-radii of all such triangles are the same, viz. $2 \sin \theta$ times the radius of the inscribed circle of the tricusp.

8. Khabaza has pointed out [1] that the altitudes of a triangle are the cuspidal tangents of the Simson-line envelope of the triangle. This is not true; for there exist, as we have seen above, ∞^2 triangles whose Simson lines touch this envelope and evidently the altitudes of all such triangles cannot be the same. Also, the cuspidal tangents of a tricusp meet at angles of 120°, which is not the case with the altitudes of the triangle, unless the triangle is equilateral. However, there exists, for a given tricusp, one and only one $\pi/2$-tangent triangle whose altitudes are the cuspidal tangents of the tricusp: it is the triangle formed by the tangents to the tricusp at the points where the inscribed circle touches it.

9. The results in (3.1) also suggest the solution of the following problem :

Given a tricusp, what is the relationship between three triangles ABC, $A'B'C'$ and $A''B''C''$ such that the given tricusp is the polar-tricusp of ABC, is the ex-tricusp of $A'B'C'$, and is the in-tricusp of $A''B''C''$?

SOLUTION. ABC can be any $\pi/2$-tangent triangle of the given tricusp and it can be chosen in ∞^2 ways, as we have seen in § 7. $A'B'C'$ should be the anti-medial triangle of one of the ∞^2 possible positions of ABC and $A''B''C''$ should be the pedal triangle of one of them. Thus, without loss of generality, we can say that the triangles ABC, $A'B'C'$, and $A''B''C''$ are respectively a $\pi/2$-tangent triangle, its anti-medial triangle, and its pedal triangle.

I am deeply indebted to late Dr. R. Vaidyanathaswamy for his valuable suggestions and encouragement throughout the preparation of this paper.

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A WRONSKIAN OF THE HERMITE POLYNOMIALS

By S. K. CHATTERJEA

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1. In connection with his proof of Turān's inequality for the Hermite functions, S. K. Lakshmana Rao [1] has shown the relation

$$\Delta_n(x) = -W(H_n(x), H_{n-1}(x)), \tag{1.1}$$

where

$$\Delta_n(x) = \begin{vmatrix} H_n(x) & H_{n-1}(x) \\ H_{n+1}(x) & H_n(x) \end{vmatrix}$$

and $\quad W(H_n(x), H_{n-1}(x)) = \begin{vmatrix} H_n(x) & H_{n-1}(x) \\ H'_n(x) & H'_{n-1}(x) \end{vmatrix}.$

We ask now : does such an equivalent Wronskian exist in the case of the determinant $\sigma_n(x)$, where

$$\sigma_n(x) \equiv \begin{vmatrix} H_n(x) & H_{n-1}(x) & \dots & H_{n-r}(x) \\ H_{n+1}(x) & H_n(x) & \dots & H_{n-r+1}(x) \\ \dots & \dots & \dots & \dots \\ H_{n+r}(x) & H_{n+r-1}(x) & \dots & H_n(x) \end{vmatrix}, \quad r \not> n?$$

The purpose of this present paper is to establish the relation

$$\sigma_n(x) = (-)^{[(r+1)/2]} . W(H_n(x), H_{n-1}(x), \dots, H_{n-r}(x)), \tag{1.2}$$

expressing the determinant $\sigma_n(x)$ of the Hermite polynomials as the Wronskian of $H_n(x), H_{n-1}(x), \dots, H_{n-r}(x)$.

2. Throughout this paper we shall adopt Erdélyi's [2] notation and regard Hermite polynomials, $H_n(x)$, as a system of orthogonal polynomials corresponding to the interval $(-\infty, +\infty)$ and the weight function $w(x) = \exp(-x^2)$. The polynomials give rise to the following relationship ([2], p. 193)

$$H_n(x) = (-)^n e^{x^2} D^n(e^{-x^2}). \tag{2.1}$$

Now before proving (1.2) we shall require the following identity :

$$H_{n+r}(x) = \sum_{m=0}^{r} (-)^m \binom{r}{m} H_{r-m}(x) D^m \{H_n(x)\}, \tag{2.2}$$

where

$$r \not> n,\ D \equiv d/dx,\ D^0\{H_n(x)\} = H_n(x) \text{ and } \binom{r}{m} = \frac{r!}{m!\,(r-m)!}.$$

Originally we had noticed that (2.2) is true for $r = 1, 2, 3$ and then the existence of (2.2) had been proved by the principle of mathematical induction. But here we shall present a simple proof as suggested by the referee.

From (2.1) and (2.2) we have

$$\sum_{m=0}^{r} (-)^m \binom{r}{m} H_{r-m}(x)\, D^m\{H_n(x)\}$$
$$= (-)^r e^{x^2} \sum_{m=0}^{r} \binom{r}{m} D^{r-m}(e^{-x^2})\, D^m\{H_n(x)\}$$
$$= (-)^r e^{x^2}.\, D^r\{e^{-x^2}.\, H_n(x)\}; \qquad \text{(by Leibnitz's theorem)}$$
$$= (-)^{n+r} e^{x^2} D^{n+r}(e^{-x^2}); \qquad \text{(by (2.1))}$$
$$= H_{n+r}(x).$$

This completes the proof of (2.2).

3. Although the relation (1.2) is an immediate consequence of the identity (2.2), yet we feel it necessary to attach a brief proof of (1.2) which is our main result. We put $r = 1$ in (2.2) and then put in succession $n = n, n-1, n-2, \ldots n-r$, whereby we obtain the equivalent expressions for the constituents $H_{n+1}(x), H_n(x), \ldots H_{n-r+1}(x)$ of the second row of $\sigma_n(x)$. We shall call these steps a 'process.' Thus after the first 'process' we have

$$\sigma_n(x) = \begin{vmatrix} H_n(x) & H_{n-1}(x) & \cdots & H_{n-r}(x) \\ \mathbf{A} H_{1-m}.\, D^m H_n & \mathbf{A} H_{1-m}.\, D^m H_{n-1} & \cdots & \mathbf{A} H_{1-m}.\, D^m H_{n-r} \\ H_{n+2}(x) & H_{n+1}(x) & \cdots & H_{n-r+2}(x) \\ \cdots & \cdots & \cdots & \cdots \\ H_{n+r}(x) & H_{n+r-1}(x) & \cdots & H_n(x) \end{vmatrix}$$

where $\mathbf{A} = \sum_{m=0}^{1} (-)^m \binom{1}{m}$

$$= - \begin{vmatrix} H_n(x) & H_{n-1}(x) & \dots & H_{n-r}(x) \\ H'_n(x) & H'_{n-1}(x) & \dots & H'_{n-r}(x) \\ H_{n+2}(x) & H_{n+1}(x) & \dots & H_{n-r+2}(x) \\ \dots & \dots & \dots & \dots \\ H_{n+r}(x) & H_{n+r-1}(x) & \dots & H_n(x) \end{vmatrix}.$$

We repeat the 'process' $(r - 1)$ times more by putting $r = 2, 3, \dots,$ r and we ultimately obtain (1.2).

The author wishes to record his gratitude to Dr. H. M. Sengupta for his kind help and encouragement during the course of this work. He also wishes to express his thanks to the referee for his valuable suggestions.

ADDED IN PROOF: The proof of (2.2) by induction appeared in *Rev. Mat. Hisp. Amer.*, Vol. 19 (1959) pp 209-212. An equivalent result of (2.2) occurs in slightly different notation in a paper by Nielsen.

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A GENERALIZATION OF THE PEDAL LINE THEOREM

By R. RAGHAVENDRAN

1. It is well known that the feet of the perpendiculars on the sides of a triangle $A_1A_2A_3$, from any point B on its circumcircle, are collinear; the line of collinearity is called the pedal line of the point B w.r.t. the triangle $A_1A_2A_3$. It is also known that if B, A_1,A_2,A_3,A_4 are points on a circle, then the feet of the perpendiculars from B on the pedal lines of B w.r.t. the triangles $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$, and $A_1A_2A_3$ are collinear; we shall call this line of collinearity, the pedal line of the point B w.r.t. the quadrangle $A_1A_2A_3A_4$.

More generally if $A_1, A_2, A_3, \dots, A_{n+1}$ are points on a circle passing through the point B, we can associate a 'pedal line,' such that the feet of the perpendiculars from B on the 'pedal lines' of B associated with each of the $(n+1)$ sets of n points obtained from $A_1, A_2, A_3, \dots, A_{n+1}$ by dropping the A's one at a time, are all collinear on the pedal line of B associated with $A_1, A_2, \dots, A_{n+1}$.

Some of the properties of the ordinary pedal lines find their analogues in the case of the generalised pedal lines also, but a few others do so only when certain constants connected with the points A_i satisfy a certain set of equations.

Two noteworthy results of the later type can be found in Theorems 2 and 3.

2. With the notation mentioned above, let us take B as the pole, and BO as the initial line, where O is the centre of the circle, and d its diameter. Let θ_i be the angle OBA_i. Then it is easily verified by induction, that the equation to the pedal line of B w.r.t. the polygon $A_1A_2A_3 \dots A_n$ is

$$r.\cos(\theta - \theta_1 - \theta_2 - \dots - \theta_n) = d \cos\theta_1 . \cos\theta_2 . \dots . \cos\theta_n .$$

3. If we change to a new pole at O, and initial line OX such that the angle $XOB = \pi + \phi$, and if the angles XOA_i are α_i, the equation of the pedal line is seen to be

$$r.\cos\left(\theta - \sigma + \frac{n-2}{2}.\phi\right) = d.\prod_{i=1}^{n}\cos\left(\frac{\alpha_i - \phi}{2}\right) - \\ -\frac{d}{2}.\cos\left(\sigma - \frac{n}{2}\phi\right), \tag{i}$$

where $2\sigma = \Sigma\alpha_i$.

4. We shall now proceed to prove some results concerning the generalized pedal lines.

5. THEOREM 1. *Given n points A_i on a circle,* (i) *through any point in the plane of the circle, there pass, in general, the pedal lines of n points B_i w.r.t the polygon (A_i), and* (ii) *if OA_i, OB_i make angles α_i, β_i with OX, then $\Sigma\beta_i = \Sigma\alpha_i + 2k\pi$, where k is a rational integer.*

PROOF. In the pedal line of a point B passes through the point (r', θ'), then ϕ is a root of the equation

$$r'.\cos\left(\theta' + \frac{n-2}{2}.\phi - \sigma\right) = d.\prod\cos\left(\frac{\alpha_i - \phi}{2}\right) - \\ -\frac{d}{2}.\cos\left(\sigma - \frac{n}{2}.\phi\right).$$

On applying the formula $2\cos A = e^{-iA}.(e^{i2A} + 1)$ wherever necessary, and multiplying by $2^n.e^{i(n\phi/2+\sigma)}$, the above equation becomes

$$(2^{n-2} - 1).d.e^{in\phi} + (2^{n-1}.r'.e^{i\theta'} - a_1).e^{i(n-1)\phi} - a_2.e^{i(n-2)\phi} - \dots \\ \dots - a_{n-2}.e^{i.2\phi} + (2^{n-1}.r'.e^{-i\theta'}.e^{i.2\sigma} - a_{n-1})e^{i\phi} + \\ + (2^{n-2} - 1).d.e^{i2\sigma} = 0, \tag{ii}$$

where

$$a_r = d.\Sigma e^{i(\alpha_1+\alpha_2+\dots+\alpha_r)}, r = 1, 2, 3, \dots, n-1.$$

This equation in $e^{i\phi}$ has n roots, and consequently there are n values of $\phi (0 \leqslant \phi < 2\pi)$, which satisfy the above equation. Hence the first part of the theorem is proved.

If ϕ_i are the roots of the equation (ii), then $\beta_i = \pi + \phi_i$, and we have

$$e^{i(\beta_1+\beta_2+\ldots+\beta_n)} = e^{i(\alpha_1+\alpha_2+\ldots+\alpha_n)}.$$

Therefore we get

$$\Sigma\beta_i = \Sigma\alpha_i + 2k\pi, \tag{iii}$$

where k is a rational integer.

6. When $n = 3$, we have $\beta_1 + \beta_2 + \beta_3 = \alpha_1 + \alpha_2 + \alpha_3 + 2k\pi$; we easily see that the pedal lines of A_1, A_2, A_3 w.r.t. triangle $B_1B_2B_3$ are concurrent. In fact, these results were given by A. Narasinga Rao in [1].

7. From (ii), we see that

$$\left.\begin{aligned}
(2^{n-2}-1).\ \Sigma e^{i(\beta_1+\beta_2)} + \Sigma e^{i(\alpha_1+\alpha_2)} &= 0\\
(2^{n-2}-1).\ \Sigma e^{i(\beta_1+\beta_2+\beta_3)} + \Sigma e^{i(\alpha_1+\alpha_2+\alpha_3)} &= 0\\
\ldots \qquad \ldots \qquad \ldots &= 0\\
\ldots \qquad \ldots \qquad \ldots &= 0\\
\ldots \qquad \ldots \qquad \ldots &= 0\\
(2^{n-2}-1).\ \Sigma e^{i(\beta_1+\beta_2+\ldots+\beta_{n-2})} + \Sigma e^{i(\alpha_1+\alpha_2+\ldots+\alpha_{n-2})} &= 0
\end{aligned}\right\} \tag{iv}$$

Conversely if β_i satisfy the $(n-3)$ equations in (iv), and the equation (iii), we see that the pedal lines of the points B_i w.r.t. the polygon (A_i) are concurrent at a point (r', θ') given by

$$2^{n-1}.\ r'.\ e^{i\theta'} = d\ (2^{n-2}-1).\ \Sigma\, e^{i\beta_r} + d.\ \Sigma\, e^{i\alpha_r}.$$

8. If α_i satisfy the $(n-3)$ equations

$$\left.\begin{aligned}
&\Sigma\, e^{i(\alpha_1+\alpha_2)} = 0,\\
&\Sigma\, e^{i(\alpha_1+\alpha_2+\alpha_3)} = 0,\\
&\ldots \quad \ldots \quad \ldots\\
&\ldots \quad \ldots \quad \ldots\\
&\Sigma\, e^{i(\alpha_1+\alpha_2+\ldots+\alpha_{n-2})} = 0,
\end{aligned}\right\} \tag{v}$$

let us call the polygon A_i 'special', and write it as (A_i) 'special'.

From equations (iv), we see that (B_i) also is 'special', if (A_i) is 'special', and that the pedal lines of the points A_i w.r.t. (B_i) are concurrent.

So we have the following

THEOREM 2 : *If (A_i) is 'special', and the pedal lines of the points $B_1, B_2 \ldots, B_n$ w.r.t. (A_i) are concurrent, then the pedal lines of $A_1, A_2, \ldots, A_n$ w.r.t (B_i) are also concurrent.*

We may also note that the necessary (and sufficient) condition for the above result to be true is that (A_i) should be 'special.'

9. When (A_i) is 'special', the equation (i) becomes

$$r.\cos\left(\theta + \frac{n-2}{2}.\varphi - \sigma\right)$$

$$= \frac{d}{2^{n-1}}.\sum \cos\left(\alpha_i + \frac{n-2}{2}.\varphi - \sigma\right) - \frac{d}{2^{n-1}}.(2^{n-2}-1).\cos\left(\frac{n}{2}.\varphi - \sigma\right).$$

If we change the pole to the point H, whose cartesian coordinates w.r.t. O are $\left(\frac{d}{2^{n-1}}.\sum \cos\alpha_i, \frac{d}{2^{n-1}}.\sum \sin\alpha_i\right)$, the above equation becomes

$$r.\cos\left(\theta + \frac{n-2}{2}.\varphi - \sigma\right) = -\frac{d}{2^{n-1}}.(2^{n-2}-1).\cos\left(\frac{n}{2}.\varphi - \sigma\right). \quad \text{(vi)}.$$

10. The straight line whose equation is

$$r.\cos(\theta - \alpha) = \mu.\cos(l\alpha - \delta), \quad \text{(vii)}$$

where $l\,(> 1), \mu, \delta$ are constants, and α is a variable parameter, envelops a hypocycloid. Now equation (vii) is simply another form of the p-Ψ equation of the hypocycloid, viz.

$$p = (a - 2b).\sin\left(\frac{a}{2b-a}.\psi\right),$$

where a is the radius of the fixed circle, and b that of the generating circle. We note that $l = a/|2b-a|$.

Equation (vi) will transform into an equation of the form (vii), on the substitution $\alpha = \sigma - (n-2)\varphi/2$. So we see that the different pedal lines w.r.t. a polygon (A_i) 'special', envelop a hypocycloid. Also since $a/|2b-a| = l = n/(n-2)$, we see that there are n cusps in the hypocycloid.

So we have the following result.

THEOREM 3. *The different pedal lines w.r.t. a polygon (A_i) 'special' envelop an n-cusped hypocycloid.*

11. We may draw attention to two cases, when (A_i) is 'special'.

(a) When (A_i) is a regular polygon. Taking α_i as $\frac{2}{n}\left\{\gamma+(i-1)\pi\right\}$, we see that α_i are the roots of the equation $e^{in\alpha}-e^{i2\gamma}=0$, and hence α_i satisfy the equations (v);

(b) when $A_1 A_2 \ldots A_{n-1}$ is a regular polygon, and A_n is any other point on the circle, we can prove that α_i satisfy the equations (v) by a method similar to above.

12. The following results about the pedal lines can be proved.

(a) If B_1B_2 subtends an angle α at a point on the circle, then an angle between the pedal lines of B_1, and B_2 is $(n-2)\,\alpha$.

(b) If BL perpendicular to the pedal line of B w.r.t. $A_2 A_3 \ldots A_n$ meets the circle again at L_1, then the pedal line of B w.r.t. $A_1A_2 \ldots A_n$ is parallel to A_1L_1.

(c) If the pedal lines of $B_1, B_2, \ldots, B_n$ w.r.t. (A_i) are concurrent at a point P, and the pedal line of A_i w.r.t. (B_i) meets PA_i at P_i, then

$$PP_i : P_iA_i = (2^{n-2}-2) : 1.$$

The following results hold, when (A_i) is 'special.'

(d) If H is the point mentioned in (9), and the pedal line of any point B w.r.t. (A_i) meets HB at C, then $HC : CB = (2^{n-2}-1) : 1$.

(e) When n is odd, the pedal lines of the extremities of a diameter of the circle, intersect at right angles, on a fixed circle.

It will be noted that (a), (b), (d) and (e) are the analogues of the properties of the ordinary pedal line.

It can be seen that Theorem 2 can be proved using (c), (d) and the theorem of Menelaus.

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ON A REMARKABLE CREMONA INVOLUTION

By S. S. SUBRAHMANYAM

1. It is known that the quadratic Cremona involution determined by point-pairs conjugate with respect to the system of conics passing through four points I_0, I_1, I_2, I_3 forming an orthocentric tetrad reduces to the system of pairs of isogonal conjugates with respect to the harmonic triangle ABC for which I_0, I_1, I_2, I_3 are the in- and ex-centres, and that the points A, B, C are the singular points for the transformation; in this case, the involution is said be the *isogonal involution defined by* I_0, I_1, I_2, I_3.

Any isogonal point-pair of ABC, any conic Σ inscribed in ABC, and also each of the four squared points $I_0^2, I_1^2, I_2^2, I_3^2$ are in-polar to all the reactangular hyperbolas passing through I_0, I_1, I_2, I_3. The points of contact of tangents to Σ from one of the four points, say I_0, constitute a conic in-polar to all these rectangular hyperbolas, as it belongs to the tangential pencil of conics determined by Σ and I_0^2; that is,

the transformation between the points of contact of tangents from I_0 *to any conic inscribed in* ABC *is the isogonal involution* Γ *defined by* I_0, I_1, I_2, I_3. (1)

The object of the present paper is to investigate the transformation between the points of contact of tangents from I_0 to any conic passing through A, B, C. This transformation has been dealt with, partially, by R. Vaidyanathaswamy [1].

2. Let P, Q be the points of contact of tangents from I_0 to a conic S passing through A, B, C, and let p, q be the tangents to S at P and Q. Apply the above transformation Γ and denote the transforms by the corresponding accented letters; we have a line S' and two conics p', q' passing through I_0, A, B, C (since I_0 is invariant for Γ) and touching the line S' at P', Q', so that P' and Q' are conjugate points with respect to all conics passing through I_0, A, B, C; that is, *the isogonal conjugates, with respect to ABC, of the two points of contact of tangents from I_0 to any conic circumscribing ABC are themselves conjugates in the quadratic involution Γ' determined by I_0, A, B, C* [1]. (2)

3. We see from result (2) that $Q \equiv \Gamma.\Gamma'.\Gamma(P)$. On applying the transformations successively, the product $\Gamma.\Gamma'.\Gamma \equiv \Gamma''$ is easily seen to be a quintic Cremona involution for which A, B, C must be quadruple singular points, two of the four tangents to any homaloidal quintic of Γ'' at each of these singular points being the two sides of the triangle ABC that pass through it; but, since the sides of the triangle ABC are removed once from the transform of each homaloidal quartic of $\Gamma'.\Gamma$ by Γ, the points A, B, C, can only be double singular points of Γ'', but of extraordinary singularities. In fact, A, B, C, are cusps on each homaloidal quintic of Γ'', so that they account for 3×8 or 24 of the interesections of any two members of the homaloidal net of quintics of Γ''. Choosing trilinear coordinates with ABC as the triangle of reference, I_0 has coordinates (1, 1, 1) and the isogonal involution Γ is given by the equations

$$x' : y' : z' = yz : zx : xy,$$

and Γ' is given by

$$x' : y' : z' = x(y + z - x) : y(z + x - y) : z(x + y - z),$$

so that Γ'' is given by

$$x' : y' : z' = x(yz + zx - xy)\ (yz - zx + xy) : \ldots : \ldots . \qquad (3)$$

This type of quintic Cremona involution with three cuspidal singular points has not been classified in H. P. Hudson's *Treatise on Cremona Transformations*.

4. A more interesting example of this quintic Cremona involution occurs in plane geometry. Replacing the trilinear coordinates (x, y, z) of a general point by the areal line-coordinates $[l, m, n]$, referring to the same triangle ABC, it is clear that I_0 will be replaced by the line at infinity in this " duality ", so that the corresponding quintic Cremona line-involution will represent the line-transformation between the asymptotes of conics inscribed in ABC.

Thus,

the line-transformation between the asymptotes of any conic inscribed in a triangle ABC is a quintic Cremona line-involution with the sides BC, CA, AB as inflexional singular lines. (4)

Incidentally, by the same "duality", we deduce from (1) that the asymptotes of conics circumscribing a triangle are isotomic conjugates with respect to the triangle.

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A NOTE ON THE ORDER OF INTEGRAL FUNCTION

By R. S. L. SRIVASTAVA

[Received March 19, 1959]

1. Let $f(z) = \sum_{0}^{\infty} c_n z^n$ be an integral function of order ρ and lower order λ. It is known [1] that $f(z)$ is of finite order ρ, if and only if,

$$\limsup_{n\to\infty} \frac{n \log n}{\log \{1/|c_n|\}} = \rho. \tag{1.1}$$

Further, Shah [2] has shown that if $|c_n/c_{n+1}|$ is a non-decreasing function of n, for $n > n_0$, then

$$\lambda = \liminf_{n\to\infty} \frac{n \log n}{\log \{1/|c_n|\}} = \liminf_{n\to\infty} \frac{\log n}{\log |c_n/c_{n+1}|}, \tag{1.2}$$

and

$$\rho = \limsup_{n\to\infty} \frac{\log n}{\log |c_n/c_{n+1}|}, \tag{1.3}$$

where $0 \leqslant \lambda \leqslant \infty$.

In this paper we derive relations between the orders of two or more integral functions and also obtain the condition that two integral functions of regular growth may be of the same finite order.

2. THEOREM 1. *If* $f_1(z) = \sum_{0}^{\infty} a_n z^n$, $f_2(z) = \sum_{0}^{\infty} b_n z^n$ *be integral functions of orders* ρ_1, ρ_2 *and lower orders* λ_1, $(0 \leqslant \lambda_1 \leqslant \infty)$ *and* λ_2, $(0 \leqslant \lambda_2 \leqslant \infty)$, *respectively, and* $|a_n/a_{n+1}|$, $|b_n/b_{n+1}|$ *be non-decreasing functions for* $n > n_0$, *then the function* $f(z) = \sum_{0}^{\infty} c_n z^n$, *where* $\log |c_n/c_{n+1}| \sim \log |a_n/a_{n+1}|\, |b_n/b_{n+1}|$, *is an integral function of order* ρ *and lower order* λ *such that*

(i)
$$\frac{1}{\rho} \geqslant \frac{1}{\rho_1} + \frac{1}{\rho_2}, \tag{2.1}$$

(ii)
$$\frac{1}{\lambda} \leqslant \frac{1}{\lambda_1} + \frac{1}{\lambda_2}. \tag{2.2}$$

PROOF. Since $|a_n/a_{n+1}|$ is a non-decreasing function, using the relations (1.2) and (1.3) for the function $f_1(z)$, we get

$$\limsup_{n\to\infty} \frac{\log |a_n/a_{n+1}|}{\log n} = \frac{1}{\lambda_1}, \tag{2.3}$$

and

$$\liminf_{n\to\infty} \frac{\log |a_n/a_{n+1}|}{\log n} = \frac{1}{\rho_1}. \tag{2.4}$$

Hence for any $\epsilon > 0$, we have for sufficiently large n,

$$\frac{\log |a_n/a_{n+1}|}{\log n} < \frac{1}{\lambda_1} + \epsilon/2, \tag{2.5}$$

and

$$\frac{\log |a_n/a_{n+1}|}{\log n} > \frac{1}{\rho_1} - \epsilon/2. \tag{2.6}$$

Similarly, for the function $f_2(z)$, we get,

$$\frac{\log |b_n/b_{n+1}|}{\log n} < \frac{1}{\lambda_2} + \epsilon/2, \tag{2.7}$$

and

$$\frac{\log |b_n/b_{n+1}|}{\log n} > \frac{1}{\rho_2} - \epsilon/2. \tag{2.8}$$

From (2.5) and (2.7), we get,

$$\frac{\log |a_n/a_{n+1}|\,|b_n/b_{n+1}|}{\log n} < \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \epsilon$$

for any $\epsilon > 0$ and n sufficiently large.

If now $\log |c_n/c_{n+1}| \sim \log |a_n/a_{n+1}|\,|b_n/b_{n+1}|$, we have,

$$\limsup_{n\to\infty} \frac{\log |c_n/c_{n+1}|}{\log n} = \limsup_{n\to\infty} \frac{\log |a_n/a_{n+1}|\,|b_n/b_{n+1}|}{\log n} \leqslant \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

or

$$\frac{1}{\lambda} \leqslant \frac{1}{\lambda_1} + \frac{1}{\lambda_2}.$$

Similarly, from (2.6) and (2.8), we get,

$$\frac{1}{\rho} \geqslant \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

COROLLARY 1. *If $f_1(z)$, $f_2(z)$ are of regular growth then so is $f(z)$ and* $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$.

This is evident from (2.1) and (2.2), since for functions of regular growth $\lambda_1 = \rho_1$, $\lambda_2 = \rho_2$ and so $\lambda = \rho$.

COROLLARY 2. *If* $f_k(z) = \sum_0^\infty a_n^{(k)} z_n$, $(k = 1, 2, \ldots, m)$, *be m integral functions of orders* ρ_k *and lower orders* λ_k, $(0 \leqslant \lambda_\kappa \leqslant \infty)$ *respectively and each of the functions* $|a_n^{(k)}/a_{n+1}^{(k)}|$ *be non-decreasing for* $n > n_0$, *then the function* $f(z) = \sum_0^\infty c_n z^n$, *where*

$$\log |c_n/c_{n+1}| \sim \log |a_n^{(1)}/a_{n+1}^{(1)}| \ldots |a_n^{(m)}/a_{n+1}^{(m)}|,$$

is an integral function of order ρ *and lower order* λ, *such that*

$$\frac{1}{\rho} \geqslant \frac{1}{\rho_1} + \frac{1}{\rho_2} + \ldots + \frac{1}{\rho_m},$$

and

$$\frac{1}{\lambda} \leqslant \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \ldots + \frac{1}{\lambda_m}.$$

COROLLARY 3. *If the m functions* $f_k(z)$, $(k = 1, 2, \ldots, m)$, *be of regular growth then so is the function* $f(z)$ *and*

$$\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2} + \ldots + \frac{1}{\rho_m}.$$

The Corollaries 2 and 3 follow as immediate generalizations of Theorem 1 and Corollary 1, respectively.

3. THEOREM 2. *The integral functions,* $f_1(z) = \sum_0^\infty a_n z^n$ *and* $f_2(z) = \sum_0^n b_n z^n$, *which are of regular growth and such that* $|a_n / a_{n+1}|$, $|b_n / b_{n+1}|$ *are non-decreasing functions for* $n > n_0$, *will be of the same finite order* ρ, *if and only if,*

$$\log |a_n / a_{n+1}| \, |b_{n+1} / b_n| = o(\log n), \qquad (3.1)$$

as $n \to \infty$.

PROOF. In view of (1.1) and (1.2), it follows that if $f_1(z)$ is of regular growth, (i.e. $\rho_1 = \lambda_1$), then it will be of finite order ρ_1, if and only if,

$$\lim_{n\to\infty} \frac{\log n}{\log |a_n / a_{n+1}|} = \rho_1, \tag{3.2}$$

and $f_2(z)$ will be of order ρ_2, if and only if,

$$\lim_{n\to\infty} \frac{\log n}{\log |b_n/b_{n+1}|} = \rho_2. \tag{3.3}$$

Hence, if $\rho_1 = \rho_2 = \rho$ (say), we have,

$$\lim_{n\to\infty} \frac{\log |a_n/a_{n+1}|}{\log n} = \frac{1}{\rho} = \lim_{n\to\infty} \frac{\log |b_n/b_{n+1}|}{\log n},$$

so that

$$\lim_{n\to\infty} \frac{\log |a_n/a_{n+1}| \, |b_{n+1}/b_n|}{\log n} = 0.$$

Again, if (3.1) holds, we have,

$$\lim_{n\to\infty} \frac{\log |a_n/a_{n+1}| - \log |b_n/b_{n+1}|}{\log n} = 0$$

or

$$\frac{1}{\rho_1} = \lim_{n\to\infty} \frac{\log |a_n/a_{n+1}|}{\log n}$$

$$= \lim_{n\to\infty} \frac{\log |b_n/b_{n+1}|}{\log n} = \frac{1}{\rho_2},$$

since both the limits exist if $f_1(z)$ and $f_2(z)$ are of regular growth and $|a_n/a_{n+1}|$, $|b_n/b_{n+1}|$ are non-decreasing functions. Hence the theorem is proved.

I wish to express my thanks to Dr. S. K. Bose for the suggestions and the guidance he has given me in the preparation of this paper.

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BOUNDS OF $\sigma(N)$

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In this paper upper and lower bounds of $\sigma(N)$ depending upon the number of distinct prime factors, are obtained. Also inequalities connecting $\sigma(N)$ and $d(N)$ are established. Here $\sigma(N)$ denotes the sum of divisors of N including 1 and itself and $d(N)$ denotes the number of divisors of N.

