



MADURAI KAMARAJ UNIVERSITY

(University with Potential for Excellence)

DISTANCE EDUCATION



B.Sc., MATHEMATICS

Second Year

Paper - V

ANALYTICAL GEOMETRY OF THREE DIMENSION, VECTOR CALCULUS AND NUMERICAL ANALYSIS

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**ANALYTICAL GEOMETRY OF THREE DIMENSION,
VECTOR CALCULUS AND NUMERICAL ANALYSIS**
Paper-V

Dear Student,

We welcome you as a student of the second year B.Sc., degree course in Mathematics

This Paper-V deals with Analytical Geometry of Three Dimension, Vector calculus and Numerical Analysis. The learning material for this paper will be supplemented by contact seminars.

Learning through the Distance Education mode, as you are all aware, involves self-learning and self-assessment and in this regard you are expected to put in disciplined and dedicated effort.

On our part, we assume of our guidance and support.

With best wishes.

SYLLABUS

B.Sc., mathematics – Second Year – Paper-V

ANALYTICAL GEOMETRY OF THREE DIMENSION, VECTOR

CALCULUS AND NUMERICAL ANALYSIS

- Unit 1 :** The plane, Angle between two planes, Length of perpendicular, Bisecting plane, Distance between two planes.
- Unit 2 :** The straight line, Symmetric form, Image of point, Image of a line about a plane, The plane and the straight line, Angle between a plane and straight line.
- Unit 3 :** Coplanar lines, shortest distance between two lines, skew line.
- Unit 4 :** The sphere, Equation of the sphere, Equations of the tangent plane, simple problems.
- Unit 5 :** Vector Differentiation, Gradient, divergence, Curl, properties, Vector Integration, Line Integrals
- Unit 6 :** Numerical solutions of Algebraic and Transcendental equations, Iteration Method, Newton's method, Method of false positions, Solutions of Simultaneous linear equations, Gauss method, Gauss Jordan method, Gauss Iteration method, Gauss Jacobi method.
- Unit 7 :** Finite difference, Forward difference and backward difference, Finite difference, operators, Relations, properties, Finding missing terms, Inverse operators, Factorial notation.

Unit 8 : Interpolation, Newton's Forward and Backward formulae, divided differences and properties, Newton's divided difference formula, Gauss formula, Stirling formula, Bessel formula, Laplace and Everett's formula, Simple problems, Inverse interpolation using Lagrange formulation.

Unit 9 : Numerical differentiation, finding the first and second order derivatives, Maximum and minimum values of a function for a given data.

Unit 10 : Numerical Integration, Newton's Cote's formula, Trapezoidal rule, Simpson's one third rule, Simpson's three eighth rule, Weddle's rule.

Text Books:

1. Analytical geometry of three dimensions and Vector calculus, by Dr. S. Arumugam and Issac, New Gamma Publications, Reprint 2006.
Chapters: 2, 3, (3.1, 3.2, 3.3), 4, 5, 7(7.1)
2. Numerical Analysis, by Dr. S. Arumugam, A. Issac and A. Somasundaram, New Gamma Publications, Edition 2003.
Chapters: 1, 2, 3 (3.1, 3.2), 4, 5, 6.

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UNIT 1

THE PLANE

Space for Hints

Section 1.0. Introduction

The analytical geometry of three dimensions (3-D) is a normal extension of the analytical geometry of two dimensions (2-D). In order to apply the algebraic methods to study of curves and surfaces that lie in three dimensional space, we first establish a co-ordinate system.

Section 1.1 Basic concepts

Rectangular Cartesian co-ordinates

We choose a point O and call it the origin. Let OX, OY and OZ be three mutually perpendicular lines called the x-axis, y-axis, and z-axis respectively (co-ordinate axes). These lines determine three mutually perpendicular planes xoy, yoz and zox which are called xy, yz and zx co-ordinate planes respectively. Now, let P be any point in space. Let x,y,z denote the perpendicular distances from P to yz, zx and xy co-ordinate planes respectively. Then the three real x,y,z are called the rectangular Cartesian Co-ordinates of P and the point P is represented by the ordered triple (x,y,z). Conversely, any ordered triple of real numbers (x,y,z) represents a unique point in space. Thus the set of points in space can be identified with the set

$$R^3 = \{(x,y,z) / x,y,z \in R\}$$

Note. 1 The space R^3 is divided into eight octants by the co-ordinate planes.

Note. 2 The points on the xy plane are of the form (x,y,0), the point on the yz plane are of the form (0,y,z) and the points on zx plane are of the form (x,0,z).

Note. 3 The points on the x-axis are of the form (x,0,0) and the points on the y-axis are of the form (0,y,0) and the points are of the form (0,0,z).

Theorem 1.1 If P (x_1, y_1, z_1) and Q(x_2, y_2, z_2) are two points, then

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Theorem 1.2 If point of division R of the line joining P (x_1, y_1, z_1) and Q(x_2, y_2, z_2) internally in the ratio $l:m$ is

$$R \left(\frac{lx_2 + mx_1}{l+m}, \frac{ly_2 + my_1}{l+m}, \frac{lz_2 + mz_1}{l+m} \right)$$

Corollary. 1 If R divides the line joining P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) externally in the ratio $l:m$ then R is $\left(\frac{lx_2 - mx_1}{l-m}, \frac{ly_2 - my_1}{l-m}, \frac{lz_2 - mz_1}{l-m} \right)$

Corollary.2 The midpoint of the line joining P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$

Corollary. 3 The centroid of the triangle whose vertices are (x_i, y_i, z_i), $i = 1, 2, 3$ is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right)$

Section 1.1.1 Direction Cosines

Definition: Let α, β, γ be the angles made by a straight line with the positive directions of the co-ordinate axes. These angles are called the direction angles and the cosines of these angles are called the direction cosines (d.c) of the line.

Note. 1 The d.c's of a line are usually denoted by l, m, n so that $l = \cos \alpha, m = \cos \beta$ and $n = \cos \gamma$

Note. 2 The direction cosines of the x, y and z -axis respectively 1,0,0; 0,1,0 and 0,0,1

Theorem 1.3 If l, m, n are the d.c's of a line then $l^2 + m^2 + n^2 = 1$.

Definition. Direction Ratios Any three numbers a, b, c which are proportional to the d.c of a line are called the direction ratios (d.r) (or direction numbers at the line).

Hence $l = ak; m = bk; n = ck$ where k is a non zero constant.

Section 1.2 The Relation between direction cosines and direction ratios.

If we know the direction ratios a, b, c of a line then we can find the direction cosines as follows:

We have $l = ak; m = bk; n = ck$ for some $k \neq 0$.

$$\text{Now } l^2 + m^2 + n^2 = 1$$

$$\text{Hence } k^2(a^2 + b^2 + c^2) = 1$$

$$\Rightarrow k = \pm \frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

$$\therefore dc = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Theorem 1.4 The d.r.'s of the line joining $P(x_2, y_2, z_2)$ and $Q(x_1, y_1, z_1)$ are two points then the d.c. of the line PQ are

$$\pm \frac{x_2 - x_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \pm \frac{y_2 - y_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \pm \frac{z_2 - z_1}{\sqrt{\Sigma(x_2 - x_1)^2}}$$

$$\text{Where } \Sigma(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

Theorem 1.5 The angle between the two lines whose d.c.'s are l, m, n and l_1, m_1, n_1

$$\text{Corollary 1. } \sin \theta = \sqrt{(lm_1 - l_1m)^2 + (mn_1 - m_1n)^2 + (nl_1 - n_1l)^2}$$

Corollary 2. If a, b, c and a_1, b_1, c_1 are the d.c.'s of the lines, then the angle between the lines, then the angle between the lines is given by

$$\cos \theta = \frac{aa_1 + bb_1 + cc_1}{\sqrt{\Sigma a^2} \sqrt{\Sigma a_1^2}}$$

$$\text{and } \sin \theta = \frac{\sqrt{(ab_1 - a_1b)^2 + (bc_1 - b_1c)^2 + (ca_1 - c_1a)^2}}{\sqrt{a^2 + b^2 + c^2} \sqrt{a_1^2 + b_1^2 + c_1^2}}$$

Corollary 4. Two lines whose d.r. are a, b, c and a_1, b_1, c_1 are perpendicular iff $aa_1 + bb_1 + cc_1 = 0$

Corollary 5. Two lines with direction cosines l, m, n and l_1, m_1, n_1 are parallel iff

$$\frac{l}{l_1} = \frac{m}{m_1} = \frac{n}{n_1} \text{ and consequently iff } \frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$$

Section 1.2. Equation of Plane

In this chapter we derive several forms of the equation of a plane in R^3

Definition A plane in R^3 is defined to be the locus of a point (x, y, z) satisfying a linear equation of the form $ax + by + cz + d = 0$ where a, b, c are not all zero.

Theorem 1.2.1

Equation of a plane passing through a given point (x_1, y_1, z_1) and having a normal whose d.r are a, b, c is given by $a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$.

Proof. Let $A(x_1, y_1, z_1)$ be a given point on the plane. Let LM be a normal to the plane. The d.r of LM are a, b, c . Let $P(x, y, z)$ be any point on the plane. Then AP is perpendicular to LM .

Also d.r of AP are $x-x_1, y-y_1, z-z_1$

$$\therefore a(x-x_1) + b(y-y_1) + c(z-z_1) = 0. \dots\dots\dots 1$$

– Since $P(x, y, z)$ is arbitrary (1) represents the equation of the plane.

Theorem 1.2.2

The equation of the plane passing through the Points $A(x_1, y_1, z_1)$ $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \dots\dots\dots (1)$$

Proof. Let the equation of the plane be $ax + by + cz + d = 0 \dots\dots\dots (2)$

Since A, B, C lie on this plane we have

$$ax_1 + by_1 + cz_1 + d = 0 \dots\dots\dots (3)$$

$$ax_2 + by_2 + cz_2 + d = 0 \dots\dots\dots (4)$$

$$ax_3 + by_3 + cz_3 + d = 0 \dots\dots\dots (5)$$

Eliminating the constants a, b, c from (2)(3)(4) and (5) we get the result (1).

Note. In numerical problems it is convenient to solve the three equations (3), (4) and (5) in terms of one directly and get the equation of the plane on substitution in (2).

Aliter. The equation of any plane passing through (x_1, y_1, z_1) can be written in the form $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ (1)

If this passes through (x_2, y_2, z_2) and (x_3, y_3, z_3) also, we have

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0 \text{ (2)}$$

$$a(x_3 - x_1) + b(y_3 - y_1) + c(z_3 - z_1) = 0 \text{ (3)}$$

Eliminating a, b, c from (1) (2) and (3) we have

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

which is the equation of the plane passing through (x_1, y_1, z_1) , (x_2, y_2, z_2) and (x_3, y_3, z_3) .

Note: To verify whether four points are coplanar we have to find the equation of the plane passing through any three points and check whether the fourth point lies on it or not. Equivalently the four points are coplanar if

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Theorem 1.2.3 (Intercepts form) The equation of the plane having intercepts a, b, c with the coordinate axes is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Proof. Let the equation of the plane be

$$Ax + By + Cz + D = 0 \text{ (1)}$$

Space for Hints

Let this plane meet the coordinate axes OX, OY, OZ at P, Q, R respectively.

$$\therefore OP = a; OQ = b; OR = c$$

$$\therefore P \text{ is } (a, 0, 0); Q \text{ is } (0, b, 0) \text{ and } R \text{ is } (0, 0, c).$$

Since P, Q, R lie on the plane we have

$$Aa + D = 0; Bb + D = 0; Cc + D = 0$$

$$\therefore A = -\frac{D}{a}; B = -\frac{D}{b}; C = -\frac{D}{c}$$

$$\therefore \text{Equation (1) becomes } -\frac{D}{a}x - \frac{D}{b}y - \frac{D}{c}z + 0$$

$$(i.e.) \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \quad \text{Hence the theorem}$$

Theorem 1.2.4 (Normal form) The equation of a plane can be written as $lx + my + nz = p$ where l, m, n are the d.c of the normal to the plane and p is the length of the perpendicular from the origin to plane.

Proof. Let the plane meet the coordinate axes at A, B, C with intercepts a, b, c .

$$\therefore \text{The equation of the plane is } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \dots \dots \dots (1)$$

Let the length of the perpendicular OD from O to the plane be p .

$$\text{Now, } l = \cos \angle DOA = \frac{OD}{OA} = \frac{p}{a}$$

$$\therefore a = \frac{p}{l}. \text{ Similarly } b = \frac{p}{m} \text{ and } c = \frac{p}{n}.$$

$$\therefore \text{Equation (1) of the plane becomes } \frac{lx}{p} + \frac{my}{p} + \frac{nz}{p} = 1$$

$$(i.e.) lx + my + nz = p$$

Note. The above equation of the plane can also be written $x \cos \alpha + y \cos \beta + z \cos \gamma = p$ where α, β, γ are the angles which the normal to the plane makes with the coordinate axes.

Transformation to the normal form.

The general equation of the plane $ax + by + cz + d = 0$ (1)

where $a^2 + b^2 + c^2 \neq 0$ can be transformed to the normal form $lx + my + nz = p$ (2)

Equations (1) and (2) represent the same plane if

$$\frac{a}{l} = \frac{b}{m} = \frac{c}{n} = \frac{-d}{p} = k \text{ (say)}$$

$$\therefore l = \frac{a}{k}, m = \frac{b}{k}, n = \frac{c}{k} \text{ and } p = -\frac{d}{k}$$

Since $l^2 + m^2 + n^2 = 1$ we get $k = \pm \sqrt{a^2 + b^2 + c^2}$

$$\therefore l = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}}; m = \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}};$$

$$n = \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}} \text{ and } p = \pm \left(\frac{-d}{\sqrt{a^2 + b^2 + c^2}} \right) \text{ (3)}$$

Now we choose the sign of k opposite to that of d so as to make p positive. Substituting (3) in (2) we obtain the required normal form.

Note 1. For the plane $ax + by + d = 0$ a, b, c are the d.r of the normal to

the plane and $\pm \frac{a}{\sqrt{\sum a^2}}, \pm \frac{b}{\sqrt{\sum b^2}}, \pm \frac{c}{\sqrt{\sum c^2}}$ (with suitable sign so that P is

always positive) denote the d.c. of the normal to the plane.

Example. The d.r. of the normal to the plane $2x - 3y + 6z + 7 = 0$ are

$2, -3, 6$. Hence the direction cosines are $-\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$ and the length of the

normal from the origin to the plane is $\frac{7}{7} = 1$.

Section 1.3 Angle Between Two Planes

Definition. Angle between two planes is defined to be the angle between the normal to them from any point.

Theorem 1.2.5 The angle θ between the planes $ax + by + cz + d = 0$

and $a_1x + b_1y + c_1z + d_1 = 0$ is given by $\cos \theta = \pm \left[\frac{aa_1 + bb_1 + cc_1}{\sqrt{(\sum a^2)} \sqrt{(\sum a_1^2)}} \right]$

Proof. The d.c. of the normal to plane $ax + by + cz + d = 0$ are

$$\frac{a}{\sqrt{(\sum a^2)}}, \frac{b}{\sqrt{(\sum a^2)}}, \frac{c}{\sqrt{(\sum a^2)}}$$

The d.c. of the normal to the plane $a_1x + b_1y + c_1z + d_1 = 0$

are $\frac{a_1}{\sqrt{(\sum a_1^2)}}, \frac{b_1}{\sqrt{(\sum a_1^2)}}, \frac{c_1}{\sqrt{(\sum a_1^2)}}$

$\therefore$ By theorem 2.5 the angle between the planes is given by $\cos$

$$\theta = \pm \left[\frac{aa_1 + bb_1 + cc_1}{\sqrt{(\sum a^2)} \sqrt{(\sum a_1^2)}} \right]$$

Corollary 1. The planes $ax + by + cz + d = 0$; $a_1x + b_1y + c_1z + d_1 = 0$ are at right angles iff $aa_1 + bb_1 + cc_1 = 0$ (refer corollary 2 of 2.3).

Corollary 2. The planes $ax + by + cz + d = 0$ and

$a_1x + b_1y + c_1z + d_1 = 0$ are parallel iff $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$.

Hence the equation of plane parallel to $ax + by + cz + d = 0$ is of the form $ax + by + cz + k = 0$.

Theorem 1.2.6. Length of the perpendicular from a point A (x_1, y_1, z_1) to the plane

$$ax + by + cz + d = 0 \text{ is } \left[\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right]$$

Proof. Let $lx + my + nz = p$ be the normal form of the plane

$$ax + by + cz + d = 0.$$

$$\therefore l = \pm \frac{a}{\sqrt{\sum a^2}}, m = \pm \frac{b}{\sqrt{\sum a^2}}, n = \pm \frac{c}{\sqrt{\sum a^2}} \text{ and } p = \pm \frac{d}{\sqrt{\sum a^2}}$$

Now equation of the plane through the given point $A = (x_1, y_1, z_1)$ and parallel to the plane $lx_1 + my_1 + nz_1 = p_1$ 1

Where p_1 is the length of the perpendicular from the origin to the plane.

Since (x_1, y_1, z_1) lies on (1) we have $lx_1 + my_1 + nz_1 = P_1$.

Now, the length of the perpendicular from (x_1, y_1, z_1) to the given plane $= p - P_1 = p - lx_1 - my_1 - nz_1$.

$$= \pm \frac{d}{\sqrt{(\sum a^2)}} - \left\{ \pm \frac{ax_1}{\sqrt{(\sum a^2)}}, \pm \frac{by_1}{\sqrt{(\sum a^2)}}, \pm \frac{cz_1}{\sqrt{(\sum a^2)}} \right\}$$

$$= \left[\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{(\sum a^2)}} \right] \text{ Hence the result.}$$

Theorem 1.2.7 Equation of a plane through the line of intersection of two given planes $\pi_1=0$ and $\pi_2=0$ is $\pi_1 + \lambda\pi_2 = 0$ (λ any constant)

Proof. Let $\pi_1 = a_1x + b_1y + c_1z + d_1 = 0$ 1

$\pi_2 = a_2x + b_2y + c_2z + d_2 = 0$ 2

be the equations of the two planes.

Consider the equation $\pi_1 + \lambda\pi_2 = 0$

(i.e. $(a_1x + b_1y + c_1z + d_1) + (a_2x + b_2y + c_2z + d_2) = 0$ 3

Equation (3) is of the first degree in x, y and z and hence represents a plane. Further any point (x_1, y_1, z_1) satisfying (1) and (2) also satisfies (3). Hence (3) passes through the line of intersection of (1) and (2).

Two sides of a plane. Consider a plane and two points A and B not lying in the plane. Then the points A, B may lie on (i) opposite sides of the plane or (ii) in the same side of the plane.

If A, B lie on either side of the plane, the segment AB has common point with the plane whereas if A, B lie in the same side of the plane the segment AB does not have a common point with the plane.

We proceed to find a criterion for two given points to lie on the same or different sides of a given plane.

Theorem 1.2.8 Two points A (x_1, y_1, z_1) , B (x_2, y_2, z_2) lie on the same or different sides of the plane $ax + by + cz + d = 0$ according as $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of the same or different signs.

Proof. Let the line AB meet the given plane at P. Let P divide AB in the ratio $k:1$. if k is positive P divides AB internally and if k is negative P divides AB externally.

(i.e.) if k is positive A, B lie on opposite sides of the plane and if k is negative A, B lie on the same side of the plane.

P is $\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}, \frac{kz_2 + z_1}{k+1} \right)$ and P lies on the plane. Hence we

$$\text{have } a \left(\frac{kx_2 + x_1}{k+1} \right) + b \left(\frac{ky_2 + y_1}{k+1} \right) + c \left(\frac{kz_2 + z_1}{k+1} \right) + d = 0$$

$$\therefore k(ax_2 + by_2 + cz_2 + d) + (ax_1 + by_1 + cz_1 + d) = 0.$$

$$\therefore k = - \left(\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} \right)$$

k is negative if $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of same signs.

k is positive if $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of opposite signs. Hence the result follows.

Example 1. The origin and (2, -3, 7) lie on the same side of the plane $2x - 3y + 2z + 8 = 0$.

For, by substituting the two points in the expression $2x - 3y + 2z + 8$ the values are of same signs.

Example 2. (2, 1, 1) and (2, 5, -1) lie on different sides of the plane $x - 2y + 3z + 4 = 0$. (verify).

Section 1.4 Angle Bisectors of Two Planes

As we have two bisectors between two given lines we also have two bisectors between two given planes. Here the bisectors are two planes.

Theorem: 1.2.9. The equation of the bisectors of the angle between the planes.

$$ax + by + cz + d = 0$$

$$a_1x + b_1y + c_1z + d_1 = 0 \quad \dots\dots\dots 1$$

$$\text{are } \left(\frac{ax + by + cz + d}{\sqrt{a^2 + b^2 + c^2}} \right) = \pm \left(\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} \right)$$

Proof: The bisector is the locus of a point which moves such that it is equidistant from the two given planes (1).

Let P (x_1, y_1, z_1) be any point on the locus.

Note: There are two bisecting planes one for positive sign and another for negative sign. Of these two planes one bisects the acute angle and the other obtuse angle between the given planes. The bisector of the acute angle makes with either of the plane an angle which is less than 45° and the bisector of obtuse angle makes with either of them an angle which is greater than 45° .

Solved problems:

Problem 1. Find the equation of the plane passing through $(1, 1, 0)$, $(1, 2, 1)$ and $(-2, 2, -1)$.

Solution. Let the equation of the required plane be

$$ax + by + cz + d = 0 \dots\dots 1$$

Since the given points lie on it we have

$$a + b + d = 0 \dots\dots\dots 2$$

$$a + 2b + c + d = 0 \dots\dots\dots 3$$

$$-2a + 2b - c + d = 0 \dots\dots\dots 4$$

$$(2) - (3) \Rightarrow -b - c = 0 \dots\dots\dots 5$$

$$(3) - (4) \Rightarrow 3a + 2c = 0 \dots\dots\dots 6$$

From (5) and (6) we have $\frac{a}{-2} = \frac{b}{-3} = \frac{c}{3} = k$ (say)

$$\therefore a = 2k; b = -3k; c = 3k$$

Substituting in (2) we have $d = 5k$.

$$\therefore \text{Equation (1) becomes } -2x - 3y + 3z + 5 = 0.$$

Aliter 1. The equation of any plane passing through the point $(1, 1, 0)$ is given by

$$a(x-1) + b(y-1) + c(z-0) = 0. \dots\dots\dots 1$$

The other two points also lie on it.

$$\therefore b + c = 0 \dots\dots\dots 2$$

$$-3a + b - c = 0 \dots\dots\dots 3$$

$$\therefore \frac{a}{-2} = \frac{b}{-3} = \frac{c}{3} = k \text{ (say) and } k \neq 0.$$

$$\therefore a = -2k; b = -3k; c = 3k.$$

$$\therefore \text{Equation of the plane (1) is } -2(x-1) - 3(y-1) + 3(z-0) = 0.$$

$$\text{(i.e.) } -2x - 3y + 3z + 5 = 0.$$

Aliter 2. The equation of the plane passing through the given three points is

$$\begin{vmatrix} x-1 & y-1 & z-0 \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0. \text{ (i.e.) } -2x - 3y + 3z + 5 = 0$$

Space for Hints

Problem 2. Find the equation of the plane passing through (2, 2, 1) and (9, 3, 6) and perpendicular to the plane $2x + 6y + 6z = 9$.

Solution. Equation of the plane passing through (2, 2, 1) is a $(x-2) + b(y-2) + c(z-1) = 0$... 1
where a, b, c are the d.r of the normal to the plane to be determined.

Since (9, 3, 6) lies on this plane we have $7a + b + 5c = 0$ 2

Since the plane (1) is perpendicular to $2x + 6y + 6z = 9$

we have $2a + 6b + 6c = 0$ 3

Solving (2) and (3) we have $\frac{a}{-24} = \frac{b}{-32} = \frac{c}{40}$

$$\therefore \frac{a}{3} = \frac{b}{4} = \frac{c}{-5} = k \text{ (say)}$$

$$\therefore a = 3k, b = 4, c = -5k$$

Substituting in (1) we get $3(x-2) + 4(y-2) - 5(z-1) = 0$.

$$\therefore 3x + 4y - 5z - 9 = 0.$$

Problem 3. Find the equation of the plane through (2, 3, -4) and (1, -1, 3) and parallel to the x-axis.

Solution. Equation of the plane passing through (2, 3, -4) is

$$a(x-2) + b(y-3) + c(z+4) = 0 \text{ 1}$$

where a, b, c are the d.r of the normal to the plane which is to be determined.

Since (1, -1, 3) also lies on the plane we have

$$-a - 4b + 7c = 0 \text{ 2}$$

Since the plane (1) is parallel to the x-axis its normal is perpendicular to the x-axis whose d.r. are 1, 0, 0.

$$\therefore a.1 + b.0 + c.0 = 0 \Rightarrow a = 0 \dots\dots\dots 3$$

From (2) and (3) we get $a = 0$; $b = 7k$ and $c = 4k$.

Substituting in (1) we get the equation of the required plane as $7k(y-3) + 4k(z+4) = 0$.

$$\therefore 7y + 4z - 5 = 0 \text{ is the equation of the required plane.}$$

Problem 4. Find the equation of the plane which passes through the point (3, -2, 4) and is perpendicular to the line joining the points (2, 3, 5) and (1, -2, 3).

Solution. Since the plane is perpendicular to the line joining A (2,3,5) and B (1, -2, 3) the line AB is normal to the plane.

The d.r. of the normal AB are 1, 5, 2

$$\therefore \text{The equation of the required plane is } 1(x-3) + 5(y+2) + 2(z-4) = 0$$

$$\text{(i.e.) } x + 5y + 2z - 1 = 0$$

Problem 5. Find the equation of the plane which passes through the point (1, -2, 1) and is perpendicular to each of the planes $3x+y+z-2 = 0$ and $x-2y+z+4 = 0$.

Solution. Let the equation of the plane be $ax + by + cz + d = 0 \dots\dots\dots(1)$

It passes through (1, -2, 1). Hence we get $a-2b+c+d = 0 \dots\dots\dots(2)$

Since (1) is perpendicular to the planes $3x + y + z - 2 = 0$ and $x-2y+z+4 = 0$

$$\text{we have } 3a + b + c = 0 \dots\dots\dots(3)$$

$$a - 2b + c = 0 \dots\dots\dots(4)$$

$$\therefore \frac{a}{3} = \frac{b}{-2} = \frac{c}{-7} = (k \text{ say})$$

$$\therefore a = 3k; b = -2k; c = -7k$$

Substituting in (2) we get $d = 0$

$$\therefore \text{Equation of the required plane is } 3x - 2y - 7z = 0.$$

Problem 6. Find the angle between the planes $2x - y + z = 6$ and $x + y + 2z = 7$.

Solution. The d.c. of the normal to the plane $2x - y + z - 6 = 0$ is $\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}$.

The d.c. of the normal to the plane $x + y + 2z - 7 = 0$ is $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$.

$\therefore$ The angle θ between the two planes is given by $\cos \theta = \frac{2-1+2}{\sqrt{6} \cdot \sqrt{6}} = \frac{1}{2}$.

$$\therefore \theta = \cos^{-1} \left(\frac{1}{2} \right) = 60^\circ$$

Problem 7. The foot of the perpendicular from the origin to a plane is $(2, -1, 2)$. Find the equation of the plane.

Solution. Let $ax + by + cz + d = 0$ be the equation of the required plane.

We know a, b, c are the d.r of the normal to the plane. Since $P(2, -1, 2)$ is the foot of the perpendicular from the origin O to the plane. OP is normal to that plane. Hence the d.r of normal to the plane are $2, -1, 2$.

$\therefore$ The equation of the plane becomes $2x - y + 2z + d = 0$.

Since $(2, -1, 2)$ lies on it we have $4 + 1 + 4 + d = 0$ is the required equation of the plane.

Aliter. Let the equation of the plane in the normal form be $lx + my + nz = p$ where l, m, n are the d.c. of the normal the plane from the origin and p is the length of the normal.

$$\therefore p = \sqrt{(2-0)^2 + (-1-0)^2 + (2-0)^2} = 3$$

The d.c. of the normal are $\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}$

$\therefore$ The equation of the plane becomes $\frac{2}{3}x - \frac{1}{3}y + \frac{2}{3}z = 3$.

$$(i.e.) 2x - y + 2z - 9 = 0$$

Problem 8. Find the coordinates of the foot of the perpendicular drawn from the origin to the plane $2x - 3y + z - 7 = 0$.

Solution. Let P (x_1, y_1, z_1) be the foot of the perpendicular from the origin.

Since P (x_1, y_1, z_1) lies on the plane we have $2x_1 - 3y_1 + z_1 = 7$ 1

The direction ratios of OP are $x_1 - 0, y_1 - 0, z_1 - 0$.

(i.e.) d.r. of OP are x_1, y_1, z_1 .

OP is normal to the given plane whose direction ratios are 2, -3, 1.

$$\therefore \frac{x_1}{2} = \frac{y_1}{-3} = \frac{z_1}{1} = k \text{ (say)}$$

$$\therefore x_1 = 2k; y_1 = -3k; z_1 = k$$

Substituting in (1) we get $4k + 9k + k = 7$. hence $k = \frac{1}{2}$.

$$\therefore x_1 = 1, y_1 = -\frac{3}{2}, z_1 = \frac{1}{2}.$$

$$\therefore \text{The foot of the perpendicular is } \left(1, -\frac{3}{2}, \frac{1}{2}\right),$$

Problem 9. A plane meets the coordinates axes at A, B, C such that the centroid of the ΔABC is the point (α, β, γ) . Show that the equation of

the plane is $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$

Solution. Let the equation of the plane be $ax + by + cz + d = 0$... 1

Since it meets the x-axis at A we get $A = \left(-\frac{d}{a}, 0, 0\right)$

Similarly $B = \left(0, -\frac{d}{b}, 0\right)$ and $C = \left(0, 0, -\frac{d}{c}\right)$

Centroid of ΔABC is $\left(-\frac{d}{3a}, -\frac{d}{3b}, -\frac{d}{3c}\right)$

But centroid of ΔABC is given to be (α, β, γ)

$$\therefore -\frac{d}{3a} = \alpha; -\frac{d}{3b} = \beta \text{ and } -\frac{d}{3c} = \gamma.$$

$$\therefore a = -\frac{d}{3\alpha}; b = -\frac{d}{3\beta} \text{ and } c = -\frac{d}{3\gamma}$$

$\therefore$ Equation of the required plane (1) becomes

$$-\frac{d}{3\alpha}x - \frac{d}{3\beta}y - \frac{d}{3\gamma}z + d = 0. \text{ i.e. } \frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$$

Problem 10. A moving plane passes through a fixed point (α, β, γ) and intersects the coordinate axes at A, B, C. Show that the locus of the centroid of the ΔABC is $\frac{\alpha}{x} + \frac{\beta}{y} + \frac{\gamma}{z} = 3$

Solution. Let A be $(a, 0, 0)$, B $(0, 0, b)$ and C $(0, 0, c)$.

Let (x_1, y_1, z_1) be the centroid of the triangle ABC.

$$\therefore x_1 = a/3; y_1 = b/3, z_1 = c/3 \dots\dots\dots 1$$

Now the equation of the plane ABC is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

It passes through the fixed point (α, β, γ)

$$\therefore \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \text{ Hence } \frac{\alpha}{3x_1} + \frac{\beta}{3y_1} + \frac{\gamma}{3z_1} = 1 \text{ (by (1))}$$

$$\text{i.e. } \frac{\alpha}{x_1} + \frac{\beta}{y_1} + \frac{\gamma}{z_1} = 3$$

$$\therefore \text{The locus of } (x_1, y_1, z_1) \text{ is } \frac{\alpha}{x} + \frac{\beta}{y} + \frac{\gamma}{z} = 3$$

Problem 11. Find the equation of the plane through the intersection of the planes $3x - y - 2z - 4 = 0$ and $x + y + z - 2 = 0$ and passing through the point $(2, 2, 1)$.

Solution. The required plane is $3x - y + 2z - 4 + \lambda (x + y + z - 2) = 0$ where λ is to be determined.

Since $(2, 2, 1)$ lies on it, we have $6 - 2 + 2 - 4 + \lambda (2 + 2 + 1 - 2) = 0$.

$$\therefore 2 + 3\lambda = 0. \text{ Hence } \lambda = -\frac{2}{3}$$

$\therefore$ Equation of the required plane is

$$3x - y + 2z - 4 - \frac{2}{3} (x + y + z - 2) = 0$$

$$\text{i.e. } 7x - 5y + 4z - 9 = 0.$$

Problem 12. Find the equation of the required plane through the intersection of the planes $x + 3y - z = 4$ and $2x + 2y + 2z = 1$ which is perpendicular to the plane $x + y - 4z = 0$.

Solution. The equation of the required plane is of the form

$$x + 3y - z - 4 + \lambda (2x + 2y + 2z - 1) = 10.$$

$$\therefore (1 + 2\lambda)x + (3 + 2\lambda)y + (-1 + 2\lambda)z - 4 - \lambda = 0.$$

Since it is perpendicular to $x + y - 4z = 0$ we have

$$(1 + 2\lambda) \cdot 1 + (3 + 2\lambda) \cdot 1 + (-1 + 2\lambda) \cdot (-4) = 0$$

$$\therefore -4\lambda + 8 = 0. \text{ Hence } \lambda = 2.$$

$\therefore$ Equation of the required plane is

$$x + 3y - z - 4 + 2(2x + 2y + 2z - 1) = 0$$

$$\text{i.e. } 5x + 7y + 3z - 6 = 0.$$

Problem 13. Find the equation of the plane which is the rotation by an angle α of $lx + my = 0$ about its line of intersection with $z = 0$.

Solution. The required plane is the plane passing through the intersection of the two planes $lx + my = 0$ and $z = 0$ and hence its equation is $lx + my + \lambda z = 0$ for some λ to be determined.

Give α is the angle between the planes

$$lx + my = 0 \text{ and } lx + my + \lambda z = 0$$

$$\therefore \cos \alpha = \frac{l^2 + m^2 + \lambda(0)}{\sqrt{l^2 + m^2} \sqrt{l^2 + m^2 + \lambda^2}}$$

$$\therefore \cos^2 \alpha (l^2 + m^2) (l^2 + m^2 + \lambda^2) = (l^2 + m^2)^2$$

$$\therefore \left[\cos^2 \alpha (l^2 + m^2 + \lambda^2) \right] = (l^2 + m^2)$$

$$\therefore \lambda^2 \cos^2 \alpha = (l^2 + m^2) (1 - \cos^2 \alpha)$$

$$\therefore \lambda^2 = (l^2 + m^2) \tan^2 \alpha. \text{ Hence } \lambda = \pm \sqrt{l^2 + m^2} \tan \alpha$$

$$\therefore \text{Equation of the required planes are } lx + my \pm \left[\sqrt{l^2 + m^2} \tan \alpha \right] z = 0$$

Problem 14. Find the distance between the parallel planes

$$2x - 2y + z + 3 = 0 \text{ and } 4x - 4y + 2z + 5 = 0$$

Solution. The plane $2x - 2y + z + 3 = 0$ meets the z -axis at $(0, 0, -3)$ (it is got by putting $x=0$ and $y=0$). The perpendicular distance from $(0, 0, -3)$ to the plane $4x - 4y + 2z + 5 = 0$ is the distance between the parallel

planes and it is $= \pm \left[\frac{-6+5}{\sqrt{4^2 + (-4)^2 + 2^2}} \right] = \pm \left[\frac{-1}{6} \right] = \frac{1}{6}$

Problem 15. Prove that the homogeneous second degree equation.

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \dots\dots\dots 1$$

represents a pair of planes if $abc + 2gh - af^2 - bg^2 - ch^2 = 0$

Also prove that the angle between the planes is

$$\tan^{-1} \left[\frac{2(f^2 + g^2 + h^2 - ab - bc - ca)^{1/2}}{a + b + c} \right]$$

Solution. Let (1) represent a pair of planes. Let the equations of the separate planes be

$$lx + my + nz = 0 \quad \dots\dots\dots 2$$

$$l'x + m'y + n'z = 0 \quad \dots\dots\dots 3$$

(since (1) is homogeneous (2) and (3) can not contain constant term)

$$\therefore ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \equiv (lx + my + nz)(l'x + m'y + n'z)$$

Equating like terms we get

$$a = ll' \quad 2f = m'n + mn'$$

$$b = mm' \quad 2g = ln' + l'n$$

$$c = nn' \quad 2h = lm' + l'm$$

Consider the identity $\begin{vmatrix} l & l' & 0 \\ m & m' & 0 \\ n & n' & 0 \end{vmatrix} \times \begin{vmatrix} l' & l & 0 \\ m' & m & 0 \\ n' & n & 0 \end{vmatrix} = 0$

i.e. $\begin{vmatrix} ll' + l'l & lm' + l'm & ln' + l'n \\ l'm + m'l & mm' + m'm & mn' + m'n \\ nl' + n'l & nm' + n'm & nn' + n'n \end{vmatrix} = 0$

$$\text{i.e. } \begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$

$$\therefore 8(abc + 2fgh - af^2 - bg^2 - ch^2) = 0$$

$$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

If θ is the angle between the planes

$$\cos \theta = \frac{ll' + mm' + nn'}{\sqrt{(l^2 + m^2 + n^2)} \sqrt{(l'^2 + m'^2 + n'^2)}}$$

$$\therefore \tan \theta = \frac{[(mn' - m'n)^2 + (nl' + n'l)^2 + (lm' - l'm)^2]^{1/2}}{ll' + mm' + nn'}$$

$$\therefore \tan \theta = \frac{[(mn' - m'n)^2 + (nl' + n'l)^2 + (lm' - l'm)^2]^{1/2}}{ll' + mm' + nn'}$$

$$= \frac{(4f^2 - 4bc + 4g^2 - 4ac + 4h^2 - 4ab)^{1/2}}{a + b + c}$$

$$\therefore \tan \theta = \frac{2(f^2 - g^2 + h^2 - ab - bc - ca)^{1/2}}{a + b + c}$$

Hence the result

Problem 16. Find the bisector of the acute angle between the planes.

$$3x + 4y - 5z + 1 = 0 \quad \dots\dots\dots 1$$

$$5x + 12y - 13z = 0 \quad \dots\dots\dots 2$$

Solution. The equation of the two bisecting planes are given by

$$\frac{3x + 4y - 5z + 1}{\sqrt{3^2 + 4^2 + (-5)^2}} = \pm \left[\frac{5x + 12y - 13z}{\sqrt{5^2 + 12^2 + (-13)^2}} \right]$$

$$\text{(i.e.) } \frac{3x + 4y - 5z + 1}{\sqrt{50}} = \pm \left[\frac{5x + 12y - 13z}{\sqrt{328}} \right]$$

$$\text{(i.e.) } \frac{3x + 4y - 5z + 1}{5\sqrt{2}} = \pm \left[\frac{5x + 12y - 13z}{13\sqrt{2}} \right]$$

$$\therefore 13(3x + 4y - 5z + 1) = \pm (5x + 12y - 13z)$$

$$\therefore \text{The two bisecting planes are } 14x - 8y + 13 = 0 \dots\dots\dots 3$$

$$64x + 112y - 130z + 13 = 0 \text{ (verify) } \dots\dots 4$$

Let θ be the angle between (1) and (3).

$$\therefore \cos \theta = \frac{42 - 32 + 0}{\sqrt{3^2 + 4^2 + (-5)^2} \sqrt{14^2 + (-8)^2 + 0^2}}$$

$$= \frac{10}{\sqrt{50}\sqrt{260}} = \frac{1}{\sqrt{5}\sqrt{26}}$$

$$\therefore \tan^2 \theta = \sec^2 \theta - 1 = 5 \times 26 - 1 = 129$$

$\therefore 14x - 8y + 13 = 0$ bisects the obtuse angle between the given planes (1) and (2)

Hence $64x + 112y - 130z + 13 = 0$ is the acute angle bisector of the planes (1) and (2).

Problem 17. The plane $x + 35y - 10z - 156 = 0$ is a bisector of the angle between two planes one of which is $4x - 3y + 12z + 13 = 0$. Find the equation of the other plane.

Solution. Let $\pi : x + 35y - 10z - 156 = 0 \dots\dots\dots 1$

$$\pi_1 : 4x - 3y + 12z + 13 = 0 \dots\dots\dots 2$$

Equation of the plane passing through the intersection of $\pi = 0$ and $\pi_1 = 0$ is $\pi + \lambda \pi_1 = 0$

$$\text{(i.e.) } (1+4\lambda)x + (35-3\lambda)y + (-10+12\lambda)z - 156 + 13\lambda = 0 \dots\dots\dots 3$$

Let us suppose (3) represents the required plane for some λ where λ to be determined. Since (1) is angle bisector of (2) and (3) we have to find λ such that angle between (1) and (2) is equal to the angle between (1) and (3).

$$\begin{aligned} \therefore & \frac{(1 \times 4) + (35 \times (-3)) + (-10)(12)}{\sqrt{1^2 + 35^2 + (-10)^2} \sqrt{4^2 + (-3)^2 + 12^2}} \\ &= \frac{(1+4\lambda) + (35-3\lambda)35 + (12\lambda-10)(-10)}{\sqrt{1^2 + 35^2 + (-10)^2} \sqrt{(1+4\lambda)^2 + (35-3\lambda)^2 + (12\lambda-10)^2}} \\ & \frac{-221}{13} = \frac{1+4\lambda+1225-105\lambda-120\lambda+100}{\sqrt{(1+4\lambda)^2 + (35-3\lambda)^2 + (12\lambda-10)^2}} \\ & -17 \sqrt{(1+4\lambda)^2 + (35-3\lambda)^2 + (12\lambda-10)^2} = -221\lambda+1326 \end{aligned}$$

$$\sqrt{(1+4\lambda)^2 + (35-3\lambda)^2 + (12\lambda-10)^2} = 13\lambda - 78$$

Squaring on both sides we get

$$1+16\lambda-8\lambda+1225+9\lambda+210\lambda+144\lambda+100+240\lambda=169(\lambda^2-12\lambda+36)$$

$$-442\lambda + 1326 = -2028\lambda + 6084$$

$$\therefore 1586\lambda = 4758. \text{ Hence } \lambda = 3.$$

Substituting in (3) we get the equation of the required plane $13x + 26y + 26z - 117 = 0$ (i.e.) $x + 2y + 2z - 9 = 0$.

Exercise

1. Find the equation of the plane.

i) through the point (1,3,4) and parallel to the plane $2x+3y-4z+8=0$.

ii) through the origin and parallel to $11x+13y-7z+7=0$.

iii) through the point (1, -1, 2) and parallel to the xy-plane.

iv) through (α, β, γ) and parallel to $ax+by+cz=0$.

