MADURAI KAMARAJ UNIVERSITY

(University with Potential for Excellence)

DIRECTORATE OF DISTANCE EDUCATION

B.Sc., PHYSICS

FIRST YEAR

ANCILLARY MATHEMATICS

PAPER - II

UNIT : 1 - 5

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B.Sc.,

ANCILLARY MATHEMATICS - PAPER - II

Dear Student,

We welcome you as a Student of the B.Sc degree Course.

This paper deals with Algebra, Calculus, Trigonometry, Analytical Geometry of Three Dimensions, Vector Calculus, Differential Equations, Applications of Differential Equations. The learning material for this paper will be supplemented by Contact seminars.

Learning through the Distance Education mode, as you are all aware, involves self – learning and self – assessment and in this regard you are expected to put in disciplined and dedicated effort.

On our part, we assume of our guidance and support.

With best wishes.

SYLLABUS

ANCILLARY MATHEMATICS – PAPER – II

Unit I: (Statistics)

Correlation Coefficient – Rank Correlation Coefficient – Interpolation – Lagrange and Newton's Methods – Attributes – Index numbers.

Unit II: (Algebra)

Matrices – Rank – Consistency of Equations – solutions of Equations – Characteristic equations – Eigen values and Eigen Vectors.

Unit III: (Modern Algebra)

Groups – subgroups – permutation groups – Homomorphisms and Isomorphisms.

Unit IV: (Calculus)

Fourier Series – Trignometric Series – Even and odd functions – Half Range Fourier Series.

Unit V: (Linear Programming)

Definition of Standard Linear Programming Problem – Feasible solutions – optimum solution – basic feasible solutions – optimum basic feasible solution – degenerate solution of a linear programming problem.

CONTENTS

UNIT-I
STATISTICS

Objective

In this unit, we are going to discuss how to find the correlation between two variables and then to find the relationship between two variables.
After the completion of this unit one may able to fit

- Correlation coefficient between two variables.
- Regression equations.
- Rank correlation coefficient.

1.0 Introduction

Correlation is a statistical measure for finding the degree of association between two or more variables. Here "association" mean that the tendency of the variables moves together. If two variables x and y are so related that movements in one, tend to be accomplished by the corresponding movements in the other variables, then two variables are correlated.

1.1 Correlation coefficient

Let x be a variable having the values $x_1, x_2, x_3, \ldots, x_n$ and y be the second variable having values $y_1, y_2, y_3, \ldots, y_n$. If there is a change in one variable corresponding to a change in the other variable we say that the variables are connected.

If the two variables deviate in the same in the same direction the correlation is said to be direct or positive. If they always deviate in the opposite direction the correlation is said to be inverse or negative.

Definition : The covariance between two variable X and Y is defined by

$$\frac{\sum(x-\bar{x})(y-\bar{y})}{n}$$

$$\boxed{\text{cov}(x,y)=\frac{\sum(x-\bar{x})(y-\bar{y})}{n}} \quad \text{(i.e)}$$

Definition : Karl Pearson's coefficient correlation between two variable X and Y is denoted by γ_{xy} and is defined by $\frac{\text{cov}(x,y)}{\sigma_x\sigma_y}$

$$\text{(i.e)} \qquad \gamma_{xy}=\frac{\text{cov}(x,y)}{\sigma_x\sigma_y}$$

$$\boxed{\gamma_{xy}=\frac{\sum(x-\bar{x})(y-\bar{y})}{n\sigma_x\sigma_y}} \quad \text{(i.e)}$$

Note: Two variables x and Y are independent if γ_{xy} =0

Theorem 1.1

$$\gamma_{xy}=\frac{n\sum xy-(\sum x)(\sum y)}{\sqrt{n\sum x^2-(\sum x)^2}\sqrt{n\sum y^2-(\sum y)^2}}$$

Proof:

we know that covariance of X and Y is $\text{cov}(x,y)$

$$=\frac{\sum(x-\bar{x})(y-\bar{y})}{n}$$

$$=\frac{1}{n}\left[\sum(xy-x\bar{y}-\bar{x}y+\bar{x}\bar{y})\right]$$

$$=\frac{1}{n}\left[\sum xy-n\bar{x}\bar{y}-n\bar{x}\bar{y}+n\bar{x}\bar{y}\right]$$

Space for Hint

$$=\frac{1}{n}\left[\sum xy - n\frac{\sum x}{n}\frac{\sum y}{n}\right]$$

$$=\frac{1}{n^2}\left[n\sum xy - \left(\sum x\right)\left(\sum y\right)\right]$$

and $\sigma_x = \sqrt{\frac{1}{n}\sum x^2 - \left(\frac{\sum x}{n}\right)^2}$

(i.e) $\sigma_x = \sqrt{\frac{n\sum x^2 - (\sum x)^2}{n^2}}$

(i.e) $\sigma_x = \frac{1}{n}\sqrt{n\sum x^2 - \left(\sum x\right)^2}$

Similarly $\sigma_y = \frac{1}{n}\sqrt{n\sum y^2 - \left(\sum y\right)^2}$

Thus $\gamma_{xy} = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y}$

(i.e) $\gamma_{xy} = \frac{n\sum xy - \left(\sum x\right)\left(\sum y\right)}{\sqrt{n\sum x^2 - \left(\sum x^2\right)}\sqrt{n\sum y^2 - \left(\sum y\right)^2}}$

This prove the theorem.

Theorem 1.2 : The correlation coefficient is independent of the change of origin and scale.

Proof :

We know that correlation coefficient is $\gamma_{xy} = \frac{\sum\left(x-\bar{x}\right)\left(y-\bar{y}\right)}{n\sigma_x\sigma_y}$

Let $u_i = \frac{x_i - A}{h}$ and $v_i = \frac{y_i - B}{k}$ where $h, k > 0$

Now $u_i = \frac{x_i - A}{h}$ and $v_i = \frac{y_i - B}{k}$

Thus $x_i = A + hu_i$ and $y_i = B + kv_i$

Hence $\bar{x} = A + h\bar{u}$ and $\bar{y} = B + k\bar{v}$

$\therefore x_i - \bar{x} = h(u_i - \bar{u})$ and $y_i - \bar{y} = k(v_i - \bar{v})$

and $\sigma_x = h\sigma_u, \sigma_y = k\sigma_v$

$$\text{Hence } \gamma_{xy} = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y}$$

$$= \frac{\sum (x - \bar{x})(y - \bar{y})}{n\sigma_x\sigma_y}$$

$$= \frac{\sum h(u - \bar{u})k(v - \bar{v})}{nh\sigma_u k\sigma_v}$$

$$= \frac{\sum (u - \bar{u})(v - \bar{v})}{n\sigma_u\sigma_v}$$

$$= \gamma_{uv}$$

Thus $\gamma_{xy} = \gamma_{uv}$

(i.e) the correlation coefficient is independent of the change of origin and scale

This proves the theorem.

Theorem 1.3 $-1 \leq \gamma \leq 1$

Proof: $$\gamma_{xy} = \frac{\sum (x - \bar{x})(y - \bar{y})}{n\sigma_x\sigma_y}$$

$$= \frac{\sum (x - \bar{x})(y - \bar{y})}{n\sqrt{\frac{1}{n}(x_i - \bar{x})^2}\sqrt{\frac{1}{n}(y_i - \bar{y})^2}}$$

$$= \frac{\sum (a_i b_i)}{\sqrt{\sum a_i^2}\sqrt{\sum b_i^2}}$$

(i.e) $\gamma_{xy} = \frac{\left(\sum a_i b_i\right)^2}{\left(\sum a_i^{\ 2}\right)\left(\sum b_i^{\ 2}\right)}$ --(2.1)

We know that from Schwartz inequality $\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^{\ 2}\right)\left(\sum b_i^{\ 2}\right)$

Thus (1.1) $\Rightarrow \gamma^2_{xy} \leq 1$

(i.e) $|\gamma_{xy}| \leq 1$

(i.e) $-1 \leq \gamma \leq 1$

This proves the theorem.

Note:

(i) This correlation γ is said to be perfectly positive if $\gamma = 1$

(ii) This correlation γ is said to be perfectly negative if $\gamma = -1$

Theorem 1.4 $\gamma_{xy} = \frac{\sigma^2_x + \sigma^2_y - \sigma^2_{x-y}}{2\sigma_x \sigma_y}$

Proof: We know that $\sigma^2_{x-y} = \frac{1}{n}\sum\left[(x_i - y_i) - (\bar{x} - \bar{y})\right]^2$

$$= \frac{1}{n}\sum\left[(x_i - \bar{x}) - (y_i - \bar{y})\right]^2$$

$$= \frac{1}{n}\left[\sum(x_i - \bar{x})^2 - 2\sum(x_i - \bar{x})(y_i - \bar{y}) + \sum(y_i - \bar{y})^2\right]$$

$$= \sigma^2_x - 2\gamma_{xy}\sigma_x\sigma_y + \sigma^2_y$$

$$\therefore \gamma_{xy} = \frac{\sigma^2_x + \sigma^2_y - \sigma^2_{x-y}}{2\sigma_x\sigma_y}$$

This proves the theorem.

Space for hint

Example 1.1:

Find the correlation to the following data :

x	:	10	12	18	24	23	27
y	:	13	18	12	25	30	10

Solution:

Given that

x	:	10	12	18	24	23	27
y	:	13	18	12	25	30	10

We know that $\gamma_{xy} = \frac{\sum\left(x-\bar{x}\right)\left(y-\bar{y}\right)}{n\sigma_x\sigma_y}$

	x	y	$x-\bar{x}$	$\left(x-\bar{x}\right)^2$	$y-\bar{y}$	$\left(y-\bar{y}\right)^2$	$\left(x-\bar{x}\right)\left(y-\bar{y}\right)$
	10	13	-9	81	-5	25	45
	12	18	-7	49	0	0	0
	18	12	-1	1	-6	36	6
	24	25	5	25	7	49	35
	23	30	4	16	12	144	48
	27	10	8	64	-8	64	-64
total	114	108	0	236	0	318	70

Now $\bar{x} = \frac{\sum x}{n}$

$= \frac{144}{6}$

$=19$

and $\bar{y} = \frac{\sum y}{n}$

$= \frac{108}{6}$

$=18$

and $\sigma_x = \sqrt{\frac{1}{n}\sum(x-\bar{x})^2}$

$= \frac{236}{6}$

$=6.272$

and $\sigma_y = \sqrt{\frac{1}{n}\sum(y-\bar{y})^2}$

$= \frac{318}{6}$

$=7.280$

Now $\gamma_{xy} = \frac{\sum(x-\bar{x})(y-\bar{y})}{n\sigma_x\sigma_y}$

$= \frac{70}{6(6.272)(7.280)}$

$=0.256$

Thus the correlation coefficient between X and Y is 0.256

Example 1.2 :

Find the correlation to the following data:

Age of husband	:	23	27	28	29	30	31	33	35	36	39
Age of wife	:	18	22	23	24	25	26	28	29	30	32

Solution:

Given that

Age of husband	:	23	27	28	29	30	31	33	35	36	39
Age of wife	:	18	22	23	24	25	26	28	29	30	32

We know that $\gamma_{xy} = \dfrac{n\sum xy - (\sum x)(\sum y)}{\sqrt{n\sum x^2 - (\sum x)^2}\sqrt{n\sum y^2 - (\sum y)^2}}$

	x	y	x^2	y^2	xy
	23	18	529	324	414
	27	22	729	484	594
	28	23	784	529	644
	29	24	841	576	696
	30	25	900	625	750
	31	26	961	676	806
	33	28	1089	784	924
	35	29	1225	841	1015
	36	30	1296	900	1080
	39	32	1521	1024	1248
Total	311	257	9875	6763	8171

$$\text{Now } \gamma_{xy} = \frac{n\sum xy - (\sum x)(\sum y)}{\sqrt{n\sum x^2 - (\sum x)^2}\sqrt{n\sum y^2 - (\sum y)^2}}$$

$$= \frac{10(8171) - (311)(257)}{\sqrt{10(9875) - (311)^2}\sqrt{10(6763) - (257)^2}}$$

$$= \frac{1783}{(45.044)(39.762)}$$

$$= 0.996$$

Thus the correlation coefficient between X and Y is 0.996

Example 1.3:

Find the correlation coefficient to the following data:

Marks in Mathematics	:	65	66	67	67	68	69	70	72
Marks in Statistics	:	67	68	65	68	72	72	69	71

Solution:

Given that

Marks in Mathematics	:	65	66	67	67	68	69	70	72
Marks in Statistics	:	67	68	65	68	72	72	69	71

We know that $\gamma_{xy} = \frac{n\sum uv - (\sum u)(\sum v)}{\sqrt{n\sum u^2 - (\sum u)^2}\sqrt{n\sum v^2 - (\sum v)^2}}$ where u=x-A and v=y-B

Choose A=67 and B=68

	x	y	$u = x - 67$	$v = y - 68$	u^2	v^2	uv
	65	67	-2	-1	4	1	2
	66	68	-1	0	1	0	0
	67	65	0	-3	0	9	0
	67	68	0	0	0	0	0
	68	72	1	4	1	16	4
	69	72	2	4	4	16	8
	70	69	3	1	9	1	3
	72	71	5	3	25	9	15
Total	544	552	8	8	44	52	32

Now $$\gamma_{xy} = \frac{n\sum uv - \left(\sum u\right)\left(\sum v\right)}{\sqrt{n\sum u^2 - \left(\sum u\right)^2}\sqrt{n\sum v^2 - \left(\sum v\right)^2}}$$

$$= \frac{8(32) - (8)(8)}{\sqrt{8(44) - (8)^2}\sqrt{8(52) - (8)^2}}$$

$$= \frac{192}{(16.971)(18.762)}$$

$$= 0.603$$

Thus the correlation coefficient between X and Y is o.603

Check Your Progress:

(1) Find the correlation coefficient to the following data:

X	:	300	350	400	450	500	550	600	650	700
Y	:	800	900	1000	1100	1200	1300	1400	1500	1600

(2) Find the correlation coefficient to the following data:

Father's height	:	67	68	64	67	72	70	70	69	70
Son's height	:	65	66	67	68	68	69	71	72	72

Example 1.4:

Space for Hint

If $\sum x = 71,\ \sum y = 70,\ \sum x^2 = 555,\ \sum y^2 = 526,$ $\sum xy = 527$ and

n=100, then find γ_{xy}

Solution:

Given that $\sum x = 71,\ \sum y = 70,\ \sum x^2 = 555,\ \sum y^2 = 526,\ \sum xy = 527$, n=100.

We know that $$\gamma_{xy} = \frac{n\sum xy - (\sum x)(\sum y)}{\sqrt{n\sum x^2 - (\sum x)^2}\sqrt{n\sum y^2 - (\sum y)^2}}$$

$$= \frac{47730}{(224.63)(218.40)}$$

$$=0.9729$$

Example 1.5:

Given $n = 1000$, $\bar{x} = 65,\ \bar{y} = 83,\ \sigma_x = 4.5,\ \sigma_y = 3.6$ and the sum of the products

of the deviations from the mean of x and y is 4800. Find the correlation co-efficient between x and y.

Solution:

Given that $n = 1000$, $\bar{x} = 65,\ \bar{y} = 83,\ \sigma_x = 4.5,\ \sigma_y = 3.6$ and

$\sum (x - \bar{x})(y - \bar{y}) = 4800$

We know that $$\gamma_{xy} = \frac{\sum (x - \bar{x})(y - \bar{y})}{n\sigma_x\sigma_y}$$

$$\therefore \gamma_{xy} = \frac{4800}{1000(4.5)(3.6)}$$

$=0.2963$

Example 1.6 :

If $z = ax + by$ and γ is the correlation coefficient between x and y. Show that

$\sigma^2{}_z = a^2\sigma^2{}_x + b^2\sigma^2{}_y + 2ab\gamma\sigma_x\sigma_y$. Hence deduce that $\gamma = \dfrac{\sigma^2{}_x + \sigma^2{}_y - \sigma^2{}_{x-y}}{2\sigma_x\sigma_y}$.

Proof:

Given that $z = ax + by$

$\therefore \overline{z} = a\overline{x} + b\overline{y}$

Now $z - \overline{z} = a(x - \overline{x}) + b(y - \overline{y})$

and $(z - \overline{z})^2 = a^2(x - \overline{x})^2 + b^2(y - \overline{y})^2 + 2ab(x - \overline{x})(y - \overline{y})$

We know that $\gamma_{xy} = \gamma$

$$= \frac{\sum (x - \overline{x})(y - \overline{y})}{n\sigma_x\sigma_y}$$

$$\therefore \frac{1}{n}\sum (x - \overline{x})(y - \overline{y}) = \gamma\sigma_x\sigma_y$$

Thus $\sigma^2{}_z = \frac{1}{n}\sum (z - \overline{z})^2$

$$= \frac{1}{n}\sum \left(a^2(x - \overline{x})^2 + b^2(y - \overline{y})^2 + 2ab(x - \overline{x})(y - \overline{y})\right)$$

$$= a^2\frac{1}{n}\sum (x - \overline{x})^2 + b^2\frac{1}{n}\sum (y - \overline{y})^2 + 2ab\frac{1}{n}\sum (x - \overline{x})(y - \overline{y})$$

$$= a^2\sigma^2{}_x + b^2\sigma^2{}_y + 2ab\gamma\sigma_x\sigma_y \quad \text{------------------------------}(2.2)$$

Deduction:

Putting a = b = 1 in (2.2), we get,

$$\sigma^2{}_{x-y} = \sigma^2{}_x + \sigma^2{}_y - \gamma 2\sigma_x\sigma_y$$

$$\text{Thus } \gamma = \frac{\sigma^2{}_x + \sigma^2{}_y - \sigma^2{}_{x-y}}{2\sigma_x\sigma_y}.$$

This proves the problem.

Example 1.7 :

If x and y are discrete variables and if $\sigma^2{}_y = \sigma^2{}_y = \sigma$ and $\operatorname{cov}(x, y) = \frac{1}{2}\sigma^2$.

Find (i) σ_{2x-3y} and (ii) $\sigma_{2x+3,2y-3}$

Solution : (i)

Let $u = 2x - 3y$

$\therefore \bar{u} = 2\bar{x} - 3\bar{y}$

and $u - \bar{u} = 2(x - \bar{x}) - 3(y - \bar{y})$

Hence $(u - \bar{u})^2 = 4(x - \bar{x})^2 + 9(y - \bar{y})^2 - 12(x - \bar{x})(y - \bar{y})$

$$= 4\sigma^2 + 9\sigma^2 - 12 \times \frac{1}{n} \times \sigma^2$$

$$= 7\sigma^2$$

Thus $\sigma_u = \sqrt{7}\sigma$

(i.e) $\sigma_{2x-3y} = \sqrt{7}\sigma$

Solution : (ii)

Let $u = 2x + 3$ and $v = 2y - 3$

$\therefore \bar{u} = 2\bar{x} + 3$ and $\bar{v} = 2\bar{y} - 3$

And $(u-\bar{u})(v-\bar{v}) = 4(x-\bar{x})(y-\bar{y})$

(i.e) $\operatorname{cov}(u, v) = 4\operatorname{cov}(x, y)$

$$= 4 \times \frac{1}{2}\sigma^2$$

$$= 2\sigma^2$$

and $\sigma_u^2 = 4\sigma_x^2$, $\sigma_v^2 = 4\sigma_y^2$

(i.e) $\sigma_u^2 = 4\sigma^2$, $\sigma_v^2 = 4\sigma^2$

Thus $\gamma_{uv} = \dfrac{\operatorname{cov}(u, v)}{\sigma_u \sigma_v}$

$$= \frac{2\sigma^2}{4\sigma^2}$$

$$= \frac{1}{2}$$

Example 1.8:

A computer while calculating the correlation coefficient between two variables x and y obtained in the following constants.

$n = 25$, $\sum x = 125$, $\sum y = 100$, $\sum x^2 = 650$, $\sum y^2 = 460$ and $\sum xy = 508$. It was later found that at the time of checking that operator had copied down two pairs of observations $(x_i . y_i)$ as $(6,14)$ and $(8,6)$ instead of the correct values $(8,12)$ and $(6,8)$. Obtain the correct value of the correlation coefficient between x and y.

Solution:

Given that ,

$Wrong(x_i, y_i)$	$Correct(x_i y_i)$
(6,14)	(8,12)
(8,6)	(6,8)

Correct $\sum x$ =wrong $\sum x -$ (sum of wrong items)

+ (sum of correct items)

$$= 125 - (6+8) + (8+6)$$
$$= 125$$

Correct $\sum y$ =wrong $\sum y -$ (sum of wrong items)

+ (sum of correct items)

$$= 100 - (14+6) + (12+8)$$
$$= 100$$

Correct $\sum x^2$ =wrong $\sum x^2 -$ (sum of squares of wrong items)

+(sum of squares of correct items)

$$= 650 - (6^2 + 8^2) + (8^2 + 6^2) = 650$$

Correct $\sum y^2$ = wrong $\sum y^2 -$ (sum of squares of wrong items)

+ (sum of squares of correct items)

$$= 460 - (14^2 + 6^2) + (12^2 + 8^2)$$
$$= 436$$

Correct $\sum xy$ =wrong $\sum xy -$ (sum of product of wrong items)

+ (sum of product of correct items)

$$= 508 - (6\times 14 + 8\times 6) + (8\times 12 + 6\times 8)$$
$$= 520$$

Thus correct correlation coefficient= γ_{xy}

$$= \frac{n\sum xy - \left(\sum x\right)\left(\sum y\right)}{\sqrt{n\sum x^2\left(\sum x\right)^2}\sqrt{n\sum y^2 - \left(\sum y\right)^2}}$$

$$= \frac{25(520) - (125)(100)}{\sqrt{25(650) - (125)^2}\sqrt{25(436) - (100)^2}}$$

$$= \frac{500}{25\times 30}$$

$$= 0.667$$

Hence the correlation coefficient is 0.667.

Check Your Progress:

(1) Find the correlation coefficient for the data.

x:	120	110	120	119	140	125	127	119	140	160
y:	240	250	260	266	232	245	255	267	268	239

(2) Find the correlation coefficient for the data.

x:	300	350	400	450	500	550	600	650	700
y:	800	900	1000	1100	1200	1300	1400	1500	1600

$(answers\ : (1) 0.373, (2) 1)$

1.2 Rank Correlation Coefficient

Definition: Rank correlation coefficient is defined as $\rho = 1 - \frac{6\sum(x-y)}{n\left(n^2-1\right)}$

Note: If two or more individuals get the same rank then the rank correlation formula is modified as $\rho = 1 - \frac{6\left(\sum (x-y)^2 + CF\right)}{n(n^2-1)}$ where

CF= correction factor and it is obtained from

$$CF = \frac{1}{12} m_1 (m_1^2 - 1) + \frac{1}{12} m_2 (m_2^2 - 1) + \ldots,$$

Here $m_1, m_2, \ldots$ is the number of times that ranks be repeated.

Example 1.1:

Find the rank correlation coefficient for the following data.

x:	5	2	8	1	4	6	3	7
y:	4	5	7	3	2	8	1	6

Solution:

Given that

x:	5	2	8	1	4	6	3	7
y:	4	5	7	3	2	8	1	6

We know that rank correlation

$$\rho = 1 - \frac{6\sum (x-y)^2}{n(n^2-1)}$$

x	y	$x-y$	$(x-y)^2$
5	4	1	1
2	5	-3	9
8	7	1	1
1	3	-2	4
4	2	2	4
6	8	-2	4
3	1	-2	4
7	6	1	1
Total			28

Thus $\rho = 1 - \frac{6 \times 28}{8 \times (8^2 - 1)}$

$= 1 - \frac{168}{504}$

$= 0.667$

(i.e) the rank correlation coefficient is 0.667

Example 1.2:

Find the rank correlation coefficient for the following data.

X :	78	65	36	98	25	75	82	90	62	39
Y :	84	53	51	91	60	68	62	86	58	47

Solution:

Given that

X :	78	65	36	98	25	75	82	90	62	39
Y :	84	53	51	91	60	68	62	86	58	47

$$\rho = 1 - \frac{6\sum (x-y)^2}{n(n^2-1)}$$

X	Y	Rank of $X(x)$	Rank of $Y(y)$	$x-y$	$(x-y)^2$
78	84	4	3	1	1
65	53	6	8	-2	4
36	51	9	9	0	0
98	91	1	1	0	0
25	60	10	6	4	16
75	68	5	4	1	1
82	62	3	5	-2	4
90	86	2	2	0	0
62	58	7	7	0	0
39	47	8	10	-2	4
Total					30

Thus $\rho = 1 - \frac{6 \times 30}{10 \times (10^2 - 1)}$

$= 1 - \frac{180}{990}$

$= 0.818$

(i.e) the rank correlation coefficient is 0.818.

Example 1.3:

Find the rank correlation coefficient for the following data.

X :	48	58	38	28	60	38	54	60	40	56
Y :	74	70	32	52	46	54	38	40	32	22

Solution: Given that

X :	48	58	38	28	60	38	54	60	40	56
Y :	74	70	32	52	46	54	38	40	32	22

We know that rank correlation

$$\rho = 1 - \frac{6\left(\sum (x-y)^2 + CF\right)}{n(n^2 - 1)}$$

Where $CF = \frac{1}{12} m_1 (m_1^2 - 1) + \frac{1}{12} m_2 (m_2^2 - 1) + \dots,$

X	Y	Rank of $X(x)$	Rank of $Y(y)$	$x-y$	$(x-y)^2$
48	74	6.0	1.0	5.00	25.00
58	70	3.0	2.0	1.00	1.00
38	32	8.5	9.5	-1.00	1.00
28	52	10.0	4.0	6.00	36.00
60	46	1.5	5.0	-3.50	12.25
38	54	8.5	3.0	5.50	30.25
54	38	5.0	8.0	-3.00	9.00

60	40	1.5	7.0	-5.50	30.25
40	32	7.0	9.5	-2.50	6.25
56	22	4.0	11.0	-7.00	49.00
22	42	11.0	6.0	5.00	25.00
		Total			225.00

Now $CF = \frac{1}{12}m_1(m_1^2 - 1) + \frac{1}{12}m_2(m_2^2 - 1) + \ldots,$

$$= \frac{1}{12}2(2^2 - 1) + \frac{1}{12}2(2^2 - 1) + \frac{1}{12}2(2^2 - 1)$$

$$= 1.5$$

Thus $\rho = 1 - \frac{6\left(\sum (x-y)^2 + CF\right)}{n(n^2 - 1)}$

$$= 1 - \frac{6 \times (225 + 1.5)}{11(11^2 - 1)}$$

$$= 1 - \frac{1359}{11(11^2 - 1)}$$

$$= 1 - \frac{1359}{1320}$$

$$= -0.0295$$

(i.e) the rank correlation coefficient is -0.0295

Example 1.4

Find the rank correlation coefficient for the following data

X :	115	109	112	87	98	98	120	100	98	118
Y :	75	73	85	70	76	65	82	73	68	80

Solution:

Given that

$X:$	115	109	112	87	98	98	120	100	98	118
$Y:$	75	73	85	70	76	65	82	73	68	80

We know that rank correlation = ρ

$$= 1 - \frac{6\left(\sum (x-y)^2 + CF\right)}{n\left(n^2-1\right)}$$

Where $CF = \frac{1}{12}m_1\left(m_1^2-1\right)+\frac{1}{12}m_2\left(m_2^2-1\right)+\ldots,$

X	Y	Rank of $X(x)$	Rank of $Y(y)$	$x-y$	$(x-y)^2$
115	75	3	5.0	-2.00	4.00
109	73	5	6.5	-1.50	2.25
112	85	4	1.0	3.00	9.00
87	70	10	8.0	2.00	4.00
98	76	8	4.0	4.00	16.00
98	65	8	10.0	-2.00	4.00
120	82	1	2.0	-1.00	1.00
100	73	6	6.5	-0.50	0.25
98	68	8	9.0	-1.00	1.00
118	80	2	3.0	-1.00	1.00
		Total			42.50

Now $CF = \frac{1}{12}m_1\left(m_1^2-1\right)+\frac{1}{12}m_2\left(m_2^2-1\right)+\ldots,$

$$= \frac{1}{12}3\left(3^2-1\right)+\frac{1}{12}2\left(2^2-1\right)$$

=2.5

Thus $\rho = 1 - \frac{6\left(\sum (x-y)^2 + CF\right)}{n\left(n^2 - 1\right)}$

$= 1 - \frac{6 \times (42.5 + 2.5)}{10\left(10^2 - 1\right)}$

$= 1 - \frac{42.50}{990}$

$= 0.7273$

(i.e) the rank correlation coefficient is 0.7273

Example 1.5:

Ten competitors in a beauty contest were ranked by three judges in the following order:

Judge I:	1	6	5	10	3	2	4	9	7	8
Judge II:	3	5	8	4	7	10	2	1	6	9
Judge III:	6	4	9	8	1	2	3	10	5	7

Solution:

Step 1: To find the rank correlation coefficient between Judges I and II

Now

Judge I (x):	1	6	5	10	3	2	4	9	7	8
Judge II (y):	3	5	8	4	7	10	2	1	6	9

Now we shall find $\sum (x-y)^2$

x	y	$x-y$	$(x-y)^2$
1	3	-2	4
6	5	1	1
5	8	-3	9

10	4	6	36
3	7	-4	16
2	10	-8	64
4	2	2	4
9	1	8	64
7	6	1	1
8	9	-1	1
Total			195

We know that rank correlation = ρ

$$=1-\frac{6\left(\sum (x-y)^2\right)}{n\left(n^2-1\right)}$$

$$= 1-\frac{6\times 195}{10\times\left(10^2-1\right)}$$

$$=1-\frac{1170}{990}$$

$$=-0.18182$$

(i.e) the rank correlation coefficient is -0.182

Step 2:

To find the rank correlation between judges II and III

Now

Judge II (y):	3	5	8	4	7	10	2	1	6	9
Judge III (z) :	6	4	9	8	1	2	3	10	5	7

Now we shall find $\sum (y-z)^2$

y	z	$y-z$	$(y-z)^2$
3	6	-3	9
5	4	1	1
8	9	-1	1
4	8	-4	16
7	1	6	36
10	2	8	64
2	3	-1	1
1	10	-9	81
6	5	1	1
9	7	2	4
Total			204

We know that rank correlation = ρ

$$=1-\frac{6\left(\sum (y-z)^2\right)}{n(n^2-1)}$$

$$= 1-\frac{6\times 204}{10\times(10^2-1)}$$

$$=1-\frac{1224}{990}$$

$$=-0.23636$$

(i.e) the rank correlation coefficient is -0.236.

Step 3:

To find the rank correlation between judges III and I.

Now

Judge III (z):	6	4	9	8	1	2	3	10	5	7
Judge I (x):	1	6	5	10	3	2	4	9	7	8

Now we shall find $\sum(x-z)^2$

x	z	$x-z$	$(x-z)^2$
6	1	5	25
4	6	-2	4
9	5	4	16
8	10	-2	4
1	3	-2	4
2	2	0	0
3	4	-1	1
10	9	1	1
5	7	-2	4
7	8	-1	1
Total			31

We know that rank correlation

$$= \rho = 1 - \frac{6\left(\sum(x-z)^2\right)}{n(n^2-1)}$$

$$= 1 - \frac{6 \times 31}{10 \times (10^2 - 1)}$$

$$= 1 - \frac{186}{990}$$

$$= 0.812121$$

(i.e) the rank correlation coefficient is 0.812.

Step 4:

Clearly from step 3, the judge I and Judge III have a positive rank correlation and there these judges have taste in beauty.

Check your progress:

(1) Find the rank correlation coefficient to the following data.

X :	92	89	87	86	83	77	71	63	53	50
Y:	86	83	91	77	68	85	52	82	37	57

(2) Find the rank correlation coefficient to the following data.

X :	30	50	25	30	60	70	80	65	75	98
Y:	50	60	20	40	70	40	90	60	40	80

(3) Find the rank correlation coefficient to the following data.

X :	48	33	40	9	16	16	65	24	16	57
Y:	13	13	24	6	15	4	20	9	6	19

(answers: (1) 0.73, (2) 0.61, (3) 0.73)

1.3 Interpolation

1.3.0 Introduction

Interpolation is the process of finding the most appropriate estimate for missing data. It is the "art of reading between the lines of a table". For making the most probable estimate it requires the following assumptions.

(i) The frequency distribution is normal and not marked by sudden ups and downs.

(ii) The changes in the series are uniform within a period. Interpolation technique is used in various disciplines like economics, business, population studies, price determination etc. It is used to fill in the gaps in the statistical data for the sake of continuity of information. For example, if we know the records for the past five years except the third year which is not available due to unforeseen conditions the interpolation technique helps to estimate the record for that year too under the assumption that the changes in the records over these five years have been uniform.

It is also possible that we may require information for future in which case the process of estimating the most appropriate value is known as extrapolation.

Given a set of tabular values of a function $y = f(x)$ where the explicit nature of the function is not known, then $f(x)$ is replaced by a simpler function $\phi(x)$ such that $f(x)$ and $\phi(x)$ agree with the set of tabulated points. Any other value may be calculated from $\phi(x)$. This function $\phi(x)$ is known as an interpolating function. In particular if $\phi(x)$ is a polynomial then the process is called polynomial interpolation and $\phi(x)$ is called an interpolating polynomial. The existence of an interpolating polynomial is supported by weierstrass approximation theorem which asserts that any continuous function on a closed interval can be approximated by a polynomial. In this chapter we introduce various interpolation polynomials using the concepts of forward difference, backward difference and central difference.

1.3.1 Newton's Interpolation Formula

Theorem 1.1.

(Newton's Forward interpolation formula for equal intervals).

Let the function $y = f(x)$ take the values $y_0, y_1, \ldots, y_n$ at the points $x_0, x_1, \ldots, x_n$ where $x_i = x_0 + ih$. . Then Newton's forward interpolation polynomial is given by

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots + \frac{p(p-1)\ldots(p-\overline{n}-1)}{n!}\Delta^n y_0$$

where $x = x_0 + ph$.

Proof

Let $\phi(x)$ be an interpolating polynomial of degree n which represents $y = f(x)$ in $x_0 \le x \le x_0 + nh$.

Then $\phi(x)$ can be written in the form

$$\begin{aligned}\phi(x) = {} & a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_0-h) \\ & + a_3(x-x_0)(x-x_0-h)(x-x_0-2h) + \ldots + \\ & a_n(x-x_0)(x-x_0-h)\ldots(x-x_0-(n-1)h) \rightarrow \quad ①\end{aligned}$$

By definition of interpolating polynomial we have $y_0 = \phi(x_0)$.

Putting $x = x_0$ in ①, we get $a_o = y_0$.

Putting $x = x_0 + h$ in ①, we get $a_o = y_0$

$$a_1 = \frac{1}{h}(y_1 - a_0)$$

$$= \frac{1}{h}(y_1 - y_0)$$

$$\therefore a_1 = \frac{\Delta y_0}{h}.$$

Putting $x = x_0 + 2h$ in ①, we get $a_o = y_0$

$$a_2 = \frac{1}{2h^2}(y_2 - a_0 - 2ha_1)$$

$$= \frac{1}{2h^2}(y_2 - y_0 - 2\Delta y_0)$$

$$= \frac{1}{2h^2}(y_2 - 2y_1 + y_0)$$

$$a_2 = \frac{\Delta^2 y_0}{2!h^2}.$$

Similarly substituting $x = x_0 + 3h, \ldots, c_0 + \overline{n-1h}$ we get

$$a_3 = \frac{\Delta^3 y_0}{3!h^3}, \ldots, a_n = \frac{\Delta^n y_0}{n!h^n}$$

substituting these values in ① we get

$$\phi(x) = y_0 + (x - x_0)\frac{\Delta y_0}{1!h} + (x - x_0)(x - x_0 - h)\frac{\Delta^2 y_0}{2!h^2} + \ldots +$$

$$(x - x_0)(x - x_0 - h)\ldots(x - x_0 - \overline{n-1h})\frac{\Delta^n y_0}{n!h^n}$$

This gives the Newton's forward interpolation polynomial
Since $\phi(x)$ is the interpolating polynomial which represents $y = f(x)$ then function $\phi(x)$ can be written as y.

$$\therefore y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots +$$

$$\frac{p(p-1)\ldots(p - \overline{n-1})}{n!}\Delta^n y_0 \quad \text{where } x = x_0 + ph. \quad \text{ie, } p = \frac{x - x_0}{h}$$

interpolation formula is used to interpolate the values of y near the end of the set of tabulated values or for extrapolating values of y to the right of the last tabulated value y_n.

Solved Problems

Problem 1 ***Space for Hints***

If $y(75)=246,\ \ y(80)=202,\ \ y(85)=118,\ y(90)=40,$ find $y(79)$.

Solution Hence $x_0=75,\ h=5$ and we have to find the value of

y at $x=79$.

Newton's forward interpolation formula is

$$y_p=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\ldots+\frac{p(p-1)\ldots(p-(n-1))}{n!}\Delta^n y_0$$

where $p=\dfrac{x-x_0}{h}$

$\therefore\ p=\dfrac{79-75}{5}=0.8$.

we now form the forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
75	246			
		-44		
80	202		-40	
		-84		46
85	118		6	
		-78		
90	40			

using the values of $\Delta y_0,\ \Delta^2 y_0,\ \Delta^3 y_0$ in ① we get,

$$y_{0.8}=246+0.8(-44)+\frac{(0.8)(0.8-1)}{2!}(-40)+\frac{(0.8)(0.8-1)(0.8-2)}{3!}(46)$$

$$=215.472$$

Problem 2 Find a cubic polynomial which takes the following Values

x	0	1	2	3
$f(x)$	1	2	1	10

Solution Let us form the difference table first

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

By Newton's – Gregory formula,

$$f(x) = f(x_0)+(x-x_0)\frac{\Delta f(x_0)}{1!h}+(x-x_0)(x-x_0-h)\frac{\Delta^2 f(x_0)}{2!h^2}+\dots$$

here $x_0 = 0$ and $h = 1$.

$$\therefore\ f(x)=1+(x-0)\frac{1}{1!}+(x-0)(x-1)\frac{(-2)}{2!}+(x-0)(x-1)(x-2)\frac{12}{3!}$$

$$=1+x-x(x-1)+2x(x-1)(x-2)$$

$$=1+x-x^2+x+2x^3-6x^2+4x$$

$$=2x^3-7x^2+6x+1.$$

$$\therefore\ f(x)=2x^3-7x^2+6x+1$$

Problem 3 A function $y = f(x)$ is given by the following table. Find $f(0.2)$ by a suitable formula.

x	0	1	2	3	4	5	6
$y=f(x)$	176	185	194	203	212	220	229

Solution

Since the value $x=0.2$ is near the beginning of the table we use Newton's forward interpolation formula.

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots + \frac{p(p-1)\ldots p(p-(n-1))}{n!}\Delta^n y_0 \quad \text{①}$$

where $p = \dfrac{x - x_0}{h}$.

Here $x_0 = 0$, $h = 1$ and we want to find the value of $f(x)$ at $x = 0.2$

$$\therefore\ p = \frac{0.2-0}{1} = 0.2$$

Now we form the forward difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
0	176						
		9					
1	185		0				
		9		0			
2	194		0		0		
		9		0		-1	
3	203		0		-1		5
		9		-1		4	
4	212		-1		3		
		8		2			
5	220		1				
		9					
6	229						

$\therefore$ ① becomes,

$$y_{0.2} = 176 + 0.2\times 9 + \frac{(0.2)(0.2-1)(0.2-3)(0.2-4)}{5!}(-1) + \frac{(0.2)(0.2-1)(0.2-2)(0.2-3)(0.2-4)(0.2-5)}{6!}(5)$$

$$= 177.67232.$$

Hence $f(0.2) = 177.67$.

Space for Hints

Problem 4 Construct Newton's forward interpolation polynomial for the following data

x	4	6	8	10
y	1	3	8	16

use it to find the value of y for $x = 5$.

Solution Here $x_0 = 4$ and $h = 2$. The Newton's forward interpolation formula is

$$y = y_0 + \frac{(x - x_0)}{1!h}\Delta y_0 + (x - x_0)(x - x_0 - h)\frac{\Delta^2 y_0}{2!h^2} + \ldots \longrightarrow ①$$

we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

$$\therefore\ y = 1 + \frac{(x-4)\times 2}{1!2} + \frac{(x-4)(x-6)\times 3}{2!\ 2^2} + \frac{(x-4)(x-6)(x-8)\times 0}{3!\ 2^3}$$

$$= 1 + (x-4) + \frac{3(x-4)(x-6)}{8}$$

$\therefore y = 1 + (x-4) + \frac{3}{8}(x^2 - 10x + 24)$ is the required interpolating polynomial, when $x = 5$, $y_5 = 1 + 1 + \frac{3}{8}(5^2 - 50 + 24) = 1.625.$

Problem 5 Find the value of y from the following data at $x = 2.65$ ***Space for Hints***

x	-1	0	1	2	3
y	-21	6	15	12	3

Solution Since the value of $x = 2.65$ is near the end of the table we use Newton's backward interpolation formula.

The formula is $y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \ldots$

where $p = \frac{x - x_n}{h}$

Here $x = 2.65$, $x_n = 3$ and $h - 1$.

$$\therefore p = \frac{2.65 - 3}{1} = -0.35$$

$$\therefore p = -0.35$$

To find ∇y_n, $\nabla^2 y_n$ etc we from the backward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-1	-21				
		27			
0	6		-18		
		9		6	
1	15		-12		0
		-3		6	
2	12		-6		
		-9			
3	3				

$$\therefore y_{-0.35} = 3 + (-0.35)(-9) + \frac{(-0.35)(-0.35+1)}{2!}(-6) + \frac{(-0.35)(-0.35+1)(-0.35+2)}{3!} \times 6$$

$= 6.4571.$

Problem 6 The following data gives the melting point of an alloy of zinc and lead, θ is the temperature and x is the percentage of lead. Using Newton's interpolation formula find (i) θ when $x = 48$
(ii) θ when $x = 84$

x	40	50	60	70	80	90
θ	184	204	226	250	276	304

Solution (i) Since $x = 48$ is near the beginning of the table we use Newton's forward interpolation formula.

The formula can be written as

$$\theta_p = \theta_0 + p\Delta\theta_0 + \frac{p(p-1)}{2!}\Delta^2\theta_0 + \ldots$$

$$\text{where } p = \frac{x - x_0}{h} = \frac{48-40}{10} = 0.8$$

To find $\Delta\theta_0$, $\Delta^2\theta_0$ etc we form forward difference table.

x	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$	$\Delta^5\theta$
40	184					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

$$\theta_p = 184 + 0.8 \times 20 + \frac{(0.8)(0.8-1)}{2!} \times 2$$

$$= 199.84 \simeq 200.$$

$\therefore \theta_p = 199.84 \simeq 200.$

Space for Hints

(ii) Since $x = 84$ is nearer to the end of the table we use Newton's backward interpolation formula

$$\theta_p = \theta_n + p\nabla\theta_n + \frac{p(p+1)}{2!}\nabla^2\theta_n + \ldots$$

where $p = \frac{x - x_n}{h} = \frac{84-90}{10} = -0.6$

The values of $\theta_n, \nabla\theta_n, \nabla^2\theta_n$ etc. Can be obtained from the forward difference table by sloping backwards with respect to the increasing direction of x.

$$\therefore \theta_n = 304,\ \nabla\theta_n = 28,\ \nabla^2\theta_n = 2, \nabla^3\theta_n = \nabla^4\theta_n = \nabla^5\theta_n = 0.$$

Hence we have

$$\theta - 0.6 = 304 + (-0.6)\times 28 + \frac{(-0.6)(-0.6-1)}{2!}(2)$$

$$= 286.96 \cong 287.$$

$$\therefore \theta - 0.6 = 286.96 \cong 287.$$

Problem 7 From the data given below, find the number of student whose weight is between 60 and 70.

weight	0-40	40-60	60-80	80-100	100-120
No.of students	250	120	100	70	50

Solution The less than cumulative frequency table of the given data is as shown below.

weight less than x	40	60	80	100	120
No.of students	250	370	470	540	590

we now form the difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
40	250				
		120			
60	370		-20		
		100		-10	
80	470		-30		20
		70		10	
100	540		-20		
		50			
120	590				

Number of students whose weight is between 60 and 70 is got from $y_{70} - y_{60}$.

we have $y_{60} = 370.$

Now we shall find y_{70} by Newton's forward interpolation formula.

$$y_p = y_0 + p\nabla y_0 + \frac{p(p-1)}{2!}\nabla^2 y_0 + \ldots + \frac{p(p-1)\ldots(p-(n-1))}{n!}\nabla^n y_0$$

where $p = \dfrac{x - x_0}{h}$.

Here $x_0 = 40,\ h = 20$ and $x = 70.$

Hence $p = \dfrac{70-40}{20} = 1.5$

Now,

$$y_{70} = y_{1.5} = 250 + 1.5 \times 120 + \frac{(1.5)(0.5)}{2!}(-10) + \frac{(1.5)(0.5)(-0.5)(-1.5)}{4!}(20)$$

$$= 423.59375$$

$$y_{70} = 424 \text{ (approximately).}$$

∴ Number of persons whose weight is between 60 and 70 is 424 – 370 =54.

Exercise Problems

(1)For the following data find $f(9)$

(2) using Newton's forward interpolation formula

Space for Hints

x	8	10	12	14	16
$f(x)$	1000	1900	3250	5400	8950

(2) Find $f(2.5)$ using Newton's forward difference formula for the given data.

x	1	2	3	4	5	6
y	0	1	8	27	64	125

From the following table find the value of tan 45°15'

$x°$	45	46	47	48	49	50
$\tan x°$	1.00000	1.0355 3	1.0723 7	1.11061	1.1503 7	1.1917 5

(4) From the table given below find Sin 52° by using Newton's forward interpolation formula.

x	45	50	55	60
Sinx	0.7071	0.7660	0.8192	0.8660

(5) The following data is taken from the steam table. Find the pressure at temperature t =142° and t = 175°

Temperature c°	140	150	160	170	180
Pressure kg/cm^2	3.685	4.854	6.302	8.075	10.225

(6) The following table gives the census population of a town for the years 1931-1991. Estimate the population (i) for the year 1965 (ii) for the year 1933 by using appropriate interpolation formula.

Year	1931	1941	1951	1961	1971
Population in lakhs	36	66	81	93	101

(7) Estimate the value of $f(22)$ and $f(42)$ from the following data

x	20	25	30	35	40	45
$f(x)$	354	332	291	260	231	204

(8) Using Newton's forward interpolation formula find a polynomial of degree four which takes the values

(i)

x	2	4	6	8	10
y	0	0	1	0	0

(ii)

x	4	6	8	10
y	1	3	8	16

Hence find $f(5)$.

(9) Using Newton's backward formula find the polynomial of degree three passing through (3,6) (4,24) (5,60) and (6,120).

Hence find the ordinate when the abscissa is 5.5.

10) Given the following data, express y as a function of x.

Hence find the value of y at $x=0.5$.

x	0	1	2	3	4
y	3	6	11	18	27

Interpolation with unequal intervals

The various interpolation formulae we have derived so far are applicable only when the values of the function are given at equally spaced points. We now proceed to develop interpolation formulae for unequally spaced points. In this section we study lagrange's interpolation formula and in the next section Newton's divided difference formula which are applicable for unequally spaced values of x.

1.3.2 Lagrange's Interpolation formula

Theorem 1.3 (Lagrange's Interpolation formula).

Let $y_0, y_1, \ldots, y_n$ be the values of $f(x)$ at $x_0, x_1, \ldots x_n$ (not necessarily at equally interval) then an interpolating polynomial $\phi(x)$ for $f(x)$ is given by

$$\phi(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1$$

$$+\cdots+\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}y_n.$$

Proof

Since n values for $f(x)$ are given we can assume $f(x)$ to be a polynomial of degree n-1.

Let $\phi(x)=A_1(x-x_1)(x-x_2)\cdots(x-x_n)+A_2(x-x_0)(x-x_2)\cdots(x-x_n)$
$+\cdots+A_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$ ⟶ ①

when $x=x_0$, we get $y_0=A_1(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)$

$$\therefore A_1=\frac{y_0}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}$$

Similarly when $x=x_1,x_2,\ldots,x_n$ we get

$$A_2=\frac{y_1}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}$$

$$A_n=\frac{y_n}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}$$

Substituting these values in ① we get Lagrange's formula.

Note Since $\phi(x)$ is the interpolation polynomial which represents $f(x)$, the function $\phi(x)$ can be replaced by $f(x)$ in ①.

Hence Lagrange's formula becomes

$$y=f(x)=\frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1$$
$$+\cdots+\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}y_n.$$

Solved Problems

Problem 1 Use Lagrange's formula to find the value of y at $x=6$ from the following data.

x	3	7	9	10
y	168	120	72	63

Solution

The Lagrange's formula for four set of data is

$$y=\left[\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right]y_0+\left[\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\right]y_1$$

$$+\left[\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\right]y_2.+\left[\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\right]y_3 \rightarrow ①$$

Here $x_0 = 3, \quad x_1 = 7, \quad x_2 = 9, \quad x_3 = 10$

$y_0 = 168, \; y_1 = 120, \; y_2 = 72, \; y_3 = 63$

Space for Hints

To find $y(6)$ put $x = 6$ in ①

$$y(6) = \left[\frac{(6-7)(6-9)(6-10)}{(3-7)(3-9)(3-10)}\right]\times 168 + \left[\frac{(6-3)(6-9)(6-10)}{(7-3)(7-9)(7-10)}\right]\times 120$$

$$+\left[\frac{(6-3)(6-7)(6-10)}{(9-3)(9-7)(9-10)}\right]\times 72 + \left[\frac{(6-3)(6-7)(6-9)}{(10-3)(10-7)(10-9)}\right]\times 63$$

$= 12 + 180 - 72 + 27$

$= 147.$

Problem 2 The Lagrange's formula to fit a polynomial to the data.

x	0	1	3	4
y	-12	0	6	12

find the value of y when x=2.

Solution

The Lagrange's formula for four set of data is

$$y = \left[\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right]y_0 + \left[\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\right]y_1$$

$$+\left[\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\right]y_2.+\left[\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\right]y_3$$

Here $x_0 = 0, \quad x_1 = 1, \quad x_2 = 3, \quad x_3 = 4$ and

$y_1 = 12, \; y_2 = 0, \; y_3 = 6, \; y_4 = 12$

$$\because y = \left[\frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)}\right]\times(-12) + \left[\frac{x(x-3)(x-4)}{1(-2)(-3)}\right]\times 0 + \left[\frac{x(x-1)(x-4)}{3\times 2\times(-1)}\right]\times 6$$

$$+\left[\frac{x(x-1)(x-3)}{4\times 3\times 1}\right]\times 12$$

$$= (x-1)(x-3)(x-4) - x(x-1)(x-4) + x(x-1)(x-3)$$

$$= (x-1)[(x-3)(x-4) - x(x-4) + x(x-3)]$$

$$= (x-1)(x^2 - 6x + 12)$$

$$= x^3 - 7x^2 + 18x - 12$$

when $x = 2$, $y = y(2) = 2^3 - 7x2^2 + 18 \times x2 - 12 = 4.$

Space for Hints

Problem 3 Find the form of the function y for the following data.

Hence find $y3$

x	0	1	2	5
y	2	3	12	147

Solution

Here $x_0 = 0;\quad x_1 = 1;\quad x_2 = 2;\quad x_3 = 5$

$y_0 = 2;\quad y_1 = 3;\quad y_2 = 12;\quad y_3 = 147$

Applying Lagrange's formula we get

$$y = \left[\frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}\right] \times 2 + \left[\frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}\right] \times 3 + \left[\frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}\right] \times 147$$

$$= \frac{(x^3 - 8x^2 + 17x - 10)}{5} + \frac{3(x^3 - 7x^2 + 10x)}{4} - 2(x^3 - 6x^2 + 5x) + \frac{x^3 - 3x^2 + 2x}{60} \times 127$$

$$= \frac{1}{60}\left[60x^3 + 60x^2 - 60x + 120\right]$$

$$\therefore y = x^3 + x^2 - x + 2$$

At $x = 3$, $y(3) = 3^3 + 3^2 - 3 + 2$

$= 35.$

Problem 4

The values of $U(x)$ are known at a, b, c. Show that maximum (or) minimum of Lagrange's interpolation polynomial is attained at

$$x = \frac{\sum Ua(b^2 - c^2)}{2\sum Ua(b-c)}$$

s

Solution

By Lagrange's formula for 3 set of values we have

$$U(x)=\frac{(x-b)(x-c)}{(a-b)(a-c)}\times Ua+\frac{(x-a)(x-c)}{(b-a)(b-c)}\times Ub+\frac{(x-a)(x-b)}{(c-a)(c-b)}\times Uc$$

$$=\sum\left[\frac{x^2-(b+c)x+bc}{(a-b)(a-c)}\right]Ua$$

$U(x)$ attains its maximum (or) minimum when $\frac{dU}{dx}=0$.

Now, $\frac{dU}{dx}=0\Rightarrow\sum\left[\frac{2x-(b+c)}{(a-b)(a-c)}\right]U=0$

$$\Rightarrow\left[\frac{2x-(b+c)}{(a-b)(a-c)}\right]Ua+\left[\frac{2x-(c+a)}{(b-c)(b-a)}\right]Ub+\left[\frac{2x-(a+b)}{(c-a)(c-b)}\right]Uc=0$$

$$\Rightarrow(b-c)[2x-(b+c)]Ua+(c-a)+Uc(c-a)[2x-(c+a)]Ub$$
$$+(a-b)[2x-(a+b)Uc=0]$$

$$\Rightarrow 2x[Ua(b-c)+Ub(c-a)Uc+(a-b)]=(b^2-c^2)Ua$$
$$+(c^2-a^2)Ub+(a^2-b^2)Uc$$

$$\therefore x=\frac{(b^2-c^2)Ua+(c^2-a^2)Ub+(a^2-b^2)Uc}{2[Ua(b-c)+Ub(c-a)+Uc(a-b)]}$$

$$=\frac{\sum Ua(b^2-c^2)}{2\sum Ua(b-c)}.$$

Exercises

(1) Use Lagrange's interpolation formula to find the value of y when $x=10$ if the following values of x and y are given.

x	5	6	9	11
Y	12	13	14	16

Space for Hints

(2) Use Lagrange's formula to find $f(x)$ when $x=0$, given the following data.

x	-1	-2	2	4
$f(x)$	-1	-9	11	69

(3) Find U_5 given that $U_1=4$; $U_2=7$; $U_4=13$ and $U_7=30$.

(4) The following table gives the normal weight of a baby during first 6 months of life.

Age of month	0	2	3	5	6
Weight in *lbs*	5	7	8	10	12

(5) The amount A of a substance remaining in a reacting system after an interval of time t in a certain chemical experiment is given in the following table.

t	2	5	8	14
A	94.8	87.9	81.3	68.7

(6) Determine by Lagrange's formula the percentage of number of criminals under 35 years

Age	% No. of Criminals
Under 25 years	52.0
Under 30 years	67.3
Under 40 years	84.1
Under 50 years	94.4

(7) Use Lagrange's formula find a polynomial to the data.

x	0	1	3	4
$U(x)$	-12	0	6	12

(8) Use Lagrange's formula to find a polynomial to the following data.

x	-1	0	2	3
$U(x)$	-8	3	1	12

(9) Find the form of $U(x)$ given that $U_0 = 8;\ U_1 = 11;\ U_4 = 78;\ U_5 = 123.$

(10) Apply Lagrange's formula to find $f(x)$ for the following data.

x	1	2	-4
$f(x)$	3	-5	4

(11) From the following table find the value of y when $x = 10$.

x	5	6	9	11
y	12	13	14	16

Hence find $f(2)$.

1.4 Attributes

Objectives

In this unit, we are going to discuss how to find the relationship between two attributes, independency of two attributes.

After the completion of this unit one may able to find the

- Relationship between attributes
- Independency of two attributes

1.4.0 Introduction

In the earlier units we have discussed those data related to quantitative. But we cannot measure those data having qualitative nature. Qualitative characteristics of a population are called attributes and they cannot be measured by numeric quantity. .

Suppose a population is divided into two classes according to the possession or non–possession or presence or absence of a single attribute. The class in which a particular characteristics is present is called positive class and it is denoted by upper case of English alphabets and the absence of the characteristics is called negative class and it is denoted by the lower case of Greek letters. That is
A,B,C,...
denotes the presence of characteristics and α,β,γ,... denotes the corresponding absence of the characteristics.

For example, if attribute A represents *rich* and B represents *literate* then α refers *poor* and β refers *illiterate*. Further AB represents the possession of both rich and literate; Aβ represents rich and illiterate; αB represents poor and literate and αβ represents poor and illiterate.

The above example can be represent as table form as

Attribute	B	β
A	AB	Aβ
A	αB	Aβ

Space for

hint

A class represented by n attributes is called a class of n^{th} order. Thus A,B, α,β are called first order , AB, Aβ, αB, αβ are called class of second order, ABC, ABγ, Aβγ,... are called class of third order.

Now the number of individuals possess the attribute A is called frequency of the attribute A and it is denoted by (A). Hence (αB) stands the number of individuals possessing the attributes α and B.
The total frequency in a population is denoted by N.

Note:

Class frequencies of the type (A), (AB), (ABC), ... are called positive class frequencies.

Class frequencies of the type (α), (αβ), (αβγ), ... are called negative class frequencies.

Class frequencies of the type (Aβ), (αB), (AβC),... are called contrary class frequencies.

1.4.1.Attributes

The frequency classes for two attributes can be represented in the form of a table which is given below.

Attribute	B	β	Total
A	AB	Aβ	(A)
A	αB	Aβ	(α)
Total	(B)	(β)	N

The relationship between the class frequencies of various orders are given below.

For class frequency of order 2:
Space for

N= (A) + (α) = (B) + (β),
hint

(AB) + (Aβ) = (A),

(αB) + (αβ) = (α),

(AB) + (αB) = (B),

(Aβ) + (αβ) = (β),

(AB) + (Aβ) + (αB) + (αβ) = N

For class frequency of order 3

(ABC) + (ABγ) + (AβC) + (Aβγ) = (A),

(ABC) + (Aβγ) + (αBC) + (αBγ) = (B),

(ABC) + (AβC) + (αBC)+ (αβC) = (C),

(ABC) + (ABγ) = (AB),

(ABC) + (Aβγ) = (Aγ) etc.,

Theorem 1.5

For n attributes

(a)total number of class frequencies is 3^n

(b)total number of positive class frequencies is 2^n

(c)total number of negative class frequencies is $2^n - 1$

Proof:

Let n attributes be given.