THEOREM I. *If N is odd, $\sigma(N) < N(3/2)^r$, where r is the number of distinct prime factors.*

$$\frac{\sigma(p^\alpha)}{p^\alpha} = \left(1 - \frac{1}{p^{\alpha+1}}\right)\left(\frac{p}{p-1}\right) < \frac{3}{2}, \text{ if } p \geqslant 3. \qquad \text{(A)}$$

If $N = p_1^{\alpha_1}.p_2^{\alpha_2}\ldots p_r^{\alpha_r}$, then from (A) and from the multiplicative property of $\sigma(N)$ we have

$$\frac{\sigma(N)}{N} < \left(\frac{3}{2}\right)^r.$$

THEOREM II. *If N is even, $\sigma(N) < N.2.(3/2)^r$, where r is the number of distinct odd prime factors.*

$$\frac{\sigma(2^\beta)}{2^\beta} = \left(1 - \frac{1}{2^{\beta+1}}\right)/2 < 2.$$

So, if $N = 2^\beta.\,p_1^{\alpha_1}.\,p_2^{\alpha_2}.\ldots p_r^{\alpha_r}$ where p's are odd prime factors, it follows from (A) that

$$\frac{\sigma(N)}{N} < 2.\left(\frac{3}{2}\right)^r.$$

THEOREM III. $\dfrac{\sigma(N)}{N} < x^r$, *where r is the number of distinct prime factors and $x > \dfrac{2p}{2p-1}$, p being the lowest prime factor of N.*

Consider

$$f(p) = (x-1)\,p^{\alpha+1} - xp^\alpha + 1.$$

$f(p)$ is an increasing function of p if

$$f'(p) = (\alpha + 1)(x - 1)\, p^{\alpha} - \alpha x p^{\alpha-1} > 0,$$

i.e. if

$$p > \frac{x\,\alpha}{(x-1)(\alpha+1)} = \frac{x}{x-1}\left[1 - \frac{1}{\alpha+1}\right]$$

but $f(1) = 0$.

Hence, if

$$p > \frac{x}{x-1}\left[1 - \frac{1}{\alpha+1}\right], \quad f(p) > 0,$$

i.e.

$$\frac{p^{\alpha+1} - 1}{p^{\alpha}(p-1)} < x. \tag{B}$$

If $N = p_1^{\alpha_1}.\ p_2^{\alpha_2}\ \ldots .\ p_r^{\alpha_r}$, from (B) it is evident that

$$\frac{\sigma(N)}{N} < x^r. \tag{C}$$

If every prime factor is greater than $\frac{x}{x-1}\left[1 - \frac{1}{\alpha+1}\right]$, where α is the corresponding power of the prime, since the lowest value of α is 1, (C) is true if the lowest prime factor $p > \frac{1}{2} \cdot \frac{x}{x-1}$.

This implies that

$$x > \frac{2p}{2p-1}.$$

Corollary. *If N is odd, it is evident from Theorems I and III, that x lies between $\frac{2p}{2p-1}$ and $\frac{3}{2}$.*

If N is even, x is greater than $\frac{4}{3}$.

Theorem IV. $\frac{\sigma(N)}{N} > x^r$, *where $x < \frac{p}{p-1}$, p being the greatest prime factor of N and $x > 1$.*

As in Theorem III, we can show that

$$\frac{p^{\alpha+1}-1}{p^{\alpha}(p-1)} > x \quad \text{if} \quad p < \frac{x}{x-1}\left[1-\frac{1}{\alpha-1}\right],$$

i.e. if

$$p < \frac{x}{x-1}.$$

Hence $\frac{\sigma(N)}{N} > x^r$, if every prime factor of N is less than $\frac{x}{x-1}$.

This is true if the greatest prime factor p is less than $\frac{x}{x-1}$.

This implies that $x < \frac{p}{p-1}$.

THEOREM V. *If N is odd $\sigma(N) < N\,d(N)$.*

Let

$$N = p_1^{\alpha_1}.\ p_2^{\alpha_2}.\ \ldots .\ p_r^{\alpha_r}$$

and

$$d(N) = (\alpha_1+1)\ (\alpha_2+1), \ldots, (\alpha_r+1).$$

Since

$$\alpha_s + 1 > \frac{3}{2}, \quad s = 1, 2, \ldots, r$$

$$d(N) > \left(\frac{3}{2}\right)^r.$$

From Theorem I it follows that $d(N) > \frac{\sigma(N)}{N}$.

THEOREM VI. *If N is even, $\sigma(N) < \frac{4}{3} N\,d(N)$.*

Now $d(N) > \frac{3}{2}\cdot\left(\frac{3}{2}\right)^r$, where r is the number of distinct odd prime factors of N. From Theorem II it follows that $d(N) > \frac{3}{2}\cdot\frac{\sigma(N)}{2N}$.

Thus

$$\sigma(N) < \frac{4}{3} N\,d(N).$$

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A new formula for Genocchi numbers

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IN this note we prove the following formula for Genocchi numbers.

$$G_r = \sum_{i=1}^{r} a_i/i!\ \Delta^i\, 0^r, \tag{1}$$

where $a_1 = -1$, $a_2 = 0$, and all other a's are given by

$$a_{2N} = S_{2N}^{2N}\, G_{2N} + S_{2N-2}^{2N}\, G_{2N-2} + \dots + S_2^{2N}\, G_2 + S_1^{2N}\, G_1, \tag{2}$$

$$a_{2N+1} = S_{2N}^{2N+1}\, G_{2N} + S_{2N-2}^{2N+1}\, G_{2N-2} + \dots + S_2^{2N+1}\, G_2 + S_1^{2N+1}\, G_1, \tag{3}$$

where S's are the Stirling's numbers of the first kind defined by

$$N(N-1)\dots(N-r+1) = \sum_{i=1}^{r} S_i^r N^i. \tag{4}$$

Δ is the usual notation of difference calculus, i.e. $\Delta f(x) = f(x+1) - f(x)$ and G_r are the Genocchi numbers defined by

$$2t/(e^t+1) = \sum_{r=1}^{\infty} (-)^{r+1}\, G_r/r!\ t^r. \tag{5}$$

PROOF. It is known [4, p. 147] that $G_r/2$ is the coefficient of N in K_r, where N is even and K_r is expressed as a polynomial in N, with

$$K_r(N) = 1^r - 2^r + 3^r - \dots - N^r. \tag{6}$$

We know that [2]

$$t^r = \{t\Delta + t(t-1)/2!\ \Delta^2 + \dots + t(t-1)\dots(t-r+1)/r!\,\Delta^r\}\, 0^r. \tag{7}$$

Giving t values $t = 1, 2, \dots, N$ (N even) and alternatively adding and subtracting we get

$$K_r(N) = \Big\{ \Delta \sum_{x=1}^{N} (-)^{x+1}\, x + \Delta^2/2! \sum_{x=2}^{N} (-)^{x+1}\, x(x-1) + \dots +$$

$$+ \Delta^r/r! \sum_{x=N}^{N} (-)^{x+1}\, x(x-1)(x-2)\dots(x-r+1) \Big\}\, 0^r. \tag{8}$$

or,

$$K_r(N) = \Big[-N/2.\Delta - N^2/2.\Delta^2/2! + \Delta^3/3! \sum_{x=3}^{N} (-)^{x+1} S_i^3 x^i +$$

$$+ \Delta^r/r! \sum_{x=N}^{N} (-)^{x+1} \sum_{x=1}^{N} S_i^N x^i \Big] 0^r. \quad (9)$$

Now rewrite (9) in the form

$$K_r(N) = \Big[-N/2\,\Delta - N^2/2\,\Delta^2/2! +$$

$$+ \Delta^3/3! \left(\sum_{x-1}^{N} (-)^{x+1} \sum_{i=1}^{3} S_1^3 x^i - \sum_{x=1}^{2} (-)^{x+1} \sum_{i=1}^{3} S_i^3 x^i \right) \dots +$$

$$+ \Delta^r/r! \left(\sum_{x=1}^{N} (-)^{x+1} \sum_{i-1}^{N} S_i^N x^i - \sum_{x=1}^{N-1} (-)^{x+1} \sum_{i=1}^{N} S_i^N x^i \right) \Big] 0^r. \quad (10)$$

We can prove that

$$\sum_{x=i}^{r} (-)^{x+1} S_i^{r+1} x^i = 0. \quad (11)$$

As $x(x-1)(x-2)\dots(x-r) = \sum_{i=1}^{r+1} S_i^{r+1} x^i$, from (4) (12)

multiplying both sides of (12) by $(-)^{x+1}$ and summing from $x = 1$ to $x=r$, we get

$$\sum_{i=1}^{r+1} \sum_{x=1}^{r} (-)^{x+1} S_i^{r+1} x^i = \sum_{x=1}^{r} (-)^{x+1} x(x-1)(x-2)\dots(x-r) = 0.$$

Hence (10) can be written as

$$K_r(N) = \Big[-N/2\,\Delta - N^2/2\,\Delta^2/2! + \Delta^3/3! \sum_{x=1}^{N} (-)^{x+1} \sum_{i=1}^{3} S_i^3 x^i +$$

$$+ \Delta\,/r! \sum_{x=1}^{N} (-)^{x+1} S_i^N x^i \Big] 0\,. \quad (13)$$

Whence equating the coefficients of N, we get the required result.

Results similar to (1) for Bernoulli's numbers are given by Cayley [2] and Garabedian [3].

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3. Garabedian H. L. : A new formula for the Bernoulli's numbers *Bull. American Math. Soc.* 46 (1940), 531-533.

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On the non-existence of odd perfect numbers of a certain form

By M. Perisastri

Various kinds of results have been obtained regarding the odd perfect numbers. In this paper I prove the following

Theorem. *If s is the least of the positive integers α_r where $r = 1, 2, \ldots, k$ and*

$$\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k} > \log \left[\frac{2^{s+1}-1}{2^s} \zeta(s+1) \right],$$

where the p's are all odd primes and $\zeta(s) = \sum_1^\infty \frac{1}{m^s}$, then $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ is not an odd perfect number.

Proof. Write $n = \prod_{r=1}^{k} p_r^{\alpha_r}$, $3 \leqslant p_1 < p_2 < \ldots < p_k$. Let $\sigma(n)$ denote the sum of the divisors of n so that if n is a perfect number, $\sigma(n) = 2n$.

It is known that $\frac{\sigma(n)}{n} = \pi \left[\left(1 - \frac{1}{p_r^{\alpha_r+1}}\right) \Big/ \left(1 - \frac{1}{p_r}\right) \right]$. If n is a perfect number, we get

$$2=\prod\left[\left(1-\frac{1}{p_r^{\alpha_r+1}}\right)\Big/\left(1-\frac{1}{p_r}\right)\right] \tag{1}$$

Now let q_k denote the k^{th} odd prime, $q_1=3, q_2=5, \ldots$. Then clearly $p_r \geqslant q_r$.

Hence
$$\prod_{r=1}^{k}\left(1-\frac{1}{p_r^{\alpha_r+1}}\right) \geqslant \prod_{r=1}^{k}\left(1-\frac{1}{q_r^{s+1}}\right)$$
$$> \prod_{q}^{k}\left(1-\frac{1}{q^{s+1}}\right)\frac{2^{s+1}}{(2^{s+1}-1)},$$

where q runs through all the primes, 2, 3, 5, ...

But it is well known that $\prod_{q}\left(1-\frac{1}{q^{s+1}}\right)=\frac{1}{\zeta(s+1)}$.

erefore, we have

$$\prod_{r=1}^{k}\left(1-\frac{1}{p_r^{\alpha_r+1}}\right) > \frac{2^{s+1}}{2^{s+1}-1}\,\frac{1}{\zeta(s+1)}.$$

Therefore, (1) gives

$$2\prod_{r=1}^{k}\left(1-\frac{1}{p_r}\right) > \frac{1}{\zeta(s+1)}\;2^s/(2^{s+1}-1). \tag{2}$$

But

$$e^{-x} > 1-x, \text{ if } x>0.$$

Therefore, (2) gives

$$2\left[\exp\left(-\sum_{r=1}^{k}\frac{1}{p_r}\right)\right] > \frac{2^{s+1}}{2^{s+1}-1}\cdot\frac{1}{\zeta(s+1)}.$$

Therefore

$$\sum_{r=1}^{k}\frac{1}{p_r} < \log\left[\frac{2^{s+1}-1}{2^s}\cdot\zeta(s+1)\right],$$

which is contrary to the hypothesis. Hence the theorem is proved.

On the equation $13^x - 3^y = 10$

By A. MAKOWSKI, *Warsaw*

The purpose of this note is to solve in non-negative integers the equation

$$13^x - 3^y = 10. \tag{1}$$

I shall prove the following

THEOREM. *The equation* (1) *has exactly two solutions in non-negative integers* : $x = y = 1$ *and* $x = 3,\ y = 7$.

PROOF. It is evident that $y \geqslant 1$. For $y = 1$ we get $x = 1$. Suppose $y \geqslant 2$. Taking 1 mod (9) we obtain $4^x \equiv 13^x - 3^y \equiv 10 \equiv 1 \pmod 9$, therefore $3|x$. Since $3 \equiv -1 \pmod 4$ and $13 \equiv 1 \pmod 4$, from (1) it follows that $1 - (-1)^y \equiv 2 \pmod 4$, hence y is odd: $y = 2v + 1$. Multiplying both parts of (1) by 3^3 and substituting $3^{v+2} = Y$, $3.13^{\frac{1}{3}x} = x$ we get the equation

$$Y^2 + 270 = X^3. \tag{2}$$

O. Hemer proved [see On the Diophantine equation $y^2 - k = x^3$, Uppsala 1952 (dissertation), table 3] that the equation (2) has only one solution in positive integers: $X = 39$, $Y = 243$. From this solution we obtain $v = 3$, therefore $x = 3$, $y = 7$.

Thus the theorem is proved.

I conjecture that the equation $13^x - 3^y = 10^z$ has no solution with $z \neq 1$.

A note on the bilinear transformation

By T. V. LAKSHMINARASIMHAN, *Madras Christian College, Tambaram*

A problem proposed by Vaidyanathaswamy many years ago [*J. Ind. Math. Soc.* 17 (1927), *Notes and Questions*, Question 1471, p. 32], whose solution I have been unable to locate in these pages, is presented below in a more general form, and solved in that form, so as to bring out a property of the bilinear transformation.

PROBLEM. $A_1, A_2, \ldots, A_m$ *are centres of force in the Argand plane, of which* A_r *attracts according to the law* $a_r/$ *distance* $(r = 1, 2, \ldots, m)$. *If* $a_1 + a_2 + \ldots + a_m = 0$, *show that the positions of equilibrium in the plane for this field of force, together wilh the centres of force, from a configuration whose character is invariant under any bilinear transformation generally, i.e. the centres of force, when subjected to any such transformation generally result in the equilibrium positions being subjected to the same transformation.*

SOLUTION. If A_r is the point $z_r = x_r + iy_r$, then a necessary and sufficient condition for the point $z = x + iy$ to be a position of equilibrium is that the components X, Y, parallel to the real and the imaginary axes respectively, of the total force at z or (x, y), due to the given field, should separately vanish, i.e. that

$$X \equiv -\sum_{r=1}^{m} \frac{a_r(x-x_r)}{(x-x_r)^2+(y-y_r)^2} = 0,\ Y \equiv -\sum_{r=1}^{m} \frac{a_r(y-y_r)}{(x-x_r)^2+(y-y_r)^2} = 0$$

or

$$Z \equiv X + iY \equiv -\sum_{r=1}^{m} \frac{a_r}{\bar{z}-\bar{z}_r} = 0,\ \bar{Z} \equiv -\sum_{r=1}^{m} \frac{a_r}{z-z_r} = 0, \quad (1)$$

where, as usual, a complex number with a bar above it denotes the conjugate of that number. It follows from (1) that there are in general $m-1$ positions of equilibrium. It also follows that *the configuration consisting of the centres of force and the equilibrium positions is invariant in character for inversion generally.* For, suppose that the inversion is with respect to the circle $|z - z_0| = k$. Then the inverses with respect to the circle of the equilibrium positions are still equilibrium positions in the field of force due to the inverses of the centres of attraction, provided that (1) continues to hold when the point z is replaced by its inverse $z_0 + k^2(\bar{z} - \bar{z}_0)^{-r}$ and the point z_r by its inverse $z_0 + k^2(\bar{z}_r - \bar{z}_0)^{-1}$, i.e. provided that

$$-\sum_{r=1}^{m} \frac{a_r}{k^2(\bar{z} - \bar{z}_0)^{-1} - k^2(\bar{z}_r - \bar{z}_0)^{-1}} = 0,$$

or

$$-\sum_{r=1}^{m} \left\{ \frac{a_r(\bar{z}_r - z_0)}{(\bar{z}_r - \bar{z}_0) - (\bar{z} - \bar{z}_0)} - a_r \right\} = 0,$$

or

$$-(\bar{z} - \bar{z}_0) \sum_{r=1}^{m} \frac{a_r}{\bar{z} - \bar{z}_r} = 0,$$

or

$$(\bar{z} - \bar{z}_0) z = 0. \tag{2}$$

Tacitly excluding the case $z_0 = z_r$ $(r = 1, 2, \ldots, m)$, as indeed, we have done in the above argument, we see that (2) follows from (1) and hence that the italicized statement preceding the argument is true.

Again, *the configuration of the centres of force and the equilibrium positions is invariant in character for translation as well as linear magnification combined with reflexion in a straight line through the origin.* For, a translation in which z, z_r are changed to $z + \gamma$, $z_r + \gamma$ respectively leaves (1) unaffected, and a linear magnification in the ratio $1 : k$ combined with reflexion in the line $am\, z = \theta$ which changes z, z_r to $k\,\bar{z}\, e^{i2\theta}$, $k\bar{z}_r\, e^{i2\theta}$ respectively also leaves (1) unaffected.

Now any bilinear transformation is geometrically a combination of an inversion, a translation and a linear magnification together with a reflexion. Hence the two italicized statements above, taken together, complete the solution of our problem. If the bilinear transformation is

$$w = \frac{az + b}{cz + d},$$

the phrase "inversion generally" in the first italicized statement is to be understood to mean the exclusion of the case in which the centre of inversion is a centre of force or the point $z = -d/c$ is a centre of force.

As Vaidyanathaswamy has stated, the solution of our problem contains the proof of Bôcher's theorem that, if the zero of the polynomials $F_m(z)$ and $F_n(z)$ are represented by the points $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_n$ in the Argand plane, then the zeros of their Jacobian

$$\begin{vmatrix} F_m(z) & F_n(z) \\ \dfrac{dF_m(z)}{dz} & \dfrac{dF_n(z)}{dz} \end{vmatrix}$$

are the equilibrium positions in the field due to centres of force at each of $A_1, A_2, \ldots, A_m$ attracting according to the law of $(\text{distance})^{-1}$ and centres of force at each of $B_1, B_2, \ldots, B_n$ repelling according to the law of $(\text{distance})^{-1}$.

For, if A_r is the point z_r, and B_r is the point ζ_r, then

$$F_m(z) \equiv C(z - z_1)(z - z_2) \ldots (z - z_m),$$

$$F_n(z) \equiv D(z - \zeta_1)(z - \zeta_2) \ldots (z - \zeta_n),$$

while the Jacobian of $F_m(z)$ and $F_n(z)$ equated to zero is the same as the equation

$$\sum_{r=1}^{m} \frac{1}{z - z_r} - \sum_{r=1}^{n} \frac{1}{z - \zeta_r} = 0,$$

which, interpreted in the light of (1), is Bôcher's theorem.

Concurrent θ-normals planes and generalized Brocard points

By A. M. GNANADOSS, *Madras Christian College*

Vaidyanathaswamy's theorem on concurrent θ-normals at any three points of a conic [R. VAIDYANATHASWAMY : 'On the θ-normals of a conic' *Math. Student* 2 (1933), 121-130; C. T. RAJAGOPAL: 'On the intersections of a central conic and its principal hyperbolas' *Math. Gazette,* 35 (1951), 97-104] which was generalized for any plane analytic curve and given a simple proof in a previous note by the author ["Concurrent θ-normals' *Math. Student* 23 (1958), 182] is further generalized in the present note.

DEFINITION. *Given any ordinary point on a three dimensional analytic curve, the plane through the binormal making an angle θ with the rectifying plane is called the θ-normal plane at that point.*

THEOREM 1. *Given four ordinary points on a three dimensional analytic curve, there exist four values of θ for which the θ-normals at the four points are concurrent.*

PROOF. Let the binormals at the four points be a, b, c, d and the rectifying planes α, β, γ, δ. Then the locus of the line of intersection of the planes through a, b making equal angles with α, β respectively is a quadric Q_{ab} passing through a, b. The locus of the point of intersection of the three planes through a, b, c making equal angles with α, β, γ respectively, is the intersection of two quadrics, Q_{ab} and Q_{bc}, which is a twisted cubic C, (excluding the common generator b). The planes through a, b, c, d making equal angles with α, β, γ, δ respectively, meet at one of the four points at which the cubic C meets the quadric Q_{cd} (excluding the two points where C meets c).

NOTE 1. Contrasted with the two dimensional case, in general, the four solutions are non-trivial ; for the planes through a, b, c, d making an angle $\tan^{-1} i$ (or $\tan^{-1}(-i)$) with $\alpha, \beta, \gamma, \delta$ do not, in general, have a common point.

NOTE 2. Theorem 1 may be re-stated thus :

Let a, b, c, d be four lines on four planes α, β, γ, δ *and* α', β', γ', δ' *a set of concurrent planes through a, b, c, d. Then, there are four sets for which*

$$\widehat{\alpha\alpha'} = \widehat{\beta\beta'} = \widehat{\gamma\gamma'} = \widehat{\delta\delta'}.$$

This leads to the analogue of the Brocard points.

THEOREM 2. *Given a skew-quadrilateral ABCD, there exist two sets of four points* Ω_i ; Ω'_i $(i = 1, 2, 3, 4)$ *such that the angles between pairs of planes BCD,* $BC\Omega_i$ $(i = 1, 2, 3, 4)$ *are equal and the angles between the pairs of planes BCD,* $\Omega_i'CD$ $((i = 1, 2, 3, 4)$ *are equal.*

The theorem of Note 2 may further be generalized into

THEOREM 3. *Let a, b, c, ... be* $(n+1)$ *general secunda on* $(n+1)$ *general primes*[†] α, β, γ, ... *in a space of n dimensions ; and* α', β', γ',... *a set of* $(n+1)$ *concurrent primes through a, b, c, ... Then there are* $(n+1)$ *such sets for which*

$$\widehat{\alpha\alpha'} = \widehat{\beta\beta'} = \widehat{\gamma\gamma'} = \dots$$

† A prime is the locus represented by a first degree equation and a secundum is the intersection of two primes. (e.g. see SEMPLE and ROTH : ' Algebraic Geometry ', Chap. 1)

CLASSROOM NOTES

A new proof for a theorem of classical dynamics

By S. A. NAIMPALLY, *Indian Institute of Technology, Kharagpur*

RAMSEY [1, p. 116] has proved the following theorem.

Principal axes at any point P are the normals to the quadrics passing through P, confocal with the ellipsoid of gyration at the centre of gravity of the system.

The following is an alternate proof.

Let A, B, C, be the principal moments of inertia at the centre of gravity of a mass distribution, whose total mass is unity. Let P have co-ordinates (a, b, c) with respect to principal axes and let $\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$ where $l^2+m^2+n^2=1$ be any straight line through P. The moment of inertia of the system about the parallel straight line through the origin is $Al^2+Bm^2+Cn^2$. Since the distance of the line from the origin is $[a^2+b^2+c^2-(al+bm+cn)^2]$, by the theorem of parallel axes, the moment of inertia of the system about the line is

$$[a^2+b^2+c^2-(al+bm+cn)^2+Al^2+Bm^2+Cn^2],$$

i.e.
$$a^2+b^2+c^2-\lambda,$$

where
$$\lambda=[(al+bm+cn)^2-(Al^2+Bm^2+Cn^2)].$$

To find the principal moments of inertia at P, we have to find the extremum values of I, i.e. of λ subject to the condition $l^2+m^2+n^2=1$. Using Lagrange's multiplier K, this amounts to solving the following equations

$$a(al+bm+cn)-Al+Kl=0,$$
$$b(al+bm+cn)-Bm+Km=0,$$
$$c(al+bm+cn)-Cn+Kn=0.$$

It follows that $K = -\lambda$. Therefore, the equations reduce to

$$a(al + bm + cn) = 1\ (A + \lambda) \text{ etc.}$$

Hence, the direction cosines l, m, n are proportional to

$$\frac{a}{A+\lambda},\ \frac{b}{B+\lambda},\ \frac{c}{C+\lambda},$$

where λ satisfies the equation

$$\frac{a^2}{A+\lambda} + \frac{b^2}{B+\lambda} + \frac{c^2}{C+\lambda} = 1.$$

This shows that the principal axes are normals to the quadrics

$$\frac{x^2}{A+\lambda} + \frac{y^2}{B+\lambda} + \frac{z^2}{C+\lambda} = 1$$

which are confocal to the ellipsoid of gyration

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 1$$

and pass through P.

The principal moments of inertia at P are $OP^2 - \lambda_r$ ([$r = 1, 2, 3$], where (λ_r) are the roots of the equation

$$\frac{a^2}{A+\lambda} + \frac{b^2}{B+\lambda} + \frac{c^2}{C+\lambda} = 1$$

i.e. λ_r are the parameters of the three confocals passing through P.

REFERENCE

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NOTES AND DISCUSSIONS

By R. VENKATACHALAM IYER

In connection with "Class-Room Notes" (*Mathematics Student* 1959—page 57) the following may be of interest.

The problem has been treated by Victor Thébault in his book, *Les Recreations Mathematiques,* Gautheir Villars, Paris, (1952.) 162-164.

Let $(x)_r$ stand for the number

$$x,\ x,\ x,\ \ldots,\ x \text{ repeated } r \text{ times.}$$

For all bases of numeration B, let

$$N = 1\ 0\ \overline{B-1}_r\ \ \overline{B-2}\ \ \overline{B-1},$$

Then, $\quad KN = K\ \overline{K-1}\ \ \overline{B-1}_r\ \ \overline{B-K-1}\ \ \overline{B-K}$, where $K < B$

and, $N(B-K) = \overline{B-K}\ \ \overline{B-K-1}\ \ \overline{B-1}_r\ \ \overline{K-1}\ K.$

$$\text{Hence, } \frac{B-K}{K} = \frac{\overline{B-K}\ \ \overline{B-K-1}\ \ \overline{B-1}_r\ \ \overline{K-1}\ \ K}{K\ \ \overline{K-1}\ \ \overline{B-1}_r\ \ \overline{B-K-1}\ \ \overline{B-K}}$$

Examples

(1) When $B = 10$, with $K = 1$ and 2, we have,

$$9 = \frac{989_r\,01}{109_r\,89} \text{ and } 4 = \frac{879_r\,12}{219_r\,78}.$$

(2) When $B = 12$, with $K = 1, 2, 3, 4$, we have

$$11 = \frac{\overline{11}\ \ \overline{10}\ \ \overline{11}_r\ \ 0\ \ 1}{1\ \ 0\ \ \overline{11}_r\ \ \overline{10}\ \ \overline{11}},\quad 5 = \frac{\overline{10}\ \ 9\ \ \overline{11}_r\ \ 1\ \ 2}{2\ \ 1\ \ \overline{11}_r\ \ 9\ \ \overline{10}},$$

$$3 = \frac{9\ \ 8\ \ \overline{11}_r\ 23}{3\ \ 2\ \ \overline{11}_r\ 89},\quad 2 = \frac{8\ \ 7\ \ \overline{11}_r\ \ \ 34}{4\ \ 3\ \ \overline{11}_r\ \ \ 78}.$$

(3) In the Sexagesimal Scale $B = 60$, with $K = 1, 2, 3, 4, 5, 6, 10, 12, 15, 20$, we obtain 10 integers $\overline{B-K}/K$, namely 59, 19, 14, 11, 9, 5, 4, 3 and 2 which can be expressed as fractions as envisaged in the problem.

BOOK REVIEWS

Differential Geometry, By A. V. Pogorelov, Translated from the first Russion edition by Leo F. Boron, Groningen, P. Noordhoff, N. V. (1959), pp. ix + 171, $ 3.90.

THE basic material of this book is formed by the lectures delivered by the author on Differential Geometry in the Physics-Mathematics Department of the Kharkov State University in Russia. The book consists of two parts. Part I deals with the 'Theory of Curves' in three chapters entitled 'The Concept of Curve;' 'Concepts for curves which are related to the concept of contact' and 'Fundamental concepts for curves which are related to the concepts of curvature and torsion.' Part II deals with the 'Theory of Surfaces' in six chapters entitled 'Concept of surface;' 'Fundamental concepts for surfaces which are related to the concept of contact;' 'First quadratic form of a surface and concepts related to it;' 'Second quadratic form of a surface and questions about surface theory related to it;' 'Fundamental equations of the theory of surfaces' and 'Intrinsic geometry of surfaces.'

The book contains a rigorous discussion of the fundamentals of differential geometry and of the methods of investigation which are characteristic of this branch of mathematics, without disturbing well-established tradition in the process. A large amount of factual material concerning differential geometry has been relegated to problems and theorems at the end of each chapter, the solutions of which ought to be considered obligatory for serious students of geometry.

The book is written in good style and contains neat figures as illustrations. It will be a welcome addition to any public or private library and will be found quite useful both by teachers and students of Differential Geometry.

RAM BEHARI

Problems in Euclidean Space—Application of Convexity, By H. G. Esgleston: Pergamon Press, London (1957) viii + 165.

THIS book constitutes the Adams Prize Essay of the University of Cambridge, 1955-56; and treats 10 problems in finite-dimentional Euclidean spaces.

PROBLEM 1 characterises those sub-sets of the plane (or of any finite-dimentional Euclidean space) that are intersections of a decreasing sequence of connected open sets. Its interest is increased by the fact that such sets are precisely the sets of points which are infinitely repeated values of a suitable meromorphic function defined in the unit circle $|Z| < 1$.

PROBLEM 2 (suggested by Ulam) shows that every homeomorphism of the plane on to itself can be approximated arbitrarily closely by the product of homeomorphisms each of which leaves invariant every horizantal (or vertical) line. This result is true if approximation is taken in the point-wise Topology ($T_n \to T$ means $T_nP \to TP$ for every P), but not in the uniform Topology. But the analogous result in the case of a closed square (in which case naturally we restrict attention to those homeomorphisms which leave each boundary point invariant) is true also in the uniform Topology.

PROBLEM 3 derives some bounds for the ratio $\mu(X)/\Lambda(X)$ associated with a plane set where $\mu(X)$ is the linear measure (in the sense of Hausdorff) of the set X, and $\Lambda(X)$ is the lower bound of the linear measures of the projections of X in various lines in the plane.

PROBLEM 4 provides a positive answer in case $n = 3$ to Borsuk's conjecture that every point-set in n-dimensional Euclidean space whose diameter is D can be covered by at most $n + 1$ sets each of diameter less than D.

PROBLEM 5 shows that a convex set in the plane can be approximated equally closely by (i) convex polygons P_n with n or less vertices and by (ii) such of these polygons P_n which are contained in the given set, the measure of approximation of two convex sets

X, Y being taken the difference between the perimeters of $X \cup Y$ and $X \cap Y$. It is also shown that $T_A(X, n)$ is a convex function of n where $T_A(X, n)$ is the lower bound of the measures of possible approximation of the convex set by polygons with not more than n vertices—the measure now being the area—difference of $X \cup Y$ and $X \cap Y$. The convexity of the analogous function $T_p(X, n)$ for the earlier perimeter-difference case is left unsettled, while that of all other analogous functions introduced has been settled.

PROBLEM 6 characterises the triangle as the unique convex set with the property that through every point in the interior of the region bounded by the curve, there pass precisely three lines each meeting the curve in a pair of points with parallel support lines; and discusses many allied results.

PROBLEM 7 establishes a conjecture of Besicovitch, that the convex body bounded by three equal circular arcs cutting each other at 120° is in a sense the most asymmetric of a certain class of convex curves.

PROBLEM 8 shows that every plane convex set of minimal width contains a convex set of constant width where $c = 3 - \sqrt{3}$ but that the result is false for any larger value of c.

PROBLEM 9 gives a number of "best possible" inequalities concerning the geometric elements of a convex plane set and a circumscribing triangle. Every convex plane set of perimeter l can be circumscribed by a triangle of minimum width $\leqslant l/\sqrt{3}$; and every such convex plane set of area A can be circumscribed by a triangle of area $\leqslant 2A$.