2. Find the equation of the plane

i) passing through the points (-1, 1, 1) and (1, -1, 1) and perpendicular to the plane $x+2y+2z=5$

ii) passing through the points (1,2,3) and (-4,1,-2) and perpendicular to the plane $7x+2y-z+3=0$.

iii) passing through (3, 1, -3) and (1, -1, 1) and parallel to the line whose direction ratios are 1, 1, -2.

iv) passing through the point (-1, 3, 2) and is perpendicular to each of the two planes $x+2y+2z=5$; $3x+3y+2z=8$.

v) passing through the point (-1, 3, 4) and perpendicular to

3. Find the equation of the plane

i) through (-2, -2, 2), (1, 1, 1) and (1, -1, 2)

ii) through (7, 0, 6); (2, 2, 1) and (3, 4, 2)

iii) through (3, 1, 2); (4, -2, -1) and (1, 2, 4)

4. Prove that the following four points are coplanar.

i) (-6, 3, 2); (3, -2, 4); (5, 7, 3) and (-13, 17, -1)

ii) (0, 1, 3); (1, -2, 2); (-1, 2, 2) and (2, 0, 6)

5. Find the equation of the plane

i) having x, y, z intercepts 2, $\frac{3}{2}$, $\frac{4}{3}$

ii) having x and y intercepts 2 and $\frac{3}{2}$ and passing through the point

(1, 2, -5)

6. Find the intercepts made on the coordinate axes by the following planes. Also find the length and the direction cosines of the perpendicular from the origin on these planes.

i) $x + 2y - 2z = 9$. ii) $6x - 3y + 2z = 14$.

7. Find the coordinates of the foot of the perpendicular drawn from the origin to the plane $12x - 4y - 3z = 169$.

8. Find the angle between the following pair of planes.

i] $2x - y + z = 6$;

$x + y + 2z = 7$

ii] $2x + 2y + z = 5$;

$3x - 2y - z = 6$

iii] $6x - 3y - 2z = 7$;

$x + y + 2z + 9 = 0$.

9. Prove that the planes $x + 2y + 2z = 0$; $2x + y - 2z = 0$ are perpendicular. Also find another plane through the origin which will be perpendicular to each of the planes.

10. Find the equation of the plane through the line of intersection of the planes $x + y + z - 2 = 0$ and also passing through the origin.

11. Find the equation of the plane through the intersection of the planes $3x + 4y + 5z + 18 = 0$ and $2x + 3y - 5z + 11 = 0$ and passing through (1, 2, -1)

12. Find the equation of the plane which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$; $2x + y - z + 5 = 0$ and is perpendicular to the plane $5x + 3y + 6z + 8 = 0$.

13. Find the equation of the plane passing through the line of intersection of the planes $2x + y + 3z - 4 = 0$ and $4x - y + 5z - 7 = 0$ and which is perpendicular to the yz plane.

14. The plane $4x - y + 5z - 7 = 0$ is rotated through a right angle about its line of intersection with the plane $2x - 3y + 3z - 5 = 0$. Find the equation of this plane in its new position.

15. Prove that the equation of the plane through the line of intersection of the planes $ax + by + cz + d = 0$; $a_1x + b_1y + c_1z + d_1 = 0$ and perpendicular to the xy plane is $(ac_1 - a_1c)x + (bc_1 - b_1c)y + (dc_1 - d_1c) = 0$.

Space for Hints

16. Find the equation of the plane through the line of intersection of the planes $x + 2y + 3z + 2 = 0$ and $2x + 3y - z + 3 = 0$ and parallel to the line whose direction ratios are 1, 1, 1.
17. The plane $ax + by + cz + d = 0$ rotates about its line of intersection with $a_1x + b_1y + c_1z + d_1 = 0$ through 90° . find its equation in the new position.
18. Find the equation of the plane passing through the line of intersection of the planes $2x - y + 5z - 3 = 0$ and $4x + 2y - z + 7 = 0$ and parallel to the z-axis.
19. A variable plane is at a constant distance p from the origin and meets the axes in A, B, C. Show that (i) the locus of the centroid of the triangle is $x^{-2} + y^{-2} + z^{-2} = 9p^{-2}$. ii) The locus of the centroid of the tetrahedron OABC is $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$.
20. A moving plane passes through a fixed point (α, β, γ) and intersects the coordinate axes at A, B, C. Show that the locus of the centroid of the tetrahedron OABC is $\frac{\alpha}{x} + \frac{\beta}{y} + \frac{\gamma}{z} = 4$
21. A plane moves such that the sum of its x, y, z intercepts is a constant k. Find the locus of the foot of the perpendicular drawn from the origin to it.
22. Two systems of rectangular axes have the same origin. if a plane cuts them at distances a, b, c and a_1, b_1, c_1 from the origin show that $a^{-2} + b^{-2} + c^{-2} = a_1^{-2} + b_1^{-2} + c_1^{-2}$
Hint: Consider planes in intercepts form and find the perpendicular distances from the origin to the planes.
23. A variable plane which remains at constant distance $3p$ from the origin cuts the coordinate axes at A, B, C. Show that the locus of the centroid of the triangle ABC is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$
24. Prove that $2x^2 + 6y^2 + 12z^2 + 18yz + 2zx + xy = 0$ represents a pair of planes and find the angle between them.
25. Find the length of the perpendicular from the point (4, 3, 0) to the plane $2x + 3y + \sqrt{3}z + 3 = 0$.

26. Find the distance between the parallel planes.

Space for Hints

i] $x - 2y + 2z - 8 = 0$; $x - 2y + 2z + 19 = 0$

ii] $2x - 2y + z + 1 = 0$; $4x - 4y + 2z + 3 = 0$

iii] $12x + 3y - 4z + 15 = 0$; $12x + 3y - 4z + 2 = 0$.

27. Show that the points $(1, 1, 1)$ and $(-3, 0, 1)$ lie on opposite sides of the plane $3x + 4y - 12z + 13 = 0$.

28. Find the bisector of the acute angle between the planes.

i) $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$.

ii) $7x + 4y + 4z + 3 = 0$; $2x + y + 2z + 2 = 0$.

29. Find the bisector of the obtuse angle between the planes

$3x + 4y - 5z + 1 = 0$; $5x + 12y - 13z = 0$.

30. Find the equations of the planes bisecting the angles between the planes $x + 2y + 2z - 3 = 0$ and $3x + 4y + 12z + 1 = 0$ and specify the one which bisects the acute angle.

31. The plane $13x + 7y + 10z + 9 = 0$ is a bisector of the angle between the planes one of which is $2x + y + 2z + 2 = 0$. Find the equation of the other plane.

UNIT 2 STRAIGHT LINES

Space for Hints

Section 2. 1 Equation of a Straight Line

We obtain different forms of equations of a straight line in space.

1. **Non-symmetric form.** We know that two planes in general intersect in a line. Hence a line in space can be represented by two linear equations.

$$\pi_1 : a_1x + b_1y + c_1z + d_1 = 0 \text{ and } \pi_2 : a_2x + b_2y + c_2z + d_2 = 0.$$

2. **Symmetric form:** We can write the equations of a line if we know its direction cosines and a point on it.

Let A (x_1, y_1, z_1) be a given point on the line. Let l, m, n be the d.c of the line. Let P(x, y, z) be any point on the line.

The direction ratios of AP are $x - x_1, y - y_1, z - z_1$

Since the d.c are l, m, n we have $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \dots\dots 1.$

Hence $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r$ represents the equations of

the given line.

Note 1. Any point on the line (1) is $(x_1 + lr, y_1 + mr, z_1 + nr)$.

Note 2. The equations of a line passing through (x_1, y_1, z_1) and having direction ratios (a, b, c) are also given by $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

Note 3. The equations of a line given in non symmetric form can be converted to symmetric form as follows.

Let the equations of the two planes be $\pi_1 : a_1x + b_1y + c_1z + d_1 = 0$ and $\pi_2 : a_2x + b_2y + c_2z + d_2 = 0$. Let $\pi_1 = 0$ and π_2 intersect along a line L: $\pi_1 = \pi_2$. Let the d.r. of L be l, m, n .

Since the line L lies on both $\pi_1 = 0$ and $\pi_2 = 0$ the normals to the planes are perpendicular. Hence $a_1l + b_1m + c_1n = 0$; $a_2l + b_2m + c_2n = 0$

$$\therefore \frac{l}{b_1c_2 - b_2c_1} = \frac{m}{c_1a_2 - c_2a_1} = \frac{n}{a_1b_2 - a_2b_1}$$

$\therefore$ The d.r of L are $b_1c_2 - b_2c_1, c_1a_2 - c_2a_1, a_1b_2 - a_2b_1$.

We now find a point A on L by considering the point where it meets the plane $z = 0$ (xy-plane), (say). It is got from the equations $a_1x + b_1y + d_1 = 0$ and $a_2x + b_2y + d_2 = 0$.

$$\therefore \frac{x}{b_1 d_2 - b_2 d_1} = \frac{y}{d_1 a_2 - d_2 a_1} = \frac{z}{a_1 b_2 - a_2 b_1}$$

$$\therefore A \text{ is } \left(\frac{b_1 d_2 - b_2 d_1}{b_1 d_2 - a_2 b_1}, \frac{d_1 a_2 - d_2 a_1}{a_1 b_2 - a_2 b_1}, 0 \right)$$

Hence the equation of the line L in symmetric form is

$$\frac{x - \left(\frac{b_1 d_2 - b_2 d_1}{b_1 d_2 - a_2 b_1} \right)}{b_1 c_2 - b_2 c_1} = \frac{\frac{d_1 a_2 - d_2 a_1}{a_1 b_2 - a_2 b_1}}{c_1 a_2 - c_2 a_1} = \frac{z - 0}{a_1 b_2 - a_2 b_1}$$

Instead of finding the point where the given line meet the plane $z = 0$ we can also find the point where L meets plane $x = 0$ or $y = 0$.

3. Two-points form. Equation of straight line can be obtained when two points on the line are known.

If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points on a line then the direction ratios of the line are $x_2 - x_1, y_2 - y_1, z_2 - z_1$

The equation of the line is
$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Solved problems

Problem 1. Find the direction cosines of the line

$$\frac{2x+1}{3} = \frac{4y-3}{1} = \frac{2z-3}{0} \quad \text{Also find a point on it.}$$

Solution. The given line can be rewritten as

$$\frac{2\left(x + \frac{1}{2}\right)}{3} = \frac{4\left(y + \frac{3}{4}\right)}{1} = \frac{2\left(z + \frac{3}{2}\right)}{0}$$

$$\text{(i.e.) } \frac{x + \frac{1}{2}}{3/2} = \frac{y + \frac{3}{4}}{1/4} = \frac{z + \frac{3}{2}}{0}. \quad \text{Hence the direction ratios are } \left(\frac{3}{2}, \frac{1}{4}, 0 \right)$$

$$\therefore \text{The direction cosines are } \left(\frac{6}{\sqrt{37}}, \frac{1}{\sqrt{37}}, 0 \right)$$

$$\text{A point on the line is } \left(-\frac{1}{2}, \frac{3}{4}, \frac{3}{2} \right)$$

Problem 2. Find the value of k so that the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2} \text{ and } \frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5} \text{ may be perpendicular to}$$

each other.

Solution. The direction ratios of the lines are $-3, 2k, 2$ and $3k, 1, -5$

Since the lines are perpendicular we have $(-3) 3k + (2k) 1 + 2(-5) = 0$.

Hence $k = -10/7$.

Problem 3. Prove that the lines $x + y - z = 5$; $9x - 5y + z = 4$ and $6x - 8y + 4z = 3$; $x + 8y - 6z + 7 = 0$ are parallel.

Solution. Let a, b, c be the direction ratios of the line determined by the planes $x + y - z = 5$ and $9x - 5y + z = 4$.

Since the line is perpendicular to the normals of the above two planes we have $a + b - c = 0$

$$9a - 5b + c = 0$$

$$\therefore \frac{a}{-4} = \frac{b}{-10} = \frac{c}{-14} \text{ i.e. } \frac{a}{2} = \frac{b}{5} = \frac{c}{7}$$

The direction ratios of the first line are $2, 5, 7$.

Similarly we can prove that the direction ratios of the second line are $2, 5, 7$.

Hence the direction ratios of the two lines are proportional. Hence they are parallel.

Problem 4. Find the angle between the lines

$$x - 2y + z = 0 = x + y - z - 3$$

$$x + 2y + z - 5 = 0 = 8x + 12y + 5z.$$

Solution. The first line is the intersection of the two planes $x - 2y + z = 0$ and $x + y - z - 3 = 0$. Let a, b, c be direction ratios of the line. Since the line is perpendicular to the normals of the above two planes we have $a - 2b + c = 0$ (1)

$$a + b - c = 0 \text{(2)}$$

$$\therefore \frac{a}{1} = \frac{b}{2} = \frac{c}{3}$$

$\therefore$ Direction ratios of the first line are $1, 2, 3$.

Let the direction ratios of the second line be a_1, b_1, c_1 .

$$\therefore a_1 + 2b_1 + c_1 = 0$$

$$8a_1 + 2b_1 + 5c = 0. \text{ Hence } \frac{a_1}{-2} = \frac{b_1}{3} = \frac{c_1}{-4}$$

$\therefore$ The d.r of the second line are $-2, 3, -4$.

The angle θ between the two lines is given by $\cos \theta$

$$= \frac{aa_1 + bb_1 + cc_1}{\sqrt{(\sum a^2)} \sqrt{(\sum a_1^2)}}$$

$$= \frac{-2+6-12}{\sqrt{14}\sqrt{29}} = \frac{-8}{\sqrt{14}\sqrt{29}}$$

$$\therefore \theta = \cos^{-1} \left[\frac{8}{\sqrt{406}} \right]. \text{ (acute angle).}$$

Problem 5. Find in symmetrical form the equations of the line given by $x + 5y - z = 7$; $2x - 5y + 3z + 1 = 0$.

Solution. The required line is the intersection of the planes

$$x + 5y - z - 7 = 0 \quad \dots\dots\dots (1)$$

$$2x - 5y + 3z + 1 = 0 \quad \dots\dots\dots (2)$$

Let a, b, c be the direction ratios of the line. Since the line is perpendicular to the normal of (1) and (2) we have

$$a + 5b - c = 0 \quad \dots\dots\dots (3)$$

$$2a - 5b + 3c = 0 \quad \dots\dots\dots (4)$$

$$\therefore \frac{a}{10} = \frac{b}{-5} = \frac{c}{-15} \quad \text{Hence the d.r are } 2, -1, -3.$$

We now find one point on the line.

The line meets the xy - plane $z = 0$ at the point $(x, y, 0)$ where (x, y) satisfy the equations $x + 5y = 7 \quad \dots\dots\dots (5)$

$$2x - 5y = -1 \quad \dots\dots\dots (6)$$

Solving (5) and (6) we get $x = 2, y = 1$.

$\therefore$ A point on the line is $(2, 1, 0)$.

$$\therefore \text{ Equations of the lines are } \frac{x-2}{2} = \frac{y-1}{-1} = \frac{z}{-3}$$

Problem 6. Find the coordinates of the point of intersection of the line

$$\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4} = \frac{z+3}{4} \text{ with the plane } 2x + 4y - z + 1 = 0.$$

Solution. Any point on the given line is P (1+2r, 2-3r, -3+4r)

If P lies on the plane $2x + 4y - z + 1 = 0$ then

$$2(1+2r) + 4(2-3r) - (-3+4r) + 1 = 0.$$

$$\therefore 4r - 12r - 4r + 2 + 8 + 3 + 1 = 0.$$

$$\therefore -12r = -14. \text{ Hence } r = 7/6.$$

$$\therefore P \text{ is } \left(1 + \frac{7}{3}, 2 - \frac{7}{2}, -3 + \frac{14}{3}\right)$$

$$\therefore P \text{ is } \left(\frac{10}{3}, \frac{-3}{2}, \frac{5}{3}\right)$$

Problem 7. Find the perpendicular distance of the point P (1, 1, 1) from

the line $\frac{x-2}{3} = \frac{y+3}{2} = \frac{z}{-1}$. Also find the foot of the perpendicular.

Solution. Let A be the foot of the perpendicular from P (1, 1, 1) on the line.

$\therefore$ AP is perpendicular to the line.

The direction ratios of the line are 3, 2, -1

The coordinates of A can be taken as (2+3r, -3+2r, -r)

$\therefore$ The direction of ratios of AP are 1+3r, -4+2r, -r-1.

Since AP is perpendicular to the line we have

$$(1+3r)3 + (-4+2r)2 + (-r-1)(-1) = 0.$$

$$\therefore 14r = 4. \text{ Hence } r = \frac{2}{7}$$

$$\therefore A \text{ is } \left(2 + \frac{6}{7}, -3 + \frac{4}{7}, -\frac{2}{7}\right) = \left(\frac{20}{7}, -\frac{17}{7}, -\frac{2}{7}\right)$$

$$\therefore \text{The foot of the perpendicular is } \left(\frac{20}{7}, -\frac{17}{7}, -\frac{2}{7}\right)$$

$$\therefore AP^2 = \left(\frac{20}{7} - 1\right)^2 + \left(-\frac{17}{7} - 1\right)^2 + \left(-\frac{2}{7} - 1\right)^2$$

$$= \left(\frac{13}{7}\right)^2 + \left(-\frac{24}{7}\right)^2 + \left(-\frac{9}{7}\right)^2 = \frac{118}{7}$$

$$\therefore Ap = \sqrt{\left(\frac{118}{7}\right)}$$

Problem 8. Find the foot of the perpendicular from the origin on the

$$3x - y - z - 4 = 0 = 4x - 3y - 2z + 2.$$

Solution. Let L be the line of intersection of the given planes. Let a, b, c be d.r. of the line L. Since L is perpendicular to the normal of both the planes we have

$$3a - b - c = 0; 4a - 3b - 2c = 0.$$

$$\therefore \frac{a}{-1} = \frac{b}{2} = \frac{c}{-5} \text{ Hence d.r. of the L are } -1, 2, -5.$$

Let A be the point of intersection of L with the xy-plane, $z = 0$. the coordinates of A are obtained by solving $3x - y = 4$ and $4x - 3y = -22$.

$$\therefore A \text{ is } (14/5, 22/5, 0).$$

$$\text{Hence equation of the line L is } \frac{x - (14/5)}{-1} = \frac{y - (22/5)}{2} = \frac{z}{-5}$$

Any point P on the L line is $P(-r + 14/5, 2r + 22/5, -5r)$

The d.r of OP are $-r + 14/5, 2r + 22/5, -5r$.

Suppose P is the foot of the perpendicular from O to the line L.

Then OP is perpendicular to L gives

$$-1(-r + 14/5) + 2(2r + 22/5) - 5(-5r) = 0.$$

$$\therefore 30r = -6. \text{ Hence } r = -1/5.$$

$$\therefore P \text{ is } (3, 4, 1).$$

Problem 9. Find the image of the point (2, 3, 4) under the reflection in the plane $x - 2y + 5z = 6$.

Solution. Let P (2, 3, 4) be the given point and let P' be its image under the reflection in the plane $x - 2y + 5z = 6$.

The d.r of the normal to the plane are 1, -2, 5.

$$\therefore \text{The d.r of } PP' \text{ are also } 1, -2, 5.$$

$$\text{Hence the equations of the line are } \frac{x - 2}{1} = \frac{y - 3}{-2} = \frac{z - 4}{5}$$

$$\therefore P' \text{ is of the form } (2 + r, 3 - 2r, 4 + 5r).$$

Mid point of PP' is Q $(2 + r/2, 3 - r, 4 + 5r/2)$

Since Q lies on the plane (1) we have

$$(2 + r/2) - 2(3 - r) + 5(4 + 5r/2) = 6.$$

$$\therefore 2 + r/2 - 6 + 2r + 20 + 25r/2 = 6.$$

$$\therefore 15r = -10. \text{ Hence } r = -23.$$

$$\therefore P \text{ is } (2, -2/3, 3 + 4/3, 4 - 10/3).$$

$$\text{i.e. } P \text{ is } \left(\frac{4}{3}, \frac{13}{3}, \frac{2}{3} \right)$$

Problem 10. Find the image of the point (1, 3, 4) under the reflection in the plane $2x - y + z + 3 = 0$. Hence prove that the image of the line

$$\frac{x-1}{1} = \frac{y-3}{-2} = \frac{z-4}{-3} \text{ is } \frac{x+3}{1} = \frac{y-5}{-5} = \frac{z-2}{-10}$$

Solution. Let P (1, 3, 4) be the given point and P' be its image in the plane.

$$2x - y + z + 3 = 0 \quad \dots\dots\dots (1)$$

The direction ratios of normal to the plane are 2, -1, 1.

$\therefore$ The direction ratios of PP' are also, 2, -1, 1.

$$\text{Hence the equations of the line PP' are } \frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1}$$

The coordinates of P' are (1 + 2r, 3 - r, 4 + r) for some r.

$$\text{Mid point PP' is } Q \left(\frac{1+2r+1}{2}, \frac{3-r+3}{2}, \frac{4+r+4}{2} \right)$$

$$\text{(i.e.) } Q \text{ is } (r+1, 3-r/2, 4+r/2).$$

This point Q lies on the plane $2x - y + z + 3 = 0$.

$$\therefore 2(r+1) - (3-r/2) + (4+r/2) + 3 = 0.$$

$$\therefore 3r = -6. \text{ Hence } r = -2.$$

$$\therefore P' \text{ is } (1 - 4, 3 + 2, 4 - 2) \text{ (i.e.) } P' \text{ is } (-3, 5, 2).$$

$\therefore$ The image of (1, 3, 4) under reflection is (-3, 5, 2).

$$\text{We now find the point where the line } \frac{x-1}{1} = \frac{y-3}{-2} = \frac{z-4}{-3} \dots\dots\dots (2)$$

meets the plane $2x - y + z + 3 = 0$.

Any point on the line (2) is R (1 + 4r, 3 - 2r, 4 - 3r) and it lies on the plane (1). Hence $2(1 + 4r) - (3 - 2r) + (4 - 3r) + 3 = 0$.

$$\therefore r = -6. \text{ Hence R is } (-5, 15, 22).$$

Since R lies in the plane (1) the image of R in the plane (1) is itself

∴ The line RP is the image of the line (2) and its equations are

$$\frac{x+1}{-3+5} = \frac{y-5}{5-15} = \frac{z-2}{2-22} \text{ i.e. } \frac{x+3}{2} = \frac{y-5}{-10} = \frac{z-2}{-20}$$

$$\text{(i.e.) } \frac{x+3}{1} = \frac{y-5}{-5} = \frac{z-2}{-10}$$

Exercises.

1. Find the equations of the straight line

(i) passing through the point (a, b, c) and having the direction ratios a, b, c.

(ii) passing through (1, -2, 3) and parallel to the line $\frac{x-4}{2} = \frac{y}{3} = \frac{z-1}{4}$

(iii) passing through the point (a, b, c) and parallel to the x-axis

(iv) passing through the point (2, -1, 4) and perpendicular to the plane $3x + 4y - 5z + 6 = 0$.

(v) Passing through the points (1, -2, 3) and (2, -1, 6).

(vi) passing through (1, 2, 3) and making equal angles with the coordinate axes.

2. Find the equation of the four diagonals.

i) of a rectangular parallel piped ii) of a cube

3. Find the direction ratios of the line and a point on it for the following lines. i) $\frac{x-4}{3} = \frac{2y-1}{5} = 3z-2$.

$$\text{ii) } \frac{2x-1}{3} = \frac{3y-1}{4} = \frac{3z-1}{0}$$

4. Find the point where the line $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-1}{2}$ meets the plane $x + y + z - 7 = 0$.

5. Find the point where the line joining (-1, 2, -3) and (3, 4, 5) meets the plane $x - 2y + 3z - 4 = 0$

6. Find the coordinates of the foot of the perpendicular drawn from the point (2, 3, 1) to the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$.

7. Find in the symmetrical form the equations of the line given by

$$\text{i) } 3x + 2y - z - 4 = 0 = 4x + y - 2z + 3$$

$$\text{ii) } 4x + 4y - 5z - 12 = 0 = 8x + 12y - 13z - 32.$$

Space for Hints

8. Prove that the lines $2x + 3y - 4z = 0 = 3x - 4y + z$ and $5x - y - 3z = 0 = x - 7y + 5z - 6$ are parallel.
9. Find the equation of the line through the point (1, 2, 3) parallel to the line $3x + y + z - 6 = 0; x - y + 2z = 0$.
10. Show that the lines $2x + y + 3z - 7 = 0 = x - 2y - z - 5$ and $4x + 4y - 8z = 0 = 10x - 8y + 7z$.
11. Find the length of the perpendicular from the point (5, 4, -1) to the line $\frac{x-1}{2} = \frac{y}{9} = \frac{z}{5}$.
12. Show that the perpendicular distance of the point (3, 0, 5) from the line $x - 2y + 2z - 4 = 0 = x + 3z + 11$ is 3.
13. Find the perpendicular distance of an angular point of a cube from a diagonal which does not pass through that angular point.
14. Find the image of the point (1, -2, 3) under the reflection in the plane $2x - 3y + 2z + 3 = 0$.
15. Show that the image of the line $\frac{x+5}{1} = \frac{y+7}{6} = \frac{z}{1}$ under the reflection in the plane $2x - y + z + 3 = 0$ is the line $\frac{x-1}{3} = \frac{y-3}{5} = \frac{z-4}{2}$.

Section 2.2. A Plane and a Line.

Theorem 2.1 Let $L : \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ (1)

be a line and $\pi : ax + by + cz + d = 0$ (2)

be a plane. The condition that the line L i) be parallel to the plane π is $ax_1 + by_1 + cz_1 + d \neq 0$ and $al + bm + cn = 0$. ii) the line L to lie in the plane π is $ax_1 + by_1 + cz_1 + d = 0$ and $al + bm + cn = 0$.

Proof. (i) The coordinates of any point on the line (1) are $(x_1 + lr, y_1 + mr, z_1 + nr)$

Suppose this point lies on the plane (2) then

$$a(x_1 + lr) + b(y_1 + mr) + c(z_1 + nr) + d = 0$$

$$\text{i.e. } r(al + bm + cn) + (ax_1 + by_1 + cz_1 + d) = 0 \dots\dots\dots 3$$

If the line is parallel to the plane, no point of the line lies on the plane.

$\therefore$ No value of r satisfies (3). Hence $(al + bm + cn) = 0$ and $ax_1 + by_1 + cz_1 + d \neq 0$

iii) The line (1) will lie in the plane (2) if every point on the line lies in the plane. Hence (3) must be satisfied by all values of r .

$\therefore al + bm + cn = 0$ and $ax_1 + by_1 + cz_1 + d \neq 0$

Theorem 2.2

The angle between the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the plane

$ax + by + cz + d = 0$ is given by $\sin \theta = \frac{al + bm + cn}{\sqrt{(a^2 + b^2 + c^2)} \sqrt{(l^2 + m^2 + n^2)}}$

Proof. Let θ be the angle between the given plane and the straight line

$\therefore 90 - \theta$ is the angle between the line and the normal to plane

The direction ratios of the normal to the plane are a, b, c and the direction ratios of the line are l, m, n .

$\therefore \cos (90 - \theta) = \frac{al + bm + cn}{\sqrt{(a^2 + b^2 + c^2)} \sqrt{(l^2 + m^2 + n^2)}}$. Hence the theorem.

Corollary. The line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ is parallel to the plane

$ax + by + cz + d = 0$ iff $al + bm + cn = 0$

Theorem 2.3 The equation of the plane containing two lines

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \dots\dots\dots (1)$$

$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \dots\dots\dots (2)$$

$$\text{is } \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Proof. The equation of the plane containing the line (1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \dots\dots\dots (A)$$

$$\text{subject to } al_1 + bm_1 + cn_1 = 0 \dots\dots\dots (B)$$

Also the line (2) is perpendicular to the normal to the plane (A).

$$\therefore al_2 + bm_2 + cn_2 = 0 \dots\dots\dots (C)$$

Eliminating a, b and c from (A), (B) and (C) we get the required equation

Theorem 2.4 Length of perpendicular from a point $P(x_1, y_1, z_1)$ to the line

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \dots\dots\dots(1) \text{ is}$$

$$\left[(x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2 - \frac{[l(x_1 - \alpha) + m(y_1 - \beta) + n(z_1 - \gamma)]^2}{l^2 + m^2 + n^2} \right]^{1/2}$$

Proof let PL be the perpendicular from P to the line (1)

Then L is $(\alpha + lr, \beta + mr, \gamma + nr)$ for some r

$\therefore$ The direction ratios of PL are

$$x_1 - \alpha - lr, y_1 - \beta - mr, z_1 - \gamma - nr$$

Since PL is perpendicular to the line (1) we have

$$l(x_1 - \alpha - lr) + m(y_1 - \beta - mr) + n(z_1 - \gamma - nr) = 0$$

$$l(x_1 - \alpha) + m(y_1 - \beta) + n(z_1 - \gamma) = r(l^2 + m^2 + n^2) \dots(2)$$

$$\therefore r = \frac{l(x_1 - \alpha) + m(y_1 - \beta) + n(z_1 - \gamma)}{l^2 + m^2 + n^2}$$

Now $PL^2 =$

$$(x_1 - \alpha - lr)^2 + (y_1 - \beta - mr)^2 + (z_1 - \gamma - nr)^2 = (x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2 - 2r[l(x_1 - \alpha) + m(y_1 - \beta) + n(z_1 - \gamma)] + r^2(l^2 + m^2 + n^2)$$

$$\therefore -2r[l(x_1 - \alpha) + m(y_1 - \beta) + n(z_1 - \gamma)] + r^2(l^2 + m^2 + n^2)$$

$$= (x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2 - 2r^2(l^2 + m^2 + n^2) + r^2(l^2 + m^2 + n^2)$$

$$= (x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2 - r^2(l^2 + m^2 + n^2)$$

Hence the result.

UNIT- 3 COPLANAR LINES

Theorem 3.1 Condition for two lines

Space for Hints

$$\frac{(x-x_1)}{l_1} = \frac{(y-y_1)}{m_1} = \frac{(z-z_1)}{n_1} \dots\dots\dots(1)$$

and $\frac{(x-x_2)}{l_2} = \frac{(y-y_2)}{m_2} = \frac{(z-z_2)}{n_2} \dots\dots\dots(2)$

to be coplanar is
$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Proof. Two coplanar lines must either parallel or intersecting. The lines

(1) and (2) are parallel if $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Suppose the lines are not parallel

$\therefore$ Let the lines intersect at P (say)

$\therefore$ The coordinates of any point on the line (1) are

$$(x_1 + l_1 r, y_1 + m_1 r, z_1 + n_1 r).$$

The coordinates of any point on the line (2) are

$$(x_2 + l_2 r_1, y_2 + m_2 r_1, z_2 + n_2 r_1).$$

Since P is a common point for (1) and (2) we have

$$x_1 + l_1 r = x_2 + l_2 r_1; y_1 + m_1 r = y_2 + m_2 r_1; z_1 + n_1 r = z_2 + n_2 r_1 \text{ for}$$

some values for r and r_1 .

$$\therefore (x_1 - x_2) + l_1 r - l_2 r_1 = 0$$

$$(y_1 - y_2) + m_1 r - m_2 r_1 = 0$$

$$(z_1 - z_2) + n_1 r - n_2 r_1 = 0$$

Eliminating r and r_1 from the above three equations we get

$$\begin{vmatrix} x_1 - x_2 & l_1 & l_2 \\ y_1 - y_2 & m_1 & m_2 \\ z_1 - z_2 & n_1 & n_2 \end{vmatrix} = 0 \text{ (i.e)}$$

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Further the above condition is satisfied even if $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$ and hence it is the required condition for the given lines to be coplanar. Hence the theorem.

Note. If the line given by (1) lies in the plane

$$ax + by + cz + d = 0 \quad \dots\dots\dots(1)$$

then we have $ax_1 + by_1 + cz_1 + d = 0 \quad \dots\dots\dots(2)$

$$\text{and } al + bm + cn = 0.$$

From (1) and (2) the equation of the plane containing the line can be written as $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ together with the condition $al + bm + cn = 0$

3.2 Skew lines

Definition. Two straight lines in space which are not coplanar are called skew lines

Note. There is only one straight line which is perpendicular to both the skew lines

Definition : The shortest distance between two skew lines is the length of the common perpendicular drawn to the lines

Theorem 3.2 Shortest distance between two skew lines

$$L_1: \frac{(x - x_1)}{l_1} = \frac{(y - y_1)}{m_1} = \frac{(z - z_1)}{n_1} \quad \dots\dots\dots(1)$$

$$L_2: \frac{(x - x_2)}{l_2} = \frac{(y - y_2)}{m_2} = \frac{(z - z_2)}{n_2} \quad \dots\dots\dots(2)$$

is given by $S.D = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\Sigma(l_2 m_2 - l_2 m_1)^2}}$

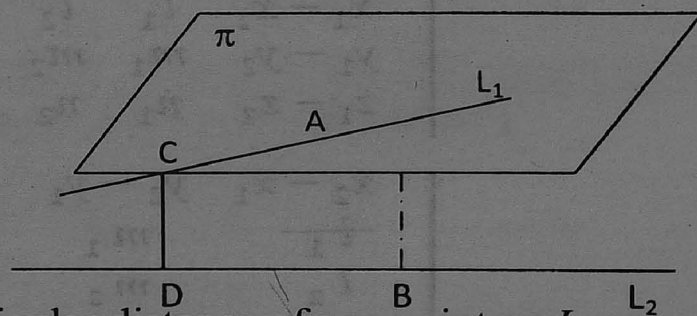
Proof. Let A (x_1, y_1, z_1) be a point on the line L_1 and B (x_2, y_2, z_2) be a point on L_2 .

Consider a Plane π containing the Line L_1 and parallel to L_2 .

The every point on L_2 will be equidistant from this plane.

Hence the short distance is the perpendicular distance of any point on L_2 to the plane π .

$\therefore$ The shortest distance is CD (refer figure) Equation of the plan π can be taken as $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \quad \dots\dots(3)$



$$\text{subject to } al_1 + bm_1 + cn_1 = 0 \quad \dots\dots\dots(4)$$

Since the plane is parallel to L_2 we have

$$al_2 + bm_2 + cn_2 = 0 \quad \dots\dots\dots(5)$$

$$\text{From (4) and (5) we get } \frac{a}{m_1n_2 - m_2n_1} = \frac{b}{n_1l_2 - n_2l_1} = \frac{c}{l_2m_2 - l_2m_1}$$

$\therefore$ The equation of the plane π is

$$(m_1n_2 - m_2n_1)(x - x_1) + (n_1l_2 - n_2l_1)(y - y_1) + (l_2m_2 - l_2m_1)(z - z_1) = 0 \quad \dots\dots(6)$$

$\therefore$ The shortest distance S.D = the perpendicular distance from $B(x_2, y_2, z_2)$

To the plane (6)

$$\therefore \text{S.D} = \frac{(m_1n_2 - m_2n_1)(x_2 - x_1) + (n_1l_2 - n_2l_1)(y - y_1) + (l_2m_2 - l_2m_1)(z_2 - z_1)}{\sqrt{\sum (m_1n_2 - m_2n_1)^2}}$$

Hence the shortest distance can be expressed in the form as given in the theorem.

Now, we find the equation of the line of shortest distance CD. CD is the intersection of the planes ACD and BDC. Let l, m, n

$\therefore$ The equation of the plane ACD is

$$\pi_1: \begin{vmatrix} x_1 - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \quad \dots\dots\dots(1)$$

The equation of the plane BDC is

$$\pi_2: \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0 \quad \dots\dots\dots(2)$$

Since CD is perpendicular to L_1 and L_2 we have

$$ll_1 + mm_1 + nn_1 = 0 \text{ and } ll_2 + mm_2 + nn_2 = 0$$

$$\text{Hence } \frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_2m_2 - l_2m_1} \quad \dots\dots\dots(3)$$

$\therefore$ The equations of the shortest distance CD are given by $\pi_1 = 0$ and $\pi_2 = 0$ where l, m, n are given by (3)

Note. If the shortest distance between the two lines L_1 and L_2 is zero then the lines are coplanar

Solved problems.

Problem 1. Find the equation of the plane containing the point $(-1, 7, 2)$

and the line $\frac{x+3}{2} = \frac{y+2}{3} = \frac{z-2}{-2}$ (1)

Solution. The equation of the plane containing the line (1) is

$$a(x+3) + b(y+2) + c(z-2) = 0 \text{(2)}$$

subject to $2a + 3b + 2c = 0$ (3)

Since the plane passes through $(-1, 7, 2)$ we have from (2)

$$2a + 9b = 0 \text{(4)}$$

From (3) and (4) we have $\frac{a}{18} = \frac{b}{-4} = \frac{c}{24}$

$$\therefore a = 18k, b = -4k, c = 24k$$

substituting in (2) we get the equation of the plane as

$$9x - 2y + 12z = 1$$

Problem . 2 Find the equation of the plane which contains the two parallel lines

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3} \text{(1)}$$

$$\frac{x-3}{1} = \frac{y+2}{2} = \frac{z+4}{3} \text{(2)}$$

Solution. The equation of the plane containing the line (1) is given by

$$a(x-1) + b(y-2) + c(z-3) = 0 \text{(3)}$$

subject to $a + 2b + 3c = 0$ (4)

Since the line (2) lies on the plane (3) the point $(3, -2, 4)$ lies on it

From (3) we have $2a - 4b - 7c = 0$ (5)

From (4) and (5) we have $\frac{a}{-2} = \frac{b}{13} = \frac{c}{-8}$

$$\therefore a = -2k, b = 13k, c = -8k$$

The equation of the plane is $-2x + 13y - 8z = 0$.

Problem 3. Find the condition that the lines

$$a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2 \quad \dots\dots(1)$$

$$\text{and } a_3x + b_3y + c_3z + d_3 = 0 = a_4x + b_4y + c_4z + d_4 \quad \dots\dots(2)$$

may be coplanar

Solution Let the lines represented by the equations (1) and (2) be coplanar

Then (1) and (2) will have a common point say (x_1, y_1, z_1)

Hence it lies on all the four planes which determine the two lines

$$\therefore a_1x_1 + b_1y_1 + c_1z_1 + d_1 = 0$$

$$a_2x_1 + b_2y_1 + c_2z_1 + d_2 = 0$$

$$a_3x_1 + b_3y_1 + c_3z_1 + d_3 = 0$$

$$a_4x_1 + b_4y_1 + c_4z_1 + d_4 = 0$$

Eliminating x_1, y_1, z_1 from the above four equations we get the

required condition as
$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0$$

Problem 4 Prove that the lines

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}; \frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1}; \frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2}$$

are coplanar if
$$\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Solution : Obviously the first two lines intersect at $(0, 0, 0)$. Hence they determine a plane.

Now, the equation of the plane containing these two lines is

$$\begin{vmatrix} x & y & z \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$$

$$\text{(i.e.) } (mn_1 - m_1n)x - (ln_1 - l_1n)y + (lm_1 - l_1m)z = 0 \quad \dots\dots(1)$$

Suppose the three lines are coplanar. Then the third line also lies on (1)

$\therefore$ The normal to the plane (1) is perpendicular to the third line

$$\therefore l_2(mn_1 - m_1n) - m_2(ln_1 - l_1n) + n_2(lm_1 - l_1m)z = 0$$

$$(i.e) \begin{vmatrix} l_2 & m_2 & n_2 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} = 0 \quad (i.e) \begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Problem 5. Show that the lines through the origin having direction ratios $-1, 2, 5; 2, 3, -4; 0, 7, 6$ lie in a plane.

Solution. The equation of the three lines are

$$\frac{x}{-1} = \frac{y}{2} = \frac{z}{5}; \frac{x}{2} = \frac{y}{3} = \frac{z}{-4}; \frac{x}{0} = \frac{y}{7} = \frac{z}{6}$$

$$\text{Since } \begin{vmatrix} -1 & 2 & 5 \\ 2 & 3 & -4 \\ 0 & 7 & 6 \end{vmatrix} = 0 \text{ (verify) the lines are coplanar (refer problem 4).}$$

Problem 6. Find in symmetrical form, the equations of the orthogonal projection of the line

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-4}{4} \dots\dots\dots(1)$$

$$\text{on the plane } 3x + 4y + 5z + 0. \dots\dots\dots(2)$$

Solution. The Orthogonal projection of the line (1) is the intersection of plane (2) and the plane containing the line (1) perpendicular to the plane (2).

The equation of the plane containing the line (1) is

$$a(x-1) + b(y-2) + c(z-4) = 0 \dots\dots\dots(3)$$

$$\text{subject to } 2a + 3b + 4c = 0. \dots\dots\dots(4)$$

The plane (3) is perpendicular to (2). Hence $3a + 4b + 5c = 0$.

$$\text{From (4) and (5) we have } \frac{a}{-1} = \frac{b}{2} = \frac{c}{-1} = k \text{ (say)}$$

$$a = -k; b = 2k; c = -k.$$

$$\therefore \text{Equation of the plane (3) is } x - 2y + z - 1 = 0 \dots\dots\dots(6)$$

$\therefore$ The required line is the intersection of the plane (2) and (6)

Now, we get the equations of the line in the symmetrical form.

Let α, β, γ be the direction ratios of the line.

$$\therefore 3\alpha + 4\beta + 5\gamma = 0 \text{ and } \alpha - 2\beta + \gamma = 0$$

$$\therefore \frac{\alpha}{14} = \frac{\beta}{2} = \frac{\gamma}{-10}$$

Hence the direction ratios are 7, 1, -5.

The line meets the xy plane at $z=0$. Hence $x - 2y = 1$ and $3x + 4y = 0$

Solving the two equations we get the point as $\left(\frac{2}{5}, \frac{-3}{10}, 0\right)$

$$\therefore \text{The equation of the line is } \frac{x - \frac{2}{5}}{7} = \frac{y + \frac{3}{10}}{1} = \frac{z}{-5}$$

Problem 7. Show that the lines $\frac{x-2}{1} = \frac{y-4}{2} = \frac{z-5}{2}$ and

$\frac{x-5}{2} = \frac{y-8}{3} = \frac{z-7}{2}$ are coplanar and find the equation of the plane containing them.

Solution. Here $(x_1, y_1, z_1) = (2, 4, 5)$ and $(x_2, y_2, z_2) = (5, 8, 7)$

We know that the lines are coplanar if

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$$

$$\text{Now } \begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} = \begin{vmatrix} -3 & -4 & -2 \\ 1 & 2 & 2 \\ 2 & 3 & 2 \end{vmatrix} = 0 \text{ (verify)}$$

$$\text{Equation of the plane containing the line is } \begin{vmatrix} x-2 & y-4 & z-5 \\ 1 & 2 & 2 \\ 2 & 3 & 2 \end{vmatrix} = 0$$

$$\text{(i.e.) } 2x - 2y + z - 1 = 0$$

Problem 8. Prove that the lines

$$\frac{x-3}{2} = \frac{y-2}{-5} = \frac{z-1}{3} \quad \dots\dots\dots(1)$$

$$\text{and } \frac{x-1}{-4} = y + 2 = \frac{z-6}{2} \quad \dots\dots\dots(2)$$

are coplanar. Find the point of intersection. Also find the equation of the plane determined by the lines

Solution. Here the condition $\frac{l}{l_1} = \frac{m}{m_1} = \frac{n}{n_1}$ is not satisfied. Hence the lines are not parallel. Hence if the lines are to be coplanar they must intersect.