Thus the number of ways of choosing r attributes from the given set of n attributes is $\binom{n}{r}$.

Since each attribute gives two symbols (one for positive class and another for negative class),the number of class frequencies of order r that can be obtained from r attributes is 2^r.

Hence the total number of class frequencies of order r is $\binom{n}{r}2^r$.

Thus the total number of all class frequencies=$\sum_{r=0}^{n}\binom{n}{r}2^r$

$$=1+\binom{n}{1}2^1+\binom{n}{2}2^2+\binom{n}{3}2^3+\ldots+\binom{n}{n}2^n$$

$$=(1+2)^n$$

$$=3^n$$

(ii)Further any collection of r attributes have only one positive class frequency of order r.

Hence the total number of positive class frequencies of all orders$=\sum_{r=0}^{n}\binom{n}{r}$

$$=1+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\dots+\binom{n}{n}$$

$$=(1+1)^n$$

$$=2^n$$

(iii)Clearly there is no negative class frequency of order 0.

Then any collection of r attributes gives rise to one negative class frequency of order r.

Hence the total number of negative class frequencies of all orders

$$=\sum_{r=0}^{n}\binom{n}{r}-1$$

$$=2^n-1$$

Dichotomization:

Dichotomization is the process of dividing a collection of objects into two classes according to the possession or non-possession of an attribute.

Notations: We use the following notations in the theory of attribute.

The total number of objects in a population is N.

If A is an attribute then N = (A) + (α) and it is written as (A) =A.N and (α) = α.N.

Thus N = (A) + (α)

=A.N + α.N

=(A + α) .N

A+α=1.

Hence

Thus in symbolic expression A can be replaced by $1-\alpha$ and α by $1-A$.

Example 1.1

Prove that $(BC) = (ABC) + (\alpha BC)$

Proof:

Now $(\alpha BC) = \alpha BC.N$

$$= (1 - A)BC.N$$

$$= BC.N - ABC.N$$

$$= (BC) - (ABC)$$

Hence $(BC) = (ABC) + (\alpha BC)$

Example 1.2

Prove that for two attributes A and B, prove that

$N = (AB) + (A\beta) + (\alpha B) + (\alpha\beta)$

Proof:

Let A and B be two attributes.

$$\therefore N = (A) + (\alpha)$$

$$= (AB) + (A\beta) + (\alpha B) + (\alpha\beta)$$

Hence $N = (AB) + (A\beta) + (\alpha B) + (\alpha\beta)$

Example 1.3

For any three attributes, prove that

$$N = (ABC) + (AB\gamma) + (A\beta C) + (A\beta\gamma) + (\alpha BC) + (\alpha B\gamma) + (\alpha\beta C) + (\alpha\beta\gamma).$$

Proof:

$$L.H.S = N$$

$$= (AB) + (A\beta) + (\alpha B) + (\alpha\beta)$$

$$= (ABC) + (AB\gamma) + (A\beta C) + (A\beta\gamma) + (\alpha BC) + (\alpha B\gamma) + (\alpha\beta C) + (\alpha\beta\gamma)$$

$$= R.H.S$$

Example 1.4

Prove that for any two attributes negative class frequencies can be expressed in terms of positive class frequencies and converse also true.

Proof:

Consider two attributes A and B.

$$\text{Now } (\alpha\beta) = \alpha\beta.N$$

$$= (1 - A)(1 - B).N$$

$$= (1 - A - B + AB).N$$

$$= N - A.N - B.N + AB.N$$

$$= N - (A) - (B) + (AB)$$

$(\alpha\beta) = N - (A) - (B) + (AB)$ Hence

Thus negative class frequencies can be expressed in terms of positive class

Thus negative class frequencies can be expressed in terms of positive class

frequencies.

and (AB)= AB.N

$$= (1 - \alpha)(1 - \beta).N$$

$$= (1 - \alpha - \beta + \alpha\beta).N$$

$$= N - \alpha.N - \beta.N + \alpha\beta.N$$

$$= N - (\alpha) - (\beta) + (\alpha\beta)$$

Hence $$\boxed{(AB) = N - (\alpha) - (\beta) + (\alpha\beta)}$$

Thus positive class frequencies can be expressed in terms of negative class frequencies.

Example 1.5

Show that n attributes $A_1, A_2, \ldots, A_n$,

$$(A_1, A_2, A_3, \ldots, A_n) \geq (A_1) + (A_2) + \ldots + (A_n) - (n-1)N$$

Proof:

We shall prove the result using induction on n.

If n=2 then $(\alpha_1\alpha_2) = N - (A_1) - (A_2) + (A_1A_2)$

We know that $(\alpha_1\alpha_2) \geq 0$

i.e.) $N - (A_1) - (A_2) + (A_1A_2) \geq 0$

(i.e.) $(A_1A_2) \geq (A_1) + (A_2) - N$

(i.e.) $(A_1A_2) \geq (A_1) + (A_2) - (2-1)N$

Thus the result is true for n=2.

Assume the result is true for n=k.

(i.e.) $(A_1,A_2,A_3,\ldots,A_k) \geq (A_1)+(A_2)+\ldots+(A_n) - (k-1)N$

Now we shall prove the result is true for n=k+1

Now $(A_1,A_2,A_3,\ldots,A_{k+1}) \geq (A_1)+(A_2)+\ldots+(A_{k-1})+(A_kA_{k+1}) - (k-1)N$

(i.e.) $(A_1,A_2,A_3,\ldots,A_{k+1}) \geq (A_1)+(A_2)+\ldots+(A_{k-1})+(A_k)+(A_{k+1}) - N - (k-1)N$

(i.e.) $(A_1,A_2,A_3,\ldots,A_{k+1}) \geq (A_1)+(A_2)+\ldots+(A_{k-1})+(A_k)+(A_{k+1}) - kN$

(i.e.) $(A_1,A_2,A_3,\ldots,A_{k+1}) \geq (A_1)+(A_2)+\ldots+(A_{k-1})+(A_k)+(A_{k+1}) - (\overline{k+1}-1)N$

(i.e.)the result is true for n=k+1

∴ by mathematical induction, the result is true for all positive integer n.

(i.e.)for $n \in Z^+$, $(A_1,A_2,A_3,\ldots,A_n) \geq (A_1)+(A_2)+\ldots+(A_n) - (n-1)N$.

This proves the problem.

Example 1.6

Given frequencies (A) =1150, (α) =1120, (AB) =1075 , (αβ) =985.Find the remaining class frequencies and the total number of the observations.

Solution:

Given that (A)=1150, (α)=1120, (AB)=1075 , (αβ)=985.

We shall form the contingency table as follows.

Attribute	B	B	Total

A	1075	75	1150
A	135	985	1120
Total	1210	1060	2270

From the above table it is clear that (Aβ) =75, (αβ) =135,(β)=985,(B)=1210 and total number of observations=N=2270.

Example 1.7

Given the following class frequencies, find the frequencies of the positive and negative classes and the total number of observations.

(AB) =733, (Aβ) =840,(αB) =699, (αβ) =783.

Solution: Given that (AB)=733, (Aβ)=840,(αB)=699, (αβ)=783.

We shall form the contingency table as follows.

Attribute	B	B	Total
A	733	840	1573
A	699	783	1482
Total	1432	1623	3055

From the above table it is clear that,

negative class frequencies : (α)= 1482 and (β)=1623,

positive class frequencies: (A)= 1573 and (B)=1432

total number of observations= N =3055.

Example 1.8

A survey reveals that out of 1000 people in a locality 800 like coffee; 700 like tea; 660 like both coffee and tea. Find how many people like neither coffee nor tea.

Solution:

Let A be attribute to denote the coffee liker.

Let B be attribute to denote the tea liker.

We shall form the contingency table as follows.

Attribute	B	B	Total
A	660	140	800
A	40	160	200
Total	700	300	1000

From the above table, the number of people like neither coffee nor tea is $(\alpha\beta) = 160$.

Example 1.9

100 children took three examinations.40 passed the first, 39 passed the second and 48 passed the third, 10 passed all three.21 failed all three, 9 passed the first two and failed the third, 19 failed the first two and passed the third. Find how many children passed at least two examinations.

Solution:

Let A be attribute to denote a child passed in the first examination.

Let B be attribute to denote a child passed in the second examination.

Let C be attribute to denote a child passed in the third examination.

Given that N=100, (A) =40, (B) =39, (C) =48,

(ABC) =10, $(\alpha\beta\gamma)$ =21, $(AB\gamma)$ =9, $(\alpha\beta C)$=19.

We know that (ABC) + (αBC) + $(A\beta C)$

=(C) – $(\alpha\beta C)$

=48 – 19

=29

∴number of children passed in at least two examination

= (ABC) + (αBC) + $(A\beta C)$ + $(AB\gamma)$

=29+9

=38

Example 1.10

A survey conducted among T.V. viewers in a city revealed the following results.

850 see Doordharshan T.V. programmes; 780 see Star T.V. programmes; 326 see Cable T.V. programmes; 50 see all three programmes; 200 see Doordharshan T.V. programme and Star T.V. programmes but not Cable T.V. programmes;110 do not see Doordharshan and Star T.V. Programmes but see Cable T.V. programmes.

(i)Find how many people see Doordharshan and Star T.V. programmes

(ii)Find how many people see at least two T.V. programmes.

Solution:

Let A be attribute to denote those see Doordharshan T.V. programmes.

Let B be attribute to denote those see Star T.V. programmes.

Let C be attribute to denote those see Cable T.V. programmes.

Given that (A) = 850, (B) = 70, (C) = 326, (ABC) =200, $(\alpha\beta C)$=21.

(i)Now the number of people see Doordharshan and Star T.V. programmes

$$=(AB)$$

$$=(ABC) + (AB\gamma)$$

$$=50+200$$

$$=250$$

(ii)We know that $(ABC) + (\alpha BC) + (A\beta C) = (C) - (\alpha\beta C)$

$$=326 - 110$$

$$=216$$

The number of people see at least two T.V. programmes

$$=(ABC)+(\alpha BC)+(A\beta C)+(AB\gamma)$$

$$=216+200 = 416$$

1.4.2 Consistency of Data

Consider a population with the attribute A and B. For the data observed in the same Population (AB) cannot be greater than (A). Hence the figures (A) = 200 and (AB) = 250 are Inconsistent and $(\beta) = -800$ *is also inconsistent.*

Definition : A set of class frequencies is said to be consistent if none of them is negative.

The following are a set of criteria for testing the consistency in the case of single attribute,

Two attributes and three attributes.

Attribute	Condition for consistency	Equivalent positive class condition	Number of conditions
A	$(A) \geq 0$ $(\alpha) \geq 0$	$(A) \geq 0$ $(N) \leq N$	2
A, B	$(AB) \geq 0$ $(A\beta) \geq 0$ $(\alpha B) \geq 0$ $(\alpha\beta) \geq 0$	$(AB) \geq 0$ $(AB) \leq (A)$ $(AB) \leq (B)$ $(AB) \geq (A)+(B) - N$	2^2
A,B,C	$(ABC) \geq 0$ $(AB\Upsilon) \geq 0$ $(A\beta C) \geq 0$ $(\alpha BC) \geq 0$ $(A\beta\Upsilon) \geq 0$ $(\alpha B\Upsilon) \geq 0$ $(\alpha\beta C) \geq 0$ $(\alpha\beta\Upsilon) \geq 0$	(1) $(ABC) \geq 0$ (2) $(ABC) \leq (AB)$ (3) $(ABC) \leq (AC)$ (4) $(\alpha BC) \leq (BC)$ (5) $(ABC) \geq (AB) + (AC) - (A)$ (6) $(ABC) \geq (AC) + (BC) - (C)$ (7) $(ABC) \geq (AC) + (BC) - (C)$ (8) $(ABC) \leq (AB) + (BC) + (AC) - (A) - (B) - (C) + N$ (9) $(AB) + (BC) + (AC) \geq (A) + (B) + (C) - N$	2^3

		(10) (AC) +(BC) –(AB) ≤ (C) (11) (AB) + (BC) –(AC) ≤ (B) (12) (AB)+ (AC) –(BC) ≤ (A)	

Example 1.11

Test the consistency of the data when (A) = 800, (B) = 700, (AB) = 660, N = 1000

Solution :

For the given data, test the consistency.

(A) = 800, (B) = 700, (AB) = 50, N =1000.

Solution :

Given that (A) = 800, (B) = 700, (AB) = 50, N = 1000.

We shall find the other class frequencies

Attribute	B	B	Total
A	660	140	800
α	40	160	200
Total	700	300	1000

Since all class frequencies are positive, therefore the given data is consistent

Example 1.12

Test the consistency of the data when (A) =600, (B) = 500, (AB) = 50, N =1000.

Solution :

For the given data, test the consistency.

(A) = 600, (B) = 500, (AB) = 50, N = 1000.

Solution :

Given that (A) = 600, (B) = 500, (AB) = 50, N = 1000.

We shall find the other class frequencies.

Attribute	B	β	Total
A	50	550	600
A	450	-50	400
Total	500	500	1000

Since the class frequencies (αβ) is negative, therefore the given data is Inconsistent.

Example 1.13

A market investigator returns the following data. OF 2000 people consulted 1754 liked chocolates, 1872 liked toffee and 572 liked biscuits, 676 liked chocolates and toffee, 286 liked chocolate and biscuits, 270 liked toffee and biscuits, 114 Liked all the three.Verify that the information given by the investigator is consistent.

Solution :

Let A be the attribute that liking chocolate, B be the attribute that liking toffee And C be the attribute that liking biscuit.

Given that N = 2000, (A) = 1754, (B) = 1872, (C) = 572, (AB) = 676, (AC) =286

(BC) =270, (ABC) = 114.

We know that $(\alpha\beta\Upsilon) = N - (A) - (B) - (C) + (AB) + (AC) + (BC) - (ABC)$

$= 2000 - 1754 - 1872 - 572 + 676 + 286 + 270 - 114$

$= -1080$

Since the class frequency $(\alpha\beta\Upsilon)$ is negative, therefore the given data is Inconsistent.

Example 1.14

The following summary appears in a report on a survey covering 1000 fields. Scrutinize the numbers and point out if there is any mistake or misprint in Them.

Manure fields	510
Irrigate fields	490
Fields growing improved varieties	427
Field both irrigated and manure	189
Field both manure and growing improved varieties	140
Field both irrigated and growing improved varieties	85

Solution :

Let A be the attribute of Manured fields, B be the attribute of Irrigated fields And C be the attribute of Fields growing improved varieties.

Given that N = 1000, (A) = 510, (B) = 490, (C) = 427, (AB) = 189, (AC) =85, (BC) = 140.

We know that $(\alpha\beta Y) = N - (A) - (B) - (C) + (AB) + (AC) + (BC) - (ABC)$

$$= 1000 - 510 - 490 - 427 + 189 + 140 + 85 - (ABC)$$

$$= -13 - (ABC)$$

Now $(\alpha\beta Y) \geq 0$

(i.e.) $-13 - (ABC) \geq 0$

(i.e.) $(ABC) \leq -13 \cdot 0$

Since the class frequency (ABC) is negative, therefore there is a mistake in the data.

Example 1.15

If (A) = 50, (B) = 60, (C) = 50, $(A\beta) = 5$, $(AY) = 20$ and N = 100. Find the least and greatest values of (BC).

Solution :

Given that (A) = 50, (B) = 60, (C) = 50, $(A\beta) = 5$, $(AY) = 20$ and N = 100 .

Now $(AB) = (A) - (A\beta)$

$$= 50 - 5$$

$$=45$$

And $(AC) = (A) - (AY)$

$$= 50 - 20$$

$$= 30$$

Now $(AB) + (BC) + (AC) \geq (A) + (B) + (C) - N$

(i.e.) $45 + (BC) + 30 \geq 50 + 60 + 50 - 100$

$(BC) \geq -15$ --(1.1)

Again $(AB) + (AC) - (BC) \leq (AB)$

$45 + 30 - (BC) \leq 50$

$(BC) \geq 25$ --(1.2)

Now $(AB) + (BC) - (AC) \leq (B)$

$= 45 + (BC) - 30 \leq 60$

$= (BC) \leq 45$ --
(1.3)

Now $(AC) + (BC) - (AB) \leq (C)$

$= 30 + (BC) - 45 \leq 50$

$= (BC) \leq 65$ --
(1.4)

From (1.1),(1.2),(1.3) and (1.4), we have $-13 \leq 25 \leq (BC) \leq 45 \leq 65$

$= 25 \leq (BC) \leq 45$.

Check your progress

(1) N = 120, (A) =60,(B) = 90, (C) =30, (BC) = 15, (AC) = 15, find the limits between which (AB) lie.

(2) Find the least and greatest values of (ABC) if (A) =50, (B) = 60,(C)= 80, (AB) =35, (AC) = 45, (BC) = 42.

(3) Show that there is some error in the following data: 50% of peopleAre wealthy and healthy, 35% are wealthy but not healthy, 20% are healthy but wealthy.

(4) Of 2000 people consulted 1854 speak Tamil, 1507 speak Hindi , 572 speak English, 676 speak Tamil and Hindi, 286 speak Tamil and English, 270 speak Hindi and English, 114 speak Tamil,Hindi and English. Check whether the information is correct?

1.4.3 Independence and Association of Data

Two attributes A and B are said to be independent if there is same Propotion of A's amongst β's or vice versa.

Two attributes A and B are independent if one of the following is true.

(1) $\frac{(AB)}{(B)} = \frac{(A\beta)}{(\beta)}$

(2) $\frac{(AB)}{(B)} = \frac{(\alpha B)}{(\alpha)}$

(3) $(AB) = \frac{(A)(B)}{N}$

(4) $(A\beta) = \frac{(A)(\beta)}{N}$

(5) $(\alpha\beta) = \frac{(\alpha)(\beta)}{N}$

(6) $(\alpha B) = \frac{(\alpha)(B)}{N}$

(7) $(AB)(\alpha\beta) = (A\beta)(Ab)$

Note :

(1) (AB) $= \frac{(A)(B)}{N}$, we say that A and B are associated.

(2) (AB) $> \frac{(A)(B)}{N}$, we say that A and B are positively associated.

(3) (AB) $\cdot \frac{(A)(B)}{N}$, we say that A and B are negatively associated.

To measure the intensity of two attribute A and B is called coefficient of association.

Most commonly used coefficient of association is Yule's coefficient of association or coefficient of colligation.

The Yule's coefficient of association is defined as

$$Q = \frac{(AB)(\alpha\beta)-(A\beta)(\alpha B)}{(AB)(\alpha\beta)+ (A\beta)(\alpha B)}$$

And the coefficient of association is defined as

$$Y = \frac{1-\sqrt{\frac{(A\beta)(\alpha B)}{(AB)(\alpha\beta)}}}{1+\sqrt{\frac{(A\beta)(\alpha B)}{(AB)(\alpha\beta)}}}$$

Note :

(1) If Q =Y= 0 T then the attributes A and B are independent.

(2) If Q= Y= 1 then the attributes A and B are perfectly associated.

(3) If Q= Y= -1 then the attributes A and B are perfectly associated.

Example 1.16

State and prove the relationship between coefficient of association and coefficient of colligation.

Solution:

Statement: The relationship between coefficient of association and coefficient Of collігation is $Q = \frac{2Y}{1+Y^2}$

Proof :

x Let $x = \frac{(A\beta)(\alpha B)}{(AB)(\alpha\beta)}$

Then $Y = \frac{1-\sqrt{\frac{(A\beta)(\alpha B)}{(AB)(\alpha\beta)}}}{1+\sqrt{\frac{(A\beta)(\alpha B)}{(AB)(\alpha\beta)}}}$

$Y = \frac{1-\sqrt{x}}{1+\sqrt{x}}$

$\therefore Y^2 = \left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^2$

$= \frac{1-2\sqrt{x}+x}{(1+\sqrt{x})^2}$

Thus $1+Y^2 = 1+ \frac{1-2\sqrt{x}+x}{(1+\sqrt{x})2} = \frac{2(1+x)}{(1+\sqrt{x})^2}$

Hence $\frac{Y^2 2Y}{1+Y2} = \frac{2\frac{1-\sqrt{x}}{1+\sqrt{x}}}{2\frac{1+x}{(1+\sqrt{x})2}} = \frac{1-x}{1+x}$

$= \frac{1-\frac{(A\beta)(\alpha B)}{(AB)(\alpha\beta)}}{1+\frac{(A\beta)(\alpha B)}{(AB)(\alpha\beta)}}$

$= \frac{(AB)(\alpha B)-(A\beta)(\alpha B)}{(AB)(\alpha B)-(A\beta)(\alpha B)}$

$= Q$

Therefore $Q = \frac{2Y}{1+Y^2}$

Example 1.17

Find the association between A and B when N = 1000, (A) = 470, (B) = 620, (AB) = 320.

Solution :

Given that N = 1000, (A) = 470, (B) = 620, (AB) = 320.

Now $\frac{(A)(B)}{N} = \frac{470 \ X \ 620}{1000}$

= 291.4

▪ 470

= (AB)

(i.e.) $\frac{(A)(B)}{N}$ ▪ (AB)

Therefore attributes A and B are positively associated.

Example 1.18

Find the association between A and B when (AB) =90, (Aβ) =65, (αB) = 260, (αβ) = 110.

Solution :

Given that (AB) =90, (Aβ) =65, (αB) = 260, (αβ) = 110.

Attribute	B	β	Total
A	90	65	155
A	260	110	370
Total	350	175	525

Now $\frac{(A)(B)}{N} = \frac{155X \ 350}{525}$

= 103.33

>90

$=(AB)$

(i.e.) $\frac{(A)(B)}{N} > (AB)$

Therefore attributes A and B are negatively associated.

Example 1.19

IN an examination in Tamil and English 245 of the candidates passed in Tamil, 147 passed In both , 285 failed in Tamil and 190 failed in Tamil but passed in English . How far is the Knowledge in the two subjects associated?

Solution :

Let A be an attribute that a candidate passed in Tamil subject. Let B be an attribute that a candidate passed in English subject. Given that (A) = 245, (AB) = 147, ($\alpha\beta$) = 285, (αB) = 190. The remaining class frequencies can be obtained from the following table.

Attribute	B	β	Total
A	147	98	245
A	190	285	475
Total	337	383	720

Now $\frac{(A)(B)}{N} = \frac{245 \text{ X } 337}{720}$

$= 114.67$

< 147

$= (AB)$

(i.e.) $\frac{(A)(B)}{N} < (AB)$

Therefore A and B are positively associated.

(i.e.) the knowledge in the two subjects is positively associated

Example : 1.20

Calculate Yule's coefficient of association between marriage and failure of studies.

Attribute	B	β	Total
A	90	65	155
α	260	110	370
Total	350	175	525

Solution:

Let A be an attribute that denote marriage and
Let B be an attribute that denote failure of studies,
Given that

Attribute	B	β	Total
A	90	65	155
α	260	110	370
Total	350	175	525

Therefore Yule's coefficient of association between marriage and failure of studies =Q

$$=\frac{(AB)(\alpha\beta)-(A\beta)(\alpha B)}{(AB)(\alpha\beta)+(A\beta)(\alpha B)}$$

$$=\frac{(65)(260)-(90)(110)}{(65)(260)+(90)(110)}$$

$$=\frac{7000}{26800}$$

$=0.2612$

Example 1.21

Investigate the association between darkness of eye colour in father and son from the following

Data:

Father with dark eyes and sons with dark eyes	78
Father with dark eyes and sons with not dark eyes	122
Father with not dark eyes and sons with dark eyes	96
Father with not dark eyes and sons with not dark eyes	704

Can we infer eye color is hereditary?

Solution:

Let A be the attribute that denote dark eye to a father and

Let B be the attribute that denote dark eye to a son.

Thus we have the following

Attribute	B	β	Total
A	78	122	200
α	96	704	800

Total	174	826	1000

Therefore Yule's coefficient of association between marriage and failure of studies = Q

$$\frac{(AB)(\alpha\beta) - (A\beta)(\alpha B)}{(AB)(\alpha\beta) + (A\beta)(\alpha B)}$$

$$= \frac{(78)(704)-(122)(96)}{(78)(704)+(122)(96)}$$

$$= \frac{43200}{66624}$$

$$= 0.648$$

Example 1.22

From the figures given in the following table compare the association between literacy and unemployment in rural and urban areas and give reasons for the different of any.

	Urban	Rural
Total adult males	25 lakhs	200 lakhs
Literate males	10 lakhs	40 lakhs
Unemployed males	5 lakhs	12 lakhs
Literate and unemployed males	3 lakhs	4 lakhs

Solution :

Let A,B denote literacy of males and unemployed males respectively.

Step 1

First we shall find the coefficient of association between A and B in Urban.

Thus we have

Attribute	B	β	Total
A	7	3	10
A	13	2	15
Total	20	5	25

Thus $Q = \frac{(AB)(\alpha\beta)-(A\beta)(\alpha B)}{(AB)(\alpha\beta)+(A\beta)(\alpha B)}$

$$\frac{(7)(2)-(13)(3)}{(7)(2)+(13)(3)} = \frac{14-39}{14+39}$$

$= -0.472$

Step 2:

First we shall find the coefficient of association between A and B in rural.

Thus we have

Attribute	B	β	Total
A	36	4	40
A	152	8	160
Total	188	12	200

Thus $Q = \frac{(AB)(\alpha\beta)-(A\beta)(\alpha B)}{(AB)(\alpha\beta)+(A\beta)(\alpha B)}$

$$= \frac{(36)(8)-(152)(4)}{(36)(8)+(152)(4)}$$

$= \frac{-320}{896}$

$= -0.357$

Step 3 :

From step 1 and step 2 it is clear that the association between the literacy and Unemployed males in urban is greater than that in the urban.

Example 1.23

160 plants are characterized as per the nature of the leaves and color of the flower.

	Normal leaves	Abnormal leaves
White leaves	99	36
Black leaves	20	5

Examine the statement "the color of the flowers and the nature of the leaves are Independent."

Solution:

Let A be an attribute that denote color of the flower and

Let B be an attribute that denote nature of leaves.

Given that

Attribute	B	β
A	99	36
α	260	110

Therefore Yule's coefficient o association between marriage and failure of studies Q =

$$\frac{(AB)(\alpha\beta) - (A\beta)(\alpha B)}{(AB)(\alpha\beta) + (A\beta)(\alpha B)}$$

$$= \frac{(99)(5)-(20)(36)}{(99)(5)+(20)(36)}$$

$$= \frac{-225}{1215}$$

=-0.185

Thus the colour of the flower and nature of leaves are not independent but they are negatively associated.

Check your progress

(1) Calculate the coefficient of association between intelligence father and son from the

following data.

Intelligent father with intelligent sons	200
Intelligent fathers with dull sons	50
Dull fathers with intelligent sons	110
Dull fathers with dull sons	600

Comment on the result.

(2) From the following data compare the association between marks in Physics and Chemistry in two universities X and Y.

	X	Y
Total number of candidates	200	1600
Pass in Physics	80	320
Pass in Chemistry	40	90
Pass in Physics and Chemistry	20	30

(3) Of 500 students appeared for a competitive examination 350 were successful 280 had attended coaching class and of these 220 cm out successful. Does the coaching help in success?

1.5 Index Numbers

1.5.0 Introduction

For comparing many business and economic problems, relative numbers obtained by reducing the data are used. These relative numbers comparing any two situations comprising of same type of variables are called 'index numbers'. Index numbers are used when one is trying to compare series of numbers of vastly different size. It is a way to standardize the measurement of numbers so that they are directly comparable. Students you all would have come across this tool used in macro economies for stating inflation, consumer price index, wholesale price index, growth index, industrial output index etc. These indexes are statistical numbers for decision making and information.

Objectives

The objectives of the unit are as follows

1. To understand the various types of index numbers

2. To construct consumer price index.

3. To know about the various tests of perfection for index numbers.

These index numbers are defined as follows

Index numbers are devices for mitigating deceptions caused by changes in the value of money-Marris.R. Index numbers represents the general level of magnitude of changes between two or more situations of a number of variables taken as a whole.-Karmel.P.H.

Index numbers are used to measure the changes in some quantity which can be observed directly which we know to have a definite influence on many other qualities which we can so observe, tending to increase all or diminish all, while this influence is canceled by many causes affecting the separate quantities in various ways.-Bowley.

From the above definition we can know index numbers which only measures the relative change is magnitude. Index numbers relate a variable or variables to the same variable or variables pertaining to another period. In this comparison of variable the time period which is used as a basis for comparison is called as 'base period' and the other period is called as 'current period'. The Base year is of two types. They are

1. Fixed Base year and 2. Chain Base year

Fixed base year is a particular year which is used as a basis for comparison of a number of succeeding years. So, sometimes the base year becomes a remote period.

On the other hand, chain base year is a varying one and is the year immediately preceding the current year.

1.5.1. Characteristics of Index Numbers

1) Index numbers are specialized averages helps in comparing the changes in variables which are in different units.
2) Index numbers are expressed in percentages to show the extent of change without using percentage sign(%).
3) Relative variations are measured with the help of Index numbers.
4) Index numbers are for comparison they compare changes taking place over time or between places and like categories.

1.5.2 Uses of Index Numbers

1) The index numbers are used as end results in many situations. For example index numbers are often cited in new reports and acting as general indicators of economic condition.

2) Index numbers are used as part of an intermediate computation to understand other information latter.
3) Sales indices were used to modify and improve the estimates of future.
4) Consumer price index is used to determine the "real" buying power of money.

1.5.3 Types of index numbers

There are various types of index numbers. The most important types are as follows,

1. Price index (plural: "price indices" or "price indexes") is a normalized average (typically a weighted average) of prices for a given class of goods or services in a given region, during a given interval of time. It is a statistic designed to help to compare how these prices, taken as a whole, differ between time periods or geographical locations.

Price indices have several potential uses. For particularly broad indices, the index can be said to measure the economy's price level or a cost of living. More narrow price indices can help producers with business plans and pricing. Sometimes, they can be useful in helping to guide investment.

2. Quantity index numbers study the changes in the volume of goods produced or consumed; for instance industrial production, agricultural production, import, export, etc. they are useful and helpful to study the output in an economy.

3. Value index numbers compare the total value of a certain period with the total value of the base period. Here the total value is equal to the price of each, multiplied by the quantity; for instance, indices of profits, sales, inventories, etc.

1.5.4 Problems Related to Index Numbers

1) Finding suitable data is difficult: Ex. If sales of a company are reported only on annual basis, it would be unable to determine the seasonal sale pattern.
2) Incomparability of indices which occur when attempts are made to compare one index with another.
3) Inappropriate weighting of factors: in developing consumer price index, proper attention should be paid to the changes in some variables which are more important than others. This is practically very difficult and lead to it appropriate weighting.
4) Selection of improper base: Selecting the base year may be affected by individual interest or by routine method base selection. Sometimes high sales may occur due to some reasons if it is selected as a base. Then it will be a wrong representation. So proper care should be taken to select the base.

1.5.5 Construction of Index Numbers

The various methods of construction of index numbers are given below:

Notations: In this chapter we use the following notations.

1. p_0 = price of the commodity in the base year
2. p_1 = price of the commodity in the current year
3. q_0 = quantity of the commodity in the base year
4. q_1 = quantity of the commodity in the current year

1.5.5.1 Un-weighted (simple) Index Numbers

1. Simple Aggregate Method: This is the simplest method of constructing the index numbers. The prices of the different commodities of the current year are added and the total is divided by the sum of the prices of the base year commodity and multiplied by 100.

Symbolically $p_{01} = \Sigma p_1 / \Sigma p_1 \text{x} 100$

Example 1.1

From the following data construct the aggregate index number for 2001 taking 2000 as the base.

Commodities	Price in 2000	Price in 2001
A	50	70
B	40	60
C	80	90
D	110	120
E	20	20

Solution:

We know that aggregate index number= $p_{01} = \Sigma p_1 / \Sigma p_1 \text{x} 100$

Commodities	Price in 2000	Price in 2001
A	50	70
B	40	60
C	80	90
D	110	120
E	20	20
Total	300	360

Now $p_{01} = \Sigma p_1 / \Sigma p_1 \text{x} 100$

$= 360/300 \text{x} 100$

=120

Example 1.2

From the following data construct an index number for 2005 taking 2004 as the base.

Commodity (unit)	Price(Rs.) 2004	Price(Rs.) 2005
Butter (kg.)	110	120
Cheese (kg.)	75	80
Milk (lt.)	13	13
Bread (1)	9	9
Eggs (doz)	18	20
Ghee (1 tin)	850	860

Solution:

We know that aggregate index number= $p_{01} = \Sigma p_1 / \Sigma p_1 x100$

Commodity (unit)	Price(Rs.) 2004	Price(Rs.) 2005
Butter (kg.)	110	120
Cheese (kg.)	75	80
Milk (lt.)	13	13
Bread (1)	9	9
Eggs (doz)	18	20
Ghee (1 tin)	850	860
Total	1075	1102

Now $p_{01} = \Sigma p_1 / \Sigma p_1 x100$

=1102/1075x100

=102.51

Check Your Progress

1) From the following data construct an index number for 2010 taking 2009 as the base.

Commodity (unit)	Price(Rs.) 2009	Price(Rs.) 2010
Rice	7	8
Wheat	3.5	3.75
Oil	40	45
Gas	78	85
Flour	4.5	5.25

(Answer: p_{01}=110.5)

2) From the following data find the index number of the price relatives taking 2007 as the base year using (i)arithmetic mean and (ii)geometric mean.

Commodity (unit)	Price(Rs.) 2007	Price(Rs.) 2008
Rice	7	8
Wheat	3.5	3.75
Oil	40	45
Gas	78	85
Flour	4.5	5.25

(Answer:

(i) using arithmetic mean p_{01}=111.92

(ii) using geometric mean p_{01}=111.87)

3) From the following data of the whole sale price of rice for the 5 years construct the index numbers taking (i) 1987 as the base and (ii)1990 as the base.

Years	1987	1988	1989	1990	1991	1992
Price of Rice per kg.	5.00	6.00	6.50	7.00	7.50	8.00

1.5.6 Average of Price Relative Method

The ratio of the prices p_1/p_2 is called the price relative.

The formula for finding index number for the current year

$$=p_{01}$$

$$= p_1/p_0 \text{ x}100$$

In the average of price relative method the average of price relatives for various items is calculated by using any one of the measures of central tendencies such as arithmetic mean, geometric mean, harmonic mean etc., Among, various central tendencies arithmetic mean and geometric mean are very common averages used in this method.

(1) The Arithmetic mean index number is p_{01}

(i.e)

$$p_{01} = \Sigma\, p_1/p_0 \times 100/n$$

(2) The geometric mean index number is $p_{01} =$

$$p_{01} = (\Pi\, p_1/p_0)^{1/n} \times 100$$

(i.e)

$$p_{01} = \text{antilog}\,[1/n\ \Sigma(\,p_1/p_0 \times 100)]$$

Example 1.3

From the following data compute price index by simple average method based on (i) arithmetic mean and (ii) geometric mean for 2005 taking 2004 as the base.

Commodity (unit)	Price(Rs.) 2004	Price(Rs.) 2005
Butter (kg.)	110	120
Cheese (kg.)	75	80
Milk (lt.)	13	13
Bread (1)	9	9
Eggs (doz)	18	20
Ghee (1 tin)	850	860

Solution:

(i)First we shall find the price index based on arithmetic mean method.

Commodity(unit)	Price(Rs.)2004	Price(Rs.)2005	$p_1/p_0 \times 100$
Butter (kg.)	110	120	109.09
Cheese (kg.)	75	80	106.67
Milk (lt.)	13	13	100.00
Bread (1)	9	9	100.00
Eggs (doz)	18	20	111.11
Ghee (1 tin)	850	860	101.18
Total	1075	1102	628.05

Thus the price index using arithmetic mean method

$= p_{01}$

$= \Sigma\, p_1/p_0 \times 100/n$

$=628.05/6$

$=104.67$

(ii) First we shall find the price index based on geometric mean method

Commodity (unit)	Price(Rs.) 2004	Price(Rs.) 2005	p_1/p_0 x100	Log P
Butter (kg.)	110	120	109.09	2.0378
Cheese (kg.)	75	80	106.67	2.0280
Milk (lt.)	13	13	100.00	2.0000
Bread (1)	9	9	100.00	2.0000
Eggs (doz)	18	20	111.11	2.0458
Ghee (1 tin)	850	860	101.18	2.0051
Total				12.1167

Thus the price index using geometric mean method

$= p_{01}$

$= \text{antilog} [1/n \sum (p_1/p_0 x100)]$

$= \text{antilog} (12.1167/6)$

$=104.5785$

$=104.58$

Example 1.4

From the following data compute price index by simple average method based on (i) arithmetic mean and (ii) geometric mean and (iii) simple aggregative method.

Commodities	Price(Rs.)2004	Price(Rs.)2005
Rice	158	272
Cholam	168	326
Cambu	157	309
Ragi	155	304

Solution:

(i)First we shall find the price index based on arithmetic mean method.

Commodities	Price(Rs.) 2004	Price(Rs.) 2005	p_1/p_0 x100
Rice	158	272	172.15

Cholam	168	326	194.05
Cambu	157	309	196.82
Ragi	155	304	196.13
Total			759.1438

Thus the price index using arithmetic mean method

$=p_{01}$

$=\sum p_1/p_0 \times 100/n$

$=759.1438/4$

$=189.79$

(ii)First we shall find the price index based on geometric mean method.

Commodities	Price(Rs.) 2004	Price(Rs.) 2005	$p_1/p_0 \times 100$	Log P
Rice	158	272	172.15	2.2359
Cholam	168	326	194.05	2.2879
Cambu	157	309	196.82	2.2941
Ragi	155	304	196.13	2.2925
		Total		9.1104

Thus the price index using geometric mean method

$=p_{01}$

$=\text{antilog}\ [1/n \sum (p_1/p_0 \times 100)]$

$=\text{antilog}\ (9.1104/4)$

$=\text{antilog}\ (2.2776)$

$=189.50$

(iii)First we shall find the price index based on geometric mean method.

Commodities	Price(Rs.) 2004	Price(Rs.) 2005
Rice	158	272
Cholam	168	326
Cambu	157	309
Ragi	155	304
Total	638	1211

Thus the price index using geometric mean method

$=p_{01}$

$= \Sigma p_1/\Sigma p_0 x100$

$=1211/638$

$=189.81$

Example 1.5

From the data given below calculate the index numbers taking (i)2001 as the base year and (ii) 2005 as base year.

Years	2001	2002	2003	2004	2005	2006	2007	2008	2009
Price of wheat per kg.	4	5	6	7	8	10	9	10	11

Solution:

Construction of index numbers taking 2001 as base year

Years	Price of Wheat per kg.	Index Numbers
2001	4	100
2002	5	4/5x100 =125
2003	6	6/5x100 =150
2004	7	7/5x100 =175
2005	8	8/5x100 =200
2006	10	10/5x100=250
2007	9	9/5x100 =225
2008	10	10/5x100=250
2009	11	11/5x100=275

Construction of index numbers taking 2001 as base year

Years	Price of Wheat per kg.	Index Numbers
2001	4	4/8x100 =50
2002	5	5/8x100 =62.50
2003	6	6/8x100 =75
2004	7	7/8x100 =87.50
2005	8	8/8x100 =100
2006	10	10/8x100 =125

2007	9	9/8x100 =112.50
2008	10	10/8x100 =125
2009	11	11/8x100 =137.50

Example 1.6

Construct the whole sale price index number for 2001 and 2002 from the following data by considering 2000 as the base year.

Commodity	Whole Sale Price In Rs. per quintal		
	2000	2001	2002
Rice	700	750	825
Wheat	540	575	600
Ragi	300	325	310
Cholam	250	280	295
Flour	320	330	335
Ravai	325	350	360

Solution:

Construction of index numbers taking 2001 as base year

Commodity	Whole Sale Price in Rs.per quintal			Relative for 2001	Relative for 2002
	2000 p_0	2001 p_1	2002 p_1		
Rice	700	750	825	750/700X100 =107.14	825/700X100 =117.86
Wheat	540	575	600	575/540X100 =106.48	600/540X100 =111.11
Ragi	300	325	310	325/300X100 =108.33	310/300X100 =103.33

Cholam	250	280	295	280/250X100 =112.00	295/250X100 =118.00
Flour	320	330	335	330/320X100 =103.13	325/325X100 =104.69
Ravai	325	350	360	350/325X100 =107.69	360/325X100 =110.77
Total				644.77	665.76
Index number using A.M.				107.46	110.96

Thus the index number for 2001 as base year 2000=107.46
And the index number for 2002 as base year 2000=110.96

Example 1.7
From the following data find (i) fixed base index numbers with 2003 as the base year and (ii) chain base index numbers.

commodity	Price in Rs.				
	2001	2002	2003	2004	2005
I	2	3	5	7	6
II	8	10	12	4	18
III	4	3	7	9	12

Solution:
Construction of fixed base index numbers taking 2001 base year

Commodity	Price in Rs.				
	2001	2002	2003	2004	2005
I	100	150	250	350	300
II	100	125	150	50	225
III	100	75	175	225	300
Total	300	350	575	625	825
Index number (A.M.)	100	116.67	191.67	208.33	275

Workings:

Fixed base index number for the years 2002,2003,2004 and 2005 based on 2001 for the commodity I be calculated as
3/2x100=150,5/2x100=250,7/2x100=350 and6/2x100=300.
Similarly for the years 2002, 2003,2004 and 2005 based on 2001 for the commodity II be calculated as
10/8x100=125,12/8x100=150,4/8x100=50,18/8x100=225

Construction of chain base index numbers.

Commodity	Price in Rs.

	2001	2002	2003	2004	2005
I	100	3/2x100 =150	5/3x100 =166.67	7/5x100 =140	6/7x100 =85.71
II	100	10/8x100 =125	12/10x100 =120	4/12x100 =33.33	18/4x100 =450
III	100	3/4x100 =75	7/3x100 =233.33	9/7x100 =128.57	12/9x100 =133.33
Total	300	350	520	301.90	669.05
Index number (A.M.)	100	116.67	173.33	100.63	223.02

1.5.7 Weighted Index Number

The unweighted index numbers studied in the earlier sections are not unweighted in the true sense of the term. Actually we assign equal importance to all the items included in the index and as such they are in reality weighted, weights being implicit rather than explicit.

Weighted index numbers are of two types. The purpose of weighting is to make the index numbers more perspective and to give more important to them. They are (i)Weighted Aggregate Index Numbers and (ii) Weighted Average of Price Relatives.

1.5.7.1 Weighted Aggregative Index Number

Weighted aggregative index numbers are of the simple aggregative type with the fundamental difference that weights are assigned to the various items included in the index. There are various methods of assigning weights and consequently a large number of formulae for constructing index numbers have been devised. Some of the important index numbers are

1. Laspeyre's index number
2. Paasche's index number
3. Dorbish and Bowley's index number

4. Fisher's index number

5. Marshall-Edgeworth index number

6. Kelly's index number

Laspeyre's index number:

In this method the price of the commodities in the base year as well as the current year are known and the they are weighted by the quantity used in the base year.

(i.e)

$$P_{01} = \frac{\Sigma p_1 q_0}{\Sigma p_o q_0} \times 100$$

Paasche's index number:

In this method the price of the commodities in the base year as well as the current year are known and the they are weighted by the quantity used in the current year.

(i.e)

$$P_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_o q_1} \times 100$$

Dorbish and Bowley's method:

This is an index number got by the arithmetic mean of Laspeyre's and Paasche's index numbers.

(i.e)

$$P_{01} = \frac{1}{2}\left[\frac{\Sigma p_1 q_0}{\Sigma p_o q_0} + \frac{\Sigma p_1 q_1}{\Sigma p_o q_1}\right] \times 100$$

Fisher's Ideal method:

Fisher's price index number is given by the geometric mean of Laspeyre's and Paasche's index numbers.

(i.e)

$$P_{01}=\sqrt{\frac{\Sigma p_1q_0}{\Sigma p_oq_0}+\frac{\Sigma p_1q_1}{\Sigma p_oq_1}}\times 100$$

Marshall-Edgeworth method:

In Marshall- Edgeworth's index number, the arithmetic mean of base year and current year quantities are taken as the weights.

(i.e)

$$P_{01}=\frac{\Sigma p_1(q_0+q_1)}{\Sigma p_0(q_0+q_1)}\times 100$$

Kelly's method:

Kelly's index number uses quantities of some period (which is neither the base year nor the current year) as weights. This weight is kept contant for all periods.

i.e)

$$P_{01}=\frac{\Sigma p_1q}{\Sigma p_oq}\times 100$$

Where $q=\frac{q_0+q_1}{2}$

Example:1.8

Construct the index number of prices from the following data using

i. Laspeyre's index number
ii. Paasche's index number

iii. Bowley's index number

iv. Fisher's index number

v. Mrshall-Edgeworth index number

Commodity	Base year		Current year	
	Price	Quantity	Price	Quantity
A	2	8	4	6
B	5	10	6	5
C	4	14	5	10
D	2	19	2	13

Solution:

Now

Commodity	Base year		Current year					
	Price	Quantity	Price	Quantity	p_0q_0	p_1q_0	p_0q_1	p_1q_1
	p_0	q_0	p_1	q_1				
A	2	8	4	6	16	32	12	24
B	5	10	6	5	50	60	25	30
C	4	14	5	10	56	70	40	50
D	2	19	2	13	38	38	26	26
				Total	160	200	103	130

Laspeyre's index number

$$=P_{01} = \frac{\sum p_1q_0}{\sum p_oq_0}\times 100$$

$$=\frac{200}{160}\times 100 \qquad = 125$$

And Paasche's index number

$$=P_{01} = \frac{\sum p_1q_1}{\sum p_oq_1}\times 100$$

$$=\frac{130}{103}\times 100$$

$$=126.21$$

And Bowley's index number

$=P_{01}$

$$=\frac{1}{2}\left[\frac{\sum p_1q_0}{\sum p_oq_0}+\frac{\sum p_1q_1}{\sum p_oq_1}\right]\times 100$$

$$=\frac{1}{2}\left[\frac{200}{160}+\frac{130}{103}\right]\times 100$$

$=125.61$

And Fisher's index number

$=P_{01}$

$$=\sqrt{\frac{\sum p_1q_0}{\sum p_oq_0}\times\frac{\sum p_1q_1}{\sum p_oq_1}}\times 100$$

$$=\sqrt{\frac{200}{160}\times\frac{130}{103}}\times 100$$

$=125.61$

And Marshall-Edgeworth index number

$=P_{01}$

$$=\frac{\sum p_1q_0+\sum p_1q_1}{\sum p_0q_o+\sum p_0q_1}\times 100$$

$$=\frac{1}{2}\left[\frac{200}{160}+\frac{130}{103}\right]\times 100$$

$=125.48$

Example:1.9

For the given data find the different weighted index numbers

Commodity	Base year		Current year	
	Price	Quantity	Price	Quantity
A	6	50	10	56
B	2	100	2	120
C	4	60	6	60
D	10	30	12	24
E	8	40	12	26

Solution:

Commodity	Base year		Current year		p_0q_o	p_1q_o	p_0q_1	p_1q_1
	Price p_0	Quantity q_0	Price p_1	Quantity q_1				
A	6	50	10	56	300	500	336	56o
B	2	100	2	120	200	200	240	240
C	4	60	6	60	240	360	240	360
D	10	30	12	24	300	360	240	288
E	8	40	12	26	320	480	208	312
				Total	1360	1900	1264	1760

Now Laspeyre's index number

$=P_{01}$

$$=\frac{\sum p_1 q_0}{\sum p_o q_0}\times 100$$

$$=\frac{1900}{1360}\times 100$$

$=139.71$

And Paasche's index number

$=P_{01}$

$$=\frac{\sum p_1 q_1}{\sum p_o q_1}\times 100$$

$$=\frac{1760}{1264}\times 100$$

$=139.24$

And Bowley's index number

$= P_{01}$

$$=\frac{1}{2}\left[\frac{\sum p_1 q_0}{\sum p_o q_0}+\frac{\sum p_1 q_1}{\sum p_o q_1}\right]\times 100$$

$$=\frac{1}{2}\left[\frac{1900}{1360}+\frac{1760}{1264}\right]\times 100$$

$=139.47$

And Fisher's index number

$=P_{01}$

$$=\sqrt{\frac{\sum p_1 q_0}{\sum p_o q_0}\times\frac{\sum p_1 q_1}{\sum p_o q_1}}\times 100$$

$$=\sqrt{\frac{1900}{1360}\times\frac{1760}{1264}}\times 100$$

$= 139.47$

And Marshall- Edgeworth index number

$$= P_{01}$$

$$= \frac{\sum p_1 q_0 + \sum p_1 q_1}{\sum p_0 q_0 + \sum p_0 q_1} \times 100$$

$$= \frac{1}{2}\left[\frac{1900}{1360} + \frac{1760}{1264}\right] \times 100$$

$= 139.48$

Example:1.10

For the given data find the different weighted index numbers

Commodity	Base year		Current year	
	Price	Value	Price	Value
A	10	100	12	144
B	15	75	20	120
C	8	80	10	110
D	20	60	25	50
E	50	500	60	540

Solution:

Commodity	Base year		Current year		p_0q_0	p_1q_0	p_0q_1	p_1q_1
	Price p_0	Quantity q_0	Price p_1	Quantity q_1				
A	10	10	12	12	100	120	120	144
B	15	5	20	6	75	100	90	120
C	8	10	10	11	80	100	88	110
D	20	3	25	2	60	75	40	50
E	50	10	60	9	500	600	450	540
				Total	815	995	788	964

Now Laspeyre's index number

$$=P_{01}$$

$$=\frac{\Sigma p_1q_0}{\Sigma p_oq_0}\times 100$$

$$=\frac{788}{815}\times 100$$

$$=122.09$$

And Paasche's index number

$$=P_{01}$$

$$=\frac{\Sigma p_1q_1}{\Sigma p_oq_1}\times 100$$

$$=\frac{964}{995}\times 100$$

$$=122.34$$

And Bowley's index number

$= P_{01}$

$$=\frac{1}{2}\left[\frac{\sum p_1q_0}{\sum p_oq_0}+\frac{\sum p_1q_1}{\sum p_oq_1}\right]\times 100$$

$$=\frac{1}{2}\left[\frac{788}{815}+\frac{964}{995}\right]\times 100$$

$=122.21$

And Fisher's index number

$=P_{01}$

$$=\sqrt{\frac{\sum p_1q_0}{\sum p_oq_0}\times\frac{\sum p_1q_1}{\sum p_oq_1}}\times 100$$

$$=\sqrt{\frac{788}{815}\times\frac{964}{995}}\times 100$$

$= 122.21$

And Marshall- Edgeworth index number

$= P_{01}$

$$=\frac{\sum p_1q_0+\sum p_1q_1}{\sum p_0q_o+\sum p_0q_1}\times 100$$

$$=\left[\frac{788}{815}+\frac{964}{995}\right]\times 100$$

$=122.21$

Example:11

Find the value of x in the following data if the ratio between

Laspyre's andPaasche's index numbers is 28:27

Commodity	Base year		Current year	
	Price	Value	Price	Value
A	1	10	2	5
B	1	5	x	2

Solution:

Commodity	Base year		Current year		p_0q_o	p_1q_o	p_0q_1	p_1q_1
	Price p_0	Quantity q_0	Price p_1	Quantity q_1				
A	1	10	2	5	10	20	5	10
B	1	5	x	2	5	5x	2	2x
				Total	15	20+5x	7	10+2x

Laspeyre's index number

$=L_{01}$

$=\frac{\sum p_1q_0}{\sum p_oq_0}\times 100$

$=\frac{20+5x}{15}\times 100$

Paasche's index number

$= P_{01}$

$=\frac{\sum p_1q_1}{\sum p_oq_1}\times 100$

$=\frac{10+2x}{7}\times 100$

Given that $L_{01}:P_{01}=28:27$

(i.e) $\frac{20+5x}{15}\times\frac{7}{10+2x}=\frac{28}{27}$

(i.e) 180+45x=200+40x

(i.e) x=4

Thus the required value is 4.

Check Your Progress

(1) For the given data find the different weighted index numbers.

Commodity	Base year		Current year	
	Price	Quantity	Price	Quantity
A	10	25	12	30
B	8	21	9	25
C	4.5	28	6.5	35
D	3.5	16	4	20

(2) For the given data find the different weighted index numbers

Commodity	Base year		Current year	
	Price	Quantity	Price	Quantity
A	2	10	3	30
B	5	16	6.5	11
C	3.5	18	4	16
D	7	21	9	25
E	3	11	3.5	20

(3) For the given data find the different weighted index numbers

Commodity	Base year		Current year	
	Price	Value	Price	Value
A	8	200	65	1950
B	20	1400	30	1650
C	5	80	20	900
D	10	360	15	300
E	27	2160	10	600

1.5.7.2 Test for Perfection

Several formulae have been suggested for construction index numbers and the problem is that of selecting the most appropriate one in a given situation. The following test are suggested for choosing an appropriate index number:

(1) Time reversal test, (2) Factor reversal test.

1.Time Reversal Test:

Reversibility is an important property that an index number should posses. A good index number should satisfy the time reversal tests. In the

Words of Irving Fisher, "The formula for calculating an index number should be such that it gives the same ration between one point of comparison and the other , no matter which of the two is taken as the base: or putting it in another way , the index number reckoned forward should be reciprocal of the one reckoned backward." One of the advantages claimed in favor of Fisher's formula is that it makes the index number reversible. The time reversal test shows that the following equations hold good

$$P_{01} \times P_{10} = 1$$

where P_{01} is the index for time "1" on time "0" as base and P_{10} is the index for time "0" on time "1" as base, If the product is not unity, then we can say there is a time bias in the method.

Example 1. 12

Verify that Laspeyre's index number satisfies time reversal test.

Solution :

We know that Laspeyre's index number

$$= P_{01}$$

$$= \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0}$$

$$\text{and } P_{10} = \frac{\Sigma p_0 q_0}{\Sigma p_1 q_1}$$

$$\text{Now } P_{01} \times P_{10} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_0 q_0}{\Sigma p_1 q_1}$$

$$\neq 1.$$

Therefore Laspeyre's index number does not satisfy time reversal test.

Example 1. 13

Verify that Paache's index number satisfies time reversal test.

Solution :

We know that Paache's index number

$$= P_{01}$$

$$= \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}$$

$$\text{and } P_{10} = \frac{\Sigma p_0 q_0}{\Sigma p_1 q_0}$$

$$\text{Now } P_{01} \times P_{10} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times \frac{\Sigma p_0 q_0}{\Sigma p_1 q_0}$$

$$\neq 1$$

Therefore Paache's index number does not satisfy time reversal test.

Example 1. 14

Verify that Fisher's index number sastisfies time reversal test.

Solution :

We know that Fisher's index number

$$= P_{01}$$

$$= \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}}$$

$$\text{and } P_{10} = \sqrt{\frac{\Sigma p_0 q_1}{\Sigma p_1 q_1} \times \frac{\Sigma p_0 q_0}{\Sigma p_1 q_0}}$$

$$\text{Now } P_{01} \times P_{10} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times \sqrt{\frac{\Sigma p_0 q_1}{\Sigma p_1 q_1} \times \frac{\Sigma p_0 q_0}{\Sigma p_1 q_0}}$$

Therefore Fisher's index number satisfies time reversal test.

2. Factor Reversal Test:

Another basic test is that the formula for index number ought to permit to interchanging the prices and quantities without giving inconsistent results i.e., the two results multiplies together should give the true value ratio.

A good index number should satisfy not only the time reversal test, but also the factor reversal test. A good index number should allow time reversibility, interchange of the base year and the current year, without giving inconsistent results.

$$P_{01} \times Q_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} \quad \text{(i.e)}$$

Where P_{01} is the index for time "1" on time "0" as base and Q_{01} is the quantity index for time "1" on time "0" as base.

If the product is not unity, then we can say there is a time bias in the method.

Example 1.15

Verify that Fisher's index number satisfies factor reversal test.

Solution:

We know that Fisher's index number

$$= P_{01}$$

$$= \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}}$$

and $$Q_{01} = \sqrt{\frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}}$$

Now $$P_{01} \times Q_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times \sqrt{\frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}}$$

$$= \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}$$

Therefore Fisher's index number satisfies time reversal test.

Note : An index number satisfies both time and factor reversal tests then that Index number is called **ideal index number**

Example1.16 For the given data verify that Fisher's index number in an ideal number

Solution :

Commodity	Base year		Current year	
	Price	Value	Price	Value
A	10	100	12	144
B	15	75	20	120
C	8	80	10	110
D	20	60	25	50
E	50	500	60	540

Commodity	Base year		Current Year		p_0q_0	p_1q_0	p_0q_1	p_1q_1
	Price P_0	Quantity q_0	Price P_1	Quantity q_1				
A	10	10	12	12	100	120	120	144
B	15	5	20	6	75	100	90	120
C	8	10	10	11	80	100	88	110
D	20	3	25	2	60	75	40	50
E	50	10	60	9	500	600	450	540
				Total	815	995	788	964

First we shall prove that Fisher's index satisfies time reversal test.

Now the Fisher's index number, $p_{01} = \sqrt{\frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times \frac{\Sigma p_1q_1}{\Sigma p_0q_1}}$

$= \sqrt{\frac{995}{815} \times \frac{964}{788}}$

$= 1.22$

and $P_{10} = \sqrt{\frac{\Sigma q_1p_0}{\Sigma q_0p_0} \times \frac{\Sigma q_1p_1}{\Sigma q_0p_1}}$

$= \sqrt{\frac{788}{964} \times \frac{815}{995}}$

$= 0.82$

Now $p_{01} \times p_{10} = 1.22 \times 0.82$

$= 1$

Hence Fisher's index number satisfies time reversal test.

First we shall prove that Fisher's index number satisfies factor reversal test.