PROBLEM 10 solves a problem of Besicovitch on the packing of an equilateral triangle, by a sequence of inscribed equilateral triangles with sides parallel to those of the given triangle, but "with vertices hanging downwards" and shows that the dimension of the complement of the parts so covered must be $\geqslant \log 3 / \log 2$ and that in that dimension, it must have measure $\geqslant 1/3$. The equalities are attained for a suitable packing.

The book constitutes fascinating though not easy reading ; and provides a host of open problems for the interested worker with ingenuity and skill.

M. VENKATARAMAN

General Degree Applied Mathematics By S. L. Green and J. E. C. Gliddon. University Press Limited, London. 1959, pp. 346, price 18 Sh.

This book provides an introductory course in applied mathematics. There are eleven chapters. The first two chapters deal with algebraic properties of vectors, differentiation of vectors, gradient, divergence and curl. There is a discussion of Gauss theorem, Stokes theorem, Green's theorem and curvilinear coordinates. Chapter three is concerned with the principle of virtual work for a system of particles and stability of equilibrium. The fourth and fifth chapters deal with bending of beams under various conditions, flexible strings and chains. Generalized coordinates are introduced in chapter six, together with Lagrange's equations of motion for a dynamical system and the normal modes of the system. There is a general discussion of the transverse vibrations of a string about a position of equilibrium in the next chapter.

The chapters eight and nine deal with a number of topics in hydrodynamics : the equation of continuity, Euler's equations of motion, Bernoulli's equation, circulation, sources, sinks, doublets, images and the motion of a sphere in a liquid. There is a discussion of motion of a liquid in two dimensions, followed by the circle theorem and the theorem of Blasius. Chapter ten deals with the theory of gravitational attractions and potentials, and chapter eleven with simple theorems and problems in electrostatics and magnetostatics.

The authors have successfully dealt with a large number of topics in Applied mathematics and as such this book should prove useful to undergraduate students studying for Mathematics Honours.

F. C. AULUCK

REPORT OF THE TWENTYFIFTH CONFERENCE OF THE INDIAN MATHEMATICAL SOCIETY

THE INDIAN MATHEMATICAL SOCIETY

TWENTY-FIFTH CONFERENCE, ALLAHABAD, DECEMBER 25-27, 1959

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On the Ground : S. N. SRIVASTAVA, S. N. BHARGAVA, G. D. DIXIT, NAWAL KISHORE, F. M. KHAN, Miss JALAJ KUMARI, T. SINGH, Miss K. D. PENDHARKAR, Mrs. P. L. BHATNAGAR, Miss E. P. ZACHARIAH, Miss SHASHI GOEL, Mrs. N. PRAKASH, Mrs. S. D. CHOPRA, Miss BIMLA GUPTA, Miss RATNA, Miss G. K. RAJESWARI, Miss K. SAVITRI, Miss G. KHANNA, Miss PRAMILA SRIVASTAVA, Miss SULAXANA KUMARI, Mrs. S. L. NIGAM, Miss SARALA SHARMA, S. R. SINHA, S. N. BHATT.

Sitting : K. R. GUNJIKAR, M. VENKATARAMAN, G. L. CHANDRATREYA, R. P. BAMBAH, P. L. BHATNAGAR, RAM BEHARI, C. ORLOFF, B. N. PRASAD (*Local Secretary*), B. S. MADHAVA RAO (*President*), JAWAHARLAL NEHRU (*Prime Minister*), SHRI RANJAN (*Vice-Chancellor*), S. MAHADEVAN (*Secretary*), W. HAHN, D. S. KOTHARI, R. S. VARMA, H. GUPTA, F. C. AULUCK, D. M. PATEL, R. S. MISHRA, U. N. SINGH.

Standing 1st Row : S. K. LAKSHMAN RAO, S. SWAMINATHAN, T. N. SINHA, M. PARAMESWARA IYER, M. R. PARAMESWARAN, G. LAL, N. SANKARAN, B. N. TAGORE, R. K. MISHRA, O. P. GUPTA, LAKSHMIKANT, C. N. SRINIVASIENGAR, P. C. CONSUL, A. CHAUDHURY, K. CHANDRASHEKAR, H. G. S. SHARMA, M. M. LAL, A. MUKERJEE, J. M. GANDHI, J. DUTTA, O. A. SIDDIQI, M. N. BHATT, K. R. CHAUDHURY, B. S. GREWAL, R. P. SRIVASTAVA, K. M. GARG, V. D. THAWANI, T. PATI, S. M. A. KAZIM RIZVI, R. P. AGRAWAL, S. D. CHOPRA, V. N. SRIVASTAVA, R. S. GUPTA, R. GUPTA, K. N. SRIVASTAVA.

Standing 2nd Row : G. SANKARANARAYANAN, B. R. BHONSLE, A. G. LELE, G. B. KHANWALKAR, J. MEDHI, R. S. KULKURNI, G. K. PATWARDHAN, N. D. GUPTA, PRAKASH CHAND, S. C. MALIK, P. C. JAIN, L. RADHAKRISHNA, L. MASOOD, B. D. AGRAWAL, T. N. SRIVASTAVA, A. C. SHAMIHOKE, SATYA PRAKASH, B. B. MEHRA, MOHAN LAL, KANTI SWARUP, G. C. NIWAS, K. B. SHAH, N. K. MEHTA, G. BANDOPADYAYA, P. C. VAIDYA, V. PICHAIRAJAN, D. N. MISRA, A. M. VAIDYA.

Standing 3rd Row : C. S. RAGHAVAN, BRIJ MOHAN, S. N. KAWALGIKAR, N. D. GAUTAM, B. R. LUTHRA, R. C. GUPTA, R. K. JAIN, B. B. CHAKRABORTY, H. C. SAXENA, B. S. JAIN, R. K. JHA, G. L. SARAN, AFZAL AHMED, D. R. KAPREKAR, SAHIB SINGH, K. M. SHAH, K. K. MATHUR, V. N. DIXIT, GURUPRASAD SINGH, SHANTI NARAYAN, A. SHARMA, K. D. BHATTARAI, K. B. LAL, P. SUBBA RAO, K. K. GOSOWN, M. H. OBERAI, Z. U. AHMED, V. B. BUCH, B. CHINNARAJ, R. SRINIVASAN, N. YEGNANARAYANAN, G. V. KRISHNA RAO.

Standing 4th Row: D. N. VERMA, M. S. SUBRAMANYAM, B. VISWANATHAN, RAMANAND, V. K. HANDA, G. C. GOEL, V. K. GANGAL, S. H. DWIVEDI, S. N. AGRAWALA, D. N. HUDDAR, K. G. MITTAL, W. H. ABDI, A. N. MEHRA, J. B. L. SRIVASTAVA, HARI SHANKAR, JAGANATH PRASAD, S. M. MAZHAR, H. P. DIKSHIT, H. N. SHUKLA, K. N. PANDE, A. P. BIUR, S. P. PANDEY, SHRI KRISHNA SINGH, K. N. GUPTA, V. G. RAI, S. N. MAHAPATRA.

PROGRAMME

December 25, 1959

11-00 A. M.—Inauguration (Senate Hall)

12-45 P. M.—Photograph (Senate Hall Campus)

Interval For Lunch

2-30 P. M.—Symposium (1) :

Partition Functions and Statistical Mechanics (Physics Lecture Theatre)

4-00 P. M.—At Home (Muir College Quadrangle)

4-30 P. M.—Opening of the Mathematical Exhibition (J. K. Institute)

5-45 P. M.—Popular Lecture by Prof. D. S. Kothari

6-45 P. M.—Invited one-hour address by Prof. C. Orloff (Senate Hall)

8-00 P. M.—Dinner

9-00 P. M.—Entertainment Programme (Dramatic Hall)

December 26, 1959

7-30 A. M.—Breakfast

8-30 A. M.—Meeting of the Council (Vizianagaram Hall)

9-15 A. M.—Reading of Papers (Physics Lecture Theatre)

11-00 A. M.—Coffee Break (Maths. Deptt. Verandah)

11-15 A. M.—Reading of Papers (Physics Lecture Theatre)

12-30 P. M.—Invited half-an-hour address by Dr. T. Pati (Physics Lecture Theatre)

Interval For Lunch

2-30 P. M.—Symposium (2) :
Generalized Functions (Physics Lecture Theatre)

3-45 P. M.—Meeting of the General Body (Physics Lecture Theatre)

4-30 P. M.—At Home (Muir Collge Quardangle)

5-45 P. M.—Invited half-an-hour address by Dr. S. D. Chopra

6-15 P. M.—Invited half-an-hour address by Prof. W. Hahn

8-00 P. M.—Dinner

9-00 P. M.—Entertainment Programme (Dramatic Hall)

December 27, 1959

7-30 A. M.—Breakfast

9-30 A. M.—Reading of Papers (Physics Lecture Theatre)

10-00 A. M.—Invited one-hour address by Prof. E. Saibel (Physics Lecture Theatre)

11-00 A. M.—Coffee Break (Maths. Deptt. Verandah)

11-15 A. M.—Symposium (3) :
Non-Newtonian and visco-elastic media (Physics Lecture Theature)

Interval For Lunch

3-30 P. M.—Local Excursion and at Home (Swadeshi Cotton Mills, Naini)

7-30 P. M.—Banquet (Senate Hall)

REPORT OF THE CONFERENCE

THE twenty-fifth conference of the Indian Mathematical Society met at Allahabad on December, 25-27, 1959, under the auspices of the University of Allahabad. Shri Jawahar Lal Nehru, Prime Minister of India, inaugurated the Conference and Professor B. S. Madhava Rao, President of the Society, presided over all the sessions.

The inauguration session on December 25, 1959, started with the welcome address by Dr. Shri Ranjan, Vice-Chancellor, Allahabad University and Chairman of the Reception Committee. The Prime Minister of India, then, formally inaugurated the session. After the inauguration, the President delivered his address (printed elsewhere). This session ended after a report by the Secretary on the activities of the Society and the vote of thanks by the local secretary.

The Mathematical programme of the Conference consisted of presentation of papers, invited addresses, symposia and a popular talk. The invited addresses were delivered by Prof. C. Orloff, Prof. E. Saibel, Prof. W. Hahn, Dr. T. Pati and Dr. S. D. Chopra. The symposium on Partitions and Statistical Mechanics was organised by Prof. F. C. Auluck, that on Generalized Functions by Prof. U. N. Singh and the one on " Non-Newtonian and Visco-elastic media " by Professor B. R. Seth. Prof. D. S. Kothari gave a popular talk on atomic explosions. There was also a Mathematical Exhibition.

NARASINGA RAO GOLD MEDAL

The Narasinga Rao medal for the year 1959 was awarded to Shri M. R. Parmeswaran.

SECRETARY'S REPORT

By Prof. S. MAHADEVAN

MR. PRIME MINISTER, MR. VICE-CHANCELLOR, LADIES AND GENTLEMEN,

On behalf of the Indian Mathematical Society, I have great pleasure in welcoming you all to this historic place. Although we have been meeting in various university centres, this is the first time we meet here. This has been made possible by the kind invitation extended to us by the Vice-Chancellor and by Prof. B. N. Prasad, Head of the Department of Mathematics. I wish to thank them for this. We are specially grateful to you, Sir, for having found time amidst your multifarious duties, to inaugurate this conference.

Before I proceed I wish to refer to the death of one of our members Sri N. Lakshmanamurti who retired recently from the Andhra University. I wish to convey our condolences to the bereaved family.

This society was founded in 1907 for the express purpose of encouraging research in mathematics, the basic science and is the oldest all-India scientific body in India. Recently we celebrated the Golden Jubilee in Poona which had been our headquarters for nearly 40 years.

Our activities are threefold : maintaining a good Library, convening of conferences and running of research periodicals. The Society began its activities by organizing a library containing books and a number of foreign periodicals. These were circulated to members to make them familiar with the trends in research. The first Librarian was Prof. R. P. Paranjpye of Poona. We have been developing the library with whatever money we could spare. Now we have about 1000 books, 4000 bound volumes of periodicals and an almost complete run of foreign journals. The library is now housed in the Ramanujan Institute of Mathematics to be of help to the research students working there and is looked after by the Director, Professor C. T. Rajagopal. Our ambition is to make this an up-to-date library with all the latest and important books and all the mathematical periodicals. This cannot be achieved with our meagre resources and I appeal to the Government of India to give us a special grant for this purpose.

It was only in 1916 that we held our first Conference in Madras with diffidence. After its success we were emboldened to meet once every two years in important centres. From 1950 we have been meeting annually. These Conferences draw large people, and workers in different fields discuss their problems with co-workers and hold group discussions. Besides the reading of technical papers, we have invited addresses from eminent mathematicians, Symposia on selected topics and popular lectures to make mathematics familiar to lay people.

Coming to our periodicals, we started the Journal in 1909 under the editorship of M. T. Naraniengar of Bangalore. It is a pleasure to recall that Ramanujan's papers appeared in the Journal from 1911 and his first paper 'On Bernoulli's numbers' attracted great attention then. After the Silver Jubilee in 1932, the Journal was bifurcated and a new one styled the Mathematics Student was started. The Mathematics Student contains short papers, mathematical notes, Questions and solutions, Reviews etc., and is the official organ of the Society; and this is of great use to young research workers and teachers in Colleges. Both these are now under the editorship of Professor R. P. Bambah of Panjab and Prof. P. L. Bhatnagar of Bangalore. To commemorate the Golden Jubilee we are publishing a Jubilee Volume containing about 700 pages of invited papers from different mathematicians of the world and will be the cross section of the latest trends of research in different fields. We request the Government of India to help us by a special grant for this purpose. The work is in the press and it is hoped that it will be ready by the middle of next year. This Society is awarding annually a gold medal for the best paper appearing in our Journals. This year we are awarding it to Sri M. R. Parameswaran of the Ramanajun Institute for his papers in the Journal.

The history of the Society is closely linked up with the history of research in mathematics, the basic science. Our Society is trying its utmost to foster this research. Some of the Universities are developing mathematical research to a limited extent. But in order that research should flourish, it should be undertaken by institutes specially devoted to mathematics alone. There is the Tata Institute of Fundamental Research which is doing good work and which has been recognized by the Government of India as the national centre for research. There is also the Ramanujan Institute of Mathematics in Madras which is also doing good work with its limited resources. I appeal to the Government of India which is now maintaining this Institute to develop it fully by increasing its contribution, appointing adequate teaching staff and by increasing the number of research scholars. The Society has requested the Government to open two more research institutes one in Delhi and the other in Calcutta. It is a pity that research in applied science is neglected and research scholars for this specific purposes should be allowed to work in various Universities and Institutes. In this connection I wish to thank the University Grants Commission for having appointed a Committee to review the mathematics syllabus and teaching in all the Universities. We hope that the efforts of the Committee will go a long way towards increasing the output of research in Indian Universities.

Our financial position is not to be envied. There have been no endowments or foundations as we find in other countries. We live mostly on the dues from members and subscribers and it is very difficult to balance the budget. Some Universities like Madras, Osmania and Bombay, the National Institute of Sciences of India and the Government of India are giving us annual grants. But these are not sufficient and we request the Government to increase their aid so that we may live above want.

We are unable to fulfil many of the objectives we have in view owing to lack of funds. We want to found studentships to deserving students for research in special fields, to start summer schools in different centres to encourage and stimulate research, to invite foreign mathematicians of repute to lecture to us, to institute prizes and medals, to organize seminars and group discussions as frequently as possible in different places, to print monographs on select topics and to organize conferences at least twice a year. All these can be done when we have large resources and only the Government of India should come to our help in this great national enterprise.

PRESIDENTIAL ADDRESS

By B. S. MADHAVA RAO

1. INTRODUCTION

ONE OF the most remarkable features of modern scientific development is the continued and increasing use of diverse mathematical methods for the formulation of existing and new problems of the natural sciences, specially physics. Attempts inspired by such formulation have often created whole new mathematical entities thus enriching mathematics itself. These creations along with other free mathematical creations, sometimes prophetically anticipating the actual patterns of physical relations, and in addition the already existing mathematical disciplines have so widened the domain of mathematics as to give rise to the fear that the organic unity of the subject itself may be jeopardised. But I think there are no grounds for such a fear since the vital force of mathematics is the premise of the indissolubility of its parts based on secure logical foundations.

With this notion of the universality and organic unity of mathematics clearly in mind, I wish to subdivide the subject in the usual manner into its four constituent parts, arithmetic, algebra, analysis and geometry for the specific purpose of comparing the roles they have played in several branches of theoretical physics. For this purpose, I shall divide the problems of the latter also into the four types of interactions so far observed in Nature, viz. the nuclear interactions which are short range forces in nucleii between protons and neutrons through the intermediary of pions ; secondly, the electromagnetic interactions typified by forces between charged particles and radiation of light by atoms; thirdly, the recently discovered weak decay interactions typified, for example, by the β-decay of the neutron into a proton, a negative electron, and an anti-neutrino; and fourthly the gravitational interaction which is dominant for large bodies on the astronomical scale. Estimates of the strengths of these interactions have been obtained, and taking the strength of the first as unity, the strengths of the others are respectively in the ratios $10^{-2} : 10^{-14} : 10^{-39}$.

The first part of mathematics, viz. arithmetic, specially the higher arithmetic including the theory of numbers has perhaps not had much application in physics unless, of course, we consider under the category of arithmetic the remarkable contributions of John von Neumann to the theory of automata including large scale computing machines of various types used for theoretical experimentation of the problems of mathematical physics. Such a consideration, however, appears inappropriate since these contributions have been motivated by symbolic logic including chiefly Gödel-type questions relating to the Turing machine theory and decision procedures. For the rest, analysis has of course been the backbone of theoretical physics, and has played a fundamental role in the development, up to a certain stage, of the understanding of all the four types of interactions mentioned above. It is a powerful tool of multipurpose type, and was in the earlier years almost the supreme mathematical technique which could enable the understanding of the several discoveries of the physical sciences, in particular those related to the second type of interactions, but the revolutions that have taken place in the physics of the twentieth century have however shown that classical analysis by itself is inadequate to obtain an insight into these new complex phenomena, and it is necessary to consider mathematics stemming from set theory, and specially modern algebra. Thus, algebra so successfully applied to the development of the theory of elementary particles culminating in the notion of parity non-conservation, may well be considered to have been singularly useful in achieving an understanding of the third type of interaction. Equally so, it appears likely that a similar role will be played by geometry in future in relation to the fourth type of interactions. This does not imply that geometry has so far not played any role in general relativity. In fact, classical general relativity which deals with gravitation is itself a pure geometrical theory. The suggestion made is that future attempts to understand the fourth type of interaction in relation to the other three, by employing the techniques of quantum mechanics will call for the use of complex geometrical notions, perhaps including those from the domain of topology.

2. Role of geometry in other branches of physics

Talking of geometry, it is perhaps not out of place to mention briefly the role that it has played in other branches of physics. The treatment of classical dynamics in the Hamiltonian form is of a purely geometrical nature introducing generalized Euclidean spaces like phase space, configuration space and momentum space, wherein the notions of general and infinitesimal contact transformations specify the equations of motion of the dynamical system. Recent workers like J. A. Wheeler have gone even further, and attempted to set up the whole of classical physics as pure geometry. Equally so in statistical mechanics, such geometrical notions have played an important role. But most striking of all is the introduction of the geometry of an infinite-dimensional unitary space, i.e. the Hilbert space as an adequate basis for the physical considerations of quantum theory. This fulfils a part of Hilbert's program of axiomatisation by setting up an isomorphism between a physical theory, and the corresponding mathematical system. This geometrical formulation of quantum mechanics whereby states of the physical system are described as vectors in Hilbert space, and measurable quantities by Hermitian operators acting upon these vectors, has unified the matrix and wave aspects of quantum mechanics, and placed the earlier Dirac-Jordan transformation theory attempting this unification on a proper axiomatic basis. The essentials of this geometric treatment have survived the two great extensions which quantum theory has undergone, viz. the relativistic quantum mechanics, and the quantum theory of fields. Specially valuable has been the fundamental work of J. von Neumann on different type of Hilbert spaces, in particular the separable type, on different types of linear operators in Hilbert space like the bounded and unbounded Hermitian ones, and the all important hypermaximal symmetric one, and on linear manifolds in Hilbert space corresponding to continuous geometries and geometries without points. This very important work has thrown light on Dirac's δ-function, on canonical commutation rules, and above all on a proper understanding of the statistical aspects of quantum theory.

3. General relativity in the older physics

The general theory of relativity has long occupied a position of isolation with respect to the rest of contemporary physics and despite the elegance of its concepts has not exhibited any real relation whatever to quantum physics. On the other hand, the special theory of relativity has been intimately amalgamated with quantum physics, and this fusion is the basis on which the theory of elementary particles has been built. Almost all the successes of general relativity have been on the cosmic scale as typified by the three crucial tests of the theory, viz. the advance of the perihelion of the planet Mercury, the deflection of light in the gravitational field of the Sun, and the gravitational shift of spectral lines. As regards these three tests, opinion has recently been expressed that they do not really lend strong support to the general theory of relativity in as much as the first two tests also result from several other theories differing from Einstein's in major respects, and the third could be explained on the basis of elementary considerations of energy conservation involving photons. Thus, in the usual notation, the red shift is given by $\frac{\Delta\lambda}{\lambda} = \frac{GM}{rc^2}$, and this can be derived if one assumes that the photon has mass given by $m = \frac{h}{\lambda c^2}$, and also that the potential V is the ordinary Newtonian gravitational potential given by $V = \frac{GM}{r}$ for, the reddening of the spectral lines can then be viewed as the loss of energy by photons as they leave the region of high gravitational potential. General relativity has however had the most fruitful and stimulating effect on the field of cosmology which is now growing very rapidly due to the recent extra-ordinary achievements of astronomy. In particular, a proper use of the principles of general relativity, taking into account the more than doubling of the distance scale of distant nebulae, has satisfactorily disposed of the discrepancy which existed for a long time in the value of the Hubble constant leading to the disquieting conclusion that the age of the Universe appeared smaller than the age of the earth's crust as determined from radioactive measurements. Outside of cosmology the impact of general relativity on the rest of physics has not been so great as that of special relativity. This is perhaps a consequence of the extreme weakness of the gravitational interaction in comparison with the other three types of interactions; in fact, even taking the order of the weak β-decay interaction as unity, the gravitational interaction is of the order 10^{-25} ! The smallness of this interaction does not however prove that considerations of general relativity will be unimportant for microscopic phenomena. It is hard to believe that the principle of general covariance that the laws of physics have the same form in all co-ordinate systems, which is the fundamental postulate of general relativity, has no counterpart in microphysics. Einstein felt that on the side of the quantum physicists the importance of the claim of general relativity in the search for the laws of the micro-world was usually underestimated, but his own standpoint that the quantal description of nature was essentially incomplete was perhaps responsible at that time for this underestimation.

4. General relativity in modern physics

In recent years, there have been indications that general relativity is beginning to assume a greater importance in contemporary physics, and is destined to play a far more significant role therein than it has done so far. There are two important lines of indication which tend to confirm this conclusion. The first one has its origin in recent researches in microphysics itself relating to interactions of several types between the host of elementary particles so far discovered. Even earlier, Landau's theoretical work on the quantum theory of fields, specially quantum electrodynamics using a "smoothing out" process for point particles introducing a fundamental length of the order of 10^{-13} cm., had shown that a crisis in the notion of an electric charge appears in the range of high energies where the gravitational interaction becomes comparable to the electromagnetic. Coming to more recent work on elementary particles, a closer examination of the interactions responsible for the β-decay, based on the classical and the newer parity experiments, has shown that this interaction is dominantly of the (V-A) type, and not of the (S-T) type as originally supposed, and has made the idea of the universal (V-A) Fermi interaction again acceptable. Advances in the theory of the neutrino have resulted in showing the validity of the two-component theory, and of the law of conservation of leptons, and have raised the question as to whether there is any connection between the beta interactions, and the gravitational forces. Again, other experiments recently undertaken to measure the magnetic moment of the muon have shown that this particle is a pure Dirac particle just like the electron with electro-dynamic coupling of the conventional form. Also the weak couplings of electron and muon are identical in form and strength both involving the same neutrino, and as far as is known both the electron and muon lack any other interaction whatever except the gravitational. Here appears the only known difference between these two otherwise identical particles, viz. the difference between their masses as lending support to the idea of the existence of gravitational interactions in the field of elementary particles. For, the alternative plausible explanation that the differences in mass among these particles are always due to differences in interaction does not hold in this case. The neutrino involved in the beta decay has itself been considered as being perhaps responsible for gravitational interactions also, and attempts have been made to construct a rigorous neutrino theory of gravitation. While such an attempt does not appear promising, the interaction of neutrinos with a gravitational field would yield interesting result about neutrinos as well as about gravitation, since this is the only force in which neutrinos are subject to simple analysis. Such interactions have recently been considered by J. A. Wheeler, and many interesting results obtained. In particular, the consideration of the statistical mechanics of an ensemble of neutrinos supports the idea that the gravitational contraction of a star after the stage where thermonuclear energy has been exhausted is due to the emission of neutrinos. The interaction cross-section of neutrinos with matter being of the order 10^{-43} cm^2, a neutrino once created would have a life time of the order of 10^{26} years in our galaxy, and correspondingly longer in intergalactic space. But, as pointed out by Wheeler, the density of matter inside a star in the late stages of gravitational contraction may reach a high value of 10^{38} nucleons/cm^3 and then the mean free path of the neutrino would be of the order of 1 km., so that the opacity of matter makes itself felt even for the neutrino. Other results obtained relate to neutrino pair-creation processes not depending on beta interactions, the contribution of neutrinos to the stress-energy tensor of the gravitational field, the response of neutrinos to gravitational fields, and most interesting of all, the creation of gravitational fields by neutrinos. Mention should finally be made of the recent creation of a new working hypothesis based on the work of the Russian physicist Ivavenko using quantum methods according to which there exist fundamental particles of gravitation called gravitons, or gavitational quanta. Since the quantisation of general relativity has not so far been achieved in a satisfactory manner, it is not clear what exactly is the method of quantisation adopted to derive this particle aspect of gravitational field theory. Perhaps, a simplification of the type suggested by S. N. Gupta of treating Einstein's theory as a theory of gravitation in flat space is used to enable the quantisation being carried through. As Gupta himself has shown, such a process can actually be carried out, and leads to gravitational quanta of vanishing rest mass, and spin 2, and also makes it possible to calculate the interaction of gravitons with other particles in the usual way. I might mention, in this connection, that if the wave equation of the graviton obtained on this theory be put in the Dirac form, the commutation rules satisfied by the corresponding matrices in the Dirac equation are the same as those which I have derived some years back for particles of spin 2. The work of Ivavenko also takes interaction of gravitons with other particles into account, and in particular, shows that two gravitons may interact, and form another pair of elementary particles, for instance, an electron and a positron. In view of the work of Wheeler mentioned above, it would be interesting to examine on the basis of this type of quantum theory, the interactions between neutrinos and gravitons, and the possibility of neutrinos interacting so as to produce gravitons. This Russian work on gravitons has also shown that there exists a similarity between the conceptions of the graviton nature of gravitation and hydrodynamics. It is noteworthy that this similarity exists in Newton's theory of gravitation also, and finds its expression in the analogy which shows the identity of certain equations of hydrodynamics, and Newton's theory of the potential.

The second line of indication showing that general relativity is assuming greater importance in modern physics is provided by development on the macroscopic level based on the recent launching of earth satellites, and also contemporary developments relating to the " maser " or microwave amplification by stimulated emission of radiation. As mentioned earlier, the three tests proposed for general relativity cannot be considered sufficiently conclusive in favour of the theory, and artificial satellites may soon provide ways of such verification, and may suggest other methods of testing the theory by observation. Thus, in addition to the usual perihelion shift of a planet during its revolution round the Sun, general relativity also predicts a further, but a much smaller, shift that is induced by the rotation of the Sun on its axis. This secondary shift is small to be measured even in the case of Mercury. On the other hand, the earth with a much higher rate of rotation has an angular velocity 25 times greater than that of the Sun. As a result of this and the nearness of an artificial satellite, we can look for a rotation effect upon its orbit 5,000 times greater in magnitude. For close artificial satellites the effect may amount to 50 seconds of arc per century, as great as the total relativistic effect for Mercury. Satellites may also help us to obtain precise measurements of the gravitational frequency shift by the use of identical atomic clocks placed in a distant satellite, and on the earth, and comparing the two clock readings. A more ambitious programme would be to test cosmological results derived from general relativity by devising astronomical observations made possible by elaborate equipment assembled by sputniks in interplanetary space. The second development mentioned above regarding masers appears to provide more practicable types of experiments to test general relativity. One such experiment being planned at present aims at providing a direct test for the principle of equivalence. On this principle, the effect of an external gravitational field in a local co-ordinate system can be eliminated completely by letting the co-ordinate system free in that gravitational field ; and such a system is then locally equivalent to a Lorentz frame, so that light rays must travel in the system with the same velocity in all directions. Applying this result to the gravitational field of the Sun, the above statement concerning the local propagation of light leads to the result that the velocity of light in the direction of the line joining the earth and the Sun should be the same as that in a direction perpendicular to it and the earth's radius. Nowadays, maser techniques are approaching an accuracy $\frac{\delta c}{c} \doteq 10^{-12}$, ($c$ = velocity of light) in comparing the two velocities. If a discrepancy is found to this accuracy, it would imply a deviation the principle of equivalence in first order of the dimentionless quantity $\frac{GM}{C^2R} \doteq 9 + 10^{-9}$, where M is the mass of the Sun, G the gravitational constant, and R the distance of the earth from the Sun.

5. Geometry and General Relativity

In order to understand better the relationship between geometry and gravitation, we have to look more closely into the basic principles of general relativity, examine some of the proposed unified field theories, and consider recent attempts made towards the quantisation of covariant field theories. The basic principles of general relativity constituted by (i) general covariance stating that the laws of physics have the same form in all co-ordinate systems, (ii) local validity of special relativity stating that the laws of this theory hold locally in a co-ordinate system with vanishing gravitational field, and (iii) the precise form of the field equations, give to the theory a purely geometric form with the Minkowskian interval given by

$$ds^2 = g_{ik}\, dx^i dx^k, \; (g_{ik} = g_{ki}) \tag{1}$$

the components g_{ik} of the metric tensor being functions of the co-ordinates. Thus (i) makes the Riemann geometry of space-time fundamental for general relativity. The principle of the geodesic line which is a consequence of (ii) enables the derivation of the equations of the world line of a test particle in a purely gravitational field by means of the affine connections Γ^i_{jk} defined in terms of the derivatives of the g_{ik}. The g_{ik} themselves act as the potentials of the gravitational field, and thus appear in their dual role describing both the metrical properties of the geometry of space-time, and the dynamical action of gravity. Using (i), equations of special relativity can be translated into general relativity in an unambiguous way if we assume that no new quantities in addition to the metric tensor are introduced into the system, and that no derivatives of g_{ik} of order higher than the first appear in the general relativistic equations. As regards (iii), it can be shown, by considering the above equations of the world line in the limiting Newtonian case, that the field equations expressed in terms of the energy-momentum tensor of the sources, must be in the form

$$T_{ij} = a_1 R_{ij} + a_2\, g_{ij} R + a_3\, g_{ij}\,, \; (a_1, a_2, a_3 \text{ consts.}), \tag{2}$$

where R_{ij} is the Ricci contracted curvature tensor of the Riemann space, and R the curvature scalar, which again are of a purely geometrical significance. (2) can be further simplified by dropping the last purely cosmological term, and using the fact that the covariant derivative of T_{ij} should vanish in a general co-ordinate system, which follows as a consequence of (ii), and the energy-momentum conservation law in a Lorentz frame. The simplification is then an immediate consequence of the purely geometrical Bianchi identities expressing the vanishing of the covariant derivative of the tensor G_{ij},

where

$$G_{ij} = R_{ij} - \tfrac{1}{2} g_{ij} R \tag{3}$$

and gives the field equations in the form

$$G_{ij} = -KT_{ij} \tag{4}$$

with $K = \dfrac{8\pi\gamma}{c^4}$ (γ = Newtonian gravitational constant) if T_{ij} has the dimension an energy density. The covariance of the field equations yield interesting information, in the case of a pure gravitational field, about continuation properties of the g_{ik}, and one could further derive conservation laws by noting that Einstein's gravitational equations (4) can be derived from a variational principle

$$\delta \int L d^4 x = 0, \tag{5}$$

where $L = |g|^{\frac{1}{2}} R$, g the determinant of the g_{ik} and $|g|$ its absolute value, and the variation is of the g_{ik}. By applying a suitable coordinate transformation one can derive, for example, the conservation laws for energy and momentum in the form

$$\frac{\partial G_i^j}{\partial x^j} = 0, \text{ with } G_i^j = T_i^j + t_i^j,$$

the first term on the right being the energy momentum density of the field producing gravitation, and the second may be considered as the gravitational part of the total energy-momentum density. These considerations are enough to show the geometrical nature of Einstein's general theory of relativity.