The coordinates of any point on the line (1) are

$$P(2r + 3, -5r + 2, 3r + 1) \dots\dots\dots(3)$$

The coordinates of any point on the line(2) are

$$Q(-4r_1 + 1, r_1 - 2, 2r_1 + 6) \dots\dots\dots(4)$$

the two lines intersect if $2r + 3 = -4r + 1$;

$$-5r + 2 = r_1 - 2 \text{ and } 3r + 1 = 2r_1 + 6 \text{ for some values of } r, r_1$$

$$\therefore 2r + 4r_1 = 2 \dots\dots\dots(5)$$

$$-5r - r_1 = -4 \dots\dots\dots(6)$$

$$3r - 2r_1 = 5 \dots\dots\dots(7)$$

Solving (5) and (6) we get $r = 1$ and $r_1 = -1$.

These values satisfy the equation (7) also

Hence the two lines intersect. The point of intersection P is $(5, -3, 4)$ (from (3))

The equation of the plane containing the two lines (1) and (2) is

$$\text{Given by } \begin{vmatrix} x-3 & y-2 & z-1 \\ 2 & -5 & 3 \\ -4 & 1 & 2 \end{vmatrix} = 0$$

$$(i.e) 13x + 16y + 18z - 89 = 0$$

Problem 9. Find the distance of the point $(3, 4, 5)$ from the point of intersection of $\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$ with the plane $x + y + z = 2$

Solution. We note the point $A(3, 4, 5)$ is a point on the line.

Any point on the line is $P(r + 3, 2r + 4, 2r + 5)$

If P is the point of intersection of the line with the plane, then P lies on the plane

$$\therefore (r + 3) + (2r + 4) + (2r + 5) = 2$$

$$\therefore 5r = -10. \text{ hence } r = -2$$

$$\therefore P \text{ is } (1, 0, 1).$$

$\therefore$ Hence the required distance

$$\therefore AP = \sqrt{(3-1)^2 + (4-0)^2 + (5-1)^2} = \sqrt{36} = 6.$$

Problem 10. Find the shortest distance between the lines

Space for Hints

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \quad \text{and} \quad \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$$

Are they coplanar ?

Solution.

$$S.D = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\sum (l_1 m_2 - l_2 m_1)^2}}$$

Here $(x_1, y_1, z_1) = (1, 2, 3)$ and $(x_2, y_2, z_2) = (2, 3, 4)$

$$\text{Now } \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

$$\therefore S.D = 0$$

$\therefore$ Hence the lines are coplanar

Problem 11. Find the shortest distance and the equation of the line of

shortest distance between the straight lines $\frac{x+3}{-4} = \frac{y-6}{6} = \frac{z}{2}$ and

$$\frac{x+2}{-4} = \frac{y}{1} = \frac{z-7}{1}$$

Solution.

$$S.D = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\sum (l_1 m_2 - l_2 m_1)^2}}$$

Here $(x_1, y_1, z_1) = (-3, 6, 0)$ and $(x_2, y_2, z_2) = (-2, 0, 7)$

$(l_1, m_1, n_1) = (-4, 6, 2)$ and $(l_2, m_2, n_2) = (-4, 1, 1)$

$$\text{Now } \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} 1 & -6 & 7 \\ -4 & 6 & 2 \\ -4 & 1 & 1 \end{vmatrix} = 168 \text{ (verify)}$$

$$\therefore \sum (m_1 n_2 - m_2 n_1)^2 = (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2$$

$$+ (l_1 m_2 - l_2 m_1)^2$$

$$= (6 - 2)^2 + (-8 + 4)^2 + (-4 + 24)^2$$

$$= 16 + 16 + 400 = 432$$

$$\therefore S.D = \frac{160}{\sqrt{432}} + \frac{160}{12\sqrt{3}} = \frac{14}{\sqrt{3}} \text{ Equation of the line of shortest distance is}$$

Space for Hints

$$\begin{vmatrix} x & x_1 & y & y_1 & z & z_1 \\ l_1 & & m_1 & & n_1 & \\ l & & m & & n & \end{vmatrix} = 0 = \begin{vmatrix} x & x_2 & y & y_2 & z & z_2 \\ l_2 & & m_2 & & n_2 & \\ l & & m & & n & \end{vmatrix}$$

$$\text{Where } \frac{l}{m_1 n_2 - m_2 n_1} = \frac{m}{n_1 l_2 - n_2 l_1} = \frac{n}{l_1 m_2 - l_2 m_1}$$

$$\frac{1}{4} = \frac{m}{-4} = \frac{n}{20} \quad (\text{i. e}) \quad \text{and} \quad \frac{l}{1} = \frac{m}{-1} = \frac{n}{5}$$

Hence the equation of the shortest distance is

$$\begin{vmatrix} x+3 & y-6 & z \\ -4 & 6 & 2 \\ 1 & -1 & 5 \end{vmatrix} = 0 = \begin{vmatrix} x-2 & y & z-7 \\ -4 & 1 & 1 \\ 1 & -1 & 5 \end{vmatrix}$$

$$\therefore 16x + 11y - z - 18 = 0 = 2x + 7y + z - 3 \quad (\text{verify})$$

Problem 12. Find the shortest distance and the equation of the line of shortest distance in symmetrical form of the lines

$$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7} \quad \text{and} \quad \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$$

Solution

Coordinates of any point P on the first line are

$$(3r+8, -16r-9, 7r+10)$$

Coordinates of any point Q on the second line are

$$(3s+15, 8s+29, -5s+5)$$

Let PQ be the shortest distance

Direction ratios of PQ are $(3s-3r+7, 8s+16r+38, -5s-7r-5)$.

Since PQ is perpendicular to both the lines we have

$$3(3s-3r+7) - 16(8s+16r+38) + 7(-5s-7r-5) = 0$$

$$\text{and } 3(3s-3r+7) + 8(8s+16r+38) - 5(-5s-7r-5) = 0$$

$$(\text{i. e}) \quad 77s + 157r = -311 \quad \dots\dots\dots(1)$$

$$\text{and } 7s + 11r = -25 \quad \dots\dots\dots(2)$$

Solving [1] and [2] we get $r = -1$ and $s = -2$

$$\therefore P \text{ is } (5, 7, 3) \text{ and } Q \text{ is } (9, 13, 15)$$

$$\begin{aligned} \therefore PQ &= \sqrt{(9-5)^2 + (13-7)^2 + (15-3)^2} = \sqrt{16+36+144} \\ &= \sqrt{196} = 14. \end{aligned}$$

The equations of the line of shortest distance are

$$\frac{x-5}{5-9} = \frac{y-7}{7-13} = \frac{z-3}{3-15}$$

$$(i.e) \frac{x-5}{-4} = \frac{y-7}{-6} = \frac{z-3}{-12}$$

$$(i.e) \frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}$$

Problem 13. Find the shortest distance between the lines L_1 and L_2 if

$$L_1 : \frac{x-5}{3} = \frac{y-6}{-4} = z-9$$

$$L_2 : 2x - 2y + z - 3 = 0 = 2x + y + 2z - 9$$

Solution. Let π be the plane through L_2 and parallel to L_1 . Then the shortest distance between L_1 and L_2 is the perpendicular distance from $(5, 6, 9)$ to the plane π is of the form

$$2x - 2y + z - 3 + \lambda (2x - y + 2z - 9) = 0$$

$$(i.e) (2 + 2\lambda)x + (-2 - \lambda)y + (1 + 2\lambda)z - 9(3 + \lambda) = 0 \quad \dots\dots(1)$$

Since π is parallel to L_1 the normal to π is perpendicular to L_1

$$\therefore 3(2 + 2\lambda) - 4(-2 - \lambda) + 1(1 + 2\lambda) = 0 \quad \text{hence } \lambda = -\frac{5}{4}$$

$$\therefore \text{From (1) the equation of } \pi \text{ is } 2x + 3y + 6z - 33 = 0$$

$$\therefore \text{S.D} = \frac{2(5) + 3(6) + 6(9) - 33}{\sqrt{(2^2 + 3^2 + 6^2)}} = 7$$

Problem 14. Find the shortest distance between the lines

$$2x - 2y + 3z - 12 = 0 = 2x + 2y + z \quad \dots\dots\dots(1)$$

$$2x - z = 0 = 5x - 2y + 9 \quad \dots\dots\dots(2)$$

Solution. The equation of a plane containing the line (1) is

$$2x - 2y + 3z - 12 + \lambda (2x - 2y + z) = 0$$

$$(i.e) (2 + 2\lambda)x + (-2 + 2\lambda)y + (3 + \lambda)z - 12 = 0 \quad \dots\dots\dots(3)$$

The equation of a plane containing the line (2) is

$$(2x - z) + \mu(5x - 2y + 9) = 0$$

$$(i.e) (2 + 5\mu)x - 2\mu y - z = 9\mu = 0 \quad \dots\dots\dots(4)$$

We find the values of λ and μ such that the planes (3) and (4) are parallel.

$$\text{We have } \frac{2+2\lambda}{2+5\mu} = \frac{-2+2\lambda}{2\mu} = \frac{3+\lambda}{-1} \quad \dots\dots(5)$$

Taking the first two ratios in (5) we get

$$2\lambda - 3\mu + 7\lambda\mu - 2 = 0 \quad \dots\dots(6)$$

Taking the second and third ratios in (5) we get

$$-\lambda - 3\mu + 7\lambda\mu - 2 = 0 \quad \dots\dots(7)$$

$$\text{From (7) we get } \lambda = \frac{1+3\mu}{1-\mu} \quad \dots\dots(8)$$

Substituting (8) in (6) we get

$$2\left(\frac{1+3\mu}{1-\mu}\right) - 3\mu + 7\left(\frac{1+3\mu}{1-\mu}\right)\mu - 2 = 0$$

$$\text{(i.e) } 2 + 6\mu - 3\mu + 3\mu^2 + 7\mu + 21\mu^2 - 2 + 2\mu = 0$$

$$\text{(i.e) } 24\mu^2 + 12\mu = 0 \text{ hence } 12\mu(2\mu + 1) = 0.$$

$$\therefore \mu = 0 \text{ or } \mu = -\frac{1}{2}$$

Hence from (8) we get $\lambda = 1$ or $\lambda = -\frac{1}{3}$

When $\lambda = 1, \mu = 0$ does not satisfy [5]

Hence we take $\lambda = -\frac{1}{3}$ and $\mu = -\frac{1}{2}$

Hence the planes (3) and (4) become

$$x - 2y + 2z - 9 = 0 \quad \dots\dots\dots(9)$$

$$x - 2y + 2z + 9 = 0 \quad \dots\dots\dots(10)$$

The point of intersection of (9) with the z axis is $(0, 0, \frac{9}{2})$

$\therefore$ The required shortest distance is the perpendicular distance from

$$(0, 0, \frac{9}{2}) \text{ to the plane (10)} = \pm \left(\frac{18}{\sqrt{1^2 + (-2)^2 + 2^2}} \right) = 6$$

Exercise.

[1] prove that the line $\frac{x-1}{2} = \frac{y+3}{4} = \frac{z-7}{-1}$ is parallel to the plane $4x + 3y + 20z = 5$. Does it lie on the plane?

[2] Prove that the line $\frac{2x-1}{4} = \frac{y-4}{0} = \frac{z-7}{3}$ lie on the plane $6x + 0y - 4z - 63 = 0$

[3] Find the equation of the plane which passes through $(1, 2, -1)$ and which contains the line $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z+2}{-1}$

[4] Find the equation to the plane which contains the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ and is perpendicular to the plane $x + 2y + z = 12$

[5] Find the equation of the plane determined by the parallel lines

$$[i] \frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3} \text{ and } \frac{x-3}{1} = \frac{y-2}{2} = \frac{z-1}{3}$$

$$[ii] \frac{x-3}{1} = \frac{y-2}{-4} = \frac{z-1}{5} \text{ and } \frac{x-1}{1} = \frac{y+1}{-4} = \frac{z-2}{5}$$

[6] Prove that the following lines are coplanar and find the equation of the plane in which they lie

$$[i] \frac{x-3}{2} = \frac{y-2}{-5} = \frac{z-1}{3}; \frac{x-1}{-4} = \frac{y+2}{1} = \frac{z-6}{2}$$

$$[ii] x + 1 = \frac{y+2}{2} = z - 1; x - 2y + 2z - 3 = 0 = x - 4y + 5z - 8$$

[7] Prove that the lines $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and

$$ax + by + cz + d = 0 = a_1x + b_1y + c_1z + d_1 \text{ are coplanar if } \frac{ax_1 + by_1 + cz_1 + d}{al + bm + cn} = \frac{a_1x_1 + b_1y_1 + c_1z_1 + d_1}{a_1l + b_1m + c_1n}$$

[8] Show that the following lines intersect each other and find the coordinates of the point of intersection.

$$[i] \frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}; \frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$$

$$[ii] \frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}; \frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$$

[9] Verify whether the lines given below intersect or not. If intersect find the coordinates of the point of intersection and the equation of the plane in which they lie.

$$[i] \frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}; \frac{x-2}{1} = \frac{y-4}{-3} = \frac{z-6}{5}$$

$$[ii] \frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}; \frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$$

$$[iii] \frac{x+3}{-4} = \frac{y-6}{6} = \frac{z}{2}; \frac{x+2}{-2} = \frac{y}{1} = \frac{z-7}{1}$$

$$[iv] 2x - y - z - 3 = 0 = x - 3y - 2z - 4 \\ x - y + z - 2 = 0 = 4x + y - 5z - 3$$

[10] Show that the lines

$$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7} \quad \text{and} \quad \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \quad \text{are skew lines}$$

[11] Find the shortest distance and equation of the shortest distance between the following skew lines

$$[i] \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}; \quad \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$$

$$[ii] \frac{x+2}{4} = \frac{y+6}{3} = \frac{z-34}{-10}; \quad \frac{x+6}{4} = \frac{y-7}{-3} = \frac{z-7}{-2}$$

$$[iii] \frac{x-3}{-3} = \frac{y-3}{1} = \frac{z-3}{-1}; \quad \frac{x+3}{3} = \frac{y+7}{-2} = \frac{z-6}{-4}$$

[12] Find the shortest distance between the lines

$$\frac{x-7}{2} = \frac{y-2}{3} = \frac{z-3}{4}; \quad \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5} \quad \text{are they coplanar?}$$

[13] Find the length of the shortest distance between the lines

[i]

$$x - y + z = 0 = 2x - 3y + 4z \quad \text{and} \quad x + y + 2z - 3 = 0 = 2x + 3y + 3z$$

Section 3.3 Equations of two skew lines in a simple form

Theorem 3.7 The equations of two skew lines in a simple form is given by $y = mx, z = c$ and $y = -mx, z = -c$

Proof. Let AB and CD be the two given skew lines.

Let EF be the line of shortest distance between AB and CD and let $EF = 2c$.

Let O be the mid point of EF.

Take O as the origin and

z-axis along FE. Through O draw

lines OL, OM parallel to AB and CD respectively

Since OL and OM are perpendiculars to EF, the

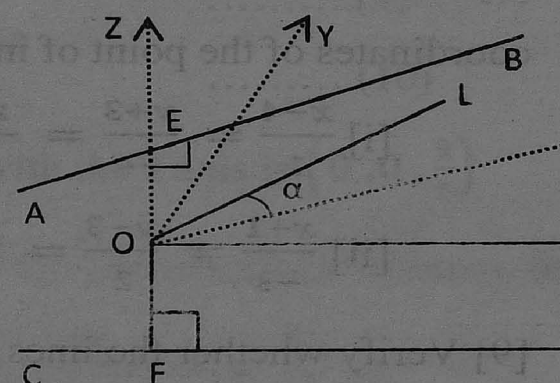
x-axis and y-axis will lie in the plane OLM.

Take the internal and external and external bisectors of $\angle MOL$ as

x-axis and y-axis respectively Let $\angle MOL = 2\alpha$, hence $\angle xOL = \alpha$.

$\therefore$ The line OL makes angles $\alpha, \pi/2 - \alpha, \pi/2$ with the coordinate axes respectively.

$\therefore$ d.c of OL are $\cos \alpha, \sin \alpha, 0$



$\therefore$ d.r of AB are $\cos \alpha, \sin \alpha, 0$

Since E is a point on the

z -axis such that $OE = c$, E is $(0, 0, c)$

$$\therefore \text{Equation of AB} = \frac{x-0}{\cos \alpha} = \frac{y-0}{\sin \alpha} = \frac{z-c}{0}$$

Taking the first 2 ratios we have $y = x \tan \alpha$ and $z = c$ (1)

The line OM makes angles $\alpha, \pi/2 + \alpha, \pi/2$ with the axes

$\therefore$ d.c of OM are $\cos \alpha, -\sin \alpha, 0$

$\therefore$ d.r of CD are $\cos \alpha, -\sin \alpha, 0$

Since F is the point on CD it is $(0, 0, -c)$

$$\text{Equation of CD is } \frac{x-0}{\cos \alpha} = \frac{y-0}{-\sin \alpha} = \frac{z+c}{0}$$

$$\therefore y = -x \tan \alpha \text{ and } z = -c \text{(2)}$$

Taking $m = \tan \alpha$ the equation of the skew lines (1) and (2) can be written in a simple form as $y = mx, z = c$; $y = -mx, z = -c$.

Note 1. Equations of skew lines can also be written as

$$\frac{x}{1} = \frac{y}{m} = \frac{z-c}{0}; \frac{x}{1} = \frac{y}{-m} = \frac{z+c}{0}$$

Note 2. The coordinates of any point on the skew lines AB and CD may be taken as (r, mr, c) and $(r_1, -mr_1, -c)$ respectively for various r and r_1

Solved problems

Problem 1. Show that the surface generated by a straight line which meets the given skew lines at the same angles is $yz = mcx$ where m and c have the usual meanings with regard to the skew lines in simple form.

Solution. The equation of 2 skew lines in simple form are

$$\frac{x}{1} = \frac{y}{m} = \frac{z-c}{0}; \frac{x}{1} = \frac{y}{-m} = \frac{z+c}{0}$$

Any point on the first line be $P(r, mr, c)$ and any point on the second line be $Q(r_1, -mr_1, -c)$

$$\therefore \text{Equation of PQ is } \frac{x-r}{r-r_1} = \frac{y-mr}{m(r+r_1)} = \frac{z-c}{2c} \text{[1]}$$

Given that PQ makes same angle with the 2 given skew lines

$$\therefore \frac{(r-r_1)+m^2(r+r_1)}{\sqrt{1+m^2} \sqrt{(r-r_1)+m^2(r+r_1)^2+4c^2}} = \frac{(r-r_1)-m^2(r+r_1)}{\sqrt{1+m^2} \sqrt{(r-r_1)+m^2(r+r_1)^2+4c^2}}$$

$$\therefore (r-r_1)+m^2(r+r_1) = (r-r_1)-m^2(r+r_1)$$

$$\therefore 2m^2 r + 2m^2 r_1 = 0$$

$$\text{Hence } 2m^2 (r+r_1) = 0$$

$$\therefore r+r_1 = 0 \text{ Hence } r = -r_1 \quad \dots\dots\dots(2)$$

Using (2) in (1) we get

$$\therefore \frac{x-r}{2r} = \frac{y-mr}{0} = \frac{z-c}{2c}$$

$$\therefore \frac{x-r}{2r} = \frac{z-c}{2c} \quad \dots\dots\dots(3)$$

$$y = mr \quad \dots\dots\dots(4)$$

$$\text{From (4) we get } \frac{x-\left(\frac{y}{m}\right)}{2\left(\frac{y}{m}\right)} = \frac{z-c}{2c}$$

$$\therefore \frac{mx-y}{2y} = \frac{z-c}{2c} \text{ hence } (mx-y)c = y(z-c)$$

$$\therefore yz = mcx$$

Problem 2. A point moves so that the line joining the feet of the perpendiculars from it two given lines subtends a right angle at the mid point of their shortest distance. Show that its locus is $x^2 - m^2 y^2 = \frac{c^2(1+m^2)2}{1-m^2}$ where m and c have the usual meaning with

regard to the skew lines in simple form.

Solution .

The equation of the skew lines in simple form are

$$L_1 : \frac{x}{1} = \frac{y}{m} = \frac{z-c}{0} ; L_2 : \frac{x}{1} = \frac{y}{-m} = \frac{z+c}{0}$$

Let the coordinates of any point A on the first line be (r, mr, c)

and B on the second line be $(r_1, -mr_1, -c)$

Let P (r_0, y_0, z_0) be any point on the locus such that PA is perpendicular to the first line and PB is perpendicular to the second line and $\angle AOB = 90^\circ$.

The d.r of AP are $x_0 - r, y_0 - mr, z_0 - c$

The d.r of the first line are $l, m, 0$

The d.r of BP are $(x_0 - r_1, y_0 + mr_1, z_0 + c)$

The d.r of the second line are $l, m, 0$

The d.r of OA are r, mr, c

The d.r of OB are

$r_1, -mr_1, -c$

PA is perpendicular to the first line
 $\Rightarrow x_0 - r + (y_0 - mr)m + (z_0 - c)0 = 0$.

$$\therefore \text{Hence } r_1 = \frac{x_0 - y_0 m}{1 + m^2} \dots\dots(2)$$

$$\therefore \text{Now OA is perpendicular to OB} \Rightarrow rr_1 - m^2 rr_1 - c^2 = 0$$

$$\therefore \text{Hence } rr_1 (1 - m^2) = c^2$$

$$\therefore \text{Substituting (1) and (2) in (3) we get } \left(\frac{x_0 + my_0}{1 + m^2} \right) \left(\frac{x_0 + my_0}{1 + m^2} \right) (1 - m^2) = c^2$$

$$\therefore (x_0^2 - m^2 y_0^2)(1 - m^2) = c^2 (1 + m^2)^2$$

$\therefore$ Locus of P (x_0, y_0, z_0) is

$$x^2 - m^2 y^2 = \frac{c^2 (1 + m^2)^2}{1 - m^2}$$

Problem 3. Find the locus of a point which moves such that the perpendiculars drawn from it to given skew lines are at right angles.

Solution. Take any point

P (x_0, y_0, z_0) and points

A and B on the two lines are

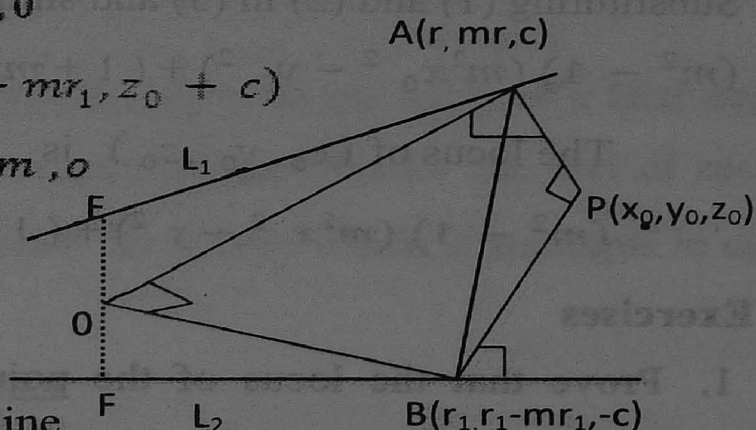
(r, mr, c) and $(r_1, -mr_1, -c)$

respectively We have to find the locus of P such that PA is perpendicular to L_1 : PB is perpendicular to L_2 and AP is perpendicular to BP

$$\text{PA is perpendicular to } L_1 \rightarrow r = \frac{x_0 + my_0}{1 + m^2} \text{ (refer problem 2) } \dots[1]$$

$$\text{PA is perpendicular to } L_2 \Rightarrow r = \frac{x_0 + my_0}{1 + m^2} \text{ (refer problem 2) } \dots[2]$$

Now AP is perpendicular to BP



Space for Hints

$$\rightarrow (x_0 - r)(x_0 - r_1) + (y_0 - mr)(y_0 - mr_1)(z_0 - c)(z_0 + c) = 0 \dots\dots(2)$$

Substituting (1) and (2) in (3) and simplifying we get

$$(m^2 - 1)(m^2 x_0^2 - y_0^2) + (1 + m^2)^2 (z_0^2 - c^2) = 0$$

$\therefore$ The locus of (x_0, y_0, z_0) is

$$\therefore (m^2 - 1)(m^2 x^2 - y^2) + (1 + m^2)^2 (z^2 - c^2) = 0$$

Exercises

1. Prove that the locus of the point P which moves such that its distance from the origin is equal to the distance AB between the feet A, B of the perpendiculars drawn from P to the lines

$$L_1 : \frac{x}{1} = \frac{y}{m} = \frac{z-c}{0} \text{ and}$$

$$L_2 : \frac{x}{1} = \frac{y}{-m} = \frac{z+c}{0} \text{ is } (1 - m^2)(x^2 - y^2) = (1 + m^2)^2 x(4c^2 - z^2)$$

(Hint $OP^2 = AB^2$; PA is perpendicular to L_1 ; PB is perpendicular to L_2)

If A and B are points on two skew lines L_1, L_2 as given in problem 1 and if A, B move along the lines such that AB is parallel to a fixed plane prove that the locus of the mid point of AB is a straight line

(Hint: taking the fixed plane as $px + qy + rz + s = 0$ prove that locus is $z = 0, qm^2 + py = crm = 0$)

4. Prove that the locus of a point P which is equidistant from the feet of perpendiculars drawn from p to the two given straight line L_1 and L_2 is $(1 + m^2)cz + mxy = 0$

(Hint : PA = PB where P is any point on the locus and A is (r, mr, c) and B is $(r_1, -mr_1, -c)$ with PA is perpendicular to L_1 and PB is perpendicular to L_2)

Show that the locus of the mid points of lines of constant length which have their extremities on two given lines is an ellipse whose centre bisects the shortest distance and whose axes are equally inclined to the lines.

UNIT- 4

THE SPHERE

Section 4.0 Introduction

In unit 3 we have studied the plane as the locus of a first degree equation in x, y and z . We now proceed to consider the loci of second degree equations. We first consider the sphere which is analogous to circle in plane geometry.

Section 4.1 Equation of a sphere

Definition. A sphere is the locus of a point in space which moves such that its distance from a fixed point is constant. The fixed point is called the centre of the sphere and the fixed distance is called the radius of the sphere.

We now proceed to find several forms of the equation of a sphere.

1. Centre radius form.

Theorem 4.1. The equation of the sphere with centre $C(a, b, c)$ and radius r is given by $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$.

Proof. Let $P(x_0, y_0, z_0)$ be any point on the sphere.

$$\text{Hence } CP^2 = r^2.$$

$$\therefore (x_0 - a)^2 + (y_0 - b)^2 + (z_0 - c)^2 = r^2.$$

$$\therefore \text{Locus of } (x_0, y_0, z_0) \text{ is } (x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

Corollary. Equation of the sphere with centre origin and radius r is $x^2 + y^2 + z^2 = r^2$.

2. General form of a sphere

Theorem 4.2. The equation

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere with centre $(-u, -v, -w)$ and radius $\sqrt{u^2 + v^2 + w^2 - d}$.

Proof. The given equation can be written as

$$(x + u)^2 + (y + v)^2 + (z + w)^2 = u^2 + v^2 + w^2 - d.$$

This represents the locus of a point (x, y, z) which moves such that its distance from the point $C(-u, -v, -w)$ is equal to the constant

$$\sqrt{u^2 + v^2 + w^2 - d}.$$

Space for Hints

Note 1. The equation

$ax^2 + ay^2 + az^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere with centre $\left(-\frac{u}{a}, -\frac{v}{a}, -\frac{w}{a}\right)$; radius $\sqrt{\left(\frac{u^2}{a^2} + \frac{v^2}{a^2} + \frac{w^2}{a^2} - \frac{d}{a}\right)}$

Note 2. The equation of the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ can be denoted as $S = 0$ where $S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d$.

2. Diameter form

Theorem 4.3. The equation of the sphere described on the line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ as diameter is given by

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

Proof. Let $P(x, y, z)$ be any point on the sphere with AB as diameter.

$\therefore$ The direction ratios of AP are $x - x_1, y - y_1, z - z_1$ and the direction ratios of BP are $x - x_2, y - y_2, z - z_2$.

Consider the circle passing through A, B and P . This circle also has AB as diameter and hence $\angle APB = 90^\circ$. [i.e.] AP is perpendicular to BP .

$$\therefore (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

Since this is true for any point (x, y, z) on the sphere it represents the equation of the required sphere.

Section 4.2 Tangent Line and Tangent Plane

Definition. The straight line joining two points P and Q on a surface is called a chord of the surface. When Q moves along the surface and ultimately coincides with P the limiting position of PQ touches the surface at P and is called a tangent line of the surface.

In the case of a sphere with centre C there are many tangent lines at a point P on it, all of them being perpendicular to the radius CP . All these tangents lie on the plane through P perpendicular to CP . This plane is called the tangent plane of the sphere at P .

Theorem 4.4. The equation of the tangent plane to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ at $P(x_1, y_1, z_1)$ is

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$$

Proof. The centre of the given sphere is $C(-u, -v, -w)$. The tangent plane to the sphere at $P(x_1, y_1, z_1)$ passes through P and has CP as its normal.

The direction ratios of CP are $x_1 + u, y_1 + v, z_1 + w$.

Hence the equation of the tangent plane at P is

$$(x_1 + u)(x - x_1) + (y_1 + v)(y - y_1) + (z_1 + w)(z - z_1) = 0$$

$$(i.e) xx_1 + yy_1 + zz_1 + ux + vy + wz = x_1^2 + y_1^2 + z_1^2 + ux_1 + y_1 + z_1$$

$$(i.e) xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d$$

$$= x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d$$

$$= 0. \text{ [Since } (x_1, y_1, z_1) \text{ lies on the sphere]}$$

Hence the result.

Angle of intersection of two spheres

The angle of intersection of two spheres at a common point is the angle between the tangent planes to them at that point.

Since the angle between the two tangent planes at the common point is same as the angle between their normals at that point we note that the angle between the two spheres is same as the angle between the radii of the two spheres at the common point. Also we note that the angle of intersection at every common point of the spheres is the same.

Suppose the two spheres $S = 0$ and $S_1 = 0$ with centre A and B and radii r and r_1 respectively cut orthogonally then $AB^2 = AP^2 + BP^2$ where P is the common point and $AP = r$ and $BP = r_1$.

Theorem 4.5: The condition for two spheres

$$S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{and}$$

$$S_1 = x^2 + y^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0 \quad \text{to cut orthogonally is}$$

$$2uu_1 + 2vv_1 + 2ww_1 = d + d_1$$

Proof. The centre of $S = 0$ is $A(-u, -v, -w)$ and radius

$$r = \sqrt{u^2 + v^2 + w^2 - d}$$

The centre of $S_1 = 0$ is $B(-u_1, -v_1, -w_1)$ and radius

$$r_1 = \sqrt{u_1^2 + v_1^2 + w_1^2 - d_1}$$

Let P be a common point. The two spheres cut orthogonally if

$$r^2 + r_1^2 = AB^2$$

$$\begin{aligned}
 (u^2 + v^2 + w^2 - d) + (u_1^2 + v_1^2 + w_1^2 - d) &= (u_1 - u)^2 + (v_1 - v)^2 \\
 &+ (w_1 - w)^2 \\
 &= u^2 + u_1^2 - 2uu_1 + v^2 + v_1^2 - 2vv_1 + w^2 + w_1^2 - 2ww_1 \\
 \therefore 2uu_1 + 2vv_1 + 2ww_1 &= d + d_1.
 \end{aligned}$$

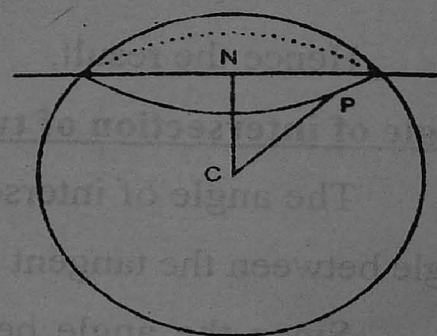
Section 4.3. Section of a Sphere

Theorem 4.6 A plane section of a sphere is a circle.

Proof. Let a plane π cut a sphere of radius r and centre C .

Let P be a point on the plane section. We claim that the locus of P is a circle.

Let N be the foot of the perpendicular drawn from C to the plane π .



$$\therefore CN \perp NP$$

$$\therefore NP = \sqrt{CP^2 - CN^2} = \sqrt{r^2 - CN^2}$$

which is a constant. Hence the locus of P is a circle with centre N and radius $\sqrt{r^2 - CN^2}$.

Note 1. The section of a sphere by a plane through its centre is known as a great circle and the centre and radius of a great circle are the same as that of the sphere.

Note 2. The curve of intersection of a sphere by a plane is a circle. Hence a circle can be represented by two equations one being the equation of a sphere S and the other a plane π . Hence $S = 0$ and $\pi = 0$ given together represent a circle.

Note 3. Let $S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ and $\pi = ax + by + cz + d = 0$. Then $S + \lambda\pi = 0$ (where λ is a constant) represents the equation of a sphere passing through the circle given by $S = 0$ and $\pi = 0$. For, $S + \lambda\pi = 0$ represents a sphere. Further $S + \lambda\pi = 0$ is satisfied by the points common to $S = 0$ and $\pi = 0$.

Note 4. Two intersecting spheres also determine a circle. For, if $S = 0$ and $S_1 = 0$ represent two spheres then $S - S_1$ is a first degree equation in x, y, z and hence represents a plane.

Hence $(S = 0 \text{ and } S - S_1 = 0)$ or $(S_1 = 0 \text{ and } S - S_1 = 0)$ determine a circle.

Solved problems

Problem 1. Find the equation of the sphere with centre $(1, -1, 2)$ and radius 3.

Solution. The required equation is $(x - 1)^2 + (y + 1)^2 + (z - 2)^2 = 3^2$.

$$\text{[i.e.]} \quad x^2 + y^2 + z^2 - 2x + 2y - 4z - 3 = 0$$

Problem 2. Find the centre and radius of the sphere

$$2x^2 + 2y^2 + 2z^2 - 2x + 2y - 4z - 5 = 0$$

Solution. The given equation can be written as

$$x^2 + y^2 + z^2 - x + y - 2z - \frac{5}{2} = 0$$

$$\therefore \text{Centre is } \left(\frac{1}{2}, -\frac{1}{2}, 1\right) \text{ and radius} = \sqrt{\frac{1}{4} + \frac{1}{4} + 1 + \frac{5}{2}} = 2.$$

Problem 3. Obtain the equation of the sphere having its centre at the point $(6, -1, 2)$ and touching the plane $2x - y + 2z = 0$.

Solution. Since the plane touches the sphere, the radius r is the perpendicular distance from the centre $(6, -1, 2)$ to the plane $2x - y + 2z - 2 = 0$.

$$\therefore r = \pm \left[\frac{(2 \times 6) - (-1) + (2 \times 2) - 2}{\sqrt{[12^2 + (-1)^2 + 2^2]}} \right] = \pm \left[\frac{12 + 1 + 4 - 2}{\sqrt{9}} \right] = \frac{15}{3} = 5.$$

$$\therefore \text{Equation of the sphere is } (x - 6)^2 + (y + 1)^2 + (z - 2)^2 = 5^2.$$

$$\text{(i.e.) } x^2 + y^2 + z^2 - 12x + 2y - 4z + 16 = 0.$$

Problem 4. Find the equation of the sphere passing through the points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Solution: Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$$

It passes through the origin. Hence $d = 0$.

$(1,0,0)$ lies on the sphere. Hence $1 + 2u + d = 0$. Hence $u = -1/2$.

$(0,1,0)$ lies on the sphere. Hence $1 + 2v + d = 0$. Hence $v = -1/2$.

$(0,0,1)$ lies on the sphere. Hence $1 + 2w + d = 0$. Hence $w = -1/2$.

$\therefore$ The equation of the sphere is $x^2 + y^2 + z^2 - x - y - z = 0$

Problem 5. Find the equation of the sphere which has the line joining the points $(2,7,5)$ and $(8,-5,1)$ as diameter.

Solution. The equation of the required sphere is

$$(x-2)(x-8) + (y-7)(y+5) + (z-5)(z-1) = 0.$$

$$(i.e) x^2 + y^2 + z^2 - 10x - 2y - 6z - 14 = 0.$$

Problem 6. Find the equation of the sphere passing through the points $(1,1,-2)$, $(-1,1,2)$ and having the centre of the sphere on the line $x + y - z - 1 = 0 = 2x - y + z - 2$.

Solution. Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$$

It passes through the points $(1,1,-2)$, $(-1,1,2)$.

$$\therefore 1 + 1 + 4 + 2u + 2v - 4w + d = 0$$

$$2u + 2v - 4w + d = -6 \quad \dots\dots\dots[1]$$

$$\text{Similarly } -2u + 2v + 4w + d = -6 \quad \dots\dots\dots[2]$$

The centre $(-u, -v, -w)$ lies on the line determined by the two planes $x + y - z - 1 = 0$ and $2x - y + z - 2 = 0$.

$$\therefore -u - v - w = 1 \quad \dots\dots\dots[3]$$

$$-2u + v - w = 2 \quad \dots\dots\dots[4]$$

$$\therefore [1] - [2] \Rightarrow 4u - 8w = 0 \quad \dots\dots\dots[5]$$

From [5] we get $u = 2w$.

$$\therefore [3] \text{ and } [4] \text{ become } -w - v = 1 \quad \dots\dots\dots[6]$$

$$-5w + v = 2 \quad \dots\dots\dots[7]$$

From [6] and [7] we get $w = -1/2$ and $v = -1/2$.

From [3] we get $u = -1$ and from [1] we get $d = -5$.

$\therefore$ The equation of the sphere becomes

$$x^2 + y^2 + z^2 - 2x - y - z - 5 = 0.$$

Problem 7. Find the condition for the plane $lx + my + nz = p$ to be a tangent plane to the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

Solution. The centre of the sphere $S = 0$ is $(-u, -v, -w)$ and the radius is $\sqrt{u^2 + v^2 + w^2 - d}$.

The plane $lx + my + nz = p$ is a tangent plane to the sphere if the perpendicular distance from the centre $(-u, -v, -w) = \text{radius}$.

$$\therefore \frac{-lu - mv - nw - p}{\sqrt{l^2 + m^2 + n^2}} = \sqrt{u^2 + v^2 + w^2 - d}$$

$$\therefore (lu + mv + nw + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d).$$

This is the required condition.

Problem 8. Obtain the equation of the tangent plane at the origin to the sphere $x^2 + y^2 + z^2 + 8x - 6y + 4z = 0$.

Solution: We note that the origin $(0,0,0)$ lies on the sphere.

Equation of the tangent plane at $(0,0,0)$ is

$$x(0) + y(0) + z(0) + 4(x + 0) - 3(y + 0) + 2(z + 0) = 0.$$

$$\text{(i.e.) } 4x - 3y + 2z = 0.$$

Problem 9. Show that the plane $2x - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 + 8x - 6y + 4z = 0$. Also find the point of contact.

Solution: The centre of the sphere is $(1, 2, -1)$ and the radius 3. (verify)

The perpendicular distance from $(1, 2, -1)$ to the given plane

$2x - 2y + z + 12 = 0$ is

$$\frac{2 - 4 - 2 + 12}{\sqrt{2^2 + (-2)^2 + 1^2}} = \frac{9}{3} = 3$$

Thus the perpendicular distance from the centre to the plane = radius of the sphere.

$\therefore 2x - 2y + z + 12 = 0$ is a tangent plane to the given sphere.

$\therefore$ The direction ratios of the normal to the plane are 2, -2, 1.

The equation of the perpendicular from the centre to the plane

$$\text{is } \frac{x-1}{2} = \frac{y-2}{-2} = \frac{z+1}{1}$$

Any point on this line is given by $F(2r+1, -2r+2, r-1)$.

This point P is the point of contact if it lies on the plane

$$2x - 2y + z + 12 = 0.$$

$$\therefore 2(2r+1) - 2(-2r+2) + (r-1) - 12 = 0$$

$$\therefore 9r = -9. \text{ Hence } r = -1.$$

The point of contact P is $(-1, 4, -2)$.

Problem 10. Find the equation of the sphere through the circle $x^2 + y^2 + z^2 + 2x + 3y + 5z = 0$; $2x + 6y + 5z - 6 = 0$ and passing through the centre of the sphere $S = x^2 + y^2 + z^2 - 2x - 4y + 6z + 1 = 0$.

Solution. Centre of the sphere $S = 0$ is $(1, 2, -3)$

The equation of the required sphere is of the form

$$x^2 + y^2 + z^2 + 2x + 3y + 5z + \lambda(2x + 6y + 5z - 6) = 0 \dots\dots[1]$$

It passes through $(1, 2, -3)$.

$$\text{Hence } 1 + 4 + 9 + 2 + 6 - 15 + \lambda(2 + 13 - 6) = 0$$

$$\therefore -7\lambda + 7 = 0. \text{ Hence } \lambda = 1.$$

$\therefore$ From [1] the equation of the sphere is

$$x^2 + y^2 + z^2 + 4x + 9y - 10z - 6 = 0.$$

Problem 11. Find the equation of the sphere through the circle $S = x^2 + y^2 + z^2 - 4 = 0$ and

$S_1 = x^2 + y^2 + z^2 + 4x - 2y + 4z + 10 = 0$ and through the point $(2, 1, 1)$.

Solution. The plane π determined by $S = 0$ and $S_1 = 0$ is

$$S - S_1 = -4x + 2y - 4z + 6 = 0$$

$$\therefore \pi = 2x - y + 2z - 3 = 0.$$

Now, the equation of the required sphere is

$$S + \lambda\pi = x^2 + y^2 + z^2 - 4 + \lambda(2x - y + 2z - 3) = 0 \dots\dots\dots[1]$$

It passes through the point $(2, 1, 1)$.

$$\text{Hence } 4 + 1 + 1 - 4 + \lambda(4 - 1 + 2 - 3) = 0.$$

$$\therefore 2\lambda = -2. \text{ Hence } \lambda = -1.$$

$\therefore$ From [1] the equation of the sphere is

$$x^2 + y^2 + z^2 - 2x + y - 2z - 1 = 0.$$

Problem 12. Find the equation of the sphere passing through the circle $x^2 + y^2 + z^2 - 4 = 0$, $2x + 4y + 6z - 1 = 0$ and having its centre on the plane $x + y + z = 6$.

Solution: Equation of the sphere passing through the circle determined by the sphere and the plane is given by

$$x^2 + y^2 + z^2 - 4 + \lambda(2x + 4y + 6z - 1) = 0.$$

$$(i.e) x^2 + y^2 + z^2 + 2\lambda x + 4\lambda y + 6\lambda z - 4 - \lambda = 0 \dots\dots\dots[1]$$

Its centre is $(-\lambda, -2\lambda, -3\lambda)$.

This centre lies on the plane $x + y + z = 6$.

$$\therefore -\lambda - 2\lambda - 3\lambda = 6. \text{ Hence } -6\lambda = 6. \text{ Hence } \lambda = -1.$$

Equation of the required sphere is got from [1] as

$$x^2 + y^2 + z^2 - 2x - 4y - 2z - 3 = 0.$$

Problem 13. The circle on the sphere $x^2 + y^2 + z^2 - 6y - 10z + 23 = 0$ has centre $(1, 2, -2)$. Find its equation.