Now the Fisher's index number, $p_{01} = \sqrt{\frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times \frac{\Sigma p_1q_1}{\Sigma p_0q_1}}$

$$= \sqrt{\frac{995}{815} \times \frac{964}{788}}$$

$$= 1.22$$

and $P_{10} = \sqrt{\frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}}$

$$= \sqrt{\frac{788}{815} \times \frac{964}{995}}$$

$$= 0.97$$

Now $P_{01} \times Q_{01} = 1.22 \times 0.97$

$$= 1.18$$

and $\frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} = \frac{964}{815}$

$$= 1.18$$

Thus $P_{01} \times Q_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}$

Hence Fisher's index number satisfies factor reversal test.

Thus Fisher's index number is an ideal index number.

Example 1.17

For the given data verify that Fisher's index number is an ideal number.

Commodity	Base year		Current year	
	Price	Value	Price	Value
A	6	50	10	56
B	2	100	2	120
C	4	60	6	60
D	10	30	12	24
E	8	40	12	26

Solution :

Commodity	Base year		Current year		P_0q_0	P_1q_0	P_0q_1	P_1q_1
	Price P_0	Quantity q_0	Price P_1	Quantity q_1				

A	6	50	10	56	300	500	336	560
B	2	100	2	120	200	200	240	240
C	4	60	6	60	240	360	240	360
D	10	30	12	24	300	360	240	288
E	8	40	12	26	320	480	208	312
				Total	1360	1900	1264	1760

First we shall prove that Fisher's index number satisfies time reversal test.

Now the Fisher's index number, $P_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}}$

$= \sqrt{\frac{1900}{1360} \times \frac{1760}{1264}}$

$= 1.39$

and $P_{10} = \sqrt{\frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}}$

$= \sqrt{\frac{1264}{1760} \times \frac{1360}{1900}}$

$= 0.72$

Now $P_{01} \times P_{10} = 1.39 \times 0.72$

$= 1$

Hence Fisher's index number satisfies time reversal test.

First we shall prove that Fisher's index number satisfies factor reversal test.

Now the Fisher's index number, $P_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}}$

$$= \sqrt{\frac{1900}{1360} \times \frac{1760}{1264}}$$

$$= 1.39$$

and $P_{10} = \sqrt{\frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}}$

$$= \sqrt{\frac{1264}{1360} \times \frac{1760}{1264}}$$

$$= 1.14$$

Now $P_{01} \times Q_{01} = 1.39 \times 1.14$

$$= 1.59$$

and $\frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} = \frac{1760}{1360}$

$$= 1.59$$

Thus $p_{01} \times Q_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}$

Hence Fisher's index number satisfies factor reversal test.

Thus Fisher's index number is an ideal index number.

Check your Progress

Verify the Fisher's index number is an ideal index number for the following data

Commodity	Base year		Current year	
	Price	Value	Price	Value
A	8	60	12	66
B	4	120	4	160
C	6	60	8	90
D	15	45	14	52
E	10	50	14	36

1.5.7.3 Weighted Average of Price Relative Method

In the weighted aggregative methods discussed in the earlier sections Price relatives were not computed. However, like unweighted relatives method it is possible to compute weighted average of relatives. For purposes of averaging we use either arithmetic mean or the geometric mean.

The formula for weighted average of relative index number using arithmetic mean is

$$P_{01} = \frac{\Sigma pv}{\Sigma v}$$

where P = price relative and V = value weights.

Note : In the above formula p can be calculated from $\frac{p_1}{p_0} \times 100$

(i.e) $P = \frac{p_1}{p_0} \times 100$ and $V = p_0 q_0$.

The formula for weighted average of relative index number using geometric Mean is

$$P_{01} = \text{antilog}\{\frac{\Sigma v \log p}{\Sigma v}\}$$

Where P = price relative and V = value weights.

Note : In the above formula P can be calculated from $\frac{P_1}{P_0} \times 100$

(i.e) $P = \frac{P_1}{P_0} \times 100$ and $V = p_0 q_0$

Example 1.18

From the following data compute price index by applying weighted average of Price relative method using (i) arithmetic mean and (ii) geometric mean.

Commodity	P_0	q_0	P_1
Sugar	28	20	38
Flour	22	40	34
Milk	30	10	36

Solution :

Index number using weighted arithmetic mean of price relatives.

Commodity	P0	q0	P1	V = p0q0	$P=\frac{p_1}{p_0}\times 100$	PV
Sugar	28	20	38	560	111.11	62222
Flour	22	40	34	880	116.67	102667
Milk	30	10	36	300	106.67	32000
	Total			1740		196889

Now $p_{01} = \frac{\Sigma PV}{\Sigma V}$

$= \frac{196889}{990}$

Index number using weighted geometric mean of price relatives.

Commodity	P_0	q_0	P_1	$V= p_0q_0$	$P = \frac{p_1}{p_0}$	**Vlog p**	PV
Sugar	28	20	38	560	111.11	2.0458	1146
Flour	22	40	34	880	116.67	2.0669	1819
Milk	30	10	36	300	106.67	2.0280	608
	Total			1740			3573

Now $P_{01} = \text{antilog}\{\frac{\Sigma v \log p}{\Sigma v}\}$

$= \text{antilog}\{\frac{3573}{990}\}$

= 113.09

Check Your Progress

From the following data compute price index by applying weighted average of Price relative method using (i) arithmetic mean and (ii) geometric mean.

commodity	Weights	Price per unit 2004	Price per unit 2006

Commodity	P_0	q_0	P_1
Sugar	18	20	28
Flour	12	40	24
Milk	20	10	26

1.5.8 Consumer Price Index Number

Consumer index numbers are useful for wage negotiations and wage contracts. Dearness allowance is calculated based on the cost of living index numbers. Even for family budgets cost of living index numbers are calculated taking into consideration the importance of the consumption of articles. Generally items on which the money is spent are classified into certain heads such as Food, clothing, fuel, rent etc., each of these groups can further subdivided. Suitable weights can be associated as per the relative importance of the items. Formula to find the cost of living index numbers based on

(i) Aggregate expenditure method

Cost of living index number = I_{01}

(i.e) $$I_{01} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

(ii) Family budget method

Cost of living index number = I_{01}

A	40	80	85
B	25	60	55
C	5	345	50
D	20	35	40
E	10	25	20

where P = price relative = $\frac{p_1}{p_0} \times 100$

and V = value weights = $p_0 q_0$

Also

$$I_{01} = \frac{\sum PW}{\sum W}$$

$$I_{01} = \frac{\sum PV}{\sum V}$$

Where P = price relative = $\frac{p_1}{p_0} \times 100$

and W = value weights = $p_0 q_0$

Example 1.19

Calculate the index number of prices for 2006 on the basis of 2004 from the data given below.

Solution :

Now consumer price index number $= \frac{\Sigma PW}{\Sigma W}$

	Food	Rent	Clothing	Fuel	Misc
Weights	35%	15%	20%	10%	20%
Prices in 2001	1800	500	600	100	500
Prices in 2002	2000	700	900	130	550

$= \frac{10362.68}{539.26}$

$= 103.63$

Commodity	Weights (W)	Price per unit 2004	Price per unit 2006	$P = \frac{p_1}{p_0} \times 100$	*PW*
A	40	80	85	106.25	4250.00
B	25	60	55	91.67	2291.67
C	5	34	50	147.06	735.29
D	20	35	40	114.29	2285.71
E	10	25	20	80.00	800.00
Total	100			539.26	10362.68

Example 1.20

An enquiry into the budgets of the middle class families in a city in India gave the following information.

What changes in cost of living index of 2002 as compared with that of 2001 are seen ?

Solution

We know that cost of living index number $= \frac{\Sigma PW}{\Sigma W}$

Now

	Food	Rent	Clothing	Fuel	Misc
Weight	40%	20%	20%	10%	10%
Prices in 2008	2800	1500	1600	1100	1500
Prices in 2009	3000	1700	1900	1130	1550

Check your progress

An enquiry into the budgets of the middle class families in a city in India gave the following information.

What changes in cost of living index of 2009 as compared with tha of 2008 are seen ?

	Weights	Prices in 2001	Prices in 2002	*P*	*W*	*PW*
Food	35%	1800	2000	111.11	35	3888.89
Rent	15%	500	700	140.00	15	2100.00
Clothing	20%	600	900	150.00	20	3000.00
Fuel	10%	100	130	130.00	10	1300.00
Misc	20%	500	550	110.00	20	2200.00
Total					100	12488.89

Thus the cost of living index number $= \frac{\Sigma PW}{\Sigma W}$

$$= \frac{12488.89}{100.00}$$

$$= 124.90$$

Thus the prices in 2002 compared with the price in 2001 has risen to 24.9%.

1.5.8.1 CONVERSION OF CHAIN BASE INDEX NUMBER INTO FIXED BASE INDEX NUMBER AND CONVERSLY

The fixed base index number shows the changes in the level of phenomenon as compared to a fixed year as base year. But the chain base index number compares the level from the preceding year. In some situations it is necessary that to convert the chain base index number into a fixed base index number directly without referring to the actual prices.

Thus fixed base index of the current year

$$\begin{matrix}\text{Fixed base index of}\\ \text{the current year}\end{matrix} = \frac{\begin{pmatrix}\text{chain base index of}\\ \text{the current year}\end{pmatrix} \times \begin{pmatrix}\text{fixed base index of}\\ \text{the preceding year}\end{pmatrix}}{100}$$

and

$$\begin{matrix}\text{Chain base index number}\\ \text{of the current year}\end{matrix} = \frac{\begin{pmatrix}\text{fixed base index of}\\ \text{the current year}\end{pmatrix} \times 100}{\begin{matrix}\text{fixed base index of}\\ \text{the previous year}\end{matrix}}$$

Example 1.21

Years	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002
Chain index	100	112.8	86.4	102.6	120.5	105.3	103.3	109.8	88.4	75.8

Convert the following chain base index numbers into fixed base index numbers.

Solution:

We know that

$$\begin{matrix}\text{Fixed base index of}\\ \text{the current year}\end{matrix} = \frac{\begin{pmatrix}\text{chain base index of}\\ \text{the current year}\end{pmatrix} \times \begin{pmatrix}\text{fixed base index of}\\ \text{the preceding year}\end{pmatrix}}{100}$$

Now the third column of the following table show the fixed base index number

Years	Chain index	Fixed base index number
1993	100	100
1994	112.8	$\frac{112.8 \times 100}{100} = 100$
1995	86.4	$\frac{112.8 \times 86.4}{100} = 97.5$
1996	102.6	100.0
1997	120.5	120.5
1998	105.3	126.9
1999	103.3	131.1
2000	109.8	143.9
2001	88.4	127.2
2002	75.8	96.4

Example 1.22

Convert the following chain base index numbers into fixed base index numbers.

Years	1996	1997	1998	1999	2000
Chain index	80	110	120	90	140

Solution :

We know that

$$\text{Fixed base index of the current year} = \frac{\begin{pmatrix}\text{chain base index of}\\ \text{the current year}\end{pmatrix} \times \begin{pmatrix}\text{fixed base index of}\\ \text{the preceding year}\end{pmatrix}}{100}$$

Now the third column of the following table show the fixed base index number

Years	Chain index	Fixed base index number
1996	80	80
1997	110	$\frac{110 \times 80}{100} = 88$
1998	120	$\frac{120 \times 88}{100} = 105.6$
1999	90	95.04
2000	140	133.06

Check your progress

Convert the following chain base index number into fixed base index number

Years	2000	2001	2002	2003	2004	2005
Chain index	90	105	102	98	120	125

Example1.23

From the fixed base index numbers for the data given below prepare chain base index numbers.

Years	2000	2001	2002	2003	2004	2005
Chain index	94	98	102	95	98	100

Solution:

We know that

$$\text{Chain base index number of the current year} = \frac{\begin{pmatrix}\text{fixed base index of} \\ \text{the current year}\end{pmatrix} \times 100}{\text{fixed base index of the previous year}}$$

Now the third column of the following table shows the chain base index numbers.

Years	Fixed base index number	Chain index
2000	94	100.00
2001	98	104.26
2002	102	104.08
2003	95	93.14
2004	98	103.16
2005	100	102.04

Example 1.24

From the fixed base index numbers for the data given below prepare chain base index numbers.

Years	2000	2001	2002	2003	2004	2005
Fixed base index number	100	112.8	97.4	100	120.5	126.1

Solution :

We know that

$$\text{Chain base index number of the current year} = \frac{\left(\text{fixed base index of the current year}\right) \times 100}{\text{fixed base index of the previous year}}$$

Now the third column of the following table show the chain base index numbers.

Years	Fixed base index number	Chain index
2000	100	100.00
2001	112.8	112.80
2002	97.4	86.35
2003	100	102.67
2004	120.5	120.50
2005	126.1	104.65

1.5.8.2 Limitations of Index Numbers

Even though index numbers are very important in business and economic activities, they have their own limitations; they are:

1. There may be error in each stage of the construction of the index number, namely, selection of commodities, selection of base period, selection of weight, etc.
2. Index numbers may not represent the exact change in price level, because they are based on sample data.
3. Tastes, habits and customs of people change in course of time and may make the weighting not suitable for the present data.

Space for Hints

UNIT – 2
ALGEBRA

2.0 Introduction

In chapter 2 we have introduced m x n matrices and we have represented linear transformations by these matrices. In this chapter we shall develop the general theory of matrices. Throughout this chapter we deal with matrices whose entries are from the field F of real or complex numbers.

2.1 Matrices

We have already seen that an m x n matrix A is an array of mnnumbers a_{ij} where $1 \le i \le m, 1 \le j \le n$ arranged in m rows and n columns as follows:

$$\begin{pmatrix} a_{11} & a_{12} & \dots\dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots\dots & a_{2n} \\ . & . & \dots\dots\dots & . \\ . & . & \dots\dots\dots & . \\ . & . & \dots\dots\dots & . \\ a_{m1} & a_{m2} & \dots\dots\dots & a_{mn} \end{pmatrix}$$

We shall denote this matrix by the symbol (a_{ij}). If $m = n$, A is called a **square matrix** of order n.

Definition: 2.1.1

Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be **equal** if A and B have the same number of rows and columns and the corresponding entries in the two matrices are same.

Addition of matrices:

We have already defined the addition of two m x n matrices $A = (a_{ij})$ and $B = (b_{ij})$ by $A + B = (a_{ij} + b_{ij})$.

We note that we can add two matrices iff they have the same number of rows and columns.

Space for Hints

Example:

$$\text{If } A=\begin{pmatrix}1 & 2\\ 3 & 4\\ 9 & 5\end{pmatrix} \text{ and } \begin{pmatrix}0 & 4\\ 2 & 1\\ -1 & 0\end{pmatrix} \text{ then } A+B=\begin{pmatrix}1 & 6\\ 5 & 5\\ 8 & 5\end{pmatrix}$$

Remark:

The set of all m x n matrices is an abelian group under matrix addition. The m x n matrix with each entry 0 is **the zero matrix** and is denoted by **0** and the additive inverse of matrix $A=(a_{ij})$ is $(-a_{ij})$ and is denoted by $-A$.

If $A=(a_{ij})$ is any matrix and α is any number (real or complex) we have defined the matrix αA by

$$\alpha A=(\alpha a_{ij}).$$

The set of all m x n matrices over the field **R** under matrix addition and scalar multiplication defined above is a vector space. This result is true if **R** is replaced by **C** or by any field F.

We now proceed to define multiplication of matrices. We have already defined the multiplication of 2 x 2 matrices, which we generalise in the following definition.

Definition: 2.1.2

Let $A=(a_{ij})$ be an m x n matrix and $B=(b_{ij})$ be an n x p matrix. We define the **product AB** as the m x p matrix (c_{ij}) where the ij^{th} entry e_{ij} is given by

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\dots\dots+a_{in}b_{nj}=\sum_{k=1}^{n} a_{ik}b_{kj}.$$

Note: 1

The product AB of two matrices is defined only when the number of columns of A is equal to the number of rows of B.

Note: 2

Hints

The entry c_{ij} of the product AB is found by multiplying i^{th} row of A and the j^{th} column of B. To multiply a row and a column, we multiply the corresponding entries and add.

Examples:

1. Let $A=\begin{pmatrix}1&0&2&3\\0&2&1&1\\1&0&0&1\end{pmatrix}$ and $B=\begin{pmatrix}1&1\\1&5\\3&2\\1&0\end{pmatrix}$. A is a 3×4 matrix and B is a 4×2 matrix. Hence the product AB is a 3×2 matrix and

$$AB=\begin{pmatrix}1&0&2&3\\0&2&1&1\\1&0&0&1\end{pmatrix}\begin{pmatrix}1&1\\1&5\\3&2\\1&0\end{pmatrix}=\begin{pmatrix}10&5\\6&12\\2&1\end{pmatrix}$$

Note that in this example the product BA is not defined. Even if the product BA is defined. AB need not be equal to BA.

2. Let $A=\begin{pmatrix}2&4&0\\9&3&1\\4&7&2\end{pmatrix}$ and $I=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$. Then $AI = IA = A$. (Verify)

3. Consider the square matrix of order n given by

$$I_n=\begin{pmatrix}1&0&0&\cdots\cdots\cdots&0\\0&1&0&\cdots\cdots\cdots&0\\.&.&.&\cdots\cdots\cdots&.\\.&.&.&\cdots\cdots\cdots&.\\.&.&.&\cdots\cdots\cdots&.\\0&0&0&\cdots\cdots\cdots&1\end{pmatrix}$$

Let A be any $m \times n$ matrix. Then $I_nA = A$. Also if A is an $m \times n$ matrix, $AI_n = A$. If A is any $n \times n$ *matrix*, $AI_n = A$. If A is any $n \times n$ matrix, $AI_n = I_nA = A$. I_n is called the **identity matrix** of order n. We shall denote the identity matrix of any order by the symbol I.

Problems:

Problem: 1

Show that the matrix $A = \begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}$ satisfies the equation $A(A - I)(A + 2I) = 0$.

Solution:

$$A - I = \begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 1 \\ 3 & 0 & 3 \\ -5 & 2 & -5 \end{pmatrix}$$

$$A + 2I = \begin{pmatrix} 4 & -3 & 1 \\ 3 & 3 & 3 \\ -5 & 2 & -2 \end{pmatrix}$$

Now,

$A(A - I)(A + 2I)$

$$= \begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}\begin{pmatrix} 1 & -3 & 1 \\ 3 & 0 & 3 \\ -5 & 2 & -5 \end{pmatrix}\begin{pmatrix} 4 & -3 & 1 \\ 3 & 3 & 3 \\ -5 & 2 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -12 & -4 & -12 \\ -9 & -3 & -9 \\ 21 & 7 & 21 \end{pmatrix}\begin{pmatrix} 4 & -3 & 1 \\ 3 & 3 & 3 \\ -5 & 2 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= 0$$

Hence $A(A - I)(A + 2I) = 0$.

Problem: 2

Prove that $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$.

Solution:

We prove this result by induction on n.

When $n = 1$ result is obviously true.

Let us assume that the result is true for $n = k$.

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^k = \begin{pmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{pmatrix}$$

$\therefore$ $$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^k \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^{k+1} & \lambda^k + k\lambda^k \\ 0 & \lambda^{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^{k+1} & (k+1)\lambda^k \\ 0 & \lambda^{k+1} \end{pmatrix}$$

$\therefore$ The result is true for $n = k + 1$

Hence the result is true for all positive integers n.

Exercises:

1. Write down six pairs of matrices A and B such that the product AB is defined and in each case compute the product AB.
2. (a) Show that if A is an m x n matrix, then AB and BA are both defined iff B is an n x m matrix.

   (b) Write down six pairs of matrices A and B such that both AB and BA are defined and compute the products AB and BA.
3. If A and B are two matrices such that AB and A + B are both defined, show that A, B are square matrices of the same order.
4. Let $A = \begin{pmatrix} 1 & -2 & 4 \\ -3 & 0 & 2 \\ 7 & 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 3 & -3 \\ 0 & 0 & 1 \end{pmatrix}$.

   Compute A, B^2, AB and BA.
5. If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ show that $A^2 - 4A - 5I = 0$.
6. If $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$ prove that $A^3 - 6A^2 + 7A + 2I = 0$.
7. Prove that if $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$, then $A^k = \begin{pmatrix} 1+2k & -4k \\ k & 1-2k \end{pmatrix}$ for any positive integer k.

8. Decide which of the following statements are true and which are false.

(a) For any two matrices A and B, A + B is defined.

(b) AB is defined $\Rightarrow$BA is defined.

(c) For any matrix A, A^2 is defined.

(d) For any square matrix A, A^2 is defined.

(e) Matrix addition is commutative.

(f) Matrix addition is associative.

(g) Matrix multiplication is commutative.

(h) If A and B are 3 x 3 matrices then $(A+B)^2 = A^2 + 2AB + B^2$.

(i) If A and B are 3 x 3 matrices then $(A+B)(A-B) = A^2 - B^2$.

(j) (h) and (i) are true if AB = BA.

Answers:

8. (a) F (b) F (c) F (d) T (e) T (f) T (g) F (h) F (i) F (j) T

Result: 2.1

Let A be an m x n matrix, B an n x p matrix and C a p x q matrix. Then $A(BC) = (AB)C$.

Definition: 2.1.3

Let $A = (a_{ij})$ be an m x n matrix. Then the n x m matrix $B = (b_{ij})$ where $b_{ij} = a_{ji}$ is called the **transpose** of the matrix A and it is denoted by A^T. Thus A^T is obtained from the matrix A by interchanging its rows and columns and the $(i, j)^{th}$ entry of $A^T = (j,i)^{th}$ entry of A.

For example, if $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & 1 \\ 0 & 3 & 1 & 5 \end{pmatrix}$ then $A^T = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 3 & 0 & 1 \\ 4 & 1 & 5 \end{pmatrix}$

Clearly if A is an m x n matrix, then A^T is an n x m matrix.

Result: 2.3

Space for Hints

Let A and B be two m x n matrices. Then

(i) $(A^T)^T = A$.

(ii) $(A+B)^T = A^T + B^T$.

Result: 2.4

Let A be an m x n matrix and B be an n x p matrix. Then $(AB)^T = B^T A^T$.

Definition: 2.1.4

Let $A = (a_{ij})$ be a matrix with entries from the **field of complex numbers.**

The **conjugate** of A, denoted by $\overline{A}$, is defined by $\overline{A} = \overline{(a_{ij})}$.

$\overline{A}^T$ is called the **conjugate transpose** of the matrix A.

For example

If $A = \begin{pmatrix} 2 & 2+i & -i \\ 1+i & -3 & 4+3i \end{pmatrix}$ then $\overline{A} = \begin{pmatrix} 2 & 2-i & i \\ 1-i & -3 & 4-3i \end{pmatrix}$.

Result: 2.5

Let A and B be matrices with entries from C. Then

(i) $\overline{(\overline{A})} = A$.

(ii) $\overline{A+B} = \overline{A} + \overline{B}$.

(iii) $\overline{kA} = \overline{k}\ \overline{A}$, where $k \in C$.

(iv) $A = \overline{A} \Leftrightarrow$ all entries of A are real.

(v) $\overline{AB} = \overline{A}\ \overline{B}$ provided AB is defined.

(vi) $(\overline{A})^T = \overline{A^T}$.

Exercises:

1. Let $A = \begin{pmatrix} 3 & 4 & 6 \\ -1 & 7 & 2 \\ 4 & 3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 2 \\ -2 & 0 & 0 \\ 3 & 4 & 1 \end{pmatrix}$. Find $A^T, B^T, (A+B)^T, (AB)^T$ and $B^T A^T$.

2. Let $A = \begin{pmatrix} 2i & 3+4i & 0 \\ 1+i & 1-i & i \\ 3 & 2i & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 & 6 \\ -1 & 4 & 6 \\ 2 & 0 & 2 \end{pmatrix}$.

Find $\overline{A}, \overline{A+B}, \overline{AB}, \overline{A}\ \overline{B}, \overline{A}^T, \overline{A^T}, \overline{A^T} \cdot \overline{B}^T, \overline{A}^T B$ and $A\overline{B}^T$.

Types of Matrices:

Definition: 2.1.5

An $1 \times n$ matrix is called a **row matrix**. Thus a row matrix consists of 1 row and n columns.

It is of the form $(a_{11}, a_{12}, a_{13}, \ldots\ldots, a_{1n})$.

Definition: 2.1.6

An $m \times 1$ matrix is called a **column matrix**. Thus a column matrix consist of m rows and 1 column and it is of the form $\begin{pmatrix} a_{11} \\ a_{21} \\ \ldots\ldots \\ a_{m1} \end{pmatrix}$

Definition: 2.1.7

Let $A = (a_{ij})$ be a square matrix. Then the elements $a_{11}, a_{22}, \ldots\ldots, a_{nm}$ are called the diagonal elements of A and the diagonal elements constitute what is known as the **principal diagonal** of the matrix A. A square matrix is called a **diagonal matrix** if all the entries which do not belong to the principal are zero. Hence in a diagonal matrix $a_{ij} = 0$ if $i \neq j$.

For example $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is a diagonal matrix.

Definition: 2.1.8

A diagonal matrix in which all the entries of the principal diagonal are equal is called a **scalar matrix**.

For example $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ is a scalar matrix.

Definition: 2.1.9

A square matrix (a_{ij}) is called an **upper triangular matrix** if all the entries above the principal diagonal are zero.

Hence $a_{ij} = 0$ whenever $i < j$ in an upper triangular matrix.

Definition: 2.1.10

A square matrix (a_{ij}) is called a **lower triangular matrix** if all the entries below the principal diagonal are zero.

Hence $a_{ij} = 0$ whenever $i > j$ in an lower triangular matrix.

Space for Hints

For example, $\begin{pmatrix}1 & 2 & 3\\0 & 2 & 1\\0 & 0 & 3\end{pmatrix}$ is lower triangular $\begin{pmatrix}1 & 0 & 0 & 0\\1 & 2 & 0 & 0\\0 & 2 & 3 & 0\\2 & 3 & 2 & 4\end{pmatrix}$ is upper triangular.

Clearly a square matrix is a diagonal matrix iff it is both lower triangular and upper triangular.

Definition: 2.1.11

A square matrix $A = (a_{ij})$ is said to be **symmetric** if $a_{ij} = a_{ji}$ for all i, j.

Example:

$\begin{pmatrix}a & b\\b & a\end{pmatrix}, \begin{pmatrix}a & h & g\\h & b & f\\g & f & c\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\2 & 0 & 0 & 5\\3 & 0 & 6 & 7\\4 & 5 & 7 & 8\end{pmatrix}$ are symmetric matrices.

Result: 2.6

A square matrix A is symmetric iff $A = A^T$.

Result: 2.7

Let A be any square matrix. Then $A + A^T$ is symmetric.

Result: 2.8

Let A and B be symmetric matrices of order n. Then

(i) $A + B$ is symmetric.

(ii) AB is symmetric iff $AB = BA$.

(iii) $AB + BA$ is symmetric.

(iv) If A is symmetric, then kA is symmetric where $k \in F$.

Definition: 2.1.12

A square matrix $A = (a_{ij})$ is said to be skew symmetric if $a_{ij} = -a_{ji}$, for all i, j.

Note:

Let A be a skew symmetric matrix. Then $a_{ii} = -a_{ii}$. Hence $2a_{ii} = 0$ (ie) $a_{ii} = 0$, for all i. Thus in a skew symmetric matrix all the diagonal entries are zero.

$\begin{pmatrix}0 & -a\\a & 0\end{pmatrix}, \begin{pmatrix}0 & -2 & 1\\2 & 0 & -3\\-1 & 3 & 0\end{pmatrix}$ are examples of skew symmetric matrices.

Result: 2.9

A square matrix A is skew symmetric matrix iff $A = -A^T$.

Space for Hints

Result: 2.10

Let A be any square matrix. Then $A - A^T$ is skew symmetric.

Result: 2.11

Any square matrix A can be expressed uniquely as the sum of a symmetric matrix and a skew symmetric matrix.

Result: 2.12

Let A and B be skew symmetric matrices of order n. Then

(i) $A + B$ is skew symmetric.

(ii) kA is skew symmetric, where $k \in F$.

(iii) A^{2n} is a symmetric matrix and A^{2n+1} is a skew symmetric matrix where n is any positive integer.

Definition: 2.1.13

A square matrix $A = (a_{ij})$ is said to be a **Hermitian matrix** if $a_{ij} = \overline{a}_{ji}$ for all i, j. A is said to be a **skew Hermitian matrix** iff $\overline{a_{ij}} = -\overline{a}_{ji}$ for all i, j.

Example:

$$\begin{pmatrix} 1 & -1+2i & 3+4i \\ -1-2i & -2 & 3 \\ 3-4i & 3 & 2 \end{pmatrix}$$ is a Hermitian matrix.

$$\begin{pmatrix} 0 & -a+ib \\ a+ib & 0 \end{pmatrix}, \begin{pmatrix} ib & c+id \\ -c+id & ib \end{pmatrix}$$ are skew Hermitian matrices.

Note:

1. Any Hermitian matrix over **R** is a symmetric matrix and any skew Hermitian matrix over **R** is a skew symmetric matrix.
2. Let $A = (a_{ij})$ be a Hermitian matrix. Then $a_{ii} = \overline{a}_{ii}$ and hence a_{ii} is real for all i.

Space for Hints

3. Let $A=(a_{ij})$ be a skew Hermitian matrix. Then $a_{ii}=-\overline{a}_{ii}$ and hence $a_{ii}=0$ or purely imaginary for all i.

Result: 2.13

Let A be a square matrix.

(i) A is Hermitian iff $A=\overline{A}^{T}$.

(ii) A is skew Hermitian iff $A=-\overline{A}^{T}$.

Result: 2.14

Let A and B be square matrices of the same order. Then

(i) A, B are Hermitian $\Rightarrow A+B$ is Hermitian.

(ii) A, B are skew Hermitian $\Rightarrow A+B$ is skew Hermitian.

(iii) A is Hermitan $\Rightarrow iA$ is skew Hermitian.

(iv) A is skew Hermitian $\Rightarrow iA$ is Hermitian.

(v) A is Hermitian and k is real $\Rightarrow kA$ is Hermitian.

(vi) A is skew Hermitian and k is real $\Rightarrow kA$ is skew Hermitian.

(vii) A, B are Hermitian $\Rightarrow AB+BA$ is Hermitian.

(viii) A, B are Hermitian $\Rightarrow AB-BA$ is skew Hermitian.

Result: 2.15

Let A be any square matrix.

Then

(i) $A+\overline{A}^{T}$ is Hermitian.

(ii) $A-\overline{A}^{T}$ is skew Hermitian.

Result: 2.16

Any square matrix A can be uniquely expressed as the sum of a Hermitian matrix and a skew Hermitian matrix.

Definition: 2.1.14

A real square matrix A is said to be **orthogonal** if $AA^{T}=A^{T}A=I$.

Example:

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$ is an orthogonal matrix (verify).

Result: 2.17

Let A and B be orthogonal matrices of the same order. Then

(i) A^T is orthogonal.

(ii) A B is orthogonal.

Definition: 2.1.15

A square matrix A is said to be an **unitary matrix** if $A\overline{A}^T = \overline{A}^T A = I$.

For example $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is unitary.

Note:

Any unitary matrix over **R** is an orthogonal matrix.

Result: 2.18

If A and B are unitary matrices of the same order, then AB is also an unitary matrix.

Exercises:

1. A square matrix A is called an **idempotent matrix** if $A^2 = A$.

   Show that $\begin{pmatrix} 2 & -3 & 5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix}$ and $\begin{pmatrix} -1 & 3 & 5 \\ 1 & -3 & 5 \\ -1 & 3 & 5 \end{pmatrix}$ are idempotent matrices.

2. Show that if $AB = A$ and $BA = B$ then A and B are idempotent matrices.

3. Show that if A is an idempotent matrix, then $B = I - A$ is also an idempotent matrix and $AB = BA = 0$.

4. A square matrix A is said to be **nilpotent** if $A^n = 0$ for some positive integer n. Show that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$ are nilpotent.

5. A square matrix A is said to be **involutory** if $A^2 = I$.

Show that $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ are involutory.

6. Show that a square matrix A is involutory iff $(I+A)(I-A)=0$.

7. Show that $\frac{1}{2}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}$ is an orthogonal matrix.

8. If (ℓ_i, m_i, n_i) where $i=1,2,3$ are the direction cosines of three mutually perpendicular lines referred to an orthogonal cartesian coordinate system, then show that $\begin{pmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{pmatrix}$ is an orthogonal matrix.

9. Show that $\frac{1}{2}\begin{pmatrix} \ell+i & -1+i \\ 1+i & 1-i \end{pmatrix}$ is an unitary matrix.

The Inverse of a Matrix

A 2 x 2 matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an inverse iff $|A|=ad-bc \neq 0$ and the inverse of A is given by $\frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Such matrices are called **non – singular.**

In this section we shall describe the method of finding the inverse of any non – singular matrix of order n.

Determinants:

We can associate with any $n \times n$ matrix $A=(a_{ij})$ over a field F an element of F given by the determinant $\begin{vmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ . & . & \cdots\cdots & . \\ . & . & \cdots\cdots & . \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{vmatrix}$.

Its value can be determined in the usual way and it is denoted by $|A|$

For example,

(i) If $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $|A| = ad - bc$.

(ii) If $A=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ then $|A| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 1$.

Definition: 2.1.16

A square matrix A is said to be **singular** if $|A| = 0$.

A is called a **non – singular matrix** if $|A| \neq 0$.

Remark:

The rule for multiplying two matrices is same as the rule for multiplying two determinants.

Hence if A and B are two n x n matrices.

$|AB| = |A||B|$.

Result: 2.19

The product of any two non – singular matrices is non – singular.

Definition: 2.1.17

Let $A = (a_{ij})$ be an n x n matrix. If we delete the row and the column containing the element a_{ij} we obtain a square matrix of order $n-1$ and the determinant of this square matrix is called the **minor** of the element a_{ij} and is denoted by M_{ij}.

The minor M_{ij} multiplied by $(-1)^{i+j}$ is called the **cofactor** of the element a_{ij} and is denoted by A_{ij}.

$$\therefore \quad A_{ij} = (-1)^{i+j} M_{ij}.$$

Example:

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Corresponding to the 9 elements a_{ij}, we get 9 minors of A. For example, the minor of a_{11} is

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \text{ and the minor of } a_{23} \text{ is}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}.$$

The cofactor of a_{11} is $A_{11} = (-1)^2 M_{11} = M_{11}$.

The cofactor of a_{23} is $A_{23} = (-1)^{2+3} M_{23} = -M_{23}$.

Definition: 2.1.18

Let $A = (a_{ij})$ be a square matrix. Let A_{ij} denote the co – factor of a_{ij}. The transpose of the matrix (A_{ij}) is called the **adjoint** or **adjugate** of the matrix A and is denoted by adj A.

Thus the $(i, j)^{th}$ entry of adj A is A_{ji}.

Note:

If A is a square matrix of order n then adj A is also a square matrix of order n.

Example:

$$\text{Let } A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ -2 & 1 & 3 \end{pmatrix}$$

$$\text{Then } A_{11} = \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} = 4.$$

$$A_{12} = -\begin{vmatrix} 3 & -1 \\ -2 & 3 \end{vmatrix} = -7 .$$

Similarly other co-factors can be calculated and we get

$$\text{adj}A = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} 4 & 2 & -2 \\ -7 & 7 & 7 \\ 5 & -1 & 1 \end{pmatrix}$$

We notice that

$$A(\text{adj}A) = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 4 & 2 & -2 \\ -7 & 7 & 7 \\ 5 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$

$$= (\text{adj}A)A \text{ . (verify)}$$

Exercises:

1. Write down six square matrices A and calculate adj A, A(adjA) and (adjA)A .
2. Prove that $\text{adj}\, A^T = (\text{adj}\, A)^T$.
3. If A is symmetric prove that adj A is symmetric.

Result: 2.20

Let A be any square matrix of order n. Then $(\text{adj}\, A)A = A(\text{adj}\, A) = |A| I$ where I is the identity matrix of order n.

Definition: 2.1.19

Let A be a square matrix of order n. A is said to be **invertible** in there exists a square matrix B of order n such that $AB = BA = I$ and B is called the **inverse** of A and is denoted by A^{-1}.

Note:

The invertible matrices are precisely the units of the ring $M_n(F)$

Result: 2.21

A square matrix A of order n is non - singular iff A is invertible.

Problems:

Problem: 1

Compute the inverse of the matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$.

Solution:

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{vmatrix} = -1.$$

Since $|A| \neq 0$, A is non – singular.

Hence A^{-1} exists and is given by $A^{-1} = \dfrac{\text{adj } A}{|A|}$.

Now, we find $\text{adj } A = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$ where $A_{ij}, (i, j = 1,2,3)$ are cofactors of a_{ij}

.

$$A_{11} = \begin{vmatrix} 6 & -5 \\ -2 & 2 \end{vmatrix} = 2;$$

$$A_{12} = -\begin{vmatrix} -15 & -5 \\ 5 & 2 \end{vmatrix} = 5$$

$$A_{13} = \begin{vmatrix} -15 & 6 \\ 5 & -2 \end{vmatrix} = 0;$$

$$A_{21} = -\begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} = 0$$

$$A_{22} = \begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} = -1;$$

$$A_{23} = -\begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} = -1$$

$$A_{31} = \begin{vmatrix} -1 & 1 \\ 6 & -5 \end{vmatrix} = -1$$

$$A_{32} = -\begin{vmatrix} 2 & 1 \\ -15 & -5 \end{vmatrix} = -5$$

$$A_{33} = \begin{vmatrix} 2 & -1 \\ -15 & 6 \end{vmatrix} = -3.$$

Hence $\text{adj}A = \begin{pmatrix} 2 & 0 & -1 \\ 5 & -1 & -5 \\ 0 & -1 & -3 \end{pmatrix}$

$$\therefore \qquad A^{-1} = \frac{1}{-1} = \begin{pmatrix} 2 & 0 & -1 \\ 5 & -1 & -5 \\ 0 & -1 & -3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 1 \\ -5 & 1 & 5 \\ 0 & 1 & 3 \end{pmatrix}$$

Problem: 2

If $\omega = e^{2\pi i/3}$ find the inverse of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$.

Solution:

We note that $\omega^3 = 1$.

$$\therefore \qquad |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 1[\omega^2 - \omega^4] - 1[\omega - \omega^2] + 1[\omega^2 - \omega]$$

$$= \omega^2 - \omega - \omega + \omega^2 + \omega^2 - \omega = 3(\omega^2 - \omega).$$

Since $|A| \neq 0$, A is non – singular. Hence A^{-1} exists and is given by $A^{-1} = \frac{adjA}{|A|}$.

To find: adj A

$$adjA = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Where $A_{ij}, (i, j = 1,2,3)$ are cofactors of a_{ij}

$$A_{11} = \begin{vmatrix} 1 & \omega \\ 1 & \omega^2 \end{vmatrix} = \omega^2 - \omega$$

$$A_{12} = -\begin{vmatrix} 1 & \omega^2 \\ 1 & \omega \end{vmatrix} = -(\omega - \omega^2) = \omega^2 - \omega$$

$$A_{13} = \begin{vmatrix} 1 & \omega \\ 1 & \omega^2 \end{vmatrix} = \omega^2 - \omega$$

$$A_{21} = \begin{vmatrix} 1 & 1 \\ \omega^2 & \omega \end{vmatrix} = -(\omega - \omega^2) = \omega^2 - \omega$$

$$A_{22} = \begin{vmatrix} 1 & 1 \\ 1 & \omega \end{vmatrix} = \omega - 1$$

$$A_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & \omega^2 \end{vmatrix} = -(\omega^2 - 1) = 1 - \omega^2$$

$$A_{31} = \begin{vmatrix} 1 & 1 \\ \omega & \omega^2 \end{vmatrix} = \omega^2 - \omega$$

$$A_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & \omega^2 \end{vmatrix} = -(\omega^2 - 1) = 1 - \omega^2$$

$$A_{33} = \begin{vmatrix} 1 & 1 \\ 1 & \omega \end{vmatrix} = \omega - 1$$

$$\therefore \text{adj } A = \begin{pmatrix} \omega^2-\omega & \omega^2-\omega & \omega^2-\omega \\ \omega^2-\omega & \omega-1 & 1-\omega^2 \\ \omega^2-\omega & 1-\omega^2 & \omega-1 \end{pmatrix} \qquad \text{i} \qquad \text{(verify)}$$

$$\therefore \qquad A^{-1} = \frac{1}{3(\omega^2-\omega)} \begin{pmatrix} \omega^2-\omega & \omega^2-\omega & \omega^2-\omega \\ \omega^2-\omega & \omega-1 & 1-\omega^2 \\ \omega^2-\omega & 1-\omega^2 & \omega-1 \end{pmatrix}$$

$$= \frac{1}{3\omega} \begin{pmatrix} \omega & \omega & \omega \\ \omega & 1 & -1-\omega \\ \omega & -1-\omega & 1 \end{pmatrix}$$

Problem: 3

Show that a square matrix A is orthogonal iff $A^{-1} = A^T$.

Solution:

Suppose A is orthogonal. Then $AA^T = I$.

$\therefore \quad |AA^T| = |I| = 1$.

$\therefore \quad |A||A^T| = 1$.

$\therefore \quad |A||A| = 1$.

$\therefore \quad |A| \neq 0$ and hence A is non – singular.

$\therefore \quad A^{-1}$ exists.

Now, $A^{-1}(AA^T) = A^{-1}I$.

$\therefore \quad (A^{-1}A)A^T = A^{-1}$.

$\therefore \quad IA^T = A^{-1}$

$\therefore \quad A^T = A^{-1}$.

Conversely, let $A^T = A^{-1}$.

Then $AA^T = AA^{-1} = I$. Similarly $A^TA = I$.

Hence A is orthogonal.

Problem: 4

Show that a square matrix A is involutory iff $A = A^{-1}$.

Solution:

Suppose A is involutory. Then $A^2 = I$. Hence $|A^2| = 1$.

$\therefore$ $|A|^2 = |A||A| = 1$.

$\therefore$ $|A| \neq 0$ and hence A is non – singular.

$\therefore$ A^{-1} exists.

Now, $A^{-1}(AA) = A^{-1}I$.

$\therefore$ $(A^{-1}A)A = A^{-1}$.

$\therefore$ $IA = A^{-1}$.

$\therefore$ $A = A^{-1}$

Conversely, let $A = A^{-1}$.

Then $A^2 = AA - AA^{-1} = I$

$\therefore$ A is involutory.

Exercises:

1. Compute the inverse of each of the following matrices.

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 2 & -3 \\ -3 & 2 & 2 \\ 2 & -3 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Answers:

1.

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(b) $-\frac{1}{31}\begin{pmatrix} -9 & 4 & 11 \\ -8 & 7 & -4 \\ -2 & -6 & -1 \end{pmatrix}$

(c) $\frac{1}{5}\begin{pmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Elementary Transformations

Definition: 2.2.1

Let A be an m x n matrix over a field F. An elementary row – operation on A is of any one of the following three types.

1. The interchange of any two rows.
2. Multiplication of a row by a non – zero element c in F.
3. Addition of any multiple of one row with any other row.

Similarly we define an elementary column operation on A as any one of the following three types:

1. The interchange of any two columns.
2. Multiplication of a column by a non – zero element c in F.
3. Addition of any multiple of one column with any other column.

Example:

Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & -1 \end{pmatrix}, A_1 = \begin{pmatrix} 3 & -1 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} 2 & 2 \\ 4 & 1 \\ 6 & -1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 2 \\ 5 & 7 \\ 3 & -1 \end{pmatrix}$. A_1 is obtained from A by interchanging the first and third rows.

A_2 is obtained from a by multiplying the first column of A by 2.

A_3 is obtained from A by adding to the second row the multiple by 3 of the first row.

Notation:

We shall employ the following notations for elementary transformations.

(i) Interchanging of i^{th} and j^{th} rows will be denoted by $R_i \leftrightarrow R_j$.

(ii) Multiplication of i^{th} row by a non – zero element $c \in F$ will be denoted by $R_i \to cR_i$.

(iii) Addition of k times the j^{th} row to the i^{th} row will be denoted by $R_i \to R_i + kR_j$.

The corresponding column operations will be denoted by writing C in the place of R.

Definition: 2.2.2

An $m \times n$ matrix B is said to be **row equivalent (column equivalent)** to an $m \times n$ matrix A if B can be obtained from A by a finite succession of elementary row operations (column operations).

A and B are said to be **equivalent** if B can be obtained from A by a finite succession of elementary row or column operations.

If A and B are equivalent. We write $A \sim B$.

Exercise:

Prove that row equivalence, column equivalence and equivalence are equivalence relations in the set of all m x n matrices.

Definition: 2.2.3

A matrix obtained from the identity matrix by applying a single elementary row or column operation is called an elementary matrix.

For example, $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$ are elementary matrices obtained from the identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ by applying the elementary operations $R_1 \leftrightarrow R_2$, $R_1 \rightarrow 4R_1, R_3 \rightarrow R_3 + 2R_2$ respectively.

Exercise:

Give examples of elementary matrices of order 4.

Result: 2.22

Any elementary matrix is non – singular.

Result: 2.23

Let A be an m x n matrix and B be an n x p matrix. Then every elementary row (column) operation of the product AB can be obtained by subjecting the matrix A (matrix B) to the same elementary row (column) operation.

Result: 2.24

Each elementary row operation on an m x n matrix A is equivalent to **pre – multiplying the matrix** A by the corresponding elementary m x m matrix.

Canonical form of a matrix:

We now use elementary row and column operations to reduce any matrix to a simple form, called the canonical form of a matrix.

Result: 2.25

By successive applications of elementary row and column operations, any non – zero m x n matrix A can be reduced to a diagonal matrix D in which the diagonal entries are either 0 or 1 and all the 1's proceeding all the zeros on the diagonal. In other words, any non – zero m x n matrix is equivalent to a matrix of the form $\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$ where I_r is the r x r identity matrix and O is the zero matrix.

Problems:

Problem: 1

Reduce the matrix $A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \\ 2 & 4 & -2 \end{pmatrix}$ to the canonical form.

Solution:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \\ 2 & 4 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 + C_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad C_3 \to C_3 + 3C_2$$

$$\sim\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} \quad R_2 \to -R_2$$

Problem: 2

Find the inverse of the matrix $A=\begin{pmatrix}1 & 0 & 2\\ 3 & 1 & -1\\ -2 & 1 & 3\end{pmatrix}$

Solution:

$$\begin{pmatrix}1 & 0 & 2\\ 3 & 1 & -1\\ -2 & 1 & 3\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.A$$

$$\Rightarrow\begin{pmatrix}1 & 0 & 2\\ 0 & 1 & -7\\ 0 & 1 & 7\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ -3 & 1 & 0\\ 2 & 0 & 1\end{pmatrix}.A,$$

$$R_2 \to R_2 - 3R_1$$

$$R_3 \to R_3 + 2R_1$$

$$\Rightarrow\begin{pmatrix}1 & 0 & 2\\ 0 & 1 & -7\\ 0 & 0 & 14\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ -3 & 1 & 0\\ 5 & -1 & 1\end{pmatrix}.A,$$

$$R_3 \to R_3 - R_2$$

$$\Rightarrow\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}\frac{2}{7} & \frac{1}{7} & -\frac{1}{7}\\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ \frac{5}{14} & -\frac{1}{14} & \frac{1}{14}\end{pmatrix}$$

$$R_1 \to R_1 - \frac{1}{7}R_3$$

$$R_2 \to R_2 + \frac{1}{2} R_3$$

$$R_3 \to \frac{1}{14} R_3$$

$$\Rightarrow A^{-1} = \begin{pmatrix} \frac{2}{7} & \frac{1}{7} & -\frac{1}{7} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{5}{14} & -\frac{1}{14} & \frac{1}{14} \end{pmatrix}$$

Exercises:

1. Write down six matrices (not necessarily square matrices) and reduced them to the canonical form.
2. Find the inverse of the following matrices by using elementary operations.

(a) $\begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Answers:

2. (a) $\begin{pmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{pmatrix}$ (b) $\frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

Definition: 2.2.4

Let A and B be two square matrices of order n. B is said to be **similar** to A if there exists a n x n non – singular matrix P such that $B = P^{-1}AP$.

Problems:

Problem: 1

Similarity of matrices is an equivalence relation in the set of all $n \times n$ matrices.

Proof:

Let S be the set of all $n \times n$ matrices.

Let $A \in S$.

Since $A = I^{-1}AI$ and I is non – singular, A is similar to A.

Hence similarity of matrices is reflexive.

Now, let $A, B \in S$ and let A be similar to B.

$\therefore$ $A = P^{-1}BP$ where $P \in S$ is a non – singular matrix.

Now, $P^{-1}BP = A \Rightarrow PP^{-1}BPP^{-1} = PAP^{-1}$

$$\Rightarrow B = PAP^{-1}$$

$$\Rightarrow B = (P^{-1})^{-1}A(P^{-1}).$$

Since P is non – singular $P^{-1} \in S$ is also non – singular.

$\therefore$ B is similar to A.

Hence similarity of matrices is symmetric.

Now, let $A, B, C \in S$.

Let A be similar to B to B be similar to C. Hence there exist non – singular matrices $P, Q \in S$ such that $A = P^{-1}BP$ and $B = Q^{-1}CQ$.

Now,

$$A = P^{-1}BP$$

$$= P^{-1}(Q^{-1}CQ)P$$

$$= (P^{-1}Q^{-1})CQP$$

$=(QP)^{-1}C(QP)$.

Since $P, Q \in S$ are non – singular, $QP \in S$ is also non – singular.

Hence A is similar to C.

$\therefore$ Similarity of matrices is transitive.

Hence similarity of matrices is an equivalence relation.

Problem: 2

If A and B are similar matrices show that their determinants are same.

Solution:

Let A and B be two similar matrices.

$\therefore$ there exists a non – singular matrix P such that $vB = P^{-1}AP$.

Now, $|B| = |P^{-1}AP|$

$= |P^{-1}||A||P|$

$= |A|$ (since $|P^{-1}| = \frac{1}{|P|}$)

Hence the result.

2.2 Rank

We now proceed to introduce the concept of the rank of a matrix.

Definition: 2.2.1

Let $A = (a_{ij})$ be an m x n matrix. The rows $R_i = (a_{i1}, a_{i2}, \ldots\ldots, a_{in})$ of A can be thought of as elements of F^n. The subspace of F^n generated by the m rows of A is called the **row space** of A.

Similarly, the subspace of F^m generated by the n columns of A is called the **column space** of A.

The dimension of the row space (column space) of A is called the **row rank (column rank)** of A.

Result: 2.26

Any two row equivalent matrices have the same row space and have the same row rank.

Result: 2.27

Any two column equivalent matrices have the same column rank.

Result: 2.28

The row rank and the column rank of any matrix are equal.

Definition: 2.2.2

The **rank** of a matrix A is the common value of its row and column rank.

Note: 1

Since the row rank and the column rank of a matrix are unaltered by elementary row and column operations, **equivalent matrices** have the same rank. In particular if a matrix A is reduced to its canonical form $\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$, then rank of $A = r$.

Thus to find the rank of a matrix A, we reduce A to the canonical form and find the number of non – zero entries in the diagonal.

Note that in the canonical form of the matrix A, there exists an r x r sub – matrix, namely, I_r, whose determinant is not zero.

Further every $(r+1)x(r+1)$ sub – matrix contains a row of zeros and hence its determinant is zero.

Also under any elementary row or column operation the value of a determinant is either unaltered or multipled by a non – zero constant.

Hence the matrix A is also such that

(i) There exists an r x r sub – matrix whose determinant is non zero.

(ii) The determinant of every $(r+1)x(r+1)$ sub – matrix is zero.

Hence one can also define the rank of a matrix A to be r if A satisfies (i) and (ii).

Note: 2

Any non – singular matrix of order n is equivalent to the identity matrix and hence its rank is n.

Note: 3

The rank of a matrix is not altered on multiplication by non – singular matrices, since pre multiplication by non – singular matrices, since pre multiplication by a non – singular matrix is equivalent to applying elementary row operations and post – multiplication by a non – singular matrix is equivalent to applying elementary column operations.

Problems:

Problem: 1

Find the rank of the matrix $A = \begin{pmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 7 \end{pmatrix}$

Solution:

$$A = \begin{pmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 7 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 4 & 3 \\ 4 & 3 & 6 & 7 \\ 0 & 1 & 2 & 7 \end{pmatrix} C_1 \leftrightarrow C_3$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & -5 & -10 & -5 \\ 0 & 1 & 2 & 7 \end{pmatrix} \begin{matrix} C_1 \to C_2 - 2C_1 \\ C_3 \to C_3 - 4C_1 \\ C_4 \to C_4 - 3C_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & -10 & -5 \\ 0 & 1 & 2 & 7 \end{pmatrix} R_1 \to R_2 - 4R_1$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 1 & 0 & 6 \end{pmatrix} \begin{matrix} C_3 \to C_3 - 2C_2 \\ C_4 \to C_4 - C_2 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} R_3 \to R_3 + \frac{1}{5} R_2$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 6 & 0 \end{pmatrix} C_2 \leftrightarrow C_3$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{matrix} R_2 \to -\frac{1}{5} R_2 \\ R_3 \to \frac{1}{6} R_3 \end{matrix}$$

$\therefore$ Rank of A = 3.

Problem: 2

Find the rank of the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 4 & 2 \end{pmatrix}$ by examining the determinant minors.

Solution:

$$\begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & 0 \\ 0 & 3 & 4 \end{vmatrix} = 0 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 4 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & 2 \\ 0 & 3 & 2 \end{vmatrix} = 0 = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 0 & 2 \\ 0 & 4 & 2 \end{vmatrix}$$

$\therefore$ Every 3 x 3 sub matrix of A has determinant zero.

Also, $\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} = -3 \neq 0$

$\therefore$ Rank of A = 2.

Exercises.

1. Determine the rank of any six matrices of your choice.
2. Find the rank of the following matrices,

(a) $\begin{pmatrix} 3 & -1 & 2 \\ 0 & 1 & -3 \\ 6 & -1 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 & 2 & 1 \\ 2 & -3 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 3 & 1 & 1 & -1 & 2 \\ 4 & 0 & 1 & 0 & 3 \\ 9 & -1 & 2 & 3 & 7 \end{pmatrix}$

3. Find the column rank of the matrices

(a) $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 1 & -5 & -1 \\ 1 & -2 & 1 & -5 \\ 1 & 5 & -7 & 2 \end{pmatrix}$

(Hint: Row rank = rank of the matrix = column rank)

4. Find the row rank of the matrix

$$\begin{pmatrix} 1 & 3 & 1 & -2 \\ 1 & 4 & 3 & -1 \\ 2 & 3 & -4 & -7 \\ 3 & 8 & 1 & -7 \end{pmatrix}$$

Answers:

2. (a) 2 (b) 3 (c) 3

3. (a) 2 (b) 3

2.3 Solving Linear Equations

In this section we shall apply the theory of matrices developed in the preceding sections to study the existence of solutions of simultaneous linear equations.

Matrix form of a set of linear equations

Consider a system of m linear equations in n unknowns $x_1, x_2, \ldots\ldots, x_n$ given by

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n = b_2$$

....

....

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots a_{mn}x_n = b_m$$

Using the concept of matrix multiplication and equality of matrices this system can be written as $AX = B$ where,

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots & a_{2n} \\ . & . & \ldots\ldots\ldots & . \\ . & . & \ldots\ldots\ldots & . \\ a_{m1} & a_{m2} & \ldots\ldots\ldots & a_{mn} \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{pmatrix}$$

The m x n matrix A is called the **coefficient matrix**.

Definition: 2.4.1

A set of values of $x_1, x_2, \ldots\ldots\ldots x_n$ which satisfy the above system of equations is called a **solution** of the system. The system of equations is said to be **consistent** if it has at least one solution. Otherwise the system is said to be **inconsistent**.

The $m \times (n+1)$ matrix given by

$$\begin{pmatrix} a_{11} & \cdots\cdots & a_{1n} & b_1 \\ a_{21} & \cdots\cdots & a_{2n} & b_2 \\ . & \cdots\cdots & . & . \\ . & \cdots\cdots & . & . \\ a_{m1} & \cdots\cdots & a_{mn} & b_m \end{pmatrix}$$

is called the **augmented matrix** of the system and is denoted by (A, B).

Thus the augmented matrix (A, B) is obtained by annexing to A to the column matrix B, which becomes the $(n+1)^{th}$ column in (A, B).

Note:

Since every column in A appears in (A, B) the column space of the matrix A is a subspace of the column space of the matrix (A, B).

Hence the rank of A $\leq$ rank of (A, B).

Result: 2.29

The system of linear equations $AX = B$ is consistent iff rank of A = rank of (A, B).

Problems:

Problem: 1

Show that the equations

$$x + y + z = 6$$

$$x + 2y + 3z = 14$$

$$x + 4y + 7z = 30$$

are consistent and solve them.

Solution:

The given system of equations can be put in the matrix form

$$AX = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \\ 30 \end{pmatrix} = B.$$

$\therefore$ The augmented matrix is given by

$$(A, B) = \begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{pmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix} R_3 \to R_3 - 3R_2$$

Hence rank of A = rank of $(A, B) = 2$.

Hence the given system is consistent.

Also the given system of equations reduces to

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix}$$

$$\therefore \quad x + y + z = 6$$

$$y + 2z = 8.$$

Putting $z = c$ we obtain the general solution of the system as $x = c - 2, y = 8 - 2c, z = c$

Problem: 2

Verify whether the following system of equations is consistent. If it is consistent, find the solution.

$$x-4y-3z=-16$$

$$4x-y+6z=16$$

$$2x+7y+12z=48$$

$$5x-5y+3z=0.$$

Solution:

The matrix form of the system is given by

$$\begin{pmatrix}1 & -4 & -3\\ 4 & -1 & 6\\ 2 & 7 & 12\\ 5 & -5 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}-16\\ 16\\ 48\\ 0\end{pmatrix}$$

∴ The augmented matrix is given by

$$(A,B)=\begin{pmatrix}1 & -4 & -3 & -16\\ 4 & -1 & 6 & 16\\ 2 & 7 & 12 & 48\\ 5 & -5 & 3 & 0\end{pmatrix}$$

$$\sim\begin{pmatrix}1 & -4 & -3 & -16\\ 0 & 15 & 18 & 80\\ 0 & 15 & 18 & 80\\ 0 & 15 & 18 & 80\end{pmatrix}\begin{matrix}R_2\to R_2-4R_1\\ R_3\to R_3-2R_1\\ R_4\to R_4-5R_1\end{matrix}$$

$$\sim\begin{pmatrix}1 & -4 & -3 & -16\\ 0 & 15 & 18 & 80\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{matrix}R_3\to R_3-R_2\\ R_4\to R_4-R_2\end{matrix}$$

∴ Rank of A = Rank of (A, B) = 2 and hence the system is consistent. Also the system of equations reduces to

$$\begin{pmatrix} 1 & -4 & -3 \\ 0 & 15 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -16 \\ 80 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ $x - 4y - 3z = -16$ and $15y + 18z = 80$.