The field equations (4) for the case of the pure gravitational field ($T_{ij} = 0$), viz. $G_{ij} = 0$ are quite unambiguously determined by the geometry alone, or in other words, only the left-hand side is determined in (4). But general relativity can virtually accommodate any additional field (with T_{ij} as its energy-momentum tensor), but it provides no method for choosing among the possibilities. For example, if T_i be the electromagnetic stress-energy tensor acting as a source in the gravitational field, we can use the Lagrangian principle (5) by generalising L to $|g|^{\frac{1}{2}} R-Ls$, where Ls the electromagnetic part of the Lagrangian is given, for the case of pure electromagnetism, by

$$L_s = \tfrac{1}{2}\,|g|^{\frac{1}{2}} f_{ik} f^{ik}\,;\, f_{ik} = \frac{\partial \phi_i}{\partial x^k} - \frac{\partial \phi_k}{\partial x^i}, \tag{6}$$

with the usual notation for the electromagnetic fields, and the vector potentials. The variation of the g_{ik} yields the gravitational equations, while the variation of the ϕ_i yields the Maxwell equations, the theory thus resulting being called the Einstein-Maxwell theory. But this theory arising out of the coupling of the electromagnetic field cannot be considered a pure geometric theory, the electromagnetic energy-momentum tensor remaining as an additional non-geometric element. Many attempts have been made towards attaining such a purely geometrical or unified field theory by trying to deduce all physical interactions from one law, and to modify the field equations in such a way that they would admit solutions corresponding to the "particles" of the coupled field. With the advent of quantum-mechanical ideas, such unified theories have had to meet a further restriction, viz. that the sources of the fields corresponding to the "particles" like masses, and electric charges for example, be not accounted for in a classical way, but the statistical interpretation of quantum mechanics be adopted in describing them and their properties. This restriction has led to a further ambiguity of interpreting a unified field theory either as a c-number theory later to be quantised, or as a field theory which implicitly contains the quantum laws already. Also, most of the attempts made so far have confined themselves to the coupling of only gravitational and electromagnetic fields on the assumption that these are the only fundamental ones. But with the advent of quantum theory, and on the basis of more recent work on elementary particles, such an assumption is no longer valid. In fact, as mentioned in the Introduction, there are two other types of interaction fields equally fundamental. While in general relativity the principle of general covariance and the equivalence principle lead directly to the metric structure of space-time, and also the explicit form of the field equations, no such general principles are available for a unified field theory. Thus, to a large extent the work of building unified field theories has been concerned with a mathematical structure, or a geometry based on a space more general than the four-dimensional Riemannian space-time S_4.

6. Unified field theories

We will briefly review some of these attempts at unified field theories. One of the best known is the five-dimensional theory of Kaluza-Klein which is based on the observation that the ten g_{ik}, the four ϕ_i, and the fourteen field equations describing the gravitational and electromagnetic fields in S_4 may be interpreted in terms of a suitable five-dimensional Riemann space S_5. This space is further characterised by the property that in an appropriate co-ordinate system all components of its metric tensor are independent of the fifth co-ordinate x^5. The theory developed on this basis is found merely to provide a re-interpretation of the Einstein-Maxwell theory in quasi-geometrical terms without changing its content in any way. The Kaluza-Klein theory has been slightly modified by Jordan by deriving an additional fifteenth field equation for the gravitational constant which is interpreted as variable in consonance with the idea originally advanced by Dirac. Jordan's theory which includes only the pure electromagnetic field has been shown not to lead to an unambiguous interpretation of g_{ik} as the true metric tensor. Klein has further generalised the original Kaluza-Klein theory in an entirely different way by dropping the condition that the field quantities be independent of the fifth co-ordinate x^5, and developing a truly five-dimensional theory with x^5 having a quantum theoretical significance. Quite in contrast to the theories are the non-symmetric ones of Einstein and Schrödinger operating in S_4 only, and based on Levi-Civita's geometrical concept of parallel displacement of vectors or the affine connection (affinity). Such a parallel displacement of a vector a_i is defined by

$$\delta a^i = -\Gamma^i_{jk}\, a^j\, dx^k \tag{7}$$

and affine geometry developed on this basis remains applicable even if no metric is defined, and the curvature tensor, in particular, can be defined in terms of the Γ's. In such a geometry, the symmetric part of an affinity is an affinity, while the anti-symmetric part is a tensor, and thus Riemannian geometry is a special type of affine geometry in which the affinity is symmetric and the length of a vector remains unchanged by parallel displacement. Eddington was the first to base a unified field theory on affinity instead of on the metric, but he did not postulate specific field equations. The theory was later developed by Einstein by using both a non-symmetric affinity, and a non-symmetric metric tensor as building stones, Schrödinger also developed the theory in a different way by introducing only the Γ's as primary field variables, and obtaining the metric tensor as a derived quantity. Both the theories can be derived from invariant variational principles, and have succeeded in leading to conservation laws, but, in view of the prohibitive mathematical difficulties, it has not been possible to derive much more from the field equations. Thus Einstein's unified field theory can, in a sense, be considered as geometry which is a little too pure, even the physical interpretation being based on geometry rather than on physics. More recently V. Hlavaty has provided such a detailed geometrical background for the physical interpretation. Further, Einstein's theory leads to conceptual difficulties by regarding the variables as classical field variables which are not to be quantised.

7. Wheeler's geometrodynamics

Of quite a different sort is the unified theory very recently developed by J. A. Wheeler and his collaborators, of which many striking applications have been made. This theory which considers the coupling of only the electromagnetic field is set up in a purely geometric way, and can be called a geometry of gravitation and electromagnetism; in fact, Wheeler has called it "Geometrodynamics". This development is based on a remarkable result discovered nearly 35 years ago by Rainich, and recently rediscovered by Wheeler and Misner, that in regions where electromagnetism is the only contributor to the stress-energy tensor and where the electromagnetic field itself is free of sources, one can replace the entire content of the Einstein-Maxwell theory by a theory which is purely geometrical. In this way both the gravitational field and the electromagnetic field are entirely determined by the curvature of space-time. The fundamental equations are given by

$$R_{ij} R^{jk}_{\cdot} = \delta^k_i (R_{mn} R^{mn})/4, \tag{8}$$

$$R = 0, \tag{9}$$

$$\alpha_{n,m} - \alpha_{m,n} = 0, \tag{10}$$

where α_m is defined by

$$\alpha_m \equiv \frac{ig^{-\frac{1}{2}} \epsilon_{mnrs} R^{nj,r} R^s_j}{R_{ik} R^{ik}_{\cdot}}, \tag{11}$$

with the further requirement, in order to ensure that the energy density is positive, that

$$R_{00} > 0. \tag{12}$$

In the above equations, the indices take values 1, 2, 3, 0; the comma denotes covariant differentiation, and ϵ_{mnrs} is a covariant totally anti-symmetric tensor density of weight-1, taking the value + 1 when *mnrs* is an even permutation of 1, 2, 3, 0. It can be shown that the equations (8)-(12) containing solely geometrical elements are entirely equivalent to the usual Einstein-Maxwell theory. The curl condition (10) guarantees the existence of a scalar $\alpha(x)$ the "complexion" of the electromagnetic field, such that for $\alpha = n\pi$, $n = 0, 1, 2 ..,$ one has a pure electric field, and for $\alpha = \frac{1}{2}n\pi$ $(n = 1, 3, 5, ..)$ the field is pure magnetic. The complexion together with the value of the Ricci curvature tensor at a point completely determine the electromagnetic field f_{ij} and hence also the stress-energy tensor. The equations (8), (9) and (12) express a two-way connection between the f_{ij} and R_{ij} in that any electromagnetic field f_{ij} produces a Ricci curvature R_{ij} satisfying these equations, and conversely, given a R_{ij} satisfying these equations we can solve for f_{ij}. Further, it can be shown that the geometrodynamics set up above can also be derived from a variational principle with the aid of a Lagrangian density which is a function of the metric tensor g^{ij}, its derivatives up to the second order, and the path of integration occurring in the definition of the complexion α. A deeper analysis made of the geometrical meaning of the above system of equations, specifically of the two way connection between the f_{ij} and ${}_{a}R_{ij}$, has shown that one has to use topological notions like the topology of fibre bundles in order to explain the configuration of the Ricci vierbein at a point in relation to the local light cone.

A striking application of geometrodynamics has been the creation by Wheeler of new types of "gravitational electromagnetic entities" or "*geons*", which constitute self-consistent solutions of the field equations of geometrodynamics. These geons may be described as arising out of an electromagnetic field of appropriate character and sufficient energy density as can hold itself together for a time long in comparison with the characteristic periods of the field oscillations. In other words, a gravitational field sufficiently strong can guide an electromagnetic wave, and confine its energy to a bounded region of space, thus giving rise to a geon. Alternatively, a geon can be formed by a standing electromagnetic wave holding itself together when its energy is great enough, and has therefore enough mass to provide the guiding gravitational field all by itself. Thus a geon results from a classical field theory for fields of zero rest mass based on purely geometric considerations. Wheeler has shown that the geon can be endowed with a mass, has a characteristic decay rate in the free state, moves through space like a Newtonian entity when subjected to fields that vary sufficiently slowly in space and time, and undergoes transmutation when the fields are much stronger. He has also shown that in order to be subject to pure classical analysis only (without bringing in quantum considerations) geons must have a mass greater than 10^{38} gm., and radius over 10^{11} cm. if they be spherical geons. Obviously, the concept of these purely classical geometrical geons has only a heuristic value, and they appear to have no connection with observational science. Some collaboraters of Wheeler have constructed consistent solutions of the field equations of geometrodynamics corresponding to "linear" and toroidal" geons by considering the cases where the electromagnetic field energy is spread over an infinitely long line, and those in which it is concentrated in a toroidal region of space respectively. In principle, the coupled electromagnetic field of geometrodynamics can be replaced by any other field, for example a neutrino field, or a mixed field containing both the electromagnetic field and the other field introduced. Wheeler has considered, in this direction, the question relating to the comparison of the energy levels of an electron in electrostatic and gravitational fields, and the neutrino and the photon in a gravitational field, and obtained interesting results using the metric of a thin shell spherical geon. Geometrodynamics has been further generalized by replacing the simply connected space S_4 by a multiply connected space with a given topology, and, in particular, such a space with the topology of the torus admits a zero frequency solution of the fields equations corresponding to Maxwell's equations. Comparing this solution with that of the two-component Dirac equation for the neutrino space, Wheeler has deduced another interesting result that there is no reason to expect for the neutrino field a concept of charge having any direct analogy to electric charge. Lastly, mention may be made of the "*wormhole*" picture of elementary particles that results on the introduction of a multiply connected space as indicated above. These wormholes are the "handles" corresponding to the Betti numbers of the associated topologcal 3-space, and to each wormhole can be associated a "charge" which stays constant in time consequent on the Maxwell-type equations regardless of the way in which the metric changes with time as long as the topology does not change. Wheeler arrives at a figure of 10^{60} wormholes in the volume of space filled by a single electron, thereby giving a picture of elementary particles not as simple structures but as some sort of collective modes. This proposal is no doubt a fresh attack on the structure of elementary particles, but it is doubtful if it can be made to yield results capable of experimental verification.

8. Quantisation of General Relativity

Finally, we come to the question of a quantum theory of the gravitational field which is necessary since it is hard to reconcile oneself to the idea that classical and quantum fields can exist side by side. One principal argument in favour of the quantisation of general relativity is that it appears unlikely that a mass point which gives rise to a classical Schwarzschild field is itself subject to Heisenberg's uncertainity relations. Another is the hope expressed by Pauli that quantising the metric tensor may ameliorate the infinities of the other field propagators along the light cone, and that it may make a contribution to the theory of elementary particles. So far, two kinds of approaches have been made towards quantisation, viz. (i) those starting from a Lagrangian or Hamiltonian formulation of the classical theory using the so called true observables whose values are independent of the choice of the co-ordinate system, guage frame, and the like, and (ii) those which attempt a pathintegral quantisation of the Feynman type, either introducing true observables as a prerequisite or working with formal expressions in terms of the original variables, hoping that these expressions will automatically be useful propagators.

Perhaps the most successful attempt in the first direction is the one made by Dirac who, with a sort of prevision, undertook nearly ten years ago the question of formulating the classical theory of gravitation in Hamiltonian form as a preliminary to quantisation. The usual procedure for the transition from the Lagrangian to the Hamiltonian form requires the momenta to be independent functions of the velocities. This condition is not satisfied in the case of the relativistic field theory, and so, Dirac first of all gave an alternative procedure for this transition based on a direct solution of the equations provided by the consistency requirements, and showed that, under certain conditions, one can eliminate some of the degrees of freedom, and so make a substantial simplification of the Hamiltonian formalism. Applying this procedure specifically to Einstein's theory of gravitation, Dirac assumes that the fundamental concept in Hamiltonian theory of the state of a system at a given time is to be interpreted in a relativistic theory as the state on a general three-dimensional space-like surface in space-time. On the basis of this interpretation a simple scheme is adopted by choosing the system of co-ordinates x^m ($m = 0, 1, 2, 3$) such that the surfaces $x^0 =$ const. are all space-like, and using x^1, x^2, x^3 as parameters to label the points on these surfaces. The actual carrying through of the procedure shows that one can make a change in the action density, not affecting the equations of motion, and obtain a Hamiltonian for the gravitational field in interaction with matter, involving only the six g_{rs} ($r, s = 1, 2, 3$) thus obtaining a simplification. Using this simplification into six degrees of freedom for the case of the weak field approximation, one can make a Fourier resolution of the field quantities, and one then gets a clean separation of those degrees of freedom corresponding to the true variables from those corresponding to variables depending on the co-ordinate system used. There are two of the former and four of the latter for each Fourier component, the two former corresponding to gravitational waves with two independent states of polarisation. One of the latter is responsible for the Newtonian attraction between the masses, and also gives a negative gravitational self-energy for each mass. In must be emphasised, however, that sincet hese positive results are obtained by using the scheme of taking the surfaces $x^0 =$ const. as space-like, the adoption of Hamiltonian methods expressed in their simplest form therefore force one to abandon four-dimensional symmetry. This giving up of four-dimensional symmetry may, at first sight, look like an apparent divorce between geometry and gravitation, but a closer examination shows that this entails the setting up of a really more intimate relation between gravitation, and the topological notions associated with the space-like 3-surfaces used to set up the simplified Hamiltonian. In fact, as Dirac has pointed out, only individual solutions of Einstein's gravitational equations exhibit four-dimensional symmetry, while a physical state does not correspond to an individual solution, but to a family of solutions all related to the same Hamilton's principal function. It is such a family that corresponds to a wave function in the quantum theory, while the individual solution has no quantum analogue. Thus, the adoption of Hamiltonian methods, with a view to quantisation, forces the dropping of the idea of four-dimensional symmetry which may not after all be a fundamental property of the physical world, and replacing it by deeper geometrical notions expressing the basic assumption of Dirac's theory that the space-like surface on which the state of the system is defined shall always remain space-like. Still more recently, Dirac has considered the question of attaching a meaning to the energy of a gravitational field based on his Hamiltonian method of approach, but has been able to give only one example where an unambiguous definition of the energy density can be given, viz. the special case where there is no matter present, and the gravitational field consists, to the first order of accuracy, only of waves moving in one direction. The actual quantisation of the Hamiltonian theory of gravitation has not been carried out so far but only indications have been given of how one could pass over to the quantum theory using devices such as weakly vanishing quantities, and modified Poisson brackets. Mention should also be made in this connection of the attempts made by Belinfante and co-workers towards the quantisation of the interacting field of electrons, electro-managetism and gravity. The reason for taking the first two fields in interaction with gravity is that the theory of these fields is representative for the complications due to gauge-invariance as well as those arising from the spinor character and the Fermi-Dirac quantisation of a fermion field. Once these difficulties will have been solved for this representative case, they expect that quantisation of other fields interacting with the gravitational field will not meet with serious obstacles. These workers have also tried to modify the two devices mentioned above in connection with the Dirac quantisation, by interpreting weak equations as restrictions on the Hilbert space of the corresponding quantum mechanics, and avoiding the use of modified Poisson brackets by introducing new variables satisfying canonical commutation rules. No positive results have, however, been obtained to show that the quantisation of these interactions has been successfully carried through.

Coming now to the methods using path-integral quantisation, we will first mention Misner's Feynman quantisation of general ralativity, indicating briefly the several steps leading to this quantisation, viz. (i) introduction of a 4-manifold M of points, (ii) use of subsets of these points called hypersurfaces being three-dimensional sub-manifolds of M, (iii) the concept of a metric, (iv) the field history ds^2, (v) notion of field configuration if ds^2 be specified only on a single hypersurface, (vi) state functionals $\psi\sigma$ associated with a hypersurface σ, (vii) selection by using the Schrödinger equation to derive admissable ones called states from among all conceivable families of state functionals, (viii) use of the Feynman principle in place of the Schrödinger equation as a dynamical principle, (ix) the Feynman propagator defined as a functional integration over field histories, (x) subsidiary conditions defining physical states satisfying guage invariance, (xi) topological invariance of general relativity, and (xii) the ideas of the Hamilton, Schrödinger, and Heisenberg pictures to be incorporated into quantised general relativity by analogy. The Feynman method is viewed as a Hyughen's principle by taking the Feynman propagator in the general form

$$\int \exp \{ (i/h) \ (\textit{Einstein action}) \} \ d \ (\textit{field histories}) \tag{13}$$

as suggested by Wheeler. An interesting result obtained is that in any topologically invariant theory the Hamiltonian operator vanishes, and for this conclusion to hold it is necessary that the quantum theory, not just the classical theory, be also topologically invariant. The actual quantisation has not been carried through to completion.

The last case we will consider is the quantisation of geometrodynamics, or quantum geometrodynamics as called by him. Geometrodynamics describes the interacting fields of electromagnetism and gravitation, and a quantisation of the combined theory may perhaps throw more light than an independent quantisation of the two separate fields. As we have seen earlier, geometrodynamics can be derived from a Lagrangian variational principle, and Wheeler has attempted to put it in the Hamiltonian form by using techniques more general than those of Dirac, and to quantise the resulting theory on the basis of the Feynman propagator (13). A positive result obtained is to show the possibility of the existence of gravitational waves.

9. Conclusion

Thus we see that the quantisation of general relativity is still a problem for the future, and will perhaps be solved only when a deeper relation is set up between geometry and gravitation by using topological notions in Hilbert space also. As it is, the fusion of special relativity and quantum mechanics which has been successfully achieved has resulted in the discovery of a bewildering variety of elementary particles. General relativity even without being quantised has given rise, consequent on the topological point of view, to a Universe full of holes, loops, and knots, and pictures of elementary particles themselves as complex structures. What actually the results of quantisation will be had better be left to the imagination.

HYDRODYNAMIC THEORY OF LUBRICATION, A BRIEF SURVEY*

EDWARD SAIBEL

SUMMARY. The basic equations governing hydrodynamic lubrication are discussed as well as the application of these equations to various types of problems. Current work in the field and some unsolved problems are mentioned.

Introduction. It is the object of this paper to review briefly the theory of hydrodynamic lubrication, to point out the present state of knowledge, and to indicate directions which appear to be useful and fruitful for future work.

Hydrodynamic lubrication is said to take place when a film of liquid or gas separating two surfaces in relative motion, is thick enough so that no contact between these surfaces takes place. Under the proper design and operating conditions this film can be maintained and pressures can be developed which are high enough to support the static and/or dynamic loads tending to force the surfaces together. If the film is maintained it is comparatively easy for the surfaces to slide relative to each other, friction is small, wear is reduced and satisfactory operating conditions prevail.

The basic problem from a theoretical point of view is to predict the pressure distribution in the lubricant film, the oil flow, the friction forces, as well as other factors which may interest the designer. Only in relatively simple conditions of operation has it been possible to do this up to the present time. Many important problems are still unsolved. However, with the advent of the electronic digital computer, problems which appeared to be hopeless a few years ago are now being considered and numerical solutions are being obtained.

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Discussion. Theoretical solutions are generally based on the Navier-Stokes equations for the motion of viscous fluids. These are [8]:

$$\rho \frac{Du}{Dt} = X - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\, div\, \underline{w}\right)\right] + \\ + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \\ + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] \tag{1}$$

with similar equations in the y and z directions, where

ρ is density

μ is viscosity

u, v, w are components of velocity

x, y, z are coordinates

t is time

X, Y, Z are components of body force

p is pressure

$\frac{D(\ )}{Dt}$ is the material derivative

$\underline{w}$ is the vector velocity,

and $$\text{div}\, \underline{w} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.$$

It should be noted that the Navier-Stokes equations are based on the concept of a Newtonian fluid and that the simple relationship, shear stress is proportional to velocity gradient, is valid.

In the case of a non-Newtonian lubricant it is evident that other equations must form the basis of the theoretical development.

In addition to the Navier-Stokes equations (1) which are obtained from Newton's Second Law of Motion, there is the equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0. \tag{2}$$

If the process is not incompressible, it is necessary to introduce the pertinent thermodynamic relationship. For example for a perfect gas

$$p - \rho g R T = 0, \tag{3}$$

where R is the gas constant and T is absolute temperature. If in addition the process is not isothermal, the energy equation representing the balance between heat and mechanical energy must be introduced. This furnishes a differential equation for the temperature distribution. This equation takes the form

$$\rho g \frac{D}{Dt}(c_p T) = \frac{Dp}{Dt} + \left\{ \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) \right\} + \mu \phi, \tag{4}$$

where ϕ the dissipation function has the form

$$\phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2, \tag{5}$$

where c_p is the specific heat at constant pressure and k is the thermal conductivity. The final equation of the system is the law relating viscosity to pressure and temperature

$$\mu = \mu(p, T). \tag{6}$$

In general then, there are the seven equations : Equations (1) (which constitute three of them), (2), (3), (4) and (6) for the seven unknowns u, v, w, p, ρ, T, and μ.

Naturally these general equations are greatly simplified if certain assumptions are made such as thickness of the film small and pressure constant across the film. In addition if such problems are considered as incompressibility, isothermal behavior, adiabatic behavior, constant density, constant viscosity, one dimensional flow, etc. simplifications of one sort or another are introduced in the equations.

The first problems to have received considerable attention were treated starting from the Reynolds' equation [9], an approximation which can be derived either from the Navier-Stokes equations or directly [3] from an analysis based on simplifying assumptions.

The Reynold's equation is generally expressed in the form

$$\frac{\partial}{\partial x}\left(h^3 \frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(h^3 \frac{\partial p}{\partial y}\right) = 6\mu\, U \frac{\partial h}{\partial x}, \tag{7}$$

where h is the film thickness and U is the velocity of the slider, the other symbols are as defined above. In the case of a slider-bearing, the boundary conditions are usually taken to be $p = 0$ on the edges of the slider.

Various solutions of (7) have been given for h linear, parabolic, and exponential [4], when the viscosity is considered constant. These solutions for the two-dimensional case were first obtained in series form but the coverage of cases of interest has been meager.

Extensions of solutions of the slider-bearing have been made to the following cases :

(a) Viscosity as a function of pressure [4]; in this case Charnes and Saibel have shown that by using the transformation

$$\mu^{-1} = \mu_0^{-1} \frac{d\phi(p)}{dp} \tag{8}$$

and knowing the relationship between μ and p, the problem may be reduced to the previously solved one for constant viscosity. In particular using the empirical relationship

$$\mu = \mu_0\, e^{\alpha p} \tag{9}$$

it was found that the actual pressure p could be found from the solution for the corresponding case of constant viscosity, $\hat{p}$, by means of the relationship

$$p = -\frac{1}{\alpha} \log\,(1 - \alpha\hat{p}), \tag{10}$$

(b) Viscosity as a function of temperature and pressure has also been solved [4]. This solution is manageable if side flow is neglected but if side flow is to be considered, it appears as though a direct numerical solution on a calculating machine from the differential equation would be preferable. A method for doing this would have to be carefully examined if machine capabilities are not to be exceeded. From preliminary calculations carried out by the author it appears that the effect of temperature on the viscosity is much greater than the effect of pressure.

Reference [4] gives an interesting comparison of a slider-bearing in which viscosity was considered constant, and when it was treated as a function of temperature and pressure, the latter is being worked out in two different ways.

(c) The effect of elasticity of slider and/or bearing surfaces on the load carrying capacity and on the pressure distribution has been worked out [5].

This solution however assumes that the deformation caused by the pressure which is developed, is small in comparison with the film thickness. It also neglects the deformation of the surfaces due to temperature effects.

The importance of the deformation of surfaces due to pressure as well as the distortion of surfaces due to temperature can hardly be over estimated. Yet very little work has been done along these lines. This is due in part to the complexity of the equations which arise, but to a large extent it has been due to lack of recognition of the importance of the problem. The basic equations are known and even though analytic methods for finding solutions may not be at hand, numerical methods may be employed. The means for carrying out such solutions are available.

(d) The effect of lubricant inertia in ordinary applications has been shown to be small [6, 7]. However, in the case of high speed, lightly loaded bearings may be important. One such application may well be to the gas-lubricated bearing, a subject that is receiving considerable attention at the present time. So far the results of theoretical analysis have not proved very satisfactory in explaining experimental findings. Also involved in the analysis of gas-lubricated bearings is the question of dynamical stability. This is a subject of great interest which needs much more attention. Of prime importance here is also the effect of elasticity of bearing and slider surfaces as mentioned above. Because of the non-linearity, many types of instability can occur.

(e) The effect of turbulence on hydrodynamic lubrication has just recently been considered [1]. Little is known at the present time about this subject either experimentally or theoretically. It is even doubtful whether in some of the published experimental data, turbulence was actually achieved. Failure to recognize this has introduced a certain amount of confusion of thought on this topic. The approach to the problem by Chou and Saibel was based on the mixing-length theory of Prandtl. In this paper, the pressure distribution, average velocity, and friction force are developed in terms of a single parameter k. The results of a theoretical calculation for the pressure distribution for two values of k in the turbulent case, and the pressure distribution in the laminar case, for contrast are given in the paper. It may be seen that substantially higher pressures arise in turbulence, but at the expense of increasing the friction forces. Nevertheless, for large load carrying capacity it may turn out to be desireable in certain applications to design in the turbulent regime.

(f) As a final example of recent work, consider the effect of the conduction of heat in the lubricating fluid and the conduction of heat to the surrounding surfaces. This very important aspect of the theory of hydrodynamic lubrication is just now receiving attention [9] and results are as yet very limited. The ultimate problem will have to include the interaction of fluid flow of the lubricant, heat conduction in the lubricant and to the surroundings, and the elastic behavior of the slider and bearing or journal and bearing.

It will have to take into account in many cases compressibility, variability of viscosity with temperature and pressure, dynamic loading conditions as well as the steady state conditions which have been discussed above. The stability of the system, thermal distortion and its effect on the pressure development, inertia of the lubricant, and turbulence are some of the other phenomena which need a great deal of attention.

Finally, one more topic that has received too little attention is the effect of non-Newtonian behavior of the lubricant on all of the above. The physical and chemical properties which are being demanded of a lubricant are so varied and complex that in many instances simple Newtonian liquids will not suffice. The result is that the problem of a theoretical analysis due to use of non-Newtonian fluids has become acute.

All of the foregoing problems plus non-linearity of behavior makes for a tremendously difficult but interesting situation. Although the above remarks have been directed toward the slider-bearing, for the most part, the situation is very much the same for the sector thrust bearing and for the journal bearing, except that the equations governing the latter case are somewhat more difficult to handle due to the form which h, the film thickness takes.

Various approximate methods have been devised for handling these problems but on the whole the results have not been too satisfactory, particularly in the case of the journal bearing. This latter case has certain unique problems and difficulties which need considerable attention. In particular, difficulties arise because of the prediction of negative pressures, and the possibilities and consequences of cavitation form in themselves a vast field for future work.

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SOME RESULTS OF THE MODERN THEORY OF STABILITY*

By W. HAHN

As I AM going to give you a report on some results of the modern stability theory, it will be necessary to characterize at first the subject of the theory, and in particular to explain the conception of stability. Let us consider a mechanical system described by a system of differential equations

$$\dot{x}_i = F_i (x_1, \dots , x_n, t) \qquad (i = 1, 2, \dots , n) \tag{1}$$

or written in the self-explaining vector notation

$$\dot{x} = F (x, t) \qquad (x = \{x_1, \dots , x_n\}) . \tag{2}$$

Let the solution which gets the values $x_{10}, \dots , x_{n0}$ at the moment t_0 be denoted as

$$f (t ; x_0, t_0) \qquad (f (t_0, x_0, t_0) = x_0). \tag{3}$$

We assume that it is existent and unique. We further assume that the system permits the so-called trivial solution $x = 0$ as a particular solution, i.e. that

$$F(0, t) = 0. \tag{4}$$

We are interested to know whether any arbitrary solution is bounded for all values of t or moreover whether it approaches zero, that means the trivial solution, if t is increasing infinitely. To write these properties of the general solution (3) in a convenient manner, we introduce the following definition : The trivial solution is called *stable* if given any $\epsilon > 0$ one can find a $\delta > 0$ so that

$$| f (t ; x, t_0) | < \epsilon \tag{5}$$

provided that $| x_0 | < \delta$. (Of course, the modulus $| x |$ of the vector x is defined by

$$| x |^2 = x_1^2 + \dots + x_n^2 .)$$

* Invited Address, delivered at the Twentyfifth Conference of the Indian Mathematical Society, Allahabad, (1959).

That means that the mechanical system in question gets bounded oscillations, the maximum amplitude of which depends on the amount of x_0 and may be kept arbitrarely small through suitable choice of $| x_0 |$. If the trivial solution is stable and moreover if all solutions (3) approach zero provided that x_0 belongs to a certain neighbourhood $| x_0 | < \rho$ of the origin, the trivial solution is called *asymptotically stable*. To write it exactly : we have $\lim f(t, x_0, t_0) = 0$ or, given any $\eta > 0$, one can find a $T > 0$ so that

$$| f(t_0 + T ; x_0, t_0) | < \eta \; (| x_0 | < \rho). \tag{6}$$

These definitions have been given by Ljapunov [3] in his famous foundation of the stability theory written nearly 70 years ago. I mention that the conception of stability is a very important one to many applications, for instance to the theory of automatic control systems. The shortness of time does not permit me to go into the particulars.

Now, we may formulate the *stability problem*. Given a system as (1), how to find out whether or not trivial solution is stable or asymptotically stable respectively? This question may be answered if the general solution is known and if its form is not too complicated. For example, if the system is a linear one with constant coefficients, the general solution is a linear combination of terms of the form e^{at} and its behaviour for large t is completely known if the exponents a are known. But mostly, the general solution is either too complicated to be used or it is not known at all. Therefore, we need a method for solving the problem which does not use the solutions but the equations. Such a method has been developed by Ljapunov and is called the second or direct method. I shall try to explain it by means of a simple example. We are given the system

$$\begin{aligned} \dot{x} &= -y + a(x^2 + y^2)x, \\ \dot{y} &= x + a(x^2 + y^2)y. \end{aligned} \tag{7}$$

Let us consider the function

$$v = x^2 + y^2, \tag{8}$$

which becomes a function of t only if we substitute a solution

$$x = x(t),\ y = y(t)$$

of (7). That function $v(t)$ may be derived with regard to t. We get

$$\frac{dv}{dt} = \frac{\partial v}{\partial x}\dot{x} + \frac{\partial v}{\partial y}\dot{y} = 2a\,(x^2 + y^2)^2 \tag{9}$$

because $\dot{x}$ and $\dot{y}$ are known from (7). Now, v may be interpreted as the distance between the point (x, y), representing the solution in the moment t, and the origin. We learn from (9) that the derivative $\dot{v}$ is positive in case a is positive. Then, the distance is an increasing function of t and the point (x, y) will never approach the origin: the trivial solution cannot be stable. But if a is negative, the distance is decreasing, and the trivial solution is expected to be asymptotically stable.