Solution: The centre of the sphere $C(0, -3, 5)$. The plane section of the sphere is a circle whose centre is $N(1, 2, -2)$.

$\therefore NC$ is the normal to the intersecting plane. Hence d.r. of NC are $1, 5, -7$.

$\therefore$ The equation of the intersecting plane takes the form $x + 5y - 7z + d = 0$. It passes through $(1, 2, -2)$.

$$\therefore 1 + 10 + 14 + d = 0 \Rightarrow d = -25$$

Hence its equation is $x + 5y - 7z - 25 = 0$

$\therefore$ The equation of the circle is given by

$$x^2 + y^2 + z^2 - 6y - 10z + 23 = 0 = x + 5y - 7z - 25.$$

Problem 14. Find the centre and radius of the circle determined by the sphere $S = x^2 + y^2 + z^2 + 10y - 4z - 8 = 0$ and plane $\pi = x + y + z - 3 = 0$.

Solution. Centre of the sphere is $C(0, -5, 2)$ and radius

$$R = \sqrt{0^2 + 5^2 + (-2)^2 + 8} = \sqrt{37}.$$

Let O be the centre of the circle of the circle determined by $S = 0$ and $\pi = 0$.

$\therefore CO$ is perpendicular to the plane $x + y + z - 3 = 0$.

$\therefore$ Direction ratios of CO are $(1, 1, 2)$.

Space for Hints

Hence equations of CO are $\frac{x}{1} = \frac{y+5}{1} = \frac{z-2}{1}$.

Any point on CO is $(r, r-5, r+2)$.

If this point lies on the plane $x + y + z - 3 = 0$ we have

$$r + (r-5) + (r+2) - 3 = 0.$$

$\therefore 3r = 6$. Hence $r = 2$. Hence O is $(2, -3, 4)$

$$\text{Now } CO^2 = (O, -2)^2 + (-5 + 3)^2 + (2 - 4)^2 = 12.$$

$$\text{Radius of the circle} = \sqrt{R^2 - OC^2} = \sqrt{37 - 12} = \sqrt{25} = 5.$$

Problem 15. If r is the radius of the circle given by

$$S : x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0;$$

$\pi : lx + my + nz = 0$ prove that

$$(r^2 + d)(l^2 + m^2 + n^2) = (mw - nv)^2 + (nu - lw)^2 + (lv - mu)^2$$

Solution. Centre of the sphere $S = 0$ is $C(-u, -v, -w)$ and radius

$$R = \sqrt{u^2 + v^2 + w^2 - d}.$$

Let A be the centre of the circle determined by $S = 0$ and $\pi = 0$.

Then CA is perpendicular to the plane $\pi = 0$.

The d.r of CA are the normal to the plane $\pi = 0$ and they are l, m, n .

$$\therefore \text{Equation of the line CA are } \frac{x+u}{l} = \frac{y+v}{m} = \frac{z+w}{n}.$$

The point A is $(kl - u, km - v, kn - w)$ for some k .

Since A lies on the plane π , $l(kl - u) + m(km - v) + n(kn - w) = 0$.

$$k(l^2 + m^2 + n^2) = lu + mv + nw.$$

$$\therefore k = \frac{lu + mv + nw}{l^2 + m^2 + n^2}$$

$$\text{Now } r^2 = R^2 - AC^2$$

$$= (u^2 + v^2 + w^2 - d) - [(kl)^2 + (km)^2 + (kn)^2]$$

$$= u^2 + v^2 + w^2 - d - k^2(l^2 + m^2 + n^2)$$

$$= u^2 + v^2 + w^2 - d - \left(\frac{(lu + mv + nw)^2}{(l^2 + m^2 + n^2)^2} \right) (l^2 + m^2 + n^2)$$

$$\therefore r^2(l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)(l^2 + m^2 + n^2) - (lu + mv + nw)^2$$

$$\therefore (r^2 + d)(l^2 + m^2 + n^2)(u^2 + v^2 + w^2)(l^2 + m^2 + n^2) - (lu + mv + nw)^2$$

$$= (mw - nv)^2 + (nu - lw)^2 + (lv - mu)^2.$$

Problem 16. Show that the sphere

$$S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \text{ will cut the sphere}$$

$$S_1 = x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0 \text{ in a great circle if}$$

$$2uu_1 + 2vv_1 + 2ww_1 - (d + d_1) = 2r_1^2 \text{ where } r_1 \text{ is the radius of the later sphere.}$$

Solution. The plane π determined by $S = 0$ and $S_1 = 0$ is $S - S_1 = 0$ and it is $\pi: 2(u - u_1)x + 2(v - v_1)y + 2(w - w_1)z + (d - d_1) = 0$.

The intersection of the spheres $S = 0$ and $S_1 = 0$ will be a great circle if the centre $(-u_1, -v_1, -w_1)$ of $S_1 = 0$ lies on the plane $\pi = 0$.

$$\therefore -2(u - u_1)u_1 - 2(v - v_1)v_1 - 2(w - w_1)w_1 + (d - d_1) = 0$$

$$\therefore 2uu_1 - 2u_1^2 + 2vv_1 - 2v_1^2 + 2ww_1 - 2w_1^2 - d + d_1 = 0$$

$$\therefore 2uu_1 + 2vv_1 + 2ww_1 = 2(u_1^2 + v_1^2 + w_1^2) + d - d_1$$

$$= 2(r_1^2 + d_1) + d - d_1 \text{ (since } r_1^2 = u_1^2 + v_1^2 + w_1^2 - d_1)$$

$$= 2r_1^2 + d + d_1$$

$$\therefore 2uu_1 + 2vv_1 + 2ww_1 - (d + d_1) = 2r_1^2$$

Problem 17. Find the equations of the spheres which pass through the circle $x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0$; $2x + y + z - 4 = 0$ and touch the plane $3x + 4y - 14 = 0$.

Solution. Let $S = x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0$

$$\pi = 2x + y + z - 4 = 0$$

Then $S + \lambda\pi = 0$ represents a sphere passing through the circle determined by $S = 0$, and $\pi = 0$.

$$\therefore S + \lambda\pi = x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 + \lambda(2x + y + z - 4) = 0.$$

$$(i.e) x^2 + y^2 + z^2 - 2x(1 - \lambda) + y(2 + \lambda) + z(4 + \lambda) - (3 + 4\lambda) = 0 \quad \dots\dots(1)$$

Center is $\left(1 - \lambda, -\frac{2+\lambda}{2}, -\frac{4+\lambda}{2}\right)$ and radius is

$$= \sqrt{(1 - \lambda)^2 + \left(\frac{2+\lambda}{2}\right)^2 + \left(\frac{4+\lambda}{2}\right)^2 + (3 + 4\lambda)}$$

Since the sphere touches the plane $3x + 4y - 14 = 0$ the perpendicular distance from the centre of the sphere to this plane is equal to the radius of the sphere.

$$\frac{(3(1-\lambda) - 2(2+\lambda) - 14)}{\sqrt{(3^2 + 4^2)}} = \sqrt{(1-\lambda)^2 + \left(\frac{2+\lambda}{2}\right)^2 + \left(\frac{4+\lambda}{2}\right)^2 + (3+4\lambda)}$$

$$(i.e) -2(5\lambda + 15) = 5\sqrt{4(1-\lambda)^2 + (2+\lambda)^2 + (4+\lambda)^2 + 4(3+4\lambda)}$$

$$100(\lambda+3)^2 - 25[(4+4\lambda^2 - 8\lambda) + (4+\lambda^2 + 4\lambda) + (16+\lambda^2 + 8\lambda) + (12+16\lambda)]$$

$$(i.e) 4(\lambda^2 + 6\lambda + 9) = 6\lambda^2 + 20\lambda + 36$$

$$(i.e) 2\lambda^2 - 4\lambda = 0$$

$$\therefore \lambda(\lambda - 2) = 0. \text{ Hence } \lambda = 0 \text{ or } \lambda = 2.$$

Using these values of λ in (1) we get the equations of the spheres as

$$x^2 + y^2 + z^2 - 2x + y + z - 3 = 0; x^2 + y^2 + z^2 + 2x + 4y + 6z - 1$$

Problem 18. Obtain the equation of the sphere having the circle $S = x^2 + y^2 + z^2 - 3x + 4y - 2z - 5 = 0; \pi = 5x - 2y + 4z + 7 = 0$.

Solution. Equation of the sphere passing through the circle determined by the sphere $S = 0$ and $\pi = 0$ is

$$S + \lambda\pi = x^2 + y^2 + z^2 - 3x + 4y - 2z - 5 + \lambda(5x - 2y + 4z + 7) = 0$$

$$x^2 + y^2 + z^2 - x(3 - 5\lambda) + 2y(2 - \lambda) - 2z(1 - 2\lambda) + (5 - 7\lambda) = 0 \dots (1)$$

$$\text{Its centre is } \left(\frac{3-5\lambda}{2}, \lambda-2, 1-2\lambda\right)$$

Since the cross section is a great circle this centre lies on the plane $5x - 2y + 4z + 7 = 0$.

$$\therefore 5\left(\frac{3-5\lambda}{2}\right) - 2(\lambda-2) + 4(1-2\lambda) + 7 = 0.$$

$$\therefore 15 - 25\lambda - 4\lambda + 8 + 8 - 16\lambda + 14 = 0.$$

$$\therefore 45\lambda = 45. \text{ Hence } \lambda = 1.$$

$\therefore$ From (1) the equation of the sphere is

$$x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0.$$

Problem 19. Prove that the circles

$$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0, 5y + 6z + 1 = 0 \text{ and}$$

$$x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0, x + 2y - 7z = 0 \text{ lie on the}$$

same sphere and find its equation.

Solution. The equation of any sphere through the first circle is

$$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 + \lambda(5y + 6z + 1) = 0$$

$$(i.e) x^2 + y^2 + z^2 - 2x + y(3 + 5\lambda) + 2z(2 + 3\lambda) - 5 + \lambda = 0 \quad \dots(1)$$

The equation of any sphere through the second circle is

$$x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 + \lambda'(x + 2y - 7z) = 0$$

$$(i.e) x^2 + y^2 + z^2 - x(3 - \lambda') - 2y(2 - \lambda') + z(5 - 7\lambda') - 6 = 0.$$

(1) and (2) will represent the same sphere if

$$3 - \lambda' = 2; -2(2 - \lambda') = 3 + 5\lambda; 5 - 7\lambda' = 2(2 + 3\lambda); -6 = -5 + \lambda.$$

(i.e) $\lambda' = 1$ and $\lambda = -1$ (from the first and the last equations).

Also we observe that these values of λ and λ' satisfy the other two equations also. Hence the two circles lie on the same sphere and its equation is $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$.

Problem 20. Prove that the two sphere

$$S_1 = x^2 + y^2 + z^2 - 2x + 4y - 4z = 0; S_2 = x^2 + y^2 + z^2 + 10x + 2z + 10 = 0$$

touch each other and find the point of contact.

Solution. Centre of $S_1 = 0$ is $C_1(1, -2, 2)$ and radius is

$$r_1 = \sqrt{(-1)^2 + 2^2 + (-2)^2} = 3$$

Centre of $S_2 = 0$ is $C_2(-5, 0, -1)$ and radius is

$$r_2 = \sqrt{5^2 + 1^2} = \sqrt{26} = 5$$

$$\text{Distance between the centres } C_1 C_2 = \sqrt{36 + 4 + 9} = 7 = r_1 + r_2.$$

Hence the two spheres touch each other externally.

The point of contact P is the point of division of the line joining C_1 and C_2 in the ratios 3:4 internally.

$$\text{Hence } P = \left(\frac{3(-5) + 4(1)}{3+4}, \frac{3(0) + 4(-2)}{3+4}, \frac{3(-1) + 4(2)}{3+4} \right) = \left(-\frac{11}{7}, -\frac{8}{7}, \frac{5}{7} \right).$$

Problem 21. A sphere of constant radius r always passes through the origin and meets the coordinate axes in A, B, C. Prove that the locus of the centroid of the triangle ABC is the sphere

$$9(x^2 + y^2 + z^2) = 4r^2.$$

Solution. Let the equation of the sphere OABC be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0. \quad \dots\dots(1)$$

Since it passes through the origin $= 0$.

$$\text{Hence } u^2 + v^2 + w^2 = r^2.$$

The sphere meets the x -axis at A.

To find the x -coordinate of A we put $y = z = 0$ in (1) and we get $x^2 + 2ux = 0$. Hence $x = -2u$.

Hence A is $(-2u, 0, 0)$. Similarly B is $(0, -2v, 0)$ and C is $(0, 0, -2w)$.

The centroid of the triangle ABC is

$$\left(-\frac{2u}{3}, -\frac{2v}{3}, -\frac{2w}{3}\right) = (x_1, y_1, z_1) \text{ (say)}$$

$$\therefore x_1 = -\frac{2u}{3}; y_1 = -\frac{2v}{3}; z_1 = -\frac{2w}{3}.$$

$$\therefore \text{The locus of } (x_1, y_1, z_1) \text{ is } 9(x^2 + y^2 + z^2) = 4r^2.$$

Problem 22. A moving plane intersects the coordinate axes in A, B, C. If the plane always passes through a fixed point (a, b, c) prove that the locus of the centre of the sphere OABC is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$.

Solution. Let the sphere OABC be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0. \quad \dots\dots(1)$$

Since it passes through O we get $d = 0$

The centre of the sphere is $(-u, -v, -w)$.

The sphere intersects the x -axis at A.

To find the x -coordinate of A we put $y = z = 0$ in (1) and we get $x^2 + 2ux = 0$. Hence $x = -2u$.

Hence A is $(-2u, 0, 0)$. Similarly B is $(0, -2v, 0)$ and C is $(0, 0, -2w)$.

$$\text{The equation of the ABC is } \frac{x}{-2u} + \frac{y}{-2v} + \frac{z}{-2w} = 1.$$

Since it passes through the fixed point (a, b, c) we have

$$\frac{a}{-2u} + \frac{b}{-2v} + \frac{c}{-2w} = 1. \quad \dots\dots(2)$$

Now, let $(x_0, y_0, z_0) = (u, v, w)$.

$$\therefore u = -x_0, v = -y_0, w = -z_0.$$

Substituting in (2) we get $\frac{a}{2x_0} + \frac{b}{2y_0} + \frac{c}{2z_0} = 1$.

$$\therefore \text{The locus of } (x_0, y_0, z_0) \text{ is } \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2.$$

Problem 23. Find the equations of tangent planes of the sphere $x^2 + y^2 + z^2 - 4x - 4y - 4z + 10 = 0$ which are parallel to the plane $x - z = 0$.

Solution. Let (x_1, y_1, z_1) be the point on the sphere at which the tangent plane is drawn. The equation of the tangent plane at (x_1, y_1, z_1) is

$$xx_1 + yy_1 + zz_1 - 2(x + x_1) - 2(y + y_1) - 2(z + z_1) + 10 = 0$$

$$\text{(i.e.) } (x_1 - 2)x + (y_1 - 2)y + (z_1 - 2)z - 2x_1 - 2y_1 - 2z_1 + 10 = 0 \quad \dots(1)$$

This plane is parallel to $x - z = 0$.

$$\therefore \frac{x_1 - 2}{1} = \frac{y_1 - 2}{1} = \frac{z_1 - 2}{-1} = k \quad (\text{say})$$

$$\therefore x_1 = k + 2, y_1 = 2, z_1 = -k + 2.$$

(2)

Since (x_1, y_1, z_1) lies on the sphere we have

$$(k + 2)^2 + 2^2 + (-k + 2)^2 - 4(k + 2) - 8 - 4(-k + 2) + 10 = 0.$$

$$\therefore k^2 + 4k + 4 + 4 + k^2 - 4k + 4 - 4k - 8 - 8 + 4k - 8 + 10 = 0.$$

$$2k^2 - 2 = 0. \text{ Hence } k = \pm 1.$$

$\therefore$ From (2), the points are $(3, 2, 1)$ and $(1, 2, 3)$.

$\therefore$ From (1), the equation of the tangent planes are

$$x - z - 2 = 0 \text{ and } -x + z - 2 = 0.$$

Problem 24. A sphere touches the plane $x - 2y - 2z - 7 = 0$ at $(3, -1, -1)$ and passes through $(1, 1, -3)$. Find its equation.

Solution. Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$$

It passes through $(1, 1, -3)$.

$$\text{Hence we get } 2u + 2v - 6w + d = -11. \quad \dots(1)$$

Since the plane $x - 2y - 2z - 7 = 0$ touches the sphere at $(3, -1, -1)$ it's the tangent plane $(3, -1, -1)$.

But the tangent plane at $(3, -1, -1)$ is

$$3x - y - z + u(x + 3) + v(y - 1) + w(z - 1) + d = 0.$$

$$(i.e) x(3 + u) + y(-1 + v) + z(-1 + w) + 3u - v - w + d = 0.$$

$$\therefore \frac{3+u}{1} = \frac{-1+v}{-2} = \frac{-1+w}{-2} = \frac{3u-v-w+d}{-7} = k$$

$$\therefore u = k - 3; v = -2k + 1; w = -2k + 1.$$

$$\therefore d = -7k - 3u + v + w$$

$$\therefore d = -14k + 11 \quad (\text{using 2})$$

Using (2) and (3) in (1) we get $2k - 6 - 4k + 2 + 12k - 6 - 14k + 11 = -11$.

$$\therefore -4k = -12. \text{ Hence } k = 3.$$

$$\therefore u = 0; v = -5; w = -5; d = -31.$$

$\therefore$ The equation of the sphere is

$$x^2 + y^2 + z^2 - 10y - 10z - 31 = 0.$$

Problem 25. Prove that the two spheres

$$x^2 + y^2 + z^2 + 6y + 2z + 8 = 0 \text{ and}$$

$$x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0 \text{ intersect each other}$$

orthogonally.

Solution. From the equation of the spheres we have

$$u = 0, v = 3, w = 1, d = 8; u_1 = 3, v_1 = 4, w_1 = 2, d_1 = 20.$$

$$\text{Here } 2uu_1 + 2vv_1 + 2ww_1 - (d + d_1) = 0 + 24 + 4 - (8 + 20) = 0.$$

Hence the two spheres intersect orthogonally.

Problem 26. Find the equation of the sphere that passes through the two points $(0, 3, 0), (-2, -1, -4)$ and cuts orthogonally the two spheres

$$S: x^2 + y^2 + z^2 + x - 3z - 2 = 0, S_1: 2(x^2 + y^2 + z^2) + x + 3y + 4 = 0$$

Solution. Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0. \dots(1)$$

$$(0, 3, 0) \text{ lies on the sphere gives } 6v + d = 0 \dots(2)$$

$$(-2, -1, -4) \text{ lies on the sphere gives } -4u - 2v - 8w + d = -21 \dots(3)$$

The spheres $S = 0$ is cut orthogonally by the sphere given by (1) gives

$$2u\left(\frac{1}{2}\right) + 2v(0) + 2w\left(-\frac{3}{2}\right) = d - 2$$

$$u_1 u - 6w - d = -2 \dots\dots(4)$$

The sphere $S_1 - 0$ cut orthogonally gives

$$2u\left(\frac{1}{4}\right) + 2v\left(\frac{3}{4}\right) + 2w(0) = d + 4$$

$$(i.e) u + 3v - 2d = 4 \dots\dots(5)$$

Hence the equation of the sphere is

$$x^2 + y^2 + z^2 + 2x - 2y - 4z - 3 = 0.$$

Exercises.

1. Find the equation of the sphere whose centre is

(i) $(1, 4, 2)$ and radius 3 units.

(ii) $\left(\frac{1}{2}, -1/2, 1\right)$ and radius 2 units.

2. Find the centre and radius of the following spheres

(i) $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$

(ii) $2x^2 + 2y^2 + 2z^2 - 2x + 4y + 2z + 3 = 0$

(iii) $16(x^2 + y^2 + z^2) - 16x - 8y - 16z - 55 = 0$

3. Find the equations of the sphere through the four points.

(i) $(1, 1, 1); (1, 2, 1); (1, 1, 2); (2, 1, 1)$

(ii) $(0, 0, 0); (0, 1, -1); (-1, 2, 0); (1, 2, 3)$

(iii) $(0, 0, 0); (-a, b, c); (a, b, -c)$

4. Find the equation of the sphere passing through the origin and making intercepts a, b, c on the axes.

5. Find the equation of the sphere which passes through the points $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ and has its centre on the plane $x + y + z = 6$.

6. Find the equation of the sphere having its centre on the line $5z + 2x = 0 = 2y - 3z$ and passing through the points $(-4, 0, -2)$ and

7. Find the equation of the sphere on AB as diameter where A is $(1, 2, 3)$ and B is $(3, 2, 1)$.

8. Find the equation of the sphere

(i) through the circle $x^2 + y^2 + z^2 = 9$; $2x + 3y + 4z = 5$ and the point $(1, 2, 3)$

(ii) which has its centre on the plane $5x + y - 4z + 3 = 0$ and passing through the circle

$$x^2 + y^2 + z^2 - 3x - 4y - 2z + 8 = 0; 4x - 5y + 3z - 3 = 0.$$

(9) through the circle $x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$ Obtain the equation of the circle lying on the sphere.

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 3 = 0 \text{ and having its centre at } (2, 3, -4)$$

(10) Show that the radius of the circle

$$x^2 + y^2 + z^2 - x + y - z - 4 = 0; x + y + z = 0 \text{ is } 2$$

(11) Find the centre and radius of the circle of intersection of the sphere

$$x^2 + y^2 + z^2 - 2y + 4z - 11 = 0 \text{ and the plane } x + 2y + 2z = 15$$

(12) Prove that the centre and radius of the circle of intersection of the

$$\text{sphere } x^2 + y^2 + z^2 - 2x - 4y - 6z - 2 = 0 \text{ and the plane}$$

$$x + 2y + 2z = 20 \text{ are } (2, 4, 5) \text{ and } \sqrt{7}$$

(13) Find the equation of the sphere for which the circle

$$2x + 3y + 4z = 8; x^2 + y^2 + z^2 - 7y - 2z + 2 = 0 \text{ is a great circle}$$

(14) Prove that the circles

$$x^2 + y^2 + z^2 - 2x + 3y - 4z - 5 = 0; 5y + 6z + 1 = 0$$

$$x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0; x + 2y - 7z = 0 \text{ lie on the same sphere and find its equation.}$$

(15) Find the equation of the sphere for which passes through the circle

$$x^2 + y^2 + z^2 - 2x - 4y = 0; x + 2y + 3z = 8 \text{ and touch the plane } 4x + 3y = 25$$

(16) Find the equation of the sphere for which passes through the circle

$$x^2 + y^2 + z^2 = 5; x + 2y + 3z = 3 \text{ and touch the plane } 4x + 3y = 15$$

(17) Show that $x^2 + y^2 + z^2 = 25$; $x^2 + y^2 + z^2 - 18x - 24y - 40z + 225 = 0$

Touch and find the coordinates of the common point.

- (18) Find the condition for the plane

$lx + my + nz = p$ to be a tangent plane to the sphere

$$x^2 + y^2 + z^2 = r^2$$

- (19) Find the condition for the plane

$lx + my + nz = p$ to be a tangent plane to the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vz + d = 0$$

- (20) Find the equation of the tangent plane to the sphere

$$3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0 \text{ at the point } (1, 2, 3)$$

- (21) Show that the plane $2x - 2y + z + 4 = 0$ touches the sphere

$$x^2 + y^2 + z^2 - 2x - 4y + 2z = 3 \text{ and find the point of contact.}$$

- (22) Find the value of a for which the plane $x - y + z = a\sqrt{3}$ touches

the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$

- (23) A sphere of constant radius r moves always passing through the origin O . if it cuts the x, y, z axes at A, B, C

Prove that the locus of the centroid of the tetrahedron $OABC$ is the

$$\text{sphere } x^2 + y^2 + z^2 = \frac{1}{2}r^2.$$

Prove that the locus of the foot of the perpendicular from O to the

$$\text{plane } ABC \text{ is } (x^2 + y^2 + z^2)^2 (x^{-2} + y^{-2} + z^{-2}) = 4r^2$$

- (24) Find the equation of the sphere that passes through the circle

$$x^2 + y^2 + z^2 - 2x + 3y - 4z + 6 = 0; 3x - 4y + 5z - 1 = 0$$

and the cuts the sphere $x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$

orthogonally.

- (25) Find the equation of the sphere which touches the plane

$$3x + 2y - z + 2 = 0 \text{ at the point } (1, 2, 1) \text{ and cuts}$$

$$\text{orthogonally the sphere } x^2 + y^2 + z^2 - 4x + 6y + 4z - 0$$

- (18) Find the condition for the plane $lx + my + nz = p$ to be a tangent plane to the sphere $x^2 + y^2 + z^2 = r^2$.
- (19) Find the condition for the plane $lx + my + nz = p$ to be a tangent plane to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.
- (20) Find the equation of the tangent plane to the sphere $3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$ at the point $(1, 2, 3)$.
- (21) Show that the plane $2x - 3y + z + 4 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z = 3$ and find the point of contact.
- (22) Find the value of a for which the plane $x + y + z = a\sqrt{3}$ touches the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$.
- (23) A sphere of constant radius r moves always passing through the origin O , if it cuts the x, y, z axes at A, B, C . Prove that the locus of the centroid of the tetrahedron $OABC$ is the sphere $x^2 + y^2 + z^2 = \frac{1}{4}r^2$.
- Prove that the locus of the foot of the perpendicular from O to the plane ABC is $(x^2 + y^2 + z^2)(x^2 + y^2 + z^2 - 4r^2) = 4r^2$.
- (24) Find the equation of the sphere that passes through the circle $x^2 + y^2 + z^2 - 2x + 3y - 4z + 6 = 0, 3x - 4y + 5z - 1 = 0$ and cuts the sphere $x^2 + y^2 + z^2 + 2x + 2y + 2z + 11 = 0$ orthogonally.
- (25) Find the equation of the sphere which touches the plane $3x + 2y - z + 12 = 0$ at the point $(1, 2, 1)$ and cuts orthogonally the sphere $x^2 + y^2 + z^2 - 4x + 6y + 1 = 0$.

Section 5.0 Introduction

In this chapter we deal with the theory of vector calculus. With the help of a standard vector differential operator we introduce concepts like gradient of a scalar valued function, divergence and curl of a vector valued function and discuss briefly the properties arising out of these concepts. We assume that the reader is familiar with the basic results in vector algebra.

We give a brief summary of these results in the next section. We denote vectors by bold face Roman letters.

Section 5.1 Vector Algebra

Through this chapter i j k stand for unit vectors along the coordinate axes ox , oy , oz respectively. If $P(x, y, z)$ is any point its position vector is given by $OP = r = xi + yj + zk$.

The modulus of r is given by $|r| = r = \sqrt{x^2 + y^2 + z^2}$

Definition. Let a and b be two vectors. The scalar product or dot product of a and b is defined to be $a \cdot b = ab \cos \theta$ where θ is the angle between the two vectors when drawn from a common origin.

Note. i) $a \cdot b = b \cdot a$ (i.e) dot product is commutative

ii) $a \cdot a = |a|^2$

iii) $a \cdot b = 0$ if a and b are perpendicular

iv) $a \cdot b = 0 \Rightarrow a = 0$ or a and b are perpendicular vectors.

v) $a \cdot (b+c) = a \cdot b + a \cdot c$

vi) $i \cdot i = j \cdot j = k \cdot k = 1$

vii) $i \cdot j = j \cdot k = k \cdot i = 0$

viii) If $a = a_1i + a_2j + a_3k$ and $b = b_1i + b_2j + b_3k$ then

$$a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$$

Definition. Let a , b be two non zero vectors. Then the vector product or cross product of a and b is a vector perpendicular to both a and b with magnitude $ab \sin \theta$ where $0 \leq \theta \leq \pi$ is the angle between a and b and whose direction is along a unit vector n such that a , b , n form a right handed system.

Thus $a \times b = ab \sin \theta n$.

Space for Hints

Note. i) $|a \times b|$ = area of the parallelogram with a, b as adjacent sides.

ii) $a \times b = b \times a$ (i.e) cross product is not commutative.

iii) $a \times b = 0$ if a and b are parallel

iv) $i \times i = j \times j = k \times k = 0$

v) $i \times j = k, j \times k = i, k \times i = j$

vi) If $a = a_1i + a_2j + a_3k$ and $b = b_1i + b_2j + b_3k$ then

$$a \times b = (a_2b_3 - a_3b_2) i + (a_3b_1 - a_1b_3) j + (a_1b_2 - a_2b_1) k$$

$$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Definition. The scalar triple product or box product of three vectors a, b, c is defined to be the scalar $a \cdot (b \times c)$. It is sometimes denoted by $[abc]$

It can be easily verified that $a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Note 1. $a \cdot (b+c)$ represents the volume of the parallelepiped formed by the coterminous edges a, b, c .

Note 2. $[a b c] = [b c a] = [c a b]$

Note 3. $[a b c] = -[b a c] = -[c b a] = -[a c b]$

Note 4. The vectors a, b, c are coplanar iff $[a b c] = 0$

Results. 1. $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$

2. $(a \times b) \times c = (a \cdot c)b - (b \cdot c)a$

$$3. (a \times b) \cdot (c \times d) = \begin{vmatrix} a \cdot c & a \cdot d \\ b \cdot c & b \cdot d \end{vmatrix}$$

$$(a \times b) \times (c \times d) = [abd]c - [abc]d.$$

Section 5.2 Differentiation of vectors

Definition. Let $r=r(t) = x(t)i + y(t)j + z(t)k$ be a vector valued function of a scalar variable t .

r is said to be differentiable if $\lim_{\Delta t \rightarrow 0} \frac{r(t + \Delta t) - r(t)}{\Delta t}$ exists and in this case

we write $\frac{dr}{dt} = \lim_{\Delta t \rightarrow 0} \frac{r(t + \Delta t) - r(t)}{\Delta t}$

Theorem 5.1. Let $r = r(t) = x(t)i + y(t)j + z(t)k$ be a differentiable function. Then $\frac{dr}{dt} = x'(t)i + y'(t)j + z'(t)k$.

Proof. $\frac{dr}{dt} = \lim_{\Delta t \rightarrow 0} \frac{r(t + \Delta t) - r(t)}{\Delta t}$

$$= \lim_{\Delta t \rightarrow 0} \left[\frac{x(t + \Delta t) - x(t)}{\Delta t} i + \frac{y(t + \Delta t) - y(t)}{\Delta t} j + \frac{z(t + \Delta t) - z(t)}{\Delta t} k \right]$$

$$= x'(t)i + y'(t)j + z'(t)k.$$

Theorem 5.2. If $u(t) = x(t)i + y(t)j + z(t)k$ and $v(t) = X(t)i + Y(t)j + Z(t)k$ then $\frac{d}{dt}(u \cdot v) = u \cdot \frac{dv}{dt} + \frac{du}{dt} \cdot v$

Proof. $u \cdot v = x(t)X(t) + y(t)Y(t) + z(t)Z(t)$
 $= [x(t)X'(t) + y(t)Y'(t) + z(t)Z'(t)] + [x'(t)X(t) + y'(t)Y(t) + z'(t)Z(t)]$
 $= u \cdot \frac{dv}{dt} + \frac{du}{dt} \cdot v$

Theorem 5.3 $\frac{d}{dt}(u \times v) = u \times \frac{dv}{dt} + \frac{du}{dt} \times v$

Proof. $u \times v = \begin{vmatrix} i & j & k \\ x(t) & y(t) & z(t) \\ X(t) & Y(t) & Z(t) \end{vmatrix}$

$$= [y(t)Z(t) - Y(t)z(t)]i - [x(t)Z(t) - X(t)z(t)]j + [x(t)Y(t) - X(t)y(t)]k$$

$$\therefore \frac{d}{dt}(u \times v) = [(y(t)Z'(t) + y'(t)Z(t)) - (Y(t)z'(t) + Y'(t)z(t))]i$$

$$- [(x(t)Z'(t) + x'(t)Z(t)) - (X(t)z'(t) + X'(t)z(t))]j$$

$$+ [(x(t)Y'(t) + x'(t)Y(t)) - (X(t)y'(t) + X'(t)y(t))]k$$

Space for Hints

$$= \begin{vmatrix} i & j & k \\ x(t) & y(t) & z(t) \\ X'(t) & Y'(t) & Z'(t) \end{vmatrix} + \begin{vmatrix} i & j & k \\ x'(t) & y'(t) & z'(t) \\ X(t) & Y(t) & Z(t) \end{vmatrix}$$

$$= \mathbf{u} \times \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt} \times \mathbf{v}$$

Theorem 5.4 $\frac{d(f\mathbf{u})}{dt} = f \frac{d\mathbf{u}}{dt} + \frac{df}{dt} \mathbf{u}$ where f is a scalar valued function

$f(t)$

Proof is left as an exercise.

$$\textbf{Theorem 5.5} \quad \frac{d}{dt}[fgh] = \left[fg \frac{dh}{dt} \right] + \left[f \frac{dg}{dt} h \right] + \left[\frac{df}{dt} gh \right]$$

$$\textbf{Proof.} \quad \frac{d}{dt}[fgh] = \frac{d}{dt}\{f \cdot (g \times h)\} = \frac{df}{dt} \cdot (g \times h) + f \cdot \frac{d}{dt}(g \times h)$$

$$= \frac{df}{dt} \cdot (g \times h) + f \cdot \left(g \times \frac{dh}{dt} + \frac{dg}{dt} \times h \right)$$

$$= \left[\frac{df}{dt} gh \right] + \left[f \frac{dg}{dt} h \right] + \left[fg \frac{dh}{dt} \right]$$

Solved problems.

Problem 1. If $\mathbf{r} = a \cos \omega t + b \sin \omega t$ where a, b are constant vectors and

$$\omega \text{ is a constant prove that } \mathbf{r} \times \frac{d\mathbf{r}}{dt} = \omega (\mathbf{a} \times \mathbf{b}) \text{ and } \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} + \omega^2 \mathbf{r} = 0$$

$$\textbf{Solution.} \quad \frac{d\mathbf{r}}{dt} = -a \omega \sin \omega t + b \omega \cos \omega t$$

$$\frac{d^2\mathbf{r}}{dt^2} = -a \omega^2 \cos \omega t + b \omega^2 \sin \omega t$$

$$\frac{d^2\mathbf{r}}{dt^2} = -\omega^2 (a \cos \omega t + b \sin \omega t)$$

$$\therefore \frac{d^2\mathbf{r}}{dt^2} + \omega^2 \mathbf{r} = 0$$

$$\text{Now, } \mathbf{r} \times \frac{d\mathbf{r}}{dt} = (a \cos \omega t + b \sin \omega t) \times (-a \omega \sin \omega t + b \omega \cos \omega t)$$

$$\begin{aligned}
 &= \omega a \times b \cos^2 \omega t - \omega b \times a \sin^2 \omega t \\
 &= \omega a \times b \cos^2 \omega t + \omega b \times a \sin^2 \omega t \\
 &= \omega (a \times b)
 \end{aligned}$$

Problem 2. If $u(t)$ is a vector which is constant in magnitude prove that

$$\frac{du}{dt} = 0 \text{ or } \frac{du}{dt} \text{ is perpendicular to } u.$$

Solution. $u \cdot u = c$ (a constant) $\Rightarrow \frac{du}{dt} \cdot u + u \cdot \frac{du}{dt} = 0$. Hence $u \cdot \frac{du}{dt} = 0$.

$$\therefore \frac{du}{dt} = 0 \text{ or } \frac{du}{dt} \text{ is perpendicular to } u.$$

Exercises

1. If $r = ae^{\omega t} i + be^{-\omega t} j$ show that $\frac{d^2 r}{dt^2} - \omega^2 r = 0$ where a and b are constant vectors.

2. Differentiate $\left(r \cdot \frac{dr}{dt} \right)$ with respect to t .

3. Expand $\frac{d}{dt} (p \times (q \times r))$.

4. Show that $\frac{d}{dt} \left(\frac{r}{r} \right) = \frac{1}{r} \frac{dr}{dt} - \frac{1}{r^2} \frac{dr}{dt} r$.

5. If $\frac{da}{dt} = \omega \times a$ and $\frac{db}{dt} = \omega \times b$ show that $\frac{d}{dt} (a \times b) = \omega \times (a \times b)$.

6. Show that $\frac{d}{dt} \left(r \times \frac{dr}{dt} \right) = r \times \frac{d^2 r}{dt^2}$.

7. If $f = 5t^2 i + tj - t^3 k$ and $g = \sin t i - \cos t j$ find

i) $\frac{d}{dt} (f \cdot g)$ ii) $\frac{d}{dt} (f \times g)$ iii) $\frac{d}{dt} (f \cdot f)$

In differential calculus we have introduced the operator $\frac{d}{dx}$. When applied to a differentiable function $f(x)$ it yields another function $\frac{df}{dx}$. In this section we introduce another operator ∇ (to be read as **del**) given by

$$\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$$

Definition: Let $\phi(x, y, z)$ be a real valued function having continuous first order partial derivatives. We define

$$\nabla \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} = \sum i \frac{\partial \phi}{\partial x}$$

$\nabla \phi$ is called gradient of ϕ and is denoted by $\text{grad } \phi$. Thus the gradient of a scalar function ϕ is a vector valued function.

Example. If $\phi(x, y, z) = xy^2 + yz^3$ then

$$\begin{aligned} \text{grad } \phi = \nabla \phi &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (xy^2 + yz^3) \\ &= y^2 \mathbf{i} + (2xy + z^3) \mathbf{j} + 3yz^2 \mathbf{k} \end{aligned}$$

Geometrical interpretation

Let $\phi(x, y, z)$ be scalar valued function having continuous partial derivatives. Let $P(x_0, y_0, z_0)$ be any point. Let $\phi(x_0, y_0, z_0) = c$.

Then the equation $\phi(x, y, z) = c$ represents a surface. Obviously (x_0, y_0, z_0) lies on this surface. Along this surface $\phi = c$.

$$\text{(i.e.) } \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0$$

$$\text{(i.e.) } \nabla \phi \cdot d\mathbf{r} = 0 \text{ where } d\mathbf{r} = \mathbf{i} dx + \mathbf{j} dy + \mathbf{k} dz.$$

$\therefore \nabla \phi$ is perpendicular to $d\mathbf{r}$ as long as $d\mathbf{r}$ represents a change from P to Q where Q remains on the surface $\phi(x, y, z) = c$.

$\therefore \nabla \phi$ is normal to all the tangents to the surface at $P(x_0, y_0, z_0)$.

Hence $\nabla \phi$ represents the normal to the surface $\phi(x, y, z) = c$.

Hence the unit normal $\mathbf{n}$ to the surface $\phi(x, y, z) = c$ is given by $\mathbf{n} = \frac{\nabla \phi}{|\nabla \phi|}$

Definition. Let $\mathbf{a}$ be a unit vector. The component of the vector $\nabla \phi$ in the direction $\mathbf{a}$ is given by $\mathbf{a} \cdot \nabla \phi$ and is called the **directional derivative** of ϕ in the direction by $\mathbf{a}$. This can be interpreted as the rate of change of ϕ at (x, y, z) in the direction $\mathbf{a}$.

Note. Let $P = (x, y, z)$ and $Q = (x + \Delta x, y + \Delta y, z + \Delta z)$ be two neighbouring points and Δs be the distance between P and Q .

$$\text{Then } \frac{d\phi}{ds} = \frac{\partial \phi}{\partial x} \frac{dx}{ds} + \frac{\partial \phi}{\partial y} \frac{dy}{ds} + \frac{\partial \phi}{\partial z} \frac{dz}{ds} = \frac{dr}{ds} \cdot \nabla \phi$$

Since $\frac{dr}{ds}$ is a unit vector $\frac{dr}{ds} \cdot \nabla \phi$ is the directional derivative of ϕ in the direction of $\frac{dr}{ds}$

$\therefore \frac{d\phi}{ds} = \frac{dr}{ds} \cdot \nabla \phi$ has a maximum value when $\nabla \phi$ and $\frac{dr}{ds}$ have the same directions.

$\therefore$ The maximum value of the directional derivative takes place in the direction of $\nabla \phi$ and its magnitude is $|\nabla \phi|$.

Equation of the tangent plane to the surface $\phi(x, y, z) = c$ at a point $A(x_0, y_0, z_0)$.

Let $P(x, y, z)$ be any point on the tangent plane whose position vector is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

A is the point of contact of the tangent plane with the surface whose position vector is $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$

Then $\mathbf{r} - \mathbf{r}_0$ is a vector on the tangent plane. $(\nabla \phi)$ at (x_0, y_0, z_0) is the normal to the surface and hence perpendicular to the tangent plane.

$$\text{At } (x_0, y_0, z_0) \quad (\mathbf{r} - \mathbf{r}_0) \cdot (\nabla \phi) = 0 \dots\dots\dots(1)$$

Since it is true for all points $\mathbf{r}$ on the tangent plane (1) represents the equation of the tangent plane.

Equation of the normal line.

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be any point on the normal line at $A(x_0, y_0, z_0)$ whose position vector is $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$. Hence $\mathbf{r} - \mathbf{r}_0$ lies along the normal line at A . Hence $\nabla \phi$ at (x_0, y_0, z_0) is parallel to $\mathbf{r} - \mathbf{r}_0$ so that

$$\text{at } (x_0, y_0, z_0) \quad (r-r_0) \times \nabla \phi = 0. \quad \dots\dots\dots(2)$$

Since it is true for all points r on the normal line (2) represents the equation of the normal.

Equation of the (i) tangent line (ii) normal plane at a given point $A(x_0, y_0, z_0)$ of the curve which is the intersection of the two surfaces $\phi(x, y, z) = c_1$ and $\psi(x, y, z) = c_2$.

i) Let C be the curve along which the two surfaces intersect.

Let $A(x_0, y_0, z_0)$ be a point on C whose position vector is $r_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$. Let $r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be any point on the tangent line at A to the curve C .

$\nabla \phi$ at (x_0, y_0, z_0) and $\nabla \psi$ at (x_0, y_0, z_0) represent the normals to the surface $\phi = C_1$ and $\psi = C_2$ respectively and both these are perpendicular to the tangent line at A .

$\therefore (r-r_0)$ is parallel to $(\nabla \phi \times \nabla \psi)$ at (x_0, y_0, z_0) so that at (x_0, y_0, z_0) , $(r-r_0) \times (\nabla \phi \times \nabla \psi) = 0$ represents the equation of the tangent lines at A .