Putting $z = c$ we obtain the general solution of the systems as $x = -(9c/5) + (16/3)$,

$y = -(6c/5) + (16/3)$;

$z = c$.

Problem: 3

For what values of η the equations

$$x + y + z = 1$$

$$x + 2y + 4z = \eta$$

$$x + 4y + 10z = \eta^2$$ are consistent?

Solution:

The matrix form of the system is given by

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ \eta \\ \eta^2 \end{pmatrix}$$

$\therefore$ The augmented matrix is given by

$$(A,B) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \eta \\ 1 & 4 & 10 & \eta^2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \eta - 1 \\ 0 & 3 & 9 & \eta^2 - 1 \end{pmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \quad R_3 \to R_3 - 3R_2$$

∴ The given system is consistent iff $\eta^2 - 3\eta + 2 = 0$.

∴ $\quad \eta = 2$ or 1.

Problem: 4

Show that the system of equations

$x + 2y + z = 11$

$4x + 6y + 5z = 8$

$2x + 2y + 3z = 19$ is inconsistent.

Solution:

The matrix form of the system is given by

$$\begin{pmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ 8 \\ 19 \end{pmatrix}$$

The augmented matrix is given by

$$(A, B) = \begin{pmatrix} 1 & 2 & 1 & 11 \\ 4 & 6 & 5 & 8 \\ 2 & 2 & 3 & 19 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 11 \\ 0 & -2 & 1 & -36 \\ 0 & -2 & 1 & -3 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 11 \\ 0 & -2 & 1 & -36 \\ 0 & 0 & 0 & 33 \end{pmatrix} R_3 \to -R_3 - R_2$$

∴ Rank of A = 2 and rank of (A, B) = 3.

$\therefore$ The given system is inconsistent.

Exercises:

1. Solve or prove the inconsistency of the following systems of equations.

   (a) $2x - y + 3z = 8$

   $$x - 2y - z = -4$$
   $$3x + y - 4z = 0$$

   (b) $x + 2y - 5z = 0$

   $$3x + 4y + 6z = 0$$
   $$x + y + z = 0$$

   (c) $x + 2y - 5z = -9$

   $$3x - y + 2z = 5$$
   $$2x + 3y - z = 3$$
   $$4x - 5y + z = -3$$

   (d) $x + y + z = 1$

   $$x + 2y + 3z = 1$$
   $$x + 3y + 5z = 7$$
   $$x + 4y + 7z = 10$$

   (e) $x - 2y - z - t = -1$

   $$3x - 2z + 3t = -4$$
   $$5x - 4y + t = -3$$

(f) $x + y + z = 7$

$$x + 2y + 3z = 8$$
$$y + 2z = 6$$

2. For what values of λ and μ the system of equations

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + \lambda z = \mu$$

is (a) inconsistent (b) consistent (c) consistent and the solution is unique.

3. Show that the set of all solutions of the system of homogeneous equations $AX = 0$ forms a vector space.

4. Show that a system of n equations in n unknowns given by $AX = Y$ has a unique solution if the n x n matrix A is non – singular.

Answers:

1.

(a) Consistent; $x = y = z = 2$

(b) Consistent; $x = y = z = 0$;

(c) Consistent; $x = \frac{1}{2}, y = \frac{3}{2}, z = \frac{5}{2}$;

(d) Consistent; $x = c - 2, y = 3 - 2c; z = c$

(e) Inconsistent;

(f) Inconsistent;

(g) Inconsistent.

2. If $\lambda = 3$ and $\mu \neq 10$, inconsistent.

If $\lambda = 3$ and $\mu = 10$, consistent.

If $\lambda \neq 3$, consistent and the solution is unique.

2.4 Cayley's Hamilton Theorem

Definition: 2.4.1

An expression of the form $A_0 + A_{1x} + A_2x^2 + \dots\dots + A_nx^n$ where $A_0, A_1, \dots\dots, A_n$ are square matrices of the same order and $A_n \neq 0$ is called a **matrix polynomial** of degree n.

For example, $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} x + \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} x^2$ is a matrix polynomial of degree 2 and it is simply the matrix $\begin{pmatrix} 1 + x + 2x^2 & 2 + x \\ 2x + 3x^2 & 3 + x + x^2 \end{pmatrix}$.

Definition: 2.4.2

Let A be any square matrix of order n and let I be the identity matrix of order n. then the matrix polynomial given by $A - xI$ is called the **characteristic matrix** of A.

The determinant $| A - xI |$ which is an ordinary polynomial in x of degree n is called the **characteristic polynomial** of A.

The equation $| A - xI | = 0$ is called the **characteristic equation** of A.

Example: 1

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Then the characteristic matrix of A is $A - xI$ given by

$$A - xI = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-x & 2 \\ 3 & 4-x \end{pmatrix}$$

$\therefore$ The characteristic polynomial of A is

$$|A - xI| = \begin{vmatrix} 1-x & 2 \\ 3 & 4-x \end{vmatrix}$$

$$= (1-x)(4-x) - 6$$

$$= x^2 - 5x - 2$$

$\therefore$ The characteristic equation of A is $|A - xI| = 0$

$\therefore$ $x^2 - 5x - 2 = 0$ is the characteristic equation of A.

Example: 2

Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$.

The characteristic matrix of A is $A - xI$ given by

$$A - xI = \begin{pmatrix} 1-x & 0 & 2 \\ 0 & 1-x & 2 \\ 1 & 2 & -x \end{pmatrix}$$

The characteristic polynomial of A is $|A - xI| = \begin{vmatrix} 1-x & 0 & 2 \\ 0 & 1-x & 2 \\ 1 & 2 & -x \end{vmatrix}$

$$= (1-x)[(1-x)(-x) - 4] - 2(1-x)]$$

$$= -x(1-x)^2 - 4(1-x) - 2 + 2x$$

$$= -x^3 + 2x^2 - x - 4 + 4x - 2 + 2x$$

$$= -x^3 + 2x^2 + 5x - 6$$

$\therefore$ The characteristic equation of A is

$$-x^3 + 2x^2 + 5x - 6 = 0$$

(i.e) $x^3 - 2x^2 - 5x + 6 = 0$

Result: 2.30 (Cayley's Hamilton Theorem)

Any square matrix A satisfies its characteristic equation.

(i.e.) if $a_0 + a_1x + a_2x^2 + ... + a_nx^n$ is the characteristic polynomial of degree n of A then

$$a_0I + a_1A + a_2A^2 + ... + a_nA^n = 0.$$

Proof:

Let A be a square matrix of order n.

Let $|A - xI| = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ (1)

be the characteristic polynomial of A.

Now, $adj(A - xI)$ is a matrix polynomial of degree n – 1 since each entry of the matrix $adj(A - xI)$ is a cofactor of $A - xI$ and hence is a polynomial of degree $\le n-1$.

$\therefore$ Let $adj(A - xI) = B_0 + B_1x + B_2x^2 + ... + B_{n-1}x^{n-1}$. (2)

Now, $(A - xI)adj(A - xI) = |A - xI| I$.

$(\because \quad (adjA)A = A(adjA) = |A| I)$

$\therefore \quad (A - xI)(B_0 + B_1x + ... + B_{n-1}x^{n-1})$

$= (a_0 + a_{1x} + ... + a_nx^n)I$ using (1) and (2).

$\therefore$ Equating the coefficients of the corresponding powers of x we get

$AB_0 = a_0I$

$AB_1 - B_0 = a_1I$

$AB_2 - B_1 = a_2I$

....

....

$AB_{n-1} - B_{n-2} = a_{n-1}I$

$-B_{n-1} = a_nI$

Pre – multiplying the above equations by $I, A, A^2, ..., A^n$ respectively and adding we get

$$a_0I + a_1A + a_2A^2 + ... + a_nA^n = 0.$$

Note:

The inverse of a non – singular matrix can be calculated by using the Cayley Hamilton theorem as follows.

Let $a_0 + a_1x + a_2x^2 + ... + a_nx^n$ be the characteristic polynomial of A.

Then by result 2.30 we have

$$a_0I + a_1A + a_2A^2 + ... + a_nA^n = 0. \quad (3)$$

Since $|A - xI| = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ we get $a_0 = |A|$ (by putting x=0).

$\therefore \quad a_0 \ne 0 (\because A$ is a non singular matrix.)

$\therefore \qquad I = -\frac{1}{a_0}\left[a_1A + a_2A^2 + \ldots\ldots + a_nA^n\right]$(by 3)

$$\therefore \qquad \boxed{A^{-1} = -\frac{1}{a_0}\left[a_1I + a_2A + \ldots\ldots\ldots + a_nA^{n-1}\right]}$$

Problems:

Problem: 1

Find the characteristic equation of the matrix

$$A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$$

Solution:

The characteristic equation of A is given by $|A - \lambda I| = 0$.

(i.e.) $\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$.

$(8-\lambda)[(7-\lambda)(3-\lambda)-16] + 6[-6(3-\lambda)+8] + 2[24-2(7-\lambda)] = 0$

(i.e.) $(8-\lambda)(\lambda^2 - 10\lambda + 5) + 6(6\lambda - 10) + 2(2\lambda + 10) = 10$

(i.e.) $(8\lambda^2 - 80\lambda + 40 - \lambda^3 + 10\lambda^2 - 5\lambda) + (36\lambda - 60) + (4\lambda + 20) = 0$

(i.e.) $\lambda^3 - 18\lambda^2 + 45\lambda = 0$, which represents the characteristic equation of A.

Problem: 2

Show that the non – singular matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ satisfies the equation $A^2 - 2A - 5I = 0$. Hence evaluate A^{-1}.

Solution:

The characteristic polynomial of A is $|A - xI| = \begin{vmatrix} 1-x & 2 \\ 3 & 1-x \end{vmatrix} = x^2 - 2x - 5$.

$\therefore$ By Cayley – Hamilton theorem $A^2 - 2A - 5I = 0$.

$\therefore I = \frac{1}{5}(A^2 - 2A)$.

$\therefore A^{-1} = \frac{1}{5}(A - 2I)$

$= \frac{1}{5}\left[\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$

$= \frac{1}{5}\begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}$.

Problem: 3

Show that the matrix $A = \begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}$ satisfies the equation $A(A - I)(A + 2I) = 0$.

Solution:

The characteristic polynomial of A is $|A - \lambda I| = \begin{vmatrix} 2-\lambda & -3 & 1 \\ 3 & 1-\lambda & 3 \\ -5 & 2 & -4-\lambda \end{vmatrix}$

$= (2 - x)[(1 - \lambda)(-4 - \lambda) - 6] + 3[3(-4 - \lambda) + 5(1 - \lambda) + 1[6 + 5(1 - \lambda)]$

$= (2-\lambda)[-4-\lambda+4\lambda+\lambda^2-6)+3[-12-3\lambda+5-5\lambda]+6+5-\lambda$

$= (2-\lambda)[\lambda^2+3\lambda-10]+3[-8\lambda-7]+11-\lambda$

$= -\lambda^3 - \lambda^2 + 2\lambda$ (after simplification).

$\therefore$ By Cayley – Hamilton theorem $-A^3 - A^2 + 2A = 0$.

(i.e.) $A^3 + A^2 - 2A = 0$. Hence $A(A^2 + A - 2I) = 0$.

$\therefore \quad A(A + 2I)(A - I) = 0$.

Problem: 4

Using Cayley – Hamilton theorem find the inverse of the matrix

$$\begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix}$$

Solution:

Let $A = \begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix}$.

The characteristic polynomial of $A = |A - xI|$

$$= \begin{vmatrix} 7-x & 2 & -2 \\ -6 & -1-x & 2 \\ 6 & 2 & -1-x \end{vmatrix}$$

$= (7-x)[(1+x)^2-4]-2[6(1+x)-12]-2[-12+6(1+x)]$

$= (7-x)(x^2+2x-3)-12(x-1)-12(x-1)$

$= 7x^2+14x-21-x^3-2x^2+3x-12x+12-12x+12$

$= -x^3+5x^2-7x+3$.

$\therefore$ By Cayley – Hamilton theorem

$$-A^3+5A^2-7A+3I_3=0.$$

$$\therefore A^3-5A^2+7A-3I_3=0.$$

$$\therefore 3I_3=A^3-5A^2+7A.$$

$$\therefore I_3=\frac{1}{3}(A^3-5A^2+7A).$$

Pre (or post) multiplying by A^{-1} on both sides we get

$$A^{-1}=\frac{1}{3}\left[A^2-5A+7I_3\right] \qquad (1)$$

$$\text{Now, } A^2=\begin{pmatrix}7 & 2 & -2\\ -6 & -1 & 2\\ 6 & 2 & -1\end{pmatrix}\begin{pmatrix}7 & 2 & -2\\ -6 & -1 & 2\\ 6 & 2 & -1\end{pmatrix}$$

$$=\begin{pmatrix}25 & 8 & -8\\ -24 & -7 & 8\\ 24 & 8 & -7\end{pmatrix} \text{ (verify).}$$

$\therefore$ From (1)

$$A^{-1}=\frac{1}{3}\left[\begin{pmatrix}25 & 8 & -8\\ -24 & -7 & 8\\ 24 & 8 & -7\end{pmatrix}-\begin{pmatrix}35 & 10 & -10\\ -30 & -5 & 10\\ 30 & 10 & -5\end{pmatrix}+\begin{pmatrix}7 & 0 & 0\\ 0 & 7 & 0\\ 0 & 0 & 7\end{pmatrix}\right]$$

$$=\frac{1}{3}\begin{pmatrix}-3 & -2 & 2\\ 6 & 5 & -2\\ -6 & -2 & 5\end{pmatrix}$$

Problem: 5

Find the inverse of the matrix $\begin{pmatrix}3 & 3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{pmatrix}$ using Cayley – Hamilton theorem.

Solution:

The characteristic polynomial of $A = |A-xI| = \begin{vmatrix}3-x & 3 & 4\\ 2 & -3-x & 4\\ 0 & -1 & 1-x\end{vmatrix}$

$$=-x^3+x^2+11x-11 \text{ (verify).}$$

$\therefore$ By Cayley – Hamilton theorem

$$-A^3+A^2+11A-11I_3=0.$$

$$\therefore \quad A^3-A^2-11A+11I_3=0.$$

Hence $11I_3 = -(A^3 - A^2 - 11A)$.

$$\therefore \qquad I_3 = -\frac{1}{11}\left[A^3 - A^2 - 11A\right].$$

Pre (post) multiplying by A^{-1} on both sides we get

$$A^{-1} = -\frac{1}{11}\left[A^2 - A - 11I_3\right]$$

$$= -\frac{1}{11}\left[\begin{pmatrix} 15 & -4 & 28 \\ 0 & 11 & 0 \\ -2 & 2 & -3 \end{pmatrix} - \begin{pmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix} - 11\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} -\frac{1}{11} & \frac{7}{11} & -\frac{24}{11} \\ \frac{2}{11} & -\frac{3}{11} & \frac{4}{11} \\ \frac{2}{11} & -\frac{3}{11} & \frac{15}{11} \end{pmatrix}$$

Problem: 6

Verify Cayley Hamilton's theorem for the matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

Solution:

The characteristic equation of A is $|A - \lambda I| = 0$.

$$\therefore \qquad \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$\therefore \qquad (1-\lambda)(3-\lambda) - 8 = 0$$

$$\therefore \qquad \lambda^2 - 4\lambda - 5 = 0.$$

By Cayley Hamilton's theorem A satisfies its characteristic equation.

$\therefore$ We have $A^2 - 4A - 5I = 0$.

Now, $A^2 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 9 & 8 \\ 16 & 17 \end{pmatrix}$$

$4A = \begin{pmatrix} 4 & 8 \\ 16 & 12 \end{pmatrix}$ and $5I = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

$$\therefore \qquad A^2 - 4A - 5I = \begin{pmatrix} 9 & 8 \\ 16 & 17 \end{pmatrix} - \begin{pmatrix} 4 & 8 \\ 16 & 12 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

Thus Cayley Hamilton's theorem is verified.

Problem: 7

Using Cayley Hamilton's theorem for the matrix $A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}$

Find (i) A^{-1} (ii) A^4.

Solution:

The characteristic equation of A is $|A-\lambda I|=0$.

$$\therefore \begin{vmatrix} 1-\lambda & 0 & -2 \\ 2 & 2-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

(i.e) $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$. (Verify)

$\therefore$ By Cayley Hamilton's theorem

$$A^3 - 5A^2 + 8A - 4I = 0 \qquad (1)$$

$4I = A^3 - 5A^2 + 8A$

I. To find A^{-1} pre multiplying by A^{-1} we get

$$4A^{-1} = A^{-1}A^3 - 5A^{-1}A^2 + 8A^{-1}A$$

$$= A^2 - 5A + 8I$$

$$\therefore A^{-1} = \frac{1}{4}\left[A^2 - 5A + 8I\right] \qquad (2)$$

$$\text{Now, } A^2 = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{pmatrix}$$

From (2)

$$A^{-1} = \frac{1}{4}\left[\begin{pmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 0 & -10 \\ 10 & 10 & 20 \\ 0 & 0 & 10 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}\right]$$

$$= \frac{1}{4}\begin{pmatrix} 4 & 0 & 4 \\ 4 & 2 & -8 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & \frac{1}{2} & -2 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

II. To find A^4.

From (1) $A^3 = 5A^2 - 8A + 4I$

$$\therefore \quad A^4 = 5A^3 - 8A^2 + 4A$$

$$= 5[5A^2 - 8A + 4I] - 8A^2 + 4A \qquad \text{(using (1))}$$

$$= 17A^2 - 36A + 20I$$

$$= 17\begin{pmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{pmatrix} - 36\begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix} + 20\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 17 & 0 & -102 \\ 102 & 68 & 204 \\ 0 & 0 & 68 \end{pmatrix} - \begin{pmatrix} 36 & 0 & -72 \\ 72 & 72 & 144 \\ 0 & 0 & 72 \end{pmatrix} + \begin{pmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{pmatrix}$$

$$\therefore A^4 = \begin{pmatrix} 1 & 0 & -30 \\ 30 & 16 & 60 \\ 0 & 0 & 16 \end{pmatrix}$$

Exercises:

1. Obtain the characteristic polynomial for the following matrices.

   (i). $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

   (ii). $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

2. Find the characteristic equation of the following matrices.

   (i) $\begin{pmatrix} -b & -c \\ 1 & 0 \end{pmatrix}$

   (ii) $\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

   (iii) $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -8 \\ 2 & -4 & 3 \end{pmatrix}$

   (iv) $\begin{pmatrix} -b & -c & -d \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

3. Verify Cayley – Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and hence find A^{-1}.

4. If $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, prove that $A^3 - 2A^2 - 5A + 6I = 0$.

5. Verify Cayley – Hamilton theorem for A and hence find A^{-1}

(a) $A = \begin{pmatrix} 2 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

(d) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

6. If $A = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ find A^3 and A^{-3}.

7. Verify that the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{pmatrix}$ satisfies its own characteristic equation and hence find A^{-1} and A^4.

2.5 Eigen Values and Eigen Vectors

Definition: 2.5.1

Let A be an $n \times n$ matrix. A number λ is called an **eigen value** of A if their exists a non – zero vector $X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{pmatrix}$ such that $AX = \lambda X$ and X is called an **eigen vector** corresponding to the eigen value λ.

Remark: 1

If X is an eigen vector corresponding to the eigen value λ of A, then αX where α is any non – zero number, is also an eigen vector corresponding to λ.

Remark: 2

Let X be an eigen vector corresponding to the eigen value λ of A. Then $AX = \lambda X$ so that $(A - \lambda I)X = 0$. Thus X is a non – trivial solution of the system of homogeneous linear equations $(A - \lambda I)X = 0$. Hence $|A - \lambda I| = 0$, which is characteristic polynomial of A.

Let $|A - \lambda I| = a_0\lambda^n + a_1\lambda^{n-1} + \ldots\ldots + a_n$.

The roots of this polynomial give the eigen values of A. Hence eigen values are also **characteristic roots.**

Properties of Eigen Values

Property: 1

Let X be an eigen vector corresponding to the eigen values λ_1 and λ_2. Then $\lambda_1 = \lambda_2$.

Proof:

By definition $X \neq 0, AX = \lambda_1 X$ and $AX = \lambda_2 X$

$\therefore \quad \lambda_1 X = \lambda_2 X$

$\therefore \quad (\lambda_1 - \lambda_2)X = 0$

Since $X \neq 0$, $\lambda_1 = \lambda_2$.

Property: 2

Let A be a square matrix.

Then (i) the sum of the eigen values of A is equal to the sum of the diagonal elements (trace) of A.

(ii) Product of eigen values of A is $|A|$.

Poof:

(i) Let $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & . & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$

The eigen values of A are the roots of the characteristic equation

$$|A - \lambda I| = \begin{pmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{pmatrix} = 0. \quad (1)$$

Let $|A - \lambda I| = a_0\lambda^n + a_1\lambda^{n-1} + \dots\dots + a_n$ (2)

From (1) and (2) we get

$a_0 = (-1); a_1 = (-1)^{n-1}(a_{11} + a_{22} + \dots\dots + a_{nn});$ (3)

Also by putting $\lambda = 0$ in (2) we get $a_n = |A|$

Now let $\lambda_1, \lambda_2, \dots\dots, \lambda_n$ be the eigen values of A.

$\therefore$ $\lambda_1, \lambda_2, \dots\dots, \lambda_n$ be the eigen values of A.

$\therefore$ $\lambda_1 + \lambda_2 + \dots\dots + \lambda_n = -\frac{a_1}{a_0} = a_{11} + a_{22} + \dots\dots + a_{nn}$ (using (3))

$\therefore$ Sum of the eigen values = trace of A.

(ii) Product of the eigen values = product of the roots

$$= \lambda_1 \lambda_2, \ldots\ldots, \lambda_n$$

$$= (-1)^n \frac{a_n}{a_0} = \frac{(-1)^n a_n}{(-1)^n}$$

$$= a_n$$

$$= |A|.$$

Property: 3

The eigen values of A and its transpose A^T are the same.

Proof:

It is enough if we prove that A and A^T have the same characteristic polynomial. Since for any square matrix $M, |M| = |M^T|$ we have,

$$|A - \lambda I| = |(A - \lambda I)^T| = |A^T - (\lambda I)^T|$$

$$= |A^T - \lambda I|.$$

Hence the result.

Property: 4

If λ is an eigen value of a non singular matrix A then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

Proof:

Let X be an eigen vector corresponding to λ. Then $AX = \lambda X$. Since A is non singular A^{-1} exists.

$$\therefore \quad A^{-1}(AX) = A^{-1}(\lambda X)$$

$$IX = \lambda A^{-1} X$$

$$\therefore \quad A^{-1}X = \left(\frac{1}{\lambda}\right)X.$$

$\therefore$ $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

Corollary:

If $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are the eigen values of a non singular matrix A then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots\ldots, \frac{1}{\lambda_n}$ are the eigen values of A^{-1}.

Property: 5

If λ is an eigen value of A then $k\lambda$ is an eigen value of kA where k is a scalar.

Proof:

Let X be an eigen vector corresponding to λ.

Then $AX = \lambda X$ (1)

Now,

$$(kA)X = k(AX)$$

$$= k(\lambda X) \quad \text{(by (1))}$$

$$= (k\lambda)X.$$

$\therefore$ $k\lambda$ is an eigen value of kA.

Property: 6

If λ is an eigen value of A then λ^k is an eigen value of A^k where k is any positive integer.

Proof:

Let X be an eigen vector corresponding to λ.

Then $AX = \lambda X$ (1)

Now, $A^2X = (AA)X = A(AX)$

$$= A(\lambda X) \quad \text{(by (1))}$$

$$= \lambda(AX)$$

$$= \lambda(\lambda X) \quad \text{(by (1))}$$

$= \lambda^2 X$.

$\therefore$ λ^2 is an eigen value of A^2.

Proceeding like this we can prove that λ^k is an eigen value of A^k for any positive integer.

Corollary:

If $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are eigen values of A then $\lambda_1^k, \lambda_2^k, \ldots\ldots, \lambda_n^k$ are eigen values of A^k for any positive integer k.

Property: 7

Eigen vectors corresponding to distinct eigen values of a matrix are linearly independent.

Proof:

Let $\lambda_1, \lambda_2, \ldots\ldots, \lambda_k$ be distinct eigen values of a matrix and let X_i be the eigen vector corresponding to λ_i.

Hence $AX_i = \lambda_i X_i \quad (i = 1, 2, \ldots\ldots, k)$ (1)

Now, suppose $X_1, X_2, \ldots\ldots, X_k$ are linearly dependent. Then there exist real numbers $\alpha_1, \alpha_2, \ldots\ldots, \alpha_k$, not all zero, such that $\alpha_1 X_1 + \alpha_2 X_2 + \ldots\ldots + \alpha_k X_k = 0$. Among all such relations, we choose one of shortest length, say j.

By rearranging the vectors $X_1, X_2, \ldots\ldots, X_k$ we may assume that

$$\alpha_1 X_1 + \alpha_2 X_2 + \ldots\ldots + \alpha_j X_j = 0 \qquad (2)$$

$$\therefore \quad A(\alpha_1 X_1) + A(\alpha_2 X_2) + \ldots\ldots + A(\alpha_j X_j) = 0$$

$$\therefore \quad \alpha_1 (AX_1) + \alpha_2 (AX_2) + \ldots\ldots + \alpha_j (AX_j) = 0$$

$$\therefore \quad \alpha_1 \lambda_1 X_1 + \alpha_2 \lambda_2 X_2 + \ldots\ldots + \alpha_j \lambda_j X_j = 0 \qquad (3)$$

Multiplying (2) by λ_1 and subtracting from (3), we get

$$\alpha_2(\lambda_1 - \lambda_2)X_2 + \alpha_3(\lambda_1 - \lambda_3)X_3 + \ldots\ldots + \alpha_j(\lambda_1 - \lambda_j)X_j = 0 \quad (4)$$

and since $\lambda_1, \lambda_2, \ldots\ldots, \lambda_j$ are distinct and $\alpha_2, \ldots\ldots, \alpha_j$ are non – zero we have

$$\alpha_i(\lambda_1 - \lambda_i) \neq 0; \qquad i = 2,3,\ldots\ldots,j.$$

Thus (4) gives a relation whose length is $j-1$, giving a contradiction.

Hence $X_1, X_2, \ldots\ldots, X_k$ are linearly independent.

Property: 8

The characteristic roots of a Hermitian matrix are all real.

Hence

$$A = \overline{A}^T \qquad \text{(by theorem 9.13)} \qquad (1)$$

Let λ be a characteristic root of A and let X be a characteristic vector corresponding to λ.

$$\therefore \qquad AX = \lambda X \qquad (2)$$

Now,

$$AX = \lambda X \Rightarrow \overline{X}^T AX = \lambda \overline{X}^T X$$

$$\Rightarrow (\overline{X}^T AX)^T = \lambda \overline{X}^T X \qquad \text{(since } X^T AX \text{ is a 1 x 1 matrix)}$$

$$\Rightarrow X^T A^T (\overline{X}^T)^T = \lambda \overline{X}^T X$$

$$\Rightarrow X^T A^T \overline{X} = \lambda \overline{X}^T X$$

$$\Rightarrow \overline{\overline{X^T A^T}\ X} = \overline{\lambda \overline{X}^T X}$$

$$\Rightarrow \overline{X}^T \overline{A}^T\ X = \overline{\lambda} X^T \overline{X}$$

$$\Rightarrow \overline{X}^T\ AX = \overline{\lambda} X^T \overline{X} \qquad \text{(using 1)}$$

$$\Rightarrow \overline{X}^T \lambda X = \overline{\lambda} X^T \overline{X} \qquad \text{(using 2)}$$

$$\Rightarrow \lambda(\overline{X}^T X) = \overline{\lambda}(X^T \overline{X}) \qquad (3)$$

Now,

$$\overline{X}^T X = X^T \overline{X} = \overline{x_1} x_1 + \overline{x_2} x_2 + \ldots\ldots + \overline{x_n} x_n$$

$$= |x_1|^2 + |x_2|^2 + \ldots\ldots + |x_n|^2$$

$$\neq 0$$

$\therefore$ From (3) we get $\lambda = \overline{\lambda}$

Hence λ is real.

Corollary:

The characteristic roots of a real symmetric matrix are real.

Proof:

We know that any real symmetric matrix is Hermitian. Hence the result follows from the above property.

Property: 9

The characteristic roots of a skew Hermitian matrix are either purely imaginary or zero.

Proof:

Let A be a skew Hermitian matrix and λ be a characteristic root of A.

$\therefore$ $|A - \lambda I| = 0$

$\therefore$ $|iA - i\lambda I| = 0$

$\therefore$ $i\lambda$ is a characteristic root of iA.

Since A is skew Hermitian iA is Hermitian (refer results iv theorem 9.14)

$\therefore$ By theorem 9.32 $i\lambda$ is real. Hence λ is purely imaginary or zero.

Corollary:

The characteristic roots of a real skew symmetric matrix are either purely imaginary or zero.

Proof:

We know that any real skew symmetric matrix skew Hermitian.

Hence the result follows from the above property.

Property: 10

Let λ be a characteristic root of an unitary matrix A. Then $|\lambda| = 1$ (i.e) the characteristic roots of a unitary matrix are all the unit modulus.

Proof:

Let λ be a characteristic root of an unitary matrix A and X be a characteristic vector corresponding to λ.

$\therefore \quad AX = \lambda X \quad (1)$

Taking conjugate and transpose in (1) we get $(\overline{AX})^T = (\overline{\lambda X})^T$.

$\therefore \quad \overline{X}^T \overline{A}^T = \overline{\lambda}\ \overline{X}^T \quad (2)$

Multiplying (1) and (2) we get

$$(\overline{X^T A^T})(AX) = (\overline{\lambda}\ \overline{X}^T)(\lambda X)$$

$$\therefore \quad \overline{X}^T(\overline{A}^T A)X = \overline{\lambda}\lambda(\overline{X}^T X)$$

Now, since A is an unitary matrix $\overline{A}^T A = I$.

Hence $\overline{X}^T X = (\overline{\lambda}\lambda)\overline{X}^T X$

Since X is non – zero vector $\overline{X}^T$ is also non – zero vector and $\overline{X}^r X = |x_1|^2 + |x_2|^2 + \ldots\ldots\ldots. + |x_n|^2 \neq 0$ we get $\lambda\overline{\lambda} = 1$.

Hence $|\lambda|^2 = 1$. Hence $|\lambda| = 1$.

Corollary:

Let λ be a characteristic root of an orthogonal matrix A. Then $|\lambda| = 1$.

Since any orthogonal matrix is unitary the result follows from property 10.

Property: 11

Zero is an eigen value of A if and only if A is a singular matrix.

Proof:

The eigen values of A are the roots of the characteristic equation $|A-\lambda I|=0$. Now, 0 is an eigen value of A $\Leftrightarrow |A-0I|=0$

$$\Leftrightarrow |A|=0$$

$\Leftrightarrow$ A is a singular matrix

Property: 12

If A and B are two square matrices of the same order then AB and BA have the same eigen values.

Solution:

Let λ be an eigen value of AB and X be an eigen vector corresponding to λ.

$$\therefore \quad (AB)X=\lambda X$$

$$\therefore \quad B(AB)X=B(\lambda X)=\lambda(BX).$$

$$\therefore \quad (BA)(BX)=\lambda(BX)$$

$$\therefore \quad (BA)Y=\lambda Y \quad \text{where } Y=BX.$$

Hence λ is an eigen value of BA.

Also BX is the corresponding eigen vector.

Property: 13

If P and A are n x n matrices and P is a nonsingular matrix then A and $P^{-1}AP$ have the same eigen values.

Proof:

Let $B=P^{-1}AP$.

To prove A and B have same eigen values, it is enough to prove that the characteristic polynomials of A and B are the same.

Now,

$$|B-\lambda I|=|P^{-1}AP-\lambda I|$$

$$=|P^{-1}AP-P^{-1}(\lambda I)P|$$

$= |P^{-1}(A-\lambda I)P|$

$= |P^{-1}||A-\lambda I||P|$

$= |P^{-1}P||A-\lambda I|$

$= |I||A-\lambda I|$

$= |A-I\lambda|$.

$\therefore$ The characteristic equations of A and $P^{-1}AP$ are the same.

Property: 14

If λ is a characteristic root of A then $f(\lambda)$ is a characteristic root of the matrix $f(A)$ where $f(x)$ is any polynomial.

Proof:

Let $f(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$ where $a_0 \neq 0$ and $a_1, a_2, ..., a_n$ are all real numbers.

$\therefore \quad f(A) = a_0A^n + a_1A^{n-1} + ... + a_{n-1}A + a_nI$.

Since λ is a characteristic root of A, λ^n is a characteristic root of A^n for any positive integer n (refer Property 6).

$\therefore \quad A^nX = \lambda^nX$

$A^{n-1}X = \lambda^{n-1}X$

....

....

$AX = \lambda X$

$\therefore \quad a_0A^nX = a_0\lambda^nX$

$a_1A^{n-1}X = a_1\lambda^{n-1}X$

....

....

$$a_{n-1}AX = a_{n-1}\lambda X$$

Adding the above equations we have

$$a_0A^nX + a_1A^{n-1}X + ...a_{n-1}AX$$

$$= a_0\lambda^nX + a_1\lambda^{n-1}X + ... + a_{n-1}\lambda X$$

$\therefore$ $$(a_0A^n + a_1A^{n-1} + ... + a_{n-1}A)X$$

$$= (a_0\lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda)X$$

$\therefore$ $$(a_0A^n + a_1A^{n-1} + ... + a_{n-1}A + a_nI)X$$

$$= (a_0\lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n)X$$

$\therefore$ $$f(A)X = f(\lambda)X$$

Hence $f(\lambda)$ is a characteristic root of $f(A)$.

Problems:

Problem: 1

If X_1, X_2 are eigen vectors corresponding to an eigen value λ then $aX_1 + bX_2$ (a, b non – zero scalars) is also an eigen vector corresponding to λ.

Solution:

Since X_1 and X_2 are given vectors corresponding to λ, we have

$$AX_1 = \lambda X_1 \text{ and } AX_2 = \lambda X_2.$$

Hence $A(aX_1) = \lambda(aX_1)$ and $A(bX_2) = \lambda(bX_2)$.

$\therefore$ $$A(aX_1 + bX_2) = \lambda(aX_1 + bX_2).$$

$\therefore$ $aX_1 + bX_2$ is an eigen vector corresponding to λ.

Problem: 2

If the eigen values of $A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$ are 2, 2, 3 find the eigen values of A^{-1} and A^2.

Solution:

Since 0 is not an eigen value of A, A is a non singular matrix and hence A^{-1} exists.

Eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{2}, \frac{1}{3}$ and eigen values of A^2 are $2^2, 2^2, 3^2$.

Problem: 3

Find the eigen values of A^5 when $A = \begin{pmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{pmatrix}$

Solution:

The characteristic equation of A is obviously $(3-\lambda)(4-\lambda)(1-\lambda) = 0$

Hence the eigen values of A re 3, 4, 1.

$\therefore$ The eigen values of A^5 are $3^5, 4^5, 1^5$.

Problem: 4

Find the sum and product of the eigen values of the matrix $\begin{pmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{pmatrix}$ without actually finding the eigen values.

Solution:

Let $A = \begin{pmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{pmatrix}$

Sum of the eigen values = trace of $A = 3 + (-2) + 3 = 4$.

Product of the eigen values $= |A|$.

Now, $|A| = \begin{vmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{vmatrix}$

$= 3(-6+4) + 4(3-4) - 4(-1+2)$

$= -6 - 4 - 4 = -14$.

$\therefore$ Product of the eigen values = -14.

Problem: 5

Find the characteristic roots of the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

The characteristic equation of A is given by $|A - \lambda I| = 0$.

$\therefore \quad \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$.

$\therefore \quad (\cos\theta - \lambda)^2 - \sin^2\theta = 0$.

$\therefore \quad (\cos\theta - \lambda)^2 - \sin^2\theta = 0$.

$\therefore \quad (\cos\theta - \lambda - \sin\theta)(\cos\theta - \lambda + \sin\theta) = 0$.

$\therefore \quad [\lambda - (\cos\theta - \sin\theta)][\lambda - (\cos\theta + \sin\theta)] = 0$.

$\therefore$ The two characteristic roots, (the two eigen values) of the matrix are $(\cos\theta - \sin\theta)$ and $(\cos\theta + \sin\theta)$.

Problem: 6

Find the characteristic roots of the matrix $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}$

Solution:

The characteristic equation of A is given by $|A - \lambda I| = 0$.

(i.e.) $\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ -\sin\theta & -\cos\theta - \lambda \end{vmatrix} = 0$

$\therefore \quad -(\cos^2\theta - \lambda^2) - \sin^2\theta = 0$.

$\therefore \quad \lambda^2 - (\cos^2\theta + \sin^2\theta) = 0$.

$\therefore \quad \lambda^2 - 1 = 0$.

$\therefore$ The characteristic roots are 1 and -1.

Problem: 7

Find the sum and product of the eigen values of the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ without finding the roots of the characteristic equation.

Solution:

Sum of the eigen values of A = trace of $A = a_{11} + a_{22}$.

Product of the eigen values of $A = |A| = a_{11}a_{22} - a_{12}a_{21}$.

Problem: 8

Verify the statement that the sum of the elements in the diagonal of a matrix is the sum of the eigen values of the matrix

$$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

Solution:

The characteristic equation of A is $|A - \lambda I| = 0$.

(i.e.) $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$.

(i.e.) $(-2-\lambda)[(1-\lambda)(-\lambda)-12]-2[-2\lambda-6]-3[-4+(1-\lambda)] = 0$.

(i.e.) $(-2-\lambda)(\lambda^2-\lambda-12)+4(\lambda+3)+3(\lambda+3) = 0$.

(i.e.) $-2\lambda^2+2\lambda+24-\lambda^3+\lambda^2+12\lambda+4\lambda+12+3\lambda+9 = 0$.

(i.e.) $-\lambda^3-\lambda^2+21\lambda+45 = 0$.

(i.e.) $\lambda^3+\lambda^2-21\lambda-45 = 0$.

This is a cubic equation in λ and hence it has 3 roots and the three roots are the three eigen values of the matrix.

$$\text{The sum of the eigen values} = -\left(\frac{\text{coefficient of } \lambda^2}{\text{coefficient of } \lambda^3}\right)$$

$$= -1.$$

The sum of the elements on the diagonal of the matrix

$A = -2 + 1 + 0 = -1$.

Hence the result.

Problem: 9

The product of two eigen values of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16.

Find the third eigen value. What is the sum of the eigen values of A?

Solution:

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

Given, product of 2 eigen values (say) λ_1, λ_2 is 16.

$$\therefore \qquad \lambda_1 \lambda_2 = 16$$

We know that the product of the eigen values is $|A|$.

$$\text{(i.e.)} \quad \lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$\text{(i.e.)} \quad 16\lambda_3 = 6(9-1) + 2(-6+2) + 2(2-6)$$

$$= 48 - 8 - 8$$

$$= 32$$

$$\therefore \qquad \lambda_3 = 2$$

$\therefore$ The third eigen value is 2.

Also we know that the sum of the eigen values of A = trace of $A = 6 + 3 + 3 = 12$.

Problem: 10

The product of two eigen values of the matrix $A = \begin{pmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{pmatrix}$ is -12.

Find the eigen values of A.

Solution:

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A. Given product of 2 eigen values, say, λ_1 and λ_2 is -12.

$$\therefore \quad \lambda_1\lambda_2 = -12 \qquad (1)$$

We know that the product of the eigen values is $|A|$.

$$\therefore \quad \lambda_1\lambda_2\lambda_3 = \begin{vmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{vmatrix}$$

$$\text{i.e} \quad 12\lambda_3 = -12$$

$$\therefore \quad \lambda_3 = 1 \qquad (2)$$

Also we know sum of the eigen values = Trace of A.

$$\therefore \quad \lambda_1 + \lambda_2 + \lambda_3 = 2 + 1 - 3 = 0$$

$$\therefore \quad \lambda_1 + \lambda_2 = -1 \quad \text{(using (2))} \qquad (3)$$

Using (3) in (1) we get

$$\lambda_1(-1 - \lambda_1) = -12$$

$$\lambda_1^2 + \lambda_1 - 12 = 0$$

$$(\lambda_1 + 4)(\lambda_1 - 3) = 0$$

$$\therefore \lambda_1 = 3 \quad \text{or} \quad -4$$

Putting $\lambda_1 = 3$ in (1) we get $\lambda_2 = -4$. Or putting $\lambda_1 = -4$ in (1) we get $\lambda_2 = 3$

Thus the three eigen values are $3, -4, 1$.

Problem: 11

Find the sum of the squares of the eigen values of $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$

Solution:

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A. We know that $\lambda_1^2, \lambda_2^2, \lambda_3^2$ are the eigen values of A^2.

$$\therefore \qquad A^2 = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 5 & 38 \\ 0 & 4 & 42 \\ 0 & 0 & 25 \end{pmatrix}$$

$\therefore$ Sum of the eigen values of A^2 = Trace of A^2

$$= 9 + 4 + 25$$

(i.e.) $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 38$

$\therefore$ Sum of the squares of the eigen values of A = 38.

Problem: 12

Find the eigen values and eigen vectors of the matrix.

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

Solution:

The characteristic equation of A is

$$|A - \lambda I| = 0.$$

$$\therefore \qquad \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\therefore$ $(1-\lambda)[(5-\lambda)(1-\lambda)-1] - [(1-\lambda)-3] + 3[1-3(5-\lambda)] = 0.$

$(1-\lambda)(\lambda^2 - 6\lambda + 4) + (\lambda + 2) + 3(3\lambda - 14) = 0.$

$\lambda^2 - 6\lambda + 4 - \lambda^3 + 6\lambda^2 - 4\lambda + \lambda + 2 + 9\lambda - 42 = 0$.

$\therefore$ $-\lambda^3 + 7\lambda^2 - 36 = 0$. Hence $\lambda^3 - 7\lambda^2 + 36 = 0$.

$\therefore$ $(\lambda+2)(\lambda^2 - 9\lambda + 18) = 0$.

Hence $(\lambda + 2)(\lambda - 6)(\lambda - 3) = 0$.

$\therefore$ $\lambda = -2, 3, 6$ are the three eigen values.

Case: (i)

Eigen vector corresponding to $\lambda = -2$.

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to $\lambda = -2$.

Hence $AX = -2X$.

(i.e.) $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_1 \\ -2x_2 \\ -2x_3 \end{pmatrix}$

$\therefore$ $x_1 + x_2 + 3x_3 = -2x_1$

$x_1 + 5x_2 + x_3 = -2x_2$

$3x_1 + x_2 + x_3 = -2x_3$

$\therefore$ $3x_1 + x_2 + 3x_3 = 0$ (1)

$x_1 + 7x_2 + x_3 = 0$ (2)

$3x_1 + x_2 + 3x_3 = 0$ (3)

Clearly this system of three equations reduces to two equations only. From (1) and (2) we get

$\therefore$ $x_1 = -2k; x_2 = 0; x_3 = 2k$.

$\therefore$ It has only one independent solution and can be obtained by giving any value of k say k = 1.

$\therefore$ $(-2,0,2)$ is an eigen vector corresponding to $\lambda = -2$.

Case: ii

Eigen vector corresponding to $\lambda = 3$.

Then $AX = 3X$ gives

$$-2x_1 + x_2 + 3x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

$$3x_1 + x_2 - 2x_3 = 0 .$$

Taking the first 2 equation we get

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} = k(\text{say}) .$$

$\therefore$ $x_1 = -k; x_2 = k; x_3 = -k$.

Taking $k = 1$ (say) $(-1,1,-1)$ is an eigen vector corresponding to $\lambda = 3$.

Case: iii

Eigen vector corresponding to $\lambda = 6$.

We have $AX = 6X$.

Hence $-5x_1 + x_2 + 3x_3 = 0$

$$x_1 - x_2 + x_3 = 0$$

$$3x_1 + x_2 - 5x_3 = 0$$

Taking the first two equations we get

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} = k .$$

$\therefore$ $x_1 = k; x_2 = 2k; x_3 = k$. It satisfies the third equation also.

Taking $k = 1$ (say) (1, 2, 1) is an eigen vector corresponding to $\lambda = 6$.

Problem: 13

Find the eigen values and eigen vectors of the matrix

$$A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

Solution:

The characteristic equation of A is $|A - \lambda I| = 0$.

$$\therefore \quad \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0.$$

$$\therefore \quad (6-\lambda)[(3-\lambda)^2 - 1] + 2[(2\lambda - 6) + 2] + 2(2 - 6 + 2\lambda) = 0.$$

$$\therefore \quad (6-\lambda)(8 + \lambda^2 - 6\lambda) + 4\lambda - 8 + 4\lambda - 8 = 0.$$

$$\therefore \quad 48 + 6\lambda^2 - 36\lambda - 8\lambda - \lambda^3 + 6\lambda^2 + 8\lambda - 16 = 0.$$

$$\therefore \quad -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0.$$

Hence $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$.

$$\therefore \quad (\lambda - 2)(\lambda - 2)(\lambda - 8) = 0.$$

$\therefore$ The eigen values are 2, 2, 8.

We now find the eigen vectors.

Case: i

$\lambda = 2$.

The eigen vector $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is got from $AX = 2X$.

$$\therefore \quad 6x_1 - 2x_2 + 2x_3 = 2x_1$$

$$-2x_1 + 3x_2 - x_3 = 2x_2$$

$$2x_1 - x_2 + 3x_3 = 2x_3$$

$$\therefore \quad 4x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 + x_2 - x_3 = 0$$

$$2x_1 - x_2 + x_3 = 0.$$

The above three equations are equivalent to the single equation $2x_1 - x_2 + x_3 = 0$.

The independent eigen vectors can be obtained by giving arbitrary values to any two of the unknowns x_1, x_2, x_3.

Giving $x_1 = 1; x_2 = 2$ we get $x_3 = 0$.

Giving $x_1 = 3; x_2 = 4$ we get $x_3 = -2$.

$\therefore$ Two independent vectors corresponding to $\lambda = 2$ are (1, 2, 0) and (3, 4, -2).

Case: ii

$\lambda = 8$.

The eigen vector $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is got from $AX = 8X$.

$$\therefore \quad -2x_1 - 2x_2 + 2x_3 = 0 \qquad (1)$$

$$-2x_1 - 5x_2 - x_3 = 0 \qquad (2)$$

$$2x_1 - x_2 - 5x_3 = 0 \qquad (3)$$

From (1) and (2) we get

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6} = k \text{ (say)}.$$

$$\therefore \quad x_1 = 2k; x_2 = -k; x_3 = k.$$

Giving k = 1 we get an eigen vector corresponding to 8 as (2, -1, 1).

Problem: 14

Find the eigen values and eigen vectors of the matrix

$$A = \begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$$

Solution:

The characteristic equation of A is $|A - \lambda I| = 0$.

(i.e.) $$\begin{pmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix} = 0$$

$\therefore \quad (2-\lambda)[-(1-\lambda)(1+\lambda)-3]+2[-(1+\lambda)-1]+2[3-(1-\lambda)]=0$

$\therefore \quad (2-\lambda)(\lambda^2-4)-2(2+\lambda)+2(2+\lambda)=0$

$\therefore \quad 2\lambda^2 - 8\lambda^3 - \lambda^3 + 4\lambda - 4 - 2\lambda + 4 + 2\lambda = 0$

$\therefore \quad -\lambda^3 + 2\lambda^2 + 4\lambda - 8 = 0$

Hence $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

$\therefore \quad (\lambda - 2)(\lambda^2 - 4) = 0$

Hence $(\lambda-2)(\lambda-2)(\lambda+2)=0$

$\therefore \quad \lambda = 2,2,-2$ are the three eigen values.

Case: i

$\lambda = 2$.

Let $X = (x_1, x_2, x_3)$ be an eigen vector corresponding to $\lambda = 2$, X is got from $AX = 2X$.

(i.e.) $$\begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}$$

$\therefore$ The eigen vector corresponding to $\lambda = 2$ is given by the equations

$$2x_1 - 2x_2 + 2x_3 = 2x_1$$

$$x_1 + x_2 + x_3 = 2x_2$$

$$x_1 + 3x_2 - x_3 = 2x_3$$

(i.e.) $-x_2 + x_3 = 0$ (1)

$x_1 - x_2 + x_3 = 0$ (2)

$x_1 + 3x_2 - 3x_3 = 0$ (3)

Taking (1) and (2) we get $\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1} = k$ (say).

$$\therefore \qquad x_1 = 0; x_2 = k; x_3 = k.$$

Taking $k = 1$, we get $(0,1,1)$ as an eigen vector corresponding to $\lambda = 2$.

Case: ii

$$\lambda = -2$$

Corresponding to $\lambda = -2$ we have $AX = -2X$.

$$\therefore \qquad 2x_1 - 2x_2 - 2x_3 = -2x_1$$

$$x_1 + x_2 + x_3 = -2x_2$$

$$x_1 + 3x_2 - x_3 = -2x_3$$

$$\therefore \qquad 2x_1 - x_2 + x_3 = 0$$

$$x_1 + 3x_2 + x_3 = 0$$

$$x_1 + 3x_2 + x_3 = 0$$

$\therefore$ Taking the first two equations we get,

$$\frac{x_1}{-4} = \frac{x_2}{-1} = \frac{x_3}{7} = k \text{ (say)}.$$

$\therefore \quad x_1 = -4k; x_2 = -k; x_3 = 7k$.

Taki8ng k = 1 we get (-4, -1, 7) as an eigen vector corresponding to the eigen value $\lambda = -2$.

Exercises:

1. For each of the following matrices find the characteristic vectors corresponding to each characteristic root.

(a) $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & i & 0 & 0 \\ 2 & 1/2 & -i & 0 \\ 1/3 & -i & \pi & -1 \end{pmatrix}$

2. For what value of k is 3 a characteristic root of $\begin{pmatrix} 3 & 1 & -1 \\ 3 & 5 & -k \\ 3 & k & -1 \end{pmatrix}$

3. Find the characteristic roots and the corresponding characteristic vectors of $A^3 + A^2 + A + I$ if $A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$

4. Prove that if λ is a characteristic root of the matrix A then $\lambda - k$ is a characteristic root of $A - kI$.

5. Show that the characteristic root of a triangular matrix are just the diagonal elements of the matrix.

6. Show that every orthogonal matrix of odd order should have 1 or -1 as a characteristic root.

7. Show that if λ is a characteristic root of an unitary matrix then so is $\frac{1}{\lambda}$.

Answers:

1.
   (a) 0, 3, 15. The corresponding characteristic vectors are (1, 2, 2); (2, -1, -2); (2, -2, 1).

   (b) 1, 1, 5; The characteristic vectors corresponding to 1 is a linear combination of (2, -1, 0) and (1, 0, -1). A characteristic vector corresponding to 5 is (1, 1, 1).

   (c) 1, 1, 1. A characteristic vector is (1, 0, 0).

2. $k = 2$.

UNIT – III

MODERN ALGEBRA

3.0 Groups

Introduction: 3.0.1

Modern Algebra is largely concerned with the study of abstract sets endowed with one or more binary operations. In this chapter we introduce one of the basic algebraic structures known as **groups**. A group is a set with one binary operation defined on it satisfying some natural conditions. The definition of a group is an abstraction of the familiar properties of (Z, +) given below.

(i) Addition is an associative binary operation in Z.

(ii) The element $0 \in Z$ is such that $a + 0 = 0 + a = a$ for all $a \in Z$. Hence 0 is the identity element w.r.t. addition.

(iii) Let $a \in Z$. The element $-a \in Z$ is such that $a + (-a) = (-a) + a = 0$. Hence –a is the inverse of a.

We isolate these properties in the following definition.

Definition and Examples:

Definition: 3.0.2

A non – empty set G together with a binary operation $* : G \times G \to G$ is called a **group** if the following conditions are satisfied.

(i) $*$ is **associative** (i.e.) $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$.

(ii) These exists an element $e \in G$ such that $a * e = e * a = a$ for all $a \in G$. e is called the **identity** element of G.

(iii) For any element a in G there exists an element $a' \in G$ such that $a * a' = a' * a = e$. a' is called the **inverse** of a.

Examples:

1. **Z, Q, R** and **C** are groups under usual addition.
2. The set of all 2 x 2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in R$ is a group under matrix addition.

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the identity element and $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ is the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

3. The set of all 2 x 2 non – singular matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d, \in R$ is a group under matrix multiplication.

We know that matrix multiplication is associative.

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity element.

The inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ where $|A| = ad - bc \neq 0$.

4. **N** is not a group under usual addition since there is no element $e \in N$ such that $x + e = x$.

5. The set **E** of all even integers under usual addition is a group. For $a, b \in E \Rightarrow a + b \in E$ Therefore usual addition is a binary operation in **E**.

$0 \in E$ is the identity element.

If $a \in E, -a \in E$ is the inverse of a.

6. **Q*** and **R*** under usual multiplication are groups. 1 is the identity element and the inverse of a is 1/a.

7. Q^+ is a group under usual multiplication. For $a, b \in Q^+ \Rightarrow ab \in Q^+$. Therefore usual multiplication is a binary operation in Q^+.

$1 \in Q^+$ is the identity element.

If $a \in Q^+, (1/a) \in Q^+$ is the inverse of a.

8. Z under the usual multiplication is not a group. $1 \in Z$ is the identity element. However any element other than 1 and -1 does not have an inverse.

9. Let $G = \{1,-1\}$. G is a group under usual multiplication. 1 is the identity element. The inverse of each element is itself. The Cayley table for this group is

*	1	−1
1	1	−1
−1	−1	1

10. Let $G = \{1, i, -1, -i\}$. G is a group under usual multiplication. The identity element is 1. The inverse of $1, i, -1$ and $-i$ are $1, -i, -1$ and i respectively.

The Cayley table for this group is given by

*	1	i	−1	i
1	1	i	−1	−i
i	i	−1	−i	1
−1	−1	−i	1	i
−i	−i	1	i	−1

11. Let $G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

G is a group under matrix multiplication. (Construct the Cayley table for this group).

Since $a^2 + b^2 \neq 0, 1/x \in C^*$ and is the inverse of x. Hence C^* is a group under usual multiplication.

12. Let $G = \{z / z \in C \text{ and } |z| = 1\}$. G is a group under usual multiplication.

Proof:

Let $z_1, z_2 \in G$.

Then $|z_1| = |z_2| = 1$.

$\therefore |z_1 z_2| = |z_1||z_2| = 1$ and hence $z_1 z_2 \in G$.

We know that usual multiplication of complex numbers is associative.

Also $1 = 1 + i0 \in G$ and is the identity element.

Now, let $z \in G$. Then $|z| = 1$.

Hence $|1/z| = 1/|z| = 1$.

$\therefore$ $1/z \in G$ and is the inverse of z.

Hence G is a group.

13. The set of all n^{th} roots of unity with usual multiplication is a group.

Proof:

Let $w = \cos(2\pi/n) + i\sin(2\pi/n)$. Then the n^{th} roots of unity are given by $1, \omega, \omega^2, \ldots\ldots, \omega^{n-1}$.

Let $G = \{1, \omega, \omega^2, \ldots\ldots, \omega^{n-1}\}$.

We know that $\omega^n = 1, \omega^{n+1} = \omega$ etc.

Let $\omega^r, \omega^s \in G$. Let $r + s = qn + t$ where $0 \le t < n$.

$\therefore \qquad \omega^r \omega^s = \omega^{r+s} = \omega^{qn+t} = (\omega^n)^q \omega^t = \omega^t \in G$.

We know that usual multiplication of complex number is associative.

$1 \in G$ is the identity element.

Inverse of ω^r is ω^{n-r}. Hence G is a group.

14. Let $G = \{a + b\sqrt{2} / a, b \in Z\}$. Then G is a group under usual addition.

Proof:

Let $a + b\sqrt{2}$ and $c + d\sqrt{2} \in G$. Then $(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2} \in G$. We know that usual addition is associative. $0 = 0 + 0\sqrt{2} \in G$ is the identity element. $-a - b\sqrt{2}$ is the inverse of $a + b\sqrt{2}$. Hence G is group.

15. Let G be the set of all real numbers except -1. Define $*$ on G by $a * b = a + b + ab$. Then (G, $*$) is a group.

Proof:

Let $a, b \in G$. Then $a \neq -1$ and $b \neq -1$. We claim that $a + b \neq -1$.

Suppose $a * b = -1$.

Then $a + b + ab = -1$ so that $a + b + ab + 1 = 0$. i.e., $(a + 1)(b + 1) = 0$ so that either $a = -1$ or $b = -1$ which is a contradiction. Hence $a * b \neq -1$ and thus $*$ is a binary operation on G.

$*$ is associative, for $a * (b * c)$

$= a * (b + c + bc)$

$= a + (b + c + bc) + a(b + c + bc)$

$= a + b + c + bc + ab + ac + abc.$

Also $(a * b) * c = (a + b + ab) * c$

$= a + b + ab + c + (a + b + bc)c$

$= a + b + c + ab + ac + bc + abc.$

Hence $a * (b * c) = (a * b) * c$.

0 is the identity, for $a * 0 = a + 0 + 0a = a$ and $0 * a = 0 + a + 0a = a$.

Now, let a' be such that $a * a' = 0$. Hence $a + a' + aa' = 0$ so that $a' = -a/(1+a)$.

Since $a \neq -1$, we have $a' \in R - \{-1\}$

Also $a' * a = \frac{-a}{1+a} * a$

$= \frac{-a}{1+a} + a + \frac{-a^2}{1+a} = 0.$

Hence a' is the inverse of a.