The general principle of the method may be described as follows: we are given a family of closed curves surrounding the origin which cover a certain neighbourhood of the origin completely. Their equation may be written in the form $v(x, y) \equiv \text{const}$. The stability of the trivial solution may be checked by means of the direction in which the curves of the family are intersected by the solution curves of the differential equation. The question whether or not the solution curves are going from outside to inside may be decided by means of the sign of the total derivative of v.

Of course, the corresponding theorem must be proved rigorously and without any geometrical interpretation. This can be done, and the proof can be extended to systems of more than two equations. I am going to give you the exact formulation of the so called *Second theorem of Ljapunov* concerning asymptotical stability. For this purpose, I introduce the conception of the A-function. A real function $g(r)$ which is defined for $0 \leqslant r \leqslant r_1$, continuous and monotonously increasing and getting zero for $r = 0$ $(g(0) = 0)$ may be called an A-function. (The definition is only given for formulating the theorem.) Then the theorem runs as follows:

If there exists a real function $v(x, t)$ and three A-functions so that

$$g_1(|x|) \leqslant v \leqslant g_2(|x|)\ ; \quad \dot{v}(x, t) \leqslant -g_3(|x|)\ , \tag{10}$$

where

$$\dot{v} = \frac{\partial v}{\partial x_1}\dot{x}_1 + \dots + \frac{\partial v}{\partial x_n}\dot{x}_n + \frac{\partial v}{\partial t} \tag{11}$$

denotes the total derivative of v with regard to the given system of differential equations, then the trivial solution is asymptotically stable.

Usually, the properties given by the estimations (10) and (11) have got special notations in the literature.

If we want to apply the theorem we must know a suitable "Ljapunov function" v, and the theorem would be useless unless it was possible to find such a function at least in special cases. As a matter of fact, one can construct functions in a lot of cases, very important to the applications (cf. [1]) in particular in the field of non-linear vibrations and control systems. But the theorem itself does not give such a method. It only settles *sufficient* conditions and the question whether these conditions are *necessary* too, that means whether a Ljapunov function does exist at any rate, is not answered at all. The problem of the existence of Ljapunov functions has not been solved completely up till the present. But during the last years some very interesting results have been yielded, and I am about to report on them.

At first, Massera[5] succeeded in proving one converse to the above quoted Ljapunov theorem. He proved that in the case of an autonomous or a periodical system (in (2), F is independent of t or periodical with regard of t, respectively) the conditions are necessary. Soon afterwards, Malkin[4] observed that Massera's proof did not depend on the autonomy or the periodicity of the equations but that the main point was a particular property which systems of this type share with a much more general class of non-autonomous equations. To make it clear it is necessary to refine the above given definition of stability and asymptotical stability.

Let us regard the equations

$$\dot{x} = -x \quad \text{and} \quad \dot{x} = -\frac{x}{1+t}, \tag{12}$$

where x denotes a scalar. The corresponding solutions are

$$x = x_0\, e^{t_0 - t} \quad \text{and} \quad x = x_0 \frac{1+t_0}{1+t}. \tag{13}$$

Let us assume that we are given an η occurring in (6) and let us find out the corresponding number T. We easily find in the first case

$$T > |\log x_0 - \log \eta|, \tag{14}$$

whereas in the second case we have

$$T > (1+t_0)\frac{x_0 - \eta}{\eta}. \tag{15}$$

In the first case, the number T may be chosen independently of t_0, and that is the point. Similarly, the number δ occurring in the definition of stability, may or may not depend on x_0. Now, if that δ may be chosen uniformly with regard to a given neighbourhood $|x| \leqslant \rho$ of the origin, then the stability is called *uniform*, and if besides the number T may be chosen independently of t_0, the stability is called *uniform-asymptotical.*

As a matter of fact, the conditions of the quoted Second theorem of Ljapunov grant not only the asymptotical stability but also the uniform-asymptotical stability. Therefore, the theorem cannot be inverted. In case the trivial solution is asymptotically stable but not uniform-asymptotically stable (as for instance in case of the second equation (12)) a suitable function v satisfying the conditions of the theorem cannot exist. But if we formulate the theorem completely inserting the uniformity as an additional statement the theorem may be inverted. Malkin proved : The conditions (10) are both necessary and sufficient for uniform-asymptotical stability.

The proof of the theorem is rather a complicated one. It is based on a transfinite construction. Therefore, it is merely an existence proof which cannot be utilized to get a suitable Ljapunov function in a concrete case. Meanwhile, some more proofs have been given† which differ from each other with regard to the method and also slightly with regard to the results. For instance, Massera [6] succeeded in proving that there exists a Ljapunov function v which permits partial derivation of any arbitrary order, etc.

I shall neither go into the particulars nor deal with the corresponding theorems concerning non-asymptotical stability and instability respectively which are also settled up by Ljapunov, and similar inversion problems. But I shall report on one more problem concerning asymptotical stability.

In (10), the Ljapunov function v has been characterised by means of three comparing function of a particular type. On the other hand, one may characterise the behaviour of the general solution in case of uniform-asymptotical stability by means of comparing functions too. Let a real function $h(r)$ be called a B-function if it is defined for $r \geqslant r_1$, continuous, positive, monotonously decreasing and approaching zero with $r \to \infty$. Then we may state the following theorem : *Necessary and sufficient condition for uniform-asymptotical stability is the existence of an A-function $g(r)$ and a B-function $h(r)$ so that*

$$|f(t; x_0, t_0)| \leqslant g(|x|)\, h(t - t_0). \tag{16}$$

And now, the following problem arises. Is it possible to characterise the functions g and h occurring in (16) if the functions g_1, g_2, g_3 occurring in (10) are known to belong to a given type and vice versa? This problem has been dealt with especially by some Soviet mathematicians, and I shall quote a result given by Krasovskij [3]:

If the inequality (16) is of the type

$$|f(t; x_0, t_0)| \leqslant A\, |x_0|\, e^{-\beta(t-t_0)}, \tag{17}$$

where A and β are positive constants, then a Ljapunov function v exists which may be estimated by means of powers of x, i.e. the inequalities (10) may be written as

† A survey may be found in [1].

$$|x|^{\gamma_1} \leqslant v \leqslant |x|^{\gamma_2}; \quad \dot{v} \leqslant -|x|^{\gamma_3}. \tag{18}$$

This theorem and all theorems of this type are very important to the theory of the stability of *disturbed equations*. Let us consider a system

$$\dot{x} = F(x, t), \tag{19}$$

the trivial solution of which is "exponentially stable", that means an estimation (17) holds. Then we are sure about the existence of a Ljapunov function v satisfying the inequalities (18) with certain constants γ_1, γ_2, γ_3. Besides (19), we consider the disturbed system

$$\dot{x} = F(x,t) + G(x, t). \tag{20}$$

We are asking for conditions which grant that the stability behaviour of (20) is the same as the behaviour of (19). If we write the total derivative of the function v with regard to the disturbed system (20) we get

$$\dot{v} = \frac{\partial v}{\partial x_1}(F_1 + G_1) + \dots + \frac{\partial v}{\partial x_n}(F_n + G_n) + \frac{\partial v}{\partial t},$$

i.e.

$$\dot{v} = \left[\frac{\partial v}{\partial x_1} F_1 + \dots + \frac{\partial v}{\partial x_n} F_n + \frac{\partial v}{\partial t}\right] + \left[\frac{\partial v}{\partial x_1} G_1 + \dots + \frac{\partial v}{\partial x_n} G_n\right]. \tag{21}$$

Now, the first term in (21) is surely negative and moreover, it may be estimated by $|x|^{\gamma_3}$. On the other hand, the derivatives $\partial v / \partial x_i$ may also be estimated by means of powers of $|x|$. Therefore, it is not difficult to get an estimation of $G_1, \dots, G_n$ which grants that the second term on the right hand side of (21) has no influence on the sign of $\dot{v}$ so that the Ljapunov theorem may be applied.

If, for instance, the exponents in (18) equal two and if the functions G_i have Taylor developments beginning with terms of the second order at least, then, in (21), the first term has the order two and the second the order three at least. That means, the sign of $\dot{v}$ is determined by the first term only, and the disturbed equation has the same stability behaviour as the undisturbed one. The most famous special case is the well-known method of linearisation or method of small oscillations respectively, and the outlined method permits to establish the conditions where the linearization is allowed.

Finally, I would like to mention that the Ljapunov method is not restricted to differential equations in the Euclidean phase space. It may be extended to much more general conceptions as differential equations in the Banach space or general functional equations and may be the basis of a stability theory applicable in various fields of modern analysis.

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GEOMETRICAL THEORY OF SERIES*

By C. ORLOFF

It is known that the series with constant terms has two forms of expression :

$$a_n = f(n), \tag{1}$$

$$a_n = f(a_0, a_1, \dots a_{n-1}). \tag{2}$$

Both were known in the seventeenth century and it seemed for centuries that a third could not exist. This is true only partially, the third analytical expression does not exist, probably.

In 1952 I introduced one purely geometrical definition of series suggested by the geometrical representation of geometrical progressions given by M. Milankovitch in 1909. It is obvious that

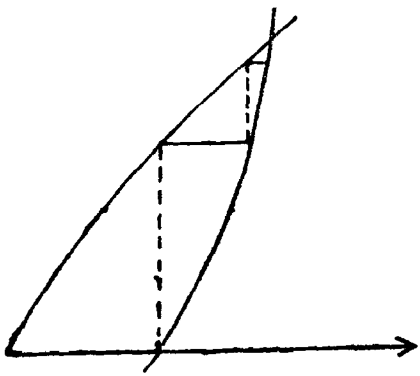


FIG. 1.

two curves like these (Fig. 1) determine one infinite sequence of straight-line portions, because the broken straight line does not stop. If we accept the values of the lengths of these portions as terms of the infinite series, then the series itself is determined by two curves. The mode of determination is obviously unique, i.e. two curves determine one series only.

* Invited address delivered at the 25th Annual Conference of the Indian Mathematical Society, Allahabad, 1959.

To make the definition more precise, let us say that a series with positive terms is determined with the help of two given lines, an axis and a given unity of measurement.

To be sure, these lines cannot be entirely arbitrary. I have suggested the following requirements which the lines should meet to be accepted as the lines determining a series :—

(i) Any perpendicular to the axis crosses the line in one point, and in one point only.

(ii) The line has one common point with the axis and one such point only.

(iii) The distance between the line and the axis is a monotonously increasing function.

These three requirements are the same for both the lines.

(iv) Both the lines are on the same side of the axis.

(v) They have different common points with the axis.

(vi) The two lines cannot coincide anywhere, they can only have discrete common points.

Obviously, these requirements are sufficient but not necessary.

These lines are called characteristic lines of a series.

As regards the mode in which the characteristic lines are given, it may be anything. A series so determined may be denoted by

$$\sum_{C_1, C_2}$$

or, if it is known which of the characteristic lines is upper and which is lower, through

$$\sum_{C_1}^{C_2}.$$

If, however, we wish to find for the terms of the series determined geometrically some analytical formula such as (1), or (2), these curves must be expressed with the help of some analytical expression.

Therefore, let us take the given axis as x-axis, and assume the origin of co-ordinates as the common point of the axis and one of the characteristic lines, and which has the smaller abscissa. In such a system of co-ordinates, the lower characteristic line C_1 will be represented by the equation :

$$\phi_1(x, y) = 0,$$

and the upper characteristic line C_2 by the equation

$$\phi_2(x, y) = 0.$$

To denote the series so determined, I have suggested the following designation :

$$\sum_{\phi_1(x,y)=0}^{\phi_2(x,y)=0} .$$

To be sure, there is no question of a system of coordinates in a purely geometrical determination of the series, as the coordinates are part of the analytical means. Since, however, we wished to correlate the originally given characteristic lines with the terms of the series obtained as a result of them, and since these terms have numerical values, it was found necessary to introduce analytical equivalents for the curves, i.e. their equations (as is done in analytical geometry; though becoming analytical, it still remains geometry). Similarly, the geometrical nature of the determination of the series does not change by any means just because we decided to represent the characteristic curves with the help of their analytical equivalents, namely equations.

I will, now, supplement the new definition by saying that the upper characteristic line may either cross the y-axis in its positive part or pass through the origin of the coordinates.

What happens now to the 6 requirements which the curves must meet to be accepted as the characteristic lines of the series ? To be sure, they all remain in force, but may be formulated in a simpler and more concise manner.

The requirements in this case will be as follow :—

1. The characteristic lines must be graphs of uniform, continuous, monotonously increasing functions determined for all values of $x \geqslant 0$.

2. They must each have one common point with one of the axes. The lower one with the positive part of the x-axis; the upper one with the positive part of the y-axis, or it shall pass through the origin of the coordinates.

3. Both characteristic lines have only discrete common points (they never coincide even in a small part).

Let us now come back to our primary problem, i.e. to establish relations between functions determined by the characteristic lines on the one hand, and the terms of the series on the other. To begin with, let us find out what forms of the equation of the characteristic lines are to be accepted in order to be able to express the relation in the simplest possible way. It turns out that it is best to have the equation of the lower characteristic line solved in respect of x, and that of the upper line solved in respect of y.

The series, when expressed in this form, hereafter referred to as normal, may be denoted as

$$\sum_{x=\psi(y)}^{y=\varphi(x)}$$

It is this form that we shall use to determine the formula for the relations in question (see Fig. 2):

$$S_n = \psi[\varphi(S_{n-1})] \qquad n = 1, 2, \ldots, \tag{3}$$

$$S_0 = a_0 = \psi(0). \tag{4}$$

Thus the required relation is found in the form of a special case of the general recurrence formula (2), because the formula (2) by substitution

$$a_n = S_n - S_{n-1} \qquad (n = 1, 2, \ldots) \qquad \text{and} \qquad a_0 = S_0$$

becomes

$$S_n = f(S_0, S_1, \ldots, S_{n-1}, n). \tag{2'}$$

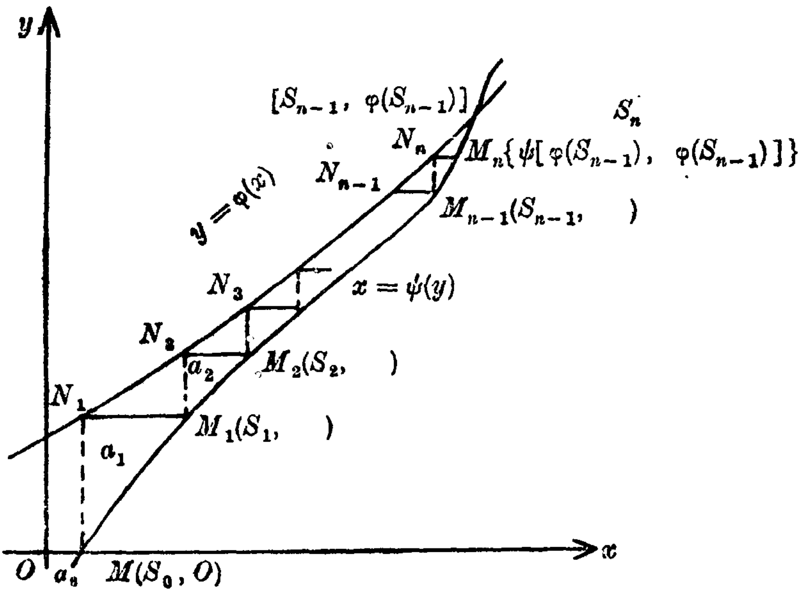


FIG 2.

The reverse problem can be successfully treated in analytical as well as in graphical way, but we do not undertake it, because of shortage of time.

Now we shall pose the question : What advantages can be obtained from this new definition of series ?

(1) Some advantages appear by calculation of the sum of the convergent series.

(2) Some others are of theoretical value, providing the proof of certain new theorems.

(3) As a whole it helps to better understanding of the theory of series.

Now we shall note some facts.

This definition of the series is true for convergent series as also for divergent. In the first case the characteristic lines have at least one common point (in the first quadrant). If there are many such common points, the point P with the smallest positive abscissa is called the point of convergence of the series. In the case of divergence the characteristic lines do not have common points (in the first quadrant) at all.

It is not possible to explain the evaluation of numerical values of the series by means of this theory and to make clear the advantages of such a treatment of series, because of the shortage of time. The very important notion of regressive series as well as the theorems of geometrical transformation of series, also, must be omitted for the same reason.

I shall give here only the notion of alternative series.

Alternative Series. I have proposed the definition of alternative series as follows from the graph (Fig. 3).

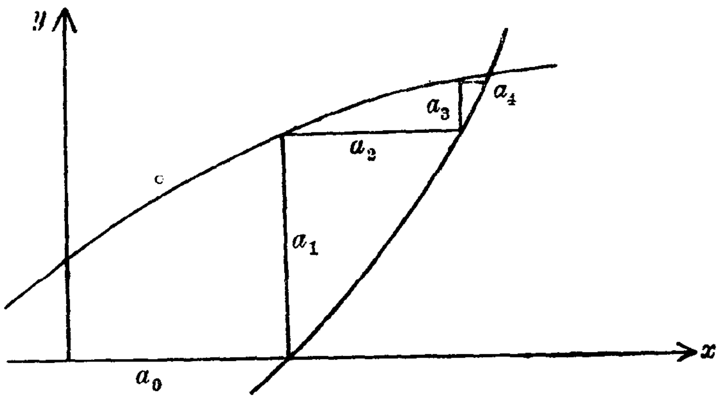


FIG. 3.

To differentiate between series with positive terms and alternative series having the same characteristic lines, I have suggested that the alternative series be denoted as

$$\sum_{\phi_1}^{\phi_2} \pm \tag{5}$$

Obviously they too can have the general and normal form, and the notions 'concharacteristic', 'conpointed' and 'conlimited' equally hold good for these series which are also amenable to geometrical transformation.

I do not think it necessary to repeat all the aforesaid regarding the series with positive terms along with those slight modifications which the alternative series may impose.

For my part, I shall only dwell on the main question which is how the terms are obtained from the equations of characteristic lines for the alternative series in their normal form. We arrive at the following formulae for the series

$$\sum_{x=\psi(y)}^{y=\varphi(x)} \pm \overrightarrow{(0, 0)}$$

$$a_0 = \psi(0), \quad a_1 = \varphi[\psi(0)], \quad S_{2K+2} = \psi[\varphi(S_{2K})], \quad S_{2K+3} = \varphi[\psi(S_{2K+1})]$$
$$K = 0, 1, 2, \dots,$$

where the following designations are used :

$$S_{2K} = \sum_{i=0}^{K} a_{2i}, \quad S_{2K+1} = \sum_{i=0}^{K} a_{2i+1}, \quad K = 1, 2, \dots .$$

In addition, I am going to set down here 5 theorems on the convergence of alternative series as proved by me in a previous paper.

THEOREM 1 : *For alternative series* (5) *to be absolutely convergent, t is necessary and sufficient that its characteristic lines have at least one point in common in the first quadrant.*

This theorem is obvious.

The sum of a series will be equal to the difference between the abscissa and the ordinate of the point of convergence P

$$S = x - y.$$

The theorem and the corresponding graph may be conveniently used in teaching the theory of series as they offer an opportunity of demonstrating vividly that the sum of an absolutely convergent alternative series does not change as a result of the commutation and association of its terms.

THEOREM 2: *For alternative series (5) to be not-absolutely convergent, it is necessary and sufficient that its characteristic lines have no common point in the first quadrant but have a commom asymptote with the equation*

$$y = x + d.$$

The sum of such a series will be $S = - d$.

As an obvious consequence of this theorem, the well-known formula for the alternative series is obtained :

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1, \text{ (if this limit exists).}$$

The proof of this theorem and its consequence is very simple. This theorem can be conveniently used in teaching the alternative series, as with the help of it, it can be vividly demonstrated that the sum of a non-absolutely convergent series may change with the commutation of the terms of the series, and that through commutation of its terms a convergent series may become divergent.

On the basis of the standpoint that the non-absolutely convergent alternative series is defined by characteristic lines, a novel classification of the alternative series can be made. For this purpose we shall introduce the notion of *monotonous asymptote*. We shall thus term an asymptote the curve of which is monotonously approaching so that the distance between a point of the curve and the asymptote is a monotonously decreasing function.

Let us now pass over to the classification of non-absolutely convergent alternative series.

Class A comprises such series for which the common asymptote is monotonous relative to both the characteristic lines, and the latter are located on both sides of the asymptote.

It will not be difficult to show that for this class of series the values a_i (starting with a_1) monotonously decrease and tend to zero.

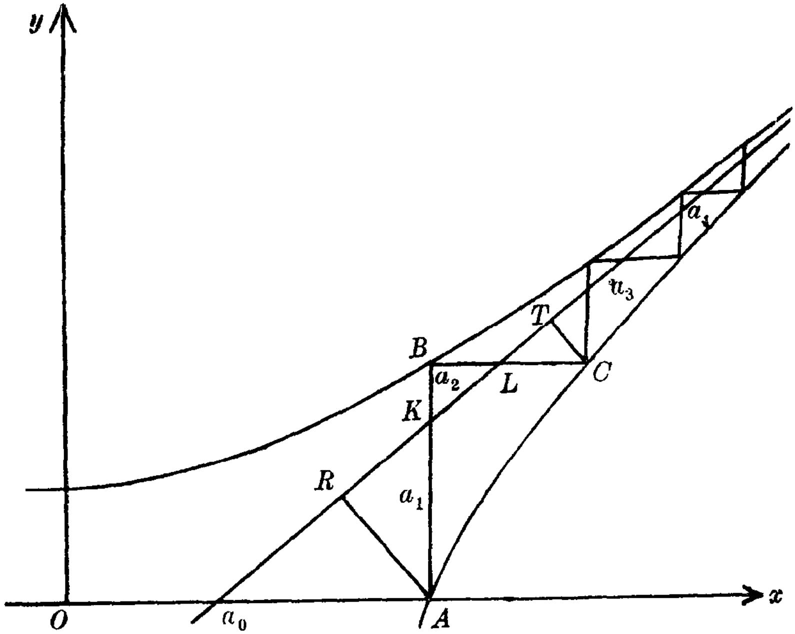


FIG. 4.

Seeing that

$$\lim_{n \to \infty} a_n = 0$$

has been found to hold good for all convergent series, hence, the series of Class A meet the requirement advanced by Leibnitz.

Class B comprises non-absolutely convergent alternative series whose characteristic lines meet the same requirements as regards the monotonous asymptote, but are located on one side of it. (Figs. 5 and 6.)

In such series, the terms do not decrease monotonously. In the first case (B_1), (Fig. 5), any odd term a_{2n+1} (taking its modulus only), for $n = 1, 2, \ldots$ is greater than the preceding a_{2n}. This is not difficult to prove. So

$$a_3 > a_2.$$

A more general formula can be obtained in a similar manner:

$$a_{2n+1} > a_{2n}.$$

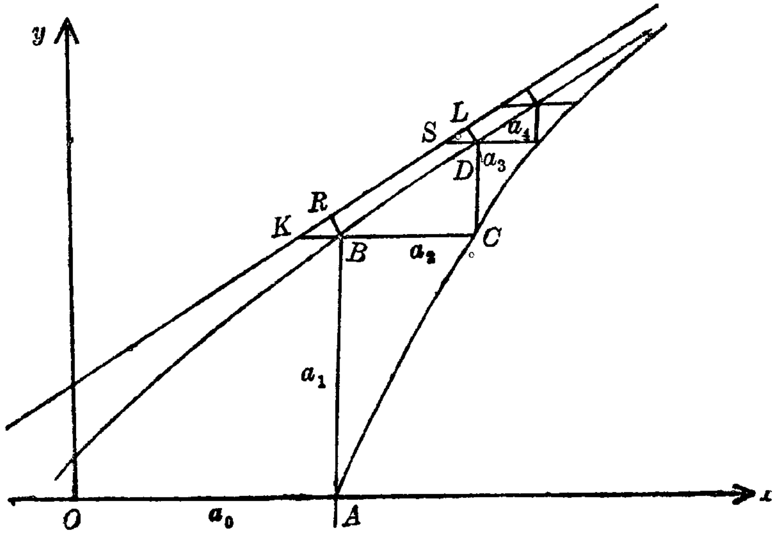
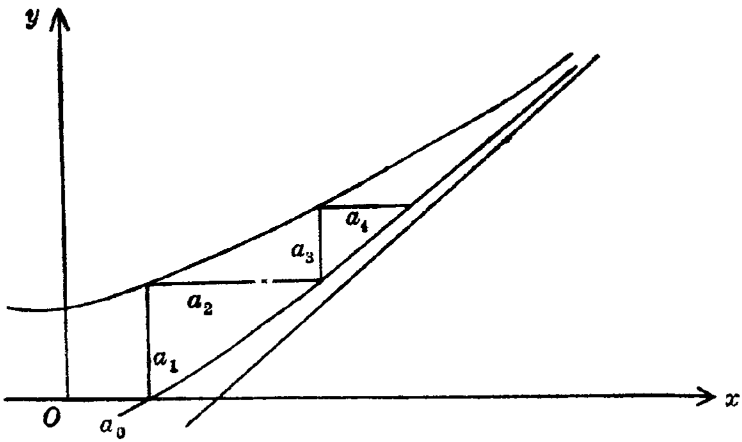


FIG. 5.



FIG, 6.

For the second case of such series (B_2), it may be shown by similar means that any even term a_{2n} (taking its modulus only) is greater than the preceding a_{2n-1} (for $n = 1, 2, \ldots$).

In the first case the sum of a series is a negative value; in the second, it is positive.

All the remaining alternative series of the not-absolutely convergent category belong to Class C.

THEOREM 3. *For alternative series (5) to be divergent, it is necessary and sufficient that its characteristic lines have neither common points in the first quadrant, nor a common asymptote with the equation* $t = x + d$.

There can be proved a great many theorems on summability of divergent series with respect to various processes of summability. I shall confine myself to two theorems on summability C_1, just for the sake of an example.

THEOREM 4. *If the characteristic lines of a divergent alternative series have each an asymptote, one* $y = x + a$, *and the other* $y = x + b$, *then the series is summable, by process* C_1, *and its sum is equal to* $(a + b)/2$.

THEOREM 5. *If the characteristic lines of a divergent alternative series lie in a part of the first quadrant plane* xOy, *limited by the straight lines having respectively equations* $y = x + a$ *and* $y = x + b$ $(b > a)$, *and by the x-and y-axes, and the series is summable by process* C_1, *then the sum S of the series is in the interval*

$$-b \leqslant S \leqslant -a.$$

These theorems can be proved without any difficulty.

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RAY THEORIES IN ELASTIC WAVES*

By S. D. CHOPRA

THE subject ' Elastic waves' includes among others the propagation of sound waves in fluids, vibrations of strings, bars, plates and shells, and the propagation of waves in elastic solid media. I propose to confine myself to a very small but important part of the subject, namely the passage of elastic waves through the earth, the most important of which are the earthquake waves and, from the applicational point of view, the waves generated by artificial explosions made at shallow depths under the ground or under the surface of water in water covered areas. Most mathematical investigations relating to the propagation of such waves through the earth regard it as a flat semi-infinite solid of perfectly elastic, isotropic, homogeneous material, which may be overlain by one or more layers, of finite thickness, of similar material.

The problems relating to the propagation of elastic waves through the earth are mainly of two types. The first type, and the simpler one, is concerned with the mere possibility of the existence of certain types of ' surface waves ' which are discernible only near the surface of the earth and ' guided waves ' or ' channel waves ' which can be propagated in wave guides formed by low-velocity layers within the earth. The second type of problems relates to the actual generation of elastic waves within the earth. In these problems natural or artificial explosive sources are simulated by idealized point and line sources. I shall discuss the application of ray theory methods of geometrical optics to problems of both these types.

Over one hundred years ago, Poisson (1829) showed that a homogeneous, isotropic, infinite elastic solid could transmit two types of waves. Later, Stokes (1849) identified Poisson's quicker

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wave as one of irrotational dilatation (longitudinal vibrations) with velocity of propagation

$$\alpha = [(\lambda + 2 \mu) / \rho]^{1/2},$$

and the slower wave as one of equivoluminal distortion (transverse vibrations) with velocity of propagation

$$\beta = (\mu / \rho)^{1/2},$$

where λ, μ are Lame's constants of the medium and ρ its density. The two types of waves are called P and S waves respectively in the language of seismology. The second type is polarized and, in the case of propagation in the earth, the symbols SV and SH are used to denote the parts polarised in the vertical and horizontal planes respectively. In elastic wave problems dealing with propagations in media with plane parallel boundaries, the symbols SV and SH denote the parts polarised in planes perpendicular and parallel respectively to the boundaries.

Rayleigh (1885), Love (1911), and Stoneley (1924) discussed the possibility of the propagation of waves now known by their respective names by formulating the questions as boundary value problems for the wave equation. The method is to assume suitable elementary solutions of the wave equation for each medium and substitute these into the boundary conditions at the interfaces and free surfaces. The condition of consistency of the resulting equations determines the phase-velocity of the waves, the existence of the waves being possible if the equation gives real values for the phase-velocity. This equation is variously called the 'phase-velocity equation' or 'period equation' or the 'frequency equation' of the waves since the equation is usually in terms of the period or frequency of the simple harmonic waves under consideration.

Since the time of the above investigations many people have played about with different models discussing the possibility of the existence of either Rayleigh-type — P and SV — waves, or Love-type — SH — waves.

Recently, a method based on the 'principle of constructive interference' of multiply reflected rays has been used for the derivation of the frequency equations of guided waves. This idea was first used by Pekeris (1948) for a two-layered liquid half-space. It was extended to the case of elastic solids by Fay and Fortier (1951) who studied the transmission of sound through steel plates immersed in water. Tolstoy and Usdin (*1953*), following Fay and Fortier, obtained the frequency equations in a number of cases. However, in their derivation of the frequency equation in the case of an infinite elastic plate which lays down the pattern to be followed in other cases treated by them, Fig. 2 seems to be inaccurate and equations (*6a*), (*6b*) seem to be valid only at the upper surface of the plate.

It is proposed to give here a clear exposition of the method by applying it to the propagation of Rayleigh-type guided waves—*P* and *SV*—in an internal stratum bounded on both sides by half-spaces of identical elastic properties and in welded contact with the stratum.

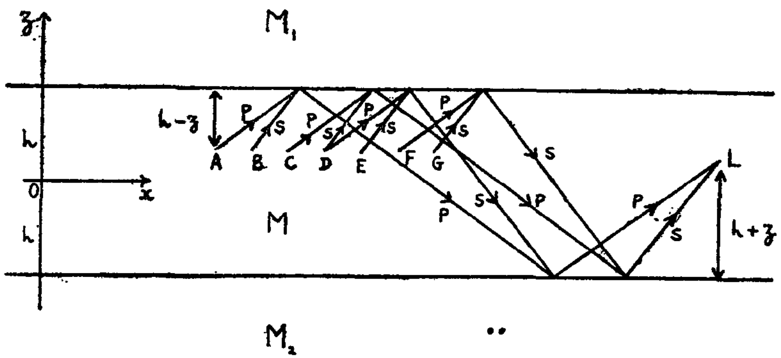

Fig. 1. Constructive interference in the internal stratum.