(ii) Also the equation of the normal plane at (x_0, y_0, z_0) is given by $(r-r_0) \times (\nabla \phi \times \nabla \psi) = 0$

Theorem 5.6 $\text{grad } (\phi \pm \psi) = \text{grad } \phi \pm \text{grad } \psi$

Proof. $\text{Grad } (\phi \pm \psi) = \nabla (\phi \pm \psi) = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (\phi \pm \psi)$

$$= \left(i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \right) \pm \left(i \frac{\partial \psi}{\partial x} + j \frac{\partial \psi}{\partial y} + k \frac{\partial \psi}{\partial z} \right)$$

$$= \nabla \phi \pm \nabla \psi$$

$$= \text{grad } \phi \pm \text{grad } \psi$$

Theorem 5.7 $\text{grad } (\phi \psi) = \phi \text{ grad } \psi + \psi \text{ grad } \phi$

Proof. $\text{Grad } (\phi \psi) = \sum i \left(i \frac{\partial}{\partial x} (\phi \psi) \right) = \sum i \left(\phi \frac{\partial \psi}{\partial x} + \psi \frac{\partial \phi}{\partial x} \right)$

$$= \sum i \phi \frac{\partial \psi}{\partial x} + \sum i \psi \frac{\partial \phi}{\partial x} = \left(\sum i \frac{\partial \psi}{\partial x} \right) \phi + \psi \left(\sum i \frac{\partial \phi}{\partial x} \right)$$

$$= \phi (\nabla \psi) + \psi (\nabla \phi) = \phi \text{ grad } \psi + \psi \text{ grad } \phi.$$

Theorem 5.8 $\text{grad} \left(\frac{\phi}{\psi} \right) = (\psi \text{ grad } \phi - \phi \text{ grad } \psi) / \psi^2$

Proof. $\text{grad} \left(\frac{\phi}{\psi} \right) = \nabla \left(\frac{\phi}{\psi} \right) = \sum i \frac{\partial}{\partial x} \left(\frac{\phi}{\psi} \right) = \sum i \left[\frac{\psi \frac{\partial \phi}{\partial x} - \phi \frac{\partial \psi}{\partial x}}{\psi^2} \right]$

$$= \frac{1}{\psi^2} \left[\sum i \psi \frac{\partial \phi}{\partial x} - \sum i \phi \frac{\partial \psi}{\partial x} \right]$$

$$= (\psi \text{ grad } \phi - \phi \text{ grad } \psi) / \psi^2$$

Solved problems

Problem 1. If r is position vector of any point $P(x, y, z)$ prove that $\text{grad } r^n = nr^{n-2}r$.

Solution $r = xi + yj + zk \dots$. Hence $r^2 = x^2 + y^2 + z^2$.

$$\therefore 2r \left(\frac{dr}{dx} \right) = 2x. \text{ Hence } \frac{dr}{dx} = \frac{x}{r}. \text{ similarly, } \frac{dr}{dy} = \frac{y}{r} \text{ and } \frac{dr}{dz} = \frac{z}{r}.$$

$$\begin{aligned} \text{Now, grad } r^n &= \nabla r^n = i \frac{dr^n}{dx} + j \frac{dr^n}{dy} + k \frac{dr^n}{dz} \\ &= i nr^{n-1} \frac{dr}{dx} + j nr^{n-1} \frac{dr}{dy} + k nr^{n-1} \frac{dr}{dz} \\ &= nr^{n-1} \left[i \left(\frac{x}{r} \right) + j \left(\frac{y}{r} \right) + k \left(\frac{z}{r} \right) \right] = nr^{n-2} (xi + yj + zk) \\ &= nr^{n-2} r. \end{aligned}$$

Problem 2. Prove that $\nabla f(r) = \left(\frac{f'(r)}{r} \right) r$.

Solution. $\nabla f(r) = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) f(r) = if'(r) \frac{\partial r}{\partial x} + jf'(r) \frac{\partial r}{\partial y} + kf'(r) \frac{\partial r}{\partial z}$

$$= f'(r) \left[i \left(\frac{x}{r} \right) + j \left(\frac{y}{r} \right) + k \left(\frac{z}{r} \right) \right] \text{ (refer problem 1)}$$

$$= \frac{f'(r)}{r} (xi + yj + zk) = \left(\frac{f'(r)}{r} \right) r.$$

Problem 3. If $\phi(x, y) = \log \sqrt{x^2 + y^2}$ show that

$$\text{grad } \phi = \frac{r - (k.r)k}{\{r - (k.r)k\} \cdot \{r - (k.r)k\}}.$$

Solution. $\text{grad } \phi = \nabla \phi = \nabla \log \sqrt{x^2 + y^2} = \frac{1}{2} \nabla \log (x^2 + y^2)$

$$= \frac{1}{2} \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \log (x^2 + y^2)$$

$$= \frac{1}{2} \left[i \left(\frac{2x}{x^2 + y^2} \right) + j \left(\frac{2y}{x^2 + y^2} \right) + k(0) \right]$$

$$= \frac{xi + yj}{(xi + yj) \cdot (xi + yj)}$$

$$= \frac{r - zk}{(r - zk) \cdot (r - zk)} \quad (\text{since } r = xi + yj + zk)$$

$$= \frac{r - (k.r)k}{\{r - (k.r)k\} \cdot \{r - (k.r)k\}} \quad (\text{since } k.r = z)$$

Problem 4. Show that $\nabla (a.r) = a$ for any constant vector a .

Solution. Let $a = a_1 i + a_2 j + a_3 k$

$$\therefore a.r = a_1 x + a_2 y + a_3 z$$

$$\nabla (a.r) = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (a_1 x + a_2 y + a_3 z) = a_1 i + a_2 j + a_3 k = a.$$

Problem 5. Obtain the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of $i + 2j + 2k$.

Solution. $\nabla \phi = y^2 i + (2xy + z^3) j + 3yz^2 k$

At $(2, -1, 1)$ we get $\nabla \phi = i - 3j - 3k$.

The unit vector of the given direction a is $(i + 2j + 2k)/3$

$\therefore$ The required directional derivative is $a \cdot \nabla \phi = \frac{-11}{3}$

Problem 6. In what direction from the point (1, 3, 2) is the directional derivative of $\phi = 2xz - y^2$ maximum? What is the magnitude of this maximum?

Solution. $\nabla \phi = 2z\mathbf{i} - 2y\mathbf{j} + 2x\mathbf{k}$.

$$\therefore \text{At } (1, 3, 2) \nabla \phi = 4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}.$$

$\therefore$ The directional derivative is maximum in the direction $4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$.

The magnitude of the maximum is $|\nabla \phi| = 2\sqrt{14}$

Problem 7. Find the unit normal to the surface $x^3 - xyz + z^3 = 1$ at (1, 1, 1)

Solution. Let $\phi = x^3 - xyz + z^3 = 1$

Let $\mathbf{n}$ denote the unit normal to the surface.

$$\text{Then } \mathbf{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

$$\text{Now } \nabla \phi = (3x^2 - yz)\mathbf{i} - xz\mathbf{j} + (3z^2 - xy)\mathbf{k}.$$

$$\therefore \nabla \phi \text{ at } (1, 1, 1) = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}. \text{ Hence } \mathbf{n} = \frac{1}{3}(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

Problem 8. If $\nabla \Phi = 2xyz^3\mathbf{i} + x^2z^3\mathbf{j} + 3x^2yz^2\mathbf{k}$ then find $\Phi(x, y, z)$ if $\Phi(1, -2, 2) = 4$.

Solution. $\nabla \Phi = \mathbf{i} \frac{\partial \Phi}{\partial x} + \mathbf{j} \frac{\partial \Phi}{\partial y} + \mathbf{k} \frac{\partial \Phi}{\partial z} = 2xyz^3\mathbf{i} + x^2z^3\mathbf{j} + 3x^2yz^2\mathbf{k}.$

$$\therefore \frac{\partial \Phi}{\partial x} = 2xyz^3 \quad \dots \dots \dots (1)$$

$$\frac{\partial \Phi}{\partial y} = x^2z^3 \quad \dots \dots \dots (2)$$

$$\frac{\partial \Phi}{\partial z} = 3x^2yz^2 \quad \dots \dots \dots (3)$$

Integrating (1), (2), (3) w.r.t. x, y, z respectively we get

$$\Phi = yz^3x^2 + f(y, z); \quad \Phi = x^2yz^3 + g(x, z); \quad \Phi = x^2yz^3 + h(x, y)$$

$$\therefore \Phi = x^2yz^3 + k \text{ where } k \text{ is a constant}$$

$$\text{Given } \phi(1, -2, 2) = 4. \text{ Hence } 4 = -16 + k. \text{ Hence } k = 20.$$

$$\therefore \Phi(x, y, z) = x^2yz^3 + 20.$$

Problem. 9. If $\nabla \Phi = (y + \sin z) \mathbf{i} + x\mathbf{j} + x \cos z \mathbf{k}$ find Φ

Solution. $\nabla \Phi = \mathbf{i} \frac{\partial \Phi}{\partial x} + \mathbf{j} \frac{\partial \Phi}{\partial y} + \mathbf{k} \frac{\partial \Phi}{\partial z} = (y + \sin z)\mathbf{i} + x\mathbf{j} + x \cos z \mathbf{k}.$

$$\therefore \frac{\partial \Phi}{\partial x} = y + \sin z \quad \dots\dots\dots (1)$$

$$\frac{\partial \Phi}{\partial y} = x \quad \dots\dots\dots (2)$$

$$\frac{\partial \Phi}{\partial z} = x \cos z \quad \dots\dots\dots (3)$$

Integrating (1), (2), (3) w.r.t.x,y,z respectively we get

$$\Phi(x,y,z) = xy + x \sin z + f(y,z) \quad \dots\dots\dots (4)$$

$$\Phi(x,y,z) = xy + g(x,z) \quad \dots\dots\dots (5)$$

$$\Phi(x,y,z) = x \sin z + h(xy) \quad \dots\dots\dots (6)$$

From (4), (5), (6) we see that $\Phi(x,y,z) = xy + x \sin z + k$ where k is a constant.

Problem 10. Find the equation of the (i) tangent plane and (ii) normal line to the surface $xyz = 4$ at the point $(1,2,2)$

Solution. Let $\phi(x, y, z) = 4$

$$\nabla \phi = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

$$\text{At } (1, 2, 2) \nabla \phi = 4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}.$$

$$\text{The position vector of } (1,2,2) \text{ is } \mathbf{r}_0 = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}.$$

$\therefore$ The equation of the tangent plane is given by $(\mathbf{r} - \mathbf{r}_0) \cdot \nabla \phi = 0$.

$$\therefore [(x-1)\mathbf{i} + (y-2)\mathbf{j} + (z-2)\mathbf{k}] \cdot (4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = 0$$

$$\therefore 4(x-1) + 2(y-2) + 2(z-2) = 0$$

$$\therefore 2x + y + z = 6.$$

ii) The equation of the normal line at $(1,2,2)$ is given by $(\mathbf{r} - \mathbf{r}_0) \times (\nabla \phi) = 0$

$$\therefore \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x-1 & y-2 & z-2 \\ 4 & 2 & 2 \end{vmatrix} = 0$$

$$[2(y-2) - 2(z-2)]\mathbf{i} - [2(x-1) - 4(z-2)]\mathbf{j} + [2(x-1) - 4(y-2)]\mathbf{k} = 0.$$

Equating the coefficients of **i, j, k** on both sides.

$(y-2) = (z-2)$; $(x-1) = 2(z-2)$; $(x-1) = 2(y-2)$ which can be written in symmetric form in rectangular Cartesian coordinates as

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-2}{1}.$$

Problem 11. Find the equation of the (i) tangent line and (ii) normal plane to the curve of intersection of the surfaces $3x^2 + y^2z + 2 = 0$; $2xz - x^2y - 3 = 0$ at the point $(1, -1, 1)$

Solution. Let $\phi = 3x^2 + y^2z + 2$; $\psi = 2xz - x^2y - 3$

$$\nabla \phi = 6xy \mathbf{i} + (3x^2 + 2yz) \mathbf{j} + y^2 \mathbf{k}; \text{ At } (1, -1, 1), \nabla \phi = 6\mathbf{i} + \mathbf{j} + \mathbf{k}.$$

$$\nabla \psi = (2z - 2xy) \mathbf{i} - x^2 \mathbf{j} + 2x \mathbf{k}; \text{ At } (1, -1, 1), \nabla \psi = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}.$$

The equation of the tangent line at $(1, -1, 1)$ is given by $(\mathbf{r} - \mathbf{r}_0) \times (\nabla \phi \times \nabla \psi) = 0$.

$$\text{Now at } (1, -1, 1) \quad \nabla \phi \times \nabla \psi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & 1 & 1 \\ 4 & -1 & 2 \end{vmatrix} = 3\mathbf{i} + 16\mathbf{j} + 2\mathbf{k}.$$

$\therefore$ Equation of the tangent line at $(1, -1, 1)$ becomes

$$\{(x-1)\mathbf{i} + (y+1)\mathbf{j} + (z-1)\mathbf{k}\} \times (3\mathbf{i} + 16\mathbf{j} + 2\mathbf{k}) = 0.$$

$$\therefore \frac{x-1}{3} = \frac{y+1}{16} = \frac{z-1}{2} \text{ is the equation of the tangent line at } (1, -1, 1) \text{ is}$$

$$(\mathbf{r} - \mathbf{r}_0) \times (\nabla \phi \times \nabla \psi) = 0$$

$$\therefore \{(x-1)\mathbf{i} + (y+1)\mathbf{j} + (z-1)\mathbf{k}\} \cdot (3\mathbf{i} + 16\mathbf{j} + 2\mathbf{k}) = 0$$

$$\text{(i.e.) } 3(x-1) + 16(y+1) + 2(z-1) = 0.$$

$$\therefore 3x + 16y + 2z + 11 = 0 \text{ is the equation of the normal plane at } (1, -1, 1).$$

Problem 12. Find the angle between the surfaces $x^2 + y^2 + z^2 = 29$ and $x^2 + y^2 + z^2 + 4x - 6y - 8z - 47 = 0$ at $(4, -3, 2)$.

$$\text{Solution Let } \phi(x, y, z) = x^2 + y^2 + z^2 = 29 \dots\dots\dots (1)$$

$$\psi(x, y, z) = x^2 + y^2 + z^2 + 4x - 6y - 8z - 47 \dots\dots\dots (2)$$

$$\nabla \phi = 2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k}; \nabla \phi \text{ at } (4, -3, 2) = 8\mathbf{i} - 6\mathbf{j} + 4\mathbf{k} \dots\dots\dots (3)$$

$$\nabla \psi = (2x + 4) \mathbf{i} + (2y - 6) \mathbf{j} + (2z - 8) \mathbf{k}$$

$$\nabla \psi \text{ at } (4, -3, 2) = 12\mathbf{i} - 12\mathbf{j} - 4\mathbf{k} \dots\dots\dots (4)$$

Space for Hints

We know that the angle between two surfaces is the angle between the tangent planes at a common point and hence the angle between the normals at that point

Equation (3) and (4) represent the normal to the surfaces (1) and (2) at (4, -3, 2) respectively.

Let θ be the angle between the normal (3) and (4) at (4, -3, 2)

$$\begin{aligned}\therefore \cos \theta &= \frac{\nabla \phi \cdot \nabla \psi}{|\nabla \phi| |\nabla \psi|} = \frac{96 + 72 - 16}{\sqrt{8^2 + (-6)^2 + 4^2} \sqrt{12^2 + (-12)^2 + (-4)^2}} \\ &= \frac{152}{\sqrt{116} \sqrt{304}} = \frac{19}{\sqrt{29} \sqrt{19}} = \sqrt{19/29} \\ \therefore \theta &= \cos^{-1} \sqrt{19/29}\end{aligned}$$

Problem 13. Determine the constants a and b so that the surface $5x^2 - 2yz - 9x = 0$ will be orthogonal to the surface $ax^2 + bz^2 = 4$ at the point (1, -1, 2).

Solution Let $\phi(x, y, z) = 5x^2 - 2yz - 9x$ and $\psi(x, y, z) = ax^2 + bz^2 - 4$.

$$\therefore \nabla \phi = (10x - 9)\mathbf{i} - 2z\mathbf{j} - 2y\mathbf{k} \text{ and } \nabla \psi = 2ax\mathbf{i} + 2bz\mathbf{k}.$$

The two surfaces will be orthogonal at (1, -1, 2) if the surface normals to the two surfaces at (1, -1, 2) are perpendicular.

$$\text{Hence } \nabla \phi \cdot \nabla \psi = 0 \text{ at } (1, -1, 2)$$

$$\therefore (\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) \cdot (-2a\mathbf{i} + 4b\mathbf{k}) = 0.$$

$$\therefore -2a - 4a + 8b = 0 \text{ (i.e.) } 8b - 6a = 0 \quad \dots\dots\dots (1)$$

Further (1, -1, 2) lies on both the surfaces.

$$\text{Taking } \psi(1, -1, 2) = 0 \text{ we have } -a + 4b = 4 \quad \dots\dots\dots (2)$$

Solving (1) and (2) we get $a = 2$ and $b = 3/2$.

Exercises

1. Find grad Φ for the following at the points indicated.

i] $\Phi(x, y, z) = zx - y^2$ at (a, b, c)

ii] $\Phi(x, y, z) = xyz$ at (x_1, y_1, z_1)

iii] $\Phi(x, y, z) = x^2 + y - z$ at (1, 1, 1)

iv] $\Phi(x, y, z) = x^3 + y^3 - z^3 - 3xyz$ at (1, 1, 1)

v] $\Phi(x, y, z) = \log(x^2 + y^2 - z^2)$ at (x_1, y_1, z_1)

vi] $\Phi(x, y, z) = 2xz^4 - x^2y$ at (2, -2, -2)

2. If $\Phi = x^2z + e^{y/x}$ and $\psi = 2z^2y - xy^2$ find $\nabla (\Phi + \psi)$ and $\nabla (\Phi - \psi)$ at $(1, 0, 2)$
3. Prove that $\nabla |r|^3 = 3r$ where $r = xi + yj + zk$.
4. If $\nabla \Phi = yzi + zxj + xyk$ find Φ
5. Find Φ if
 - i) $\nabla \Phi = (y^2 - 2xyz^3) \mathbf{i} + (3 + 2xy - x^2z^3) \mathbf{j} + (6z^3 - 3x^2yz^2) \mathbf{k}$
 - ii) $\nabla \Phi = (2x + 3y + 4xy^2) \mathbf{i} + (3x + 3y + 4x^2 + z + 1) \mathbf{j} + (y + 2z) \mathbf{k}$
6. $\nabla \Phi = 2|r|^4 r$ find Φ
7. Find the unit vector normal to the surface $x^2 - y^2 + z = 2$ at the point $(1, -1, 2)$
8. Find the unit vector which is perpendicular to the surface $z = x^2 + y^2$ at the point $(1, 2, 5)$
9. Find the unit normal to the surface $xy^2z^2 = 4$ at $(-1, -1, 2)$
10. Find the directional derivative of $\psi = xy + yz + zx$ at $(1, 2, 3)$ in the direction of $3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$.
11. Find the directional derivative of $\psi = x^2 - 2y^2 + 4z^2$ at $(1, 1, -1)$ in the direction of $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.
12. Find the directional derivative of $\psi = 4e^{2x-y+z}$ at the point $(1, 1, -1)$ in the direction towards the point $(-3, 5, 6)$.
13. In what direction from the point $(2, 1, -1)$ is the directional derivative of $\Phi = x^2yz^3$ a maximum. Find the magnitude of maximum.
14. If Φ is a differentiable function of xyz prove that $\nabla \Phi \cdot dr = d\Phi$
15. If $\mathbf{a}$ is a constant vector prove that $\nabla (\mathbf{r} \cdot \mathbf{a}) = \mathbf{a}$.
16. Prove that $\text{grad} [\mathbf{r} \cdot \mathbf{a} \times \mathbf{b}] = \mathbf{a} \times \mathbf{b}$.
17. Prove that $\nabla r^2 = 2\mathbf{r}$ where $r^2 = \mathbf{r} \cdot \mathbf{r}$
18. Find the equation of the tangent plane at the origin to the surface $x^2 + y^2 + z^2 + 8x - 6y + 4z = 0$.
19. Find the equation of the tangent plane to the surface $2xz^2 - 3xy - z^2 + 4x = 7$ at the point $(1, -1, 2)$.
20. Find the equation of the (i) tangent plane ii) normal line to the surface $x^2yz + 3y^2 - 2xz^2 + 8z = 0$ at the point $(1, 2, -1)$.

Space for Hints

21. Find the equation of the (i) tangent plane ii) normal line to the curve of intersection of the surface $x + y + z = 1$ and $x^2 + y^2 + z^2 = 1$ at $(1, 0, 0)$.
22. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.
23. Show that the surface $5x^2 - 2yz - 9z = 0$ is orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$.
24. Show that the surface $x^2 - 2yz + y^3 = 4$ is perpendicular to any member of the family of surfaces $x^2 + 1 = (2-4a)y^2 + az^2$ at the point of intersection $(1, -1, 2)$.
25. The surfaces $x + y + z = 3$ and $x^2 - y^2 + 2z^2 = 2$ intersect in a space curve. Find the equations of the (i) tangent line (ii) normal plane to this space curve at $(1, 1, 1)$.

Section 5.4 Divergence and Curl

Definition. Let $f = f_1i + f_2j + f_3k$ be a vector valued function. The divergence of f denoted by $\nabla \cdot f$ or $\text{div } f$ is defined by

$$\nabla \cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \sum i \cdot \frac{\partial f}{\partial x}$$

The curl of f denoted by $\nabla \times f$ or **curl** f is defined by

$$\text{curl } f = \sum i \times \frac{\partial f}{\partial x} = i \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) + j \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) + k \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Note 1. The divergence of a vector valued function is scalar valued function.

Note 2. The curl of a vector valued function is vector valued function.

Note 3. If $a = a_1i + a_2j + a_3k$ the symbol $a \cdot \nabla$ stands for the operator

$$a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}$$

Examples 1. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$$\text{div } \mathbf{r} = \nabla \cdot \mathbf{r} = 1 + 1 + 1 = 3 \text{ and } \text{curl } \mathbf{r} = 0$$

2. Let $\mathbf{f} = xz^3 \mathbf{i} - 2x^2yz\mathbf{j} + 2yz^4 \mathbf{k}$.

$$\text{Then } \nabla \cdot \mathbf{f} = z^3 - 2x^2z + 8yz^3 \text{ and}$$

$$\nabla \cdot \mathbf{f} = (2z^4 + 2x^2y) \mathbf{i} + 3xz^2\mathbf{j} - 4xyz \mathbf{k} \text{ (verify)}$$

Definition.

A vector $\mathbf{f}$ is called solenoidal if $\text{div } \mathbf{f} = 0$.

A vector $\mathbf{f}$ is called irrotational if $\text{curl } \mathbf{f} = 0$.

Theorem 5.9 $\text{div } (\mathbf{f} + \mathbf{g}) = \text{div } \mathbf{f} + \text{div } \mathbf{g}$

Proof. Let $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ and $\mathbf{g} = g_1\mathbf{i} + g_2\mathbf{j} + g_3\mathbf{k}$.

$$\begin{aligned} \text{div } (\mathbf{f} + \mathbf{g}) &= \frac{\partial}{\partial x} (f_1 + g_1) + \frac{\partial}{\partial y} (f_2 + g_2) + \frac{\partial}{\partial z} (f_3 + g_3) \\ &= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) + \left(\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z} \right) \\ &= \nabla \cdot \mathbf{f} + \nabla \cdot \mathbf{g} = \text{div } \mathbf{f} + \text{div } \mathbf{g} \end{aligned}$$

Theorem 5.10 Let $\mathbf{f}$ be a vector valued function and Φ a scalar valued function.

$$\text{Then } \nabla \cdot (\Phi \mathbf{f}) = (\nabla \Phi) \cdot \mathbf{f} + (\nabla \cdot \mathbf{f}) \Phi$$

$$\text{(i.e.) } \text{div } (\Phi \mathbf{f}) = (\text{grad } \Phi) \cdot \mathbf{f} + (\text{div } \mathbf{f}) \Phi$$

Proof. Let $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$

$$\begin{aligned} \nabla \cdot (\Phi \mathbf{f}) &= \nabla \cdot (\Phi f_1 \mathbf{i} + \Phi f_2 \mathbf{j} + \Phi f_3 \mathbf{k}) \\ &= \frac{\partial}{\partial x} (\Phi f_1) + \frac{\partial}{\partial y} (\Phi f_2) + \frac{\partial}{\partial z} (\Phi f_3) \\ &= \left(\frac{\partial \Phi}{\partial x} f_1 + \Phi \frac{\partial f_1}{\partial x} \right) + \left(\frac{\partial \Phi}{\partial y} f_2 + \Phi \frac{\partial f_2}{\partial y} \right) + \left(\frac{\partial \Phi}{\partial z} f_3 + \Phi \frac{\partial f_3}{\partial z} \right) \\ &= \left(\frac{\partial \Phi}{\partial x} \mathbf{i} + \frac{\partial \Phi}{\partial y} \mathbf{j} + \frac{\partial \Phi}{\partial z} \mathbf{k} \right) \cdot (f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k}) + \Phi \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) \\ &= (\nabla \Phi) \cdot \mathbf{f} + (\nabla \cdot \mathbf{f}) \Phi. \end{aligned}$$

Theorem 5.11 $\nabla \cdot (f \times g) = g \cdot (\nabla \times f) - f \cdot (\nabla \times g)$

(i.e.) $\text{div } (f \times g) = g \cdot \text{curl } f - f \cdot \text{curl } g$.

Proof. $\text{div } (f \times g) = \nabla \cdot (f \times g) = \sum \mathbf{i} \cdot \left(\frac{\partial}{\partial x} (f \times g) \right)$

$$= \sum \mathbf{i} \cdot \left(\frac{\partial f}{\partial x} \times g + f \times \frac{\partial g}{\partial x} \right) = \sum \mathbf{i} \cdot \left(\frac{\partial f}{\partial x} \times g \right) + \sum \mathbf{i} \cdot \left(f \times \frac{\partial g}{\partial x} \right)$$

$$= \sum \left(\mathbf{i} \frac{\partial f}{\partial x} \right) \cdot g - \sum \left(\mathbf{i} \times \frac{\partial g}{\partial x} \right) \cdot f$$

$$= (\nabla \times f) \cdot g - (\nabla \times g) \cdot f$$

Theorem 5.12 $\text{div grad } \Phi = \nabla \cdot \nabla \Phi$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Proof. $\nabla \Phi = \mathbf{i} \frac{\partial \Phi}{\partial x} + \mathbf{j} \frac{\partial \Phi}{\partial y} + \mathbf{k} \frac{\partial \Phi}{\partial z}$

$$\therefore \nabla \cdot \nabla \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = \nabla^2 \Phi$$

Note. The operator ∇^2 is called the Laplacian operator

When Φ is a scalar valued function $\nabla^2 \Phi$ is also a scalar valued function.

If $f = f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k}$ we define $\nabla^2 f = (\nabla^2 f_1) \mathbf{i} + (\nabla^2 f_2) \mathbf{j} + (\nabla^2 f_3) \mathbf{k}$

Theorem 5.13 $\text{curl } (f+g) = \text{curl } f + \text{curl } g$.

(i.e.) $\nabla \times (f+g) = \nabla \times f + \nabla \times g$.

Proof. $\nabla \times (f+g) = \sum \mathbf{i} \times \frac{\partial}{\partial x} (f+g) = \sum \mathbf{i} \times \frac{\partial f}{\partial x} + \sum \mathbf{i} \times \frac{\partial g}{\partial x}$

$$= \nabla \times f + \nabla \times g$$

Theorem 5.14 $\text{curl}(f \times g) = (g \cdot \nabla)f - (f \cdot \nabla)g - f \text{div } g - g \text{div } f$

Proof. $\text{curl}(f \times g) = \nabla \times (f \times g)$

$$\begin{aligned} \sum \mathbf{i} \times \frac{\partial}{\partial x} (f \times g) &= \sum \mathbf{i} \times \left[f \times \frac{\partial g}{\partial x} + \frac{\partial f}{\partial x} \times g \right] \\ &= \sum \mathbf{i} \times \left(f \times \frac{\partial g}{\partial x} \right) + \sum \mathbf{i} \times \left(\frac{\partial f}{\partial x} \times g \right) \\ &= \sum \left[\left(\mathbf{i} \cdot \frac{\partial g}{\partial x} \right) f - (\mathbf{i} \cdot f) \frac{\partial g}{\partial x} \right] + \sum \left[(\mathbf{i} \cdot g) \frac{\partial f}{\partial x} - \left(\mathbf{i} \cdot \frac{\partial f}{\partial x} \right) g \right] \\ &= f \sum \left(\mathbf{i} \cdot \frac{\partial g}{\partial x} \right) - g \sum \left(\mathbf{i} \cdot \frac{\partial g}{\partial x} \right) + \sum (\mathbf{i} \cdot g) \frac{\partial f}{\partial x} - \sum (\mathbf{i} \cdot f) \frac{\partial g}{\partial x} \\ &= f \text{div } g - g \text{div } f + (g \cdot \nabla)f - (f \cdot \nabla)g. \end{aligned}$$

Theorem 5.15 $\text{div curl } f = \nabla \cdot (\nabla \times f) = 0.$

$$\begin{aligned} \therefore \nabla \times f &= \mathbf{i} \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) - \mathbf{j} \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right) + \mathbf{k} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \\ \therefore \nabla \cdot (\nabla \times f) &= \frac{\partial}{\partial x} \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) - \frac{\partial}{\partial y} \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right) + \frac{\partial}{\partial z} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \\ &= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} - \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_1}{\partial y \partial z} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial z \partial y} = 0 \\ &= f \text{div } g - g \text{div } f + (g \cdot \nabla)f - (f \cdot \nabla)g. \end{aligned}$$

Theorem 5.16 $\text{curl grad } \Phi = \nabla (\nabla \times \Phi) = 0.$

Proof. $\nabla \Phi = \frac{\partial \Phi}{\partial x} \mathbf{i} + \frac{\partial \Phi}{\partial y} \mathbf{j} + \frac{\partial \Phi}{\partial z} \mathbf{k}$

$$\therefore \nabla \times (\nabla \Phi) = \mathbf{i} \left(\frac{\partial^2 \Phi}{\partial y \partial z} - \frac{\partial^2 \Phi}{\partial z \partial y} \right) - \mathbf{j} \left(\frac{\partial^2 \Phi}{\partial x \partial z} - \frac{\partial^2 \Phi}{\partial z \partial x} \right) + \mathbf{k} \left(\frac{\partial^2 \Phi}{\partial x \partial y} - \frac{\partial^2 \Phi}{\partial y \partial x} \right) = 0$$

Theorem 17 $\text{grad}(f \cdot g) = f \times \text{curl } g + g \times \text{curl } f + (f \cdot \nabla)g + (g \cdot \nabla)f.$

Proof. $\text{grad}(f \cdot g) = \sum \mathbf{i} \frac{\partial}{\partial x} (f \cdot g) = \sum \mathbf{i} \left[f \cdot \frac{\partial g}{\partial x} + \frac{\partial f}{\partial x} \cdot g \right]$

$$\sum f \left(\frac{\partial g}{\partial x} \right) \mathbf{i} + \sum \left(\frac{\partial f}{\partial x} - g \right) \mathbf{i} \dots \dots \dots (1)$$

$$\text{Now } f \times \left(\mathbf{i} \times \frac{\partial g}{\partial x} \right) = \left(f \times \frac{\partial g}{\partial x} \right) \mathbf{i} - (f \cdot \mathbf{i}) \frac{\partial g}{\partial x}$$

$$\therefore \left(f \cdot \frac{\partial g}{\partial x} \right) \mathbf{i} = f \times \left(\mathbf{i} \times \frac{\partial g}{\partial x} \right) + (f \cdot \mathbf{i}) \frac{\partial g}{\partial x}$$

$$\sum \left(f \cdot \frac{\partial g}{\partial x} \right) \mathbf{i} = \sum f \times \left(\mathbf{i} \times \frac{\partial g}{\partial x} \right) + \sum (f \cdot \mathbf{i}) \frac{\partial g}{\partial x} = f \times \left(\sum \mathbf{i} \times \frac{\partial g}{\partial x} \right) + \left(\sum f \cdot \mathbf{i} \times \frac{\partial g}{\partial x} \right)$$

$$f \times \left(\sum \mathbf{i} \times \frac{\partial g}{\partial x} \right) + \left(f \cdot \sum \mathbf{i} \times \frac{\partial}{\partial x} \right) g$$

$$= f \times \text{curl } g + (f \cdot \nabla) g \dots \dots \dots (2)$$

$$\text{Similarly } \sum \left(\frac{\partial f}{\partial x} \cdot g \right) \mathbf{i} = g \times \text{curl } f + (g \cdot \nabla) f \dots \dots (3)$$

Substituting (2) and (3) in (1) we get the result.

Theorem 5.18

$$\nabla \times (\Phi f) = \nabla \Phi \times f + \Phi (\nabla \times f)$$

$$(i.e) \text{curl } (\Phi f) = \text{grad } \Phi \times f + \Phi \text{curl } f$$

$$\text{Proof. } \text{curl } (\Phi f) = \nabla \times (\Phi f) \sum \left[\mathbf{i} \times \frac{\partial}{\partial x} \right] (\Phi f) = \sum \left[\mathbf{i} \times \left(\frac{\partial \Phi}{\partial x} f + \Phi \frac{\partial f}{\partial x} \right) \right]$$

$$= \left[\sum \frac{\partial \Phi}{\partial x} \mathbf{i} \right] \times f + \Phi \sum \left(\mathbf{i} \times \frac{\partial f}{\partial x} \right)$$

$$= \nabla \Phi \times f + \Phi (\nabla \times f)$$

Theorem 5.19

$$\text{curl } (\text{curl } f) = \text{grad div } f - \nabla^2 f$$

$$(i.e) \nabla \times (\nabla \times f) = \nabla (\nabla \cdot f) - \nabla^2 f$$

Proof. Let $f = f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k}$

$$\therefore \nabla \times f = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \mathbf{k}$$

$$\nabla \times (\nabla \times f) = \sum \left\{ \frac{\partial}{\partial y} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \right\}$$

$$= \sum \left\{ \left(\frac{\partial^2 f_2}{\partial y \partial x} + \frac{\partial^2 f_3}{\partial z \partial x} \right) - \left(\frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2} \right) \right\} \mathbf{i}$$

$$\begin{aligned}
&= \sum \left\{ \frac{\partial}{\partial x} \left(\frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) - \left(\frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2} \right) \right\} \mathbf{i} \\
&= \sum \left\{ \frac{\partial}{\partial x} \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) - \left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial z^2} \right) \right\} \mathbf{i} \\
&= \sum \left\{ \frac{\partial}{\partial x} (\nabla \cdot \mathbf{f}) - (\nabla^2 \cdot f_1) \right\} \mathbf{i} \\
&= \sum \left\{ \frac{\partial}{\partial x} (\nabla \cdot \mathbf{f}) \mathbf{i} \right\} - \sum (\nabla^2 f_1) \mathbf{i} \\
&= \nabla (\nabla \cdot \mathbf{f}) - \nabla^2 \mathbf{f}.
\end{aligned}$$

Definition. A vector $\mathbf{f}$ is called a harmonic vector if $\nabla^2 \mathbf{f} = 0$

Corollary. If $\mathbf{f}$ is a harmonic vector then $\nabla \times (\nabla \times \mathbf{f}) = \nabla (\nabla \cdot \mathbf{f})$

Proof. $\nabla \times (\nabla \times \mathbf{f}) = \nabla (\nabla \cdot \mathbf{f}) - \nabla^2 \mathbf{f} = \nabla (\nabla \cdot \mathbf{f})$ (since $\mathbf{f}$ is harmonic)

Solved problems.

Problem 1. Find curl curl $\mathbf{f}$ at the point $(1, 1, 1)$ if $\mathbf{f} = x^2 y \mathbf{i} + xz \mathbf{j} + 2yz \mathbf{k}$

Solution. $\text{curl } \mathbf{f} = \nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & xz & 2yz \end{vmatrix} = (2z - x) \mathbf{i} + (z - x^2) \mathbf{k}$

$$\therefore \text{curl curl } \mathbf{f} = \nabla \times (\nabla \times \mathbf{f}) = (2x + 2) \mathbf{j}.$$

$$\therefore \text{At } (1, 1, 1), \nabla \times (\nabla \times \mathbf{f}) = 4\mathbf{j}.$$

Problem 2. Prove that $\text{div } \mathbf{r} = 3$ and $\text{curl } \mathbf{r} = 0$ where $\mathbf{r}$ is the position vector of a point (x, y, z) in space.

Solution. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$$\begin{aligned}
\text{div } \mathbf{r} &= \nabla \cdot \mathbf{r} = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\
&= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3
\end{aligned}$$

$$\begin{aligned}
\text{curl } \mathbf{r} &= \nabla \times \mathbf{r} = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\
&= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0
\end{aligned}$$

Problem 3. Show that $\text{div} \left(\frac{\mathbf{r}}{r} \right) = \frac{2}{r}$

Solution. $\text{div} \left(\frac{\mathbf{r}}{r} \right) = \nabla \cdot \left(\frac{\mathbf{r}}{r} \right) = \nabla \cdot \left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{r} \right)$

$$= \frac{\partial}{\partial x} \left(\frac{x}{r} \right) + \frac{\partial}{\partial y} \left(\frac{y}{r} \right) + \frac{\partial}{\partial z} \left(\frac{z}{r} \right)$$

$$= \frac{1}{r} - \frac{x}{r^2} \frac{\partial r}{\partial x} + \frac{1}{r} - \frac{y}{r^2} \frac{\partial r}{\partial y} + \frac{1}{r} - \frac{z}{r^2} \frac{\partial r}{\partial z}$$

$$= \frac{3}{r} - \frac{x^2}{r^3} - \frac{y^2}{r^3} - \frac{z^2}{r^3} \left(\text{Since } \frac{\partial r}{\partial x} = \frac{x}{r} \text{ etc} \right)$$

$$= \frac{3}{r} - \frac{1}{r} \left(\text{Since } x^2 + y^2 + z^2 = r^2 \right)$$

$$= \frac{2}{r}$$

Problem 4. Prove that $\text{div} (r^n \mathbf{r}) = (n+3)r^n$. Deduce that $r^n \mathbf{r}$ is solenoidal iff $n=-3$

Solution. $r^n \mathbf{r} = r^n (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$.

$$\therefore \text{div } r^n \mathbf{r} = \frac{\partial}{\partial x} (x r^n) + \frac{\partial}{\partial y} (y r^n) + \frac{\partial}{\partial z} (z r^n)$$

$$= r^n + x n r^{n-1} \frac{\partial r}{\partial x} + r^n + y n r^{n-1} \frac{\partial r}{\partial y} + r^n + z n r^{n-1} \frac{\partial r}{\partial z}$$

$$= 3r^n + n r^{n-2} (x^2 + y^2 + z^2) \left(\text{since } \frac{\partial r}{\partial x} = \frac{x}{r} \text{ etc} \right)$$

$$= 3r^n + n r^{n-2} r^2$$

$$= (3+n)r^n$$

Now, $r^n \mathbf{r}$ is solenoidal iff $\text{div } r^n \mathbf{r} = 0$. (i.e) iff $(3+n)r^n = 0$

(i.e) iff $n = -3$

Problem 5. If $\mathbf{c}$ is a constant vector prove that

$$\nabla \cdot \{[(\mathbf{c} \cdot \mathbf{r}) r^n] \mathbf{r}\} = (n+4)(\mathbf{c} \cdot \mathbf{r}) r^n$$

Solution. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$$\therefore \nabla \cdot \{[(\mathbf{c} \cdot \mathbf{r}) r^n] \mathbf{r}\} = \sum \frac{\partial}{\partial x} (\mathbf{c} \cdot \mathbf{r}) r^n x$$

$$\begin{aligned}
&= \sum (c.r)r^n + nx^2(c.r)r^{n-2} + xr^n(c.i) \\
&= 3(c.r)r^n + n(c.r)r^{n-2}(x^2 + y^2 + z^2) + \\
&\quad r^n[x(c.r) + y(c.j) + z(c.k)] \\
&= 3(c.r)r^n + n(c.r)r^n + (c.r)r^n \\
&= (n+4)(c.r)r^n
\end{aligned}$$

Problem 6. Prove that $\nabla^2 f(r) = f f''(r) + (2/r)f'(r)$

Solution. $\nabla f(r) = i f'(r) \frac{\partial r}{\partial x} + j f'(r) \frac{\partial r}{\partial y} + k f'(r) \frac{\partial r}{\partial z}$

$$\begin{aligned}
&= \frac{f'(r)}{r} (xi + yj + zk) \left(\text{Since } \frac{\partial r}{\partial x} = \frac{x}{r} \text{ etc} \right) \\
&= f'(r)(r/r) \\
\therefore \nabla 2f(r) &= \nabla \cdot (\nabla f(r)) = \nabla \cdot (f'(r)(r/r)) \\
&= \nabla (f'(r))(r/r) + (\nabla \cdot r/r) f'(r) \\
&= \{f''(r)(r/r)(r/r)\} + (2/r)f'(r) \\
&= f''(r) + (2/r)f'(r)
\end{aligned}$$

Problem 7. Prove that $\nabla f(r) \times r = 0$.

Solution. $\nabla f(r) = [f'(r)/r]r$ refer problem 6

$$\therefore \nabla f(r) \times r = [f'(r)/r](r \times r) = 0$$

Problem 8. Prove that $\text{curl}(r \times a) = -2a$ where a is a constant vector.

Solution. $\text{Curl}(r \times a) = \nabla \times (r \times a) = \sum \left[i \times \frac{\partial}{\partial x} (r \times a) \right]$

$$\begin{aligned}
&= \sum \left[i \times \left(\frac{\partial r}{\partial x} \times a + r \frac{\partial a}{\partial x} \right) \right] \\
&= \sum \left[i \times \left(\frac{\partial r}{\partial x} \times a \right) \right] \quad (\text{since } a \text{ is a constant vector}) \\
&= \sum [i \times (i \times a)] \left(\text{Since } \frac{\partial r}{\partial x} = i \right) \\
&= \sum [(i.a)i - (i.i)a] = \sum [(i.a)i - a] \\
&= [(i.a)i - a] + [(j.a)j - a] + [(k.a)k - a] \\
&= (i.a)i + (j.a)j + (k.a)k - 3a = a - 3a \\
&= -2a.
\end{aligned}$$

Problem 9. Prove that $\text{div}(\mathbf{r} \times \mathbf{a}) = 0$ where $\mathbf{a}$ is a constant vector.

Solution. $\text{div}(\mathbf{r} \times \mathbf{a}) = \nabla \cdot (\mathbf{r} \times \mathbf{a}) = \sum \left[\mathbf{i} \times \frac{\partial}{\partial x} (\mathbf{r} \times \mathbf{a}) \right]$

$$= \sum \left[\mathbf{i} \times \left(\frac{\partial \mathbf{r}}{\partial x} \times \mathbf{a} \right) \right] \text{ (since } \mathbf{a} \text{ is a constant vector)}$$

$$= \sum [\mathbf{i} \cdot (\mathbf{i} \times \mathbf{a})] = 0.$$

Problem 10. Show that $\text{curl}(\mathbf{r}^n \mathbf{r}) = 0$

Solution. $\mathbf{r}^n \mathbf{r} = r^n \times \mathbf{i} + r^n y \mathbf{j} + r^n z \mathbf{k}$

$$\therefore \text{curl}(\mathbf{r}^n \mathbf{r}) = \sum \mathbf{i} \left[\frac{\partial}{\partial y} (r^n z) - \frac{\partial}{\partial z} (r^n y) \right] = \sum \mathbf{i} \left(z n r^{n-1} \frac{\partial r}{\partial y} - y n r^{n-2} \frac{\partial r}{\partial z} \right)$$

$$= n r^{n-1} \sum \mathbf{i} [z(y/r) - y(z/r)] = 0.$$

Problem 11. If $\mathbf{f} = (ax+3y+4z) \mathbf{i} + (x-3y+3z) \mathbf{j} + (3x+2y-z) \mathbf{k}$ is solenoidal find the constant a .