Thus G is a group.

16. In R^* we define $a * b = (1/2)ab$. Then $(R^*, *)$ is a group.

Proof:

Obviously $*$ is a binary operation in R^*.

Let $a, b, c \in R^*$.

Then $(a * b) * c = [(1/2)ab] * c = (1/4)abc = a * (b * c)$. Hence $*$ is associative.

Let $e \in R^*$ be such that $a * e = a$.

$\therefore$ $(1/2)ae = a$ and hence $e = 2$.

$\therefore$ $2 * a = a * 2 = a$. Hence 2 is the identity.

Let $a \in R^*$. Let $b \in R^*$ be such that $a * b = 2$. Then $(1/2)ab = 2, (i.e) b = 4/a$.

$\therefore$ $a * (4/a) = 1/2(4/a)a = 2$ i.e., (4/a) is the inverse of a.

Thus $(R^*, *)$ is a group.

17. Let G denote the set of all matrices of the form $\begin{pmatrix} x & x \\ x & x \end{pmatrix}$ where $x \in R^*$

Then G is group under matrix multiplication.

Proof:

Let $A, B \in G$. Let $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$ and $B = \begin{pmatrix} y & y \\ y & y \end{pmatrix}$.

Then $AB = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in G$.

We know that matrix multiplication is associative.

Let $E = \begin{pmatrix} e & e \\ e & e \end{pmatrix}$ be such that $AE = A$.

$$\therefore \quad \begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} 2xe & 2xe \\ 2xe & 2xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}.$$

$\therefore$ $2xe = x$. Hence $e = 1/2$.

Hence $E = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ is the identity element of G.

Let $\begin{pmatrix} y & y \\ y & y \end{pmatrix}$ be the inverse of $\begin{pmatrix} x & x \\ x & x \end{pmatrix}$. Then $\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} y & y \\ y & y \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

$\therefore \quad \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$

$\therefore \quad 2xy = 1/2$. Hence $y = x/4$.

$\therefore \quad$ Inverse of $\begin{pmatrix} x & x \\ x & x \end{pmatrix}$ is $\begin{pmatrix} x/4 & x/4 \\ x/4 & x/4 \end{pmatrix}.$

Hence G is group.

18. Let $G = \{(a,b) / a \in R^*, b \in R\}$. Then G is a group under the operation $*$ defined by $(a,b)*(c,d) = (ac, bc+d)$.

Proof:

Clearly $*$ is a binary operation defined on G. Now, let $x = (a,b), y = (c,d)$ and $z = (e,f)$ be three elements in G.

Then $x*(y*z) = (a,b)*(ce, de+f)$

$= (ace, bce, +de+f)$

Also $(x*y)*z = (ac, bc+d)*(e,f)$

$= (ace, bce+de+f)$.

$\therefore \quad x*(y*z) = (x*y)*z$, so that $*$ is associative.

Now, to find the identity element in G suppose

$(a,b)*(c,d) = (a,b)$.

Then $(ac, bc+d) = (a,b)$.

$\therefore \quad ac = a$ and $bc + d = b$.

Hence $c = 1$ and $d = 0$.

Thus $(a,b)*(1,0) = (1,0)*(a,b) = (a,b)$

$\therefore \quad (1,0) \in G$ is identity element.

Now suppose $(a,b)*(a',b') = (1,0)$.

$\therefore \quad (aa', ba'+b') = (1,0)$.

Hence $aa' = 1$ and $ba'+b' = 0$

$\therefore \quad a' = 1/a$ and $b' = -b/a$.

Thus $(a,b)*(1/a, -b/a) = (1/a, -b/a)*(a,b) = (1,0)$.

Hence $(1/a, -b/a)$ is the inverse of (a, b)

Hence (G, $*$) is a group.

19. In N we define $a * b = a$. Then (N, $*$) is not a group.

Proof:

Clearly $*$ is an associative binary operation on N. However, there is no element $e \in N$ such that $e*a = a$ for all $a \in N$. Hence there is no identity element in (N, $*$).

Hence (N, $*$) is not a group.

In the group (**Z**, +), the binary operation is commutative whereas in the group of 2 x 2 non – singular matrices, the matrix multiplication is not commutative.

Definition: 3.0.3

A group G is said to be **abelian** if $ab = ba$ for all $a, b \in G$. A group which is not abelian is called a **non – abelian** group.

Examples:

1. **Z, Q, R** and **C** under usual addition are abelian groups.
2. $(\wp(S), \Delta)$ is an abelian group since $A \Delta B = B \Delta A$ for all $A, B \in \wp(S)$.
3. Let B(R) denote the set of all bijections from R. Then B(R) is a group under the composition of functions. This group is non – abelian.

   For, consider

   $f : R \to R$ given by $f(x) = x + 3$ and

   $g : R \to R$ given by $g(x) = 2x$.

   Clearly f and g are bijections.

   Now $(f \circ g)(x) = f[g(x)] = f(2x) = 2x + 3$ and

   $(g \circ f)(x) = g[f(x)] = g(x + 3) = 2x + 6$.

   Hence $f \circ g \neq g \circ f$.

   Hence **B(R)** is non – abelian.
4. $(Z_n, \oplus)$ is an abelian group.
5. Consider the group given in example 18 of 2.0.1 Here $(2,3) * (4,5) = (8,17)$ and $(4,5) * (2,3) = (8,13)$. Thus $(2,3) * (4,5) \neq (4,5) * (2,3)$ so that this group is non – abelian.

3.0.4 Elementary Properties of a Group:

Theorem: 3.1

Let G be a group. Then

(i) Identity element of G is unique.

(ii) For any $a \in G$, the inverse of a is unique.

Proof:

(i) Let e and e' be two identity elements of G. Then $ee' = e'$ (since e' is an identity). Hence e = e'.

(ii) Let a' and a'' be two inverse of a. Hence $aa' = a'a = e$ and $aa'' = a''a = e$.

$\therefore \quad a' = a'e = a'(aa'') = (a'a)a'' = ea'' = a''$.

Note:

We denote the inverse of a by a^{-1}.

Theorem: 3.2

In a group the left and right cancellation laws holds (ie) $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow b = c$.

Proof:

$$ab = ac \quad \Rightarrow a^{-1}(ab) = a^{-1}(ac)$$

$$\Rightarrow (a^{-1}a)b = (a^{-1}a)c$$

$$\Rightarrow eb = ec$$

$$\Rightarrow b = c.$$

Similarly we can prove that $ba = ca \Rightarrow b = c$.

Theorem: 3.3

Let G be a group and $a, b \in G$. Then the equations $ax = b$ and $ya = b$ have unique solutions for x and y in G.

Proof:

Consider $a^{-1}b \in G$

Then $a(a^{-1}b) = (aa^{-1})b = eb = b$.

Hence $a^{-1}b$ is a solution of $ax = b$.

Now, to prove the uniqueness, let x_1 and x_2 be two solutions of $ax = b$. Then $ax_1 = b$ and $ax_2 = b$.

$\therefore \quad ax_1 = ax_2$ which implies $x_1 = x_2$.

$\therefore \quad x = a^{-1}b$ is the unique solution for $ax = b$.

Similarly we can prove that $y = ba^{-1}$ is the unique solution of the equation $ya = b$.

Theorem: 3.4

Let G be a group. Let $a, b \in G$. Then $(ab)^{-1} = b^{-1}a^{-1}$ and $(a^{-1})^{-1} = a$.

Proof:

$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = aea^{-1} = aa^{-1} = e$.

Similarly $(b^{-1}a^{-1})(ab) = e$.

Hence $(ab)^{-1} = b^{-1}a^{-1}$.

Proof of the second part is obvious.

Corollary:

If $a_1, a_2, \ldots\ldots, a_n \in G$ then $(a_1, a_2, \ldots\ldots, a_n)^{-1} = a_n^{-1}a_{n-1}^{-1}\ldots\ldots a_1^{-1}$

Definition: 3.1.4

Let G be a group and $a \in G$. For any positive integer n, we define $a^n = aa\ldots\ldots a$ (a written n times).

Clearly $(a^n)^{-1} = (aa\ldots\ldots a)^{-1}$

$$= (a^{-1}a^{-1}\ldots\ldots a^{-1}) = (a^{-1})^n.$$

We now define $a^{-n} = (a^{-1})^n = (a^n)^{-1}$.

Finally we define $a^0 = e$. Thus a^n is defined for all integers n.

Note:

When the binary operation on G is "+", we denote $a + a + \ldots\ldots + a$ (a written n times) as na.

Theorem: 3.5

(i) $a^m a^n = a^{m+n}, m, n \in Z$.

(ii) $(a^m)^n = a^{mn}, m, n \in Z$.

Note:

In the additive notation, the above results take the form $ma + na = (m+n)a$ and $n(ma) = (nm)a$.

Problems:

Problem: 1

Show that in a group $G, x^2 = x$ if and only if $x = e$.

Solution:

Clearly $e^2 = ee = e$.

Conversely, let $x^2 = x$.

Then $xx = xe$. Hence by cancellation law $x = e$.

Note:

An element $a \in G$ is called **idempotent** if $a^2 = a$. Thus we have shown that in a group G, the identity element is the only idempotent element.

Problem: 2

In an abelian group $(ab)^2 = a^2b^2$.

Solution:

$$(ab)^2 = (ab)(ab) = a(ba)b = a(ab)b = (aa)(bb) = a^2b^2.$$

Note:

In general for any positive integer n, $(ab)^n = a^n b^n$ (prove by using induction).

Problem: 3

Let G be a group such that $a^2 = e$ for all $a \in G$. Then G is abelian.

Solution:

$a^2 = e \Rightarrow aa = e \Rightarrow a = a^{-1}$.

Now, $ab = (ab)^{-1} = b^{-1}a^{-1} = ba$.

Hence G is abelian.

Problem: 4

Let G be a group in which $(ab)^m = a^m b^m$ for three consecutive integers and for all $a, b \in G$. Then G is abelian.

Solution:

Let $a, b \in G$.

Let $(ab)^m = a^m b^m : (ab)^{m+1} = a^{m+1} b^{m+1}$ and $(ab)^{m+2} = a^{m+2} b^{m+2}$.

Now, $(ab)^{m+1} = a^{m+1} b^{m+1}$

$\Rightarrow (ab)^m (ab) = (a^m a)(b^m b)$

$\Rightarrow (a^m b^m)(ab) = (a^m a)(b^m b)$

$\Rightarrow b^m a = ab^m$ (by cancellation law) (1)

Similarly $(ab)^{m+2} = a^{m+2} b^{m+2}$

$\Rightarrow b^{m+1} a = ab^{m+1}$

$\Rightarrow b^m ba = ab^m b$

$\Rightarrow b^m ba = b^m ab$(by(1))

$\Rightarrow ba = ab$ (by cancellation law)

$\therefore$ G is abelian.

Problem: 5

Let $(H, .)$ and $(K, *)$ be groups. We define a binary operation $\square$ on H x K by $(h_1, k_1) \square (h_2, k_2) = (h_1 h_2, k_1 * k_2)$.

Then H x K is a group.

Note:

H x K is called the **direct product** of H and K.

Solution:

First we shall prove that $\square$ is associative.

Let $(h_1,k_1),(h_2,k_2),(h_3,k_3) \in H \times K$.

$[(h_1.k_1)\square(h_2,k_2)]\ \square(h_3,k_3)$

$= (h_1h_2, k_1 * k_2) \square (h_3,k_3)$

$= ((h_1h_2)h_3, (k_1 * k_2) * k_3)$

$= (h_1(h_2h_3), k_1 * (k_2 * k_3))$

$= (h_1,k_1) \square (h_2h_3, k_2 * k_3)$

$= (h_1,k_1) \square [(h_2,k_2) \square (h_3,k_3)]$.

Let e, e_1 be the identities of the groups H and K respectively. Clearly (e,e_1) is the identity element in H x K. Also (h^{-1},k^{-1}) is the inverse of (h, k).

Hence H x K is a group.

3.1 Subgroups:

Definition: 3.1.1

Let G be a set with a binary operation $*$ defined on it. Let $S \subseteq G$. If for each $a, b \in S, a * b$ (computed in G) is in S, we say that S is **closed** with respect to the **binary operation** "$*$"

Examples:

1. (**Z**, +) is a group. The set **E** of all even integers is closed under + and further (**E**, +) is itself a group.

2. The set of G of all non – singular 2 x 2 matrices from a group under matrix multiplication. Let H be the set of all matrices of the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. H is subset of G. Also H itself is a group under matrix multiplication.

Definition: 3.1.2

A subset H of group G is called a **subgroup** of G if H forms a group with respect to the binary operation in G.

Examples:

1. Let G be any group. Then {e} and G are subgroup of G. They are called **improper subgroups of G.**
2. (**Q**, +) is a subgroup of (**R**, +) and (**R**, +) is a subgroup of (**C**, +).
3. In $(Z_8, \oplus)$, let $H_1 = \{0,4\}$ and $H_2 = \{0,2,4,6\}$. The Cayley tables for H_1 and H_2 are given by

$\oplus$	0	4
0	0	4
4	4	0

$\oplus$	0	2	4	6
0	0	2	4	6
2	2	4	6	0
4	4	6	0	2
6	6	0	2	4

It is easily seen that H_1 and H_2 are closed under $\oplus$ and $(H_1, \oplus)$ and $(H_2, \oplus)$ are groups. Hence H_1 and H_2 are subgroups of Z_8.

4. {1, -1} is a subgroup of (R*, .).
5. $\{1, i, -1, -i\}$ is a subgroup of (C*, .).
6. For any integer n we define $nZ = \{nx / x \in Z\}$. Then $(nZ, +)$ is a subgroup of $(Z, +)$.

   For, let $a, b \in nZ$. Then $a = nx$ and $b = ny$ where $x, y \in Z$.
   Hence $a + b = n(x + y) \in nZ$: Hence nZ is closed under +.
   $0 \in nZ$ is the identity element.
   Inverse of nx is $-nx = n(-x) \in nZ$.
   Hence $(nZ, +)$ is a group.
7. In the symmetric group $S_3, H_1 = \{e, p_1, p_2\} : H_2 = \{e, p_3\} : H_3 = \{e, p_4\}$; and $H_4 = \{e, p_5\}$ are subgroups.

8. A_n is a subgroup of S_n.

9. The set of permutations $\{e, p_1, p_2, p_3\}$ where

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}; \quad p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix};$$

$$p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}; \quad p_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

is a subgroups of S_4.

Note:

In all the above the examples we see that the identity element in the subgroup is the same as the identity element of the group.

Theorem: 3.6

Let H be a subgroup of G. Then

(a) The identity element of H is the same as that of G.

(b) For each $a \in H$ the inverse of a in H is the same as the inverse of a in G.

Proof:

(a) Let e and e' be the identities of G and H respectively.

Let $a \in H$. Now,

$e'a = a$ (since e' is the identity of H)

$= ea$ (since e is the identity of G and $a \in G$)

$\therefore \quad e'a = ea$.

$\therefore \quad e' = e$ (by cancellation law).

(b) Let a' and a'' be the inverse of a in G and H respectively. Since by (a), G and H have the same identity element e, we have $a'a = e = a''a$. Hence by cancellation law $a' = a''$.

Theorem: 3.7

A subset H of a group G is a subgroup of G iff

(i) It is closed under the binaty operation in G.

(ii) The identity e of G is in H.

(iii) $a \in H \Rightarrow a^{-1} \in H$.

Proof:

Let H be a subgroup of G. The result follows immediately from theorem 3.6.

Conversely let H be a subset of G satisfying conditions (i), (ii) and (iii). Then, obviously H itself is a group with respect to the binary operation in G.

Therefore H is a subgroup of G.

Theorem: 3.8

A non – empty subset H of a group G is a subgroup of G iff $a, b \in H \Rightarrow ab^{-1} \in H$.

Proof:

Let H be a subgroup of G. Then $a, b \in H \Rightarrow a, b^{-1} \in H \Rightarrow ab^{-1} \in H$.

Conversely let H be a non – empty subset of G such that $a, b \in H \Rightarrow ab^{-1} \in H$.

Since $H \neq \Phi$, there exists an element $a \in H$.

Hence $aa^{-1} \in H$. Thus $e \in H$

Also since $e, a \in H, ea^{-1} \in H$. Hence $a^{-1} \in H$. Now, let $a, b \in H$. Then $a, b^{-1} \in H$.

Hence $a(b^{-1})^{-1} = ab \in H$. Thus H is closed under the binary operation in G.

Hence by theorem 3.7 H is a subgroup of G.

Note:

If the operation is + then H is a subgroup of G iff $a, b \in H \Rightarrow a - b \in H$.

Theorem: 3.9

Let H be a non – empty finite subset of G. If H is closed under the operation in G then H is a subgroup of G.

Proof:

Let $a \in H$.

Since H is closed $a, a^2, a^3, \ldots\ldots\ldots, a^n \ldots\ldots$ are all elements of H.

But since H is finite the elements $a, a^2, a^3, \ldots\ldots$ cannot all be distinct.

Hence let $a^r = a^s, r < s$. Then $a^{s-r} = e \in H$.

Now, let $a \in H$. We have proved that $a^n = e$ for some n. Hence $aa^{n-1} = e$. Hence $a^{-1} = a^{n-1} \in H$.

Thus H is a subgroup of G.

Note:

The above theorem is not true if H is infinite. For example, **N** is an infinite subset of (**Z**, +) and **N** is closed under addition. However **N** is not a subgroup of (**Z**, +).

Theorem: 3.10

If H and K are subgroups of a group G then is $H \cap K$ is also a subgroup of G.

Proof:

Clearly $e \in H \cap K$ and hence $H \cap K$ is non – empty. Now let $a, b \in H \cap K$. Then $a, b \in H$ and $a, b \in K$. Since H and K are subgroups of G, $ab^{-1} \in H$ and $ab^{-1} \in K$.

$\therefore$ $ab^{-1} \in H \cap K$. Hence by theorem 2.8, $H \cap K$ is a subgroup of G.

Note:

1. It can be similarly proved that the intersection of any number of subgroups of G is again a subgroup of G.
2. The union of two subgroups of a group nee not be a subgroup. For example, **2 Z** and **3 Z** are subgroups of (**Z**, +) but $2Z \cup 3Z$ is not a subgroup of Z since $3,2 \in 2Z \cup 3Z$ but $3 + 2 = 5 \notin 2Z \cup 3Z$.

Theorem: 3.11

The union of two subgroups of a group G is a subgroup iff one is contained in the other.

Proof:

Let H and K be two subgroups of G such that one is contained in the other. Hence either $H \subseteq K$ or $K \subseteq H$.

$\therefore$ $H \cup K = K$ or $H \cup K = H$. Hence $H \cup K$ is a subgroup of G.

Conversely, suppose $H \cup K$ is a subgroup of G. We claim that $H \subseteq K$ or $K \subseteq H$.

Suppose that H is not contained in K and K is not contained in H. Then there exist elements a, b such that

$a \in H$ and $a \notin K$ (1)

$b \in K$ and $b \notin H$ (2)

Clearly, $a, b \in H \cup K$. Since $H \cup K$ is a subgroup of $G, ab \in H \cup K$. Hence $ab \in H$ or $ab \in K$.

Case: (i)

Let $ab \in H$. Since $a \in H, a^{-1} \in H$.

Hence $a^{-1}(ab) = b \in H$ which is a contradiction to (2).

Case: (ii)

Let $ab \in K$. Since $b \in K, b^{-1} \in K$.

Hence $(ab)b^{-1} = a \in K$ which is a contradiction to (1). Hence our assumption that H is not contained in K and K is not contained in H is false.

$\therefore$ $H \subseteq K$ or $K \subseteq H$.

Definition: 3.1.3

Let A and B be two subsets of a group G. We define $AB = \{ab / a \in A, b \in B\}$.

Note:

If A and B are two subgroups of G. AB need not be a subgroup of G.

In S_3, consider $A = \{e, p_3\}$ and $B = \{e, p_4\}$. Clearly A and B are subgroups of S_3.

Also $AB = \{ee, ep_4, ep_3, p_3p_4\} = \{e, p_4, p_3, p_2\}$.

Now $p_4p_2 = p_5 \notin AB$.

Hence AB is not a subgroup of S_3.

Theorem: 3.12

Let A and B be two subgroups of a group G. Then AB is a subgroup of G iff $AB = BA$.

Proof:

Let AB be a subgroup of G.

We claim that AB = BA.

Let $x \in AB$. Since AB is a subgroup of G, $x^{-1} \in AB$.

Let $x^{-1} = ab$ where $a \in A$ and $b \in B$.

$\therefore \quad x = (ab)^{-1} = b^{-1}a^{-1}$.

Since A and B are subgroups of G, $a^{-1} \in A$ and $b^{-1} \in B$.

$\therefore x \in BA$. Hence $AB \subseteq BA$ (1)

Now, let $x \in BA$. Then $x = ba$ where $b \in B$ and $a \in A$.

$\therefore \quad x^{-1} = (ab)^{-1} = a^{-1}b^{-1} \in AB$.

Now, since AB is a subgroup and $x^{-1} \in AB$, we have $x \in AB$.

$\therefore \quad BA \subseteq AB$ (2)

From (1) and (2) we get AB = BA.

Conversely, let AB = BA. We claim that AB is a subgroup of G. Clearly $e \in AB$ and hence AB is non-empty. Now let $x, y \in AB$. Then $x = a_1b_1$ and $y = a_2b_2$ where $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

$\therefore \quad xy^{-1} = (a_1b_1)(a_2b_2)^{-1} = a_1b_1b_2^{-1}a_2^{-1}$.

Now, $b_2^{-1}a_2^{-1} \in BA$. Since $BA = AB$, $b_2^{-1}a_2^{-1} \in AB$.

$\therefore \quad b_2^{-1}a_2^{-1} = a_3b_3$ where $a_3 \in A$ and $b_3 \in B$.

$\therefore \quad xy^{-1} = a_1b_1a_3b_3$.

Now $b_1a_3 \in BA$. Since $BA = AB, b_1a_3 \in AB$.

$\therefore \quad b_1a_3 = a_4b_4$ where $a_4 \in A$ and $b_4 \in B$.

$\therefore \quad xy^{-1} = a_1(a_4b_4)b_3 = (a_1a_4)(b_4b_3) \in AB$.

$\therefore$ AB is subgroup of G.

Corollary:

If A and B are subgroups of an abelian group G, then AB is a subgroup of G.

Proof:

Let $x \in AB$. Then $x = ab$ where $a \in A$ and $b \in B$. Since G is abelian, $ab = ba$.

$\therefore \quad x \in BA$. Hence $AB \subseteq BA$.

Similarly $BA \subseteq AB$.

$\therefore \quad AB = BA$.

Hence AB is a subgroup of G.

Problems:

Problem: 1

Let $a \in R^*$. Let $H = \{a^n / n \in Z\}$. Then H is a subgroup of R*.

Solution:

Clearly H is non – empty.

Now, let $x, y \in H$.

Then $x = a^s$ and $y = a^t$ where $s, t \in Z$.

$\therefore \quad xy^{-1} = a^s (a^t)^{-1} = a^{s-1} \in H$.

Hence H is a subgroup of R*.

Problem: 2

Let H denote the set of all permutations in S_n fixing the symbol 1. Then H is a subgroup of S_n:

Solution:

Clearly $e \in H$ and hence H is non – empty. Let $\alpha, \beta \in H$. Then α and β fix the symbol 1. Now β fixes the symbol $1 \Rightarrow \beta^{-1}$ fixes the symbol 1. Hence $\alpha\beta^{-1}$ fixes the symbol 1. Hence $\alpha\beta^{-1} \in H$.

Thus H is a subgroup of S_n.

Problem: 3

Let G be the set of all 2 x 2 matrices with entries from **R**. Then G is a group under matrix addition.

Let $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in R \right\}$. Then H is a subgroup of G.

Solution:

Let $A, B \in H$.

Then $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$

Now, $A - B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a-c & 0 \\ 0 & b-d \end{pmatrix} \in H$.

Hence H is a subgroup of G.

Problem: 4

Let G be a group.

Let $H = \{a / a \in G \text{ and } ax = xa \text{ for all } x \in G\}$. (ie) H is the set of all elements which commute with every other element. Show that H is a subgroup of G.

Solution:

Clearly $ex = xe = x$ for all $x \in G$.

Hence $e \in H$, so that H is non empty.

Now, let $a, b \in H$.

Then $ax = xa$ and $bx = xb$ for all $x \in G$.

Now,

$$bx = xb \Rightarrow b^{-1}(bx)b^{-1} = b^{-1}(xb)b^{-1}$$

$$\Rightarrow (b^{-1}b)xb^{-1} = b^{-1}x(bb^{-1})$$

$$\Rightarrow exb^{-1} = b^{-1}xe$$

$$\Rightarrow xb^{-1} = b^{-1}x. \qquad (1)$$

$\therefore \quad (ab^{-1})x = a(b^{-1}x)$

$= a(xb^{-1}) \qquad$ (by (1))

$= (ax)b^{-1}$

$= (xa)b^{-1} \qquad$ (since $ax = xa$)

$= x(ab^{-1})$.

Thus ab^{-1} commutes with every element of G.

$\therefore \quad ab^{-1} \in H$ and hence H is a subgroup of G.

Note:

The above subgroup of G is called the **center of G** and is denoted by Z(G).

Problem: 5

Let G be a group and let a be a fixed element of G.

Let $H_a = \{x / x \in G \text{ and } ax = xa\}$

(ie) H_a is the set of all elements in G which commute with a.

Show that H_a is a subgroup of G.

Solution:

Clearly $ea = ae = a$.

Hence $e \in H_a$ so that H_a is non – empty.

Now, let $x, y \in H_a$.

Then $ax = xa$ and $ay = ya$.

Now, $ay = ya \Rightarrow y^{-1}a = ay^{-1}$. (as in the previous problem) (1)

Hence $a(xy^{-1}) = (ax)y^{-1}$

$= (xa)y^{-1}$ (since $ax = xa$)

$= x(ay^{-1})$

$= x(y^{-1}a)$ (by (1))

$= (xy^{-1})a$.

Hence xy^{-1} commutes with a.

$\therefore$ $xy^{-1} \in H_a$ and hence H_a is a subgroup of G.

Note:

H_a is called the **normaliser** of a in G.

Exercises:

1. Show that $\{a+bi/a,b \in Z\}$ is a subgroup of (**C**, +).
2. Determine which of the following are subgroups of (**C**, +).

(a) R

(b) $\{a+b\sqrt{-5}/a,b \in N\}$

(c) $\{z/|z|=a\}$

(d) $\{z/\text{real part of } z \text{ is } 0\}$

(e) $\{1,i,-1,-i\}$.

3. Let G_1 and G_2 be two groups. Let e_1 and e_2 be the identity elements of G_1 and G_2 respectively. Let $G_1 \times G_2$ be the direct product of these groups. Let $H=\{(e_1,y)/y \in G_2\}$ and $K=\{(x,e_2)/x \in G_1\}$. Show that H and K are subgroups of $G_1 \times G_2$.
4. Let G be a group and let H be the centre of G. Show that H = G iff G is abelian.
5. Show that the centre of S_3 is $\{e\}$.

   (Hint: For each $a \in S_3$ and $a \neq e$, find another element $b \in S_3$ such that $ab \neq ba$).
6. Show that a proper subgroup of a non – abelian group can be abelian.

   (Hint: Consider any proper subgroup of S_3).
7. Show that any subgroup of an abelian group is abelian.
8. Let S and N be subgroups of G such that $S \cap N = \{e\}$ and $S \cup N = G$. Prove that either $S = G$ or $N = G$.
9. Find as many subgroups as you can in

(a) V_4

(b) The group of symmetries of a square.

(c) Z_6

(d) Z

3.2 Permutation Groups:

The set of all bijections B(A) from a to itself is a group under the composition of functions. In this section we make detailed study of this group when A is **finite**.

Definition: 3.2.1

Let A be a finite set. A bijection from to itself is called a **permutation** of A.

For example, if $A = \{1,2,3,4\}$ $f : A \to A$ given by $f(1) = 2$, $f(2) = 1, f(3) = 4$ and $f(4) = 3$ is permutation of A. We shall write this permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$.

An element in the bottom row is the image of the element just above it in the upper row.

Note:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 1 & 2 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

Hence any rearrangement of columns in a permutation is immaterial.

Definition: 3.2.2

Let A be a finite set containing n element. The set of all permutations of A is clearly group under the composition of functions. This group is called the **symmetric group** of degree n and denoted by S_n.

Example:

Let $A = \{1,2,3\}$. Then S_3 consists of

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}; p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}; p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}; p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix};$$

$$p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}; p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

In this group, e is the identity element. We now compute the product p_1p_2.

$$\begin{array}{lccc} & 1 & 2 & 3 \\ p_1: & \downarrow & \downarrow & \downarrow \\ & 2 & 3 & 1 \\ p_2: & \downarrow & \downarrow & \downarrow \\ & 1 & 2 & 3 \end{array} \qquad \text{Hence } p_1p_2: \begin{array}{ccc} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{array}$$

so that $p_1p_2 = e$.

$$\text{Now, } p_1p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = p_5.$$

Similarly we can compute all the other products and Cayley table for this group is given by

$\circ$	e	p_1	p_2	p_3	p_4	p_5
e	e	p_1	p_2	p_3	p_4	p_5
p_1	p_1	p_2	e	p_4	p_5	p_3
p_2	p_2	e	p_1	p_5	p_3	p_4
p_3	p_3	p_5	p_4	e	p_2	p_1
p_4	p_4	p_3	p_5	p_1	e	p_2
p_5	p_5	p_4	p_3	p_2	p_1	e

Thus S_3 is a group containing $3! = 6$ elements.

Remarks:

1. In definition 1.2.15 we have defined the composition $g \circ f$ of two functions f and g by $(g \circ f)(x) = g[f(x)]$.

   Hence to find the image of any element x under $g \circ f$, we first apply f and then g. However in forming the product of two **permutations** p_1 and p_2 we adopt a different convention. To find the image of x under the product p_1p_2, we first apply p_1 and then p_2.

2. In $S_3, p_1p_2 = p_2p_1 = e$ so that the inverse of p_1 is p_2. In general the inverse of a permutation can be obtained by interchanging the rows of the permutation.

For example, if $p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 5 & 1 \end{pmatrix}$ then the inverse of p is the permutation given by $p^{-1} = \begin{pmatrix} 3 & 4 & 2 & 5 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 2 & 4 \end{pmatrix}$

3. In $S_3, p_1p_4 = p_5$ and $p_4p_1 = p_3$. Hence $p_1p_4 \neq p_4p_1$ so that S_3 in non – abelian.

4. The symmetric group S_n contains n! elements, for let $A = \{1,2,\ldots\ldots, n\}$. Any permutation on A is given by specifying the image of each element. The image of 1 can be chosen in n different ways. Since the image of two is different from the image of 1, it can be chosen in (n – 1) different ways and so on.

Hence the number of permutations of A is n(n – 1)......... 2.1 = n!
so that the number of elements in S_n is n!.

Definition: 3.2.3

Let G be a finite group. Then the number of elements in G is called the order of G and is denoted by $|G|$ or $\circ(G)$.

Exercises:

1. Compute $\alpha\beta, \beta\alpha$ and α^{-1} if

(a) $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$

$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

(b) $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix}$

$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}$

(c) $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 7 & 2 & 5 & 6 & 1 \end{pmatrix}$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

2. Show that the permutation $e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$;

   $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$; $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$; $\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$ from a group and construct its Cayley table.

3. From the Cayley table for the group S_2.

   Consider the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$

   In this permutation $1 \to 2 \to 3 \to 4 \to 5 \to 1$. Thus the permutation maps the symbols in a cyclic order. Now consider the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}$.

   This permutation fixes the symbol 1 and maps the remaining symbols in a cyclic order.

Definition: 3.2.4

Let p be a permutation on $A = \{1,2,\ldots\ldots, n\}$. p is called a **cycle of length** r if there exist distinct symbols $a_1, a_2 \ldots\ldots\ldots a_r$ such that $p(a_1) = a_2, p(a_2) = a_3, \ldots\ldots, p(a_{r-1}) = a_r$, and $p(a_r) = a_1$, and $p(b) = b$, for all

$b \in A - \{a_1, a_2, \ldots\ldots\ldots, a_r\}$.

This cycle is represented by the symbol $(a_1, a_2, \ldots\ldots\ldots, a_r)$. Thus under the cycle $(a_1, a_2, \ldots\ldots, a_r)$ each symbol is mapped onto the following symbol except the last one which is mapped onto the first symbol and all the other symbols not in the cycle are fixed.

Example:

Let $A = \{1,2,3,4,5\}$. Consider the cycle of length 4 given by $p = (2451)$.

Then $p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$.

Obviously $(2451) = (4512) = (5124) = (1245)$.

Note:

Since cycles are special types of permutations, they can be multiplied in the usual way. The product of cycles need not be a cycle.

For example, let $p_1 = (234)$ and $p_2 = (1,5)$. Then

$$p_1p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 2 & 1 \end{pmatrix}$$ which is not a cycle.

Definition: 3.2.5

Two cycles are said to be **disjoint** if they have no symbols in common.

For example (2 1 5) and (3 4) are disjoint cycles.

Note:

If p_1 and p_2 are disjoint cycles the symbols which are moved by p_1 are fixed by p_2 and vice versa. Hence multiplication of disjoint cycles is commutative.

Examples:

1. Consider the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 3 & 5 & 6 & 7 & 4 \end{pmatrix}$. We shall write this permutation as a product of disjoint cycles. First of all 1 is moved to 2 and then 2 is moved to 1 thus giving the cycle (1 2). The element 3 is left fixed. Again starting with 4, 4 is moved to 5, 5 is moved to 6, 6 is moved to 7 and 7 is moved to 4, thus giving the cycle (4 5 6 7). Thus

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 3 & 5 & 6 & 7 & 5 \end{pmatrix} = (12)(4567) = (4567)(12).$$

2. Consider the permutation $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 7 & 5 & 4 & 1 & 6 \end{pmatrix} \in S_7$. Starting with 1 we get the cycle (1 2 3 7 6). The elements 4, 5 do not appear in it. Starting with 4 we get the cycle (4 5). Each element of the set {1, 2,.........,7} occurs is one of these two cycles.

   Thus $\alpha = (12376)(45)$.

3. Consider the permutation $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 1 & 3 & 2 & 5 \end{pmatrix}$. Clearly $\alpha = (143)(265)$.

Theorem: 3.13

Any permutation can be expressed as a product of disjoint cycles.

Proof:

Let p be a given permutation of the set $S = \{1,2,\ldots\ldots, n\}$. Let us start with any symbol $a_1 \in S$. Let $p(a_1) = a_2, p(a_2) = a_3$.........Since S is finite, these symbols cannot all be distinct and hence there exists a least positive integer r such that $1 \le r \le n$ and $p(a_r) = a_1$.

Let $c = (a_1, a_2, \ldots\ldots, a_r)$. If $r = n$ then $p = c$ so that p is a cycle. If

$r < n$, let b_1 be a symbol in S such that $b_1 \notin (a_1, a_2, \ldots\ldots, a_r)$. Starting with b_1 we can construct the cycle $d = (b_1 b_2 \ldots\ldots, b_s)$ as before. Clearly the cycles c and d are disjoint. If $r + s = n$ then $p = cd$. If $r + s < n$ we repeat the above process to obtain more cycles until all the symbols appear in one of the cycles. Thus we get a decomposition of p into disjoint cycles.

Exercise:

Express the following permutations as a procut of disjoint cycles.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$

(c) (1 2 3 4) (3 4 5)

(d) (1 3) (3 4) (4 5)

(e) (1 2 3) (1 6 5 4 3)

(f) (4 2 1 5) (3 4 2 6) (5 6 7 1)

(g) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 2 & 6 & 1 \end{pmatrix}$

(h) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 4 & 3 & 7 & 2 & 1 & 6 \end{pmatrix}$

Answers:

(a) (14) (35)

(b) (135) (24)

(c) (124) (35)

(d) (1543)

(e) (12) (3654)

(f) (16347) (25)

(g) (134256)

(h) (152476).

Note:

The decomposition of a permutation into disjoint cycles is unique except for the order of the factors.

Definition: 3.2.6

A cycle of length two is called a **transposition**. Thus a transposition (a_1a_2) interchanges the symbols a_1 and a_2 and leaves all the other elements fixed.

Theorem: 3.14

Any permutation can be expressed as a product of transpositions.

Proof:

Since any permutation is a product of disjoint cycles it is enough if we prove that each cycle is a product of transpositions.

Hence let $c = (a_1a_2, \ldots\ldots, a_l)$ be a cycle.

Clearly $(a_1a_2, \ldots\ldots, a_l) = (a_1a_2)(a_1a_3)\ldots\ldots(a_1a_r)$. This proves the theorem.

Examples:

1. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix} = (1245) = (12)(14)(15)$.

   Also (1235) = (2451) = (24) (25) (21).
   Thus the representation of a permutation as a product of transpositions is not unique.
2. (1345) (26) = (13) (14) (15) (26) = (13) (12) (12) (14) (15) (26).

   Thus in the representation of a permutation as a product of transpositions one can always insert (ab) (ab) in any place since (ab) (ab) is the identity permutation.

Theorem: 3.15

If a permutation $p \in S_n$ is a product of r transpositions and also a product of s transpositions then either r and s are both even or both odd.

Proof:

Let $p = t_1 t_2 \ldots\ldots t_r = t_1' t_2' \ldots\ldots t_s'$ where t_i, t_i' are transpositions. Now consider the polynomial in n variables $x_1, x_2, \ldots\ldots, x_n$ given by

$$\Delta = (x_1 - x_2)(x_1 - x_3)\ldots\ldots(x_1 - x_n)x(x_2 - x_3)(x_2 - x_4)\ldots\ldots(x_2 - x_n)\ldots\ldots x(x_{n-1} - x_n) = \prod_{i<j}(x_i - x_j)$$

For any permutation $p \in S_n$ we define

$$p(\Delta) = \prod_{i<j}(x_{p(i)} - x_{p(j)}.$$

Consider the transposition $t = (ij)$. Then the factor $x_i - x_j$ in Δ becomes $x_j - x_i$. Any factor $(x_k - x_\ell)$ of Δ in which neither i nor j is equal to k or ℓ is unchanged. All other factors of Δ can be paired to form products of the form $\pm(x_i - x_k)(x_k - x_j)$, the sign being determined by the relative magnitudes of i, j and k. Since t interchanges x_i and x_j any such product is unchanged. Hence the effect of the transposition t on Δ is just to change the sign of Δ ie, $t(\Delta) = -\Delta$.

$\therefore \quad p(\Delta) = (t_1 t_2 \ldots\ldots t_r)(\Delta) = (-1)^r \Delta$.

Also $p(\Delta) = (t_1' t_2' \ldots\ldots t_s')(\Delta) = (-1)^s \Delta$.

$\therefore \quad (-1)^r = (-1)^s \Rightarrow$ r and s are both even or both odd.

Definition: 3.2.7

A permutation $p \in S_n$ is called **even** or **odd** according as p can be expressed as a product of an even number of transpositions or an odd number of transpositions respectively.

Examples:

1. Consider the permutation $p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 6 & 4 & 1 & 7 & 2 & 5 \end{pmatrix}$

   $p = (134)(26)(57) = (13)(14)(26)(57)$.

   $\therefore$ p is a product of 4 transpositions.

   Hence p is an even permutation.

2. Consider the permutation $p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 4 & 3 & 6 & 1 & 7 & 9 & 8 \end{pmatrix}$

   $p = (1256)(34)(89) = (12)(15)(16)(34)(89)$

   $\therefore$ p is a product of 5 transpositions.

   Hence p is an odd permutation.

Exercises:

1. Determine which of the following permutations are odd and which of them are even.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 6 & 5 & 8 & 7 \end{pmatrix}$

(c) (1234)(356)(67)

(d) (123)(45)(5672).

2. Find all the even permutations in S_3 and show that they form a group.

3. For what values of m is a cycle of length m an even permutation?

Answer:

1.

(a) Even

(b) Odd

(c) Even

(d) Even

2. e, p_1, p_2

3. m is odd.

Theorem: 3.16.

(i). The product of two even permutations is an even permutation.

(ii). The product of two odd permutations is an even permutation.

(iii). The product of an even permutation and an odd permutation is an odd permutation.

(iv). The inverse of an even permutation is an even permutation.

(v). The inverse of an odd permutation is an odd permutation.

(vi). The identity permutation e is an even permutation.

Proof:

Let p_1, p_2 be two permutations. If p_1 is a product of r transpositions and p_2 is a product of s transpositions, then $p_1 p_2$ is a product of $r+s$ transpositions. Hence (i), (ii) and (iii) follow.

Now suppose that a permutation p is a product of r transpositions, say, $p = t_1, t_2, \ldots\ldots t_r$. Then $p^{-1} = (t_1, t_2 \ldots\ldots t_r)^{-1} = t_r^{-1} \ldots\ldots t_2^{-1} t_1^{-1} = t_r \ldots\ldots t_2 t_1$

$\therefore$ p^{-1} is also a product of r transpositions.

This proves (iv) and (v).

Now, $e = (12)(12)$ and hence e is an even permutation which proves (vi).

Theorem: 3.17

Let A_n be the set of all even permutations in S_n. Then A_n is a group containing $\frac{n!}{2}$ permutations.

Proof:

From (i), (vi) and (iv) of theorem 2.16 we see that A_n is a group.

Now let B_n be the set of all odd permutations in S_n.

Define $f : A_n \to B_n$ by $f(p) = (12)p$

f is 1 – 1, for $f(p_1) = f(p_2) \Rightarrow (12)p_1 = (12)p_2 \Rightarrow p_1 = p_2$.

f is onto, for, if $\alpha \in B_n$ then $(12)\alpha \in A_n$ and $f[(12)\alpha] = (12)(12)\alpha = \alpha$

Thus f is a bijeciton and hence the number of odd permutations in S_n = the number of even permutations in S_n. Since S_n contains $n!$ permutations, A_n has $\frac{n!}{2}$ elements.

Definition: 3.2.8

The group A_n of all even permutations in S_n is called the **alternating group on n symbols.**

Exercises:

1. Let G be a group of permutations. Show that either all the permutations in G are even or exactly half of them are even.

2. Let p be a permutation of a set A. Let $a \in A$ we say that p moves a if $p(a) \neq a$. How many elements are moved by a cycle of length r?

3. Show that the set of all permutations in S_n fixing the symbol 1 is a group.

4. Write down all the permutations of the set {1, 2, 3, 4} and determine which of them are even.

5. Determine which of the following statements are true and which of them are false.

(a) Every cycle is a permutation.

(b) Every permutation is a cycle.

(c) Product of two cycles is a cycle.

(d) Any transposition is an odd permutation.

(e) When $n \geq 3, S_n$ is nonabelian.

(f) Any permutation can be expressed as a product of cycles.

(g) The set of all odd permutations in S_n is a group.

(h) Any finite group is abelian.

Answers:

2. r elements

5.

(a) True

(b) False

(c) False

(d) True

(e) True

(f) True

(g) False

(h) False

3.3 Cyclic Groups:

Definition: 3.3.1

Let G be a group. Let $a \in G$.

Then $H' = \{a^n / n \in Z\}$ is a subgroup of G (verify). H is called the **cyclic subgroup of G generated by a** and is **denoted by** $\langle a \rangle$.

Examples:

1. In (**Z**, +), $\langle 2 \rangle = 2Z$ which is the group of even integers.
2. In the group
   $G = (Z_{12}, \oplus), \langle 3 \rangle = \{0,3,6,9\}. \langle 5 \rangle = \{0,5,10,3,8,1,6,11,4,9,2,7\} = Z_{12}$.
3. In the group $G = \{1, i-1, -i\} \langle i \rangle = \{i, i^2, i^3 ......\} = \{i, -1, -i, 1\} = G$.

Definition: 3.3.2

Let G be a group and let $a \in G$. a is called **a generator** of G if $\langle a \rangle = G$.

A group G is **cyclic** if there exists an element $a \in G$ such that $\langle a \rangle = G$.

Note:

If G is a cyclic group generated by an element a, then every element of G is of the form a^n for some $n \in Z$.

Examples:

1. (Z, +) is a cyclic group. 1 is a generator of this group. -1 is also a generator of this group. Thus a cyclic group can have more than one generator.
2. (nZ, +) is a cyclic group, n and –n are generator on this group.
3. $(Z_8, \oplus)$ is a cyclic group. 1, 3, 5, 7 are all generators of this group.
4. $(Z_n, \oplus)$ is a cyclic group of all $n \in N; 1$ is a generator of this group. In fact if $m \in Z_n$ and $(m, n) = 1$ then m is a generator of this group.
5. $G = \{1, i, -1, -i\}$ is a cyclic group under usual multiplication; i is a generator, -i is also a generator of G. However -1 is not a generator of G since $\langle -1 \rangle = \{1, -1\} \neq G$.
6. $G = \{1, \omega, \omega^2\}$ where $\omega \neq 1$ is a cube root of unity is a cyclic group. ω and ω^2 are both generators of this group.

7. In the group $G = (Z_7 - \{0\}, \odot)$, 3 and 5 are both generators. Here 2 is not a generator of G since $\langle 2 \rangle = \{2,4,1\} \neq G$.

8. Let A be a set containing more than one element. Then $(\wp(A), \Delta)$ is not cyclic; for let $B \in \wp(A)$ be any element. Then $B \Delta B = \Phi$ so that $\langle B \rangle = \{B, \Phi\} \neq \wp(A)$.

9. (**R**, +) is not a cyclic group since for any $x \in R, \langle x \rangle = \{nx / n \in Z\} \neq R$.

Exercises:

Determine which of the following groups are cyclic. If it is cyclic find all the generators of the group.

1. (6**Z**, +).
2. (**Q**, +)
3. The set of all n^{th} roots of unity under multiplication.
4. The group of symmetries of an equilateral triangle.
5. The group of symmetries of a rectangle.
6. The group of symmetries of a square.
7. $\{2^n / n \in Z\}$ under usual multiplication.
8. $(Z_4, \oplus)$
9. $(R^*, \cdot)$
10. $(Z_{11} - \{0\}, \odot)$
11. $G = \{e, p_1, p_2, p_3, p_4\}$ where

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

$$p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

$$p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$$

$$p_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix} \text{ and}$$

$$p_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$$

Answers:

1, 3, 7, 8, 10 and 11 are cyclic.

Theorem: 3.18

Any cyclic group is abelian.

Proof:

Let $G = \langle a \rangle$ be a cyclic group.

Let $x, y \in G$. Then $x = a^r$ and $y = a^s$ for some $r, s \in Z$.

Hence $xy = a^r a^s = a^{r+s} = a^{s+r} = a^s a^r = yx$.

$\therefore$ G is abelian.

Theorem: 3.19

A subgroup of cyclic group is cyclic.

Proof:

Let G be a cyclic group generated by a and let H be a subgroup of G. We claim that H is cyclic.

Clearly every element of H is of the form a^n for some integer n.

Let m be the smallest positive integer such that $a^m \in H$. We claim that a^m is a generator of H.

Let $b \in H$. Then $b = a^n$ for some $n \in Z$.

Let $n = mq + r$ where $0 \le r < m$.

Then $b = a^n = a^{mq+r} = a^{mq} a^r = (a^m)^q a^r$.

$$\therefore \qquad a^r = (a^m)^{-q} b. \qquad (1)$$

Now, $a^m \in H$. Since H is a subgroup, $(a^m)^{-q} \in H$.

Also $b \in H$.

By (1), $a^r \in H$ and $0 \le r < m$.

But m is the least positive integer such that $a^n \in H$.

$\therefore \quad r = 0$. Hence $b = a^n = a^{qm} = (a^m)^q$.

$\therefore \quad$ Every element of H is a power of a^m.

$\therefore \quad H = \langle a^m \rangle$ and hence H is cyclic.

Exercises:

1. Prove that if a is a generator of a cyclic group G then a^{-1} is also a generator of G.
2. Prove that any subgroup of (**Z**, +) is of the form n**Z** for some integer n.
3. Find the number of elements in the following cyclic subgroups.

(a) $\langle 2 \rangle$ in $(Z_{18}, \oplus)$

(b) $\langle 18 \rangle$ in $(Z_{30}, \oplus)$

(c) $\langle 5 \rangle$ in $(Z_{80}, \oplus)$

(d) $\langle i \rangle$ in C^*

4. Show that every proper subgroup of V_4 is cyclic. (However V_4 is not cyclic).
5. Show that every proper subgroup of S_3 is cyclic.
6. Give the multiplication table for the cyclic subgroup of S_5 generated by

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}$$

7. Determine which of the following statements are true and which are false.

(a) For any fixed integer n, $\{kn + 1 / k \in Z\}$ is a subgroup of (**Z**, +).

(b) Every cyclic group is abelian.

(c) Every abelain group is cyclic.

(d) Every element of a cyclic group is a generator of the group.

(e) $(\mathbf{Z}, +)$ is a cyclic group.

(f) $(\mathbf{Q}, +)$ is a cyclic group.

(g) S_3 is a cyclic group.

(h) A_3 is a cyclic group.

(i) $(Z_n \oplus)$ is a cyclic group.

(j) $(Z_n - \{0\}, \odot)$is a cyclic group.

(k) Any group of order 3 is cyclic.

(l) Any group of order 4 is cyclic.

(m) Given any positive integer n, there exists a cyclic group with n elements.

(n) Every group has cyclic subgroups.

(o) Every subgroup of a cyclic group is cyclic.

(p) If every proper subgroup of a group G is cyclic then G is cyclic.

(q) Every cyclic group has more than one generator.

Answers:

3. (a) 9 (b) 5 (c) 16 (d) 4

7. (b), (e), (h), (i), (k), (m), (n) and (o) are true.

Order of an Element:

1. Consider the group S_3 given in 3.4

$$p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix};$$

$$p_1^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = p_2 .$$

$$p_1^3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = e.$$

In this case 3 is the least positive integer such that $p_1^3 = e$. Also $\langle p_1 \rangle = \{e, p_1, p_2\}$ is a subgroup of S_3 of order 3.

2. Consider $(R^*, \cdot)$. Form the sequence of elements $2, 2^2, 2^3, \ldots\ldots, 2^n, \ldots\ldots$ In this case there is no positive integer n such that $2^n = 1$ and $\langle 2 \rangle$ contains infinite numbers of elements.

Definition: 3.3.3

Let G be a group and let $a \in G$. The least positive integer n (if it exists) such that $a^n = e$ is called the **order** of a. If there is no positive integer n such that $a^n = e$, then the order of a is said to be infinite.

In example 1, p_1 is of order 3 and 2 is of infinite order in example 2.

In (C*,.), i is an element of order 4.

Exercises:

1. Show that in any group G, e is the only element of order 1.
2. Find the order of -1 and 3 in (Z, +).
3. Find the order of -1 and 3 in (R*, .)
4. Find the order of -1 and $-i$ in $(C^*, \cdot)$.
5. Find the order of 2 and 3 in $(Z_8, \oplus)$.
6. Show that in V_4 the order of every element other than the identity is 2.
7. Show that in (**Z**, +) the order of every element other than 0 is infinite.
8. Show that in $(\wp(S), \Delta)$ the order of every element other than Φ is 2.
9. Show that in (C*, .) for every positive integer n there exists an element of order n.

Answers:

2. infinite

3. order of -1 is 2 and order of 3 is infinite.

4. order of -1 is 2 and order of $-i$ is 4.

5. order of 2 is 4 and order of 3 is 8.

Theorem: 3.20

Let G be a group and $a \in G$. Then the order of a is the same as the order of the cyclic group generated by a.

Proof:

Let a be an element of order n. Then $a^n = e$. We claim that $e, a, a^2, \ldots\ldots, a^{n-1}$ are all distinct.

Suppose $a^r = a^s$ where $0 < r < s < n$.

Then $a^{s-r} = e$ and $s - r < n$ which contradicts the definition of the order of a. Hence $e, a, a^2, \ldots\ldots, a^{n-1}$ are n distinct elements and $\langle a \rangle = \{e, a, a^2, \ldots\ldots, a^{n-1}\}$ which is of order n.

If a is of infinite order, the sequence of elements $a, a^2, \ldots\ldots, a^n, \ldots$ are all distinct and are in $\langle a \rangle$. Hence $\langle a \rangle$ is an infinite group.

Theorem: 3.21

In a finite group every elements is an finite order.

Proof:

Let $a \in G$. If a is of infinite order, then $\langle a \rangle$ is an infinite subgroup of G, which is a contradiction since G is finite. Hence the order of a is finite.

Remark:

The converse of the above theorem is not true. (ie) if G is a group in which every element is of finite order then the group G need not be finite. For example, if S is any infinite set, then $(\wp(S), \Delta)$ is an infinite group. In this group $A \Delta A = \Phi$ for every $A \in \wp(S)$ so that the order of every element other than Φ is 2.

Theorem: 3.22

Let G be a group and a be an element of order n in G. Then $a^m = e$ iff n divides m.

Proof:

Suppose n|m. Then $m = nq$ where $q \in Z$.

$\therefore \quad a^m = a^{nq} = (a^n)^q = e^q = e$.

Conversely, let $a^m = e$.

Let $m = nq + r$ where $0 \le r < n$.

$\therefore \quad a^m = a^{nq+r} = a^{nq}a^r = ea^r = a^r$.

$\therefore \quad a^r = e$ and $0 \le r < n$.

Now, since n is the smallest positive integer such that $a^n = e$, we have $r = 0$. Hence $m = nq$.

Therefore n|m.

Theorem: 3.23

Let G be a group and $a, b \in G$.

Then

(i) Order of a = order of a^{-1}.

(ii) Order of a = order of $b^{-1}ab$.

(iii) Order of ab = order of ba.

Proof:

(i) Let a be an element of order n.

Then $a^n = e$.

$\therefore \quad (a^{-1})^n = (a^n)^{-1} = e^{-1} = e$.

Now, if possible let $0 < m < n$ and $(a^{-1})^m = e$.

$\therefore \quad (a^m)^{-1} = e$. Hence $a^m = e$ which contradicts the definition of the order of a. Thus n is the least positive integer such that $(a^{-1})^n = e$

$\therefore \quad$ The order of a^{-1} is n.

(ii) We shall first prove that for any positive integer r.

$$(b^{-1}ab)^r = b^{-1}a^r b. \qquad (1)$$

(1) Is trivialy true if $r = 1$.

Now, suppose that (1) is true for $r = k$ so that $(b^{-1}ab)^k = b^{-1}a^k b$.

Then

$$(b^{-1}ab)^{k+1} = (b^{-1}ab)^k (b^{-1}ab).$$

$$= (b^{-1}a^k b)(b^{-1}ab).$$

$$= b^{-1}a^{k+1}b.$$

Hence by induction (1) is true for all positive integers.

Now, let a be an element of order n. Then $a^n = e$.

$$\therefore \quad (b^{-1}ab)^n = b^{-1}a^n b \quad \text{(by (1))}$$

$$= b^{-1}eb = e.$$

Now, if possible, let $0 < m < n$ and $(b^{-1}ab)^m = e$.

$\therefore$ $b^{-1}a^m b = e$. Hence $a^m = e$ which contradicts the definition of the order of a. Thus n is the least positive integer such that $(b^{-1}ab)^n = e$.

$\therefore$ The order of $b^{-1}ab$ is n.

(iii) The order of ab = the order of $a^{-1}(ab)\ a$ (by (ii)) = the order of ba.

Theorem: 3.24

Let G be a group and let a be an element of order n in G. Then the order of a^s, where $0 < s < n$, is n/d where d is the g.c.d of n and s.

Proof:

Let $(n/d) = k$ and $(s/d) = \ell$ so that k and ℓ are relatively prime.

Now, $(a^s)^k = a^{sk} = a^{\ell dk} = a^{\ell n} = (a^n)^\ell = e$.

Further if m is any positive integer such that $(a^s)^m = e$ then $a^{sm} = e$.

Since order of a is n, we have $n|sm$.

$\therefore$ $kd \mid \ell dm$. Hence $k \mid \ell m$

But k and ℓ are relatively prime.

Hence k/m so that $m \geq k$.

Thus k is the least positive integer such that $(a^s)^k = e$.

$\therefore$ order of $a^s = k = n/d$.

Corollary: 1

The order of any power of a cannot exceed the order of a.

Corollary: 2

Let G be a finite cyclic group of order n generated by an element a. Then a^s generates a cyclc group of order n/d where d is the g.c.d of n and s.

Corollary: 3

Let G be a finite cyclic group of order n generated by an element a.a^s is a generator of G iff s and n are relatively prime. Hence the number of generators of a cyclic group of order n is $\phi(n)$ where $\phi(n)$ is the number of positive integers less than n relatively prime to n.

For example, consider the group $(Z_{12}, \oplus)$.

$\phi(12) = 4$. Hence the group has exactly 4 generators and they are 1, 5, 7 and 11.

Problems:

Problem: 1

If G is a finite group with even number of elements then G contains at least one element of order 2.

Solution:

a is an element of order $2 \Leftrightarrow a^2 = e$

$$\Leftrightarrow a^{-1} = a .$$

Hence it is enough if we prove that there exists an element different from e in G whose inverse is itself.

Let $S = \{a / a \in G, a \neq a^{-1}\}$.

Clearly $a \in S \Rightarrow a^{-1} \in S$ and $a \neq a^{-1}$.

Hence S contains an even number of elements.

Also $e \notin S$.

Hence $S \cup \{e\}$ contains an odd number of elements. Since the order of the group is even, there exists at least one element $a \notin S \cup \{e\}$. Clearly $a = a^{-1}$.

Problem: 2

The order of a permutation p is the ℓ.c.m of the lengths if its disjoint cycles.

Solution:

Let $p = c_1 c_2 \ldots\ldots, c_r$ where the c_i's are mutually disjoint cycles of lengths ℓ_i. Now, let $p^m = e$.

Since product of disjoint cycles is commutative,

$e = p^m = (c_1 c_2, \ldots\ldots, c_r)^m = c_1{}^m c_2{}^m, \ldots, c_r^m$.

Now, since the elements moved by one cycle are left fixed by all the other cycles, $c_1^m = c_2^m = \ldots\ldots, c_r^m = e$.

Now, $c_1^m = e \Rightarrow \ell_1 \mid m$ since the order of $c_1 = \ell_1$.