Let us imagine a steady state of waves propagating in the stratum from left to right in which *P* waves are incident at the interfaces at an angle *e* and *S* waves at an angle *f* with (say)

$$\frac{\sin e}{\alpha} = \frac{\sin f}{\beta} = \frac{\sin e_1}{\alpha_1} = \frac{\sin f_1}{\beta_1} = \frac{\zeta}{\omega}, \tag{1}$$

where α, β are the velocities of propagation of the P and S waves in the stratum and α_1, β_1 the corresponding velocities in the half spaces M_1, M_2; e_1, f_1 are the angles of refraction into M_1, M_2; ω is the angular frequency of the simple harmonic waves under consideration and ζ is a quantity defined by (1) and is to be used later. Under certain circumstances ζ is allowed to become complex.

Let the displacements in the upgoing P and S waves at a point L (x, z) in the stratum be given by

$$\left.\begin{aligned} &A_P \exp\left[iw\left(t - \frac{x \sin e + z \cos e}{\alpha}\right)\right] \\ \text{and} \\ &A_S \exp\left[iw\left(t - \frac{x \sin f + z \cos f}{\beta}\right)\right] \end{aligned}\right\} \tag{2}$$

respectively and those in the downgoing waves be given by

$$\left.\begin{aligned} &A'_P \exp\left[iw\left(t - \frac{x \sin e - z \cos e}{\alpha}\right)\right] \\ \text{and} \\ &A'_S \exp\left[iw\left(t - \frac{x \sin f - z \cos f}{\beta}\right)\right]. \end{aligned}\right\} \tag{3}$$

The principle of constructive interference may be stated as follows : the various reflected P and S arrivals at any point in the stratum must add up in such a manner as to produce the vibrations in the P and S waves at that point appropriate to unattenuated propagation of waves in the stratum. Thus, when there is constructive interference, the expressions (2) and (3) for the up- and downgoing waves hold at all points in the stratum. The principle of constructive interference can be applied to the upgoing or downgoing waves at L. We apply it to the upgoing waves.

The rays reaching L after traversing the thickness of the stratum exactly twice and undergoing one reflection at each interface are :

(i) those reaching L as P,

PPP from A to L,	PSP from D to L,
SPP from B to L,	SSP from E to L;

(ii) those reaching L as S,

PPS from C to L,	PSS from F to L,
SPS from D to L,	SSS from G to L.

The points A, B, etc. are all at the same level in the stratum as L. The principle of constructive interference now means the following. The rays from A, B, D, E reaching L at time t as P must so add up as to produce the P wave at L as given by (2). Similarly, the rays from C, D, F, G reaching L at time t as S must add up to produce the S wave at L as given by (2). We require one reflection at each interface because this restores an upgoing wave to an upgoing one, and we need consider only one reflection at each interface because the argument about L can be applied to all such points as A, B, etc. Further, the above requirement for constructive interference may be applied to the displacements in the waves as given by (2) or to their components in the x- or z-direction. The final result is the same because the constituent P—or S—waves all arrive at L in the same direction. We shall work in terms of the displacements (2).

The point A has the coordinates $(x - 4h \tan e, z)$. In order that the ray PPP should arrive at L at time t, it must leave A at the time $t - 4h \sec e/\alpha$. Hence the displacement in the ray PPP at A is given by

$$\left. \begin{aligned} & A_P \exp\left[i\omega \left\{ t - \frac{4h \sec e}{\alpha} - \frac{(x - 4h \tan e) \sin e + z \cos e}{\alpha} \right\} \right] \\ & \text{or} \\ & A_P \exp\left[i\omega \left(t - \frac{x \sin e + z \cos e}{\alpha} \right) \right] \exp\left(- 4h \frac{i\omega \cos e}{\alpha} \right). \end{aligned} \right\} \quad (4)$$

As the ray PPP travels from A to L, the only places where attenuation, change of phase, or change of type can occur are at the interfaces. The attenuation and change of phase, if any, are contained in the reflection coefficient $R_{PP'}$ defined in Chopra (1958). Thus taking account of the reflections at $z = h$ and $z = -h$, the contribution by the ray PPP to the P wave at L is

$$A_P \exp\left[i\omega\left(t - \frac{x \sin e + z \cos e}{\alpha}\right)\right] \exp\left(-4h\frac{i\omega \cos e}{\alpha}\right) R^2_{PP'}. \quad (5)$$

Next, consider the contribution of the ray SPP. The point B has the coordinates $[x - (3h + z) \tan e - (h - z) \tan f,\ z]$ and the ray SPP must leave B at time $t - \dfrac{(3h + z) \sec e}{\alpha} - \dfrac{(h - z) \sec f}{\beta}$ in order to arrive at L at time t. Thus the displacement in the ray SPP at B is

$$\left.\begin{aligned} & A_S \exp\left[i\omega\left\{t - \frac{(3h + z) \sec e}{\alpha} - \frac{(h - z) \sec f}{\beta} - \right.\right. \\ & \qquad \left.\left. - \frac{[x - (3h + z) \tan e - (h - z) \tan f] \sin f + z \cos f}{\beta}\right\}\right] \\ & \text{or} \\ & A_S \exp\left[i\omega\left(t - \frac{x \sin f + z \cos f}{\beta}\right)\right] \times \\ & \qquad \times \exp\left[-(3h + z)\frac{i\omega \cos e}{\alpha} - (h - z)\frac{i\omega \cos f}{\beta}\right]. \end{aligned}\right\} \quad (6)$$

Multiplying by the reflection coefficients $R_{SP'}$ and $R_{PP'}$ for reflections at $z = h$ and $z = -h$ and taking account of the change of type at $z = h$, the contribution by the ray SPP to the P wave at L is

$$A_S \exp\left[i\omega\left(t - \frac{x \sin e + z \cos e}{\alpha}\right)\right] \times$$
$$\times \exp\left[-(3h + z)\frac{i\omega \cos e}{\alpha} - (h - z)\frac{i\omega \cos f}{\beta}\right] R_{SP'} R_{PP'}. \quad (7)$$

Similarly, we can write down the contributions of the other rays arriving at L. Equating the displacement in the P wave at L

to the sum of the displacements in the P waves arriving at L and doing the same for the S waves, we get the two equations

$$A_P\, R^2_{PP'}\, e^{-4ha} + A_P\, R_{PS'}\, R_{SP'}\, e^{-2ha-2hb} + \\ + A_S\, R_{SP'}\, R_{PP'}\, e^{-(3h+z)a-(h-z)b} + A_S\, R_{SS'}\, R_{SP'} e^{-(h+z)a-(3h-z)b} = A_P. \quad (8)$$

and

$$A_P\, R_{PP'}\, R_{PS'} e^{-(3h-z)a-(h+z)b} + A_P\, R_{PS'}\, R_{SS'}\, e^{-(h-z)a-(3h+z)b} + \\ + A_S\, R_{SP'}\, R_{PS'}\, e^{-2ha-2hb} + A_S\, R^2_{SS'}\, e^{-4hb} = A_S. \quad (9)$$

We have removed a factor $\exp\left[i\,\omega\left(t - \dfrac{x \sin e + z \cos e}{\alpha}\right)\right]$ from (8) and $\exp\left[i\,\omega\left(t - \dfrac{x \sin f + z \cos f}{\beta}\right)\right]$ from (9) and also made use of the notation introduced in (1). A consequence of (1) is

$$\left. \begin{aligned} a &= \frac{i\,\omega \cos e}{\alpha} = \left(\zeta^2 - \frac{\omega^2}{\alpha^2}\right)^{1/2}, \; a_1 = \frac{i\,\omega \cos e_1}{\alpha_1} = \left(\zeta^2 - \frac{\omega^2}{\alpha_1{}^2}\right)^{1/2}, \\ b &= \frac{i\,\omega \cos f}{\beta} = \left(\zeta^2 - \frac{\omega^2}{\beta^2}\right)^{1/2}; \; b_1 = \frac{i\,\omega \cos f_1}{\beta_1} = \left(\zeta^2 - \frac{\omega^2}{\beta_1{}^2}\right)^{1/2}. \end{aligned} \right\} \quad (10)$$

Equations (8) and (9) may be written as

$$A_P(R_{PP'}{}^2\, e^{-4ha} + R_{PS'}\, R_{SP'}\, e^{-2ha-2hb} - 1) \\ = - A_S\, R_{SP'}\, e^{-(h+z)a-(h-z)b}\,(R_{PP'}\, e^{-2ha} + R_{SS'}\, e^{-2hb}) \quad (11)$$

and

$$A_S\,(R_{SP'}\, R_{PS'}\, e^{-2ha-2hb} + R_{SS'}{}^2\, e^{-4hb} - 1) \\ = - A_P\, R_{PS'}\, e^{-(h-z)a-(h+z)b}\,(R_{PP'}\, e^{-2ha} + R_{SS'}\, e^{-2hb}). \quad (12)$$

The amplitude factors A_P, A_S and the angles e, f which determine the directions of propagation of the waves were taken quite arbitrarily in (2) except that the latter were supposed to satisfy the obvious condition of obeying Snell's Law. For given ω and ζ, (11) determines the ratio $A_P : A_S$ that is necessary for the maintenance of the P-vibrations at all points level with L. (12) similarly gives the ratio $A_P : A_S$ for the maintenance of the S-vibrations. For constructive interference *both* must be maintained. This is not possible in general, but can be achieved if ω and ζ are so related that both equations give the same value for the ratio $A_P : A_S$. Eliminating this ratio between the two equations, we get an equation which gives ζ in terms of ω, or, in other words, gives the directions of propagation, corresponding to any given ω, for which constructive interference is possible. This equation is the frequency equation of Rayleigh-type waves that can be propagated in the stratum.

Eliminating the ratio $A_P : A_S$ from (11) and (12), we get

$$(R_{PP'}^2 e^{-4ha} + R_{PS'} R_{SP'} e^{-2ha-2hb} - 1) \times \\ \times (R_{SP'} R_{PS'} e^{-2ha-2hb} + R_{SS'}^2 e^{-4hb} - 1) \\ = R_{PS'} R_{SP'} e^{-2ha-2hb} (R_{PP'} e^{-2ha} + R_{SS'} e^{-2hb})^2. \quad (13)$$

It may be observed that (13) does not involve z. If it did, then the directions of propagation necessary for constructive interference would have differed from one level to another making simultaneous constructive interference at all levels in the stratum impossible.

Substituting the values of $R_{PP'}$, etc. from Chopra (1958), (13) becomes

$$(Y_2^2 e^{-4ha} - N_5 N_6 e^{-2ha-2hb} - Y_1^2)(-N_5 N_6 e^{-2ha-2hb} + Y_3^2 e^{-4hb} - Y_1^2) \\ = -N_5 N_6 e^{-2ha-2hb} (Y_2 e^{-2ha} + Y_3 e^{-2hb})^2. \quad (14)$$

But, from Chopra (1958),

$$-N_5 N_6 = Y_1 Y_4 + Y_2 Y_3. \quad (15)$$

Using (15), (14) simplifies to

$$Y_1^2 - 2 Y_1 Y_4 e^{-2ha-2hb} + Y_4^2 e^{-4ha-4hb} - \\ - (Y_2^2 e^{-4ha} + 2 Y_2 Y_3 e^{-2ha-2hb} + Y_3^2 e^{-4hb}) = 0. \quad (16)$$

This is satisfied if

$$Y_1 - Y_2 e^{-2ha} - Y_3 e^{-2hb} - Y_4 e^{-2ha-2hb} = 0, \quad (17)$$

or

$$Y_1 + Y_2 e^{-2ha} + Y_3 e^{-2hb} - Y_4 e^{-2ha-2hb} = 0. \quad (18)$$

(17) is the frequency equation of Rayleigh-type waves with symmetric vibrations and (18) is the frequency equation of Rayleigh-type waves with antisymmetric vibrations that can be propagated

in the stratum. The fact that the symmetric and antisymmetric vibrations have become uncoupled is due to the symmetry obtaining in the medium.

In problems dealing with the actual generation of elastic waves, the wave-theoretic solutions are usually in the form of integrals which cannot be evaluated exactly. In fact, even the numerical evaluation of some of these has become possible only in very recent times with the help of electronic computers. The integrals have therefore to be evaluated approximately by using asymptotic methods. Such evaluations show that the disturbance at any point in the medium may be resolved into waves of one or more of the following types, depending upon the position of the point relative to the source of disturbance.

(i) Minimum-time-path waves predicted by the ray theory.

(ii) Minimum-time-path refracted waves which could not carry any energy according to the simple ray theory but are found to carry significant amounts of energy. (In fact, these refracted arrivals were in use in exploration seismology long before their reality was established by mathematical analysis.)

(iii) Numerous diffracted waves which can be regarded as minimum-time-path waves in an extended sense if complex angles of propagation are permitted.

(iv) Proper diffraction effects such as the Rayleigh- and Stoneley-type waves.

At short distances from the source the ray theory gives an adequate picture of the disturbance but at larger distances more and more diffracted waves have to be added to get a true picture of the disturbance.

Recently, it has been shown (Chopra, 1958) that for waves of type (i) exactly the same displacements can be obtained by the fully developed ray theory of geometrical optics as are obtained by the asymptotic evaluation of the wave-theoretic solution.

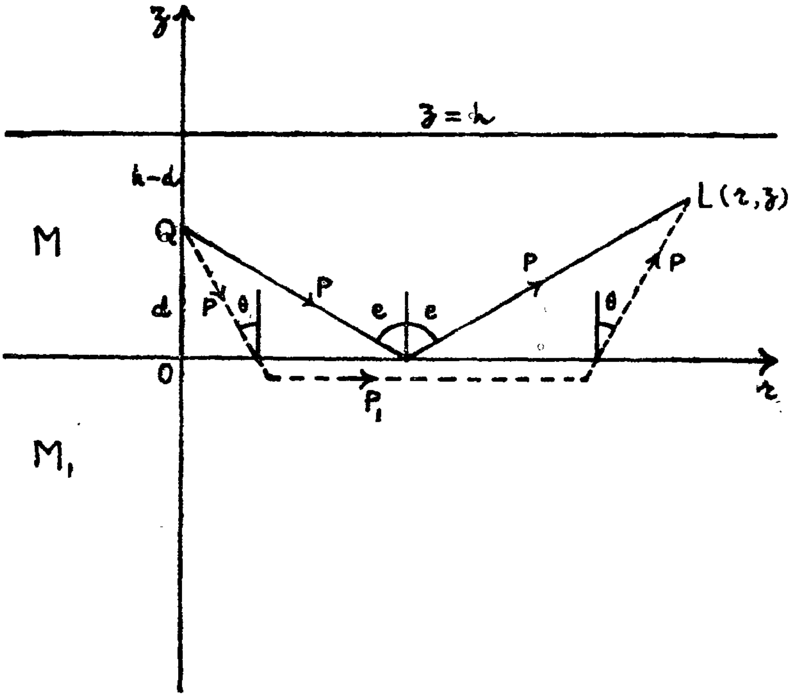

Fig. 2. Two-layer model. λ, μ are the Lame's constants, ρ the density, and α, β the velocities of dilatational and distortional waves for the layer M and λ_1, μ_1, ρ_1, α_1, β_1, the corresponding quantities for the underlying halfspace M_1. $L(r, z)$ is the point of observation. $\theta = \sin^{-1}(\alpha/\alpha_1)$. It is assumed that $\alpha_1 > \alpha > \beta_1 > \beta$.

An important advance that has taken place recently is the discovery that just as the wave-theoretic solution can be analysed to yield all the waves predicted by the ray theory, the wave-theoretic solution can be obtained by a proper synthesis of all the ray-theory-path waves using (i) the integral representation of the source as a summation of plane waves, and (ii) the plane wave reflection and refraction coefficients.

This can be done as follows.

(i) Catalogue all possible *simple* ray-theory paths.

(ii) For every such path, start with the elementary plane wave (EPW) in the integral representation (I) of the source as a summation of plane waves and obtain from it the derived plane wave (DPW) corresponding to travel by that path by multiplying it (EPW) by the appropriate reflection and refraction coefficients for each transformation at an interference and taking account of the change of type, if any, at each such transformation.

(iii) Replace the (EPW) in (I) by every (DPW) in turn, thus obtaining one integral corresponding to each path in (i).

(iv) Finally, collect the integrals thus obtained for all the P-arrivals into one series and those for S-arrivals into another. The sums of these series give the required wave solution.

For example, consider the model of Fig. 2 in which layer M of thickness h overlies the semi-infinite medium M_1. Let there be a point source of harmonic compressional waves in the layer M at a distance d from the interface between M and M_1. The displacement potential for the point source at Q can be represented by the Sommerfeld's integral (Magnus and Oberhettinger, 1954, p. 34).

$$\Phi_0 = \frac{1}{R} \exp\left\{ i\,\omega \left(t - \frac{R}{\alpha} \right) \right\}$$

$$= e^{i\omega t} \int_0^\infty \exp\left\{ -|z-d| \left(\zeta^2 - \frac{\omega^2}{\alpha^2} \right)^{1/2} \right\} J_0(r\zeta) \frac{\zeta\, d\zeta}{(\zeta^2 - \omega^2/\alpha^2)^{1/2}}, \quad (19)$$

where $R^2 = r^2 + (z-d)^2$,

$$= \frac{\omega e^{i\omega t}}{i\alpha} \int_{0,\frac{1}{2}\pi}^{\frac{1}{2}\pi + i\infty} J_0\left(\frac{\omega \sin e}{\alpha} r \right) \exp\left(-\frac{i\,\omega}{\alpha} |z-d| \cos e \right) \sin e\, de \quad (20)$$

$$\left(\text{where } \frac{\sin e}{\alpha} = \frac{\zeta}{\omega} \right)$$

$$= \frac{\omega\, e^{i\omega t}}{2\pi i \alpha} \int_{0,\frac{1}{2}\pi}^{\frac{1}{2}\pi + i\infty} \int_0^{2\pi} \exp\left[-\frac{i\omega}{\alpha} (x \sin e \cos g + y \sin e \sin g + \right.$$

$$\left. + |z-d| \cos e) \right] \sin e\, de\, dg, \quad (21)$$

which exhibits the point source as a superposition of plane waves of equal intensity whose directions of propagation make with the

z-axis not only all angles e lying between 0 and $\frac{1}{2}\pi$ but also a continued series of complex angles extending from $\frac{1}{2}\pi$ to $\frac{1}{2}\pi + i\,\infty$. g is the second polar angle of the direction of propagation of any such wave. The expression (21) is due substantially to Weyl (1919).

The displacement potential for the wave PP once reflected from the interface is given by

$$e^{i\omega t}\int_0^\infty R_{PP'}\,J_0(r\zeta)\,e^{-(d+z)a}\,\frac{\zeta d\zeta}{a}, \tag{22}$$

where $a = (\zeta^2 - \omega^2/\alpha^2)^{1/2}$ as in (10). By using the relation

$$J_0(r\zeta) = \tfrac{1}{2}\,Hs_0(r\zeta) + \tfrac{1}{2}\,Hi_0(r\zeta) \tag{23}$$

(Jeffreys and Jeffreys, 1956, Sec. 21.02), the expression (22) can be transformed to

$$\tfrac{1}{2}\,e^{i\omega t}\int_\Gamma R_{PP'}\,Hi_0(r\zeta)\,e^{-(d+z)a}\,\frac{\zeta d\zeta}{a}, \tag{24}$$

where Γ is the Sommerfeld contour (Fig. 3) which surrounds the four branch lines Re $a_1 = 0$, Re $a = 0$, Re $b_1 = 0$, Re $b = 0$, a_1, a, b_1, b being defined in (10), and the Stoneley pole, if any, relating to the interface between the layer and the underlying medium. ω is now regarded as complex with Re $\omega > 0$ and Im $\omega < 0$. The details of the transformation from (22) to (24) are given, for example, in Lapwood (1949, p. 73).

Replacing $Hi_0(r\zeta)$ in (24) by the first term of its asymptotic expansion, viz.

$$Hi_0(r\zeta) \sim \left(\frac{2}{\pi r\zeta}\right)^{1/2} e^{-i(r\zeta - \frac{1}{4}\pi)} \tag{25}$$

valid for $-2\pi < \arg(r\zeta) < \pi$ and $|r\zeta|$ large (Copson, 1935, p. 336), we get

$$\left(\frac{i}{2\pi r}\right)^{\frac{1}{2}} e^{i\omega t}\int_\Gamma R_{PP'}\,e^{-ir\zeta-(d+z)a}\,\frac{\zeta^{\frac{1}{2}}\,d\zeta}{a}. \tag{26}$$

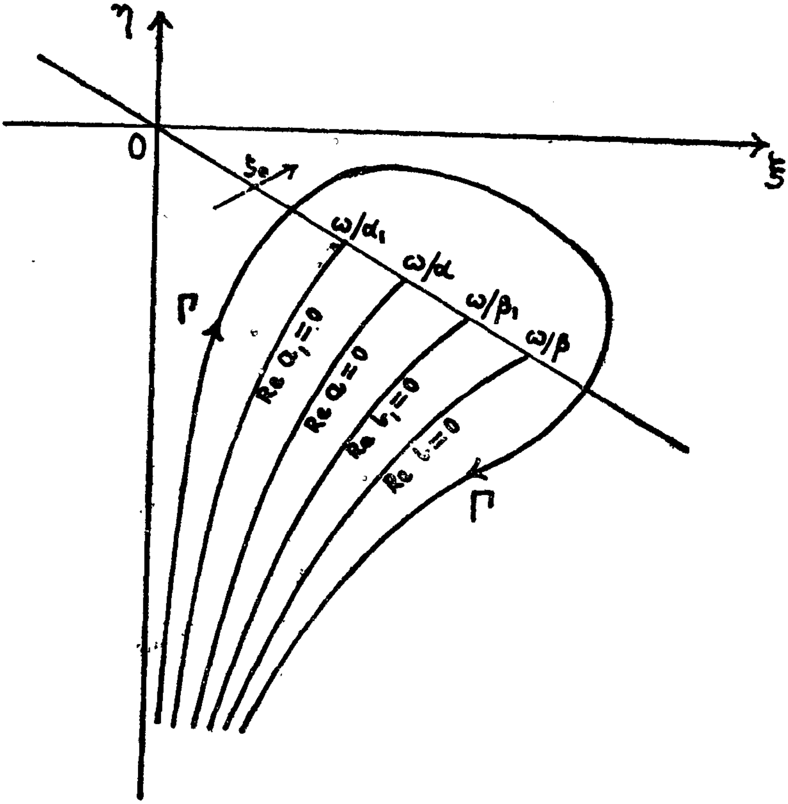

Fig. 3. The Sommerfeld Contour Γ in the ζ-plane. $\omega = c - i\,s$, $c, s > 0$. ω/a_1, etc. are branch points and the plane is cut along the lines Re $a_1 = 0$, etc. to render the integrand uniform. ζ_0 is the saddle point.

The saddle point approximation of (26) will give the accurate expression for the disturbance in the ray PP, viz.

$$\frac{-\,i\,\omega(\sin e \cos e)^{\frac{1}{2}}\,(R_{PP'})_{\zeta=\zeta_0}}{\alpha[\,r(z+d)\,]^{\frac{1}{2}}} \exp\left[\,i\,\omega\left(\,t - \frac{z+d}{\alpha}\sec e\,\right)\right], \quad (27)$$

where the saddle point S_0 is given by the equations

$$\frac{\zeta_0}{\omega} = \frac{\sin e}{\alpha}, \qquad (z+d)\tan e = r. \qquad (28)$$

The same expression can be obtained by the application of fully developed ray theory of geometrical optics.

Evaluation round the branch line Re $a = 0$ will give the minimum-time-path refracted wave PP_1P (shown dotted in Fig. 2), the phase factor for which is

$$\exp\left[i\omega\left\{t - \frac{r}{\alpha_1} - \frac{(z+d)(\alpha_1^2 - \alpha^2)^{1/2}}{\alpha\,\alpha_1}\right\}\right]. \tag{29}$$

The refracted ray PP_1P will exist only for

$$r \geqslant (z+d)\frac{\alpha}{(\alpha_1^2 - \alpha^2)^{1/2}}, \tag{30}$$

and then has a smaller travel-time than that of PP.

Evaluation round the branch line Re $a=0$ will give an approximation to (27) while the evaluations round the remaining two branch lines will give diffracted waves with exponential decay. Evaluation round the Stoneley pole, if any, will give the Stoneley wave corresponding to the interface.

It follows from the above work that we can work out the displacement in any ray without going through the tedious process of first obtaining the wave solution and its Bromwich expansion.

This address would be incomplete without a reference to the work of J. B. Keller and his colleagues at the Institute of Mathematical Sciences at the New York University. In a couple of recent papers (Seckler and Keller, 1959 *a*, *b*), they have extended the application of Fermat's principle and Huygen's principle to develop the geometrical theory of diffraction in problems in acoustics. The theory is used to determine the diffracted fields in inhomogeneous media containing smooth convex bodies. The application of the extended form of Fermat's principle yields diffracted rays and energy considerations give the field associated with each ray. This calculation introduces diffraction coefficients and decay exponents and general formulae are given for the calculation of these quantities. As a check on the theory, the results are shown to be in agreement with the asymptotic solutions provided by the wave theory methods.

In the end I wish to thank the Programme Committee of the Indian Mathematical Society for their kind invitation which has given me the opportunity to present this material to-day.

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EFFECTIVENESS OF ABSOLUTE SUMMABILITY*

By T. PATI

1. Introduction. The absolute convergence of an infinite series Σa_n is the same as the bounded variation of the sequence of its partial sums $\{s_n\}$, since

$$\Sigma \mid a_n \mid = \Sigma \mid s_n - s_{n-1} \mid .$$

By analogy, the *absolute summability* T, or summability $\mid T \mid$, of Σa_n is defined as the bounded variation of the T-transform of the sequence $\{s_n\}$: $\{t_n\}$, if T is sequence-to-sequence; $t(x)$ over an appropriate interval, if T is sequence-to-function.

The purpose of this address is to discuss the present situation with regard to questions concerning the range of applicability—technically termed 'effectiveness'—of absolute summability methods.

2. Definitions. If a method of summability T is such that the bounded variation of $\{s_n\}$ implies the bounded variation of its T-transform, then we say that the method is *absolutely conservative* or *absolute convergence preserving*, or briefly AK. If, moreover, the method is *regular*, that is to say, preserves both convergence and the sum for every convergent sequence, it is said to be *absolutely regular*, or AR. Morley [18] has remarked that if a method is AK, it is not necessary that it should be K (conservative).

An AK method of absolute summability $\mid T \mid$ will be said to be *effective*, if at least one sequence of unbounded variation is summable $\mid T \mid$. It will, accordingly, be said to be *ineffective*, if no sequence which is not of bounded variation is summable $\mid T \mid$. If every sequence which is summable $\mid A \mid$ is summable $\mid B \mid$, but the converse is not true, we say that the method $\mid A \mid$ is *less effective* than the method $\mid B \mid$. We write, adopting the notation of set-theory, $\mid A \mid \subset \mid B \mid$, $\mid A \mid \supset \mid B \mid$ and $\mid A \mid \sim \mid B \mid$, respectively, to express the facts that

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$|A|$ is *not more effective* than $|B|$, *not less effective* than $|B|$, and is *as effective as* $|B|$. The relation $|A| \sim |B|$ is obviously the resultant of the relations: $|A| \subset |B|$ and $|A| \supset |B|$. Two equally effective methods are called *equivalent.* If neither $|A| \subset |B|$ nor $|A| \supset |B|$, we shall say that the methods $|A|$ and $|B|$ are *incomparable.*

3. Some more definitions. The notion of absolute conservativeness can be extended in a natural manner by replacing absolute convergence by absolute summability in the definition. Thus, we shall say that a method H is *absolutely T-conservative* or *absolute T-summability preserving*, or briefly ATK, if the H-transform of every $|T|$-summable sequence is also $|T|$-summable. With the additional proviso that the method H in question is *T-regular,* that is to say, preserves T-summability and also the T-limit of every T-summable sequence, we construe the definition of *absolute T-regularity*, and call such a method H an ATR method. An ATK method H is characterized by the inclusion-relation $|T| \subset |TH|$, where TH denotes the iteration product method which associates with any sequence the T-transform of its H-transform, and $|TH|$ denotes absolute summability by the method TH.

4. Conditions that a method be ATK or ATR. Necessary and sufficient conditions that a sequence-to-sequence method be AK and AR were first obtained by Mears [16] and the analogous conditions for series-to-sequence and series-to-series methods have been obtained by Knopp and Lorentz [11] and Sunouchi [33], respectively. Function-to-function methods of the type:

$$\gamma_s(x) = \int_0^\infty \psi(x, t)\, ds(t),$$

of which sequence-to-sequence and sequence-to-function transformations are special cases, have been considered by Tatchell [35] and Sunouchi and Tsuchikura [34].

Morley [18] has shown that a Hausdorff method is AK if and only if it is K, and Ramanujan [30] has demonstrated that a quasi-Hausdorff method is AK if and only if it is K. The present speaker has obtained certain results regarding the conditions that, assuming T to be a regular transformation (e.g. a Wienerian transformation), a Hausdorff method be ATK. In this line, three types of problems naturally suggest themselves.

(i) To find conditions that a method which is ATK should be TK.

(ii) To find conditions that a method which is AK should be ATK.

(iii) To characterize the methods which are ATK if and only if they are TK.

5. Relative effectiveness. We shall confine our remarks only to three of the most familiar and heavily worked methods of absolute summability, viz. absolute *Abel*, *Cesàro* and *Riesz* methods.

We state the definitions for the sake of completeness.

If $\Sigma a_n x^n$ is convergent for $|x| < 1$, and the sum-function $f(x)$ is of bounded variation in $(0, 1)$, Σa_n is said to be absolutely Abel summable, or summable $|A|$. If the sequence of Cesàro-means of order α $(\alpha > -1)$ of $\{s_n\}$ is of bounded variation, we say that Σa_n is absolutely summable (C, α), or summable $|C, \alpha|$. If the Riesz-mean of *type* λ_n and *order* k, defined by

$$R_\lambda^k(\omega) = \omega^{-k} \sum_{\lambda_n \leqslant \omega} (\omega - \lambda_n)^k a_n, \quad (k \geqslant 0),$$

is of bounded variation over (h, ∞), where h is some finite positive number, then Σa_n is said to be absolutely summable (R, λ_n, k), or summable $|R, \lambda_n, k|$. The definitions of absolute Abel, Cesàro and Riesz summability are due respectively to Whittaker, Fekete and Obrechkoff [36], [5], [19].

All these methods are AR.

We have the following *negative* results.

(i) $|C|$ and (C) are incomparable if the order of absolute summability is superior to that of ordinary (Kogbetliantz [12]).

(ii) $|A|$ and $(C, 0)$ (i.e., convergence) are incomparable (Whittaker [36], Prasad [27]), even if the series considered be Fourier series.

(iii) $|A| \not\subset (C, k)$ for any positive k, however large (Fekete [6]).

(iv) $|A| \not\subset |C, k|$ for any positive k, however large, even if the series considered be Fourier series (Randels [31]).

We have the following *inclusion-relations.*

(v) $|C, k| \subset |C, k'|$ for every $k' > k > -1$. (This is called the 'consistency theorem' for $|C|$-summability; first proved by Kogbetliantz [12], and shorter proofs supplied by Morley [18] and Knopp and Lorentz [11]).

(vi) $|C, k| \subset |A|$ for every $k > -1$, however large (Fekete [6]).

We have the following *equivalence-relation.*

(vii) $|R, n, k| \sim |C, k|$ for every $k \geqslant 0$ (Hyslop [8]).

In view of its wide scope and flexibility, the problem of effectiveness of the Riesz method will be discussed here in some detail. For a proper understanding of the relative effectiveness of any two Riesz methods, it is necessary to take into account both the variation in the type and the variation in the order. Accordingly, our knowledge in this field can be broadly classified under two distinct categories.