Solution. $\text{div} \mathbf{f} = a - 2 - 1$

If $\mathbf{f}$ is solenoidal then $\text{div} \mathbf{f} = 0$.

$$\therefore a - 2 - 1 = 0. \text{ Hence } a=3.$$

Problem 12. Prove $\mathbf{f} = (x^2 - yz) \mathbf{i} + (y - zx) \mathbf{j} + (z^2 - xy) \mathbf{k}$ is irrotational

Solution. $\text{curl} \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 yz & y - zx & x^2 xy \end{vmatrix}$

$$= \mathbf{i}[-x+x] - \mathbf{j}[-y+y] + \mathbf{k}[-z+z] = 0$$

$\therefore \mathbf{f}$ is irrotational.

Problem 13. Show that the vector

$\mathbf{f} = (y^2 - z^2 + 3yz - 2x) \mathbf{i} + (3xz - 2xy) \mathbf{j} + (3xz - 2xz + 2z) \mathbf{k}$ is both in rotational and solenoidal.

Solution. Let $f = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ where $f_1 = (y^2 - z^2 + 3yz - 2x)$, $f_2 = (3xz + 2xy)$,
 $f_3 = (3xy - 2xz + 2z)$

$$\therefore \operatorname{div} f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = -2 + 2x - 2x + 2 = 0$$

Also, $\operatorname{curl} f = \mathbf{0}$ (verify). Hence f is irrotational.

Hence f is both irrotational and solenoidal

Problem 14. Find the constants a, b, c so that the vector

$f = (x+2y+az)\mathbf{i} + (bx-3y-z)\mathbf{j} + (4x+cy+2z)\mathbf{k}$ is irrotational.

Solution. $\operatorname{Curl} f = (c+1)\mathbf{i} - (4-a)\mathbf{j} + (b-2)\mathbf{k}$ (verify)

If f is irrotational $\operatorname{curl} f = \mathbf{0}$. Hence $c+1=0$; $4-a=0$; $b-2=0$.

$\therefore a=4$; $b=2$; $c=-1$.

Problem 15. If f and g are irrotational show that $f \times g$ is solenoidal.

Solution $\nabla \cdot (f \times g) = g \cdot \operatorname{curl} f - f \cdot \operatorname{curl} g$ (by theorem 7.11)

$$= g \cdot \mathbf{0} + f \cdot \mathbf{0} \text{ (since } f \text{ and } g \text{ are irrotational)}$$

$$= 0. \text{ Hence } f \times g \text{ is solenoidal.}$$

Problem 16. If $\phi(x, y, z)$ is any solution of Laplace's equation prove that $\nabla \phi$ is both solenoidal and irrotational

Solution. Since ϕ is solution of Laplace equation we have $\nabla^2 \phi = 0 \dots (1)$

Now, $\operatorname{curl} (\nabla \phi) \nabla \times (\nabla \phi) = \mathbf{0}$. (verify)

$\therefore \nabla \phi$ is irrotational. Hence the result.

Problem 17. If f is solenoidal prove that $\operatorname{curl} \operatorname{curl} \operatorname{curl} \operatorname{curl} f = \nabla^4 f$.

Solution. $\operatorname{curl} \operatorname{curl} \operatorname{curl} \operatorname{curl} f = \nabla \times \nabla \times \nabla \times \nabla \times f$

$$= \nabla \times \nabla \times [\nabla \cdot f - (\nabla^2 f)] \text{ (by theorem 3.10)}$$

$$= \nabla \times \nabla \times (-\nabla^2 f) \text{ (since } f \text{ is solenoidal } \nabla \cdot f = 0)$$

$$= \nabla \times \nabla \times g \text{ where } g = -\nabla^2 f$$

$$= \nabla (\nabla \cdot g) - \nabla^2 g$$

$$= -\nabla^2 g \text{ [since } \nabla \cdot g = \nabla \cdot (-\nabla^2 f) = -\nabla^2 (\nabla \cdot f) = 0]$$

$$= -\nabla^2 (-\nabla^2 f)$$

$$= \nabla^4 f$$

Exercises:

1. If $f = x^2z\mathbf{i} - 2y^3z^2\mathbf{j} + xy^2z\mathbf{k}$ find
[i] $\text{div } f$ [ii] $\text{curl } f$ at $(1, -1, 1)$
2. [i] If $f = (2z-3y)\mathbf{i} + (3x-z)\mathbf{j} + (y-2x)\mathbf{k}$ find $\text{div } f$ and $\text{curl } f$
[ii] If $g = (2z-3y)\mathbf{i} + (3x-2)\mathbf{j} + (y-2x)\mathbf{k}$ prove that $\text{curl } g$ is $2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$.
3. Find divergence and curl of the vector
[i] (xyz^2, yzx^2, zxy^2) [ii] $(x \cos z, y \log x, -z^2)$
4. Find the divergence $\frac{xi - yj}{x + y}$
5. If $f = 3xyz^2\mathbf{i} + 2xy^2\mathbf{j} - x^3yz\mathbf{k}$ and $\Phi = 3x^2 - yx$
Find [i] $\text{div } f$ [ii] $\text{div } (\Phi f)$ [iii] $f \cdot \text{grad } \Phi$ [iv] $\text{div } (\text{grad } \Phi)$
6. If $f = (3x^2y - z)\mathbf{i} + (xz^2 + y^4)\mathbf{j} - 2x^3z^2\mathbf{k}$ find $\nabla (\nabla \cdot f)$ at the point $(2, -1, 0)$
7. If $\Phi = 3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5$ find $\nabla^2 \Phi$
8. If $\Phi = \log r$ where $r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ prove that $\nabla^2 (\log r) = 1/r^2$.
9. Show that $\Phi = ax^2 + by^2 + cz^2$ satisfies Laplace's equation if $a+b+c=0$
10. If $f = xz^2\mathbf{i} - 2x^2yz\mathbf{j} + 2yz^4\mathbf{k}$ find $(\nabla \times f)$.
11. If c is a constant prove that $\nabla \times [c \cdot r]r^n = (c \times r)r^n$.
12. Prove that $(f \times \nabla) \times r = -2f$ where $r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
13. Prove that $\nabla \times (v \cdot v) = 2v \cdot \nabla v + 2v \times (\nabla \times v)$.
14. For what value of the constant a is the vector $(x + 3yy - 2zx + az)$ solenoidal?
15. Prove that vector (i) $(3y^4z^2, 4x^3z^2, -3x^2y^2)$ is solenoidal
(ii) $(x^2 - yz, y - zx, z^2 - xy)$ is irrotational
16. Prove that the vector $r^{-3}r$ is solenoidal.
17. Verify that $f = yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}$ is irrotational.
18. For what value of the constant a will $\text{curl } f = 0$ where $f = (axy - z^3)\mathbf{i} + (a-2)x^2\mathbf{j} + (1-a)xz^2\mathbf{k}$?
19. If $f = (3x^2 + 2y^2 + 1)\mathbf{i} + (4xy - 3y^2z - 3)\mathbf{j} + (2 - y^3)\mathbf{k}$ show that f is irrotational. Find Φ such that $f = \nabla \Phi$
20. Find the value of the constant a so that the vector $f = (axy - z^3)\mathbf{i} + (a-2)x^2\mathbf{j} + (1-a)xz^2\mathbf{k}$ will be irrotational. Hence find $\Phi(x, y, z)$ such that $f = \nabla \Phi$

21. Prove that $\mathbf{f} = e^x [2y+3z]\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}$ is irrotational. Find a function $\Phi(x, y, z)$ such that $\mathbf{f} = \text{grad } \Phi$
22. Find Φ such that $\Phi(1, 1, 1) = 3$ if $\nabla \Phi = (y+y^2+z^2)\mathbf{i} + (x+z+2xy)\mathbf{j} + (y+2zx)\mathbf{k}$.
23. Find Φ if $\nabla \Phi = (6xy + z^3)\mathbf{i} + (3x^2 - z)\mathbf{j} + (3xz^2 - y)\mathbf{k}$
24. Prove that $\text{div } \Phi (\nabla \psi) - \text{div } \psi (\nabla \Phi) = \Phi \nabla^2 \psi - \psi \nabla^2 \Phi$
25. If the vector $(\mathbf{r})$ is solenoidal prove that $\mathbf{f}(\mathbf{r}) = \frac{\mathbf{k}}{r^3}$ where $\mathbf{k}$ is a constant. Also prove that the vector is irrotational.

Section 5.5 Vector Integration

Section 5.5.1 Line Integrals

Another way of generalizing the Riemann integral $\int_a^b f(x)dx$ is by

replacing the interval $[a, b]$ by a curve in $\mathbf{R}^3$. In this generalization the integrand is a vector valued function $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$.

Definition. Let C be a curve in $\mathbf{R}^3$ described by a continuous vector valued function $\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ where $a \leq t \leq b$.

Let $\mathbf{f} = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}$ be a continuous function defined in some region which contains the curve C . The line integral of $\mathbf{f}$ over C denoted by $\int_C \mathbf{f} \cdot d\mathbf{r}$ is defined by

$$\int_C \mathbf{f} \cdot d\mathbf{r} = \int_a^b [f_1(x(t), y(t), z(t))x'(t) + f_2(x(t), y(t), z(t))y'(t) + f_3(x(t), y(t), z(t))z'(t)] dt$$

Work done by a force

A force is said to do work when its point of application moves. When a particle acted on by a force $\mathbf{f}$, moves from point $\mathbf{r}$ to a neighbouring point $\mathbf{r} + \nabla \mathbf{r}$, the work done in this small displacement is defined to be the scalar product $\mathbf{f} \cdot \nabla \mathbf{r}$. If the particle describes an arc C , then the work done is given by the line integral $\int_C \mathbf{f} \cdot d\mathbf{r}$ curve $y=x^2$ joining $(0, 0)$ and $(1, 1)$.

Solved Problems

Solution. The parametric equation of the curve can be taken as $x=t$; $y=t^2$ where $0 \leq t \leq 1$.

Problem 1.

$$\begin{aligned} \therefore \int_C f \cdot dr &= \int_0^1 [(t^2 + t^4)1 + (t^2 - t^4)2t] dt \\ &= \left[\frac{1}{3}t^3 + \frac{1}{5}t^5 + \frac{1}{2}t^4 - \frac{1}{3}t^6 \right]_0^1 = \frac{9}{10} \end{aligned}$$

Problem 2. If $f = x^2i - xyj$ and C is the straight line joining the points $(0, 0)$ and $(1, 1)$ find $\int_C f \cdot dr$

Solution. The equation of the given line is $y = x$ and its parametric equation can be taken as $x = t$, $y = t$ where $0 \leq t \leq 1$.

$$\therefore \int_C f \cdot dr = \int_0^1 (t^2 - t^2) dt = 0.$$

Problem 3. Evaluate $\int_C f \cdot dr$ where $f = (x^2 + y^2)i - 2xyj$ and the curve

C is the rectangle in the x - y plane bounded by $y = 0$, $y = b$, $x = 0$, $x = a$.

Solution. Let $O = (0, 0)$, $A = (a, 0)$, $B = (a, b)$ and $C = (0, b)$ be the vertices of the given rectangle. Hence

$$\int_C f \cdot dr = \int_{OA} f \cdot dr + \int_{AB} f \cdot dr + \int_{BC} f \cdot dr + \int_{CO} f \cdot dr$$

Now the parametric equation of OA can be taken as $x = t$, $y = 0$ where $0 \leq t \leq a$.

$$\therefore \int_C f \cdot dr = \int_0^a t^2 dt = \frac{1}{3}a^3$$

$$\int_{AB} f \cdot dr = \int_0^b (-2at) dt \text{ (since } x = a, y = t \text{ and } 0 \leq t \leq b \text{ along AB)} = -ab^2$$

$$\int_{3C} f \cdot dr = - \int_{CB} f \cdot dr = - \int_0^a (t^2 + b^2) dt \text{ (since } x = t, y = b \text{ and } 0 \leq t \leq b \text{ along CB)}$$

$$= \left(\frac{1}{3} a^3 + ab^2 \right)$$

$$\int_{CO} f \cdot dr = - \int_C f \cdot dr = \int_0^b 0 dt = 0 \text{ (since } x = 0, y = t \text{ and } 0 \leq t \leq b)$$

$$\therefore \int_C f \cdot dr = \frac{1}{3} a^3 - ab^2 - \left(\frac{1}{3} a^3 + ab^2 \right) + 0 = -2ab^2$$

Problem 4. If $f = (2y + 3) \mathbf{i} + xz \mathbf{j} + (yz - x) \mathbf{k}$ evaluate $\int_C f \cdot dr$ along the

following paths C (i) $x = 2t^2$; $y = t$; $z = t^3$ from $t = 0$ to $t = 1$

(ii) The polygonal path P consisting of the three line segments AB, BC and CD where $A = (0, 0, 0)$, $B = (0, 0, 1)$, $C = (0, 1, 1)$ and $D = (2, 1, 1)$

iii) The straight line joining $(0, 0, 0)$ and $(2, 1, 1)$

Solution. (i) $\int_C f \cdot dr = \int_0^1 [(2t + 3)4t + 2t^5 + (t^4 - 2t^2)3t^2] dt$

$$= \left[\frac{8}{3} t^3 + 6t^2 + \frac{1}{3} t^6 + \frac{3}{7} t^7 - \frac{6}{5} t^5 \right]$$

$$= \frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5} = \frac{288}{85}$$

(ii) $\int_P f \cdot dr = \int_{AB} f \cdot dr + \int_{BC} f \cdot dr + \int_{CD} f \cdot dr$

$$\int_{AB} f \cdot dr = \int_0^1 0 dt = 0 \text{ (since } x = 0; y = 0; z = t \text{ and } 0 \leq t \leq 1 \text{ along AB)}$$

$$\int_{BC} f \cdot dr = \int_0^1 0 dt = 0 \text{ (since } x = 0; y = t; z = 1 \text{ and } 0 \leq t \leq 1 \text{ along BC)}$$

$$\int_{CD} f \cdot dr = \int_0^2 5 dt = 0 \text{ (since } x = 1; y = 1; z = t \text{ and } 0 \leq t \leq 2 \text{ along CD)}$$

$$= [5t]_0^2 = 10$$

$$\text{Hence } \int_P f \cdot dr = 10$$

iii) The parametric equation of the line joining $(0, 0, 2)$ and $(2, 1, 1)$ can be taken as $x=2t, y=t, z=2-t$ where $0 \leq t \leq 1$.

$$\therefore \int f \cdot dr = \int_0^1 [(2t+3)2 + 2t^2 + (t^2 - 2t)] dt$$

$$\int_0^1 (3t^2 + 2t + 6) dt = [t^3 + t^2 + 6t]_0^1$$

$$= 8.$$

Problem 5. Find the work done by the force $F = 3xy \mathbf{i} - 5z \mathbf{j} + 10x \mathbf{k}$ along the curve C,

$$x=t^2+1; y=2t^2, z=t^3 \text{ from } t=1 \text{ to } t=2.$$

Solution $F \cdot dr = (3xy \mathbf{i} - 5z \mathbf{j} + 10x \mathbf{k}) \cdot (dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k})$
 $= 3xy dx - 5z dy + 10x dz$

Total work done $\int_C F \cdot dr = \int_C 3xy dx - 5z dy + 10x dz$

$$= \int_1^2 3(t^2+1)(2t^2)(2t) dt - 5t^3(4t) dt + 10(t^2+1)(3t^2) dt$$

$$= \int_1^2 [(12t^5 + 12t^3) - 20t^4 + (30t^4 + 30t^2)] dt$$

$$= \int_1^2 [12t^5 + 10t^4 + 12t^3 + 30t^2] dt$$

$$= [2t^6 + 2t^5 + 3t^4 + 10t^3]_1^2$$

$$= 320 - 17 = 303$$

UNIT 6

NUMERICAL SOLUTIONS OF ALGEBRAIC & TRANSCENDENTAL EQUATIONS

Section 6.0 Introduction

Space for Hints

An important problem in scientific and engineering work is to find the roots of an equation of the form $f(x)=0$. This problem is required in a number of applications and occurs frequently in modern computing.

If $f(x)$ is a quadratic expression, then we have a simple formula for the roots of $f(x)=0$. For instance, if $f(x)=ax^2+bx+c=0$, then x

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

are the roots of the above equation. The function f

(x) may be algebraic or transcendental. If $f(x)$ is a polynomial of higher degree or an expression involving transcendental functions like $a + b \sin x + \tan x + cx^2 + ae^{-x} + b \tan x + c$, $a \log x + bx^2 + c$, etc., we have no simple formula for the roots of the equation $f(x)=0$ when the coefficients of such equations are pure numbers, they are called Numerical Equations. It is possible to compute the roots of such equations to any desired degree of accuracy. In this chapter we will describe some numerical methods for the solution of the equation $f(x)=0$, where $f(x)$ is algebraic (or) transcendental.

Section 6.1

Numerical solution of Algebraic and transcendental equations

The following fundamental theorem will enable us to locate the real root of an equation $f(x)=0$

“If $f(x)$ is continuous from $x=a$ and $x=b$ and if $f(a)$ and $f(b)$ are of opposite signs, the equation $f(x)=0$ will have atleast one real root between ‘a’ and ‘b’.

In numerical computation we work with a finite number of digits and carryout the computation in finitely many steps. Hence the numerical result obtained is an approximate value of the unknown exact result

Definition If α is an approximate value of a quantity whose exact value is a , then the difference $\varepsilon = \alpha - a$ is called the absolute error of α or simply the error of α .

The relative error ε_r is defined by $\varepsilon_r = \frac{\varepsilon}{\alpha} = \frac{\alpha - a}{a}$ provided $a \neq 0$.

The percentage error ε_p is defined by $\varepsilon_p = 100\varepsilon_r$.

$r = a - \alpha = -\varepsilon$ is called the correction.

Thus $a = \alpha + r$.

ie, True value = Approximate value + correction.

We now discuss different types of errors in numerical computation.

An experimental error is an error present in the given data. Such errors may arise from measurement.

In an interactive computation method the sequence of computational steps necessary to produce an exact result is truncated after a finite number of steps. An error arising out of such truncation is called truncation error.

In the decimal notation every real number is represented in the decimal form which may contain finite or infinite number of digits. If a number contains infinite number of digits then for the purpose of numerical computation, it must be replaced by a number consisting of finitely many digits. For example a number of the form $\frac{27}{13.1} = 2.061068702$ will be

replaced by 2.06 or 2.061 or 2.06107 in a practical computation.

This process of cutting of digits and retaining required number of digits is called rounding off. Errors arising from the process of rounding off during computation are called round off errors.

Section 6.2

Space for Hints

Iteration Method (or) Method of successive approximations

Suppose we want to find the roots of the equation

$$f(x) = 0 \longrightarrow \textcircled{1}$$

Let the equation $\textcircled{1}$ can be represented as $x = \phi(x) \longrightarrow \textcircled{2}$

Let x_0 be an initial approximation to the actual root α of $\textcircled{2}$.

Putting $x = x_0$ in the RHS of $\textcircled{2}$, We get the first approximation x_1

$$x_1 = \phi(x_0)$$

$$x_2 = \phi(x_1) \dots\dots\dots x_n = \phi(x_{n-1})$$

Now it should not be taken for granted that the sequence of approximations $x_1, x_2, \dots\dots\dots x_n$ always converges to the same number and that too, to α which is a root of $x = \phi(x)$.

We have to choose the initial approximation x_0 suitably so that the successive approximations $x_0, x_1, x_2, \dots\dots\dots x_n$ converges to the root α . The choice of x_0 is determined according to the following theorem without proof.

Theorem Suppose the equation $f(x) = 0$ can be written in the form $x = \phi(x)$. Let α be a root of $f(x) = 0$ and let I be the interval containing α . Suppose $\phi(x)$ and $\phi'(x)$ are continuous in I and $|\phi'(x)| \leq A < 1$ for all $x \in I$. Then for any $x_0 \in I$ the sequence of successive approximations $x_0, x_1, x_2, x_3, \dots\dots\dots x_n, x_{n+1}, \dots\dots\dots$ converges to α .

Order of Convergence

Consider the equation $f(x) = 0$. Suppose the equation can be expressed in the form $x = \phi(x)$. Let $x_{n+1} = \phi(x_n)$ define an iteration method for solving the equation $f(x) = 0$.

Space for Hints

Let α be the root of the above equation. Let $x_n = \alpha + \epsilon_n$

Then $\epsilon_n = x_n - \alpha$ is the error in x_n . If $\phi(x)$ is differentiable in any number of times then the Taylor's formula for $\phi(x)$ is given by

$$\phi(x) = \phi(\alpha) + \frac{\phi'(\alpha)}{1!}(x - \alpha) + \frac{\phi''(\alpha)}{2!}(x - \alpha)^2 + \dots$$

$$\therefore x_{n+1} = \phi(x_n)$$

$$= \phi(\alpha) + \frac{\phi'(\alpha)}{1!} \epsilon_n + \frac{\phi''(\alpha)}{2!} \epsilon_n^2 + \dots$$

The power of ϵ_n in the 1st non vanishing term after $\phi(\alpha)$ is called the order of the iteration process.

Note The order is a measure for the speed of convergence.

For example in the case of 1st order of convergences

$$x_{n+1} = \phi(\alpha) + \phi'(\alpha) \epsilon_n \text{ [neglecting lower powers]}$$

$$/ x_{n+1} = \alpha + \phi'(\alpha) \epsilon_n$$

$$\text{ie, } x_{n+1} - \alpha = \phi'(\alpha) \epsilon_n$$

$$\therefore \epsilon_{n+1} = \phi'(\alpha) \epsilon_n$$

Similarly in the case of Second order of convergence,

$$\epsilon_{n+1} = \frac{1}{2} \phi''(\alpha) \epsilon_n^2.$$

Problem 1

Solved Problems

Use the method of iteration to find the real root lying between 1 and 2 of the equation $x^3 - 3x + 1 = 0$

Solution Let $f(x) = x^3 - 3x + 1$

$$\text{Hence } f(1) = 1^3 - (3 \times 1) + 1 = 2 - 3 = -1 \text{ and } f(2) = 2^3 - 3 \times 2 + 1 = 8 - 6 + 1 = 9 - 6 = 3$$

/ One root of $f(x) = 0$ lies between 1 and 2

Now, $f(x) = 0$ gives

$$x^3 - 3x + 1 = 0$$

$$\Rightarrow x^3 = -1 + 3x = 3x - 1$$

$$\Rightarrow x = (3x - 1)^{\frac{1}{3}} = \phi(x)$$

$$/ \phi(x) = (3x - 1)^{\frac{1}{3}}$$

$$\Rightarrow \phi'(x) = \frac{1}{(3x - 1)^{2/3}}$$

Clearly $|\phi'(x)| < 1$ for all $x \in (1, 2)$.

Hence if we take $x_0 = 2$, then the sequence of successive approximations

$x_0, x_1, x_2, \dots, x_n, \dots$ is convergent.

Let $x_0 = 2$

$$\text{Now } x_1 = \phi(x_0) = \phi(2) = [3 \times 2 - 1]^{1/3} = 1.7100$$

$$x_2 = \phi(x_1) = \phi(1.71) = [3 \times 1.71 - 1]^{1/3} = 1.6044$$

$$x_3 = \phi(x_2) = \phi(1.6044) = [3 \times 1.6044 - 1]^{1/3} = 1.5623$$

$$x_4 = \phi(x_3) = \phi(1.5623) = [3 \times 1.5623 - 1]^{1/3} = 1.5449$$

$$x_5 = \phi(x_4) = \phi(1.5449) = [3 \times 1.5449 - 1]^{1/3} = 1.5375$$

$$x_6 = \phi(x_5) = \phi(1.5375) = [3 \times 1.5375 - 1]^{1/3} = 1.5344$$

$$x_7 = \phi(x_6) = \phi(1.5344) = [3 \times 1.5344 - 1]^{1/3} = 1.5331$$

$$x_8 = \phi(x_7) = \phi(1.5331) = [3 \times 1.5331 - 1]^{1/3} = 1.5325$$

$$x_9 = \phi(x_8) = \phi(1.5325) = [3 \times 1.5325 - 1]^{1/3} = 1.5323$$

$$x_{10} = \phi(x_9) = \phi(1.5323) = [3 \times 1.5323 - 1]^{1/3} = 1.5322$$

Clearly $x_9 = x_{10} = 1.532$ (upto 3 places of decimals)

Hence the approximate value of the required root is 1.532

Problem 2 Using the method of successive approximation to solve the

$$\text{equation } 3x - \log_{10} x = 6$$

Solution Let $f(x) = 3x - \log_{10}^x - 6$

$$\begin{aligned}\text{Now } f(2) &= 6 - \log_{10} 2 - 6 = 6 - 0.3010 - 6 \\ &= -0.3010 = (-ve)\end{aligned}$$

$$f(3) = 9 - \log_{10} 3 - 6 = 3 - 0.4771 = 2.5 = +ve$$

$\therefore$ a root lies between 2 and 3

Now the given equation may be written as

$$3x - \log_{10}^x - 6 = 0$$

$$\Rightarrow 3x = 6 + \log_{10}^x$$

$$\Rightarrow x = \frac{1}{3} \left[6 + \log_{10}^x \right] = \phi(x)$$

$$\text{Now } \phi(x) = \frac{1}{3} \left[6 + \log_{10}^x \right]$$

$$\phi'(x) = \frac{1}{3} \left[\frac{\log_{10}^e}{x} \right] \left(\because \log_{10}^x = \log_e^x \times \log_{10}^e \right)$$

$$|\phi'(x)| = \frac{0.4343}{3} \left| \frac{1}{x} \right| \left(\because \log_{10}^e = 0.4343 \right)$$

$$< 1 \text{ for all } x \in (2, 3)$$

Hence if we take $x_0 = 2$, then the sequence of successive approximations $x_0, x_1, x_2, \dots, x_n, x_{n+1}, \dots$ is convergent.

$$\text{Now } x_1 = \phi(x_0) = \phi(2) = \frac{1}{3} \left[6 + \log_{10}^2 \right] = \frac{6.3010}{3} = 2.1003$$

$$x_2 = \phi(x_1) = \phi(2.1003) = \frac{1}{3} \left[6 + \log_{10}^{2.1003} \right] = \frac{6.3223}{3} = 2.1074$$

$$x_3 = \phi(x_2) = \phi(2.1074) = \frac{1}{3} \left[6 + \log_{10}^{2.1074} \right] = \frac{6.3238}{3} = 2.1079$$

$$x_4 = \phi(x_3) = \phi(2.1079) = \frac{1}{3} \left[6 + \log_{10} 2.1079 \right] = \frac{6.3239}{3} = 2.1080$$

$$x_5 = \phi(x_4) = \phi(2.1080) = \frac{1}{3} \left[6 + \log_{10} 2.1080 \right] = \frac{6.3239}{3} = 2.1080$$

Clearly $x_4 = x_5 = 2.108$ (correct to 3 places of decimals)

Hence the approximate value of the required root is 2.108.

Problem 3

Find the negative root of the equation $x^3 - 2x + 5 = 0$ using iteration method.

Solution Let $k(x) = x^3 - 2x + 5 = 0$

To find the negative root is nothing but to find the positive root of $k(-x)$

$$) = -x^3 + 2x + 5 = 0$$

$$\text{ie } -x^3 - 2x - 5 = 0$$

$$\text{Choose } f(x) = x^3 - 2x - 5 = 0$$

$$f(0) = 0 - 0 - 5 = -5 = -\text{ve}$$

$$f(1) = 1 - 2 \times 1 - 5 = 1 - 2 - 5 = -6 = -\text{ve}$$

$$f(2) = 8 - 2 \times 2 - 5 = 8 - 4 - 5 = -1 = -\text{ve}$$

$$f(3) = 27 - 2 \times 3 - 5 = 27 - 6 - 5 = 16 = +\text{ve}$$

So the root lies between 2 and 3.

The equation $f(x) = 0$ may be written as $x^3 = 2x + 5$

$$\Rightarrow x = (2x + 5)^{1/3} = \phi(x)$$

$$\text{ie, } \phi(x) = (2x + 5)^{1/3}$$

$$x_3 = \phi(x_2) = \phi(2.1074) = \frac{1}{3} \left[6 + \log_{10} 2.1074 \right] = \frac{6.3238}{3} = 2.1079$$

Space for Hints

$$\phi'(x) = \frac{1}{3} [2x+5]^{-\frac{2}{3}} \times 2 = \frac{2}{3} \times \frac{1}{(2x+5)^{\frac{2}{3}}}$$

$$|\phi'(x)| = \left| \frac{2}{3} \frac{1}{(2x+5)^{\frac{2}{3}}} \right| < 1 \text{ in the interval } (2,3)$$

So We can apply the iteration method Let $x_0 = 2.1$

$$x_1 = \phi(x_0) = \phi(2.1) = (2 \times 2.1 + 5)^{1/3} = 2.09538$$

$$x_2 = \phi(x_1) = \phi(2.09538) = (2 \times 2.09538 + 5)^{1/3} = 2.09468$$

$$x_3 = \phi(x_2) = \phi(2.09468) = (2 \times 2.09468 + 5)^{1/3} = 2.09457$$

$$x_4 = \phi(x_3) = \phi(2.09457) = (2 \times 2.09457 + 5)^{1/3} = 2.09455$$

$$x_5 = \phi(x_4) = \phi(2.09455) = (2 \times 2.09455 + 5)^{1/3} = 2.09455$$

Clearly $x_4 = x_5 = 2.09455$ (Correct to 5 places decimals)

So the +ve root of $f(x) = 0$ is 2.09455

Hence the -ve root of the given equation is -2.09455.

Problem: 4 Find the real root of the equation $\cos x = 3x - 1$ correct to four places of decimals using successive approximation method.

Solution: Let $f(x) = 3x - 1 - \cos x$

$$\text{Now } f(0) = 3 \times 0 - 1 - \cos 0 = 0 - 1 - 1 = -2 - \text{ve}$$

$$f\left(\frac{\pi}{2}\right) = \frac{3\pi}{2} - 1 - \cos \frac{\pi}{2} = \frac{3 \times 3.14}{2} - 1 - 0 = 3.714 = + \text{ve}$$

/ One root of $f(x) = 0$ lies between 0 and $\frac{\pi}{2}$

$$\text{Now } f(x) = 0 \text{ can be written as } 3x - 1 - \cos x = 0$$

$$\Rightarrow 3x = 1 + \cos x$$

$$\Rightarrow x = \frac{1}{3} [1 + \cos x] = \phi(x)$$

$$\text{Now } \phi'(x) = -\frac{\sin x}{3}$$

$$\Rightarrow |\phi'(x)| = \left| \frac{-\sin x}{3} \right| = \left| \frac{\sin x}{3} \right| < 1 \text{ for all } x \text{ in } \left(0, \frac{\pi}{2} \right)$$

Hence if we take $x_0 = 0$, the sequence of Successive approximation x_0 ,

$x_1, x_2, x_3, \dots, x_n$ is convergent

$$\text{Now } x_1 = \phi(x_0) = \phi(0) = \frac{1}{3} [1 + \cos 0] = \frac{1}{3} (2) = \frac{2}{3} = .6667$$

$$x_2 = \phi(x_1) = \phi(.6667) = \frac{1}{3} [1 + \cos(.6667)] = \frac{1}{3} [1 + .7859] = .5953$$

$$x_3 = \phi(x_2) = \phi(.5953) = \frac{1}{3} [1 + \cos(.5953)] = \frac{1}{3} [1 + 0.8280] = 0.6093$$

$$x_4 = \phi(x_3) = \phi(.6093) = \frac{1}{3} [1 + \cos(.6093)] = \frac{1}{3} [1 + 0.8200] = 0.6067$$

$$x_5 = \phi(x_4) = \phi(.6067) = \frac{1}{3} [1 + \cos(.6067)] = \frac{1}{3} [1 + 0.8215] = 0.6072$$

$$x_6 = \phi(x_5) = \phi(.6072) = \frac{1}{3} [1 + \cos(.6072)] = \frac{1}{3} [1 + 0.8212] = 0.6071 \quad x_7 =$$

$$\phi(x_6) = \phi(.6071) = \frac{1}{3} [1 + \cos(.6071)] = \frac{1}{3} [1 + 0.8214] = 0.6071$$

Clearing $x_6 = x_7 = 0.6071$ (correct to 4 places of decimals)

Hence approximate value of the required root of given equation is 0.6071.

Problem 5

Can we find a real root of the equation $x^3 + x^2 - 1 = 0$ in the interval $[0, 1]$ by the method of iteration. If so, solve it.

Solution Given $x^3 + x^2 - 1 = 0$

$$\Rightarrow x^2 [1 + x] - 1 = 0$$

$$\Rightarrow x^2[1+x]-1$$

$$\Rightarrow x^2 = \frac{1}{1+x} \Rightarrow x = \frac{1}{\sqrt{1+x}}$$

$$\text{Let } \phi(x) = x \frac{1}{\sqrt{1+x}}$$

$$\text{Then } \phi'(x) = -\frac{1}{2(1+x)^{3/2}}$$

$$\text{Clearly } |\phi'(x)| = \left| \frac{-1}{2(1+x)^{3/2}} \right| = \frac{1}{2} \left| \frac{1}{(1+x)^{3/2}} \right| < 1 \text{ for all } x \in [0,1]$$

Hence if we take $x_0 = 0$ [or any number in $[0,1]$] then the sequence of approximations $x_0, x_1, x_2, \dots, x_n, x_{n+1}, \dots$ is convergent and we can find the approximate value of the root using method of iteration.

$$\text{Now } x_1 = \phi(x_0) = \phi(.5) = \frac{1}{\sqrt{1+.5}} = 0.81649$$

$$x_2 = \phi(x_1) = \phi(0.81649) = \frac{1}{\sqrt{1+.81649}} = 0.74196$$

$$x_3 = \phi(x_2) = \phi(0.74196) = \frac{1}{\sqrt{1+.74196}} = 0.75767$$

$$x_4 = \phi(x_3) = \phi(.75767) = \frac{1}{\sqrt{1+.75767}} = 0.75427$$

$$x_5 = \phi(x_4) = \phi(.75427) = \frac{1}{\sqrt{1+.75427}} = 0.75500$$

$$x_6 = \phi(x_5) = \phi(.75500) = \frac{1}{\sqrt{1+.75500}} = 0.75485$$

$$x_7 = \phi(x_6) = \phi(.75485) = \frac{1}{\sqrt{1+.75485}} = 0.75488$$

Since the difference between x_6 and x_7 are very small, the root is 0.7548 (correct to 4 places of decimals)

Problem 6*Space for Hints*

Can we apply iteration method to find the root of the equation

$$2x = \cos x + 3 \text{ in } \left[0, \frac{\pi}{2}\right] ?$$

Solution $2x = \cos x + 3$ gives

$$x = \frac{1}{2} [\cos x + 3]$$

$$\text{Let } \phi(x) = x = \frac{1}{2} [\cos x + 3]$$

$$\text{Then } |\phi'(x)| = \left| \frac{-1}{2} \sin x \right| = \left| \frac{\sin x}{2} \right| < 1 \text{ for all } x \text{ in } \left[0, \frac{\pi}{2}\right]$$

Hence by taking $x_0 = 0$ (or any point in $\left[0, \frac{\pi}{2}\right]$) the sequence of

approximations $x_0, x_1, x_2, \dots, x_n, x_{n+1}, \dots$ root of the equation can be obtained by the method of iteration.

Exercises Solve the following equations using iterative method

1] $3x - \sqrt{1 + \sin x} = 0$

2] $x^3 + x^2 - 100 = 0$

3] $2x - \log_{10} x = 7$

4] $1 + \sin x = 2x$

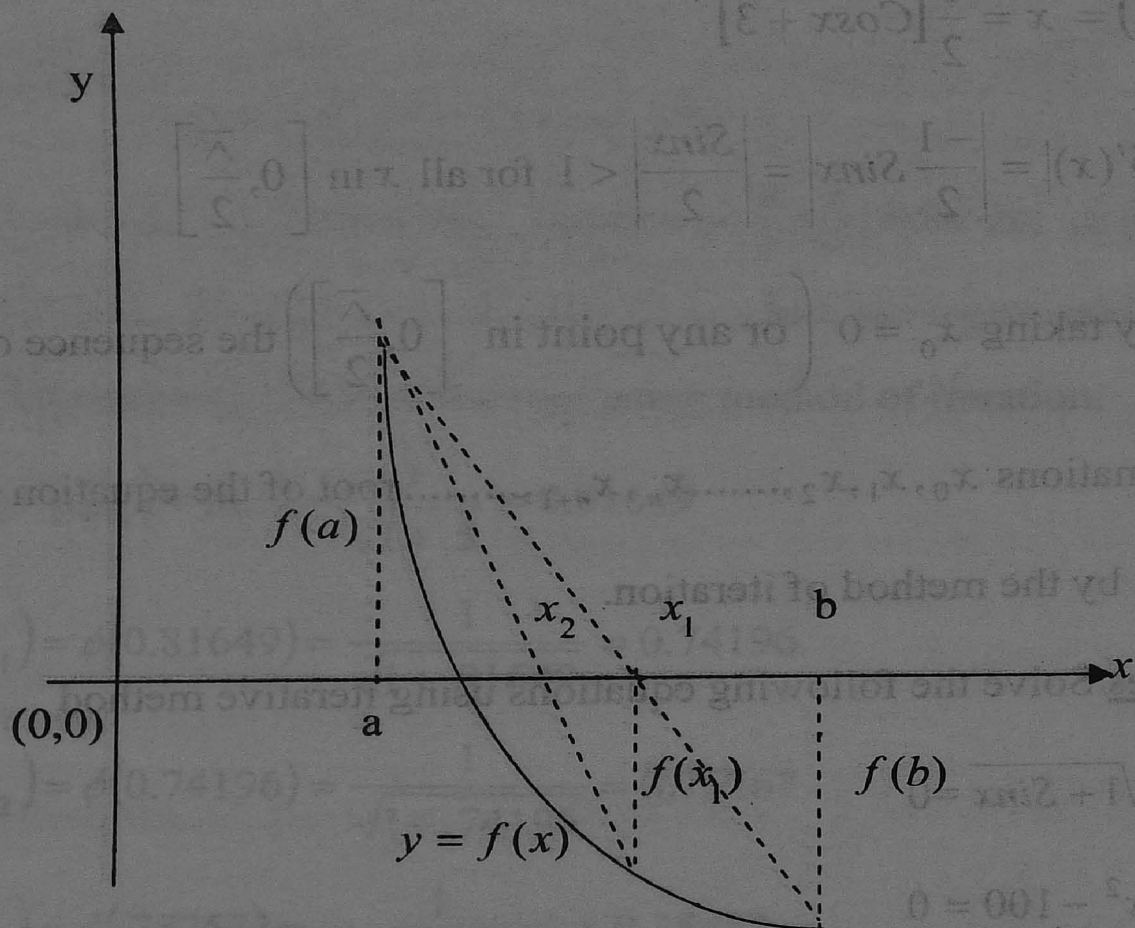
5] $e^x - 3x = 0$

6] Find a +ve real root of $x^3 - 2x - 5 = 0$

7] $3x + \sin x = e^x$

Regula- Falsi method (or) Method of false position

Consider the equation $f(x) = 0$ where $f(x)$ is a continuous function. Choose two points a, b such that $f(a)$ and $f(b)$ are of opposite signs. Hence there exists a root lying between 'a' & 'b'. In this method we approximate the curve of function $f(x)$.



Equation of the chord joining the points $(a, f(a))$ and $(b, f(b))$ is given

$$\text{by } y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a) \rightarrow \textcircled{1}$$

The point of intersection of the chord with the x axis is taken as the first approximation x_1 to the root which is obtained by putting

$$y = 0 \text{ in } \textcircled{1}$$

$$\therefore x_1 - a = \frac{-f(b)(b - a)}{f(b) - f(a)}$$

$$\Rightarrow x_1 = a - \frac{f(b)(b-a)}{f(b)-f(a)}$$

$$\Rightarrow x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)} \rightarrow \textcircled{2}$$

Now if $f(a)$ and $f(x_1)$ are of opposite signs then the root lies between a and x_1 in $\textcircled{2}$ and set the same approximation x_2 . But if $f(a)$ and $f(x_1)$ are of same sign, then $f(x_1)$ and $f(b)$ will be of opposite signs. Hence the root lies between x_1 and b . We replace a by x_1 in $\textcircled{2}$.

This process is repeated until the root is found to the desired accuracy.

Solved Problems

Problem 1 Find the real root lying between 1 and 2 of the equation $x^3 - 3x + 1 = 0$ upto 3 places of decimals by using Regula-falsi method.

Solution

$$\text{Let } f(x) = x^3 - 3x + 1$$

$$\text{Since } f(1) = 1^3 - 3 \times 1 + 1 = -1 < 0$$

$$\begin{aligned} \text{and } f(2) &= 2^3 - 3 \times 2 + 1 \\ &= 8 - 6 + 1 = 3 > 0. \end{aligned}$$

Clearly $f(1)$ and $f(2)$ are of opposite signs.

$\therefore$ One root lies between 1 and 2

Let $a = 1$ and $b = 2$.

$$\begin{aligned} \text{The 1}^{\text{st}} \text{ approximation is } x_1 &= \frac{af(b)-bf(a)}{f(b)-f(a)} \\ &= \frac{1 \times 3 - 2(-1)}{3 + 1} = \frac{5}{4} = 1.25 \end{aligned}$$

$$\begin{aligned} \text{Now } f(x_1) &= f(1.25) = (1.25)^3 - 3 \times 1.25 + 1 \\ &= -0.7969 < 0 \end{aligned}$$

Since $f(1.25)$ is negative and $f(2)$ is +ve, the root lies between 1.25

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and 2. Let $a=1.25$ and $b=2$.

The 2nd approximation is
$$x_2 = \frac{x_1 f(b) - b f(x_1)}{f(b) - f(x_1)}$$
$$= \frac{1.25 \times 3 - 2(-.7969)}{3 + 0.7969} = 1.4074$$

Now $f(1.4074) = 1.4074^3 - 3 \times 1.4074 + 1$
$$= -0.4345$$

Since $f(1.4074) < 0$ and $f(2) > 0$, the root lies between 1.4074 and 2.

Take $a = 1.4074$ and $b = 2$.