Similarly $\ell_2, \ell_3, \ldots\ldots, \ell_r$ divide m.

Thus m is a common multiple of $\ell_1, \ell_2, \ldots, \ell_r$.

$\therefore$ The order of p is the least such m which is obviously the ℓ.c.m. of $\ell_1, \ell_2, \ldots, \ell_r$

Problem: 3

If a is a generator of the cyclic group G and if there exist two unequal integers m and n such that $a^m = a^n$, prove that G is a finite group.

Solution:

Since m and n are unequal we may assume that $m > n$.

Hence $m - n$ is a positive integer.

Also $a^m = a^n \Rightarrow a^{m-n} = e$.

$\therefore$ Order of a is finite.

$\therefore$ $G = \langle a \rangle$ is a finite group (by theorem 2.20)

Exercises:

1. Show that a group G of order n is cyclic iff G contains an element of order n.
2. Find the number of generators of the cyclic groups of order 8, 24 and 60.
3. Let p and q be prime number. Find the number of generators of Z_{pq}.
4. Find the number of generators of Z_p, where p is prime.
5. Find two elements a, b in a group such that

(a) Order of ab $\neq$ (order of a) (order of b).

(b) Order of ab = (order of a) (order of b).

6. Find the order of the following permutations.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 7 & 6 & 4 & 1 & 2 & 3 \end{pmatrix}$

(c) $(12345)(67)(1657)$

(d) $(12)(23)(345)(1456)$.

7. Find all the elements of finite order in the following groups.

(a) $(\mathbf{Z}, +)$

(b) $(R^*,\cdot)$

(c) $(\wp(S),\Delta)$

8. Determine which of the following are true and which are false.

(a) In a finite group the order of every element is finite.

(b) If in a group every element is of finite order then the group is finite.

(c) In an infinite group the order of every element is infinite.

(d) The order of e is zero.

(e) The order of every element of $(R^*,\cdot)$ is finite.

Answers:

2. 4, 8 and 16.

6. (a) 4 (b) 4 (c) 7 (d) 6

7. (a) 0 (b) 1 and -1 (c) All elements.

8. (a) T (b) F (c) F (d) F (e) F

3.HOMOMORPHISMS

3.4 Introduction:

In this section we consider an important concept known as "homomorphism". The notion of homomorphism is common to all aspects of modern algebra. A homomorphism of group is a map which preserves composition.

Definition: 3.4.1

A map f from a group G into a group G' is called a **homomorphism** if $f(ab) = f(a)f(b)$ for all $a, b \in G$.

Obviously every isomorphism is a homorphism and a bijective homomorphism is an **isomorphism.**

Examples:

1. $f:(Z,+) \to (Z,+)$ defined by $f(x) = 2x$ is a homomorphism. For, $f(x+y) = 2(x+y\_ = 2x+2y = f(x)+f(y)$. Note that f is 1 – 1.
2. $f:(R^*,\cdot) \to (R^+,\cdot)$ defined by $f(x) = |x|$ is a homomorphism. For, $f(xy) = |xy| = |x||y| = f(x)f(y)$. This homorphism is onto.
3. $f: G \to G'$ defined by $f(a) = e'$, where e' is the identity in G' is a trivial homomorphism. For, $f(ab) = e' = e'e' = f(a)f(b)$.
4. $f:(Z,+) \to (C^*,\cdot)$ given by $f(n) = i^n$ is a homomorphism. For, $f(m+n) = i^{n+m} = i^n i^m = f(n)f(m)$. Note that f is neither 1 – 1 nor onto.
5. $f:(R \times R,+) \to (R,+)$ given by $f(x,y) = x$ is a homomorphism.
6. Let G be a group and N a normal subgroup of G.
   $f: G \to G/N$ given by $f(a) = Na$ is a homomorphism.
   For, $f(ab) = Nab = NaNb = f(a)f(b)$.
   f is called the **canonical homomorphism** from G to G/N. Note that f is onto.

3.5 Types of Homomorphisms:

Definition: 3.5.1

Let $f: G \to G'$ be a homomorphism.

(i) If f is onto, then it is called an **epimorphism**.

(ii) If f is 1 – 1, then it is called a **monomorphism**.

Note:

If $f: G \to G'$ is an epimorphism then G' is called a **homomorphic image** of G.

A homomorphism of a group to itself is called an **endomorphism**.

Theorem: 3.26

Let $f: G \to G'$ be a homomorphism. Then

(i) $f(e) = e'$.

(ii) $f(a^{-1}) = [f(a)]^{-1}$.

(iii) If H is a subgroup of G then $f(H)$ is a subgroup of G'.

(iv) If H is normal in G, then $f(H)$ is normal in $f(G)$.

(v) If H' is a subgroup of G', then $f^{-1}(H')$ is a subgroup of G.

(vi) If H' is normal in f(G) then $f^{-1}(H')$ is normal in G.

Proof:

(i) Let $a \in G$.

Then $f(a) = f(ae) = f(a)f(e)$.

Hence $f(e) = e'$.

(ii) $f(a)f(a^{-1}) = f(e) = e'$.

Hence $f(a^{-1}) = [f(a)]^{-1}$.

(iii) Let H be a subgroup of G.

Since H is non – empty, $f(H)$ is also non – empty.

Now, let $x, y \in f(H)$.

Then $x = f(a)$ and $y = f(b)$ where $a, b \in H$.

$\therefore \quad xy^{-1} = f(a)[f(b)]^{-1}$

$= f(a)f(b^{-1}) = f(ab^{-1})$.

Now, since H is a subgroup of G, $ab^{-1} \in H$.

$\therefore \quad xy^{-1} = f(ab^{-1}) \in f(H)$.

$\therefore \quad f(H)$ is a subgroup of G'.

(iv) Let H be normal in G. Let $x \in f(H)$ and $y \in f(G)$.

We claim that $yxy^{-1} \in f(H)$.

Now, $x = f(a)$ and $y = f(b)$ where $a \in H$ and $b \in G$.

Since H is normal in G, $bab^{-1} \in H$.

$\therefore \quad f(bab^{-1}) \in f(H)$.

$\therefore \quad f(b)f(a)f(b^{-1}) \in f(H)$.

$\therefore \quad yxy^{-1} \in f(H)$. Hence $f(H)$ is normal in $f(G)$.

(v) Since $f(e) = e' \in H'; e \in f^{-1}(H')$ and hence $f^{-1}(H') \neq \Phi$.

Now, let $a, b \in f^{-1}(H')$.

Then $f(a), f(b) \in H'$.

$\therefore \quad f(a)[f(b)]^{-1} \in H'$.

$\therefore \quad f(ab^{-1}) \in H'$(ie), $ab^{-1} \in f^{-1}(H')$.

Hence $f^{-1}(H')$ is a subgroup of G.

(vi) Let $x \in f^{-1}(H')$ and $a \in G$.

Then $f(x) \in H'$ and $f(a) \in f(G)$.

Since H' is normal in f(G), $f(a)f(x)[f(a)]^{-1} \in H'$.

$\therefore \quad f(axa^{-1}) \in H'$.

Hence $axa^{-1} \in f^{-1}(H')$.

Thus $f^{-1}(H')$ is normal in G.

Examples:

1. Consider the homomorphism $f : (Z,+) \to (Z_n, \oplus)$ which is given in the beginning of this section.

   Let $K = \{x / x \in Z, f(x) = 0\}$.

   Clearly $K = nZ$ which is a normal subgroup of Z.

2. Consider the homomorphism $f : (R^*,\cdot) \to (R^+,\cdot)$ which is given by $f(x) = |x|$.

   Let $K = \{x / x \in R^*, f(x) = 1\}$. Clearly $K = \{1, -1\}$ which is a normal subgroup of $(R^*,\cdot)$.

Definition: 4.2.2

Let $f : G \to G'$ be a homomorphism. Let $K = \{x / x \in G, f(x) = e'\}$. Then K is called the **kernel** of f and is denoted by ker f.

Theorem: 3.27

Let $f : G \to G'$ be a homomorphism. Then the kernel K of f is a normal subgroup of G.

Proof:

Since $f(e) = e', e \in K$ and hence $K \neq \Phi$.

Now, let $x, y \in K$. Then $f(x) = e' = f(y)$.

$\therefore f(xy^{-1}) = f(x)f(y^{-1}) = f(x)[f(y)]^{-1} = e'(e')^{-1} = e'e' = e'$.

Thus $xy^{-1} \in K$. Hence K is a subgroup of G.

Now, let $x \in K$ and $a \in G$.

Then

$$\begin{aligned} f(axa^{-1}) &= f(a)f(x)f(a^{-1}) \\ &= f(a)e'[f(a)]^{-1} \\ &= f(a)[f(a)]^{-1} \\ &= e'. \end{aligned}$$

$\therefore$ $axa^{-1} \in K$. Hence K is a normal subgroup of G.

Aliter:

$\{e'\}$ is a normal subgroup of $f(G)$. Hence $\ker f = f^{-1}(\{e'\})$ is a normal subgroup of G.

Problems:

Problem: 1

Let $f : G \to G'$ be a homomorphism. Then f is 1 – 1 iff $\ker f = \{e\}$.

Solution:

Obviously f is $1-1 \Rightarrow \ker f = \{e\}$.

Conversely, let $\ker f = \{e\}$.

We prove f is 1 – 1.

$$\begin{aligned} f(x) = f(y) &\Rightarrow f(x)[f(y)]^{-1} = e'. \\ &\Rightarrow f(xy^{-1}) = e'. \\ &\Rightarrow xy^{-1} \in \ker f. \\ &\Rightarrow xy^{-1} = e. \\ &\Rightarrow x = y. \end{aligned}$$

Hence f is 1 – 1.

Problem: 2

Let G be any group and H be the centre of G. Then $G/H \cong I(G)$, the group of **inner automorphisms** of G.

Solution:

Consider $f : G \to I(G)$ defined by $f(a) = \phi_a$.

Then $f(ab) = \phi_{ab} = \phi_a \circ \phi_b = f(a)f(b)$.

Hence f is a homomorphism.

Clearly f is onto.

Now, we claim that $\ker f = H$.

$$a \in \ker f \Leftrightarrow f(a) = \phi_e.$$

$$\Leftrightarrow \phi_a = \phi_e$$

$$\Leftrightarrow \phi_a(x) = x \text{ for all } x \in G$$

$$\Leftrightarrow axa^{-1} = x \text{ for all } x \in G$$

$$\Leftrightarrow ax = xa \text{ for all } x \in G$$

$$\Leftrightarrow a \in H.$$

Hence $\ker f = H$.

$\therefore$ By the fundamental theorem of homomorphism $G/H \cong I(G)$.

Problem: 3

Show that $R^*/\{1,-1\} \cong R^+$.

Solution:

Consider $f : R^* \to R^+$ defined by $f(x) = |x|$.

Clearly f is an epimorphism and $\ker f = \{1,-1\}$.

Hence by the fundamental theorem of homomorphism $R^*/\{1,-1\} \cong R^+$

Problem: 4

Any homomorphic image of a cyclic group is cyclic.

Solution:

Let G be a cyclic group and $f ; G \to G'$ be an epimorphism. Let a be a generator of G. Then $f(a)$ is a generator of G'. (refer proof of theorem 4.7).

Hence G' is cyclic.

Problem: 5

Show that the map $f : (C,+) \to (R,+)$ defined by $f(x+iy) = y$ **is** an epimorphism and $\ker f = R$. Deduce that $C/R \cong R$.

Solution:

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Then $(z_1 + z_2) = (x_1 + x_2) + i(y_1 + y_2)$.

$\therefore$ $f(z_1 + z_2) = y_1 + y_2 = f(z_1) + f(z_2)$.

Hence f is a homomorphism. Clearly f is onto.

Now, $\ker f = \{x + iy / f(x+iy) = 0\}$.

$= \{x + iy / y = 0\}$.

$=$ **R**.

$\therefore$ By the fundamental theorem of homomorphism $C/R \cong R$.

Exercises:

1. Determine which of the following maps are homomorphism. If it is a homomorphism, find the kernel.

   (a) $f:(Z,+)\rightarrow\{1,-1\}$ given by $f(n)=\begin{cases}1 & \text{if n is even}\\ -1 & \text{if n is odd}\end{cases}$.

   (b) $f:R^*\rightarrow R^*$ given by $f(x)=\begin{cases}1 & \text{if } x>0\\ -1 & \text{if } x<0\end{cases}$.

   (c) $f:(R \text{ x } R,+)\rightarrow(R,+)$ given by $f(x,y)=y$.

   (d) $f:(Z,+)\rightarrow(R^*,\cdot)$ given by $f(x)=3^x$.

   (e) $f:S_n\rightarrow\{1,-1\}$ given by

   $$f(p)=\begin{cases}1 & \text{if p is an even permutation}\\ -1 & \text{if p is an odd permutation}\end{cases}.$$

   (f) $f:R\rightarrow C$ given by $f(x)=e^{ix}$.

   (g) $f:(Z,+)\rightarrow(Z,+)$ given by $f(n)=2n$.

   (h) $f:R^*\rightarrow R^*$ given by $f(x)=-x$.

   (i) $f:Z_6\rightarrow Z_2$ given by $f(x)=$ remainder of x when x is divided by 2.

   (j) $f:C^*\rightarrow R^*$ given by $f(z)=|z|$.

   (k) $f:(R,+)\rightarrow(R,+)$ given by $f(x)=x+2$.

2. Prove that for any positive integer $n, Z/nZ\cong Z_n$.

3. If $f:G\rightarrow G'$ and $g:G'\rightarrow G''$ are homomorphisms then prove that $g\,o\,f:G\rightarrow G''$ is a homomorphism.

4. Show that any homomorphic image of an abelian group is abelian.

5. Find all the homomorphisms of $(Z,+)$ onto $(Z,+)$.

6. Show that a cyclic group of order 8 is homomorphic to

   (a) A cyclic group of order 4.

   (b) A cyclic group of order 2.

7. Prove that there exist 8 homomorphisms from (Z, +) to $(Z_8,\oplus)$. How many of these homomorphisms are onto?

8. Determine which of the following statements are true and which are false.

(a) Any isomorphism is a homomorphism.

(b) Any homomorphism is an isomorphism.

(c) An infinite group cannot be homomorphic to a finite group.

(d) Homomorphism preserves the order of an element.

(e) Any homomorphism f is a monomorphism iff ker f if $\{e\}$.

Answers:

1.

(a) Yes, $\ker f = 2Z$.

(b) Yes, $\ker f = R^+$.

(c) Yes, $\ker f = R \times \{0\}$.

(d) Yes, $\ker f = \{0\}$.

(e) Yes, $\ker f = A_n$.

(f) No

(g) Yes, $\ker f = \{0\}$.

(h) No

(i) Yes, $\ker f = \{0,2,4\}$.

(j) Yes, $\ker f = \{z / z \in C \text{ and } |z| = 1\}$.

(k) No

8. (a) T (b) F (c) F (d) F (e) T.

3.6 Isomorphism:

3.6.1 Introduction:

Let $\omega \neq 1$ be a cubic root of unity. Let $G = \{1, \omega, \omega^2\}$. G is a group under usual multiplication. The Cayley table for G is given by

.	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$(Z_3, \oplus)$ is a group and its Cayley table is given by

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

We note that these two tables for the groups of order 3 keep the same **pattern**. In fact any group of order 3 is cyclic and hence it is easily seen that all groups with 3 elements are "like" each other. Thus if two groups G and G' are "like" each other, it should be possible for us to obtain G' from G by renaming each element x in G with the name of an element x' in G'. The remaining of the elements of G can be achieved by means of a bijection $f: G \to G'$. If $x \in G$ we view $f(x)$ as a new name for x. finally if the groups are to be "like" each other, then if x and y are in G the new name for xy should be $f(x)f(y)$ so that $f(xy) = f(x)f(y)$. Note that the product of xy is computed in G and the product $f(x)(y)$ is computed in G'. Two groups which are like each other are usually called **isomorphic**. The following definition makes these ideas mathematically precise.

Definition: 3.6.1

Let G and G' be two groups. A map $f: G \to G'$ is called an **isomorphism** if

(i) f is a bijection

(ii) $f(xy) = f(x)f(y)$ for all $x, y \in G$.

Two groups G and G' are said to be isomorphic if there exists an **isomorphism** $f: G \to G'$. If two groups G and G' are isomorphic we write $G \cong G'$.

Theorem: 3.28

Isomorphism is an equivalence relation among groups.

Proof:

For any group $G, i_G : G \to G$ is clearly an isomorphism.

Hence $G \cong G$. Therefore the relation is reflexive.

Now, let $G \cong G'$ and let $f: G \to G'$ be an isomorphism.

Then f is a bijeciton.

$\therefore$ $f^{-1} : G' \to G$ is also a bijection.

Now, let $x', y' \in G'$.

Let $f^{-1}(x') = x$ and $f^{-1}(y') = y$.

Then $f(x) = x'$ and $f(y) = y'$.

$\therefore \quad f(xy) = f(x)f(y) = x'y'$.

$\therefore \quad f^{-1}(x'y') = xy = f^{-1}(x')f^{-1}(y')$.

Hence f^{-1} is an isomorphism.

Thus $G' \cong G$ and hence the relation is symmetric.

Now, let $G \cong G'$ and $G' \cong G''$.

Then there exist isomorphisms $f : G \to G'$ and $g : G' \to G''$.

Since f and g are bijections, $g \circ f : G \to G''$ is also a bijection.

Now, let $x, y \in G$. Then

$(g \circ f)(xy) = g[f(xy)]$

$= g[f(x)f(y)]$ (since f is an isomorphism)

$= g[f(x)]g[f(y)]$ (since g is an isomorphism)

$= (g \circ f)(x)(g \circ f)(y)$.

Hence $g \circ f$ is an isomorphism.

Thus $G \cong G''$ and hence the relation is transitive.

$\therefore$ Isomorphism is an equivalence relation among groups.

Examples:

1. $(Z,+) \cong (2Z,+)$.

   Consider $f : Z \to 2Z$ given by $f(x) = 2x$.
   Clearly f is a bijection. Also
   $f(x+y) = 2(x+y)$
   $= 2x + 2y = f(x) + f(y)$.
   Hence f is an isomorphism.

. Let $G = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \middle| a \in R^* \right\}$.

   G is a group under matrix multiplication.
   We claim that $G \cong (R^*, \cdot)$.

Consider $f: G \to R^*$ given by $f\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = a$.

Clearly f is a bijection.

Now, let $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and

$B = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \in G$.

Then $AB = \begin{pmatrix} ab & 0 \\ 0 & 0 \end{pmatrix}$

$\therefore \quad f(AB) = ab = f(A)f(B)$.

Hence f is an isomorphism.

3. $(R,+) \cong (R^+, \cdot)$.

Consider $f: R \to R^+$ given by $f(x) = e^x$.

Clearly f is a bijection.

Also $f(x+y) = e^{x+y} = e^x e^y = f(x)f(y)$.

Hence f is an isomorphism.

4. $G = R - \{-1\}$ is a group under $*$ defined by $a * b = a + b + ab$. We claim that $GG \cong (R*, \cdot)$.

Consider $f: G \to R^*$ given by $f(x) = x + 1$. Clearly f is a bijection.

Also $f(x * y) = f(x + y + xy)$.

$= x + y + xy + 1$.

$= (x+1)(y+1)$.

$= f(x)f(y)$.

Hence f is an isomorphism.

5. $(Z_n, \oplus)$ is a group.

Let G denote the set of all n^{th} roots of unity. G is a group under usual multiplication. We claim that $(Z_n, \oplus) \cong G$.

Consider $Z_n \to G$ given by $f(m) = \omega^m$ where

$\omega = \cos(2\pi/n) + i \sin(2\pi/n)$.

Clearly f is a bijection.

Let $a, b \in Z_n$. Let $a + b = qn + r$ where $0 \le r \le n$.

Then $a \oplus b = r$. Hence $f(a \oplus b) = \omega^r$ (1)

Also $f(a)f(b) = \omega^a \omega^b = \omega^{a+b} = \omega^{qn+r} = \omega^{qn}\omega^r = 1\omega^r = \omega^r$ (2)

From (1) and (2), we get $f(a \oplus b) = f(a)f(b)$.

Hence f is an isomorphism.

Exercises:

1. Show that $G = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \middle| a \in R \right\}$ is a group under matrix addition and prove that $G \cong (R,+)$.

2. Show that $G = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in R \text{ and } a^2 + b^2 \neq 0 \right\}$ is a group under matrix multiplication and $G \cong (C^*, \cdot)$.

3. Let $f_a : R \to R$ be the function defined by $f_a(x) = x + a$. Then $G = \{f_a / a \in R\}$ is a group under composition of functions. Show that $G \cong (R,+)$.

4. R x R is a group under + defined by $(a,b)+(c,d) = (a+c, b+d)$. Show that $(R \times R,+) \cong (C,+)$.

5. Show that $(Z,+) \cong (nZ,+)$.

6. $G = \{2^n / n \in Z\}$ is a group under usual multiplication. Show that $G \cong (Z,+)$.

7. In **Z** we define $a * b = a + b + 1$. $(Z,*)$ is a group. Show that $(Z,*) \cong (Z,+)$.

8. $G = \{a + b\sqrt{2} / a, b \in Z\}$ is a group under usual addition. **Z x Z** is a group under addition defined by $(a,b)+(c,d) = (a+c, b+d)$. Show that $G \cong Z \times Z$.

9. R* is a group under the binary operation * defined by $a * b = \frac{1}{2}ab$. Show that $(R^*,*) \cong (R^*,\cdot)$.

10. $G = \{1, i, -1, -i\}$ is a group under multiplication. Show that $f : G \to (Z_4, \oplus)$ defined by $f(1) = 0, f(i) = 1, f(-1) = 2, f(-i) = 3$ is an isomorphism.

11. Show that $(\wp(\{1,2\}),\Delta\}$ is isomorphic to the group of symmetries of a rectangle.

Theorem: 3.29

Let $f:G\to G'$ be an isomorphism. Then

(i) $f(e)=e'$ where e and e' are the identity elements of G and G' respectively. (ie). In an isomorphism identity is mapped onto identity.

(ii) $f(a^{-1})=[f(a)]^{-1}$.

Proof:

(i) To prove that $f(e)=e'$ it is enough if we prove that $a'f(e)=f(e)a'=a'$ for all $a'=a'$ for all $a'\in G'$.

Let $a'\in G'$. Since $f:G\to G'$ is a bijection, there exists such that $a\in G$ such that $f(a)=a'$.

$\therefore$ $a'f(e)=f(a)f(e)=f(ae)=f(a)=a'$.

Similarly, $f(e)a'=a'$.

$\therefore$ $f(e)=e'$.

(ii) It is enough to prove that

$$f(a)f(a^{-1})=f(a^{-1})f(a)=e'.$$

Now, $f(a)f(a^{-1})=f(aa^{-1})=f(e)=e'$

Also, $f(a^{-1})f(a)=f(a^{-1}a)=f(e)=e'$.

$\therefore$ $f(a)f(a^{-1})=f(a^{-1})f(a)=e'$.

$\therefore$ $[f(a)]^{-1}=f(a^{-1})$.

Remark:

The concept of isomorphism for groups is extremely important. Since two isomorphic groups G and G' have essentially the same structure, if one group G has an additional property (for example abelian or cyclic) then the group G' also has this additional property. This is seen in the following three theorems.

Theorem: 3.30

Let $f: G \rightarrow G'$ be an isomorphism. If G is abelian, then G' is also abelian.

Proof:

Let $a', b' \in G'$. Then there exist $a, b \in G$ such that $f(a) = a'$ and $f(b) = b'$.

Now, $a'b' = f(a)f(b) = f(ab) = f(ba) = f(b)f(a) = b'a'$.

Hence G' is abelian.

Theorem: 3.31

Let $f: G \rightarrow G'$ be an isomorphism. Let $a \in G$. Then the order of a is equal to the order of $f(a)$. (ie) Isomorphism preserves the order of each element in a group.

Proof:

Suppose the order of a is n. Then n is the least positive integer such that $a^n = e$.

Now,

$[f(a)]^n = f(a)..........f(a)$ (f(a) written n times)

$= f(a^n)$ (since f is an isomorphism)

$= f(e)$

$= e'$.

Now, if possible let m be a positive integer such that $0 < m < n$ and $[f(a)]^m = e'$.

But $f(e) = e'$. Since f is $1 - 1$ we have $a^m = e$ which contradicts the definition of the order of a.

$\therefore$ n is the least positive integer such that $[f(a)]^n = e'$.

$\therefore$ The order of f(a) is n.

Theorem: 3.32

Let $f: G \rightarrow G'$ be an isomorphism. If G is cyclic then G' is also cyclic.

Proof:

Let a be a generator of the group G. We shall prove that $f(a)$ is a generator of the group G'.

Let $x' \in G'$. Since f is a bijeciton, there exists $x \in G$ such that $f(x) = x'$.

Now, since $G = \langle a \rangle, x = a^n$ for some integer n.

Hence $x' = f(x) = f(a^n) = [f(a)]^n$.

Since $x' \in G'$ is arbitrary every element of G' is of the form $[f(a)]^n$ so that $G' = \langle f(a) \rangle$.

Hence G' is cyclic.

Problems:

Problem: 1

Show that $(R^*, \cdot)$ is not isomorphic to (R, +).

Solution:

In (R, +) every element other than 0 is of infinite order. But in $(R^*, \cdot)$ there exists an element (other than 1) of finite order. For example, -1 is of order 2 in $(R^*, \cdot)$. Hence we cannot find an isomorphism form $(R^*, \cdot)$ to $(R, +)$. (by theorem 4.6).

Problem: 2

Show that $(Z_4, \oplus)$ is not isomorphic to V_4.

Solution:

In Z_4, 1 is an element of order 4. But in V_4 every element other than e is of order 2. Hence the two groups are not isomorphic.

This can also be proved by noticing that Z_4 is cyclic and V_4 is not cyclic.

Problem: 3

If G is a group and G' is a set with a binary operation and there exists a one – one mapping f from G onto G' such that $f(ab) = f(a)f(b)$ for all $a, b \in G$ then show that G' is also a group.

Solution:

Let $a, b, c \in G'$.

Since $f : G \to G'$ is a bijection, there exists $a, b, c \in G$ such that

$f(a) = a'; f(b) = b'; f(c) = c'$.

Since G is a group, $(ab)c = a(bc)$.

$\therefore \quad f[(ab)c] = f[a(bc)]$.

$\therefore \quad f(ab)f(c) = f(a)f(bc)$ (by hypothesis)

$\therefore \quad [f(a)f(b)]f(c) = f(a)[f(b)f(c)]$.

$\therefore \quad (a'b')c' = a'(b'c')$.

$\therefore$ The binary operation in G' is associative.

Now, let $e \in G$ be the identity element.

Let $a' \in G'$. Since $f : G \to G'$ is a bijection, there exists $a \in G$ such that $f(a) = a'$.

Now, $ae = ea = a$.

$\therefore \quad f(ae) = f(ea) = f(a)$.

$\therefore \quad f(a)f(e) = f(e)f(a) = f(a)$.

$\therefore \quad a'f(e) = f(e)a' = a'$.

$\therefore \quad f(e)$ is the identitiy in G'.

Let $a' \in G'$. Since $f : G \to G'$ is a bijeciton, there exists $a \in G$ such that $f(a) = a'$.

Now, $aa^{-1} = a^{-1}a = e$.

$\therefore \quad f(aa^{-1}) = f(a^{-1}a) = f(e)$.

$\therefore \quad f(a)f(a^{-1}) = f(a^{-1})f(a) = f(e)$.

$\therefore \quad a'f(a^{-1}) = f(a^{-1})a' = f(e)$.

$\therefore \quad f(a^{-1})$ is the inverse of a' in G'.

Hence G' is a group.

Problem: 4

Let G be any group. Show that $f : G \to G$ given by $f(x) = x^{-1}$ is an isomorphism $\Leftrightarrow$ G is abelian.

Solution:

Let $f : G \to G$ given by $f(x) = x^{-1}$ be an isomorphism.

We claim that G is abelian.

Let $x, y \in G$.

Then $f(x^{-1}y^{-1}) = f(x^{-1})f(y^{-1})$. (since f is an isomorphism).

$\therefore \quad (x^{-1}y^{-1})^{-1} = (x^{-1})^{-1}(y^{-1})^{-1}$.

$\therefore \quad (y^{-1})^{-1}(x^{-1})^{-1} = (x^{-1})^{-1}(y^{-1})^{-1}$.

$\therefore \quad yx = xy$.

Hence G is abelian.

Conversely, suppose G is abelian.

Clearly $f : G \to G$ given by $f(x) = x^{-1}$ is a bijeciton.

Now,

$$f(xy) = (xy)^{-1}$$
$$= y^{-1}x^{-1}$$
$$= x^{-1}y^{-1} \text{ (since G is abelian)}$$
$$= f(x)f(y).$$

$\therefore$ f is isomorphism.

Exercises:

1. Show that any two groups of order 2 are isomorphic.
2. Show that any two groups of order 3 are isomorphic.
3. Show that any proper subgroup of (**Z**, +) is isomorphic to (**Z**, +). (Hint: Any proper subgroup of (**Z**, +) is nZ).
4. Show that (**Q**, +) is not isomorphic to $(Q*, \cdot)$.
5. Show that (**C**, +) is not isomorphic to $(C*, \cdot)$.

6. Let $f: G \to G'$ be an isomorphism. Then if H is a subgroup of G, f(H) is a subgroup of G'.
7. Prove that for any positive integer $n, (Z/nZ) \cong Z_n$.

Theorem: 3.33

Any infinite cyclic group G is isomorphic to (**Z**, +)

Proof:

Let G be an infinite cyclic group with generator a. Then $G = \{a^n / n \in Z\}$.

Define $f: Z \to G$ by $f(n) = a^n$.

Since G is infinite, $n \neq m \Rightarrow a^n \neq a^m$.

Hence f is 1 – 1. Obviously f is onto.

Now, $f(n+m) = a^{n+m} = a^n a^m = f(n)f(m)$.

Hence f is an isomorphism.

Corollary:

Any two infinite cyclic groups are isomorphic to each other.

Let G and G' be two infinite cyclic groups. By theorem 4.8, $G \cong G'$ (since $\cong$ is an equivalence relation).

Theorem: 3.34

Any finite cyclic group of order n is isomorphic to $(Z_n, \oplus)$.

Proof:

Let G be a cyclic group of order n with generator a. Then $G = \{e, a, a^2, \ldots\ldots\ldots, a^{n-1}\}$.

Define $f: Z_n \to G$ by $f(r) = a^r$.

Clearly f is a bijection.

Now, let $r, s \in Z_n$. Let $r \oplus s = t$.

Then $r + s = qn + t$, where $0 \leq t < n$.

$$\therefore \qquad f(r \oplus s) = a^{r \oplus s} = a^t \qquad (1)$$

Also,

$$f(r)f(s) = a^r a^s = a^{r+s} = a^{qn+t} = a^{qn} a^t$$

$$= (a^n)^q a^t = ea^t = a^t \qquad (2)$$

From (1) and (2), we get $f(r \oplus s) = f(r)f(s)$.

Hence f is an isomorphism.

Corollary:

Any two finite cyclic groups of the same order are isomorphic.

UNIT –IV

CALCULUS

4.1 Introduction

Before we attempt to study the Fourier Series the following basic definitions related to function, limits and continuity become indispensable.

Periodic functions

A function f(x) is said to have a period T if for all x, f(x +T) = f(x), where T is a positive constant. The least value of T > 0 is called the period of f(x).

Example 4.1.1 we know that f(x) = sin (x + 2π) = sin (x + 4π)= ··· Therefore the function has periods 2π, 4π, 6π, etc. However, 2π is the least value and therefore is the period of f(x).

Similarly cos x is a periodic function with the period 2π and tan x has period π.

Example 4.12 The period of sinnx and cosnx where n is a positive integer is $\frac{2\pi}{n}$

i.e., the period of sin 2x is π, sin 3x is $\frac{2\pi}{3}$, etc.

A **constant** has any positive number as a period.

Limit of a function

A function f(x) is said to tend to a limit 'l' as x tends to 'a' if to each givenε> 0, there exists a positive number δ such that |f(x) - 1| <ε when 0 < |x - a| <δ.

This is denoted by $\underset{x \to a}{\text{Lt}}$ f(x) = 1.

Left-hand and Right-hand limits

f(x) is said to tend to l as x tends to 'a' through values less than a, if to each ε> 0, there exists δ> 0 such that |f(x) - 1| <ε when a - δ< x < a and is denoted by f(a - 0) = $\underset{x \to a-0}{\text{Lt}}$ f(x) is called right-hand limit.

Similarly if f(x) tends to l as x tends to 'a' through values which are greater than a, if there exist $\delta > 0$ such that $|f(x) - l| < \varepsilon$ when $a < x < a + \delta$, then f(x) is said to tend to l from the right and is denoted by $f(a + 0) = \underset{x \to a+0}{Lt} f(x)$ is called right-hand limit.

To find the left-hand limit i.e., f(a - 0) we first put x = a – h in f(x) and then take the limit as $h \to 0$.

Thus $f(a - 0) = \underset{h \to 0}{Lt} f(a - h)$

To find f(a + 0) we first put x = a + h in f(x) and then take the limit as $h \to 0$.

Thus $f(a + 0) = \underset{h \to 0}{Lt} f(a + h)$

Example 4.1.3 If $f(x) = x \sin \frac{1}{x}$ find (0 - 0) and f (0 + 0).

Solution: Here f (0 - 0). is left hand limit and f(0 + 0) is right hand limit.

Now, $f(0-0) = \underset{h \to 0}{Lt} (0 - h) \sin\left(\frac{1}{0-h}\right)$

$$= \underset{h \to 0}{Lt} h \sin \frac{1}{h} = 0$$

Now, $f(0 + 0) = \underset{h \to 0}{Lt} (0 + h) \sin\left(\frac{1}{0+h}\right)$

$$= \underset{h \to 0}{Lt} h \sin \frac{1}{h} = 0$$

Example 4.1.4 If $f(x) = 2^{\frac{1}{x-1}}$ find f(1 - 0) and f(1 + 0).

Solution: $f(1 - 0) = \underset{h \to 0}{Lt} 2^{\frac{1}{1-h-1}} = \underset{h \to 0}{Lt} 2^{-\frac{1}{h}}$

(Replace x by 1 – h) $= 2^{-\infty} = 0$

$$f(1+0) = \underset{h \to 0}{Lt}\ 2^{\frac{1}{1+h-1}} = \underset{h \to 0}{Lt}\ 2^{\frac{1}{h}}$$

$$= 2^{\infty} = \infty$$

Example 4.1.5 Findf(a - 0) and f(a + 0) for the function

$$f(x) = \begin{cases} \frac{x^2}{a} - a & \text{for } 0 < x < a \\ 0 & \text{for } x = a \\ a - \frac{a^3}{x^2} & \text{for } x > a \end{cases}$$

Solution: $$f(a-0) = \underset{h \to 0}{Lt}\left[\frac{(a-h)^2}{a} - a\right] = \frac{a^2}{2} - a = 0$$

[Replace x by a - h]

$$f(a+0) = \underset{h \to 0}{Lt}\left[a - \frac{a^3}{(a+h)^2}\right]$$

$$= a - \frac{a^3}{a^2} = 0$$

Example 4.1.6 Findf(a - 0) and f(a + 0) for the function

$$f(x) = \begin{cases} -1 & \text{if } x < a \\ 1 & \text{if } x \geq a \end{cases}$$

Solution: $$f(a-0) = \underset{h \to 0}{Lt} -1 = -1$$

$$f(a+0) = \underset{h \to 0}{Lt}\ 1 = 1$$

Example 4.1.7 Findf(0 - 0) and f(0 + 0) for the function

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Solution: $f(0 - 0) = \underset{h \to 0}{Lt} -\pi \quad = -\pi$

$$f(0 + 0) = \underset{h \to 0}{Lt} (0 + h) \quad = 0$$

Continuous Function

A function f(x) is said to be continuous at x = a if given ε> 0,however small, we can find a number δ> 0 such that |f(x) – f(a)|<ε when |x - a| <δ and is denoted by $\underset{x \to a}{Lt}$ f(x) = f(a). i.e., $\underset{x \to a}{Lt}$ f(x)exists if $\underset{x \to a-0}{Lt}$ f(x) and $\underset{x \to a+0}{Lt}$ f(x) exists and are equal.

f(x) is said to be continuous in an interval (a, b) if it is continuous at every point of the interval.

Discontinuous Function

A function f(x) is said to be discontinuous at a point if it is not continuous at that point.

Piecewise Continuous Function

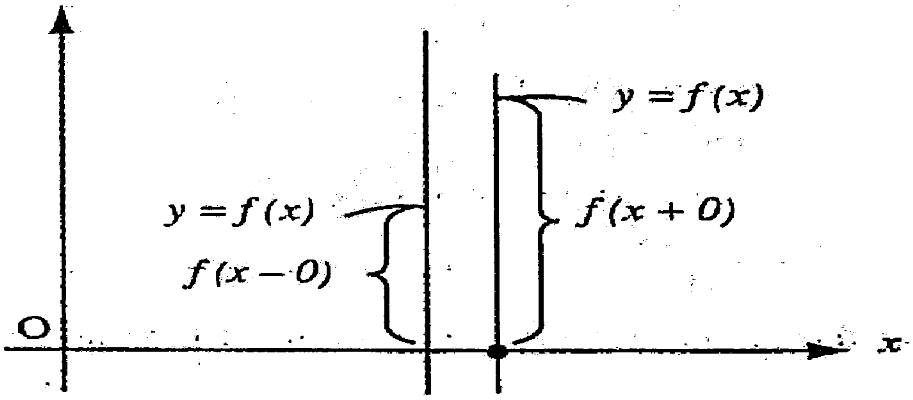


Fig.1

A function f(x) is said to be piecewise continuousin an interval if (i) the interval can be divided into a finite number of subintervals in each of which f(x) is continuous and (ii) the limits of f(x) as x approaches the end points of each subinterval are finite. Another way of stating this is to say that a piecewise continuous function is one that has at most a finite number of finite discontinuities. An example of a piecewise continuous function is shown in fig. The values of the left hand limit f(x - 0) and right hand limit f(x + 0) at the point x are as indicated in figure 1.

4.2 Fourier Series

If f(x) is a periodic function and satisfies Dirichlet conditions (to be described in subsequent article), then it can be represented by an infinite series called **Fourier Series** as

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots + a_n \cos nx$$

$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots + b_n \sin nx + \dots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Where a_0, a_n and b_n are called Fourier Coefficients.

Equation (1) is called trigonometric form of FourierSeries as we have another form called complex or exponential form of Fourier Series which will be studied at the end of this chapter.

Dirichlet Conditions

Suppose that

(i) f(x) is defined and single valued except possibly at a finite number of points in (-l, l).

(ii) f(x) is periodic with period 2l.

(iii) f(x) and $f'(x)$ are piecewise continuous in (-l, l).

Then the above series (1) converges to

(a) f(x) if x is a point of continuity

(b) $\frac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity.

Therefore the value of f(x) at any point of continuity x in(-*l*, *l*) is given by

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l}+b_n \sin\frac{n\pi x}{l}\right)$$

The value of f(x) at any point of discontinuity x in (-l, l) is given by $\frac{f(x+0)+f(x-0)}{2}$

The above conditions (i), (ii) and (iii) imposed on f(x) are sufficient but not necessary i.e., if the conditions are satisfied the convergence is guaranteed. However, if they are not satisfied the series may or may not converge. The conditions above are generally satisfied in cases which arise in science or engineering.

Determining the Fourier coefficient a_0, a_n and b_n

The Fourier Series for the function f(x) in the interval $c < x < c + 2\pi$ is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{ where}$$

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Thus values of a_0, a_n, b_n are known as **Euler's Formulae**. To establish these formulae, the following results will be required.

1. $\int_c^{c+2\pi} \cos nx\,dx = 0 \,(n \neq 0)$

2. $\int_c^{c+2\pi} \sin nx\,dx = 0 \,(n \neq 0)$

3. $\int_c^{c+2\pi} \cos mx \cos nx\,dx = 0 \,(m \neq n)$

4. $\int_c^{c+2\pi} \cos^2 nx\,dx = \pi \,(n \neq 0)$

5. $\int_c^{c+2\pi} \sin mx \cos nx\,dx = 0 \,(m \neq n)$

6. $\int_c^{c+2\pi} \sin nx \cos nx\,dx = 0$

7. $\int_c^{c+2\pi} \sin mx \sin nx\,dx = 0 \,(m \neq n)$

8. $\int_c^{c+2\pi} \sin^2 nx = \pi \,(n \neq 0)$

Proof of Euler Formulae

Let f(x) be represented in the interval (c, c +2π) by the Fourier Series:

$$f(x)= \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

To find the coefficients a_0, a_n, b_n we assume that the series (1) can be integrated term by term from x = c to x = c + 2π.

To find a_0, integrate both sides of (1) from x = c to x = c + 2π. Then,

$$\int_c^{c+2\pi} f(x)\,dx=\frac{1}{2}a_0\int_c^{c+2\pi} dx+\int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)dx+\int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)dx$$

$$=\frac{a_0}{2}[x]_c^{c+2\pi}+\sum_{n=1}^{\infty} a_n\int_c^{c+2\pi}\cos nx\,dx+\sum_{n=1}^{\infty} b_n\int_c^{c+2\pi}\sin nx\,dx$$

$$=\frac{1}{2}a_0\,[(c+2\pi-c)+0+0] \qquad \text{[by (1) and (2)]}$$

$$= a_0\pi$$

$$\therefore\ \mathbf{a_0}=\frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$

To find a_n, **multiply** each side of (1) by cosnx and integrate from x = c to x = c + 2π. Then

$$\int_c^{c+2\pi} f(x)\cos nx\,dx=\frac{1}{2}a_0\int_c^{c+2\pi}\cos nx\,dx$$

$$+\int_c^{c+2\pi}\left(\sum_{n-1}^{\infty} a_n \cos nx\,dx\right)\ \cos nx\,dx+\int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\ \cos nx\,dx$$

$$=\frac{1}{2}a_0(0)+\sum_{n=1}^{\infty} a_n\int_c^{c+2\pi}\cos nx\cos nx\,dx+\sum_{n=1}^{\infty} b_n\int_c^{c+2\pi}\sin nx\cos nx\,dx$$

$$= 0+\pi a_n+0 \qquad \text{[by integrals (1), (4) and (6)]}$$

$$\therefore\ \mathbf{a_n}=\frac{1}{\pi}\int_c^{c+2\pi} \mathbf{f(x)\cos nx\,dx}$$

To find b_n, multiply each side of (1) by sin nx and integrate from x = c to x = c + 2π. Then

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi} \sin nx\,dx +$$

$$+\int_c^{c+2\pi}\left(\sum_{n-1}^{\infty} a_n \cos nx\,dx\right)\ \sin nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\ \sin nx\,dx$$

$= 0 + 0 + \pi b_n$ [bi integrals (2), (6) and (8)]

$$\mathbf{b_n} = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

Corollary 1 : Putting c = 0, the interval becomes $0 < x < 2\pi$, and the formula (1) reduces to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Corollary 2: Putting c = -π, the intervals becomes $-\pi < x < \pi$ and the formula (1) becomes

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin nx\ \ dx$$

Note: $\cos n\pi = (-1)^n$, when 'n' is an integer.

$\sin n\pi = 0$, when 'n' is an integer.

Example 4.2.1 **Expand $f(x) = (\pi\text{-}x)^2$in $(-\pi, \pi)$.**

Solution: we know that a Fourier series for the function f(x) in the interval $(-\pi,\pi)$ is given by.

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nx + \sum_{1}^{\infty} b_n \sin x \qquad \text{..................(A)}$$

Here $f(x) = (\pi - x)^2$ in $(-\pi, \pi)$

Now $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - x)^2\ dx = \frac{1}{\pi}\left[\frac{(\pi - x)^3}{-3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[0 - \frac{(2\pi)^3}{-3}\right] = \frac{1}{\pi}\left[\frac{8\pi^3}{3}\right]$$

$$\boxed{a_0 = \frac{8\pi^2}{3}} \qquad \text{........................(1)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)^2 \cos nx\, dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2 \frac{\sin nx}{n} - 2(\pi - x)(-1)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(0+0+0) - \left(0 - 2(2\pi)(-1)\left(\frac{-\cos n\pi}{n^2}\right) + 0\right)\right]$$

$$= \frac{1}{\pi}\left[(0) - \left(\frac{-4\pi\cos n\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi\cos n\pi}{n^2}\right]$$

$$\boxed{a_n = \frac{4}{n^2}(-1)^n} \qquad \text{.......................(2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)^2 \sin nx\, dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2\left(\frac{-\cos nx}{n}\right) - 2(\pi - x)(-1)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(0 - 0 + \frac{2\cos n\pi}{n^3}\right) - \left(\frac{-(2\pi)^2\cos n\pi}{n} - 0 + \frac{2\cos n\pi}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{2\cos n\pi}{n^3}\right) - \left(\frac{-4\pi^2\cos n\pi}{n} + \frac{2\cos n\pi}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi^2\cos n\pi}{n}\right]$$

$$\boxed{b_n = \frac{4\pi}{n}(-1)^n} \qquad \text{.......................(3)}$$

From (1), (2) and (3), we get

$$a_0 = \frac{8\pi^2}{3}, \qquad a_n = \frac{4(-1)^n}{n^2}, \qquad b_n = \frac{4\pi}{n}(-1)^n$$

Substituting these values in (A), we get

$$f(x)=\frac{4\pi^2}{3}+\sum_1^\infty \frac{4(-1)^n}{n^2}\cos nx+\sum_1^\infty \frac{4\pi}{n}(-1)^n \sin nx$$

$$=\frac{4\pi^2}{3}+4\sum_1^\infty \frac{(-1)^n}{n^2}\cos nx+4\pi\sum_1^\infty \frac{(-1)^n \sin nx}{n}$$

Example 4.2.2 (a) (i) Obtain the Fourier Series for

$f(x) = 1 + x + x^2$ in $(-\pi, \pi)$.

Deduce that $\frac{1}{1^2}+\frac{1}{2^2}+\ldots\ldots\ldots\ldots\ldots\ldots=\frac{\pi^2}{6}$.

Solution: The Fourier Series of f(x) in $(-\pi, \pi)$ is

$$f(x)=\frac{a_0}{2}+\sum_1^\infty a_n \cos nx+\sum_1^\infty b_n \sin nx \qquad \ldots\ldots\ldots\ldots(1)$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x^2+x+1\right)dx=\frac{1}{\pi}\left[\frac{x^3}{3}+\frac{x^2}{2}+x\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{\pi^3}{3}+\frac{\pi^2}{2}+\pi+\frac{\pi^3}{3}-\frac{\pi^2}{2}+\pi\right]$$

$$a_0=\frac{1}{\pi}\left[\frac{2\pi^3}{3}+2\pi\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x^2+x+1\right)\cos nx\,dx$$

$$=\frac{1}{\pi}\left[\left(x^2+x+1\right)\left(\frac{\sin nx}{n}\right)-(2x+1)\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{(2\pi+1)\cos n\pi}{n^2}-\frac{(-2\pi+1)\cos nx}{n^2}\right]$$

$$=\frac{1}{n^2\pi}\left[4\pi\cos n\pi\right]=\frac{4}{n^2}(-1)^n \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin nx\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(1+x+x^2\right)\sin nx\ dx$$

$$= \frac{1}{\pi}\left[\left(x^2+x+1\right)\left(\frac{-\cos nx}{n}\right)-(2x+1)\left(\frac{-\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\left(\pi^2+\pi+1\right)\cos n\pi}{n}+\frac{2\cos n\pi}{n^3}+\frac{\left(\pi^2-\pi+1\right)\cos n\pi}{n}-\frac{2\cos n\pi}{n^3}\right]$$

$$= \frac{\cos n\pi}{n\pi}\left[-\pi^2-\pi-1+\pi^2-\pi+1\right]$$

$$b_n = \frac{(-1)^n}{n\pi}\left[-2\pi\right] = \frac{2(-1)^{n+1}}{n} \qquad \text{...................(4)}$$

Substituting (2), (3) and (4) in (1), we get

$$f(x) = \frac{1}{2\pi}\left(\frac{2\pi^3}{3}+2\pi\right)+\sum_{1}^{\infty}\frac{4}{n^2}(-1)^n\ \cos nx + \sum_{1}^{\infty}\frac{2(-1)^{n+1}}{n}\ \sin nx \quad \text{.....(5)}$$

Put x = π in (5), we get

$$\pi^2+\pi+1 = \frac{\pi^2}{3}+1+4\sum_{1}^{\infty}\frac{1}{n^2}$$

$$\text{i.e.,} \qquad \frac{2\pi^2}{3}+\pi = 4\sum\frac{1}{n^2} \qquad \text{.............(6)}$$

Put x = -π in (5), we get

$$\pi^2-\pi+1 = \frac{\pi^2}{3}+1+4\sum\frac{1}{n^2}$$

$$\frac{2\pi^2}{3}-\pi = 4\sum_{1}^{\infty}\frac{1}{n^2} \qquad \text{...........(7)}$$

$$(6)+(7) \Rightarrow \quad \frac{4\pi^2}{3} = 8\sum_{1}^{\infty}\frac{1}{n^2} \text{ i.e., } \sum_{1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Example 4.2.3 Express f(x) = x sin x as a Fourier Series in o ≤ x ≤ 2π.

Solution: we know that a Fourier Series for the function f(x) in the interval [0, 2π] is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad .....(A)$$

Here f(x) = x sin x

Now $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x\,dx$

$$= \frac{1}{\pi}\left[x(-\cos x) - 1(-\sin x)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}[-2\pi] = -2 \quad [\because \sin 2\pi = 0, \cos 2\pi = 1] \quad ....(1)$$

$$\therefore \boxed{a_0 = 2} \quad ....(1)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$\left[\because \cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]\right]$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - \left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right\}\right]$$

(Since sin 2(n + 1) π = 0, sin 2 (n - 1) π = 0 and cos 2(n + 1) π = 1, cos 2(n - 1) π = 1, whether n is odd or even.)

$$a_n = -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1} \text{ provided} n \neq 1 \quad ....(2)$$

When n = 1, we have

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx$$

$$[\because \sin 2x = 2 \sin x \cos x]$$

$$= \frac{1}{2\pi}\left\{x\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right)\right\}_0^{2\pi}$$

$$= \frac{1}{2\pi}[-\pi] = -\frac{1}{2} \quad ....(3)$$

$$\therefore \boxed{a_n = -\frac{1}{2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,[\cos(n-1)x - \cos(n+1)x]\,dx$$

$$[\because 2 \sin A \sin B = \cos(A-B) - \cos(A+B)]$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\} - 1\left\{-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$\therefore b_n = 0$ provided $n \neq 1$. ....(4)

When n = 1, we have

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,(1-\cos 2x)\,dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi(2\pi) - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \pi \qquad \text{........(5)}$$

From (A) we get

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=2}^{\infty} b_n \sin nx$$

From (1), (2), (3), (4) and (5) we get

$$a_0 = -2, \quad a_n = \frac{2}{n^2-1}, \; (n \neq 1)\, a_1 = -\frac{1}{2}, \quad b_n = 0, \; b_1 = \pi$$

Substituting these values in (A) we get,

$$f(x) = -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx$$

$$x \sin x = -1 + \pi \sin x - \frac{1}{2}\cos x + \frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x + \text{........}$$

Example 4.2.4 **Express $f(x) = (\pi - x)^2$ as a Fourier Series of period 2π in the interval $0 < x < 2\pi$. Hence deduce the sum of the series**

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \text{........}$$

[Nov.2000, Apr.89, Mech]

Solution: we know that a Fourier Series for the function f(x) in the interval $[0, 2\pi]$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \qquad ......(A)$$

Here $f(x) = (\pi - x)^2$ in $[0, 2\pi]$

Now $$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}(\pi-x)^2\,dx$$

$$=\frac{1}{\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{2\pi}=\frac{1}{\pi}\left[\frac{\pi^3}{3}+\frac{\pi^3}{3}\right]=\frac{2\pi^2}{3}$$

$$\therefore \boxed{a_0=\frac{2\pi^2}{3}} \qquad ....(1)$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}(\pi-x)^2\cos nx\,dx$$

$$=\frac{1}{\pi}\left[(\pi-x)^2\frac{\sin nx}{n}-2(\pi-x)(-1)\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[\frac{2\pi\cos 2n\pi}{n^2}+\frac{2\pi}{n^2}\right]=\frac{4}{n^2} \qquad [\because \cos 2n\pi=1]$$

$$\boxed{a_n=\frac{4}{n^2}} \qquad ....(2)$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{2\pi}(\pi-x)^2\sin nx\,dx$$

$$=\frac{1}{\pi}\left[(\pi-x)^2\left(\frac{-\cos nx}{n}\right)-2(\pi-x)(-1)\left(\frac{-\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[\frac{-\pi^2\cos 2n\pi}{n}+\frac{2\cos 2n\pi}{n^3}+\frac{\pi^2}{n}-\frac{2}{n^3}\right]$$

$$=\frac{1}{\pi}\left[\frac{-\pi^2}{n}+\frac{2}{n^3}+\frac{\pi^2}{n}-\frac{2}{n^3}\right]=0 \qquad (\because \cos 2n\pi=1)$$

$$\therefore \boxed{b_n=0} \qquad ....(3)$$

From (1), (2) and (3) we get $a_0=\frac{2\pi^2}{3}$, $a_n=\frac{4}{n^2}$, $b_n=0$

Substituting these values in (A) we get,

$$f(x)=\frac{\pi^3}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx$$

$$f(x)=(\pi-x)^2=\frac{\pi^2}{3}+4\left[\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\ldots\right] \quad \ldots..(4)$$

Putting x = 0 in (4) we have

$$f(0)=\frac{\pi^2}{3}+4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\right] \quad \ldots\ldots\ldots(5)$$

Here 0 is a point of discontinuity. Therefore the value of the function f(x) at x = 0, is given by

$$f(0)=\frac{f(0-0)+f(0+0)}{2}=\frac{\pi^2+\pi^2}{2}=\pi^2$$

i.e., $f(0)=\pi^2$

Substituting (6) in (5), we get,

$$\therefore \pi^2=\frac{\pi^2}{3}+4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots..\right]$$

$$\therefore \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots. = \frac{3\pi^2-\pi^2}{12}$$

i.e., $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots. = \frac{\pi^2}{6}$

Example 4.2.5 **Find the Fourier Series for the function f(x) = e^x defined in (-π, π).** [Nov.91, 86, Civil, Apr.89, ECE]

Solution: We know that a Fourier Series for the function f(x) in (-π, π) is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \quad \ldots\ldots..(A)$$

Here f(x) = e^x

Now $a_0 = \frac{1}{n} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x\, dx$

$$= \frac{1}{n}\left[e^x\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[e^{\pi} - e^{-\pi}\right]$$

$$\therefore \boxed{a_o = \frac{2}{\pi} \sin h\pi} \qquad \text{..........................(1)}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1^2 + n^2}(1 \cos nx + n \sin nx)\right]_{-\pi}^{\pi}$$

$$\left[\because \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)\right]$$

$$= \frac{1}{(1^2 + n^2)\,\pi}\left[e^{\pi} \cos n\pi - e^{-\pi} \cos n\pi\right]$$

$$= \frac{\cos n\pi (e^{\pi} - e^{-\pi})}{\pi(1 + n^2)} = \frac{2(-1)^n \sin h\pi}{\pi(1 + n^2)} \qquad \text{............(2)}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1 + n^2}(\sin nx - n \cos nx)\right]_{-\pi}^{\pi}$$

$$\left[\because \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)\right]$$

$$= \frac{1}{\pi(1 + n^2)}\left[-ne^{\pi} \cos n\pi + ne^{-\pi} \cos n\pi\right]$$

$$= \frac{n(-1)^n\left[e^{-\pi} - e^{\pi}\right]}{\pi(1 + n^2)} = \frac{2n(-1)^{n+1} \sin h\pi}{\pi(1 + n^2)} \qquad \text{.........(3)}$$

Now we have

$$a_0 = \frac{2 \sin h\pi}{\pi}, \quad a_n = \frac{2(-1)^n \sin h\pi}{\pi(1+n^2)}, \quad b_n = \frac{2n(-1)^{n+1} \sin h\pi}{\pi(1+n^2)}$$

Substituting these values in (A) we get,

$$f(x) = \frac{\sin h\pi}{\pi} + \frac{2 \sin h\pi}{\pi}\left[\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} \cos nx - \sum_{n=1}^{\infty} n\frac{(-1)^n}{1+n^2} \sin nx\right]$$

Example 4.2.6 Determine the Fourier Series expansion of x + x² in the interval (-π, π) and hence deduce the sum of series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$

[Anna Univ.Nov.2001]

Solution: we know that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{.........(A)}$$

Here $f(x) = x + x^2$ in $[-\pi, \pi]$

$$\text{Now} \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} + \frac{\pi^3}{3}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^3}{3}\right)\right]$$

$$= \frac{1}{\pi}\frac{2\pi^3}{3} = \frac{2}{3}\pi^2$$

$$\therefore \boxed{a_0 = \frac{2\pi^2}{3}} \qquad \text{.........(1)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x + x^2\right)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\left(x + x^2\right)\left(\frac{\sin nx}{n}\right) - (1 + 2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(1 + 2\pi)\cos n\pi}{n^2} - \frac{(1 - 2\pi)\cos n\pi}{n^2}\right]$$

$$= \frac{\cos n\pi}{n^2\pi}\left[1 + 2\pi - 1 + 2\pi\right]$$

$$\boxed{a_n = \frac{4}{n^2}(-1)^n} \quad \text{.........................(2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x + x^2\right)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\left(x + x^2\right)\left(\frac{-\cos nx}{n}\right) - (1 + 2x)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\left(\pi + \pi^2\right)(-\cos n\pi)}{n} + \frac{2\cos n\pi}{n^3} + \frac{\left(-\pi + \pi^2\right)\cos n\pi}{n} - \frac{2\cos n\pi}{n^3}\right]$$

$$= \frac{\cos n\pi}{n\pi}\left[-\pi - \pi^2 - \pi + \pi^2\right] = -\frac{2}{n}(-1)^n \quad \text{..............(3)}$$

$$\therefore \boxed{a_n = -\frac{2}{n}(-1)^n} \qquad \left[\because \cos n\pi = (-1)^n\right]$$

Hence we have $a_0 = \frac{2\pi^2}{3}$, $a_n = \frac{4}{n^2}(-1)^n$, $b_n = -\frac{2}{n}(-1)^n$.