The so-called 'first theorem of consistency' for absolute Riesz summability states that $|R, \lambda_n, k| \subset |R, \lambda_n, k'|$ for every $k' > k \geqslant 0$ [19]. Here the two methods have the same type. Since, by definition, summability $|R, \lambda_n, 0|$ is the same as absolute convergence, this result includes the assertion that summability (R, λ_n, k), $k > 0$, is AR. In other words, the first theorem of consistency asserts that the effectiveness of a method of absolute Riesz summability increases with the order, if the type remains unchanged. But when the type is made to vary, while the order remains fixed, the situation is more complicated. Inclusion relations of the type : $|R, \lambda_n, k| \subset |R, \varphi(\lambda_n), k|$ are called 'second theorems of consistency'. Indeed, while there is only one first theorem of consistency, there are quite a number of second theorems of consistency! We give here a brief resumé of the most important second theorems of consistency for absolute Riesz summability.

6. Second Theorems of Consistency. To Chandrasekharan [3] is due the first 'second theorem of consistency' for absolute summability. His result is the direct analogue of a prior result of Hardy for the case in which the order of summability is a positive integer, and asserts that $|R, \lambda_n, k| \subset |R, \mu_n, k|$, if μ_n is a logarithmico-exponential function of λ_n such that $\mu_n = O(\lambda_n^{\Delta})$ for some finite constant Δ. A generalisation of this result was obtained by Pati [23] in 1954, in the form of a theorem in which the relation of μ_n and λ_n was generalized to a functional relation of the type : $\mu_n = \varphi(\lambda_n)$, where $\varphi(t)$ is a monotonic (strictly) increasing non-negative function of t for $t \geqslant 0$, and a $(k+1)$th integral (i.e. indefinite Lebesgue integral) such that

$$t^r \varphi^{(r)}(t)/\varphi(t) \in BV(0, \infty) \quad (r = 1, 2, \ldots, k). \qquad (*)$$

In 1957 Prasad and Pati [28] gave an analogue of this result for the case of non-integral order of summability. Meanwhile, for both integral and non-integral orders of summability, Guha [7] had proved general second theorems of consistency for absolute summability. Guha's result for the case of integral order of summability is the most general result so far, in that he replaces conditions (*) by the single condition : $t^k \varphi^{(k)}(t) = O\{\varphi(t)\}$, as $t \to \infty$.

Guha's theorem for the case of non-integral order of summability is a more complicated result, involving the use of 'backward' fractional order derivative of the function : $\{1 - \varphi(t)/\varphi(\tau)\}^k$ for $\tau \geqslant t > 0$. Guha's proof of this theorem has been very recently replaced by a shorter and more direct proof by Pati [26].

Generalizing their previous work on the second theorem of consistency for the case of *non-intergal* order, Prasad and Pati [29] have recently obtained the following theorem.

If $\varphi(t)$ is a non-negative, monotonic increasing function of t for $t \geqslant 0$, steadily tending to infinity as $t \to \infty$, $\varphi(t)$ is a $(k+1)$th indefinite (Lebesgue) integral, $\varphi^{(1)}(t)$ is monotonic non-decreasing, and $t^{[k]+1} \varphi^{([k]+1)}(t) = O\{\varphi(t)\}$, as $t \to \infty$, then $|R, \lambda_n, k| \subset |R, \varphi(\lambda_n), k|$.

This theorem is a direct consequence of a more general one in which the authors assume, instead of the monotonicity of $\varphi^{(1)}(t)$, only that, uniformly in $0 < v < 1$ and $s > 0$,

$$s^k \left\{ \frac{(1-v)\,t}{\varphi(s+t) - \varphi(s+vt)} \right\}^{[k]+1-k} \frac{\{\varphi^{(1)}_{e}(s+vt)\}^{[k]+1}}{\{\varphi(s+t)\}^k} \in BV_t(0, \infty),$$

$[k]$ denoting the integral part of k.

7. Ineffectiveness of absolute summability. The ineffectiveness of a method of absolute summability may occur on account of one of the following reasons :

(i) by *definition*, that is, on account of the structure of the method ;

(ii) by virtue of *restriction on the terms of the series* ; and

(iii) by virtue of *gaps in the series.*

Theorems that assert ineffectiveness on grounds (ii) and (iii) are called 'Tauberian' and 'high-indices' theorems, respectively. We discuss these three types of ineffectiveness with respect to absolute Abel, Cesàro and Riesz summability.

(i) Absolute Cesàro summability and, more generally, absolute Riesz summability, of order zero are by definition ineffective. When the type of summability is $\exp(n)$, we know that ordinary Riesz summability is ineffective (in the sense that it implies convergence), however large the order might be. The analogue of this result has been very recently proved by Dikshit [4] by means of a technique of 'finite differences' introduced into the theory of summability by Kloosterman [10]. Indeed, for the case in which the order is unity, the ineffectiveness of absolute Riesz summability of type $\exp(n)$ had been proved earlier by Mohanty [17] and Obrechkoff [20]. Mohanty, incidentally, had proved that absolute summability by *discrete* Riesz means of type $\exp(n)$ and order unity is ineffective. We shall say that Σa_n is absolutely summable by the discrete Riesz method of type λ_n and order k, or summable $|R^*, \lambda_n, k|$, $k \geqslant 0$, if the sequence $\left\{ \frac{1}{\lambda^k_{n+1}} \sum_{\nu=1}^{n} (\lambda_{n+1} - \lambda_\nu)^k a_\nu \right\}$ is of bounded variation. By definition, summability $|R^*, \lambda_n, 0|$ is the same as absolute convergence. Pati [25] has very recently improved upon Mohanty's result by proving that even summability $|R^*, \exp(n), 2|$ is ineffective. The general problem of determining whether summability $|R^*, \exp(n), k|$ for any positive k, however large, is ineffective is yet to be solved.

A number of theorems are known, analogous, for absolute summability, to the well-known theorem of Mercer [2], [32], [15]. The following general theorem has recently been proved by Parameswaran [21].

If P is an AK method, defined by the lower-semimatrix $(p_{n,m})$, and

$$|p_{n,m}| - \sum_{k=n+1}^{\infty} \left| \sum_{i=n}^{k} (p_{k,i} - p_{k-1,i}) \right| > \lambda > 0$$

for all $n = 0, 1, 2, \ldots$, then summability $|P|$ is ineffective.

The technique of proof consists in the exploitation of the facts that the sets of matrices defining AK and R methods are isomorphic under a suitable correspondence, and that the set of matrices defining AK methods (for sequences) is a complex Banach algebra, under a suitable norm.

(ii) *Tauberian theorems.* Hyslop has shown [9] that if the sequence of $(C, 1)$ means of $\{na_n\}$ is of bounded variation, then the summability $|A|$ of Σa_n necessitates its absolute convergence. This result is evidently more general than the direct analogue for absolute summability of Littlewood's famous extension of Tauber's theorem. A generalisation of Hyslop's theorem is due to Sunouchi [33]. And, in the wake of Karamata's researches, Lorentz [14] has developed a general method for obtaining Tauberian conditions for absolute summability. He also shows that the Tauberian condition of Hyslop for absolute Abel summability cannot be improved in a certain sense [14].

On the ineffectiveness of absolute Riesz summability for series satisfying Tauberian conditions the only general results so far in the literature are the following theorems, due to Pati [22] and Bhatt [1] respectively, the latter being the more general of the two.

If Σa_n *is summable* $|R, \lambda_n, k|$, $k > 0$, $\{\lambda_n/\lambda_{n+1}\}$ *is of bounded variation, and* $\{\lambda_n a_n/(\lambda_n - \lambda_{n-1})\}$ *is of bounded variation, then* Σa_n *is absolutely convergent.*

If Σa_n *is summable* $|R, \lambda_n, k|$, $k > 0$, $\{\lambda_n/\lambda_{n+1}\}$ *is of bounded variation, and* $\left\{\frac{1}{\lambda_n} \sum_{\nu=1}^{n} \lambda_\nu a_\nu\right\}$ *is of bounded variation, then* Σa_n *is absolutely convergent.*

The Tauberian condition on a_n in the latter theorem is the discrete analogue of the summability $|R, \lambda_n, 1|$ of the sequence which is required in the former theorem to be absolutely convergent, that is of bounded variation. But the condition on $\{\lambda_n/\lambda_{n+1}\}$ is seen to be common to both the theorems. Very recently Pati [24] has established two Tauberian theorems for absolute Riesz summability in which the conditions are of a necessary and sufficient type and involve functions related to this sequence.

(iii) *High-indices theorems.* For absolute summability the first theorem of this kind seems to be the result of Zygmund [37] that summability $|A|$ is ineffective for series with gaps defined by: $a_{n \neq n_\nu} = 0$, $n_{\nu+1}/n_\nu > q > 1$.

Lorentz [13] has obtained the result that " $a_n = 0$ for $n \neq n_\nu$ $(\nu = 1, 2, \ldots)$ " is not a 'gap-condition' for the ineffectiveness of the method of absolute summability defined by the matrix $(a_{m,n})$, if

$$\lim_{\nu \to \infty} \left\{ \operatorname*{var}_{m} \sum_{n_\nu \leqslant n < n_{\nu+1}} a_{m,n} \right\} = 0,$$

and has observed that the gap-condition of Zygmund cannot be improved.

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Allahabad

SYMPOSIUM ON STATISTICAL MECHANICS AND THE PARTITION THEORY OF NUMBER

ABSTRACT OF PAPER

By D. S. KOTHARI and F. C. AULUCK

THE central problem of stastistical mechanics is the determination of the number of ways in which a given amount of energy can be shared among the different possible states of an assembly, and this problem is the same as that of determining the number of ways in which a given positive integer can be written as the sum of given summands. The theory of partitions of numbers has been extensively developed by Hardy and Ramanujan. They have shown that if $p(n|s)$ represents the number of ways of writing n as the sum of the sth powers of positive integers, the asymptotic expression for $p(n|s)$ is

$$p(n|s) \approx (2\pi)^{(-s+1)/2} \left(\frac{s}{s+1}\right)^{1/2} kn^{(1/(s+1))-(3/2)} \exp\left[(s+1)kn^{1/(s+1)}\right]$$

where

$$k = [1/s\,\Gamma(1+1/s)\,\zeta(1+1/s)]^{s/(s+1)}.$$

The same result can be obtained from thermodynamic considerations. Thus if the energy levels are

$$\epsilon = r^s, \quad r = 1, 2, 3,$$

the number of energy levels in the energy range $\epsilon, \epsilon + d\epsilon$ is

$$r^{(1-s)}s\,d\epsilon = 1/s\,\epsilon^{(1-s)/s}\,d\epsilon$$

and therefore the total energy of the system in the perfectly degeneration Bose case is

$$E = 1/s \int\limits_0^\infty \frac{t^{1/s}\,dt}{e^{t/T}-1}$$

$$= 1/s\,\Gamma(1+1/s)\,\zeta(1+1/s)\,T^{1+1/s}$$

where the temperature T is measured in energy units.

The entropy φ of the system is then given by

$$\varphi = \int \frac{dE}{T}$$
$$= (1+s)\{1/s\,\Gamma(1+1/s)\,\zeta(1+1/s)\}^{s/(s+1)}\,E^{1/(1+s}$$
$$= k(1+s)\,E^{1/(1+s)}.$$

The Boltsmann relation gives the exponential factor in $p(n\,|\,s)$ immediately, assuming n to be the energy of the system. A more detailed calculation gives the above asymptotic formula for $p(n\,|\,s)$. This approach is particularly more useful when the number of summands is restricted, and in this manner $p_m(n\,|\,s)$, the number of partitions of n into m sth powers of integers can be obtained. The Fermi-Dirac statistics correspond to the case when all the summands are different, and thus we can calculate $q_m(n\,|\,s)$. In the case $s=1$ we denote the corresponding functions by $p_m(n)$ and $q_m(n)$ respectively.

It has been shown by Szekeres that the distribution function $p_m\,(n)$ possesses a unique maximum $m > m_0$, where m_0 is given by the expression, for large values of n,

$$m_0 = c\,n^{1/2}\,L - \left(\frac{1}{4}L^2 - \frac{3}{2}L - \frac{3}{2}\right)c^2 - \frac{1}{2} + $$
$$+\,0\,(n^{-1/2}\log^4 n),$$

where

$$C = \left(\frac{6}{\pi^2}\right)^{1/2} \quad \text{and} \quad L = \log\,(Cn^{1/2}).$$

This proves a conjecture of Auluck, Chowla and Gupta regarding the uniqueness of the maximum of $p_m(n)$. In connection with Bohr's statistical theory of nuclear structure, Umeda and later Kodi Husimi obtained an expression $\overline{m}\,(n)$ for the average number of summands in a partition of n into m parts. We have for $\overline{m}(n)$, the expression

$$\overline{m}(n) = \left(c N^{1/2} + \frac{c^2}{2}\right)\left\{\log \frac{\sqrt{6N}}{\pi} + \gamma\right\} + \frac{1+c^2}{4} + \\ + O(N^{-1/2} \log N),$$

where $N = n - 1/24$ and γ is Euler's constant. Miss Luthra has calculated the second, third and fourth moments of the distributions $p_m(n)$ and $q_m(n)$. For the distribution $p_m(n)$, $\beta_1 = 1.2$ and $\beta_2 = 5.4$, and for the distribution $q_m(n)$, $\beta_1 = 0$ and $B_2 = 3$ for $n \to \infty$. In a normal distribution $\beta_1 = 0$ and $\beta_2 = 3$.

It may be noticed that for the above distributions $m_0/\overline{m} \to 1$ for large values of n, that is the mean and the most probable states are the same. That this is not true in all cases can be seen by studying the partition function $p_m(n, \lambda_1, \lambda_2, \ldots)$ of a large positive integer n into m parts $\lambda_1, \lambda_2 \ldots$ where the numbers $\{\lambda_r\}$ form a sequence of positive integers. If we take the summands to be squares of the integers 1, 2, 3, ... then, as has been shown by Haselgrove and Temperley, $m_0/\overline{m} \to 0.5504$. It can be shown that the most probable value of m is different from the average value of m if $\Sigma\, \lambda_r^{-1}$ is convergent, and that $m_0/\overline{m} \to 1$ if $\Sigma\lambda_r^{-1}$ is divergent.

The notion of partition of numbers can be extended to non-integral summands also. If we denote the number of solutions of the inequality

$$r_1 \lambda_1 + r_2 \lambda_2 + \ldots < u$$

in integers $r_i \geqslant 0$ by $P(u)$, Ingham has obtained an asymptotic formula for

$$P_h(u) = \frac{P(u) - P(u - h)}{h}.$$

This, in the case of partitions into integers, reduces to the number of partitions of the largest integer less than u, when $h = 1$. Several of the results given above can be generalized to non-integral summands by using Ingham's Theorems and their generalizations.

ABSTRACT OF TALK BY H. GUPTA

In his talk, Hansraj Gupta, showed how any partition of a j-partite number

$$N_j = (n_1, n_2, n_3, \ldots, n_j),\ n_i > 0$$

into exactly k non-degenerate parts, could be graphically represented. This method provides a very simple proof of the result

$$k!\,p(N_j, k) \sim \prod_{i=1}^{j} \binom{n_i - 1}{k - 1},$$

which is a generalization of a result due to Erdös and Lehner (*Duke Math. J.*, 8, (1941), 335-45.)

SYMPOSIUM ON NON-NEWTONIAN AND VISCO-ELASTIC MEDIA

SOME SPECIAL QUESTIONS IN THE AXIALLY SYMMETRIC FLOW OF NON-NEWTONIAN VISCOUS LIQUIDS

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In the dynamics of non-Newtonian liquids it is an important problem to examine those features of the flow that are connected to the cross viscosity. It is well known that in plane flows of such liquids with constant coefficients of viscosity and crossviscosity, the terms involving the latter coefficient drop out in the integrability relation, so that questions with direct bearing on the vorticity of the flow do not undergo any change due to cross-viscosity. However in the axially symmetric case this is not so, and the present study of such flows is motivated from this consideration. We see that quite a number of possible solutions can be chosen by a direct reference to the integrability relation, some of which coincide with the forms of solution taken in the synthetic approach to special problems.

The integrability relation is expressible in the form

$$-\frac{2}{\varpi^2}\frac{\partial \Psi}{\partial x}E^2\Psi+\frac{1}{\varpi}\frac{\partial(\Psi, E^2\Psi)}{\partial(x,\varpi)}=\nu E^2E^2\Psi+$$

$$+\nu_c\left\{-\frac{1}{\varpi^2}\frac{\partial\Psi}{\partial x}E^2E^2\Psi+\frac{4}{\varpi^3}\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial\varpi}E^2\Psi-\frac{2}{\varpi^2}\frac{\partial^2\Psi}{\partial x\,\partial\varpi}\frac{\partial}{\partial\varpi}E^2\Psi\right.$$

$$\left.+\frac{4}{\varpi^3}\frac{\partial^2\Psi}{\partial x\,\partial\varpi}E^2\Psi-\frac{2}{\varpi^2}\frac{\partial^2\Psi}{\partial x^2}\frac{\partial}{\partial x}E^2\Psi-\frac{8}{\varpi^4}\frac{\partial\Psi}{\partial x}E^2\Psi\right\}.$$

The solutions obtained are of the forms

$$\Psi=\left[\left(c_1+\int f(r)\,r^{-n}\,dr\right)r^{n+1}+\left(c_2-\int f(r)\,r^{n+1}\,dr\right)r^{-n}\right]\times$$

$$\times\sin^2\theta\left[A\,P'_n(\cos\theta)+B\,Q'_n(\cos\theta)\right]$$

and linear combinations of such forms, and are found to include many of the known solutions as particular cases.

COUETTE AND POISEVILLE FLOW IN NON-NEWTONIAN FLUIDS

By Miss S. L. RATHNA

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WEISSNBERG and Mernington have done certain experiments on Couette and Poiseville flow with highly viscous fluids. Rivlin, Reiner and Servin attempted to explain these flows theoretically by taking the coefficients of viscosity and cross-viscosity as constants. In general the coefficients of viscosity and cross-viscosity are functions of the scalar invariants of D, the rate of deformation tensor, and of the thermodynamic state of fluid. In this paper we deal with the Couette and Poiseville flows by taking the coefficients of viscosity and cross-viscosity as functions of second invariant of D. The present investigation supports the conclusion of Servin in the case of Couette flow ; the strangulation of liquid can be explained in terms of cross-viscosity.

NOTE ON THE FLOW OF VISCO-ELASTIC LIQUIDS

By S. K. SHARMA

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OLDROYD, Strawbridge and Toms (1951) experimentally verified that the rheological behaviour of a dilute (3 p.c.) solution of highly polymerized methyl metha-crulate in certain organic liquids could be represented by a stress-strain law of the form

$$\sigma + \tau \dot{\sigma} = 2\mu(\gamma + \epsilon \dot{\gamma}),$$

where σ, γ are the deviatoric stress tensor and the rate of deformation tensor, respectively, dot denoting their rate of change. μ is the coefficient of viscosity and τ and ϵ the relaxation and retardation times of the fluid.

It has been long realized that quite a number of fluids in nature possess elasticity besides viscosity. A number of generalizations of the above-stated law of stress and strain have been thus suggested by Oldroyd and others.

We have studied some flow phenomena in some of these generalized fluid models and obtained interesting results. The present note is a brief review of those results.

ON THE NON-UNIFORM ROTATION OF A NON-NEWTONIAN LIQUID FILLED IN BETWEEN TWO COAXIAL CYLINDERS OF INFINITE LENGTH

By K. G. MITHAL

Lucknow University

An incompressible non-Newtonian liquid is contained between two coaxial cylinders of radii a and b, $(a < b)$. Initially the outer cylinder is rotating with constant angular velocity Ω_2 and the inner is at rest. Unsteadiness results when the inner cylinder is given an impulsive twist such that it also begins to rotate with constant angular velocity Ω_1.

Equation of motion has been set up for the case when Θ, the coefficient of shear viscosity varies as the nth power of the distance from the axis. This has been solved for $n = 2$ with the help of the Laplace transform.

It is found that for moderate values of t, unsteadiness spread from the inner cylinder, attains its maximum value at an intermediate point and dies out as the outer cylinder is reached. Cross-viscosity affects only the pressure.

ON CERTAIN ASPECTS ON NON-NEWTONIAN FLOW IN CHANNELS WITH FLEXIBLE AND POROUS WALLS

By M. N. L. NARASIMHAN

Indian Institute of Technology, Bombay.

Certain basic properties of non-Newtonian liquids are discussed. Problems concerning the flow of such liquids are studied by introducing second-order terms in the stress-strain velocity relations of classical hydrodynamics. The problem of non-Newtonian flow through an elastic tube is discussed on the basis of such relations. The cross-viscosity effects which arise in the case of these liquids are found to be opposite in nature to that of the inertial and viscous forces during the flow through an elastic tube. Also, the problem of non-Newtonian flow in an annulus with porous walls is discussed. Velocity and pressure fields for different cross-flow Reynolds' numbers have been obtained. Both injection and suction at the walls are studied. The pressure field is compared with that in the corresponding case of a solid wall annulus. It is found that the axial pressure drops in the direction of the axis and this drop is always greater than that in the case of the solid wall annulus. Next, the problem of non-Newtonian flow between parallel porous plates is studied. It is shown that it can be deduced from the corresponding problem of the annulus flow.

FLOW OF GENERAL NON-NEWTONIAN FLUIDS IN TUBES

By SHASHI GOEL and J. N. KAPOOR

Delhi

In this paper, it has been proved that rectilinear flow of a liquid characterised by

$$T = -pI + \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_1^2 + \alpha_4 A_2^2 + \alpha_5(A_1 A_2 + A_2 A_1) + \alpha_6(A_1^2 A_2 + A_2 A_1^2) + \alpha_7(A_1 A_2^2 + A_2^2 A_1) + \alpha_8(A_1^2 A_2^2 + A_2^2 A_1^2)$$

where

$$A^r_{ij} = \| A^r_{ij} \|, r = (1, 2, \dots, n), A^1{}_{ij} = \gamma_{i,j} + \gamma_{j,i} = 2\, d_{ij}$$

$$A^{(r+1)}_{ij} = \frac{\partial}{\partial t} A^{(r)}_{ij} + \gamma c A^{(r)}_{ij,l} + A^{(r)}_{mi}\, \gamma_{m,j} + A^{(r)}_{mj}\, \gamma_{m,j}$$

I is a unit matrix and $\alpha_1, \alpha_2, \dots, \alpha_8$ are polynomials in the ten scalar invariants $t_r A_1$, $T_r A_1^2$, $t_r A_1^3$, $t_r A_2$, $t_r A_2^2$, $t_r A_2^3$, $t_r A_1 A_2$, $t_r A_1^2 A_2$, $t_r A_1 A_2^2$ and $t_r A_1^2 A_2^2$, can be maintained by a uniform pressure gradient under the same conditions as obtained by Rivlin and Green for the less general flow characterized by

$$t_{ij} = \theta d_{ij} + \psi d_{ik} d_{kj} - p\delta_{ij}$$

where θ and ψ are functions of the invariants II and III.

In the second part of this paper, the expressions for the dissipation of energy have been found in the following four cases :

(i) Rectilinear flow in a tube.

(ii) Rectilinear laminar flow,

(iii) Torsional flow between two parallel plane discs,

(iv) Helical flow in the annular region between two coaxial infinite cylinders.

Also we have obtained the conditions that

$$\alpha_1 > 0 \text{ and } \begin{vmatrix} \alpha_1 & \alpha_5 \\ \alpha_5 & 4\alpha_7 \end{vmatrix} > 0$$

for dissipation of energy to be positive.

ON AXIALLY-SYMMETRIC NON-NEWTONIAN FLOWS

J. N. KAPUR and M. M. OBERAI, *Delhi University.*

In the present work we have discussed some axially-symmetric flows and found solutions for some special cases by using cylindrical-polar co-ordinates (r, θ, z) and expressing velocity components in terms of two scalar functions ψ and Ω in the form

$$U_r = \frac{1}{r}\frac{\partial\psi}{\partial z}, \qquad U_\theta = \frac{\Omega}{r}, \qquad U_z = \frac{1}{r}\frac{\partial\psi}{\partial\gamma},$$

so that the equation of continuity is satisfied automatically.

When μ and μ_c are treated constant the results obtained are:

1. In general the three equations of motion reduce to two differential equations giving ψ and Ω; involving partial derivatives upto fourth order.

2. If Ω and ψ are considered functions of r only:

   (i) Equations for ψ and Ω are separated,

   (ii) Coefficients of $\gamma_c = \mu_c/\rho$ vanishes, so that the results are the same as for ordinary viscous fluids,

   (iii) If further the motion is considered steady we get simple solutions

$$\Omega = Ar^2 + B,$$

$$\text{and } \frac{\partial\psi}{\partial r} = D_r + Er\log r + Fr^3.$$

3. Conversely if in a steady motion $U_\theta = 0$ so that $\partial\psi/\partial z = 0$, we find that such a flow pattern would hold for all viscous fluid if $\partial\Omega/\partial z$ is also equal to zero.

4. For motion through a circular tube, $\partial p/\partial r \neq 0$, thus confirming Reiner's view that the effect of μ_c is a stress normal to the plane of shear.

5. Solutions are also obtained for flows between two co-axial cylinders and through elliptic cylinders.

6. If however μ and μ_c are also regarded as functions of r, $\partial\mu_c/\partial r$ does not occur in the equations: it is only $\partial\mu/\partial r$ that occurs.

ABSTRACTS

ALGEBRA AND THEORY OF NUMBERS

J. M. GANDHI, Patiala. *Partitions of a number and Ramanujan's τ function.*

In this paper the author has discussed some congruences for $P_R(N)$ and $G_R(N)$ defined by

$$[(1-x)(1-x^2)(1-x^3)\dots\dots]^{-R} = \sum_{N=0}^{\infty} P_R(N)\, x^N$$

and

$$[(1-x)(1-x^2)(1-x^3)\dots\dots]^{R} = \sum_{N=0}^{\infty} G_R(N)\, x^N$$

respectively.

From the general congruences for $P_R(N)$, author has obtained McMahon's congruences for $P(N)$ mod 2, by a method different from the methods adopted by McMahon (*Combinatory Analysis*, Vols. 1 and 2. Camb. Univ. Press 1916) and H. Gupta (*J.I.M.S.*—10, 1946, 32-33), where $P(N)$ denotes the unrestricted partitions of a number. The author derives also a new congruence for $P(N)$ mod 2.

From the general congruences for $G_R(N)$, some of the congruences for Ramanujan's τ function immediately follow. The author also discusses a congruence for τ function mod 11. In the end the author gives some interesting conjectures.

J. M. GANDHI, Patiala. *Generalization of Fermat's Last Theorem.*

In this paper the author makes the following conjectures, namely

1. $x^{4n} + y^{4n} = p\, z^{4n}$ has no integral solutions $\neq 0$, if x, y, z are co-prime integers.

2. The equations $px^n \pm y^n = z^n$ have no integral solutions if $p \leqslant n, n > 2$, and x, y, z are co-prime. This conjecture includes Fermat's Last Theorem as a special case when $p = 1$.

3. $n!\, x^n \pm y^n = z^n$ have no integral solutions if x, y, z are co-prime.

It is proved that

1. $x^{4n} + y^{4n} = 4p\, z^{4n}$ is impossible if $p \leqslant n$.
2. $x^4 + y^4 = 3\, z^4$ is impossible if x, y, z are co-prime.
3. $x^{2n} + y^{2n} = (2n)!\, z^{2n}$ is impossible if $n > 1$.

The author proves the above conjectures in various special cases.

The author also shows that the conjecture 2 is true for $n = 3$, 4 and 5 and quotes a large number of results from the literature to support it.

N. Sankaran, Madras. *On the completions of a half-group.*

The following theorem is proved.

In a commutative half-group the topological completion under its intrinsic uniformity and the algebraic embedding are permutable.

The main lines of argument are (i) a linearly ordered commutative divisible half-group S can be given a uniform structure with the indexing set as $(S - 0)$ itself; (ii) a linearly ordered divisible half-group can be embedded as a dense half-group of a complete half-group; (iii) a topologically complete half-group has as its algebraic embedding, a topologically complete group G^* (say); (iv) the topological completion of the group G which is the group embedding of the half-group S is a group G^{**}(say); (v) there exists a unimorphism between G^* and G^{**}.

The following is a generalization of the above. The group embedding of the topological completion of the direct product of divisible linearly ordered half-groups, with the rectangular uniformity, and the direct product of the topologically complete groups, with its rectangular uniformity, are unimorphic.

K. SAVITHRI, Baroda. *The constant $\rho(s)$ in the main theorem of Siegel for function fields.*

If T is a symmetric matrix representable by the symmetric matrix S it has been announced that the main theorem of Siegel on the representation theory of quadratic forms over the field of rational functions in one variable x over a finite field is proved upto the determination of a certain constant $\rho(s)$. It follows from the results of the same paper that $\rho(s)$ is bounded. Also $\rho(s)$ tends to one with the order of S tending to infinity if the determinant of S is fixed. Given the value of the determinant of S at $1/x$ and the order of S, $\rho(s)$ has three values. Given the determinant of S, $\rho(s)$ tends to one as the value of the determinant of S tends to infinity.

M. SUGUNAMMA, Tirupati. *On the congruence $x^2 \equiv r$ (mod m).*

It deals with the function $V(m)$, the number of values of r $(0 \leqslant r \leqslant m - 1)$ for which the congruence $x^2 \equiv r \pmod{m}$ has got at least one solution in x. It is shown here that $V(m)$ is a multipli cative function of m and

$$V(2^\alpha) = 2 + \Sigma 2^{\alpha-\beta-3} \text{ where } \beta \equiv 0 \bmod 2 \text{ and } 0 \leqslant \beta \leqslant \alpha - 3,$$

$$V(p^\alpha) = 1 + \Sigma \phi(p^{\alpha-2\beta}) \text{ where } 0 \leqslant 2\,\beta \leqslant a,$$

p being an odd prime and ϕ is Euler's totient function. Thus, $V(m)$ is completely determined in view of its multiplicative property.

R. B. SAXENA and A. SHARMA. *Some Inequalities on Polynomials.*

Recently Balars and Turan [Notes on Interpolation IV, *Acta Math.* Vol. IX, 1958] have obtained some inequalities for polynomials of degree $\leqslant 2n-1$ based on their (0, 2)-interpolation formula. Such an effort for other interpolatory polynomials seems to be called for because in the field of lacunary interpolation very little is known about the behaviour of the fundamental polynomials.

In this paper we make some applications of the interpolatory polynomials (obtained earlier) of degree $\leqslant 3n-1$, n even, where we take x_ν to be the zeros of $\pi_n(x)$ with

$$\Pi_n(x) = -n(n-1)\int_{-1}^{x} P_{n-1}(t)\,dt \equiv (1-x^2)\,P'_{n-1}(x).$$

Here $P_{n-1}(x)$ denotes the Legender polynomials of degree $n-1$ with normalisation $P_{n-1}(1) = 1$.

We prove the following:

THEOREM I. If for $\nu = 1, 2, \ldots, n$ we have for a polynomial $H_{3n-1}(x)$ of degree $\leqslant 3n-1$

$$|H_{3n-1}(x_\nu)| \leqslant A,\ |H'_{3n-1}(x_\nu)| \leqslant B,\ |(H'''_{3n-1}(x_\nu)| \leqslant C,$$

then for $-1 \leqslant x \leqslant 1$, we have

$$|H_{3n-1}(x)| \leqslant K\,An + 1.00\,7\pi\,B + \frac{7\pi}{n^2}C,$$

where K is a constant.

THEOREM II. Under the conditions of Theorem I, we have for $-1 \leqslant x \leqslant 1$,

$$|H'_{3n-1}(x)| \leqslant Ac_8\,n^{5/2} + Bc_5\,n^{3/2} + Cc_1/\sqrt{n}.$$

We have proved a similar theorem which estimates the second derivative $|H''_{3n-1}(x)|$, but we shall not give it here.