The 3rd approximation is
$$x_3 = \frac{x_2 f(b) - b f(x_2)}{f(b) - f(x_2)}$$
$$= \frac{1.4074 \times 3 - 2 \times (-0.4345)}{3 + 0.4345}$$
$$= \frac{4.2222 + 0.869}{3.4345} = 1.48424$$

Now $f(1.48424) = (1.48424)^3 - 3 \times 1.48424 + 1$
$$= 3.2576 - 4.472 + 1 = -0.1896$$

Since $f(1.48424) < 0$ and $f(2) > 0$, the root lies between 1.48424 and 2.

Let $a = 1.4824$ and $b = 2$.

The 4th approximation is
$$x_4 = \frac{x_3 f(b) - b f(x_3)}{f(b) - f(x_3)}$$
$$= \frac{1.4824 \times 3 - 2(-0.1896)}{3 + 0.1896} = 1.5132$$

Now $f(1.5132) = 1.5132^3 - 3 \times 1.5132 + 1$
$$= 3.4649 - 4.5396 + 1$$
$$= -0.0747$$

Since $f(1.5132) < 0$ and $f(2) > 0$, the root lies

between 1.5132 and 2. Let $a = 1.5132$ and $b = 2$.

The 5th approximation is $x_5 = \frac{x_4 f(b) - b f(x_4)}{f(b) - f(x_4)}$

$$= \frac{1.5132 \times 3 - 2(-0.0747)}{3 + 0.0747} = 1.525$$

Now $f(1.525) = (1.525)^3 - 3 \times 1.525 + 1$
 $= 3.5466 - 4.575 + 1 = -0.0284.$

Since $f(1.525) < 0$ and $f(2) > 0$, the root lies

between 1.525 and 2

Take $a = 1.525$ and $b = 2$

The 6th approximation is $x_6 = \frac{x_5 f(b) - b f(x_5)}{f(b) - f(x_5)}$

$$= \frac{1.525 \times 3 - 2(-0.0284)}{3 + 0.0284} = 1.5295$$

Now $f(1.5295) = (1.5295)^3 - 3(1.5295) + 1$
 $= -0.0104$

Since $f(1.5295) < 0$ and $f(2) > 0$, the root lies

between 1.5295 and 2

Take $a = 1.5295$ and $b = 2$

$\therefore$ The 7th approximation is $x_7 = \frac{x_6 f(b) - b f(x_6)}{f(b) - f(x_6)}$

$$= \frac{1.5295 \times 3 - 2 \times (-0.0104)}{3 + 0.0104} = 1.5311$$

Now $f(1.5311) = (1.5311)^3 - 3 \times (1.5311) + 1$
 $= -0.004$

Since $f(1.5311) < 0$ and $f(2) > 0$, the root lies

between 1.5311 and 2

Take $a = 1.5311$ and $b = 2$

The 8th approximation is $x_8 = \frac{x_7 f(b) - b f(x_7)}{f(b) - f(x_7)}$

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$$= \frac{1.5311 \times 3 - 2(-0.0040)}{3 + 0.0040} = 1.5317$$

$$\begin{aligned}\text{Now } f(1.5311) &= (1.5311)^3 - 3 \times (1.5311) + 1 \\ &= -0.004\end{aligned}$$

Since $f(1.5317) < 0$ and $f(2) > 0$, the root lies between 1.5317 and 2

Take $a = 1.5317$ and $b = 2$

$$\text{The 9th approximation is } x_9 = \frac{x_8 f(b) - b f(x_8)}{f(b) - f(x_8)}$$

$$= \frac{1.5317 \times 3 - 2(-0.0016)}{3 + 0.0016} = 1.5319$$

$$= 1.532 \text{ (correct to 3 places of decimals)}$$

$$\therefore x_8 = x_9 = 1.532$$

$\therefore$ The required root is 1.532 (correct to 3 places of decimals)

Problem 2 Find the smallest +ve root of $x^2 - \log_{10}^x - 12 = 0$, by Regula Falsi method.

Solution Let $f(x) = x^2 - \log_{10}^x - 12$

$$f(3) = 9 - 1.0986 - 12 = -4.0986 < 0$$

$$f(4) = 16 - 1.3863 - 12 = 2.6137$$

Since $f(3) < 0$ and $f(4) > 0$, the root lies between 3 and 4

Take $a=3$ and $b=4$

/ The 1st approximation to the root is given by

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{3f(4) - 4f(3)}{f(4) - f(3)}$$

$$= \frac{3 \times 2.6137 - 4 \times (-4.0986)}{2.6137 + 4.0986} = 3.6106$$

$$\text{Now } f(x_1) = f(3.6106)$$

$$= 13.0364 - 1.2839 - 12 = -0.2474$$

Since $f(3.6106) < 0$ and $f(4) > 0$, the root lies

between 3.6106 and 4

Take $a=3.6106$ and $b=4$

/ The 2nd approximation is

$$x_2 = \frac{x_1 f(b) - b f(x_1)}{f(b) - f(x_1)} = \frac{3.6106 \times 2.6137 - 4 \times (-0.2474)}{2.6137 + 0.2474} = 3.6443$$

Now $f(x_2) = f(3.6443)$

$$= 13.2809 - 1.2932 - 12$$

$$= -0.0123$$

Since $f(3.6443) < 0$ and $f(4) > 0$, the root lies

between 3.6443 and 4.

Take $a = 3.6443$ and $b=4$

The 3rd approximation is $x_3 = \frac{x_2 f(b) - f(x_2)b}{f(b) - f(x_2)}$

$$= \frac{3.6443 \times 2.6137 - 4(-0.0123)}{2.6137 + 0.0123} = 3.6460$$

Now $f(x_3) = f(3.646)$

$$= 13.2933 - 1.2936 - 12$$

$$= -0.0003$$

Since $f(3.646) < 0$ and $f(4) > 0$, the root lies

between 3.646 and 4.

Take $a=3.646$ and $b=4$

/ The 4th approximation is

$$x_4 = \frac{x_3 f(b) - b f(x_3)}{f(b) - f(x_3)} = \frac{3.646 \times 2.6137 - 4(-0.0003)}{2.6137 + 0.0003} = 3.6461$$

Hence the required root is 3.646 correct to 3 places of decimals

Space for Hints**Problem 3** Find a root of the equation $x^3 - 3x - 5 = 0$

by the method of false position.

Solution Let $f(x) = x^3 - 3x - 5$

$$\text{Here } f(2) = 2^3 - 3 \times 2 - 5 = -3 < 0$$

$$f(3) = 3^3 - 3 \times 3 - 5 = 13 > 0$$

Clearly $f(2)$ and $f(3)$ are of opposite signs/ The root lies between 2 and 3 Let $a=2$, and $b=3$ **The 1st approximation is**
$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{2 \times 13 - 3(-3)}{13 + 3} = 2.1875$$

$$\text{Now } f(x_1) = f(2.1875) = 10.4675 - 6.5625 - 5 = -1.095$$

Since $f(x_1) < 0$ and $f(3) > 0$.

The root lies between 2.1875 and 3

Let $a = 2.1875$ and $b=3$. **$\therefore$ The 2nd approximation is**
$$x_2 = \frac{x_1 f(b) - bf(x_1)}{f(b) - f(x_1)}$$

$$= \frac{2.1875 \times 13 - 3 \times (-1.095)}{13 + 1.095} = 2.2506$$

$$\text{Now } f(x_2) = f(2.2506) = 11.3997 - 6.7518 - 5 = -0.3521$$

Since $f(2.2506) < 0$ and $f(3) > 0$, the root lies

between 2.2506 and 3.

Let $a = 2.2506$ and $b=3$ **$\therefore$ The 3rd approximation is**
$$x_3 = \frac{x_2 f(b) - bf(x_2)}{f(b) - f(x_2)}$$

$$= \frac{2.2506 \times 13 - 3 \times (-0.3521)}{13 + 0.3521} = 2.2704$$

$$\text{Now } f(x_3) = f(2.2704) = 11.7033 - 6.8112 - 5 = -0.1079 < 0$$

Since $f(2.2704) < 0$ and $f(3) > 0$.

$\therefore$ the root lies between 2.2704 and 3.

Let $a = 2.2704$ and $b=3$.

$\therefore$ The 4th approximation is $x_4 = \frac{x_3 f(b) - b f(x_3)}{f(b) - f(x_3)}$

$$= \frac{2.2704 \times 13 - 3(-0.1079)}{13 + 0.1079} = 2.2764$$

Now $f(x_4) = f(2.2764) = 11.7963 - 6.8292 - 5 = -0.0329$

Since $f(2.2764) < 0$ and $f(3) > 0$, the root lies between

2.2764 and 3.

Take $a = 2.2764$ and $b=3$.

$\therefore$ The 5th approximation is $x_5 = \frac{x_4 f(b) - b f(x_4)}{f(b) - f(x_4)}$

$$= \frac{2.2764 \times 13 - 3 \times (-0.0329)}{13 + 0.0329} = 2.2782$$

Now $f(x_5) = f(2.2782) = 11.8243 - 6.8346 - 5 = -0.0103$

Since $f(2.2782) < 0$ and $f(3) > 0$, the root lies between

2.2782 and 3.

Take $a = 2.2782$ and $b=3$.

$\therefore$ The 6th approximation is $x_6 = \frac{x_5 f(b) - b f(x_5)}{f(b) - f(x_5)}$

$$= \frac{2.2782 \times 13 - 3(-0.0103)}{13 + 0.0103} = 2.2788$$

Now $f(x_6) = f(2.2788) = 11.8336 - 6.8364 - 5 = -0.0028$

Since $f(2.2788) < 0$ and $f(3) > 0$, the root lies between

2.2788 and 3.

Take $a = 2.2788$ and $b=3$.

$\therefore$ The 7th approximation is $x_7 = \frac{x_6 f(b) - b f(x_6)}{f(b) - f(x_6)}$

$$= \frac{2.2788 \times 13 - 3(-0.0028)}{13 + 0.0028} = 2.279$$

$$\therefore x_6 = x_7 = 2.279$$

$\therefore$ The solution is 2.279 (correct to 3 places of decimals)

Problem 4 Find by Regula falsi method the +ve root of

$$x^2 - \log_{10} x - 12 = 0$$

Solution Let $f(x) = x^2 - \log_{10} x - 12$

$$f(3) = 9 - 0.4771 - 12 = -3.4771$$

$$f(4) = 16 - 0.6021 - 12 = 3.3979$$

Since $f(3) < 0$ and $f(4) > 0$, $\therefore$ The root lies between 3 and 4.

Let $a=3$ and $b=4$.

$\therefore$ The 1st approximation is

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{3 \times 3.3979 - 4(-3.4771)}{3.3979 + 3.4771} = 3.5058$$

Since $f(3.5058) < 0$ and $f(4) > 0$, two root lies between 3.5058 and 4.

Let $a = 3.5058$ and $b=4$.

$\therefore$ The 2nd approximation is

$$x_2 = \frac{x_1 f(b) - bf(x_1)}{f(b) - f(x_1)} = \frac{3.5058 \times 3.3979 - 4(-0.2542)}{3.3979 + 0.2542} = 3.5402$$

$$\text{Now } f(x_2) = f(3.5402) = -0.016$$

Since $f(3.5402) < 0$ and $f(4) > 0$, the root lies between 3.5402 and 4.

Let $a = 3.5402$ and $b=4$

$\therefore$ The 3rd approximation is

$$x_3 = \frac{x_2 f(b) - bf(x_2)}{f(b) - f(x_2)} = \frac{3.5402 \times 3.3979 - 4(-0.016)}{3.3979 + 0.016} = 3.542$$

$$\text{Now } f(x_3) = f(3.542) = -0.0007$$

Since $f(3.542) < 0$ and $f(4) > 0$, the root lies between

3.542 and 4.

Take $a = 3.542$ and $b = 4$

$\therefore$ The 4th approximation is

$$x_4 = \frac{x_3 f(b) - b f(x_3)}{f(b) - f(x_3)} = \frac{3.5424 \times 3.3979 - 4(-0.0007)}{3.3979 + 0.0007} = 3.542$$

Hence the required root is 3.542 (correct to 3 places of decimals)

Problem 5 Solve the equation $xe^x - 2 = 0$ by Regula Falsi method

Solution: Let $f(x) = xe^x - 2$

Now $f(0) = 0 - 2 = -2 < 0$

$f(1) = 1 \cdot e^1 - 2 = e - 2 = 2.34 - 2 = 0.71828 > 0$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

Let $a = 0$ and $b = 1$.

$\therefore$ The 1st approximation is

$$\begin{aligned} x_1 &= \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0 \cdot f(1) - 1 \cdot f(0)}{f(1) - f(0)} = \frac{0 - (-2)}{0.71828 + 2} \\ &= \frac{2}{2.71828} = 0.73576 \end{aligned}$$

Now $f(x_1) = f(0.73576) = 0.73576 \times e^{0.73576} - 2 = -0.46442$

Since $f(0.73576) < 0$ and $f(1) > 0$, the root lies between 0.73576 and 1

Let $a = 0.73576$ and $b = 1$

$\therefore$ The 2nd approximation is

$$\begin{aligned} x_2 &= \frac{x_1 f(b) - b f(x_1)}{f(b) - f(x_1)} = \frac{0.73576 \times (0.71828) - 1(-0.46442)}{0.71828 + 0.46442} \\ &= 0.83952 \end{aligned}$$

Now $f(x_2) = f(0.83952) = 0.83952 \times e^{0.83952} - 2 = -0.0563$

Since $f(0.83952) < 0$ and $f(1) > 0$, the root lies between

0.83952 and 1

Let $a = 0.83952$ and $b = 1$.

Space for Hints

$\therefore$ The 3rd approximation is

$$x_3 = \frac{x_2 f(b) - b f(x_2)}{f(b) - f(x_2)} = \frac{(0.83952)(.71828) - 1(-0.0563)}{0.71828 + 0.0563}$$

$$= 0.85118$$

$$\text{Now } f(x_3) = f(0.85118) = 0.85118 \times e^{.85118} - 2 = -0.00619$$

Since $f(0.85118) < 0$ and $f(1) > 0$, the root lies between

0.85118 and 1

Let $a = 0.85118$ and $b=1$

The 4th approximation is

$$x_4 = \frac{x_3 f(b) - b f(x_3)}{f(b) - f(x_3)} = \frac{(0.85118)(.71828) - 1 \times (-0.00619)}{0.71828 + 0.00619}$$

$$= 0.85245$$

$$\text{Now } f(x_4) = f(0.85245) = (.85245)e^{.85245} - 2 = -0.00068$$

Since $f(0.85245) < 0$ and $f(1) > 0$, the root lies between

0.85245 and 1 Let $a = 0.85245$ & $b=1$

/ The 5th approximation is

$$x_5 = \frac{x_4 f(b) - b f(x_4)}{f(b) - f(x_4)} = \frac{(0.85245)(.71828) - 1 \times (-0.00068)}{0.71828 + 0.00068}$$

$$= 0.85259 \text{ (Correct to 5 places of decimals)}$$

$$\text{Now } f(x_5) = f(.85259) = (.85259)e^{.85259} - 2 = -0.00007$$

Since $f(.85259) < 0$ and $f(1) > 0$, the root lies between .85259 and 1.

Let $a = .85259$ & $b=1$.

/ The 6th approximation is

$$x_6 = \frac{x_5 f(b) - f(x_5)b}{f(b) - f(x_5)} = \frac{(0.85259)(.71828) - 1(-0.00007)}{0.71828 + 0.00007}$$

$$= 0.85260$$

Clearly $x_4 = x_5 = x_6 = 0.8525$ (Correct to 4 places of decimals)

Hence the required root is 0.8525 (Correct to 4 places of decimals)

Exercises Using Regular false method (Method of False position).

Solve the following equations.

1) $x^3 - 2x - 50 = 0$

2) $x \log_{10} x = 1.2$

3) $e^x - 3x = 0$

4) $x^3 - 4x + 1 = 0$ which lies between 0 and 1

5) $2x - 3 \sin x = 5$

6) $\cos x - 3x + 1 = 0$

7) $2x - \log_{10} x = 7$

8) $xe^x = \cos x$

Section 6.4

Newton – Raphson Method (or) Newton's method (or)

Method of tangent's

Let x_0 be an approximate root of the equation $f(x) = 0$.

Let $x_1 = x_0 + h$ be the exact root where h is very small,

+ve (or) (-)ve.

$$\therefore f(x_1) = 0 \longrightarrow \textcircled{1}$$

By Taylor's series expansion, We have

$$f(x_1) = f(x_0 + h) = f(x_0) + \frac{h}{1!} f'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots$$

Since $f(x_1) = 0$ and h is very small,

h^2 and higher powers of h can be neglected.

$$\text{Hence } f(x_0) + hf'(x_0) = 0$$

Space for Hints

$$\Rightarrow h f'(x_0) = -f(x_0)$$

$$\Rightarrow h = \frac{-f(x_0)}{f'(x_0)}, \text{ if } f'(x_0) \neq 0$$

Hence $x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$ is a first approximation.

Similarly Starting with x_1 , we set the next approximation to the root given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $n = 0, 1, 2, 3, \dots$ and it is known as

Newton - Raphson's iteration formula (or) simply Newton Raphson's formula.

Note The order of convergence of the Newton Raphson method is at least 2.

Section 6-4-1

Geometrical Interpretation of Newton – Raphson Method

Let x_0 be a point near the root α of the equation $f(x) = 0$

Let $A = (x_0, f(x_0))$

Then the equation of the tangent to the curve $y = f(x)$ at the A is given

$$\text{by } y - y_1 = m(x - x_1)$$

$$\text{ie, } y - f(x_0) = f'(x_0)(x - x_0)$$

Let $(x_1, 0)$ be the point of intersection of the tangent with the x-axis.

$$\text{Hence } -f(x_0) = f'(x_0)(x_1 - x_0)$$

$$\Rightarrow x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Which is the 1st approximation of the root.

Similarly if the tangent to the curve $A_1 = (x_1, f(x_1))$ intersects the x -axis, at $(x_2, 0)$, then x_2 gives the 2nd approximation to the root.

Repeating this process we find the successive approximations to the root.

Note Because of this geometrical interpretation Newton: Raphson method is also referred as the Method of tangents.

Solved Problems

Problem 1

Find the first approximation of the root lying between 0 and 1 of the equation $x^3 + 3x - 1 = 0$ by Newton – Raphson formula

Solution

Let $f(x) = x^3 + 3x - 1$

Hence $f'(x) = 3x^2 + 3$

Given that the root lies between 0 and 1. Take $x_0 = 0$

Newton's formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

When $n=0$, the 1st approximation is $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Now $f(x_0) = f(0) = 0^3 + 3 \times 0 - 1 = -1$

$f'(x_0) = f'(0) = 3 \times 0 + 3 = 3$, $x_1 = 0 - \frac{(-1)}{3} = \frac{1}{3} = 0.3333$

$\therefore$ The 1st approximation is 0.3333

Problem 2

Write Newton Raphson formula to obtain the curve root of N

Solution Let $x = \sqrt[3]{N} \Rightarrow x^3 = N \Rightarrow x^3 - N = 0$

Let $f(x) = x^3 - N$ and hence $f'(x) = 3x^2$

The Newton Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}; n=0,1,2$$

$$x_{n+1} = x_n - \frac{(x_n^3 - N)}{3x_n^2} \quad n = 0,1,2,3,\dots$$

Problem 3 Find the real root of $x^3 - 3x + 1 = 0$

lying between 1 and 2 upto 3 places of decimals by Newton - Raphson method.

Solution Let $f(x) = x^3 - 3x + 1$

$$\Rightarrow f'(x) = 3x^2 - 3$$

Now $f(1) = 1^3 - 3 \times 1 + 1 = -1$ and $f(2) = 2^3 - 3 \times 2 + 1 = 9 - 6 = 3$

Hence one root lies between 1 and 2

Let the initial approximation be $x_0 = 1$.

The Newton - Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \longrightarrow \textcircled{1}, n=0,1,2,3,\dots$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Now $f(x_0) = f(1) = -1$; $f'(x_0) = f'(1) = 3 - 3 = 0$

Since $f'(1) = 0$ Newton's approximation formula cannot be applied for the initial approximation as $x = 1$. Let us take $x_0 = 1.5$

New $f(x_0) = f(1.5) = (1.5)^3 - 3 \times 1.5 + 1 = -0.125$

$$f'(x_0) = f'(1.5) = 3 \times (1.5)^2 - 3 = 6.75 - 3 = 3.75$$

$$/ x_1 = 1.5 - \frac{(-0.125)}{3.75} = 1.5 + 0.0333$$

$$\therefore x_1 = 1.5333$$

$$\text{Now } f(x_1) = f(1.5333) = (1.5333)^3 - 3 \times 1.5333 + 1 = 0.0049$$

$$\text{and } f'(x_1) = f'(1.5333) = 3 \times (1.5333)^2 - 3 = 4.053$$

$$\text{Hence } x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5333 - \frac{0.0049}{4.053} = 1.5321$$

$$\text{Now } f(x_2) = f(1.5321) = (1.5321)^3 - 3 \times (1.5321) + 1 = 0$$

$$\therefore x_2 \text{ is a root of } f(x) = 0$$

Hence the required root is 1.532 correct upto 3 places of decimals.

Problems 4

Evaluate $\sqrt{12}$ to four places of decimals by Newton Raphson method.

Solution Let $x = \sqrt{12}$

$$\Rightarrow x^2 = 12 \Rightarrow x^2 - 12 = 0$$

$$\text{Let } f(x) = x^2 - 12$$

$$\text{Hence } f'(x) = 2x$$

$$\text{Now } f(3) = 3^2 - 12 = 9 - 12 = -3 \text{ and } f(4) = 4^2 - 12 = 16 - 12 = 4$$

$\therefore$ The root lies between 3 and 4

Since the numerical value of $f(3)$ is less than $f(4)$ we take the initial approximation as $x_0 = 3$.

$$\text{The Newton Raphson formula is } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \longrightarrow \textcircled{1}$$

Space for Hints

$$\text{Now } f(x_0) = f(3) = -3, f'(x_0) = f'(3) = 2 \times 3 = 6$$

$$\textcircled{1} \text{ becomes } x_1 = +3 - \frac{(-3)}{6} = 3 + \frac{1}{2} = 3.5$$

$$\text{Now } f(x_1) = f(3.5) = (3.5)^2 - 12 = 0.25$$

$$f'(x_1) = f'(3.5) = 2 \times 3.5 = 7$$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.5 - \frac{.25}{7} = 3.4643$$

$$\text{Now } f(x_2) = f(3.4643) = (3.4643)^2 - 12 = 0.0014$$

$$f'(x_2) = f'(3.4643) = 2 \times 3.4643 = 6.9286$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3.4643 - \frac{0.0014}{6.9286} = 3.4641$$

The required root is 3.4641 correct to 4 places of decimals.

Problems 5 Find the real root of $x e^x - 2 = 0$ Correct to three places of decimals using the method of tangents.

Solution Let $f(x) = x e^x - 2$

$$\therefore f'(x) = e^x + x e^x = e^x (x + 1)$$

$$\text{Now } f(0) = 0 \times e^0 - 2 = -2$$

$$f(1) = 1 \times e^1 - 2 = e - 2 = 2.7183 - 2 = 0.7183$$

$\therefore$ The root lies between 0 and 1

Since the numerical value of $f(1)$ is less than

$f(0)$, we can take the initial approximation as $x_0 = 1$

The Newton Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, 3, \dots$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \text{ Now } f(x_0) = f(1) = 0.7183$$

$$f'(x_0) = f'(1) = e^1(1+1) - 2e = 5.4366$$

$$x_1 = 1 - \frac{0.7183}{5.4366} = 1 - 0.1321 = 0.8679$$

$$\text{Now } f(x_1) = f(0.8679) = (.8679)e^{.8679} - 2 = 0.0673 \text{ and}$$

$$f'(x_1) = f'(0.8679) = e^{0.8679}(0.8679 + 1) = 4.4492$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \text{ becomes}$$

$$x_2 = 0.8679 - \frac{0.0673}{4.4492} = 0.8528$$

$$\text{Now } f(x_2) = f(0.8528) = .8528xe^{.8528} - 2 = 0.0008$$

$$f'(x_2) = f'(.8528) = e^{.8528}(1 + .8528) = 4.3471$$

$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \text{ becomes } x_3 = 0.8528 - \frac{.0008}{4.3471} = 0.8526$$

Now x_2 and x_3 are approximately equal, upto third places of decimals we can take the required root as 0.853.

The three iterations are given in the following table.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$	x_{n+1}
0	1	.7183	5.4366	.1321	.8679	x_1
1	.8679	.0673	4.4493	.0151	.8528	x_2
2	.8528	.0008	4.3470	.0002	.8526	x_3

Problem 6

Find the negative root of $x^3 - 2x + 5 = 0$ correct to three places of decimals by the Newton's Method.

Solution Let $g(x) = x^3 - 2x + 5$

We find the positive root of $g(-x) = 0$

$$\text{ie. } g(-x) = (-x)^3 - 2(-x) + 5 = -x^3 + 2x + 5 = 0$$

Space for Hints

ie, $g(-x) = x^3 - 2x - 5 = 0$

$\therefore$ We find the positive root of $f(x) = x^3 - 2x - 5 = 0$

Now $f(x) = x^3 - 2x - 5$

$\therefore f'(x) = 3x^2 - 2$

$$\Rightarrow f(2) = 2^3 - 2 \times 2 - 5$$

$$= 8 - 4 - 5 = -1 < 0$$

and $f(3) = 3^3 - 2 \times 3 - 5$

$$= 27 - 6 - 5 = 16 > 0$$

$\therefore$ One root lies between 2 and 3

Since the numerical value of $f(2)$ is less than the numerical value of $f(3)$, we can take the initial approximation as $x_0 = 2$

$\therefore$ The Newton – Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad n = 0, 1, 2, 3, \dots$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Now $f(x_0) = f(2) = -1$ and

$$f'(x_0) = f'(2) = 3 \times 2^2 - 2 = 10$$

$$\therefore x_1 = 2 - \frac{(-1)}{10} = 2 + \frac{1}{10} = \frac{21}{10} = 2.1$$

Now $f(x_1) = f(2.1) = (2.1)^3 - 2(2.1) - 5$

and $f'(x_1) = f'(2.1) = 3(2.1)^2 - 2 = 11.23$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \text{ becomes}$$

$$x_2 = 2.1 - \left(\frac{.061}{11.23} \right) = 2.0946$$

Now $f(x_2) = f(2.0946)$

$$= (2.0946)^3 - 2(2.0946) - 5 = 0.0005$$

$$f'(x_2) = f'(2.0946) = 3(2.0946)^2 - 2 = 11.1619$$

$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \text{ becomes}$$

$$x_3 = 2.0946 - \frac{.0005}{11.1619} = 2.09456$$

Clearly x_2 and x_3 are equal upto 3 places of decimals, we can take the required root to be 2.095 (correct to 3 places of decimals)

$\therefore$ The -ve root of $x^3 - 2x + 5 = 0$ is -2.095.

Problem 7

Using Newton – Raphson iterative method, find the real root of

$$x \log_{10}^x = 1.2 \text{ correct to 3 places of decimals}$$

Solution

$$\text{Let } f(x) = x \log_{10}^x - 1.2$$

$$\therefore f'(x) = x \left[\frac{\log_{10} e}{x} \right] + \log_{10}^x$$

$$= \log_{10} e + \log_{10}^x = 0.4343 + \log_{10}^x$$

$$\text{Now } f(1) = 1 \times \log_{10}^1 - 1.2 = 0 - 1.2 = -1.2$$

$$f(2) = 2 \log_{10}^2 - 1.2$$

$$= 2 \times .3010 - 1.2 = -.598$$

$$f(3) = 3 \log_{10}^3 - 1.2$$

$$= 3 \times .4771 - 1.2 = -0.2313$$

$\therefore$ One root lies between 2 and 3.

Let the initial approximation be $x_0 = 2$

Newton Raphson formula is

Space for Hints

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}; \quad n = 0, 1, 2, 3, \dots \quad \text{①}$$

The 1st approximation is obtained by putting $n = 0$ in ①

we set $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Now $f(x_0) = f(2) = -0.598$

$f'(x_0) = f'(2) = .4343 + \log_{10}^2 = .4343 + .3010 = .7353$

$\therefore x_1 = 2 - \frac{(-.598)}{.7353} = 2 + .8133 = 2.8133$

The 2nd approximation is $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

Now $f(2.8133) = f(x_1) = 2.8133 \times .4492 - 1.2 = 1.2637 - 1.2 = 0.0637$

$f'(x_2) = f'(2.8133) = 0.4343 + \log_{10}^{(2.8133)} = .4343 + .4492 = .8835$

$\therefore x_2 = 2.8133 - \left(\frac{.0637}{.8835} \right) = 2.8133 - .0721 = 2.7412$

The 3rd approximation is $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

Now $f(x_2) = f(2.7412) = 2.7412 \times 0.4349 - 1.2$
 $= 1.2004 - 1.2 = 0.004$

$f'(x_2) = f'(2.7412) = 0.4343 + 0.4379 = 0.8722$

$\therefore x_3 = 2.7412 - \left(\frac{0.004}{.8722} \right) = 2.7412 - 0.0005 = 2.7407$

Clearly x_2 and x_3 are equal upto 3 places of decimals we can take the required root to be 2.741 correct to 3 places of decimals

$\therefore$ The required root is 2.741.

Problem 8 Find by Newton – Raphson method correct to four places of decimals the root lies between 0 and 1 of the equation $3x - \cos x - 1 = 0$.

Solution

$$\text{Let } f(x) = 3x - \cos x - 1$$

$$\therefore f'(x) = 3 + \sin x$$

$$\text{Now } f(0) = 3 \times 0 - \cos 0 - 1 = -1 - 1 = -2 \text{ and}$$

$$\begin{aligned} f(1) &= 3 \times 1 - \cos 1 - 1 = 2 - \cos 1 \\ &= 2 - .5403 = 1.4597 \end{aligned}$$

Since the numerical value of $f(1)$ is less than that of $f(0)$, we can take the initial approximation as $x_0 = 1$.

The Newton - Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3, \dots$$

$\therefore$ The 1st approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\text{Now } f(x_0) = f(1) = 1.4597;$$

$$f'(x_0) = f'(1) = 3 + \sin 1 = 3 + .8415 = 3.8415$$

$$\therefore x_1 = 1 - \frac{1.4597}{3.8415} = 0.62$$

$$\text{Now the 2nd approximation is } x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\text{Here } f(x_1) = f(.62) = 3 \times .62 - \cos(.62) - 1 = .0461$$

$$f'(x_1) = f'(.62) = 3 + \sin(.62) = 3.5810$$

$$\therefore x_2 = 0.62 - \frac{0.0461}{3.581} = 0.6071$$

$$\text{The 3rd approximation is } x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\text{Now } f(x_2) = f(.6071) = 3 \times .6071 + \cos(.6071) - 1 = 0$$

$$f'(x_2) = f'(.6071) = 3 + \sin(.6071) = 3.7505$$

Space for Hints

$$\therefore x_3 = .6071 - \frac{0}{3.7505} = 0.6071$$

Since x_2 and x_3 are equal upto 4 places of decimals.

$\therefore$ The required root is 0.6071.

Problem 9 Write down Newton – Raphson formula for finding $\sqrt{a}$ when

‘a’ is a the number and hence find $\sqrt{5}$

Solution Let $x = \sqrt{a}$

$$\Rightarrow x^2 = a \Rightarrow x^2 - a = 0$$

$$\text{Let } f(x) = x^2 - a, \quad f'(x) = 2x$$

$$\begin{aligned} \text{Wkt } x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \left[\frac{x_n^2 - a}{2x_n} \right] \\ &= x_n - \frac{x_n}{2} + \frac{a}{2x_n} \\ &= \frac{x_n}{2} + \frac{a}{2x_n} \end{aligned}$$

$$\therefore x_{n+1} = \frac{1}{2} \left[x_n + \frac{a}{x_n} \right] \xrightarrow{\text{①}} \text{in the iterative formula to find } \sqrt{a}$$

To find the value of $\sqrt{5}$.

Put $a = 5$ and also $x = \sqrt{5}$ lies between 2 and 3

Let $x_0 = 2$

$$\begin{aligned} x_{n+1} &= \frac{1}{2} \left[x_0 + \frac{a}{x_0} \right] \\ x_1 &= \frac{1}{2} \left[x_n + \frac{5}{x_n} \right] \\ &= \frac{1}{2} \left[2 + \frac{5}{2} \right] = \frac{1}{2} \times \frac{9}{2} = 2.25 \end{aligned}$$

$$x_2 = \frac{1}{2} \left[x_1 + \frac{5}{x_1} \right]$$

$$= \frac{1}{2} \left[2.25 + \frac{5}{2.25} \right] = 2.2361$$

$$x_3 = \frac{1}{2} \left[x_2 + \frac{5}{x_2} \right] = \frac{1}{2} \left[2.2361 + \frac{5}{2.2361} \right] = 2.2361$$

Hence $x_2 = x_3 = 2.2361$

Hence the approximate value of $\sqrt{5} = 2.2361$ correct to 4 places of decimals.

Problem 10

Solve by Newton's method, a root of $e^x - 4x = 0$

Solution Let $f(x) = e^x - 4x$, $f'(x) = e^x - 4$

$$f(0) = e^0 - 4 \times 0 = 1 (+ve) \text{ no.}$$

$$f(1) = e^1 - 4 \times 1 = e - 4 = -1.2817 < 0.$$

$\therefore$ One root lies between 0 and 1

$\therefore$ The $|f(0)| < |f(1)|$

$\therefore$ The root is nearer to 0.

Let $x_0 = 0.3$

By Newton – Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}; \quad n = 0, 1, 2, 3, \dots$$

$\therefore$ The 1st approximation is $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$\text{Now } f(x_0) = f(0.3) = e^{0.3} - 4 \times 0.3 = 0.1499$$

$$f'(x_0) = f'(0.3) = e^{0.3} - 4 = -2.650$$

$$\therefore x_1 = 0.3 - \frac{0.1499}{-2.650} = 0.3566$$

Space for Hints

The 2nd approximation is $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$\text{Now } f(x_1) = f(.3566) = e^{.3566} - 4 \times .3566$$

$$f'(x_1) = f'(.3566) = e^{.3566} - 4$$

$$\therefore x_2 = .3566 - \frac{f(.3566)}{f'(.3566)} = 0.3574$$

The 3rd approximation is $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$= 0.3574 - \frac{f(.3574)}{f'(.3574)}$$

$$= .3574 - \left[\frac{e^{.3574} - 4(.3574)}{e^{.3574} - 4} \right] = 0.3574$$

Hence x_2 and $x_3 = 0.3574$

Hence the root is 0.3574.

Exercises

1] Find a +ve root of each of the following equations using Newton-Raphson method

(i) $x^3 + x - 1 = 0$ (ii) $x^3 - 5x - 6 = 0$

(iii) $2x^3 - 3x - 6 = 0$ (iv) $x^4 - x - 13 = 0$

(v) $x^4 - 3x + 1 = 0$ (vi) $xe^x = \cos x$

(vii) $x \sin x + \cos x = 0$ (viii) $x \cos x = 0$

(ix) $e^{-3} - \sin x = 0$ (x) $\log x - x + 3 = 0$

2] Find a positive real root of $\sqrt[3]{17}$ using Newton - Raphson method

3] Find a -ve root of $x^3 - \sin x + 1 = 0$

4] Find a -ve root of $x^3 - 5x + 11 = 0$ correct to 2 places of decimals using Newton - Raphson method

5] Using Newton - Raphson method, find the value of $\sqrt[5]{10}$ Correct to 4 places of decimals.

Simultaneous Equations

Section: 6.5.1

Introduction: Simultaneous linear algebraic equations occur in several engineering and statistical problems. In this section we deal with several numerical methods for solving such system of equations.

Section: 6.5.2

Simultaneous equations

A system of m linear equations in n unknowns

$x_1, x_2, \dots, x_n$ is given by

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_n \end{aligned} \right\} \longrightarrow \textcircled{1}$$

This set of equations can be written in the matrix

form as $AX = B$ where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} : X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

The $m \times n$ matrix A is called the coefficient matrix.

The Augmented matrix of the system of linear equation $Ax = B$

is given by

$$(A, B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right)$$

A solution of eqns $\textcircled{1}$ is a set of numbers $x_1, x_2, \dots, x_n$ which will satisfy all the ' m ' equations

Back substitution Consider a system of simultaneous linear equations given by $Ax = B$, where A is an $n \times n$ coefficient matrix.

Suppose the matrix A is upper triangular

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & a_{2n} \\ 0 & 0 & a_{33} & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \dots & a_{nn} \end{pmatrix}$$

Then the given system takes the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \dots & a_{1n} \\ 0 & a_{22} & a_{23} & a_{2n} \\ 0 & 0 & a_{33} & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\text{ie, } a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1}$$

$$a_{nn}x_n = b_n$$

$$\text{From the last equation we get } x_n = \frac{b_n}{a_{nn}}$$

Substituting the value of x_n in the previous equation we get

$$x_{n-1} = \frac{1}{a_{(n-1)(n-1)}} \left[b_{n-1} - a_{n-1,n} \left(\frac{b_n}{a_{nn}} \right) \right]$$

Proceeding like this we can find all x_i 's.

This procedure is known as Back substitution.

Similarly considering lower triangular matrix

Space for Hints

$$A = \begin{pmatrix} a_{11} & 0 & 0 \dots & 0 \\ a_{21} & a_{22} & 0 \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} \dots & a_{nn} \end{pmatrix}$$

The given system takes the form

$$a_{11}x_1 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

From the first equation we set $x_1 = \frac{b_1}{a_{11}}$.

Substituting the value of x_1 in the 2nd equation we set

$$x_2 = \frac{1}{a_{22}} \left[b_2 - a_{21} \left(\frac{b_1}{a_{11}} \right) \right]$$

Proceeding like this we can find all x_i 's.

This procedure is known as Forward substitution.

Example Solve the following equation by back substitution

$$2x + 3y + 2z = 7 \longrightarrow (1)$$

$$-3y + z = -2 \longrightarrow (2)$$

$$4z = 4 \longrightarrow (3)$$

Solution From equation (3) $4z = 4$

$$\Rightarrow z = 1$$

Substituting $z = 1$ in equation (2) we set

$$-3z + 1 = -2 \Rightarrow -3y = -3$$

$$\Rightarrow y = 1$$

Space for Hints

Substituting $y = 1$ and $z = 1$ in equation (1) we get

$$2x + 3 \times 1 + 2 \times 1 = 7$$

$$\Rightarrow 2x + 3 + 2 = 7$$

$$\Rightarrow 2x + 5 = 7$$

$$\Rightarrow 2x = 7 - 5 = 2$$

$$\Rightarrow x = 1$$

Hence

$x = 1$	$y = 1$	$z = 1$
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Section 6.6

Solution of Linear system by Gaussian Elimination and Gauss-Jordan Methods and Gauss – Jordan Method

There are two types of methods to solve simultaneous linear algebraic equations with many unknowns

I] Direct Method

- (i) Gauss elimination method
- (ii) Gauss Jordan method

II] Indirect Method (or)

Iterative Method

- (i) Gauss Jacobin method
- (ii) Gauss seidal method

Section 6.7

Gauss Elimination method

Gauss elimination method is a direct method which consists of transforming the given system of simultaneous equations to an upper triangular system. From this transformed system the required solution can be obtained by the method of back substitution.

Consider the system of n equations in ' n ' unknowns given by

$AX = B$ where A is the coefficient matrix where,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nn} \end{pmatrix}, \quad X = \begin{pmatrix} x_{11} \\ x_{2n} \\ \vdots \\ x_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Now the Augmented matrix is

$$(A, B) = \left(\begin{array}{ccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array} \right)$$

To transform the system to an equivalent upper triangular system, we use the following row operations.

a) The row operation $R_i \rightarrow R_i - \frac{a_{i1}}{a_{11}} R_1$; $i = 1, 2, 3, \dots, n$ makes all the

entries $a_{21}, a_{31}, \dots, a_{n1}$ in the first column zero.

Here the first equation is the pivotal equation.

$a_{11} \neq 0$ is called pivot element and $\frac{-a_{i1}}{a_{11}}$ for $i = 1, 2, 3, \dots, n$ are called

multipliers for first elimination.

If $a_{11} = 0$, we interchange the 1st row with another suitable row so as to have $a_{11} \neq 0$.

(b) Next we do the row operation $R_i \rightarrow R_i - \frac{a_{2i}}{a_{22}} R_2$ $i = 3, 4, 5, \dots, n$

This makes all the entries $a_{32}, a_{42}, \dots, a_{n2}$ in the second column zero.

(c) In general, the row operation

$$R_i \rightarrow R_i - \frac{a_{ik}}{a_{kk}} R_k; \quad i = k+1, k+2, \dots, n \text{ will make all the}$$

entries $a_{k+1,k}, a_{k+2,k}, \dots, a_{nk}$ in the k^{th} column zero.

Hence the given system of equations is reduced to the form

$UX = D$ where U is an upper triangular matrix. The required solution can be obtained by the method of Back substitution.

Section 6.8**Gauss - Jordan Elimination Method**

Consider the system of equations $AX = B$ If A is a diagonal matrix then, the given system reduces to

$$\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

This system reduces to the following 'n' equations

$$a_{11}x_1 = b_1 : a_{22}x_2 = b_2 \quad \dots \quad a_{nn}x_n = b_n$$

Hence we get solution directly as

$$x_1 = \frac{b_1}{a_{11}}; \quad x_2 = \frac{b_2}{a_{22}} \quad \dots \quad x_n = \frac{b_n}{a_{nn}}$$

The method of obtaining the solution of the system of equations by reducing the matrix A to a diagonal matrix is known as Gauss-Jordan elimination method.

Section 6.9**Calculation of Inverse of a Matrix:**

Let A be an $n \times n$ singular matrix

$$\text{Let } X = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix} \quad \text{be the inverse of } A$$

$\therefore AX = I$ where I is the unit matrix of order n .

$\therefore AX = I$ gives

$$\begin{pmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} & a_{nn} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \dots & x_{1n} \\ x_{21} & x_{22} \dots & x_{2n} \\ \vdots & & \\ x_{n1} & x_{n2} & x_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & & & \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This equation is equivalent to the following 'n' system of simultaneous equations

$$\begin{pmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{2n} \\ \vdots \\ x_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\text{and } \begin{pmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Each of the system of the above n-system of linear equations can be Solved by Gauss elimination method (or) Gauss Jordan method.