Substituting these values in (A), we get

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{-2}{n}(-1)^n\sin nx$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx - 2\sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n}$$

i.e., $$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{4}{n^2} \cos nx - \frac{2}{n} \sin nx\right] \qquad \text{.........(4)}$$

Putting x = π in (4) we get

$$f(\pi) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^{2n} \frac{4}{n^2}$$

Here π is a point of discontinuity. Therefore the value f(x) at x = π is given by

$$f(\pi) = \frac{f(\pi - 0) + f(\pi + 0)}{2}$$

$$= \frac{\pi + \pi^2 + \pi + \pi^2}{2} = \pi^2 + \pi$$

Hence $$\pi + \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^{2n} \frac{4}{n^2} \qquad \text{........(5)}$$

Putting x = -π in (4) we get

$$f(-\pi) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^{2n} \frac{4}{n^2}$$

Here x = -π is a point of discontinuity. Therefore

$$f(-\pi) = \frac{f(-\pi - 0) + f(-\pi + 0)}{2}$$

$$= \frac{(-\pi + \pi^2) + (-\pi + \pi^2)}{2}$$

$$= -\pi + \pi^2$$

Hence $$-\pi + \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^{2n} \frac{4}{n^2} \qquad \text{........(6)}$$

Adding (5) and (6), we get

$$2\pi^2 = \frac{2\pi^2}{3} + 2\sum_{n=1}^{\infty}(-1)^{2n}\frac{4}{n^2}$$

i.e., $\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$ $\qquad \left[\because (-1)^{2n} = 1\right]$

i.e., $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{3\pi^2 - \pi^2}{3 \times 4} = \frac{2\pi^2}{12} = \frac{\pi^2}{6}$

i.e., $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots = \frac{\pi^2}{6}$

Example 4.2.7Find the Fourier Series for f(x) if

$$\left.\begin{aligned} \mathbf{f(x)} &= -\pi \text{ in } -\pi < x < 0 \\ &= x \text{ in } 0 < x < \pi \end{aligned}\right\}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$ [Apr.86, Civil, Apr.86, Mech]

Solution: The Fourier Series for the function f(x) in (-π,π) is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \qquad \text{......(A)}$$

Now $\quad a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}f(x)\,dx+\int_{0}^{\pi}f(x)\,dx\right]$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}-\pi\,dx+\int_{0}^{\pi}x\,dx\right]$$

$$=\frac{1}{\pi}\left[(-\pi x)_{-\pi}^{0}+\left(\frac{x^2}{2}\right)_0^{\pi}\right]$$

$$=\frac{1}{\pi}\left[-\pi^2+\frac{\pi^2}{2}\right]$$

$$\therefore \boxed{a_0=-\frac{\pi}{2}} \qquad \text{...........(1)}$$

Now $\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}f(x)\cos nx\,dx+\int_{0}^{\pi}f(x)\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}-\pi\cos nx\,dx+\int_{0}^{\pi}x\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left[\left(-\frac{\pi\sin nx}{n}\right)_{-\pi}^{0}+\left\{\frac{x\sin nx}{n}-\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi}\right]$$

$$=\frac{1}{\pi}\left[\frac{1}{n^2}(\cos n\pi-1)\right]=\frac{1}{n^2\pi}\left[(-1)^n-1\right]$$

$\therefore a_n = 0$ when n is even

$= \frac{-2}{n^2\pi}$ when n is odd $\qquad$ ....(2)

$$\text{Now } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx dx + \int_{0}^{\pi} f(x)\sin nx dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nx dx + \int_{0}^{\pi} x\sin nx dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi\cos nx}{n}\right)_{-\pi}^{0} + \left\{x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{\pi}{n}(-1)^n - \frac{\pi(-1)^n}{n}\right]$$

$$b_n = \frac{1}{n}[\ 1-2(-1)^n\ ] \qquad \text{..............(3)}$$

Substituting (1),(2) and (3) in (A), We get

$$f(x) = \frac{-\pi}{4} + \sum_{n=1,3,5}^{\infty} \frac{-2}{n^2\pi}\cos nx + \sum_{n=1}^{\infty} \frac{1}{n}[1-2(-1)^n]\sin nx$$

$$\text{i.e., } f(x) = \frac{-\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots..\right)$$

$$+ 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} \ldots.. \quad (3a)$$

Putting x =0 in (3a), we get

$$f(0) = \frac{-\pi}{4} - \frac{2}{\pi}(1 + \frac{1}{3^2} + \frac{1}{5^2} + ...) \quad ....(4)$$

Here 0 is a point of discontinuity. Hence,

$$f(0) = \frac{f(0-0)+f(0+0)}{2}$$

$$= \frac{-\pi+0+0+0}{2} = \frac{-\pi}{2} \quad ....(5)$$

[Refer to left and right hand limit definition]

From (4) and (5) we get,

$$\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{2}{\pi}(1 + \frac{1}{3^2} + \frac{1}{5^2}....)$$

i,e., $\frac{-\pi}{4} = \frac{-2}{\pi}(1 + \frac{1}{3^2} + \frac{1}{5^2} + ...)$

i.e., $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + ...$

Example 4.2.8

If $f(x) = 0 \quad -\pi \le x \le 0$
$\quad\quad = \sin x \quad 0 \le x \le \pi$

Prove that $f(X) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\text{cons}2nx}{4n^2-1}$.

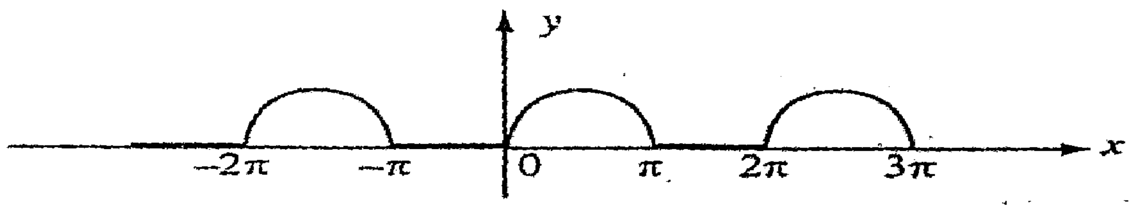
Hence show that

(i) $$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots=\frac{1}{2}$$

(ii) $$\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}+\ldots=\frac{\pi-2}{4}$$

[Anna Univ.Apr,2001]

Solution: The graph of the given function is shown below:



The Fourier Series of f (x) in (-π,π) is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx \ldots\ldots\ldots\ldots\ldots\ldots\ldots(A)$$

Now $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx+\int_{0}^{\pi} f(x)dx\right]$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx+\int_{0}^{\pi} \sin x\, dx\right]$$

$$=\frac{1}{\pi}[-\cos x]_0^{\pi}=\frac{2}{\pi}$$

$$a_0=\frac{2}{\pi} \qquad \ldots\ldots\ldots\ldots(1)$$

Now $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx dx+\int_{0}^{\pi} f(x)\cos nx dx\right]$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cos nx dx+\int_{0}^{\pi} \sin x\cos nx dx\right]$$

$$=\frac{1}{2\pi}\left[\int_{0}^{\pi} \{\sin(n+1)x+\sin(1-n)x\}dx\right]$$

$$=\frac{1}{2\pi}\left[\left(-\frac{\cos(n+1)x}{n+1}\right)_{0}^{\pi}-\left(\frac{\cos(1-n)x}{1-n}\right)_{0}^{\pi}\right]$$

$$=\frac{1}{2\pi}\left[\left(-\frac{\cos(n+1)\pi}{n+1}\right)_{0}^{\pi}+\frac{1}{n+1}+\left(\frac{\cos(n-1)\pi}{n-1}\right)-\frac{1}{n-1}\right]$$

When n is odd cos (n+1) π =1,cos (n-1) π=1

When n is even cos (n+1) π = -1,cos (n-1) π =-1

$\therefore$ When n is odd,

$$a_n=\frac{1}{2\pi}\left[-\frac{1}{n+1}+\frac{1}{n+1}+\frac{1}{n-1}-\frac{1}{n-1}\right]=0$$

$$\boxed{a_n=0}$$

When n is even .... (2)

$$a_n=\frac{1}{2\pi}\left[\frac{1}{n+1}+\frac{1}{n+1}-\frac{1}{n-1}-\frac{1}{n-1}\right]=\frac{-2}{\pi(n^2-1)}$$

i.e., = $a_n=\dfrac{-2}{\pi(n^2-1)}$ $(n\neq1)$ ....(3)

When n=1, we have

$$a_1=\frac{1}{\pi}\int_{0}^{\pi} \sin x\cos x dx$$

$$=\frac{1}{2\pi}\int_0^{\pi} \sin 2x \ dx = \frac{1}{2\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} = 0$$

ie., $\boxed{a_1=0}$ ....(4)

Now $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0} 0\sin nx dx + \int_0^{\pi} \sin x \sin nx dx\right]$$

$$=\frac{1}{2\pi}\int_0^{\pi} 2\sin nx \sin x dx$$

$$=\frac{1}{2\pi}\int_0^{\pi} [\cos(n-1)x - \cos(n+1)x] dx$$

$$=\frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi} = 0$$

$\therefore$ $\boxed{b_n=0}$ provided $n\neq 1$ ......(5)

When n=1,we have

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \sin x dx = \frac{1}{\pi}\int_0^{\pi} \sin^2 x dx$$

$$=\frac{1}{\pi}\int_0^{\pi} \frac{1-\cos 2x}{2} dx$$

$$=\frac{1}{2\pi}\left[x-\frac{\sin 2x}{2}\right]_0^{\pi}=\frac{1}{2}$$

$$\therefore \quad \boxed{b_1=\frac{1}{2}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Substituting (1),(2),(3),(4),(5) and (6) in (A), we get

$$f(x)=\frac{1}{\pi}-\frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1}+\frac{\cos 4x}{4^2-1}+\frac{\cos 6x}{6^2-1}+\ldots..\right]+\frac{1}{2}\sin x$$

$$f(x)=\frac{1}{\pi}+\frac{1}{2}\sin x-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(2n)^2-1} \qquad \ldots\ldots\ldots\ldots(7)$$

Putting x =0 in (7) we get

$$\text{i.e.,} \quad f(0)=\frac{1}{\pi}-\frac{2}{\pi}\sum_{1}^{\infty}\frac{1}{4n^2-1}$$

$$\text{i.e.,}\ 0=\frac{1}{\pi}-\frac{2}{\pi}\sum_{1}^{\infty}\frac{1}{4n^2-1}$$

$$\text{i.e.,}\ \frac{1}{2}=\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$=\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}$$

$$\text{i,e.,}\ \frac{1}{2}=\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots..$$

Putting $x=\frac{\pi}{2}$ in (7),we get,

$$f\left(\frac{\pi}{2}\right)=\frac{1}{\pi}+\frac{1}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi}{4n^2-1}$$

i.e., $1= \left(\frac{\pi}{2}\right)=\frac{1}{\pi}+\frac{1}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi}{4n^2-1}$

(Here $x=\frac{\pi}{2}$ is a point of continuity).

$$\Rightarrow \frac{1}{2}-\frac{1}{\pi}=-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}$$

$$\Rightarrow \frac{\pi-2}{4}=-\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)(2n+1)}$$

$$=-\left(\frac{-1}{1.3}+\frac{1}{3.5}-\frac{1}{5.7}+.....\right)$$

i.e., $\frac{\pi-2}{4}=\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}+.........$

Example 4.2.9If $-\pi< x <\pi$ and α is not an integer show that

$$\cos\alpha x=\frac{\sin\pi\alpha}{\pi\alpha}+\sum_{n=1}^{\infty}(-1)^n\frac{2\alpha\sin\pi\alpha}{\pi(\alpha^2-n^2)}\cos nx$$

Solution: Here $f(x)=\cos\alpha x$.

We know that

$$f(x)\ \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \qquad ....(A)$$

Now $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\alpha x dx = \frac{1}{\pi}\left[\frac{\sin\alpha x}{\alpha}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi\alpha}[\sin\alpha\pi + \sin\alpha\pi]$$

$\therefore$ $\boxed{a_0 = \frac{2\sin\alpha\pi}{\alpha\pi}}$ ....(1)

Here $\sin \alpha\pi \neq 0$, since α is not an integer.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\alpha x.\cos nx dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \{\cos(\alpha+n)x + \cos(\alpha-n)x\}dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(\alpha+n)x}{\alpha+n} + \frac{\sin(\alpha-n)x}{\alpha-n}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\sin(\alpha+n)\pi}{\alpha+n} + \frac{\sin(\alpha-n)\pi}{\alpha-n} + \frac{\sin(\alpha+n)\pi}{\alpha+n} + \frac{\sin(\alpha-n)\pi}{\alpha-n}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2\sin(\alpha+n)\pi}{\alpha+n} + \frac{2\sin(\alpha-n)\pi}{\alpha-n} + \right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(a+n)\pi}{a+n} + \frac{\sin(a-n)\pi}{a-n}\right]$$

$$=\frac{1}{\pi}\left[\frac{(\alpha-n)\sin(\alpha+n)\pi+(\alpha+n)\sin(\alpha-n)\pi}{(\alpha+n)(\alpha-n)}\right]$$

$$=\frac{\alpha}{\pi}\left[\frac{\sin(\alpha+n)\pi+\sin(\alpha-n)\pi}{\alpha^2-n^2}\right]$$

$$=\frac{\alpha}{\pi}\left[\frac{\sin\alpha\pi\cos n\pi+\cos\alpha\pi\sin n\pi+\sin\alpha\pi\cos n\pi-\cos\alpha\pi\sin n\pi}{\alpha^2-n^2}\right]$$

$$=\frac{\alpha}{\pi}\left[\frac{2\sin\alpha\pi\cos n\pi}{\alpha^2-n^2}\right] \qquad [\because \sin n\pi=0]$$

$$\therefore \quad \boxed{a_n\frac{2\alpha\sin\alpha\pi(-1)^n}{\pi(\alpha^2-n^2)}}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi} \cos\alpha x\sin nx dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\left\{\frac{\sin(\alpha+n)x-\sin(\alpha-n)x}{2}\right\}dx$$

$$=\frac{1}{2\pi}\left[\frac{-\cos(\alpha+n)x}{\alpha+n}+\frac{\cos(\alpha-n)x}{\alpha-n}\right]_{-\pi}^{\pi}$$

$$=\frac{1}{2\pi}\left[\left\{\frac{-\cos(\alpha+n)\pi}{\alpha+n}+\frac{\cos(\alpha-n)\pi}{\alpha-n}\right\}-\left\{\frac{-\cos(\alpha+n)\pi}{\alpha+n}+\frac{\cos(\alpha+n)\pi}{\alpha-n}\right\}\right]$$

$$=\frac{1}{2\pi}\times 0=0$$

$\therefore$ $\boxed{b_n = 0}$ ....(3)

Substituting (1),(2) and (3), we get

$$\cos\alpha x = \frac{\sin\alpha\pi}{\pi\alpha} + \sum_{n=1}^{\infty} \frac{2\alpha(-1)^n \sin\alpha\pi}{\pi(\alpha^2 - n^2)} \cos nx$$

$$= \frac{\sin\alpha\pi}{\pi\alpha} + \frac{2\alpha}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sin\alpha\pi}{(\alpha^2 - n^2)} \cos nx$$

Example 4.2.10If f (x) is a periodic function defined over a period(0,2π) by f (x) $=\frac{1}{12}(3x^2 \ \ -6x\pi+2\pi^2)$. Prove that f (x) $=\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ and hence show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots$$

Solution:we know that the Fourier series for f (x) in (0,2π) is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots\ldots\ldots..(1)$$

To find a_0

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$= \frac{1}{12\pi}\left[x^3 - 3x^2\pi + 2\pi^2 x\right]_0^{2\pi}$$

$$= \frac{1}{12\pi}[8\pi^3 - 12\pi^3 + 4\pi^3]$$

$a_0 = 0$ ....(2)

To find a_n

$$a_n = \int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{12}(3x^2 - 6x\pi + 2\pi^2)\cos nx\,dx$$

$$= \frac{1}{12\pi}\left[(3x^2 - 6x\pi + 2\pi^2)\left(\frac{\sin nx}{n}\right) - (6x-6\pi)\left(\frac{-\cos nx}{n^2}\right) + 6\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

Substituting (2) in (1), we get

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}\sin nx$$

$$\therefore \frac{x(\pi^2 - x^2)}{12} = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3}$$

4.3 Even and odd functions

A function f (x) is said to be even if f(-x) = f(x).

Example: x^2, $\cos x$, $\sin^2 x$, $|x|$, x sin x are even functions.

A function f(x) is said to be odd if f (-x) =-f(x).

Example: x^3, $\sin x$, $\tan^3 x$ are odd functions.

Note 1: The product of two even functions or two odd functions is an even function while the product of an even function and an odd function is an odd function.

Note 2 : $\int_{-\pi}^{\pi} f(x)dx = 0$ if f (x) is an odd function

(or)

$\int_{-\pi}^{\pi} f(x)dx = 2\int_{0}^{\pi}$ f(x)dx, if f (x) is an even function

Similarly $\int_{-l}^{l} f(x)dx = 0$, if (x) dx if f (x) is an odd function.

(or)

$\int_{-l}^{l} f(x)dx = 2\int_{0}^{l}$ f(x) dx if f (x) is an even function

Note 3: When f(x) is an even function, the Euler's coefficients becomes

(Anna Univ.Nov.2001)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nxdx$$

[∵Both f(x) and cosnx are even,the product f(x) cosnx is also even]

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = 0$$

[∵f(x) is even,sinnx is odd, the product f (x) sin nx is odd function.]

Therefore,if a function f(x) is even,its Fourier expansion contains only cosine terms.

i.e., $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$$

Note 4: If f(x) is an odd function, then its Fourier expansion contains only sine terms

i.e., $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$.

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$

Since $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0$ [$\because$ f(x) is odd]

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = 0$$

Example 4.3.1 Obtain the Fourier series of period 2π, for the function f (x) =x^2 in (-π,π). Specify the sum of the series at the end points x=π, -π Deduce the sum of the series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots \text{and } 1-\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots$$ [Nov.89,Apr.89,Civil]

Solution: Given $f(x) = x^2$. Here $f(x) = f(-x) = x^2$

Hence f(x) is an even function.

Therefore the Fourier coefficient $b_n=0$. Now the Fourier series of f(x) is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx \qquad [\because b_n=0] \qquad \text{-------(A)}$$

Now we have to find a_0 and a_n.

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx=\frac{2}{\pi}\int_0^{\pi}x^2dx$$

$$a_0=\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}=\frac{2}{3}\pi^2 \text{ -------(1)}$$

$$\boxed{a_0=\frac{2}{3}\pi^2} \qquad \text{-------(1)}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}x^2\cos nx\,dx$$

$$=\frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)-2x\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\frac{2\pi\cos nx}{n^2}\right]=\frac{4}{n^2}(-1)^n$$

i.e., $\boxed{a_n=\frac{4}{n^2}(-1)^n}$ -------(2)

Substituting (1) and (2) in (A), we get

$$f(x)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

i.e., $$x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$=\frac{\pi^2}{3}+4\left[\frac{-\cos x}{1^2}+\frac{\cos 2x}{2^2}-\frac{\cos 3x}{3^2}+\ldots\right]$$

$$\therefore x^2=\frac{\pi^2}{3}-4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\ldots\right] \text{-------(3)}$$

Putting x = $\pi$and -πin (3) we get

$$\pi^2 = \frac{\pi^2}{3} - 4\left[-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \ldots\right]$$

$$\text{i.e., } \pi^2 - \frac{\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} - \ldots\right]$$

$$\therefore \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Putting x = 0 in (3) we get

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$$

Here 0 is a point of continuity in (-π, π).

Example 4.3.2

Obtain the Fourier Series to represent the function f(x) = $|x|$, -π< x <π and deduce $\frac{1}{1^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{8}$.

Solution:

Given f(x) = $|x|$

$\therefore$ f(-x) = $|-x| = |x|$

Hence f(x) = f(-x) = $|x|$

$\therefore$ The given function f(x) = $|x|$ is an even function.

$\therefore$ The Fourier coefficient $b_n = 0$.

Hence the Fourier Series of f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad [\because b_n = 0] \qquad \text{-------(1)}$$

$$\text{Now } a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} |x|dx = \frac{2}{\pi}\int_0^{\pi} xdx$$

$$=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi = \pi$$

i.e., $\boxed{a_0 = \pi}$ -------(2)

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx = \frac{2}{\pi}\int_0^\pi |x|\cos nx dx$$

$$= \frac{2}{\pi}\int_0^\pi x\cos nx dx \qquad [\text{In } (0,\pi),\ |x|=x]$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^\pi$$

$$=\frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

i.e., $\boxed{\begin{aligned} a_n &= 0, \text{ if n is even} \\ &= \frac{-4}{n^2\pi}, \text{ if n is odd}\end{aligned}}$ -------(3)

Substituting (2) and (4) in (1) we get,

$$f(x) = \frac{\pi}{2} + \sum_{n=1,3,5}^{\infty} \frac{-4}{n^2\pi}\cos nx$$

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right)$$

$$\text{i.e., } |x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) \text{-------(4)}$$

Putting x = 0 in (4) we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$\text{i.e., } \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Note : $x = 0$ is a point of continuity in $-\pi < x < \pi$.

Example 4.3.3

Obtain the Fourier Series expansion of f(x) given by

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}, -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, 0 \le x \le \pi \end{cases}$$ **and hence deduce that**

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$$. **[Apr. 91 Civil, Nov. 87. Mech.]**

Solution: In $-\pi \le x \le 0$, i.e., $0 \le -x \le \pi$.

$$\therefore f(-x) = 1 - \frac{2(-x)}{\pi} = 1 + \frac{2x}{\pi} = f(x)$$

$\therefore$ f(x) is an even function.

Hence the Fourier coefficient b_n=0. New the Fourier Series for f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{ -------(A)}$$

$$\text{Now } a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx = \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi} = 0$$

i.e., $\boxed{a_0 = 0}$ -------(1)

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx dx$$

$$= \frac{2}{\pi}\left[\left(1 - \frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(\frac{-2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2\cos n\pi}{\pi n^2} + \frac{2}{\pi n^2}\right] = \frac{4}{n^2\pi^2}\left[1 - (-1)^n\right]$$

i.e., $a_n = 0$, when n is even

$= \frac{8}{n^2\pi^2}$, when n is odd -------(2)

Substituting (1) and (2) in (A), we get,

$$\therefore f(x) = \sum_{n=1,3,5,}^{\infty} \frac{8}{n^2\pi^2} \cos nx$$

$$\text{i.e., } f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\right] \text{-------(3)}$$

Putting x = 0 in (3) we get,

$$f(0) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\ldots\right]$$

$$\text{i.e., } 1 = \frac{8}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right]$$

$$\text{i.e., } \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots = \frac{\pi^2}{8}$$

Note: Here x = 0 is a point of continuity.

Example 4.3.4

If 'a' is neither zero nor an integer, find the Fourier series expansion of period 2π for the function f(x) = sin ax, in -π≤x ≤π.

Solution: Here f(x) = sin ax is an odd function.

Hence the Fourier coefficients $a_0 = 0$, $a_n = 0$. Therefore the Fourier Series for the function. f(x) is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx\,dx \qquad \text{--------- (A)}$$

$$\text{Now } b_n = \frac{2}{\pi}\int_0^{\pi} \sin ax \sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}[\cos(n-a)x - \cos(n+a)x]dx\right]$$

$$\frac{1}{\pi}\left[\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{n+a}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-\alpha)\pi}{n-a} - \frac{\sin(n+\alpha)\pi}{n+a}\right] \qquad [\because \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[\frac{\sin n\pi\cos a\pi - \cos n\pi \sin a\pi}{n-a} - \frac{\sin n\pi \cos a\pi}{n+a} - \frac{\cos n\pi \sin a\pi}{n+a}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(-\sin a\pi)}{n-a} + \frac{(-1)^{n+1}\sin a\pi}{n+a}\right] \quad [\because \sin n\pi = 0]$$

(Note that $\sin a\pi \neq 0$, since a is not an integer).

$$= \frac{(-1)^{n+1}\sin a\pi}{\pi}\left[\frac{1}{n-a} + \frac{1}{n+a}\right] = (-1)^{n+1}\frac{2n\sin a\pi}{\pi(n^2-a^2)}$$

i.e., $b_n = (-1)^{n+1}\dfrac{2n\sin a\pi}{\pi(n^2-a^2)}$ -------(B)

Substituting (B) in (A) we get

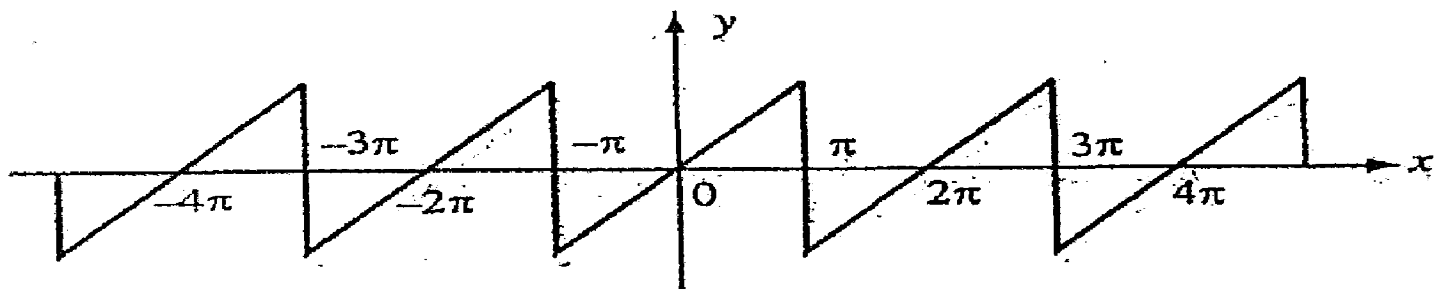
$$\therefore f(x) = \sin ax = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{n^2-a^2}\sin nx$$

Example 4.3.5

Show that the Fourier Series for f(x) = x, -π< x <π is given by

$$\mathbf{f(x) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}} \qquad \text{[Apr. 87, ECE]}$$

Solution: The graph of the given function is shown as below:



Given f(x) = x

$\therefore$ f(-x) = -x

i.e., f(x) = f(-x) =-f(x)

$\therefore$ f(x) = x is an odd function.

Hence a_0 =0, a_n = 0.

Therefore the Fourier Series for the function. f(x) is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \text{ -------(1)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n}\right) - (-\pi)\left(\frac{-\cos n\pi}{n}\right)\right]$$

$$= \frac{1}{\pi}\left[-2\cos n\pi\right] = \frac{2}{n}(-1)^{n+1}$$

i.e., $\boxed{b_n = \frac{2}{n}(-1)^{n+1}}$ -------(2)

Substituting (2) in (1), we get,

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}$$

Example 4.3.6

Find the Fourier series to represent the function f(x) = | sin x |, -π<x <π.

Solution: Here sin x is an odd function. But | sin x | is an even function. ∴ The Fourier coefficient $b_n = 0$. ∴ The Fourier series for

f(x) = | sin x | becomes

$$f(X) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{ -------(1)}$$

To find a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)dx \qquad [\because f(x) \text{ is even}]$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \, dx \qquad [\because \text{in } (0, \pi), \ |\sin x| = \sin x]$$

$$= \frac{2}{\pi}[-\cos x]_0^{\pi}$$

$$= \frac{2}{\pi}[1+1] = \frac{4}{\pi} \text{-------}(2)$$

To find a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx \ [\because f(x) \text{ is even}]$$

$$= \frac{2}{\pi}\int_0^{\pi} |\sin x| \cos nx \, dx$$

$$= \frac{2}{\pi}\int_{-\pi}^{\pi} \sin x \cos nx \, dx \ \ [\text{In } (0,\pi), \ |\sin x| = \sin x]$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}[\sin(n+1)x - \sin(n-1)x]dx$$

[Using 2 cos A sin B = sin (A+B) – sin (A-B)]

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos n\pi \cos\pi}{n+1} + \frac{\cos n\pi \cos\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \ [\because \sin n\pi = 0]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n+1} - \frac{\cos n\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \ [\because \cos\pi = -1]$$

$$= \frac{1}{\pi}\left[\frac{(n-1)\cos n\pi - (n+1)\cos n\pi + n - 1 - n - 1}{n^2-1}\right]$$

$$= \frac{1}{(n^2-1)\pi} [-2 \cos n\pi - 2]$$

$$= \frac{-2}{(n^2-1)\pi} [1 + (-1)^n]$$

$$\therefore a_n = \begin{cases} 0, & \text{when 'n' is odd} \\ \dfrac{-4}{\pi(n^2-1)}, & \text{when 'n' is even provided } n \neq 1 \end{cases} \text{-------}(3)$$

When n =1,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx$$

$$= \frac{2}{\pi} \cdot \frac{1}{2}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-1+1\right] = 0 \text{-------}(4)$$

Substituting (2), (3) and (4) in (1), we get

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{2}{\pi} + \sum_{n=2,4}^{\infty} \frac{-4}{\pi(n^2-1)} . \cos nx$$

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\left[\sum_{n=2,4}^{\infty} \frac{\cos nx}{n^2-1}\right]$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \ldots\right]$$

Example 4.3.7

Find the Fourier series for f(x) = |cos x| in the interval (-π, π).

[Nov.'88, Apr.'92]

Solution: Given f(x) = |cos x|

f(-x) = |cos (-x)|

$= |\cos x|$

$= f(x)$

$\therefore$ The given function $f(x) = |\cos x|$ is even function. Hence the Fourier series for f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{ -------(1)}$$

To find a_0

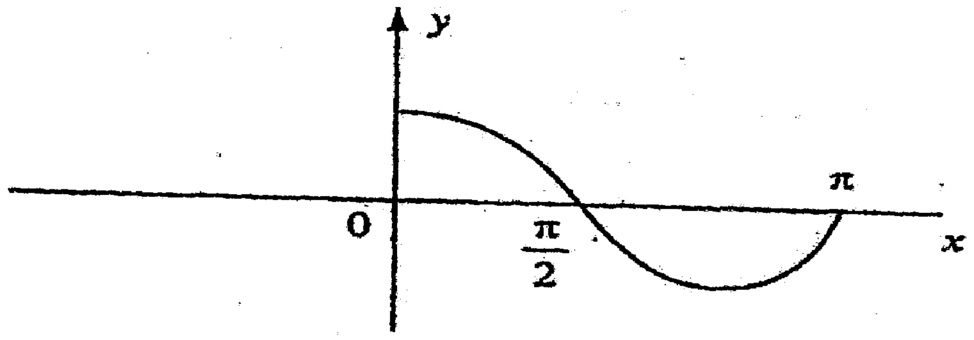
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x|\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} |\cos x|\,dx \quad [\because |\cos x| \text{ is even function}]$$

$$= \frac{2}{\pi}\left[\int_{0}^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx\right]$$

[$\because$ cos x is +ve in $0 < x < \frac{\pi}{2}$ and, cos x is –ve in $\frac{\pi}{2} < x < \pi$. See graph]



$$= \frac{2}{\pi}\left\{[\sin x]_0^{\pi/2} [-\sin x]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}[1+1] \quad [\because \sin \pi = 0 \text{ and } \sin 0 = 0]$$

$$a_0 = \frac{4}{\pi} \text{ -------(2)}$$

To find a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} |\cos x|\cos nx\,dx \quad [\because |\cos x|\cos nx \text{ is even function}]$$

$$= \frac{2}{\pi}\left[\int_{0}^{\pi/2} \cos x\cos nx\,dx + \int_{\pi/2}^{\pi} -\cos x\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_{0}^{\pi/2} \frac{1}{2}\{\cos(n+1)x + \cos(n-1)x\}dx - \int_{\pi/2}^{\pi} \frac{1}{2}\{\cos(n+1)x + \cos(n-1)x\,dx\}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_{0}^{\pi/2} - \left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1} + \frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{2\sin(n-1)\frac{\pi}{2}}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin\frac{n\pi}{2}\cos\frac{\pi}{2} + \cos\frac{n\pi}{2}\sin\frac{\pi}{2}}{n+1} + \frac{\sin\frac{n\pi}{2}\cos\frac{\pi}{2} - \cos\frac{n\pi}{2}\sin\frac{\pi}{2}}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1}\right] \quad \left[\because \sin\frac{\pi}{2} = 1, \cos\frac{\pi}{2} = 0\right]$$

$$= \frac{2}{\pi}\left[\frac{(n-1)\cos\frac{n\pi}{2} - (n+1)\cos\frac{n\pi}{2}}{n^2-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{-2\cos\frac{n\pi}{2}}{n^2-1}\right]$$

$$=\frac{-4\cos\frac{n\pi}{2}}{\pi(n^2-1)} \quad \text{[provided } n \neq 1\text{]-------(3)}$$

When n =1, we have

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi}|\cos x|\cos x\,dx = \frac{2}{\pi}\int_{0}^{\pi}|\cos x|\cos x\,dx \quad [\because |\cos x|\cos x \text{ is an even function}]$$

$$=\frac{2}{\pi}\left[\int_{0}^{\pi/2}\cos x\cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\cos x\,dx\right]$$

$$=\frac{2}{\pi}\left[\int_{0}^{\pi/2}\cos^2 x\,dx - \int_{\pi/2}^{\pi}\cos^2 x\,dx\right]$$

$$\int_{\pi/2}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx$$

$$=\frac{2}{\pi}\left[\int_{0}^{\pi/2}\left(\frac{1+\cos 2x}{2}\right)dx - \int_{\pi/2}^{\pi}dx\right]$$

$$=\frac{2}{\pi}\left[\left(\frac{x}{2}+\frac{\sin 2x}{4}\right)_0^{\pi/2} - \left(\frac{x}{2}+\frac{\sin 2x}{4}\right)_{\pi/2}^{\pi}\right]$$

$$=\frac{2}{\pi}\left[\frac{\pi}{4}-\frac{\pi}{2}+\frac{\pi}{4}\right]$$

$a_1 = 0$ -------(4)

Substituting (2), (3) and (4) in (1), we get

$$f(x) = \frac{2}{\pi}+\sum_{n=2}^{\infty}\frac{-4\cos\frac{n\pi}{2}}{\pi(n^2-1)}.\cos nx$$

$$\therefore |\cos x| = \frac{2}{\pi}-\frac{4}{\pi}\sum_{n=2}^{\infty}\frac{\cos\frac{n\pi}{2}\cos n\pi}{(n^2-1)}$$

Example 4.3.8

Prove thatsinhax $= \frac{2}{\pi}\left[\sin h\, a\pi\sum_{n=1}^{\infty}\frac{(-1)^{n-1}.n\sin nx}{n^2+a^2}\right]$ **in** $(-\pi,\pi)$

[Nov. '87, Apr. '89]

Solution: Let f(x) = sin h ax. Clearly sin h ax is an odd function.

For f(x) = sin h ax = $\frac{e^{ax} - e^{-ax}}{2}$

$$f(-x) = \frac{e^{-ax} - e^{ax}}{2}$$

$$= -\left[\frac{e^{ax} - e^{-ax}}{2}\right]$$

= -f(x)

∴ Fourier series for f(x) in (-π, π) is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ -------(1)

[∵ f(x) is an odd function. Its fourier coefficients a_0 and a_n are zero]

To find b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin h\ ax \sin nx\, dx$$

$$= \frac{1}{\pi}.2\int_{0}^{\pi} \sin h\, ax \sin nx\, dx$$ [∵ odd function x odd function = even function]

$$= \frac{2}{\pi}\int_{0}^{\pi} \frac{e^{ax} - e^{-ax}}{2} \sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{0}^{\pi} e^{ax} \sin nx\, dx - \int_{0}^{\pi} e^{-ax} \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{e^{ax}}{a^2 + n^2}(a \sin nx - n \cos nx)\right\}_0^{\pi} - \left\{\frac{e^{-ax}}{a^2 + n^2}(a \sin nx - n \cos nx)\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-n(-1)^n}{a^2 + n^2} e^{a\pi} + \frac{n}{a^2 + n^2} + \frac{n(-1)^n}{a^2 + n^2}.e^{-a\pi} - \frac{n}{a^2 + n^2}\right]$$

$$= \frac{(-1)^n.n}{\pi(n^2 + a^2)}\left[- e^{a\pi} + e^{-a\pi}\right]$$

$$= \frac{n(-1)^{n+1}}{\pi(n^2+a^2)} .2 \sin h\, a\, \pi \qquad \text{-------(2)}$$

Substituting (2) in (1), we get

$$f(x) = \sum_{n=1}^{\infty} \frac{2n(-1)^{n+1}}{\pi(n^2+a^2)} \sinh a\pi . \sin nx$$

$$\text{i.e., } \sin h\, ax = \frac{2}{\pi} \sinh a\pi \sum_{n=1}^{\infty} \frac{(-1)^{n+1}.n \sin nx}{n^2+a^2}$$

Example 4.3.9

Prove that in the range $-\pi \leq x \leq \pi$

$$\cos h\, ax = \frac{2a}{\pi} \sin h\, a\pi \left[\frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2+a^2} \right]. \text{ [Nov. '88]}$$

Solution: Let f(x) = cos h ax. Then its Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{-------(1)}$$

To find a_0

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos h\, ax\, dx$$

$$= \frac{1}{\pi} \left[\frac{\sin h\, ax}{a} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi a} [\sin h a\pi - \sin h(-a\pi)]$$

$$= \frac{1}{\pi a} [\sin h a\pi + \sin h a\pi]$$

$$= \frac{2 \sin h\, a\pi}{a\pi} \qquad \text{-------(2)}$$

To find a_n

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos h\,ax\cos nx\,dx$$

$$=\frac{1}{\pi}.\frac{1}{2}\int_{-\pi}^{\pi}(e^{ax}+e^{-ax})\cos nx\,dx$$

$$=\frac{1}{2\pi}\left\{\int_{-\pi}^{\pi}e^{ax}\cos nx\,dx+\int_{-\pi}^{\pi}e^{-ax}\cos nx\,dx\right\}$$

$$=\frac{1}{2\pi}\left\{\int_{-\pi}^{\pi}e^{ax}\cos nx\,dx+\int_{-\pi}^{\pi}e^{-ax}\cos nx\,dx\right\}$$

$$=\frac{1}{2\pi}\left[\left\{\frac{e^{ax}}{a^2+n^2}(a\cos nx+n\sin nx)\right\}_{-\pi}^{\pi}+\left\{\frac{e^{-ax}}{a^2+n^2}(-a\cos nx+n\sin nx)\right\}_{-\pi}^{\pi}\right]$$

$$=\frac{1}{2\pi}\left[\frac{ae^{a\pi}}{n^2+a^2}(-1)^n-\frac{e^{-a\pi}}{n^2+a^2}a(-1)^n-\frac{e^{-a\pi}}{n^2+a^2}.a(-1)^n+\frac{e^{a\pi}}{n^2+a^2}a(-1)^n\right]$$

$$=\frac{(-1)^n.a}{2\pi(n^2+a^2)}\left[2e^{a\pi}-2e^{-a\pi}\right]$$

$$=\frac{a(-1)^n}{\pi(n^2+a^2)}.2\sinh a\pi$$

$$=\frac{2a(-1)^n\sinh a\pi}{\pi(n^2+a^2)}\quad\text{-------(3)}$$

Since the given function f(x) = cos h ax is an even function, the Fourier coefficient

$b_n = 0$ -------(4)

[$\because$ f(x) = cos h ax = $\frac{e^{ax}+e^{-ax}}{2}$; f(-x) = $\frac{e^{-ax}+e^{ax}}{2}$ f(x). Hence f(x) = cos h ax is an even function].

Substituting (2), (3) and (4) in (1), we get

$$f(x)=\frac{\sinh a\pi}{\pi a}+\sum_{n=1}^{\infty}\frac{2a(-1)^n\sinh a\pi}{\pi(n^2+a^2)}.\cos nx$$

$$=\frac{2a}{\pi}\sin h a\pi\left[\frac{1}{2a^2}+\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2+a^2}\right]$$

Example 4.3.10

Obtain a Fourier expansion for $\sqrt{1-\cos x}$ in the interval $-\pi<x<\pi$.

[Apr. '86, Nov. '88]

Solution: Let f(x) = $\sqrt{1-\cos x}$. It is an even function.

For f(x) = $\sqrt{1-\cos x}$

$$f(-x) = \sqrt{1-\cos(-x)} = \sqrt{1-\cos x} = f(x)$$

$\therefore$ Its fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{ -------(1)}$$

To find a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{1-\cos x}\, dx \qquad [\because f(x) \text{ is an even function}]$$

$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{2}.\sin\left(\frac{x}{2}\right)dx \left[\because \frac{1-\cos x}{2} = \sin^2\frac{x}{2}\right]$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{-\cos\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi}$$

$$= \frac{4\sqrt{2}}{\pi}\left[-\cos\frac{\pi}{2}+1\right] \qquad [\because \cos\frac{\pi}{2} = 0]$$

$$a_0 = \frac{4\sqrt{2}}{\pi} \text{ -------(2)}$$

To find a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\sqrt{1-\cos x}.\cos nx\,dx$$

$$=\frac{2}{\pi}\int_{0}^{\pi}\sqrt{2}.\sin\frac{x}{2}\cos nx\,dx$$

$$=\frac{2\sqrt{2}}{\pi}\int_{0}^{\pi}\frac{1}{2}\left\{\sin\left(\frac{1}{2}+n\right)x+\sin\left(\frac{1}{2}-n\right)x\right\}dx$$

$$=\frac{\sqrt{2}}{\pi}\left[\frac{-\cos\left(\frac{1}{2}+n\right)x}{\frac{1}{2}+n}-\frac{\cos\left(\frac{1}{2}-n\right)x}{\frac{1}{2}-n}\right]_{0}^{\pi}$$

$$=\frac{\sqrt{2}}{\pi}\left[\frac{-\cos\left(\frac{1}{2}+n\right)\pi}{\frac{1}{2}+n}-\frac{\cos\left(\frac{1}{2}-n\right)\pi}{\frac{1}{2}-n}+\frac{1}{\frac{1}{2}+n}+\frac{1}{\frac{1}{2}-n}\right]$$

$$=\frac{\sqrt{2}}{\pi}\left[\frac{1}{\frac{1}{2}+n}+\frac{1}{\frac{1}{2}-n}\right]$$

$$\left[\begin{array}{l}\because \cos\left(\frac{1}{2}+n\right)\pi=\cos\frac{\pi}{2}\cos n\pi-\sin\frac{1}{2}\pi\sin n\pi\\ \qquad\qquad\qquad =0-0\left(\cos\frac{\pi}{2}=0,\sin n\pi=0\right)\\ \qquad\qquad\qquad =0\\ \cos\left(\frac{1}{2}-n\right)\pi\ =\cos\frac{\pi}{2}\cos n\pi+\sin\frac{1}{2}\pi\sin n\pi\\ \qquad\qquad\qquad =0+0=0\end{array}\right]$$

$$=\frac{\sqrt{2}}{\pi}\left[\frac{\frac{1}{2}-n+\frac{1}{2}+n}{\frac{1}{4}-n^{2}}\right]$$

$$=\frac{\sqrt{2}}{\pi}\left[\frac{4}{1-4n^{2}}\right]\text{-------}(3)$$

Substituting (2) and (3) in (1), we get

$$f(x) = \frac{2\sqrt{2}}{\pi} + \sum_{n=1}^{\infty} \frac{\sqrt{2}}{\pi} \cdot \frac{4}{1-4n^2} \cos nx$$

$$\therefore \sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{\cos nx}{4n^2-1}$$

EXERCISES

1. Determine the Fourier series expansion of f(x) = x (2π-x) in 0 < x < 2π.

   [Apr.91]

   [Ans. $f(x) = \frac{2\pi^2}{3} - 4\sum_{1}^{\infty} \frac{\cos nx}{n^2}$]

2. Expand f(x) = x sin x as a fourier series of period 2π, in the interval -π< x <π, show that

   $x \sin x = 1 - \frac{1}{2}\cos x - 2\left\{\frac{\cos 2x}{1.3} - \frac{\cos 3x}{2.4} + \ldots\right\}$ [Nov.89, Mech]

3. Find the Fourier Series for $f(x) = (\pi-x)^2$ in (-π,π)

   $f(x) = \frac{4\pi^2}{3} + \left(\frac{4}{n^2}\right)(-1)^n \cos nx + \frac{4\pi(-1)^n}{n} \sin nx$ [Apr.87, Civil]

4. Prove that in the interval -π< x <π. [Nov.87, Civil]

   $\cos x = \frac{1}{2}\sin x + 2\sum_{n=2}^{\infty} \frac{(-1)^n}{(n^2-1)} \sin nx$

5. Obtain the Fourier series of period 2π for the function

   αx(π-x) in 0 ≤ x ≤ 2π, where α is a constant.

   [Ans. $f(x) = -\frac{\alpha\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4\alpha}{n^2} \cos nx + \sum_{n=1}^{\infty} \frac{2\pi\alpha}{n} \sin nx$]

6. Obtain the Fourier series expansion of $\frac{1}{2}$ (π-x) in (0, 2π). [Nov.86, Mech., Nov. 88, Mech.]

   [Ans. $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$]

7. Expand $f(x) = x^2$, 0 < x < 2π in a Fourier series if the period is 2π.

[Ans. $f(x) = \dfrac{4\pi^2}{3} + \sum_{n=1}^{\infty} \dfrac{4}{n^2} \cos nx + \sum_{n=1}^{\infty} \dfrac{-4\pi}{n} \sin nx$]

8. Find the Fourier Seriesof the following function which is assumed to have the period $f(x) = \dfrac{x^2}{4}$, $-\pi < x < \pi$.

[Apr. 87, ECE]

Hence deduce $1 + \dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16} + \dots = \dfrac{\pi^2}{6}$

[Ans. $\dfrac{x^2}{4} = \dfrac{\pi^2}{12} - \left(\dfrac{\cos x}{1^2} - \dfrac{\cos 2x}{2^2} + \dfrac{\cos 3x}{3^2} - \dfrac{\cos 4x}{4^2} + \dots \right)$]

9. Show that the Fourier series for the function

$$f(x) = \begin{cases} x+1, & 0 < x < \pi \\ x-1, & -\pi < x < 0 \end{cases} \text{ is } \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{1-(-1)^n(1+\pi)}{n} \right] \sin nx.$$

[Anna Univ. Apr. 2001]

10. Expand the Fourier series to represent $f(x) = \dfrac{1}{4}(\pi - x)^2$, $0<x<2\pi$.

[Ans. $f(x) = \dfrac{\pi^2}{12} + \sum_{n=1}^{\infty} \dfrac{\cos nx}{n^2}$]

11. Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$. Hence show that $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \dots = \dfrac{\pi^2}{12}$

[Ans. $x - x^2 = \dfrac{-\pi^2}{3} - 4\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \cos nx - 2\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n} \sin nx$]

12. Obtain the Fourier Seriesfor $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.

[Ans. $e^{-x} = \left(\dfrac{1-e^{-2\pi}}{\pi} \right) \left\{ \dfrac{1}{2} + \sum_{1}^{\infty} \dfrac{\cos nx}{1+n^2} + \sum_{1}^{\infty} \dfrac{n \sin nx}{1+n^2} \right\}$]

13. Find the Fourier series to represent e^{ax} in the interval $-\pi < x < \pi$.

[Ans. $e^{ax} = \dfrac{\sinh a\pi}{\pi} \left\{ \dfrac{1}{a} + \sum_{n=1}^{\infty} \dfrac{2a(-1)^n \cos nx}{a^2+n^2} + \sum_{n=1}^{\infty} \dfrac{2n(-1)^n \sin nx}{a^2+n^2} \right\}$]

14. Find the Fourier series of the function $f(x) = \begin{cases} x^2, & 0 \le x \le \pi \\ -x^2, & -\pi \le x \le 0 \end{cases}$

[Ans. $f(x) = 2\left(\pi - \frac{4}{\pi}\right)\sin x - \pi \sin 2x + \frac{2}{3}\left(\pi - \frac{4}{9\pi}\right)\sin 3x - \frac{\pi}{2}\sin 4x + \ldots$]

15. Obtain the Fourier series of the following functions in the given interval.

(i) $f(x) = \begin{cases} 1 \text{ in } (0, \pi) \\ 0 \text{ in } (\pi, 2\pi) \end{cases}$

(ii) $f(x) = \begin{cases} 1 \text{ in } -\pi < x < 0 \\ 2 \text{ in } 0 < x < \pi \end{cases}$

(iii) $f(x) = \begin{cases} a \text{ in } 0 < x < \pi \\ -a \text{ in } \pi < x < 2\pi \end{cases}$

(iv) $f(x) = \begin{cases} -1 \text{ in } -\pi < x < 0 \\ 1 \text{ in } 0 < x < \pi \end{cases}$

(v) $f(x) = \begin{cases} 0 \text{ for } -\pi < x < 0 \\ \pi \text{ for } 0 < x < \pi \end{cases}$

(vi) $f(x) = \begin{cases} -x \text{ for } -\pi < x < 0 \\ 0 \text{ for } 0 < x < \pi \end{cases}$

(vii) $f(x) = \begin{cases} x \text{ in } -\pi \le x \le 0 \\ 0 \text{ in } 0 \le x \le \pi \end{cases}$

(viii) $f(x) = \begin{cases} x + \frac{\pi}{2} & -\pi \le x \le 0 \\ \frac{\pi}{2} - x & 0 \le x \le \pi \end{cases}$

(ix) $f(x) = \begin{cases} x \text{ in } -\pi < x < 0 \\ 0 \text{ in } 0 < x < \frac{\pi}{2} \\ x - \frac{\pi}{2} \text{ in } \frac{\pi}{2} < x < \pi \end{cases}$

Answers:

(i) $\frac{1}{2}+\frac{2}{\pi}\sum_{1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$

(ii) $\frac{3}{2}+\frac{2}{\pi}\sum_{1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$

(iii) $\frac{4a}{\pi}\left[\sum_{n=1,3,5}^{\infty}\frac{\sin nx}{n}\right]$

(iv) $\frac{4}{\pi}\left[\sum_{n=1,3,5}^{\infty}\frac{\sin nx}{n}\right]$

(v) $\frac{\pi}{2}+2\left[\sum_{n=1,3,5}^{\infty}\frac{\sin nx}{n}\right]$

(vi) $\frac{\pi}{4}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}+\sum_{1}^{\infty}\frac{(-1)^n\sin nx}{n}$

(vii) $-\frac{\pi}{4}+\frac{2}{\pi}\sum_{1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}+\sum_{1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}$

(viii) $\frac{4}{\pi}\sum_{n=1,3,5}^{\infty}\frac{\cos nx}{n^2}$

(ix) $\frac{-3\pi}{16}+\frac{1}{\pi}\sum_{n=1,3,5}^{\infty}\frac{\cos nx}{n^2}+\frac{2}{\pi}\sum_{n=2,6,10}^{\infty}\frac{\cos nx}{n^2}-\frac{3}{2}\sum_{n=1}^{\infty}\frac{(-1)^n\sin nx}{n}$

$$-\frac{1}{\pi}\left[\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\ldots\right]$$

4.4 Change of Interval

So far we have discussed that a given function f(x) can be expanded in a Fourier Seriesin the interval of length 2π. But in many Engineering problems, it is desired to expand a function in a Fourier Series in the interval of length 2l and not 2π. To change the length 2l to 2π we put

$\frac{x}{l} = \frac{z}{\pi}$ i.e., $z = \frac{\pi x}{l}$,so that

when x=c, $z = \frac{\pi c}{l}$ =d (say)

whenx=c+2l, $z = \frac{\pi(c+2l)}{l} = \frac{\pi c}{l} + 2\pi = d + 2\pi$

∴ The function f (x) of period 2*l*in (c,c+ 2l) is transformed to the function $f(\frac{lz}{\pi})$ =F (z),say, of period 2π in (d,d+2π) and the latter function can be expressed in Fourier Series

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \qquad \ldots\ldots\ (1)$$

where $a_0 = \frac{1}{\pi}\int_d^{d+2\pi} F(z)dz$ .... (2)

$$a_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\cos nz dz \qquad \ldots\ldots\ldots(3)$$

and $b_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\sin nz dz$ .... (4)

Now by applying the inverse substitution

$$z = \frac{\pi x}{l}, dz = \frac{\pi}{l}dx$$

whenz= d, x=c

and when z =d + 2π, x = c +2l

$\therefore$ Equation (1) becomes

$$F(z) = F\left(\frac{\pi x}{\iota}\right) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\iota} + + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\iota}$$

$$\text{where } a_0 = \frac{1}{\iota}\int_{c}^{c+21} f(x)dx \text{[by(2)]} \qquad \text{.........(5)}$$

$$a_n = \frac{1}{\iota}\int_{c}^{c+21} f(x)\cos\frac{n\pi x}{\iota} dx \text{[by(3)]} \qquad \text{........ (6)}$$

$$b_n = \frac{1}{\iota}\int_{c}^{c+21} f(x)\sin\frac{n\pi x}{\iota} dx \text{[by(4)]} \qquad \text{...........(7)}$$

$\therefore$ The Fourier Series for f(x) in the interval $c < x < c + 21$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\iota} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\iota} \qquad \text{.........(8)}$$

Where a_0, a_n and b_n are given by (5),(6) and (7).

Note 1: If we put c=0 the interval becomes $0 < x < 21$ then the above equations are reduced to

$$a_0 = \frac{1}{1}\int_{0}^{2\iota} f(x)dx$$

$$a_n = \frac{1}{1}\int_{0}^{2\iota} f(x)\cos\frac{n\pi x}{\iota} dx,$$

$$\text{and } b_n = \frac{1}{1}\int_{0}^{2\iota} f(x)\sin\frac{n\pi x}{\iota} dx$$

Note 2: If we put c= -l then the interval becomes $-l < x < l$ and the above results are reduced to

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx,$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$$

Note 3: If f(x) is an even function then we have

$$a_0 = \frac{2}{l}\int_{0}^{l} f(x)dx$$

$$a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

and $b_n = 0$

Note 4: If f (x) is an odd function then we have $a_0 = 0$, $a_n = 0$ and

$$b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}dx$$

Example 4.4.1: Find the Fourier Series expansion of period 2l for the function $f(x) = (l-x)^2$ in the range (0,2l) . Deduce the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$

[Nov.91]

Solution: The Fourier Series of f (x) in (0,2l) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\iota} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\iota} \qquad .....(1)$$

Now $a_0 = \frac{1}{\iota}\int_0^{2l} f(x)dx = \frac{1}{\iota}\int_0^{2l} (1-x)^2 dx$

$$= \left(\frac{(1-x)^3}{3\iota}\right)_0^{2\iota} = \frac{\iota^3 + 1^3}{3\iota} = \frac{2\iota^3}{3\iota} = \frac{2}{3}\iota^2$$

i.e., $\boxed{a_0 = \frac{2}{3}\iota^2}$ .....(2)

$$a_n = \frac{1}{\iota}\int_0^{2\iota} f(x)\cos\frac{n\pi x}{\iota}dx$$

$$= \frac{1}{\iota}\int_0^{2\iota} (\iota - x)^2 \cos\frac{n\pi x}{\iota}dx$$

$$= \frac{1}{\iota}\left[(\iota-x)^2\left\{\frac{\sin\frac{n\pi x}{\iota}}{\frac{n\pi}{\iota}}\right\} - 2(\iota-x)(-1)\left\{\frac{-\cos\frac{n\pi x}{\iota}}{\frac{n^2\pi^2}{1^2}}\right\} + 2\left\{\frac{-\sin\frac{n\pi x}{1}}{\frac{n^3\pi^3}{1^3}}\right\}\right]_0^{2\iota}$$

$$= \frac{1}{\iota}\left[\frac{2\ell\cos 2n\pi}{\frac{n^2\pi^2}{\ell^2}} + \frac{2\ell}{\frac{n^2\pi^2}{\ell^2}}\right] = \frac{4\ell^2}{n^2\pi^2}$$

[$\because \sin 2n\pi = 0, \cos 2n\pi = 1$]

i.e., $\boxed{a_n = \frac{4l^2}{n^2\pi^2}}$ .....(3)

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{1}{\ell}\int_0^{2\ell} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{1}{l}\left[(l-x)^2\left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}-2(l-x)(-1)\left\{\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right\}+2\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]_0^{2l}$$

$$=\frac{1}{l}\left[\frac{-l^2\cos 2n\pi}{\frac{n\pi}{l}}+\frac{2\cos 2n\pi}{\frac{n^3\pi^3}{l^3}}+\frac{l^2}{\frac{n\pi}{l}}-\frac{2}{\frac{n^3\pi^3}{\ell^3}}\right]$$

$$=\frac{1}{l}\left[\frac{-l^2}{\frac{n\pi}{l}}+\frac{l^2}{\frac{n\pi}{l}}\right]=0 \qquad [\because \cos 2n\pi = 1]$$

i.e., $b_n=0$ ....(4)

Substituting (2), (3) and (4) in (1), we get,

$$f(x)=\frac{l^2}{3}+\sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} \qquad ....(5)$$

$$=\frac{l^2}{3}+\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi x}{n^2}$$

Putting $x = 0$ in (5) we get

$$f(0)=\frac{l^2}{3}+\sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi^2}$$

Here 0 is a point of discontinuity. Hence the value of the function at x = 0 is given by

$$f(0)=\frac{f(0-0)+f(0+0)}{2}$$

$$=\frac{(l-0)^2+(l+0)^2}{2}=\frac{2l^2}{2}=l^2$$

i.e., $$l^2=\frac{l^2}{3}+\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

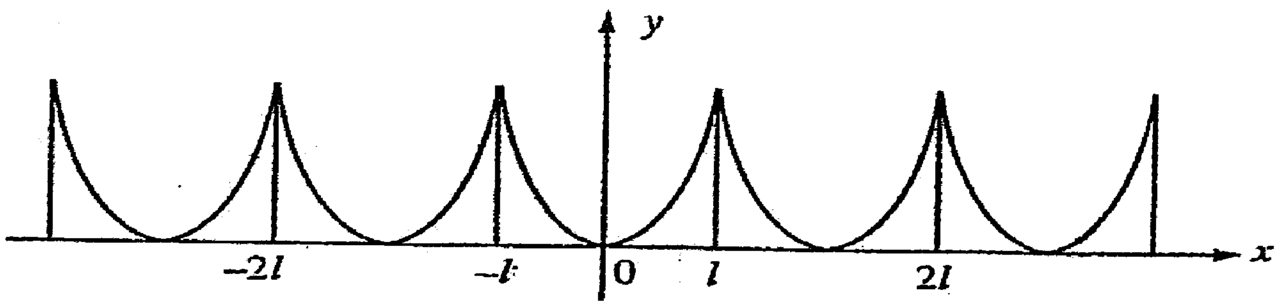
$$\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}=1-\frac{1}{3}=\frac{2}{3}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Example 4.4.2Obtain the Fourier Series to represent x^2 from

x = -l to x = l.