M. SUGUNAMMA, Tirupati. *On Eckford Cohen's generalizations of Ramanujan's trigonometrical sum C (n, r).*

This paper deals with two distinct generalizations of Ramanujan's trigonometrical sum $C(n, r)$ considered by Eckford Cohen [Duke Mathematical Journal, Vol. 16 (1949); American Mathematical Monthly, Vol. 66 No. 2 (1959)].

I. $C^{(s)}(n,r) = \Sigma \exp(2\pi\, inx/r^s)\ 0 \leqslant x < r^s, (x, r^s)_s = 1$, where $(x, r^s)_s$ denotes the greatest s-th power factor common to x and r^s.

II. $C_{(k)}(n,r) = \Sigma \exp(2\pi\, in\ (x_1 + \ldots + x_k)/r), 0 \leqslant x_i < r, (i = 1, 2 \ldots k)$, $((x_i), r) = 1$ where $((x_i), r)_s$ denotes the g.c.d. of $x_1, x_2 \ldots x_k$ and r. Here it is shown that $C^{(s)}(n^s, r) = C_{(s)}(n, r)$. This paper also deals with the function $C_{(k)}^{(s)}(n, r)$ given by $C_{(k)}^{(s)}(n,r) = \Sigma \exp\ (2\pi\, in\ (x_1 + \cdots + x_k)\, |r^s), 0 \leqslant x_i \leqslant r^s, (i = 1, \ldots, k), ((x_i), r^s)_s = 1$, where $((x_i), r^s)^s$

denotes greatest s-th power factor common to $x_1, x_2, \ldots x_k$ and r^s, which is a generalisation of both I and II. It is proved that this is a multiplicative function of r and also of both arguments n and r, and further

$$C_{(k)}^{(s)}(n, r) = \phi_{(ks)}(r)\,\mu(r|q)/\phi_{(ks)}(r|q), \text{ where } r^s = (n, r^s)_s,$$

μ is Mobius function, $\phi_{(ks)}$ is Jordan's function. The properties of these functions are used to evaluate the number of partitions of n (mod r^s) of each of the forms :

(A) $a_1 + a_2 + \ldots + a_k \equiv n \pmod{r^s}$ $((a_i), r^s)_s = 1$, $0 \leqslant a_i < r^s$, $i = 1, 2, \ldots 2k$.

(B) $b_1\{A_1\} + \ldots + b_t\{A_t\} \equiv n \pmod{r^s}$ where $\{A_i\} = (a_{i1} + a_{i2} + \ldots + a_{ik})$, $((a_{ij}), r^s)_s' = 1$, $0 \leqslant a_{ij} < r^s$, $(i = 1, \ldots, t\,;\ j = 1, \ldots k)$ and $(b_j, r) = 1$, $(j = 1, 2 \ldots t)$.

(C) $C_1\, a_1^s \{B_1\} + C_2\, a_2^s \{B_2\} + \ldots + C_t\, a_t^s \{B_t\} \equiv n \pmod{r^s}$ where $\{B_i\} = (b_{i1} + b_{i2} + \ldots + b_{ik})$, $0 \leqslant b_{ij} < r^s$, $(i = 1, \ldots, t\,;\ j = 1, 2 \ldots k)$ $0 \leqslant a_i < r$ and $(c_h, r_s) = 1$, $(h = 1, 2 \ldots t)$.

If P, Q, R, denote the number of partitions of $n \pmod{r^s}$, of types (A), (B), (C), respectively, it is shown that

$$P = r^{(k-1)s}\,\phi_{(k-1)s}(g)/(g)^{(k-1)s}, \text{ where } g^s = (n, r^s)_s,$$

$$Q = (1/r^s)\sum_{d/r}\left(C_{(k)}^{(s)}(\delta, r^s)\right)^t C^{(s)}(n, r/\delta)$$

$$R = r^{s(kt-1)}\sum_{d/r} C^{(s)}(n, d)\, d^t.$$

V. Venugopal Rao, Baroda. *Indefinite quadratic forms with integral coefficients.*

Let S be a semi integral, symmetric, non-degenerate, *indefinite* matrix of order m and $\mathbf{A}$ a rational column vector with m rows such that $2S\mathbf{A}$ is integral. The number of integral solutions $\mathbf{X}$ of the matrix equation $(\mathbf{X}+\mathbf{A})'\ S(\mathbf{X}+\mathbf{A}) = t$, $(\mathbf{X}+\mathbf{A})'$ denoting the transpose of $(\mathbf{X}+\mathbf{A})$ and t a rational number, is in general infinite. Let $M(S, \mathbf{A}, t)$ denote the "measure of representation" of such integral solutions in the sense of C. L. Siegel [*Math. Ann.* 124 (1951), 17-54] which is necessarily finite for $m > 4$ and all t rational. Let $\Gamma_{\mathbf{A}}(S)$ denote the group of all unimodular matrices U of order m satisfying $U'SU = S$ and $U\mathbf{A} \equiv \mathbf{A}$ (mod 1) and let $\mu_{\mathbf{A}}(S)$ denote the measure of the group $\Gamma_{\mathbf{A}}(S)$ in the sense of Siegel. Further let $\Sigma M(S, \mathbf{A}, t) = \mu_{\mathbf{A}}(S)x + P(S, \mathbf{A}, x)$, x being any positive real number and $0 < t \leqslant x$. The singular series for $M(S, \mathbf{A}, t)$ together with the Siegel estimate for certain generalized Gaussian sums which occur in the singular series and an estimate for trigonometric sums have for consequence the estimate $P(S, \mathbf{A}, x) = O(1)$ as $x \to \infty$. This estimate is the best possible one; in fact the estimate $P(S, \mathbf{A}, x) = \Omega(1)$ is valid as $x \to \infty$, the symbol Ω being Hardy-Littlewood symbol. These results generalize to indefinite quadratic forms some of the corresponding results for positive definite quadratic forms.

ANALYSIS

Agrawal, Bhagwan Das, Varanasi. *Theorems on self-reciprocal functions.*

During the last few decades, many theorems have been given by different authors, for deriving new self-reciprocal functions, from well-known functions, self-reciprocal in the Hankel Transforms. The object of this paper is to add a few more theorems, of a similar character. The method used is that of the Mellin's Transforms.

The main result is

Theorem. If $f(x)$ be R_μ, then the function

$$g(x) = \int_0^x x^{\frac{1}{2}-\nu} y^{\mu+\frac{1}{2}} P\{(x^2 - y^2)^{1/n}\} f(y)\, dy$$

is R_ν, where

$$P(x) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} e^{\frac{1}{2}} x^{n} s.\ S^{\mu/2-\nu/2}\ \psi(S)\ ds, \text{ if } x^n > 0, \text{ and } = 0 \text{ if } x^n < 0$$

provided that $\psi(S) = \psi(1/S)$, and $\psi(S)$ is $O(|S|^{-a/2-\delta})$ for small S, and $O(|S|^{a/2+\delta})$ for large S, for every positive δ, and $0 < a < \frac{1}{2}$.

In all there are ten theorems and eleven new self-reciprocal functions in this paper.

AHMAD, AFZAL, Hyderabad. *Extension of functionals on quaternion linear spaces.*

If a quaternion linear bounded functional is defined on a quaternion linear subspace of a quaternion Banach space, then it is shown that there exists a quanternion linear bounded functional defined on the whole space which is equal to the given functional on the subspace and whose norm on the whole space does not exceed the norm of the given functional on the subspace. This generalizes a theorem of H. F. Bohnenblust and A. Sobczyk [Bull. Amer. Math. Soc. Vol. 44 (1938), p. 91.] where they have restricted themselves to complex vector spaces. The method used is similar in nature.

V. LAXMIKANTH, Hyderabad. *On the boundedness of solutions of non-linear differential systems.*

This is a continuation of the author's work (Proc. Amer. Math. Soc. 1957) and (Proc. Nat. Acad. Sci., India, 1958) in this field. Some results for the boundedness of solutions of non-linear differential systems are obtained under a set of more liberal conditions.

V. LAXMIKANTH, Hyderabad. *On the self-reciprocal functions for double Hankel Transform.*

Defining a function of two variables for double Hankel Transform, some results have been obtained for finding self-reciprocal functions of this type. Examples are constructed to illustrate the results.

M. R. PARAMESWARAN, Madras. *Remark on the structure of the summability field of a Hausdorff matrix.*

Let c denote the set of convergent sequences and for any sequence x let $c \oplus x$ be the linear space spanned by c and x. Rhodes [Bull. Amer. Math. Soc. 65 (1959), 9-11] has raised the question whether, given an arbitrary (unbounded) sequence x, there exists a regular Hausdorff matrix (H, μ_n) whose summability field (H) is precisely $c \oplus x$. The present note shows that the answer to the question is negative, thus :

Let $x = \{x_n\}$ be the unbounded sequence of partial sums of $\Sigma\, a_n$, where $a = \{a_n\}$ is bounded, divergent and Borel-summable *e.g.*

$a_n = 1 + (-1)^n$. Then $(H) = c \oplus x$ for no Hausdorff matrix H. For, if $\lim \mu_n = 0$, then $a \in (H), a$ is not in $c \oplus x$ shows that $(H) \neq c \oplus x$; and if $\lim \mu_n \neq 0$, then $x \in (H)$ implies $a \in c$ by a result of the author [*J. Indian Math. Soc.* (1959)], which is a contradiction, so that x is not in (H).

PRAMILA SRIAVASTAVA, Allahabad. *On the summability of the Dirichlet's product of summable series.*

1. Given two infinite series Σa_n and Σb_n, we associate with these summability by Riesz means of type λ and μ respectively. We form the sequence of numbers ν_n which are numbers $\lambda_p + \mu_q$ arranged in increasing order of magnitude and associate summability by Riesz means of type ν with the series Σc_n, where $c_n = \Sigma a_p b_q (\nu_n = \lambda_p + \mu_q)$. The series Σc_n is defined to be the Dirichlet's product of Σa_n and Σb_n of type (λ, μ). If $\lambda_n = \mu_n = n$, the Dirichlet's product is precisely the Cauchy product. In the present paper we are concerned with the case $\lambda_n = \mu_n = \log n$.

2. In this paper the following theorems are proved.

THEOREM 1. If (i) $\Sigma a_n n^{-s}$ is summable $|C, r_1|$ and (ii) $\Sigma b_n n^{-s}$ is summable $|C, r_2|$, both for $\sigma > 0$, then (iii) $\Sigma c_n n^{-s}$ $(c_n = a_p b_q, pq = n)$ is summable $| C, R | r_2 \geqslant r_1 \geqslant 0, 0 \leqslant R \leqslant r_1 + r_2$, for $\sigma > 0$, where $\theta = \theta (r_1, r_2, R) = r_2 - R$, for $R \leqslant r_2 - r_1 = \frac{1}{2} (r_1 + r_2 - R)$, for $R \geqslant r_1, r_2$.

THEOREM 2. For arbitrary given values $r_2 \geqslant r_1 \geqslant 0, 0 \leqslant R \leqslant r_1 + r_2$, there exist two Dirichlet's series (i) and (ii), which in the half plane $\sigma > 0$, are summable $|C, r_1|$ and $|C, r_2|$ respectively and are such that the product series (iii) has, as abscissa of summability $|C, R|$, exactly the number θ of Theorem 1.

The analogous results for ordinary summability were obtained by Bohr (1950).

T. PATI, Allahabad. *Absolute Cesaro summability factors of infinite series.*

The object of this paper is to establish a theorem on the absolute Cesaro summability factors of power series on its circle of convergence. For this purpose the author has to appeal to a result of Hardy and Littlewood concerning strong logarithmic summability of power series on its circle of convergence (Fund. Math., 1935), and the following result of his own, established in this paper :

THEOREM 1. If $\{\lambda_n\}$ is a convex sequence such that $\Sigma n^{-1}\lambda_n < \infty$ and Σa_n is bounded $[R, \log n, 1]$, then $\Sigma \lambda_n a_n$ is summable $|C, 1|$.

The final result obtained is :

THEOREM 2. If $f(z) = \Sigma c_n z^n$ is a power series of the complex class L, such that $\int_o^t |f(e^{i\theta})| d\theta = O(|t|)$, as $t \to 0$, and $\{\lambda_n\}$ is a convex sequence such that $\Sigma n^{-1}\lambda_n < \infty$, then $\Sigma \lambda_n c_n$ is summable $|C, 1|$.

T. PATI and Z. U. AHMAD, Allahabad. *The absolute summability factors of infinite series* (I).

This is the first of a series of papers in which the authors propose, *inter alia*, to establish by a direct technique of proof a generalised version of a recent result of Bosanquet and Chow (*Jour. London Math. Soc.* 1957, pp. 73-82, Theorem B). The latter's demonstration of this result depends on a previous theorem of Chow, to which its equivalence is shown by means of a number of results more or less out of context. The following theorem is proved :

THEOREM. Let k be a positive integer, and let $s^k{}_n$ denote the nth Cesaro mean of order k of the sequence of partial sums of Σa_n, and let $s^k{}_n = O(\lambda_n)$, as $n \to \infty$, where $\{\lambda_n\}$ is a positive monotonic non-decreasing sequence. Then, if (i) $\Sigma \lambda_n |\epsilon_n| < \infty$ and (ii) $\Sigma n^k \lambda_n |\Delta^{k+1}\epsilon_n| < \infty$, the series $\Sigma a_n \epsilon_n$ is summable $|C, k|$.

R. RANGA RAO and V. S. VARADARAJAN, Calcutta. *On the decomposition of Haar measure in compact groups.*

The behaviour of singularity under convolution has always been an interesting question. In particular, it may be asked whether the convolution of singular measures is necessarily singular. However, Salem (*Trans. Amer. Math. Soc.* 54, 1943) has constructed examples of singular measures whose iterates are absolutely continuous. In this paper we examine this question in another direction. The main theorem of this paper asserts that the Haar measure on any infinite compact abelian group can always be written as the convolution of two singular measures. It is also proved that in any non-discrete locally compact abelian group there are singular measures whose convolution is absolutely continuous.

HARI SHANKER, Baroda. *Proximate orders and exceptional values of a meromorphic function.*

The author begins by indicating the existence of proximate orders for meromorphic functions of finite positive order similar to that for entire functions of finite positive order (*cf* G. Valiron, *Lectures on integral functions*) and then proceeds to prove the following two theorems concerning the exceptional values of a meromorphic function in relation to its proximate order.

THEOREM 1. If $W(z)$ be a meromorphic function of order ρ and of proximate order $\rho(r)$ then

$$\limsup_{r\to\infty} \frac{n(r, \alpha)}{r^{\rho(r)}} \geqslant \rho/3, \quad (0 \leqslant |\alpha| \leqslant \infty)$$

for every α with two possible exceptions.

THEOREM 2. If $g(z)$ be any meromorphic function (or a constant) satisfying the relation $T(r, g) = 0\left(T(r, w)\right)$ and if $n(r, w-g)$ denotes the number of zeros of $w(z)-g(z)$ in $|z| \leqslant r$, then

$$\limsup_{r\to\infty} \frac{n(r, w-g)}{r^{\rho(r)}} \geqslant \rho/3$$

for every $g(z)$ with two possible exceptions, where ρ and $\rho(r)$ are the order and proximate order of $W(z)$.

It will be seen that Theorem 2 is a generalisation of Theorem 1. The proofs are simple and are based upon the second fundamental theorem of Nevanlinna. Analogous results are also stated for entire functions.

S. M. SHAH, Evanston (Illinois) and Aligarh. *An extension of Lindelof's theorem to meromorphic functions.*

T. P. SRINIVASAN, Chandigarh. *On certain approaches to the theory of integration.*

A measure is a pair (Λ, λ), where Λ is a semi-ring of subsets and λ is a non-negative countably additive set function on Λ.

The *extended measure space* of the (Λ, λ) is pair (Λ^*, λ) where Λ^* is the class of λ^*-measurable sets and λ, the restriction of λ^* to Λ^*, λ^* being the outer measure induced by (Λ, λ).

Let Δ denote the semi-ring of half open intervals (α, β) in $R^+{}_1$ the space of non-negative reals, and δ denote the length function. The *associated product measure* of (Λ, λ) is the measure $(\bar{\Lambda}, \bar{\lambda})$ where $\bar{\Lambda} = \Lambda \times \Delta$ and $\bar{\lambda} = \lambda > \delta$ on $\bar{\Lambda}$, in the usual notation, and the *associated product measure* space of (Λ, λ) is the extended measure space $(\bar{\Lambda}^*, \bar{\lambda})$ of $(\bar{\Lambda}, \bar{\lambda})$.

The following theorems are proved :

1. The associated product measure space of (Λ, λ) is also the associated product measure space of (Λ^*, λ).

2. The extended measure spaces of two measures (Λ_i, λ_i) whose associated product measure spaces are identical, are themselves identical; in symbols, if $(\bar{\Lambda}_1{}^*, \bar{\lambda}_1) = (\bar{\Lambda}_2{}^*, \bar{\lambda}_2)$ then $(\Lambda_1{}^*, \lambda_1) = (\Lambda_2{}^*, \lambda_2)$.

The equivalence of Stone's approach [*cf.* Proc. Nat. Acad. Sci., U.S.A., 34 (1948), 336-342] with the classical simple function and ordinate set appoaches to the theory of integration is derived as a consequence, using only the standard properties of measures and no properties of integrable or measurable functions beyond their definitions.

Several consistency theorems for which separate proofs were needed in A. C. Zaanen, *An Introduction to the Theory of Integration*, Amsterdam, 1958, are derived as corollaries from our method of proof.

GEOMETRY

MANDAN, SAHIB RAM, Kharagpur. *Uni- and demi-orthocentric simplexes.*

The study of the altitudes of a simplex in an n-space grows more and more complex as n increases, nevertheless it forms an interesting theme. The present paper is a fourth one in the series on the same topic, the former three being in press for which, however, reference can be made to their abstracts published in the *Proceedings of the* 45*th* (1958), 46*th* (1959) *and* 47*th* (1960) *Session of the Indian Science Congress Association,* one after the other. Here we take up only two special types of simplexes and observe how certain special points and special hyperspheres associated with them are related to one another. The main specialities are as follow:

(i) A simplex $S_{r(n-r)}$ is said to be *uni-orthocentric* (*UoS*) when its r altitudes concur with its *special r-altitude* l at its *r-orthocentre* H' and other $n-r$ with its special $(n-r)$-altitude l' at its $(n-r)$-orthocentre H'. l, l' and its $(n+1)$th altitude lie in a plane, concurrent at its uni-orthocentre U. Evidently, to meet such simplexes, we must rise and peep into spaces higher than a solid. The *S-point* (Monge point) of the *UoS* lies at the centroid of U, H, H' for *multiples* $1, r-1, n-r-1$ respectively. The developments of an *UoS* $S_{qr(n-q-r)}$ in 6-space onwards and $S_{pqr(n-p-r)}$ in 8-space onwards are suggested.

ii) A simplex $S_{qr(n-q-r+1)}$ is said to be *demi-orthocentric* (*DoS*) when its q-altitudes concur with its special q-altitude l' at its q-orthocentre H', other r with its special r-altitude l at its r-orthocentre H and the rest with its special $(h-q-r+1)$ altitude l'' at its $(n-q-r+1)$-orthocentre H''. l, l', l'' then concur, at its *di-orthocentre* D, as the 3-altitudes of the triangle, formed of their feet in their relative $(r-1)$-, $(q-1)$- and $(n-q-r)$-faces of the *DoS*, perspective to that formed by its 3-*demi-orthocentres*, viz. H, H', H'', from D as the centre of perspectivity. Obviously, to treat such simplexes, we must enter a 5-space onwards. The s-point of the *DoS* lies at the centroid of D, H, H', H'' for multiples 1, $r-1$, $q-1$, $n-q-r$, respectively. The developments of a *DoS* $S_{pqr(n-p-q-r+1)}$ in 7-space onwards and $S_{pqrs(n-p-q-r-s+1)}$ in 9-space onwards are suggested.

NIRMALA PRAKASH AND RAM BEHARI, Delhi. *Generalisations of Peterson-Minardi-Codazzi's equations in a subspace imbedded in a Finsler manifold.*

H. Rund obtained Codazzi's equations for a hypersurface F_{n-1} imbedded in a Finsler manifold F_n. A. Eliopoules generalised Codazzi's equations for a subspace F_m of Finsler space F_n.

In the present paper $n-m$ congruences associated with subspace F_m of F_n have been considered and further generalisations of Codazzi's equations have been obtained.

Naturally the Peterson-Minardi-Codazzi's equations obtained by H. Rund and A. Eliopoules follow as particular cases of our results.

K. CHANDRASEKHAR, Delhi. *Particular solutions of Einstein's field equations.*

APPLIED MATHEMATICS

B. B. CHAKRABORTY and P. L. BHATNAGAR, Bangalore. *The stability of force-free magnetic fields.*

In a recent paper Woltzer (Woltzer, 1958 Ap. J. 128, 384) has proved the interesting result that "the axisymmetric fields, characterised by a constant ratio of magnetic field, strength and current, are stable against all axisymmetric disturbances, the normal component of which vanishes on the surface of the field containing region". In the present note it has been found that if the variation in the total pressure on the surface of the configuration is assumed to vanish, the condition for the force-free field to exercise a stabilizing influence on the medium remains unaltered in the absence of Woltzer's assumption that the normal component of the displacement at every point of the surface is zero. In fact, it has been possible to point out conditions alternative to this assumption of Woltzer under which the force-free field will increase the stability of the medium even under a displacement, the normal component of which at the surface does not necessarily vanish. In passing it has also been noted that if proper direction cosines are introduced in equation (43) of his paper, a more rigorous proof of Woltzer's theorem, that axisymmetric force-free magnetic field is stable under axisymmetric displacement whose normal component vanishes on the surface, can be given.

TOPOLOGY

S. Swaminathan, Madras. *Analysis of compact uniconvergence spaces.*

A uniconvergence space is defined by considering axiomatically convergences indexed by a fixed directed set D, so that the axioms which they satisfy correspond to those of Weil-Bourbaki for a uniform space. Defining completeness, compactness and totally boundedness in terms of the directed set D, we obtain here a generalisation of the theorem for metric spaces that a metric space is compact if and only if it is complete and totally bounded. The relations between these notions and the $\mathbf{m}$-compactness of Nobeling (*Grundlagen der Analytischen Topologic*) and the allied notions of $\mathbf{m}$-completeness and $\mathbf{m}$-totally boundedness are worked out. Finally it is established that while the Bourbaki completeness implies $\mathbf{m}$-completeness always for any $\mathbf{m}$, $\mathbf{m}$-completeness implies Bourbaki completeness provided that uniconvergence structure is indexed by a directed set of cardinal $\mathbf{m}$.

LIST OF DELEGATES

F. C. Auluck, Afzal Ahmad, M. C. Agrawal, Bhagwan Das Agrawal, M. K. Agrawal, Santosh Arora, Mohan Lal Abrol, M. Parameshwara Ayyer, R. P. Bambah, S. P. Bandyopadhyaya, Keshava Deva Bhattrai, P. L. Bhatnagar, P. B. Bhattacharya, B. R. Bhonsle, M. N. Bhat, V. B. Buch, G. Bandyopadhyaya, Kishan Chand, G. L. Chandratreya, P. C. Consul, B. B. Chakraborty, A. C. Chaudhuri, Smt. K. Chopra, S. D. Chopra, H. R. Chaudhary, K. R. Chaudhary, Jangeshwar Dutta, S. H. Dwivedi, V. N. Dikshit, S. S. Dubey, N. D. Gautam, R. N. Gupta, J. P. Ganesh, K. R. Gunjikar, J. M. Gandhi, O. P. Gupta, V. K. Gangal, M. R. Gopal, N. D. Gupta, B. S. Grewal, G. C. Goel, K. K. Gorowara, Lata Gupta, Shashi Goel, Ramesh Chand Gupta, P. D. Gupta, Hansraj Gupta, K. M. Garg, D. N. Huddar, V. K. Handa, Muthulakshmi Iyer, P. C. Jain, B. S. Jain, R. K. Jain, C. Jagannathachari, P. C. Jain, R. S. Jain, Ramakant Jha, J. N. Kapur, D. S. Kothari, Radha Krishna, Sulakxana Kumari, D. D. Kapadia, D. R. Kaprekar, V. Lakshmikant, N. R. Kulkarani, S. N. Kawalgikar, Korgaokar, G. B. Khandekar, R. N. Kesarwani, Mohan Lal, B. R. Luthra, A. G. Lele, M. M. Lal, Braj Basi Lal, S. K. Lakshmanarai, S. Mahadevan, R. S. Mishra, Sukumar Mukerji, J. Medhi, Brij Mohan, S. Masood, S. C. Malik, B. B. Mehra, K. Markendeswara Rao, A. G. Mukerji, G. K. Menon, D. N. Misra, K. G. Mithal, V. V. Narlikar, V. V. L. Narsingha Rao, G. C. Niwas, Swadesh Nijhawan, N. S. Natrajan, V. S. Nanda, Shanti Narain, B. J. Oke, C. Orloff, M. M. Oberoi, P. V. Patel, Sunanda Patel, Mahendra Pratap, Satya Prakash, K. D. M. Pendharkar, R. K. Pathria, D. M. Patel, Nirmala Prakash, Ram Behari, Sahib Ram, S. Ramakrishna, D. S. Raghavan, B. S. Madhava Rao, S. N. Rao, R. Ramachandran, V. V. Rao, P. V. Ranganathan, A. V. Rangarajan, Ramanad Rathna, A. R. Rao, H. C. Saxena, H. G. S. Sharma, N. K. Sharma, M. Sugunamma, R. Shukla, U. P. Singh, M. K. Singhal, S. S. Singh, U. N. Singh, M. V. Singhal, T. N. Sinha, J. L. Sharma, M. L. Srivastava, D. M. Sinha, K. B. Shah, J. A. Siddiqi, M. M. Subramaniam, S. N. Sankaran, Narayan Singh, A. Sharma, S. Swaminathan, K. M. Shah, G. Sankarnarayanan,

S. Swarup, B. N. Sahney, P. Subbarao, V. N. Singh, C. N. Srinivasiengar, L. V. Subramaniam, K. Savithri, H. Shanker, S. K. Sharma, C. Shah, A. C. Shamihoke, B. N. Tagore, V. D. Thawani, U. S. Upadhyaya, B. G. Verma, R. C. Verma, A. M. Vaidya, B. Vishwanatham, P. C. Vaidya, R. S. Varma, S. Visvanathan, N. Wadhwa, Yegnanarayanan, E. Zechariah.

THE RECEPTION COMMITTEE

K. L. Misra (*Chairman*), Shri Ranjan (*Chairman, R. C.*), B. N. Prasad (*Secretary*), R. N. Chaudhuri (*Treasurer*).

THE INDIAN MATHEMATICAL SOCIETY

STATEMENT OF ACCOUNTS
FOR THE YEAR ENDED 31st MARCH 1960

Balance Sheet as at

FUNDS AND LIABILITIES	Rs.	nP.	Rs.	nP.
A. General Funds :				
Balance as per last Balance Sheet	2,428	63		
Add Value of Books arrived at after revaluation per Contra	1,51,807	99		
Publications available for sale	96,357	50		
Furniture per Contra	2,349	00		
Other Liabilities (as per last Balance Sheet) ...	80	00		
			2,53,023	12
Total " A "...			2,53,023	12
B. Other Funds :				
A. Narasinga Rao Gold Medal Fund	1,450	00		
Interest accrued	50	00		
			1,500	00
C. Building Fund :				
As per last Balance Sheet	29,953	31		
Interest accrued	450	00		
			30,403	31
D. Golden Jubilee Volume Fund :				
Deposit	7,350	00		
Deposit (1959-60)	16,714	50		
			24,064	50
Total A B C D...			3,08,990	93

† Out of the deficit of Rs. 16,469·29 a sum of Rs. 10,530·27 being the Bank Balances as on 1-4-1960 may be adjusted and the balance Rs. 5,939·02 be declared as deficit for the years 1958-59 and 1959-60.

Bangalore,

Dated : 19th Dec. 1960.

MATHEMATICAL SOCIETY

31st March 1960

PROPERTIES AND ASSETS			Rs.	nP.	Rs.	nP.
A. **Library Books :**						
Value of Books on hand			1,54,288	35		
Add Purchases during the year			2,496	03		
			1,56,784	38		
Less Depreciation for the year			15,428	00		
					1,41,356	38
Publications available for Sale					96,357	50
Furniture and Equipments (per Contra) ...			2,349	00		
Add Purchase during the year			800	21		
			3,149	21		
Less Depreciation for the year			235	00		
					2,914	21
Advances outstanding :—						
With Editor	Rs.	32	92			
,, Ex. Treasurer	,,	0	20			
,, Treasurer	,,	18	02			
,, Librarian	,,	8	83			
			59	97	59	97
Bank Balances :—						
Indian Bank C/A			4,477	40		
Indian Bank S/B			47	68		
Canara Bank C/A			6,005	19		
					10,530	27
Deficit as per last Balance Sheet			1,809	73		
Income and Expenditure Statement ...			14,659	56		
					† 16,469	29
Total "A"...					2,67,687	62
B. **Other Funds :**						
Narasinga Rao Gold Medal Fund (Invested in Fixed Deposit)			900	00	900	00
C. **Building Fund :**						
Fresh Investment in N.S.C.			13,500	00		
Fixed Deposit with Indian Bank			13,903	31		
Fresh Investment on N.S.C.			3,000	00		
					30,403	31
D. **Golden Jubilee Volume Fund :**						
Fixed Deposit with Canara Bank			10,000	00	10,000	00
Total of A B C D...					3,08,990	93

Examined and Found Correct

Sd/- J. R. S. BHATTA
Internal Auditor
Indian Institute of Science,
Bangalore 12.

EXPENDITURE	Rs.	nP.	Rs.	nP.	Rs.	nP.
A. To Management Expenses :						
Misce. Printing and Stationery			995	40		
Office Postage			423	03		
Office Expenses			1,141	90		
Hon. and Remuneration			800	00		
Bank Commission			15	07		
Conveyance and Travelling			182	52		
					3,557	92
,, **Expenses on the objects of the Society :**						
Printing of Journals			10,064	68		
Railway Freight, etc.			1,181	58		
					11,246	26
,, **Library** : Binding Charges					741	97
,, **Audit Fees**					40	00
,, **Advances adjusted** ... ...Rs.	522	65				
Less ... ... ,,	87	93				
					434	72
Total of " A "...					16,020	87
B. To Interest on National Savings Certificates :						
Transferred to Narasinga Rao Gold Medal Fund			50	00	50	00
C. To —do— **Building Funds**			450	00	450	00
D. To Depreciation :						
Library Books			15,428	00		
Furniture and Equipment			235	00		
					15,663	00
Totals A B C D...					32,183	87

***Against the deficit shown Rs. 14,659·56 a sum of Rs. 10,530·17 being the Bank Balances as on 1-4-60 may be adjusted and the balance of Rs. 4,129·29 be declared as actual deficit for the year 1959-60.**

Bangalore,

Dated : *19th Dec. 1960.*

MATHEMATICAL SOCIETY

year ending 31st March 1960

INCOME	Rs.	nP.	Rs.	nP.
A. **By Grants-in Aid from :**				
National Institute of Sciences	2,000	00		
University of Madras...	150	00		
University of Bombay	200	00		
Government through Atomic Energy Commission	1,999	10		
			4,349	10
,, **Interest on Bank Balances**			27	60
,, **Income from other Sources** :				
Life Composition Fee	950	00		
Membership and Subscription	7,970	66		
Associate Membership Fee	403	50		
			9,324	16
,, **Sale of Publications**	3,323	45	3,323	45
Total of " A "...			17,024	31
B. **By Interest on Investments :**				
Narasinga Rao Gold Medal Fund ...	50	00	50	00
C. **By Building Funds**	450	00	450	00
D. **By Deficit Transferred to Balance Sheet** ...			*14,659	56
Totals A B C D...			32,183	87

Examined and Found Correct.

Sd/- J. R. S. BHATTA
Internal Auditor
Indian Institute of Science,
Bangalore 12.