Solution :



The Fourier Series for $f(x) = x^2$ in (-l, l) is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \text{.......(1)}$$

Now $a_0=\frac{1}{l}\int_{-l}^{l} f(x)dx=\frac{1}{l}\int_{-l}^{l} x^2dx=\frac{1}{l}\left[\frac{x^3}{3}\right]_{-l}^{l}$

$$=\frac{1}{l}\left[\frac{l^3}{3}+\frac{l^3}{3}\right]=\frac{2l^3}{3l}=\frac{2}{3}l^2$$

i.e., $\boxed{a_0=\frac{2}{3}l^2}$ ....(2)

$$a_n=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

$$=\frac{1}{l}\int_{-l}^{l} x^2\cos\frac{n\pi x}{l}dx$$

$$=\frac{1}{l}\left[x^2\left\{\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}-2x\left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right\}+2\left\{\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right\}\right]_{-l}^{l}$$

$$=\frac{1}{l}\left[\frac{2l\cos n\pi}{\frac{n^2\pi^2}{l^2}}+\frac{2l\cos n\pi}{\frac{n^2\pi^2}{l^2}}\right]=\frac{4l^2\cos n\pi}{n^2\pi^2} \qquad [\because \sin n\pi=0]$$

i.e., $\boxed{a_n=\frac{4l^2\cos n\pi}{n^2\pi^2}}$ ....(3)

$$b_n=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx=\frac{1}{l}\int_{-l}^{l} x^2\sin\frac{n\pi x}{l}dx$$

$$= \frac{1}{1}\left[x^2 \left\{ \frac{\cos\frac{n\pi x}{1}}{\frac{nx}{1}} \right\} - 2x \left[\frac{\sin\frac{n\pi x}{1}}{\frac{n^2\pi^2}{1^2}} \right] + 2\left\{ \frac{\cos\frac{n\pi x}{1}}{\frac{n^3\pi^3}{1^3}} \right\} \right]_{-1}^{1}$$

$$= \frac{1}{1}\left[\frac{-1^2 \cos n\pi}{\frac{n\pi}{1}} + \frac{2\cos n\pi}{\frac{n^3\pi^3}{1^3}} + \frac{1^2 \cos n\pi}{\frac{n\pi}{1}} - \frac{2\cos n\pi}{\frac{n^3\pi^3}{1^3}} \right] = 0$$

i.e., $\boxed{b_n = 0}$

Substituting (2),(3) and (4) in (1), we get,

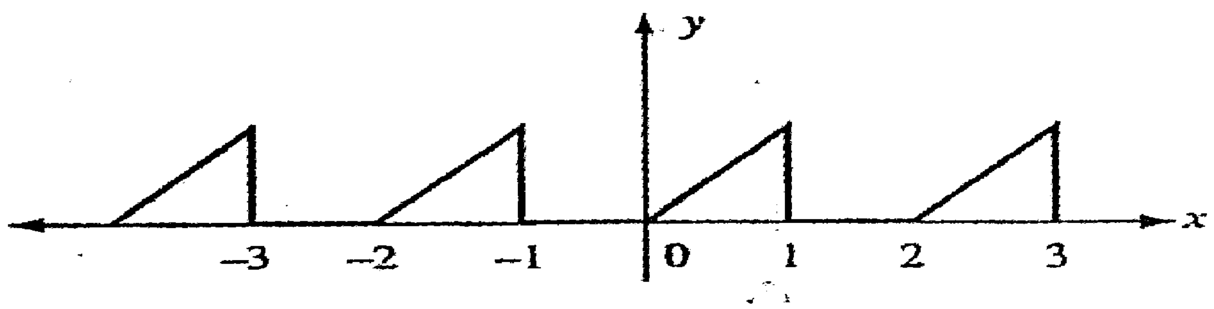
$$f(x) = \frac{\iota^2}{3} + \sum_{n=1}^{\infty} \frac{4\iota^2(-1)^n}{n^2\pi^2} \cos\frac{n\pi x}{\iota}$$

$$x^2 = \frac{\iota^2}{3} + \frac{4\iota^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n \cos\frac{n\pi x}{\iota}}{n^2}$$

Example 4.4.3 Find the Fourier Series for f (x) given

$$f(x) = \begin{cases} 0 \text{ in } -1 < x < 0 \\ 1 \text{ in } 0 < x < 1 \end{cases} \text{ and } f(x+2) = f(x) \text{ for all } x.$$

Solution:



The Fourier Series of f (x) in -1 < x< 1 is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos n\pi x+\sum_{n=1}^{\infty} b_n \sin n\pi x \qquad \ldots\ldots..(1)$$

Now $a_0 = \frac{1}{1}\int_{-1}^{1} f(x)dx$

$$=\frac{1}{1}\left[\int_{-1}^{0} o\,dx+\int_{0}^{1} 1\,dx\right]=\frac{1}{1}[x]_0^1=1$$

i.e., $\boxed{a_0=1}$ $\qquad \ldots\ldots\ldots.(2)$

$$a_n=\frac{1}{1}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{1}dx$$

$$=\frac{1}{1}\left[\int_{-1}^{0} 0.\cos\frac{n\pi x}{1}dx+\int_{0}^{1} \ell\cos\frac{n\pi x}{1}dx\right]$$

$$=\frac{1}{1}\left[\frac{\sin\frac{n\pi x}{1}}{\frac{n\pi}{1}}\right]_0^1 \quad [\because \sin n\pi=0]$$

i.e., $\boxed{a_n=0}$ $\qquad \ldots\ldots.(3)$

$$b_n=\frac{1}{1}\int_{-1}^{1} f(x)\sin\frac{n\pi x}{1}dx$$

$$=\frac{1}{1}\left[\int_{-1}^{0} 0.\sin\frac{n\pi x}{1}dx+\int_{0}^{1} \ell\sin\frac{n\pi x}{1}dx\right]$$

$$=\frac{1}{1}\left[\frac{-\cos\frac{n\pi x}{1}}{\frac{n\pi}{1}}\right]_0^1=\frac{1}{n\pi}[-\cos n\pi+1]$$

$$= \frac{1}{n\pi}[1-(-1)^n]$$

$b_n = 0$ when n is even

$= \frac{2}{n\pi}$ when n is odd ....(4)

Substituting (2),(3) and (4) in (1) we get

$$f(x) = \frac{1}{2} + \sum_{n=1,3,5}^{\infty} \frac{2}{n\pi} \sin\frac{n\pi x}{1}$$

$$f(x) = \frac{1}{2} + + \frac{2}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} \sin n\pi x$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{2}{n\pi} \frac{\sin(2n-1)\pi x}{2n-1}$$

Example 4.24 The function f (x) is defined as follows in the interval (-2,2).

$$f(x) = \begin{cases} 0 - 2 < x < -1 \\ 1 + x \quad -1 < x < 0 \\ 1 - x \quad 0 < x < 1 \\ 0 1 < x < 2 \end{cases}$$

Show that $f(x) = \frac{1}{4} + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin^2\frac{n\pi}{4} \cos\frac{n\pi x}{2}$.

Solution : The Fourier Series of the function f (x) in (-2,2) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \quad ....(1)$$

Now $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx$

$$= \frac{1}{2}\left[\int_{-2}^{-1} 0dx + \int_{-1}^{0} (1+x)dx + \int_{0}^{1} (1+x)dx + \int_{1}^{2} 0dx\right]$$

$$= \frac{1}{2}\left[\left\{\frac{(1+x)^2}{2}\right\}_{-1}^{0} + \left\{\frac{(1-x)^2}{-2}\right\}_{0}^{1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}+\frac{1}{2}\right] = \frac{1}{2}$$

i.e., $a_0 = \frac{1}{2}$ ....(2)

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx$$

$$= \frac{1}{2}\left[\int_{-2}^{-1} 0.\cos\frac{n\pi x}{2}dx + \int_{-1}^{0}(1+x)\cos\frac{n\pi x}{2}dx + \int_{0}^{1}(1+x)\cos\frac{n\pi x}{2}dx + \int_{1}^{2}0.\cos\frac{n\pi x}{2}dx\right]$$

$$= \frac{1}{2}\left[\left\{(1+x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_{-1}^{0} + \left\{(1-x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_{0}^{1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\frac{n^2\pi^2}{4}} - \frac{\cos\frac{n\pi}{2}}{\frac{n^2\pi^2}{4}} - \frac{\cos\frac{n\pi}{2}}{\frac{n^2\pi^2}{4}} + \frac{1}{\frac{n^2\pi^2}{4}}\right]$$

i.e., $a_n = \frac{4}{n^2\pi^2}\left[1-\cos\frac{n\pi}{2}\right]$ ....(3)

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx$$

$$= \frac{1}{2}\left[\int_{-2}^{-1} 0.\sin\frac{n\pi x}{2}dx + \int_{-1}^{0}(1+x)\sin\frac{n\pi x}{2}dx + \int_{0}^{1}(1-x)\sin\frac{n\pi x}{2}dx + \int_{1}^{2}0.\sin\frac{n\pi x}{2}dx\right]$$

$$= \frac{1}{2}\left[\left\{(1+x)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_{-1}^{0} + \left\{(1-x)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_{0}^{1}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{\frac{n\pi}{2}} + \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{4}} - \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{4}} + \frac{1}{\frac{n\pi}{4}}\right] = 0$$

i.e., $b_n = 0$.......(4)

Substituting (2), (3) and (4) in (1) we get

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$= \frac{1}{4} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left(1 - \cos\frac{n\pi}{2}\right)\cos\frac{n\pi x}{2}$$

$$= \frac{1}{4} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}.2\sin\frac{2n\pi}{4}\cos\frac{n\pi x}{2} \text{ [Use } 1-\cos x = 2\sin^2\frac{x}{2}\text{]}$$

$$\text{i.e., } f(x) = \frac{1}{4} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2\frac{n\pi}{4}\cos\frac{n\pi x}{2}$$

EXERCISE

1. If $f(x)=\begin{cases}\pi x, & 0\le x\le 1\\ \pi(2-x), & 1\le x\le 2\end{cases}$

show that in the interval (0,2) [Nov. 90, Civil]

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{\cos\pi x}{1^2}+\frac{\cos 3\pi x}{3^2}+\frac{\cos 5\pi x}{5^2}+\ldots\right]$$

2. Obtain the Fourier Series expansion of f(x), if

$f(x)=\begin{cases}1 \text{ for } 0\le x\le 1\\ x \text{ for } 1<x<2\end{cases}$ and f(x+2) = f(x)

[Apr.90]

3. Develop f(x) in Fourier Series in the interval (-2,2) if

$f(x)=\begin{cases}0, -2<x<0\\ 1, \quad 0<x<2\end{cases}$

[**Ans.** $f(x)=\frac{1}{2}+\frac{2}{\pi}\left(\sin\frac{\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{2}+\frac{1}{5}\sin\frac{5\pi x}{2}+\ldots\right)$]

[Apr.90, civil]

4. Find the Fourier Series for $f(x)=x^2$ in $-1<x<1$

[Nov.86,Civil]

[**Ans.** $f(x)=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n\cos n\pi x}{n^2}$]

5. In the range (0,2l), f(x) is defined by the relation

$f(x)=\begin{cases}0, 0<x<l\\ a, l<x<2l\end{cases}$ [Nov,88,Mech.]

[**Ans.** $f(x)=\frac{a}{2}-\frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{\ell}$]

6. Find the Fourier Series with period 4 to represent the function $f(x)=x^2-2$ in the interval $-2<x<2$. [Apr.90. ECE]

[**Ans.** $f(x) = -\frac{2}{3} + \sum_{n=1}^{\infty} \frac{16(-1)^n}{n^2\pi^2} \cos\frac{n\pi x}{2}$]

4.5 Half Range Expansions

In many Engineering problems it is required to expand a function f(x) in the range $(0,\pi)$ in a Fourier Series of period 2π or in the range (0,l) in a Fourier Series of period 2l. If it is required to expand f(x) in the interval (0,1), then it is immaterial what the function may be outside the range $0 < x < l$. We are free to choose it arbitrarly in the interval (-l,0).

If we extend the function f(x) by reflecting it in the Y axis so that f(-x) = f(x), then the extended function is even for which b_n=0. The Fourier expansion of f(x) will contain only cosine terms.

If we extend the function f(x) by reflecting it in the origin so that f(-x) = -f(x), then the extended function is odd for which $a_0 = x_0 =$ an =0. The Fourier expansion of f(x) will contain only sine terms.

Hence a function f(x) defined over the interval $0 < x < l$ is capable of two distinct half range series.

The half range cosines series in (0,l) is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi s}{l}dx$$

The half range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\text{where} b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

Note : (i) The half-range cosine series in $(0,\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx, a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

Note : (ii) The half-range sine series in $(0,\pi)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

$$\text{where} b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$$

Example 4.5.1. Obtain the half range sine series of the function $f(x) = kx(x-l)$ in $0 \leq x \leq l$. [Apr.86, Mech.]

Solution : We know that the half range sine series of $f(x)$ in $(0,l)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad ....(1)$$

$$\text{where} b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$\text{Now, } b_n = \frac{2}{l}\int_0^l kx(x-l)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_0^l (kx^2 - klx)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[(kx^2 - klx)\left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\} - \right.$$

$$\left[(2kx-kl)\left\{\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right\}+2k\left\{\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right\}\right]_0^l$$

$$=\frac{2}{l}\left[\frac{2k\cos n\pi}{\frac{n^3\pi^3}{l^3}}-\frac{2k}{\frac{n^3\pi^3}{l^3}}\right]$$

$$=\frac{4kl^2}{n^3\pi^3}[(-1)^n-1]$$

$\therefore b_n = 0$ when n is even $= \frac{-8kl^2}{n^3\pi^3}$ when n is odd

Substituting (2) in (1) we get,

$$f(x) = \sum_{n=1,3,5}^{\infty} \frac{-8kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}$$

i.e., $kx(x-l) = \sum_{n=1,3,5}^{\infty} \frac{-8kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}$

$$=-\frac{8kl^2}{\pi^3}\sum_{n=1,3,5}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{l}$$

$$=-\frac{8kl^2}{\pi^3}\sum_{n=1,3,5}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}$$

Example 4.5.2. Obtain the sine series for the function

$$f(x) = \begin{cases} x \text{ in } 0\le x\le\frac{1}{2} \\ 1-x \text{ in } \frac{1}{2}\le x\le 1 \end{cases}$$

[Nov.86, 91, Mech]

Solution : The sine series for the function f(x) in (0,l) is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad ....(1)$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

Now, $$b_n = \frac{2}{l}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{2}{l}\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^{l/2}\right.$$

$$\left.+\left\{(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) + 1\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\frac{\frac{-l}{2}\cos\frac{n\pi}{2}}{\frac{n\pi}{l}} + \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{l}} + \frac{\frac{l}{2}\cos\frac{n\pi}{2}}{\frac{n\pi}{l}} + \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}\right]$$

$$= \frac{2}{l}\cdot\frac{l^2}{n^2\pi^2}2\sin\frac{n\pi}{2} = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

i.e., $$\boxed{b_n = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}}$$

Substituting (2) in (1) we get

$$\therefore f(x) = \sum_{n=1}^{\infty}\frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}$$

$$= \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}$$

Example 4.5.3 : Find the half-range sine series of f(x) in $(0,\pi)$

given that $f(x) = \begin{cases} kx, & 0 \le x \le \frac{\pi}{2} \\ k(\pi - x), & \frac{\pi}{2} \le x \le \pi \end{cases}$ [Nov.90, 89, Civil]

Solution : The half-range sine series of f(x) in $(0,\pi)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{.......(1)}$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

Now, $b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} kx \sin nx\,dx + \int_{\pi/2}^{\pi} k(\pi - x)\sin nx\,dx\right]$

$$= \frac{2}{\pi}\left[\left\{kx\left(\frac{-\cos nx}{n}\right) - k\left(\frac{-\sin nx}{n^2}\right)\right\}_0^{1/2}\right.$$

$$\left. + \left\{k(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-k)\left(\frac{-\sin nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{-k\left(\frac{\pi}{2}\right)\cos\frac{n\pi}{2}}{n} + \frac{k\sin\frac{n\pi}{2}}{n^2} + \frac{k\left(\frac{\pi}{2}\right)\cos\frac{n\pi}{2}}{n} + \frac{k\sin\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{2}{\pi}\,\frac{2k\sin\frac{n\pi}{2}}{n^2}$$

$$\boxed{b_n = \frac{4k}{\pi n^2}\sin\frac{n\pi}{2}} \qquad \text{.......(2)}$$

Substituting (2) in (1), we get

$$f(x) = \sum_{n=1}^{\infty} \frac{4k}{\pi n^2} \sin\frac{n\pi}{2} \sin nx$$

$$= \frac{4k}{\pi} \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2} \sin nx}{n^2}$$

Example 4.5.4 Find a half-range sine series which represents f(x) = sin px for p not an integer on the interval 0 < x < π.

[Nov. 88, Apr.92]

Solution : The half range sine series for f(x) in (0,π) is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad ....(1)$$

Now $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin px \sin nx \, dx$$

(Here note that 'p' is not an integer but 'n' is an integer)

$$= \frac{2}{\pi} \cdot \frac{1}{2} \int_0^{\pi} [\cos(n-p)x - \cos(n+p)x] dx$$

[∵ 2 sin A sin B = cos (A-B) – cos (A+B)]

$$= \frac{1}{\pi}\left[\frac{\sin(n-p)x}{n-p} - \frac{\sin(n+p)x}{n+p}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-p)\pi}{n-p} - \frac{\sin(n+p)\pi}{n+p}\right] \qquad [\because \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[\frac{\sin n\pi \cos p\pi - \cos n\pi \sin p\pi}{n-p}\right.$$

$$\left. - \frac{\sin n\pi \cos p\pi + \cos n\pi \sin p\pi}{n+p}\right]$$

$$=\frac{-1}{\pi}\left[\frac{\cos n\pi \sin p\pi}{n-p}+\frac{\cos n\pi \sin p\pi}{n+p}\right]$$

[∵ $\sin n\pi = 0$ since n is an integer also $\sin p\pi \neq 0$ since p is not an integer]

$$=\frac{(-1)^{n+1}\sin p\pi}{\pi}\left[\frac{n+p+n-p}{n^2-p^2}\right]$$

$$=\frac{(-1)^{n+1}\sin p\pi}{\pi}.\frac{2n}{n^2-p^2}$$

i.e., $b_n = \frac{2n}{\pi}\frac{(-1)^{n+1}\sin p\pi}{n^2-p^2}$ ......(2)

Substituting (2) in (1) we get

$$f(x) \quad = \sum_{n=1}^{\infty}\frac{2n(-1)^{n+1}\sin p\pi}{\pi(n^2-p^2)}\sin nx$$

$$=\frac{2\sin p\pi}{\pi}\sum_{n=1}^{\infty}\frac{(n)(-1)^{n+1}\sin nx}{n^2-p^2}\sin nx$$

Example 4.5.5 Obtain a half-range cosine series of the function

$$\mathbf{f(x)} = \begin{cases} kx, & for\ 0 \le x < \frac{1}{2} \\ k(1-x), & \frac{1}{2} \le x \le l \end{cases}$$

Solution: We know that a half-range cosine series for f(x) in (0, l) is given by

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n \cos\frac{n\pi x}{l} \quad ......(1)$$

Now, $a_0 = \frac{2}{l}\int_0^l f(x)dx$

$$=\frac{2}{l}\left[\int_0^{l/2}kx dx+\int_{l/2}^{l}k(1-x)dx\right]$$

$$=\frac{2}{l}\left[\left(k\frac{x^2}{2}\right)_0^{1/2}+k\left(lx-\frac{x^2}{x}\right)_{1/2}^{l}\right]$$

$$=\frac{2}{l}\left[\frac{kl^2}{8}+k\left(l^2-\frac{l^2}{2}\right)-k\left(\frac{l^2}{2}-\frac{l^2}{8}\right)\right]$$

$$=\frac{2}{l}\left(\frac{kl^2}{4}\right)=\frac{kl}{2}$$

$$\text{i.e., } a_0 = \frac{kl}{2} \qquad \text{......(2)}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\left[\int_0^{l/2} kx\cos\frac{n\pi x}{l}dx+\int_{l/2}^{l} k(l-x)\cos\frac{n\pi x}{l}dx\right]$$

$$=\frac{2}{l}\left[\left\{kx\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-k\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^{l/2}\right.$$

$$\left.+\left\{k(l-x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)+k\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_{l/2}^{l}\right]$$

$$=\frac{2}{l}\left[\left(\frac{\frac{kl}{2}\sin\frac{n\pi}{2}}{\frac{n\pi}{l}}\right)+\left(k\frac{\cos\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}\right)-\frac{k}{\frac{n^2\pi^2}{l^2}}\right.$$

$$\left.-\frac{k\cos n\pi}{\frac{n^2\pi^2}{l^2}}-\frac{\frac{kl}{2}\sin\frac{n\pi}{2}}{\frac{n\pi}{l}}+\frac{k\cos\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}\right]$$

$$=\frac{2}{l}\left[\frac{2k\cos\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}-\frac{k}{\frac{n^2\pi^2}{l^2}}+\frac{k\cos n\pi}{\frac{n^2\pi^2}{l^2}}\right]$$

$\therefore a_n = \dfrac{2kl}{n^2\pi^2}\left[2\cos\dfrac{n\pi}{2} - 1 - (-1)^n\right]$ $\quad$ $[\because \cos n\pi = (-1)^n]$

When 'n' is odd, $\cos\dfrac{n\pi}{2} = 0$

$\therefore a_n = 0$ when n is odd. ......(3)

When n is even

$a_2 = \dfrac{2kl}{n^2\pi^2}[2\cos\pi - 1 - 1] = -\dfrac{8kl}{2^2\pi^2}$ .....(4)

$a_4 = \dfrac{2kl}{4^2\pi^2}[2\cos 2\pi - 1 - 1] = 0$ ......(5)

$a_6 = \dfrac{2kl}{6^2\pi^2}[2\cos 3\pi - 1 - 1]$

$= \dfrac{-8kl}{6^2\pi^2}$ and so on. .....(6)

Substituting (2), (3), (4), (5) and (6) in (1) we get

$\therefore f(x) = \dfrac{kl}{4} - \dfrac{8kl}{\pi^2}\left(\dfrac{1}{2^2}\cos\dfrac{2\pi x}{1} + \dfrac{1}{6^2}\cos\dfrac{6\pi x}{1} + ...\right)$

Example 4.5.6 **Obtain the half range cosine series for f(x) = x in (0, π) and deduce that the sum of the series**

[Apr.91,87. Mech, Apr.89, ECE]

$\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + ...$

Solution: The half-range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad ....(1)$$

Now, $a_0 = \dfrac{2}{\pi}\int_0^{\pi} f(x)\,dx = \dfrac{2}{\pi}\int_0^{\pi} x\,dx = \dfrac{2}{\pi}\left(\dfrac{x^2}{2}\right)_0^{\pi}$

$= \dfrac{2}{\pi}\left[\dfrac{\pi^2}{2}\right] = \pi$

i.e., $\boxed{a_0 = \pi}$ ....(2)

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$\therefore a_n = 0$ when n is even,

$= \dfrac{-4}{n^2\pi}$ when n is odd ....(3)

Substituting (2) and (3) in (1) we get

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1,3,5}^{\infty} \frac{-4}{n^2\pi}\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right]$$

Putting x = 0, we get

$$f(0) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right]$$

Here '0' is a point of discontinuity

$$\therefore f(0) = \frac{f(0-0) + f(0+0)}{2}$$

$$[\because f(x) = x, f(0-0) = 0 \text{ and } f(0+0) = 0]$$

$$= 0$$

$$\therefore 0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right]$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$$

Example 4.5.7 **Find a cosine series for the function**

[Apr.91]

$$f(x)=\begin{cases} x & \text{in } 0 \le x < \dfrac{\pi}{2} \\ \pi - x & \text{in } \dfrac{\pi}{2} \le x < \pi \end{cases}$$

Solution: The cosine series for the function f(x) in (0, π) is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx \qquad ....(1)$$

Now, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right]$

$$= \frac{2}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi/2} + \left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right]$$

$$\boxed{a_0 = \frac{\pi}{2}} \qquad ....(2)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left\{x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right\}_0^{\pi/2} + \left\{(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\frac{\pi}{2}\sin\frac{\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2} - \frac{1}{n^2} - \frac{\cos n\pi}{n^2} \frac{-\frac{\pi}{2}\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2\cos\frac{n\pi}{2}}{n^2} - \frac{1}{n^2} - \frac{\cos n\pi}{n^2}\right] = \frac{2}{n^2\pi}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

When n is odd $a_n = 0$, i.e., $a_1 = a_3 = a_5 = \ldots\ldots 0$ ....(3)

When n is even

$$a_2 = \frac{2}{2^2\pi}[2\cos\pi - 1 - 1] = -\frac{2}{\pi.1^2} \qquad \ldots\ldots.(4)$$

$$a_4 = \frac{2}{4^2\pi}[2\cos 2\pi - 1 - 1] = 0 \; (\because \cos 2\pi = 1) \qquad \ldots\ldots.(5)$$

$$a_6 = \frac{2}{6^2\pi}[2\cos 3\pi - 1 - 1] = -\frac{-2}{\pi.3^2} \qquad \ldots\ldots.(6)$$

and so on.

Substituting (2),(3),(4),(5) and (6) in (1) we get

$$\therefore f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \ldots\ldots\right]$$

Example 4.5.8 Using an appropriate Fourier expansion show that in the range (0, π), the function sin x can be expressed as

$$\frac{4}{\pi}\left(\frac{1}{2} - \frac{\cos 2x}{3} - \frac{\cos 4x}{15} - \frac{\cos 6x}{35} - \ldots\ldots - \frac{\cos 2nx}{4n^2 - 1}\right) \quad \text{[Apr.93]}$$

Solution: Here the range is (0, π) and the expansion contains only cosine terms. Therefore we have to expand sin x in a half range Fourier cosine series in (0, π).

We know that the half range cosine series of f(x) in (0, π) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots\ldots(1)$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x\,dx$$

$$= \frac{2}{\pi}[-\cos x]_0^{\pi}$$

$$= \frac{2}{\pi}[-\cos\pi + \cos 0]$$

$$= \frac{2}{\pi}[1-(-1)] \qquad [\because \cos\pi = -1]$$

$$a_0 = \frac{4}{\pi} \qquad \text{.........(2)}$$

Now $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x\cos nx\,dx$$

$$= \frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi}[\sin(1+n)x + \sin(1-n)x]dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= \frac{-1}{\pi}\left[\frac{\cos(1+n)\pi}{1+n} + \frac{\cos(1-n)\pi}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right]$$

When n is odd,

$$a_n = \frac{-1}{\pi}\left[\frac{1}{1+n} + \frac{1}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right]$$

$=0 \qquad [\because$ when n is odd $\cos(1+n)\pi = 1 = \cos(1-n)\pi]$

i.e., $a_n = 0$ provided $n \neq 1$ and n is odd ....(3)

when $n = 1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx = \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi}$$

$$= -\frac{1}{2\pi}[1-1] = 0$$

i.e., $a_1 = 0$ ....(4)

When n is even, (1+n) and (1-n) is odd

$$a_n = -\frac{1}{\pi}\left[\frac{-1}{1+n} - \frac{1}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right]$$

$$[\because \cos(1+n)\pi = -1; \cos(1-n)\pi = -1]$$

$$= \frac{1}{\pi}\left[\frac{2}{1+n} + \frac{2}{1-n}\right]$$

$$= \frac{2}{\pi}\left[\frac{1-n+1+n}{1-n^2}\right]$$

$$= \frac{4}{\pi}\left[\frac{1}{1-n^2}\right] \qquad ....(5)$$

Substituting (2), (3), (4) and (5) in (1), we get

$$f(x) = \frac{2}{\pi} + \sum_{n=2,4,6}^{\infty} \frac{4}{\pi(1-n^2)} \cos nx$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2,4,6}^{\infty} \frac{1}{n^2-1} \cos nx$$

$$= \frac{4}{\pi}\left[\frac{1}{2} - \sum_{n=1}^{\infty}\frac{1}{(2n)^2-1}\cos 2nx\right]$$

$$\text{i.e., } \sin x = \frac{4}{\pi}\left[\frac{1}{2} - \frac{\cos 2x}{3} - \frac{\cos 4x}{15} - \frac{\cos 6x}{35} - \ldots - \frac{\cos 2nx}{4n^2-1}\right]$$

Example 4.5.9 Expand f(x) = cos x, 0 < x <π in a Fourier sine series.

[Nov.'86, Apr.'88, Nov.'91]

Solution: We know that the Fourier sine series of f(x) in 0 < x <π is given by

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx \qquad ....(1)$$

To find a_n

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi}\frac{1}{2}\{\sin (n+1)x + \sin (n-1)x\}\, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[-\left(\frac{\cos n\pi \cos\pi - \sin n\pi\pi si\ \pi}{n+1}\right) - \left(\frac{\cos n\pi\pi cos + sinn\pi innπ}{n-1}\right) + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n+1} + \frac{\cos n\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right] \quad [\because \sin n\pi = 0]$$

$$= \frac{1}{\pi}\left[\frac{(n-1)\cos n\pi + (n+1)\cos n\pi + (n-1) + n + 1}{(n+1)(n-1)}\right]$$

$$= \frac{1}{\pi}\left[\frac{2n\cos n\pi + 2n}{n^2 - 1}\right]$$

$$= \frac{2n}{\pi}\left[\frac{1 + (-1)^n}{n^2 - 1}\right] \qquad [\text{provided } n \neq 1]$$

$$a_n = 0, \quad \text{when 'n' is odd}$$

$$\left.= \frac{4}{\pi(n^2 - 1)} \text{ when 'n' is even}\right\} \qquad \text{.........(2)}$$

Now $a_1 = \frac{2}{\pi}\int_0^{\pi} \cos x \sin x\, dx$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx$$

$$\frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} = \frac{1}{\pi}\left[-\frac{1}{2}+\frac{1}{2}\right] = 0$$

Substituting (2) and (3) in (1), we get

$$f(x) = a_1 \sin x + \sum_{n=2}^{\infty} a_n \sin nx$$

$$= 0 + \sum_{n=2,4}^{\infty} \frac{4n}{\pi(n^2-1)} \sin nx$$

$$= \frac{4}{\pi}\left[\frac{2}{3} \sin 2x + \frac{4}{15} \sin 4x + \frac{6}{35} \sin 6x + \dots\dots\dots\right]$$

$$\cos x = \frac{8}{\pi}\left[\frac{\sin 2x}{3} + \frac{2}{15} \sin 4x + \frac{3}{35} \sin 6x + \dots\dots\dots\right]$$

Example 4.5.10 Expand f(x) = $\begin{cases} \sin x, & 0 < x < \frac{\pi}{4} \\ \cos x, & \frac{\pi}{4} < x < \frac{\pi}{2} \end{cases}$ **in a series of sines.**

[Apr. '88]

Solution: We know that sine series for f(x) in $(0, \frac{\pi}{2})$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin 2nx$$

To find b_n

$$b_n = \frac{1}{\frac{1}{2}\left(\frac{\pi}{2}\right)} \int_0^{\pi/2} f(x).\sin 2nx \, dx$$

$$= \frac{4}{\pi}\left[\int_0^{\pi/4} \sin x \sin 2nx \; dx + \int_{\pi/4}^{\pi/2} \cos x \sin 2nx \; dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/4} \{\cos(2n-1)x - \cos(2n+1)x\} dx\right.$$

$$\left. + \int_{\pi/4}^{\pi/2} \{\sin(2n+1)x + \sin(2n-1)x\} dx\right]$$

$$= \frac{2}{\pi}\left[\left\{\frac{\sin(2n-1)x}{2n-1} - \frac{\sin(2n+1)x}{2n+1}\right\}_0^{\pi/4}\right.$$

$$\left. + \left\{\frac{-\cos(2n+1)x}{2n+1} - \frac{\cos(2n-1)x}{2n+1}\right\}_{\pi/4}^{\pi/2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin(2n-1)\frac{\pi}{4}}{2n-1} - \frac{\sin(2n+1)\frac{\pi}{4}}{2n+1}\right.$$

$$\left. + \frac{\cos(2n+1)\frac{\pi}{4}}{2n+1} + \frac{\cos(2n-1)\frac{\pi}{4}}{2n-1}\right]$$

$$\left[\because \cos(2n \pm 1)\frac{\pi}{2} = 0\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin\frac{n\pi}{2}\cos\frac{\pi}{4} - \cos\frac{n\pi}{2}\sin\frac{\pi}{4}}{2n-1} - \frac{\sin\frac{n\pi}{2}\cos\frac{\pi}{4} + \cos\frac{n\pi}{2}\sin\frac{\pi}{4}}{2n+1}\right.$$

$$\left.+\frac{\cos\frac{n\pi}{2}\cos\frac{\pi}{4} - \sin\frac{n\pi}{2}\sin\frac{\pi}{4}}{2n+1} + \frac{\cos\frac{n\pi}{2}\cos\frac{\pi}{4} + \sin\frac{n\pi}{2}\sin\frac{\pi}{4}}{2n-1}\right]$$

$$= \frac{2}{\pi}\left\{\frac{-\cos\frac{n\pi}{2}.\frac{1}{\sqrt{2}}}{2n-1} - \frac{\cos\frac{n\pi}{2}.\frac{1}{\sqrt{2}}}{2n+1} + \frac{1}{\sqrt{2}}.\frac{\cos\frac{n\pi}{2}}{2n+1} + \frac{1}{\sqrt{2}}.\frac{\cos\frac{n\pi}{2}}{2n-1}\right\}$$

When 'n' is even,

$$b_n = \frac{2}{\pi}[0]$$

$$= [0] \qquad \text{.........(2)}$$

When 'n' is odd

$$b_n = \frac{2}{\pi} x \frac{1}{\sqrt{2}}\left[\frac{\sin\frac{n\pi}{2} - \cos\frac{n\pi}{2}}{2n-1} - \frac{\sin\frac{n\pi}{2} + \cos\frac{n\pi}{2}}{2n+1}\right.$$

$$\left.+\frac{\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}}{2n+1} + \frac{\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}}{2n-1}\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-2\sin\frac{n\pi}{2}}{(2n+1)} + \frac{2\sin\frac{n\pi}{2}}{2n-1}\right]$$

$$=\frac{\sqrt{2}}{\pi}2\ sin\ \frac{n\pi}{2}\left[\frac{1}{2n-1}-\frac{1}{2n+1}\right]$$

$$=\frac{2\sqrt{2}}{\pi}\ sin\ \frac{n\pi}{2}\left[\frac{2n+1-2n+1}{4n^2-1}\right]$$

$$=\frac{2\sqrt{2}}{\pi}\ sin\ \frac{n\pi}{2}\left[\frac{2}{4n^2-1}\right]$$

$$b_n = \frac{4\sqrt{2}}{\pi(4n^2-1)}\ sin\ \frac{n\pi}{2} \quad \text{when 'n' is odd.} \qquad ...(3)$$

Substituting (2) and (3) in (1), we get

$$f(x) = \sum_{n=1,3,5}^{\infty} \frac{4\sqrt{2}}{(4n^2-1)\pi}.\sin\frac{n\pi}{2}.\sin 2nx$$

$$=\frac{4\sqrt{2}}{\pi}\left[\frac{sin\,2x}{3}-\frac{1}{35}sin\,6x+.....\right]$$

Example 4.5.11 Expand x sin x as a sine series in 0 < x <π

[Apr. '88]

Solution: Let $f(x) = \sum_{n=1}^{\infty} b_n\ sin\,nx$ ...(1)

Where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,sin\ nx\ dx$

$$=\frac{2}{\pi}\int_0^{\pi} x\,sin\,x\,sin\ nx\,dx \qquad ....(2)$$

$$= \frac{-2}{\pi} \cdot \frac{1}{2} \int_0^{\pi} x[\cos(n-1)x - \cos(n+1)x]\, dx$$

$$= \frac{-1}{\pi} \left\{ \int_0^{\pi} x \cos(n-1)x \; dx - \int_0^{\pi} x \cos(n+1)x \, dx \right\}$$

$$= \frac{-1}{\pi} \left[\left\{ x \frac{\sin(n-1)x}{(n-1)} - 1\left(\frac{-\cos(n-1)x}{(n-1)^2} \right)_0^{\pi} \right\} - \left\{ x \frac{\sin(n+1)x}{n+1} - 1\left(\frac{-\cos(n+1)x}{(n-1)^2} \right)_0^{\pi} \right\} \right]$$

$$= \frac{-1}{\pi} \left[\frac{\cos(n-1)\pi}{(n-1)^2} - \frac{1}{(n-1)^2} - \frac{\cos(n+1)\pi}{(n+1)^2} + \frac{1}{(n+1)^2} \right]$$

$$= \frac{-1}{\pi} \left[\frac{-\cos n\pi}{(n-1)^2} - \frac{1}{(n-1)^2} + \frac{\cos n\pi}{(n+1)^2} + \frac{1}{(n+1)^2} \right]$$

$$= \frac{-1}{\pi} \left[\frac{-(n+1)^2 \cos n\pi - (n+1)^2 + (n-1)^2 \cos n\pi + (n-1)^2}{(n-1)^2 (n+1)^2} \right]$$

$$= \frac{-1}{\pi} \left[\frac{-4n \cos n\pi - 4n}{(n^2 - 1^2)^2} \right]$$

$$= \frac{1}{\pi} \left[\frac{4n \cos n\pi - 4n}{(n^2 - 1)^2} \right]$$ [Provided $n \neq 1$]

$b_n = 0$, when ‘n’ is odd

$$\left. = \frac{8n}{\pi(n^2 - 1)^2}, \text{when 'n' is even} \right\},$$

When $n = 1$

$$b_1 = \frac{2}{\pi} \int_0^{\pi} x \sin x \sin x \; dx$$ [Putting n = 1 in (2)]

$$= \frac{2}{\pi}\int_0^{\pi} x \ sin^2 x \ dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \left(\frac{1-cos\,2x}{2}\right)dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} xdx - \int_0^{\pi} x\,cos\,2xdx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} - \left\{x\left(\frac{\sin 2x}{2}\right) - 1\left(\frac{-\cos 2x}{4}\right)\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right]$$

$$= \frac{\pi}{2} \qquad \ldots(4)$$

Substituting (3) and (4) in (1), we get

$$f(x) = b_1 \sin x + \sum_{n=2,4}^{\infty} b_n \, sinnx$$

$$= \frac{\pi}{2} sin x + \sum_{n=2,4}^{\infty} b_n \, sin nx$$

$$= \frac{\pi}{2} sin x + \sum_{n=2,4}^{\infty} \frac{8n}{\pi(n^2-1)^2} sin \ nx$$

$$\therefore x \sin x = \frac{\pi}{2} sin x + \frac{8}{\pi}\sum_{n=2,4}^{\infty} \frac{n \, sin nx}{(n^2-1)^2}$$

Example 4.5.12 Find half range sine series for the function f(x) defined as f(x)

$$= \begin{cases} x-1, & 0 \le x \le 1 \\ 1-x, & 1 \le x \le 2 \end{cases}$$

Solution: The half-range sine series of f(x) in (0, l) is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Here l = 2

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \qquad \ldots(1)$$

To find b_n

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}dx$$

$$= \int_0^1 f(x)\sin\frac{n\pi x}{2}dx + \int_0^2 f(x)\sin\frac{n\pi x}{2}dx$$

$$= \int_0^1 (x-1)\sin\frac{n\pi x}{2}dx + \int_0^2 (1-x)\sin\frac{n\pi x}{2}dx$$

$$= \left[\left\{(x-1)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_0^1\right.$$

$$\left. + \left\{(1-x)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right\}_1^2\right]$$

$$=\frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{4}}-\frac{1}{\frac{n\pi}{2}}+\frac{\cos n\pi}{\frac{n\pi}{2}}+\frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{4}}$$

$$=\frac{8\sin\frac{n\pi}{2}}{n^2\pi^2}+\frac{2}{n\pi}[(-1)^n-1]$$

$b_n = 0$, when 'n' is even

$$b_n=\frac{8\sin\frac{n\pi}{2}}{n^2\pi^2}-\frac{4}{n\pi} \text{ when 'n' is odd}$$

$$\therefore f(x)=\sum_{n=1,3,5}^{\infty}\left(\frac{8\sin\frac{n\pi}{2}}{n^2\pi^2}-\frac{4}{n\pi}\right).\sin\frac{n\pi x}{2}$$

$$f(x)=\left(\frac{8}{\pi^2}-\frac{4}{\pi}\right)\sin\frac{\pi x}{2}+\left(\frac{8}{9\pi^2}-\frac{4}{3\pi}\right)\sin\frac{3\pi x}{2}$$

$$+\left\{x\left(\frac{-\cos(1-n)x}{1-n}\right)-\left(\frac{-\sin(1-n)x}{(1-n)^2}\right)\right\}_0^{\pi}\Bigg]$$

$$=\frac{1}{\pi}\left[\frac{-\pi\cos(1+n)\pi}{1+n}-\frac{\pi\cos(1-n)\pi}{1-n}\right]$$

$$=-\left[\frac{\cos n\pi}{1+n}+\frac{\cos n\pi}{1-n}\right]$$

$$=-(-1)^n\left[\frac{1-n+1+n}{1-n^2}\right]$$

$$an = \frac{-2(-1)^n}{1-n^2} \text{ provided } n\neq 1 \qquad \ldots.(3)$$

When n = 1

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x.\sin x \cos x dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x.\sin 2x \ dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - \left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{2}\right]$$

$$a_1 = \frac{-1}{2} \qquad \ldots.(4)$$

Substituting (2), (3) and (4) in (1), we get

$$f(x) = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx$$

$$\text{i.e., } x \sin x = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{(n+1)(n-1)}\cos nx$$

Put $x = \frac{\pi}{2}$, we get

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - \frac{1}{2}\cos\frac{\pi}{2} - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{(n-1)(n+1)}\cos\frac{n\pi}{2}$$

$$\text{i.e., } \frac{\pi}{2} = 1 - 2\left[-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \ldots\ldots\ldots\right]$$

$$\text{i.e., } 1 + 2\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots\ldots\right] = \frac{\pi}{2}$$

EXERCISES

1. Obtain cosine series expansion of

$$f(x) = \begin{cases} \cos x \; in \; 0 < x < \frac{\pi}{2} \\ 0 \quad in \; \frac{\pi}{2} < x < \pi \end{cases}$$ [Apr.90, Mech],

[**Ans.** $f(x) = \frac{1}{\pi} + \frac{1}{2}\cos x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \cos 2nx}{4n^2 - 1}$]

2. Obtain cosine series expansion of $f(x) = (x-1)^3$ in (0, 1) and hence show that $\pi^2 = 8\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right)$ [Apr. 90]

[**Ans.** $f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi x}{n^2}$

3. Find the half-range cosine series for the function $f(x) = x^2$ in the range $0 \leq x \leq \pi$ and hence find the sum of the series

$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$ [Nov. 89, Mech.]

[**Ans.** $f(x) = \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\ldots\right)$]

4. Find the half – range sine series of the following functions:

(i) x^3 in $0 \leq x \leq \pi$ [Apr.91]

(ii) x^2 in $0 < x < 1$ [Apr.87]

(iii) x in $0 < x < 2$ [Apr.88, ECE]

(iv) $f(t) = \begin{cases} \frac{2kt}{l} & when \; 0 < t < \frac{1}{2} \\ \frac{2k}{t}(l - t) & when \; \frac{l}{2} < t < l \end{cases}$ [Apr.87, ECE]

Answers

(i) $$f(x) = 2\sum_{n=1}^{\infty}(-1)^n\left[\frac{6}{n^3} - \frac{\pi^2}{n}\right]\sin nx$$

(ii) $$f(x) = 2\left[\left(\frac{\pi^2 - 4}{\pi^3}\right)\sin \pi x - \frac{1}{2\pi}\sin 2x + \left(\frac{3^2\pi^2 - 4}{3^3\pi^3}\right)\sin 3x - \frac{1}{4x}\sin 4x + \ldots\ldots\right]$$

(iii) $$f(x) = \frac{-4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}.\sin\frac{n\pi x}{2}$$

(iv) $$f(t) = \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi t}{l}$$

UNIT – 5

LINEAR PROGRAMMING

5.0 Introduction:

The term Operations Research, was first coined in 1940 by McClosky and Trefthen in a small town, Bowdsey, of the United Kingdom. This new science came into existence in military context. During World War II, military management called on scientists from various disciplines and organised them into teams to assist in solving strategic and tactical problems, i.e., to discuss, evolve and suggest ways and means to improve the execution of various military projects. By their joint efforts, experience and deliberations, they suggested certain approaches that showed remarkable progress. This new approach to systematic and scientific study of the operations of the system was called the **Operation Research** or **Operational Research** (abbreviated as O.R.).

This chapter provide an overall view of the subject of operations research. It covers general ideas on the subject, thus providing a perspective. The remaining chapters deals with specific ideas and specific methods of solving O.R. problems.

OBJECTIVES

After completing this unit you will be able to

1. Understand the definition and scope of O.R.
2. Understand the different types of models in O.R.

STRUCTURE

- Nature and Definitions of O.R.
- Scope of O.R.
- Modeling in O.R.
- Standard Linear programming problem
- Keywords
- Model Questions

NATURE AND DEFINITIONS OF O.R.

Operations research, rather simply defined, is the research of operations. An operation may be called a set of acts required for the achievement of a desired outcome. Such complex, inter – related acts can be performed by four types of systems: Man, Machine, Man – Machine unit and any organization of men, machines, and man – machine units. OR is concerned with the operations of the last type of system.

Many definitions of OR have been suggested from time to time. On the other hand are put forward a number of arguments as to why it cannot be defined. Perhaps the subject is too young to be defined in an authoritative way. Some of the different definitions suggested are:

(1) OR is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control. – *Morse & Kimball*

(2) OR, in the most general sense, can be characterized as the application of scientific methods, tools and techniques to problems involving the operations of systems so as to provide those in control of the operations with optimum solutions to the problems – *Churchman, Ackoff, Arnoff.*

(3) Operations research is applied decision theory. It uses any scientific, mathematical or logical means to attempt to cope with the problems that confront the executive when he tries to achieve a thorough going rationality in dealing with his decision problems. – *Miller and Starr*

(4) Operations research is a scientific approach to problem solving for executive management. – *H.M. Wagner*

(5) Operations research is the art of giving bad answers to problems, to which, otherwise, worse answers are given. – *Thomas L. Saaty*

(6) Operations research is an aid for the executive in making his decisions by providing him with the needed quantitative information based on the scientific method of analysis. – *C. Kittel*

(7) Operations research is the systematic, method – oriented study of the basic structure, characteristics, functions and relationships of an organization to

provide the executive with a sound, scientific and quantitative basis for decision – making. – *E.L. Arnoff & M.J. Netzorg*

(8) Operations research is the application of scientific methods to problems arising from operations involving integrated systems of men, machines and materials. It normally utilizes the knowledge and skill of an interdisciplinary research team to provide the managers of such systems with optimum operating solutions. – *Fabrycky and Torgersen*

(9) Operations research is an experimental and applied science devoted to observing, understanding and predicting the behaviour of purposeful man – machine systems; and operations research workers are actively engaged in applying this knowledge to practical problems in business, government and society. – *Operations Research Society of America*

(10) Operations research is the application of scientific method by interdisciplinary teams to problems involving the control of organized (man – machine) systems so as to provide solutions which best serve the purpose of the organization as a whole. – *Ackoff and Sasieni*

(11) Operations research utilizes the planned approach (updated scientific method) and an interdisciplinary team in order to represent complex functional relationships as mathematical models for the purpose of providing a quantitative basis for decision – making and uncovering new problems for quantitative analysis. – *Thierauf and Klekamp*

(12) O.R. is the application of modern methods of mathematical science to complex problems involving management of large systems of men, machines, materials and money in industry, business, government and defence. The distinctive approach is to develop a scientific model of the system incorporating measurement of factors such as chance and risk to predict and compare the outcomes of alternative decisions, strategies or controls. – J.O.R. Society, U.K.

SCOPE OF O.R.

Having known the definition of OR, it is easy to visualize the scope of operations research. Whenever there is a problem for optimization, there is scope for the application of OR. When we broaden the scope of OR, we find that really it has been practiced for hundreds of years before World War II.

In the field of industrial management, there is a chain of problems starting from the purchase of raw material to the dispatch of finished goods. The management is interested in having an overall view of the method of optimizing profits. In order to take decision on scientific basis, OR team will have to consider various alternative methods of producing the goods and the return in each case. OR study should also point out the possible changes in the overall structure like installation of a new machine, introduction of more automation, etc. OR has been successfully applied in industry in the fields of production, blending, product mix, inventory control, demand forecast, sale and purchase, transportation, repair and maintenance, scheduling and sequencing, planning, scheduling and control of projects and scores of other associated areas.

OR has a wide scope for application in defence operations. In modern warfare the defence operations are carried out by a number of different agencies, namely airforce, army and navy. The activities performed each of them can be further divided into sub – activities viz. Operations, intelligence, administration, training and the like. There is thus a need to coordinate the various activities involved in order to arrive at optimum strategy and to achieve consistent goals. Operations research, conducted by team of experts from all the associated fields, can be quite helpful to achieve the desired results.

In both developing and developed economies, OR approach is equally applicable. In developing economies, there is a great scope of developing an OR approach towards planning. The basic problem is to orient the planning so that there is maximum growth of per capita income in the shortest possible time, by taking into consideration the national goals and restrictions imposed by the country. The basic problem in most of the countries in Asia and Africa is to remove poverty and hunger as quickly as possible. There is, therefore, a great scope for economists, statisticians, administrators, technicians, politicians and agriculture experts working together to solve this problem with an OR approach.

OR approach needs to be equally developed in agriculture sector on national or international basis. With population explosion and consequent shortage of food, every country is facing the problem of optimum allocation of land to various crops in accordance with climatic conditions and available facilities. The problem of optimal distribution of water from the various water resources is faced by each developing country and a good amount of scientific work can be done in this direction.

OR approach is equally applicable to big and small organizations. For example, whenever a departmental store faces a problem like employing additional sales girls, purchasing an additional van, etc., techniques of OR can be applied to minimize cost and maximize benefit for each such decision.

OR methods can also be applied in big hospitals to reduce waiting time of out – door patients and to solve the administrative problems.

Monte Carlo methods can be applied in the area of transport to regulate train arrivals and their running times. Queuing theory can be applied to minimize congestion and passengers' waiting time.

OR is directly applicable to business and society. For instance, it is increasingly being applied in L.I.C. offices to decide the premium rates of various policies. It has also been extensively used in petroleum, paper, chemical, metal processing, aircraft, rubber, transport and distribution, mining and textile industries.

Thus we find that OR has a diversified and wide scope in the social economic and industrial problems of today.

MODELLING IN O.R.

A model in O.R. is a simplified representation of an operation or a process in which only the basic aspects or the most important features of a typical problem under investigation and considered.

The objective of a models is to provide a means for analyzing the behaviour of the system for the purpose of improving its performance.

There are several models in each area of business, or industrial activity. For instance, an account model is a typical budget in which business accounts are referred to with the intention of providing measurements such as rate of expenses, quantity sold, etc, a mathematical equation may be considered to be a mathematical model in which a relationship between constants and variables is represented. A model which has the possibility of measuring observations may be called a quantitative model; a product, a device or any tangible thing used for experimentation may represent a physical model.

Following are the main characteristics that a good model for **Operations Research study should have:**

1. A good model should be capable of taking into account new formulations without having any significant change in its frame.
2. Assumptions made in the model should be as small as possible.
3. It should be simple and coherent. Number of variables used should be less.
4. It should be open to parametric type of treatment.
5. It should not take much time in its construction for any problem.

However, besides the above characteristics, a model has the following limitations:

(i) Models are only an attempt in understanding operations and should never be considered as absolute in any sense.

(ii) Validity of any model with regard to corresponding operation can only be verified by carrying the experiment and relevant data characteristics.

Classification of Models

Although the classification of models is a subjective problem, they may be distinguished as follows:

Models by degree of abstraction: These models are based on the past data/information of the problems under consideration and can be categories into (a) language models, and (b) case studies.

A book may be regarded as an example of a language model.

Models by function: These models consist of (a) Descriptive models, (b) Predictive models, and (c) Normative models.

(a) Descriptive models: A descriptive model simply describes some aspects of a situation based on observation, survey, questionnaire results, or other available data. The result of an opinion poll represents a descriptive model.

(b) Predictive models: Such models can answer 'what if' type of questions, i.e., they make predictions regarding certain events. For example, based on survey results, televisions networks attempt to explain and predict the election outcome before all the votes are actually counted.

(c) Normative models: Finally, when a predictive model has been repeatedly successful it can be used to prescribe a source of action. Linear programming is a normative or prescriptive model, because it prescribes what the managers ought to do.

Models by structure: These models are represented by (a) Iconic models, (b) Analogue models, and (c) Symbolic models.

Iconic or Physical models are pictorial representation of real systems and have the appearance of the real thing. Examples of such models are: city maps, houses blueprints, globe, and so on. An iconic model is said to be 'scaled – down' or 'scaled – up' according as the dimensions of the model are smaller or greater than those of the real item. For instance, in biology, the structure of a cell may be illustrated by an enlarged (scaled – up) iconic model for teaching purposes.

Iconic models are easy to observe, build and describe, but are difficult to manipulate and not very useful for the purposes of prediction. Commonly, these models represent a static event.

Analogue models are more abstract than the iconic ones for there is no 'look – alike' correspondence between these models and real life items. They are built by utilizing one set of properties to represent another set of properties. For instance, a network of pipes through which water is running could be used as a parallel for understanding the distribution of electric currents. Graphs and maps in various colours are analogue models, distribution of electric currents. Graphs and maps in various colours are analogue models, in which different colours correspond to different characteristics. A flow process chart is an analogue model which represents the order of occurrence of various events to make a product.

Mathematical or Symbolic models are most abstract in nature. They employ a set of mathematical symbols to represent the components (and relationships between them) of the real system. These models are most general and precise. However, it is not always possible to depict a real system in mathematical formulaiton, sometimes it is easier to use mathematical symbols for describing the relationship of the components, and sometimes an analogue model may express the pattern of its relationship in a better way.

Models by nature of the environment: These models can be classified into (a) Deterministic models, and (b) Probabilistic models.

In deterministic models, all the parameters and functional relationship are assumed to be known with certainty when the decision is to be made. Linear Programming and Break – even models are the examples of deterministic models.

On the other hand, models in which at least one parameter or decision variable is a random variable are called probabilistic or stochastic models. These models reflect, to some extent, the complexity of the real world and the uncertainty surrounding it.

Models by the extent of generality: These models can be categorised into (a) Specific models, and (b) General models.

When a model presents a system at some specific time, it is known as a specific model. In these models if the time factor is not considered, then they are termed as static models, and dynamic models otherwise. An inventory problem of determining economic order quantity for the next period, assuming that the demand in planning period would remain same as that of today, is an example of a static model. Dynamic Programming may be considered as an example of dynamic model.

Simulation and Heuristic models fall under the category of general models. These models are mainly used to explore alternative strategies (courses of action) which have been overlooked previously. These models do not yield any optimum solution to the problem, but give a solution to a problem depending on assumptions based on the past experience.

5.1 STANDARD LINEAR PROGRAMMING PROBLEMS

General Linear Programming Problem

The linear programming involving more than two variables may expressed as follows:

Maximize (or) Minimize $Z = c_1x_1 + c_2x_2 + c_3x_3 + \dots\dots\dots + c_nx_n$ subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots\dots\dots + a_{1n}x_n \leq \text{or} = \text{or} \geq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots\dots\dots + a_{2n}x_n \leq \text{or} = \text{or} \leq b_2$$

$$a_{31}x_1 + a_{32}x_2 + \dots\dots\dots\dots\dots + a_{3n}x_n \leq \text{or} = \text{or} \leq b_3$$

....

....

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n \leq \text{ or } = \text{ or } \geq b_m$$

and the non – negativity restrictions.

$$x_1, x_2, x_3, \ldots\ldots\ldots\ldots x_n \geq 0.$$

Note:

Some of the constraints may be equalities, some others may inequalities of ($\leq$) type and remaining ones inequalities of ($\geq$) type or of them are of same type.

5.2 Definitions

Definition:5.2.1

A set of values $x_1, x_2, \ldots\ldots, x_n$ which satisfies the constraints of LPP is called its **solution.**

Definition: 5.2.2

Any solution to a LPP which satisfies the non negativity restrictions of the LPP is called its **feasible solution.**

Definition: 5.2.3

Any feasible solution which optimizes (maximizes (or) minimizes) the objective function of the LPP is called its **optimum solution** or **optimal solution.**

Definition: 5.2.4

A basic solution is said to be a non – degenerate basic solution if none of the basic variables is zero.

Definition:5.2.5

A basic solution is said to be the degenerate basic solution if one or more of the basic variables are zero.

Definition:5.2.6

A feasible solution which is also basic is called a basic feasible solution.

KEY WORDS

Operations Research, Scientific Methods, Decision Theory, Industrial Management, Monte Carlo Methods.

MODEL QUESTIONS

1. Define O.R. and discuss its scope.
2. What are the applications of O.R?
3. "Model building is the essence of the operations research approach" Discuss.
4. Give any three definitions of operations research and explain.
5. Explain the nature of operations research and its limitation.
6. "Operation Research is a bunch of Mathematical Techniques" comment.

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