Solved Problems

Problem 1

Solve the equations $x + y = 2$; $2x + 3y = 5$ by Gauss elimination method

Solution Given that $x + y = 12 \rightarrow \textcircled{1}$

$$2x + 3y = 5 \rightarrow \textcircled{2}$$

This system can also be written as

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

ie, $A X = B$ where A is the

coefficient matrix $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and

$$B = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

The Augmented matrix is $(A, B) = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 3 & 5 \end{array} \right)$

we note that $a_{11} = 1 \neq 0$ is the pivot.

$$\text{Now } (A, B) = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 3 & 5 \end{array} \right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 3 & 5 \end{array} \right) \quad R_2 : R_2 - 2R_1.$$

$$\therefore \text{The given system of equations is } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\therefore y = 1 \quad \text{and} \quad x + y = 2$$

$$\Rightarrow x = 2 - y = 2 - 1 = 1$$

$\therefore$ By back substitution we get $x = 1$ and $y = 1$

Problem 2 Solve the following system of equations using Gauss elimination method

$$x + y + z = 9 ; 2x - 3y + 4z = 13 ; 3x + 4y + 5z = 40$$

Solution Given that $x + y + z = 9 \rightarrow \textcircled{1}$

$$2x - 3y + 4z = 13 \rightarrow \textcircled{2}$$

$$3x + 4y + 5z = 40 \rightarrow \textcircled{3}$$

The system of equations $\textcircled{1}$, $\textcircled{2}$ & $\textcircled{3}$ can also be written as

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 13 \\ 40 \end{pmatrix}$$

ie, $A X = B$

with the coefficient matrix is $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 9 \\ 13 \\ 40 \end{pmatrix}$$

The Augmented matrix is $(A, B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array} \right)$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & 15 \\ 0 & 1 & 2 & 13 \end{array} \right) \begin{array}{l} R_2 : R_2 - 2R_1 \\ R_3 : R_3 - 3R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 12 & 60 \end{array} \right) R_3 : 5R_3 + R_2.$$

$\therefore$ The given system of equations reducer to the system

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 12 & 60 \end{array} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 15 \\ 60 \end{pmatrix}$$

Hence by Back substitution method

$$12z = 60 \rightarrow \textcircled{4}$$

$$-5y + 2z = -5 \rightarrow \textcircled{5}$$

$$\text{and } x + y + z = 9 \rightarrow \textcircled{6}$$

From $\textcircled{4}$, $z = \frac{60}{12} = 5$

From $\textcircled{5}$, $5y = 2z + 5 = (2 \times 5) + 5 = 15 \Rightarrow y = 3$

Space for Hints

From ⑥, $x = 9 - y - z = 9 - 3 - 5 = 1$

∴ The solution is

$x = 1$
$y = 3$
$z = 5$

Problem 3 Solve the following equations by Gauss Jordan

method $x + y = 2$

$$2x + 3y = 5$$

Solution Given $x + y = 2 \longrightarrow \text{①}$

$$2x + 3y = 5 \longrightarrow \text{②}$$

Now equations ① & ② can be written as $A X = B$

With coefficient matrix $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $B = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

The Augmented matrix is $(A, B) = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 3 & 5 \end{array} \right)$

$$\sim \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array} \right) \quad R_2 : R_2 - 2R_1$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right) \quad R_1 : R_1 - R_2$$

$$\sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow x = 1 \text{ and } y = 1$$

∴ The solution is

$x = 1$
$y = 1$

Problem 4 Solve the following system of equations by Gauss

Jordan method $x + y + 2z = 9$

$$2x - 3y + 4z = 13$$

$$3x + 4y + 5z = 40$$

Solution Given $x + y + 2z = 9 \longrightarrow \text{①}$

$$2x - 3y + 4z = 13 \longrightarrow \text{②}$$

$$3x + 4y + 5z = 40 \longrightarrow \textcircled{3}$$

Space for Hints

The given set of equations ①, ② & ③ can also be written as

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 13 \\ 40 \end{pmatrix} \text{ ie, } AX = B \text{ with}$$

$$\text{The coefficient matrix } A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 9 \\ 13 \\ 40 \end{pmatrix}$$

$$\text{The Augmented matrix is } (A, B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array} \right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{array} \right) \begin{array}{l} R_2 : R_2 - 2R_1 \\ R_3 : R_3 - 3R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 12 & 60 \end{array} \right) \quad R_3 : 5R_3 + R_2.$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 1 & 5 \end{array} \right) \quad R_3 : R_3 / 12$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -5 & 0 & -15 \\ 0 & 0 & 1 & 5 \end{array} \right) \begin{array}{l} R_1 : R_1 - R_3 \\ R_2 : R_2 - 2R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 5 & 0 & 0 & 5 \\ 0 & -5 & 1 & -15 \\ 0 & 0 & 1 & 5 \end{array} \right) \quad R_1 : 5R_1 + R_2.$$

$\therefore$ The given system reduces to

Space for Hints

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -15 \\ 5 \end{pmatrix}$$

$$\Rightarrow z = 5 \quad \begin{cases} -5y = -15 \\ \Rightarrow y = 3 \end{cases} \quad \begin{cases} 5x = 5 \\ \Rightarrow x = 1 \end{cases}$$

$\therefore$ The solution is

$x = 1$
$y = 3$
$z = 5$

Problem 5 Solve the following system of equations by Gauss

Jordan method $5x - 2y + 3z = 18$

$$x - 7y - 3z = -22$$

$$2x - y + 6z = 22$$

Solution The given system of equations reduces to $AX=B$ where A is

the coefficient matrix $\begin{pmatrix} 5 & -2 & 3 \\ 1 & 7 & -3 \\ 2 & -1 & 6 \end{pmatrix}$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 18 \\ -22 \\ 22 \end{pmatrix}$$

The Augmented matrix is $(A, B) = \left(\begin{array}{ccc|c} 5 & -2 & 3 & 18 \\ 1 & 7 & -3 & -22 \\ 2 & -1 & 6 & 22 \end{array} \right)$

$$\sim \left(\begin{array}{ccc|c} 1 & 7 & -3 & -22 \\ 5 & -2 & 3 & 18 \\ 2 & -1 & 6 & 22 \end{array} \right) \quad R_1 \Leftrightarrow R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 7 & -3 & -22 \\ 0 & -37 & 18 & 129 \\ 0 & -15 & 12 & 66 \end{array} \right) \quad \begin{array}{l} R_2 : R_2 - 5R_1 \\ R_3 : R_3 - 2R_1 \end{array}$$

$$\sim \begin{pmatrix} 1 & 7 & -3 & -22 \\ 0 & -37 & 18 & 128 \\ 0 & 0 & 174 & 522 \end{pmatrix} \quad R_3 : 37R_3 - 15R_2$$

$$\sim \begin{pmatrix} 1 & 7 & -3 & -22 \\ 0 & -37 & 18 & 128 \\ 0 & 0 & 1 & 3 \end{pmatrix} \quad R_3 : R_3 / 174$$

$$\sim \begin{pmatrix} 1 & 7 & 0 & -19 \\ 0 & -37 & 0 & 74 \\ 0 & 0 & 1 & 3 \end{pmatrix} \quad \begin{array}{l} R_1 : R_1 + 3R_3 \\ R_2 : R_2 - 18R_3 \end{array}$$

$$\sim \begin{pmatrix} 1 & 7 & 0 & -19 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \quad R_2 : R_2 / -37$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \quad R_1 : R_1 - 7R_2$$

∴ The given system of equations reducer to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \\ 3 \end{pmatrix}$$

⇒

$x = -5$
$y = -2$
$z = 3$

Problem 6

Solve the following system of equations using

(i) Gauss elimination method

(ii) Gauss Jordan method

(a) $2x + y + 4z = 12$
 $8x - 3y + 2z = 20$
 $4x + 11y - z = 33$

(b) $x + y - z + w = 2$
 $2x + y + z - 3w = 1$
 $3x - y - z + w = 2$
 $5x + y + 3z - 2w = 7$

Solution (a) The given system of linear equations can also be written as

$$\begin{pmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 1 \\ 33 \end{pmatrix}$$

ie, $A X = B$ where A is the Coefficient matrix $\begin{pmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{pmatrix}$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 12 \\ 1 \\ 33 \end{pmatrix}$$

(i) Gauss Elimination Method

The Augmented matrix of the given system is

$$(A, B) = \left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 1 \\ 4 & 11 & -1 & 33 \end{array} \right)$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & -1 & -14 & -28 \\ 0 & 9 & -9 & 9 \end{array} \right)$$

$$R_2 : R_2 - 4R_1$$

$$R_3 : R_3 - 2R_1$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & -1 & 1 \end{array} \right)$$

$$R_2 : R_2 \cdot (-7)$$

$$R_3 : R_3 / 9$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -3 & -3 \end{array} \right)$$

$$R_3 : R_3 - R_2$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{array} \right)$$

$$R_3 : R_3 / -3$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{array} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 4 \\ 1 \end{pmatrix}$$

$\therefore$ By back substitution method we have

$$\begin{array}{l} z = 1 \quad ; \quad y + 2z = 4 \quad ; \quad 2x + y + 4z = 12 \\ \downarrow \quad \quad \downarrow \quad \quad \downarrow \\ \textcircled{1} \quad \quad \textcircled{2} \quad \quad \textcircled{3} \end{array}$$

Sub the value of z in $\textcircled{2}$ we set

$$y = 4 - 2z = 4 - (2 \times 1) = 2$$

Sub the values of y and z in $\textcircled{3}$ we set

$$2x = 12 - y - 4z$$

$$= 12 - 2 - 4 \times 1 = 6$$

$$\Rightarrow x = 3$$

$\therefore$ The solution is $x = 3 ; y = 2 ; z = 1$

(ii) Gauss Jordan Method

The Augmented matrix of the given system is

$$(A, B) = \left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \end{array} \right)$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & 4 & -12 \\ 0 & -7 & -14 & -28 \\ 0 & 9 & -9 & 9 \end{array} \right)$$

$$R_2 : R_2 - 4R_1$$

$$R_3 : R_3 - 2R_1$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & -1 & 1 \end{array} \right)$$

$$R_2 : R_2 / (-7)$$

$$R_3 : R_3 / 9$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -3 & -3 \end{array} \right)$$

$$R_3 : R_3 - R_2$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{array} \right)$$

$$R_3 : R_3 / (-3)$$

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$$\sim \left(\begin{array}{ccc|c} 2 & 1 & 0 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & -1 & 1 \end{array} \right)$$

$$R_1 : R_1 - 4R_3$$

$$R_2 : R_2 - 2R_3$$

$$\sim \left(\begin{array}{ccc|c} 2 & 0 & 0 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & -0 & 1 & 1 \end{array} \right)$$

$$R_1 : R_1 - R_2$$

$$\sim \left(\begin{array}{ccc|c} 2 & 0 & 0 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$$

$$\text{ie, } z = 1 ; y = 2 \quad \& \quad 2x = 6$$

$$\Rightarrow x = 3 ; y = 2 ; z = 1$$

The solution is

$$x = 3 ; y = 2 ; z = 1$$

(b) Given

$$x + y - z + w = 2$$

$$2x + y + z - 3w = 1$$

$$3x - y - z + w = 2$$

$$5x + y + 3z - 2w = 7$$

This system can also be written as $A X = B$, where A is the coefficient matrix,

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 1 & -3 \\ 3 & -1 & -1 & 1 \\ 5 & 1 & 3 & -2 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 7 \end{pmatrix}$$

(i) Gauss Elimination Method**Space for Hints**

The Augmented Matrix is

$$(A, B) = \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 2 & 1 & 1 & -3 & 1 \\ 3 & -1 & -1 & 1 & 2 \\ 5 & 1 & 3 & -2 & 7 \end{array} \right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 0 & -1 & 3 & -5 & -3 \\ 0 & -4 & 2 & -2 & -4 \\ 0 & -4 & 8 & -7 & -3 \end{array} \right)$$

$$R_2 : R_2 - 2R_1$$

$$R_3 : R_3 - 3R_1$$

$$R_4 : R_4 - 5R_1$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 0 & -1 & 3 & -5 & -3 \\ 0 & -2 & 1 & -1 & -2 \\ 0 & -4 & 8 & -7 & -3 \end{array} \right)$$

$$R_3 : R_3 / 2$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 0 & -1 & 3 & -5 & -3 \\ 0 & 0 & -5 & 9 & 4 \\ 0 & 0 & -4 & 13 & 9 \end{array} \right)$$

$$R_3 : R_3 - 2R_2$$

$$R_4 : R_4 - 4R_2$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 0 & -1 & 3 & -5 & -3 \\ 0 & 0 & -5 & 9 & 4 \\ 0 & 0 & 0 & 29 & 29 \end{array} \right)$$

$$R_4 : 5R_4 - 4R_3$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 0 & -1 & 3 & -5 & -3 \\ 0 & 0 & -5 & 9 & 4 \\ 0 & 0 & 0 & 29 & 29 \end{array} \right) \quad \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \quad \begin{pmatrix} 2 \\ -3 \\ 4 \\ 29 \end{pmatrix}$$

By back substitution method

$$29w = 29 \Rightarrow w = 1$$

$$-5z + 92 = 4$$

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$$\Rightarrow 5z = 9w - 4 = 9 \times 1 - 4 = 5$$

$$\Rightarrow z = 1$$

$$-y + 3z - 5w = -3$$

$$\Rightarrow y = 3z - 5w + 3 = 3 \times 1 - 5 \times 1 + 3$$

$$\Rightarrow y = 6 - 5 = 1$$

$$x + y - z + w = 2 \Rightarrow x = 2 - w - y + z \Rightarrow x = 1$$

$\therefore$ The solution is

$x = 1$	$z = 1$
$y = 1$	$w = 1$

(ii) Gauss Jordan Method

The Augmented Matrix is

$$(A, B) = \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 2 & 1 & 1 & -3 & 1 \\ 3 & -1 & -1 & 1 & 2 \\ 5 & 1 & 3 & -2 & 7 \end{array} \right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 0 & -1 & 3 & -5 & -3 \\ 0 & -4 & 2 & -2 & -4 \\ 0 & -4 & 8 & -7 & -3 \end{array} \right) \quad \begin{array}{l} R_2 : R_2 - 2R_1 \\ R_3 : R_3 - 3R_1 \\ R_4 : R_4 - 5R_1 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 0 & -1 & 3 & -5 & -3 \\ 0 & -2 & 1 & -1 & -2 \\ 0 & -4 & 8 & -7 & -3 \end{array} \right) \quad R_3 : R_3 / 2$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 0 & -1 & 3 & -5 & -3 \\ 0 & 0 & -5 & 9 & 4 \\ 0 & 0 & -4 & 13 & 9 \end{array} \right) \quad \begin{array}{l} R_3 : R_3 - 2R_2 \\ R_4 : R_4 - 4R_2 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 0 & -1 & 3 & -5 & -3 \\ 0 & 0 & -5 & 9 & 4 \\ 0 & 0 & 0 & 29 & 29 \end{array} \right) \quad R_4 : 5R_4 - 4R_3$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 0 & -1 & 3 & -5 & -3 \\ 0 & 0 & -5 & 9 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right) \quad R_4 : R_4 / 29$$

$$\sim \left(\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 1 \\ 0 & -1 & 3 & 0 & 2 \\ 0 & 0 & -5 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right) \quad \begin{array}{l} R_1 : R_1 - R_4 \\ R_2 : R_2 + 5R_4 \\ R_3 : R_3 - 9R_4 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 1 \\ 0 & -1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right) \quad R_3 : R_3 / -5$$

$$\sim \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right) \quad \begin{array}{l} R_1 : R_1 + R_3 \\ R_2 : R_2 - 3R_3 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right) \quad R_1 : R_1 - R_2$$

From the last equivalent matrix we set

$$x=1, -y=-1, z=1 \text{ \& } w=1$$

$$\text{ie, } x=1, y=1, z=1 \text{ \& } w=1$$

$\therefore$ The solution is

$x=1$	$z=1$
$y=1$	$w=1$

Hence in both methods we set the same solution.

Problem 7 Find the inverse of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} \text{ using Gaussian method}$$

Solution Let $x = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ be the inverse of the given

$$\text{matrix } A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$$

$$\therefore AX = I_3 \text{ where } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{The Augmented system is } (A : I_3) = \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array} \right)$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & +1 & 3 & -3 & 2 & 0 \\ 0 & 7 & 17 & -1 & 0 & 2 \end{array} \right) \quad \begin{array}{l} R_2 : 2R_2 - 3R_1 \\ R_3 : 2R_3 - R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -3 & 2 & 0 \\ 0 & 0 & -4 & 20 & -14 & 2 \end{array} \right) \quad R_3 : R_3 - 7R_2$$

$\therefore$ The equation $AX = I_3$ is equivalent to the following three systems

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 20 \end{pmatrix} \longrightarrow \text{(I)}$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -14 \end{pmatrix} \longrightarrow \text{(II)}$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \longrightarrow \text{(III)}$$

Applying Back substitution to (I), (II) and (III) we will get the inverse of A

From (I), $-4x_{31} = 20$

$$\Rightarrow x_{31} = \frac{20}{-4} = -5$$

$$x_{21} + 3x_{31} = -3$$

$$\Rightarrow x_{21} = -3 - 3x_{31} = -3 - 3 \times (-5)$$

$$= -3 + 15 = 12$$

$$2x_{11} + x_{21} + x_{31} = 1$$

$$\Rightarrow 2x_{11} = 1 - x_{21} - x_{31}$$

$$= 1 - 12 + 5$$

$$= 6 - 12 = -6, \therefore \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \\ -5 \end{pmatrix}$$

$$\Rightarrow x_{11} = -3$$

From (II), $-4x_{32} = -14$

$$\Rightarrow x_{32} = \frac{-14}{-4} = \frac{7}{2}$$

$$x_{22} + 3x_{32} = 2$$

$$\Rightarrow x_{22} = 2 - 3x_{32} = 2 - 3 \times \frac{7}{2}$$

$$= \frac{4 - 12}{2} = \frac{-17}{2}$$

$$2x_{12} + x_{22} + x_{32} = 0$$

$$\Rightarrow 2x_{12} = -x_{22} - x_{32}$$

$$= \frac{17}{2} - \frac{7}{2} = \frac{10}{2} = 5$$

$$\Rightarrow x_{12} = \frac{5}{2}, \quad \therefore \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ -\frac{17}{2} \\ \frac{7}{2} \end{pmatrix}$$

From (III), $-4x_{33} = 2$

$$\Rightarrow x_{33} = \frac{0}{2} - \frac{2}{4} = -\frac{1}{2}$$

$$x_{23} + 3x_{33} = 0$$

$$\Rightarrow x_{23} = -3x_{33} = -3 \times \frac{-1}{2} = \frac{3}{2}$$

$$2x_{13} + x_{23} + x_{33} = 0$$

$$\Rightarrow 2x_{13} = -x_{23} - x_{33}$$

$$= -\frac{3}{2} + \frac{1}{2} = -\frac{2}{2} = -1$$

$$\Rightarrow x_{13} = -\frac{1}{2} \quad \therefore \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}$$

The inverse of A is $X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$

$$= \begin{pmatrix} -3 & \frac{5}{2} & -\frac{1}{2} \\ 12 & -\frac{17}{2} & \frac{3}{2} \\ -5 & \frac{7}{2} & -\frac{1}{2} \end{pmatrix}$$

Problem 8 Using Gaussian elimination method find the inverse of

$$A = \begin{pmatrix} 2 & 2 & -3 \\ -3 & 2 & 2 \\ 2 & -3 & 2 \end{pmatrix}$$

Solution Let $X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ be the inverse of A

$$\therefore AX = I_3 \text{ where } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Augmented system A can be written as

$$\begin{aligned} (A, I) &= \left(\begin{array}{ccc|ccc} 2 & 2 & -3 & 1 & 0 & 0 \\ -3 & 2 & 2 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array} \right) \\ &\sim \left(\begin{array}{ccc|ccc} 2 & 2 & -3 & 1 & 0 & 0 \\ 0 & 10 & -5 & 3 & 2 & 0 \\ 0 & -5 & 5 & -1 & 0 & 1 \end{array} \right) \begin{array}{l} R_2 : 2R_2 + 3R_1 \\ R_3 : R_3 - R_1 \end{array} \\ &\sim \left(\begin{array}{ccc|ccc} 2 & 2 & -3 & 1 & 0 & 0 \\ 0 & 10 & -5 & 3 & 2 & 0 \\ 0 & -5 & 5 & 1 & 2 & 2 \end{array} \right) R_3: 2R_3 + R_2 \end{aligned}$$

$\therefore$ The equation $AX = I_3$ is equivalent to the following 3 system

$$\begin{pmatrix} 2 & 2 & -3 \\ 0 & 10 & -5 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \longrightarrow \text{(I)}$$

$$\begin{pmatrix} 2 & 2 & -3 \\ 0 & 10 & -5 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \longrightarrow \text{(II)}$$

$$\begin{pmatrix} 2 & 2 & -3 \\ 0 & 10 & -5 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \longrightarrow \text{(III)}$$

Applying back substitution to (I), (II) & (III) we will get the inverse of A

From (I), $5x_{31} = 1 \Rightarrow x_{31} = 1/5$

$$\Rightarrow 10x_{21} = 3 + 5x_{31}$$

$$= 3 + 5\left(\frac{1}{5}\right) = 4$$

$$\Rightarrow x_{21} = \frac{4}{10} = 2/5$$

$$2x_{11} + 2x_{21} - 3x_{31} = 1$$

$$\Rightarrow 2x_{11} = 1 + 3x_{31} - 2x_{21}$$

$$= 1 + 3/5 - 4/5$$

$$= 1 - \frac{1}{5} = \frac{4}{5}$$

$$\Rightarrow x_{11} = \frac{2}{5}$$

$$\therefore \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 2/5 \\ 2/5 \\ 1/5 \end{pmatrix}$$

From (II), $5x_{32} = 2 \Rightarrow x_{32} = 2/5$

$$10x_{22} - 5x_{32} = 2$$

$$\Rightarrow 10x_{22} = 2 + 5x_{32}$$

$$= 2 + 5 \times \frac{2}{5} = 4$$

$$\Rightarrow x_{22} = \frac{4}{10} = 2/5$$

$$2x_{12} + 2x_{22} - 3x_{32} = 0$$

$$\Rightarrow 2x_{12} = 3x_{32} - 2x_{22}$$

$$= 3 \times \frac{2}{5} - 2 \times \frac{2}{5} = \frac{6}{5} - \frac{4}{5} = \frac{2}{5}$$

$$\Rightarrow x_{12} = 1/5$$

$$\therefore \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 1/5 \\ 2/5 \\ 2/5 \end{pmatrix}$$

From (III), $5x_{33} = 2 \Rightarrow x_{33} = 2/5$

$$10x_{23} - 5x_{33} = 0$$

$$\Rightarrow 10x_{23} = 5x_{33}$$

$$\Rightarrow 10x_{23} = 5 \times \frac{2}{5} = 2$$

$$\Rightarrow x_{23} = \frac{2}{10} = \frac{1}{5}$$

$$2x_{13} + 2x_{23} - 3x_{33} = 0$$

$$\Rightarrow 2x_{13} = 3x_{33} - 2x_{23} = 3 \times \frac{2}{5} - 2 \times \frac{1}{5} = \frac{4}{5}$$

$$\Rightarrow x_{13} = 2/5$$

$$\therefore \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 2/5 \\ 1/5 \\ 2/5 \end{pmatrix}$$

Hence the inverse of A is

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

$$= \begin{pmatrix} 2/5 & 1/5 & 2/5 \\ 2/5 & 2/5 & 1/5 \\ 1/5 & 2/5 & 2/5 \end{pmatrix}$$

Problem 9

Solve the following system of equations by Gauss elimination and Gauss Jordan Method

$$2x + 3y = 5 ; 3x - y = 2$$

Solution The given system of equation can also be written as

$$A X = B$$

Where $A = \begin{pmatrix} 2 & 3 \\ 3 & -1 \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \end{pmatrix}$ $B = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

(i) Gauss elimination Method

The Augmented system is

$$(A ; B) = \left(\begin{array}{cc|c} 2 & 3 & 5 \\ 3 & -1 & 2 \end{array} \right)$$

$$\sim \left(\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & -8 & -11 \end{array} \right) \quad R_2 : 2R_2 - 3R_1$$

$$\text{ie, } \begin{pmatrix} 2 & 3 \\ 0 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -11 \end{pmatrix}$$

$\therefore$ by back substitution we get

$$-8y = -11 \Rightarrow y = 11/8$$

$$2x + 3y = 5$$

$$\Rightarrow 2x = 5 - 3y = 5 - \frac{33}{8} = \frac{40 - 33}{8} = \frac{7}{8}$$

$$\Rightarrow x = \frac{7}{16}$$

The solution is $x = 7/16$, $y = 11/8$

(ii) Gauss – Jordan Method

The Augmented system is

$$(A ; B) = \left(\begin{array}{cc|c} 2 & 3 & 5 \\ 3 & -1 & 2 \end{array} \right)$$

$$\sim \left(\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & -8 & -11 \end{array} \right) \quad R_2 : 2R_2 - 3R_1$$

$$\sim \left(\begin{array}{cc|c} 16 & 0 & 7 \\ 0 & -8 & -11 \end{array} \right) \quad R_1 : 8R_1 - 3R_2$$

$$\text{ie, } \begin{pmatrix} 16 & 0 \\ 0 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ -11 \end{pmatrix}$$

$$\text{we get } -8y = -11 \Rightarrow y = 11/8$$

$$16x = 7 \Rightarrow x = 7/16$$

$\therefore$ The solution is

$$x = 7/16$$

$$y = 11/8$$

$\therefore$ In both methods we get the same result.

Exercises

I] Solve the following system of equations by Gauss elimination method.

a) $x - y + z = 1; \quad -3x + 2y - 3z = -6; \quad 2x - 5y + 4z = 5$

b) $x + y + z = 6; \quad -3x + 3y + 4z = 20; \quad 2x + y + 3z = 13$

c) $x + 2y + z - w = -2; \quad 2x + 3y - z + 2w = 7;$

$-x + y + 3z - 2w = -6; \quad x + y + z + w = 6$

II] Solve the following system of equation using Gauss – Jordan method

$$10x + y + z = 12$$

$$2x + y + 4z = 12$$

(a) $2x + 10y + z = 13$

(b) $8x - 3y + 2z = 20$

$$x + y + 5z = 7$$

$$4x + 11y - z = 33$$

$$2x + 2y - z + w = 4; \quad 4x + 3y - z + 2w = 6$$

(c) $8x + 5y - 3z + 4w = 12; \quad 3x + 3y - 2y + 2w = 6$

III] Find the inverse of the following matrices using Gaussian elimination method:

$$(a) \quad A = \begin{pmatrix} 2 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix} \quad (b) \quad A = \begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}$$

Section 6.10

Indirect Method (Iterative Method)

All the preceding methods of solving systems of simultaneous equations are known as direct methods, which involve a certain amount of fixed computation. We shall now describe the Iterative (or) Indirect methods of solving these equations. Indirect methods are those in which the solution got by successive approximations. Thus in an indirect (or) iterative method, the amount of computation depends on the degree of accuracy required.

But the method of iteration is not applicable to all systems of equations. In order that the iteration may succeed, each equation of the system must contain one large coefficient (much greater than the others in three equation) and the large coefficient must be attached to a different unknown in that equation. This requirement will be when the large coefficients are along the leading diagonal of the matrix of the coefficients. When the equations are in this form, they are solvable, by the methods of successive approximations.

Section 6.10.1

Definition Diagonally Dominant

An $n \times n$ matrix A is said to be diagonally dominant if the absolute value of each leading diagonal element is greater than or equal to the sum of the absolute values of the remaining element in that row

Example (i) The matrix $A = \begin{pmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{pmatrix}$

is diagonally dominant,

$$\begin{aligned} |10| &> |-5| + |-2| \\ \text{for } |-10| &> |4| + |3| \\ |10| &> |1| + |6| \end{aligned}$$

(ii) The matrix $A = \begin{pmatrix} 2 & 3 & -1 \\ 5 & 8 & -4 \\ 1 & 1 & 1 \end{pmatrix}$ is not

diagonally dominant

$$\begin{aligned} |2| &\not> |3| + |-1| \\ \text{Since } |8| &\not> |5| + |-4| \\ |1| &\not> |1| + |1| \end{aligned}$$

Note In the system of simultaneous linear equations is 'n' unknowns $AX = B$ if A is diagonally dominant then the system is said to be a diagonal system.

Thus the system of equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3 \quad \text{is a diagonal system}$$

$$\text{if } \begin{aligned} |a_1| &\geq |b_1| + |c_1| \\ |b_2| &\geq |a_2| + |c_2| \end{aligned}$$

$$\text{and } |c_3| \geq |a_3| + |b_3|$$

The process of iteration in solving $AX = B$ will convenes quickly if the coefficient matrix A is diagonally dominant.

Section 6.11**Gauss Jacobi method (or) Jacobi method of Iteration**

Consider the system of equations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = c_2$$

$$\vdots$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = c_n$$

We assume that the coefficient matrix of this system is diagonally dominant. The above equations can be rewritten as

$$x_1 = \frac{1}{a_{11}} [c_1 - a_{12}x_2 - a_{13}x_3 \dots - a_{1n}x_n] \longrightarrow \textcircled{1}$$

$$x_2 = \frac{1}{a_{22}} [c_2 - a_{21}x_1 - a_{23}x_3 \dots - a_{2n}x_n] \longrightarrow \textcircled{2}$$

$$\vdots$$

$$\vdots$$

$$x_n = \frac{1}{a_{nn}} [c_n - a_{n1}x_1 - a_{n2}x_2 \dots - a_{nn-1}x_{n-1}] \longrightarrow \textcircled{N}$$

We start with the initial value for the variables

$$x_1, x_2, \dots, x_n \text{ to be } x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}$$

Using these values in $\textcircled{1}, \textcircled{2}, \dots, \textcircled{n}$ respectively we get

$$x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)},$$

putting $x_1 = x_1^{(1)}, x_2 = x_2^{(1)}, \dots, x_n = x_n^{(1)}$ in $\textcircled{1}, \textcircled{2}, \dots, \textcircled{n}$ respectively

we get the next approximations

$$x_1^{(2)}, x_2^{(2)}, \dots, x_n^{(2)}.$$

In general if the values of $x_1, x_2, \dots, x_n$ in the r^{th} iteration are

$$x_1^{(r)}, x_2^{(r)}, x_3^{(r)}, \dots, x_n^{(r)},$$

$$\text{Then } x_1^{(r+1)} = \frac{1}{a_{11}} [c_1 - a_{12}x_2^{(r)} - a_{13}x_3^{(r)} \dots - a_{1n}x_n^{(r)}]$$

$$x_2^{(r+1)} = \frac{1}{a_{22}} [c_2 - a_{21}x_1^{(r)} - a_{23}x_3^{(r)} \dots a_{2n}x_n^{(r)}]$$

$$x_n^{(r+1)} = \frac{1}{a_{nn}} [c_n - a_{n1}x_1^{(r)} - a_{n2}x_2^{(r)} \dots a_{nn-1}x_{n-1}^{(r)}]$$

Note In solving a specific problem in the absence of any special initial values for the variables, we usually take the initial values of the variables to be zero.

Solved problems

Problem 1 Check whether the system of equations

$$x + 6y - 2z = 5 ; 4x + y + z = 6 ; -3x + y + 7z = 5$$

is a diagonal system. If not make it to a diagonal system.

Solution Given system of linear equation is

$$x + 6y - 2z = 5$$

$$4x + y + z = 6$$

$$-3x + y + 7z = 5$$

The coefficient matrix of the given system is

$$A = \begin{pmatrix} 1 & 6 & -2 \\ 4 & 1 & 1 \\ -3 & 1 & 7 \end{pmatrix}$$

is not diagonally dominant as such

Since

$$\begin{aligned} |1| &\not\geq |6| + |-2| \\ |1| &\not\geq |4| + |1| \\ \text{but } |7| &\geq |-3| + |1| \end{aligned}$$

However effecting the operation $c_2 \Leftrightarrow c_1$ (interchanging column one and two)

$$A \sim \begin{pmatrix} 6 & 1 & -2 \\ 1 & 4 & 1 \\ 1 & -3 & 7 \end{pmatrix} \text{ which is a diagonally dominant matrix}$$

Space for Hints

$$\begin{aligned} \text{Since } |6| &\geq |1| + |-2| \\ |4| &\geq |1| + |1| \\ |7| &\geq |1| + |-3| \end{aligned}$$

Hence the given system is a diagonal system.

The system of equations can be written as

$$\begin{pmatrix} 6 & 1 & -2 \\ 1 & 4 & 1 \\ 1 & -3 & 7 \end{pmatrix} \begin{pmatrix} y \\ x \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix}$$

Problem 2 Is the system of equations diagonally dominant? If not make it diagonally dominant

$$3x + 9y - 2z = 10 ; 4x + 2y + 13z = 19 ; 4x - 2y + z = 3$$

$$\begin{array}{rcl} 3x + 9y - 2z & = & 10 \quad \xrightarrow{\quad} \textcircled{1} \\ \text{Solution Given that } 4x + 2y + 13z & = & 19 \quad \xrightarrow{\quad} \textcircled{2} \\ 4x - 2y + z & = & 3 \quad \xrightarrow{\quad} \textcircled{3} \end{array}$$

The coefficient matrix of the given system of equations

①, ② & ③ is

$$A = \begin{pmatrix} 3 & 9 & -2 \\ 4 & 2 & 13 \\ 4 & -2 & 1 \end{pmatrix}$$

$$\begin{aligned} \text{Since } |3| &\not\geq |9| + |-2| \\ |2| &\not\geq |4| + |13| \\ \text{and } |1| &\not\geq |4| + |-2| \end{aligned}$$

$\therefore A$ is not diagonally dominant matrix.

$$\text{Now } A = \begin{pmatrix} 3 & 9 & -2 \\ 4 & 2 & 13 \\ 4 & -2 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 4 & 2 & 13 \\ 3 & 9 & -2 \\ 4 & -2 & 1 \end{pmatrix} R_2 \Leftrightarrow R_1$$

$$\sim \begin{pmatrix} 13 & 2 & 4 \\ -2 & 9 & 3 \\ 1 & -2 & 4 \end{pmatrix} c_1 \Leftrightarrow c_3$$

clearly

$$\begin{aligned} |13| &\geq |2| + |4| \\ |9| &\geq |-2| + |3| \\ |4| &\geq |1| + |-2| \end{aligned}$$

This is a diagonally dominant matrix and the system of equations can be written as

$$\begin{pmatrix} 13 & 2 & 4 \\ -2 & 9 & 3 \\ 1 & -2 & 4 \end{pmatrix} \begin{pmatrix} z \\ y \\ x \end{pmatrix} = \begin{pmatrix} 19 \\ 10 \\ 3 \end{pmatrix}$$

Problem 3 Solve the following equations using Gauss Jacobi method

$$3x + 4y + 15z = 54.8 ; x + 12y + 3z = 39.66 ; 10x + y - 2z = 7.74$$

Solution Given that $3x + 4y + 15z = 54.8 \longrightarrow \textcircled{1}$

$$x + 12y + 3z = 39.66 \longrightarrow \textcircled{2}$$

$$10x + y - 2z = 7.74 \longrightarrow \textcircled{3}$$

The coefficient matrix of the given system of equations

$\textcircled{1}$, $\textcircled{2}$ & $\textcircled{3}$ is

$$A = \begin{pmatrix} 3 & 4 & 15 \\ 1 & 12 & 3 \\ 10 & 1 & -2 \end{pmatrix}$$

$$|3| \geq |4| + |15|$$

clearly $|12| \geq |1| + |3|$

$$\& \quad |-2| \geq |10| + |1|$$

$\therefore A$ is not diagonally dominant also

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Also $A \sim \begin{pmatrix} 10 & 1 & -2 \\ 1 & 12 & 3 \\ 3 & 4 & 15 \end{pmatrix} R_1 \Leftrightarrow R_3$

Now A is diagonally dominant, since

$$|10| \geq |1| + |-2|$$

$$|12| \geq |1| + |3|$$

$$|15| \geq |3| + |4|$$

Now the given system becomes

$$10x + y - 2z = 7.74 \longrightarrow \textcircled{3}$$

$$x + 12y + 3z = 39.66 \longrightarrow \textcircled{2}$$

$$3x + 4y - 15z = 54.8 \longrightarrow \textcircled{1}$$

From $\textcircled{3}$, $\textcircled{2}$ & $\textcircled{1}$ we set

$$10x = 7.74 - y + 2z$$

$$\Rightarrow x = \frac{1}{10} [7.74 - y + 2z] \longrightarrow \textcircled{4}$$

$$12y = 39.66 - x - 3z$$

$$\Rightarrow y = \frac{1}{12} [39.66 - x - 3z] \longrightarrow \textcircled{5}$$

$$15z = 54.8 - 3x - 4y$$

$$\Rightarrow z = \frac{1}{15} [54.8 - 3x - 4y] \longrightarrow \textcircled{6}$$

First iteration Let the initial values be $x_0 = y_0 = z_0 = 0$

$\therefore$ equations $\textcircled{4}$, $\textcircled{5}$ & $\textcircled{6}$ becomes

$$x_1 = \frac{1}{10} [7.74 - 0 + 0] = \frac{7.74}{10} = 0.774$$

$$y_1 = \frac{1}{12} [39.66 - 0 - 0] = \frac{39.66}{12} = 3.305$$

$$z_1 = \frac{1}{15} [54.8 - 0 - 0] = \frac{54.8}{15} = 3.6533$$

Second iteration Put $x_1 = 0.774$, $y_1 = 3.305$ & $z_1 = 3.6533$

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in ④, ⑤ & ⑥ we get

$$x_2 = \frac{1}{10} [7.74 - y_1 + 2z_1] = \frac{1}{10} [7.74 - 3.305 + 2 \times 3.6533] = 1.1742$$

$$y_2 = \frac{1}{12} [39.66 - x_1 - 3z_1] = \frac{1}{12} [39.66 - 0.774 - 1.9599] = 2.3272$$

$$z_2 = \frac{1}{15} [54.8 - 3x_1 - 4y_1] = \frac{1}{15} [54.8 - 2.322 - 13.22] = 2.6172$$

Third iteration Put $x_2 = 1.742$, $y_2 = 2.3272$ & $z_2 = 2.6172$

in ④, ⑤ & ⑥ we get

$$x_3 = \frac{1}{10} [7.74 - y_2 + 2z_2] = \frac{1}{10} [7.74 - 2.3272 + 5.2344] = 1.0647$$

$$y_3 = \frac{1}{12} [39.66 - x_2 - 3z_2] = \frac{1}{12} [39.66 - 1.1742 - 7.8516] = 2.5529$$

$$z_3 = \frac{1}{15} [54.8 - 3x_2 - 4y_2] = \frac{1}{15} [54.8 - 3.5266 - 9.3088] = 2.7979$$

Fourth iteration Put $x_3 = 1.0647$, $y_3 = 2.5529$, $z_3 = 2.7979$

in ④, ⑤ & ⑥ we get

$$x_4 = \frac{1}{10} [7.74 - y_3 + 2z_3] = \frac{1}{10} [7.74 - 2.5529 + 5.5958] = 1.0783$$

$$y_4 = \frac{1}{12} [39.66 - x_3 - 3z_3] = \frac{1}{12} [39.66 - 1.0647 - 8.3937] = 2.5168$$

$$z_4 = \frac{1}{15} [54.8 - 3x_3 - 4y_3] = \frac{1}{15} [54.8 - 3.1941 - 10.2116] = 2.7596$$

Fifth iteration Put $x_4 = 1.0783$, $y_4 = 2.5168$ & $z_4 = 2.7596$

in equation ④, ⑤ & ⑥ we get

$$x_5 = \frac{1}{10} [7.74 - y_4 + 2z_4] = \frac{1}{10} [7.74 - 2.5168 + 5.5192] = 1.0742$$

$$y_5 = \frac{1}{12} [39.66 - x_4 - 3z_4] = \frac{1}{12} [39.66 - 1.0783 - 8.2788] = 2.5252$$

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$$z_5 = \frac{1}{15} [54.8 - 3x_4 - 4y_4] = \frac{1}{15} [54.8 - 3.2349 - 10.0672] = 2.7665$$

Sixth iteration Put $x_5 = 1.0742$, $y_5 = 2.5252$ & $z_5 = 2.7665$

in equation ④, ⑤ & ⑥ we get

$$x_6 = \frac{1}{10} [7.74 - y_5 + 2z_5] = \frac{1}{10} [7.74 - 2.5252 + 5.533] = 1.0748$$

$$y_6 = \frac{1}{12} [39.66 - x_5 - 3z_5] = \frac{1}{12} [39.66 - 1.0742 - 8.2995] = 2.5239$$

$$z_6 = \frac{1}{15} [54.8 - 3x_5 - 4y_5] = \frac{1}{15} [54.8 - 3.2226 - 10.1008] = 2.7651$$

Seventh iteration Put $x_6 = 1.0748$, $y_6 = 2.5239$ & $z_6 = 2.7651$

in equation ④, ⑤ & ⑥ we get

$$x_7 = \frac{1}{10} [7.74 - y_6 + 2z_6] = \frac{1}{10} [7.74 - 2.5239 + 5.5302] = 1.0746$$

$$y_7 = \frac{1}{12} [39.66 - x_6 - 3z_6] = \frac{1}{12} [39.66 - 1.0748 - 8.2953] = 2.5242$$

$$z_7 = \frac{1}{15} [54.8 - 3x_6 - 4y_6] = \frac{1}{15} [54.8 - 3.2244 - 10.0956] = 2.7653$$

After 7 iteration the difference in 6th and 7th iterations very negligible.

Hence the solution of the system is given by

$$x_1 = 1.075 ; y = 2.524 ; z = 2.765$$

Problem 4 Check whether the following system of equations is a diagonal system (diagonally dominant). If not make it to a diagonal system by remaining the equations.

(i) $x + 5y - z = 10$; $x + y + 8z = 20$; $4x + 2y + z = 14$

(ii) $2x - y + z = 2$; $x + y + 3z = 15$; $x + y + z = 3$

Solution (i) Given that $x + 5y - z = 10$ $\longrightarrow$ ①

$$x + y + 8z = 20 \longrightarrow$$
 ②

$$4x + 2y + z = 14 \longrightarrow$$
 ③

The coefficient matrix of the given system of equations

①, ② & ③ is

$$A = \begin{pmatrix} 1 & 5 & -1 \\ 1 & 1 & 8 \\ 4 & 2 & 1 \end{pmatrix}$$

clearly $|1| \not\geq |5| + |-1|$
 $|1| \not\geq |1| + |8|$
 & $|1| \not\geq |4| + |2|$

Now $A = \begin{pmatrix} 1 & 5 & -1 \\ 1 & 1 & 8 \\ 4 & 2 & 1 \end{pmatrix}$

$$\sim \begin{pmatrix} 4 & 2 & 1 \\ 1 & 5 & -1 \\ 1 & 1 & 8 \end{pmatrix} R_1 \Leftrightarrow R_2 \Leftrightarrow R_3$$

clearly $|4| \not\geq |2| + |1|$
 $|5| \not\geq |1| + |-1|$
 $|8| \not\geq |1| + |1|$

$\therefore$ The given system of equation is diagonally dominant

The system of equation can be written as

$$\begin{pmatrix} 4 & 2 & 1 \\ 1 & 5 & -1 \\ 1 & 1 & 8 \end{pmatrix} \begin{pmatrix} z \\ x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 10 \\ 20 \end{pmatrix}$$

(ii) Given that

$$2x - y + z = 2 \longrightarrow \text{①}$$

$$x + y + 3z = 5 \longrightarrow \text{②}$$

$$x + y + z = 3 \longrightarrow \text{③}$$

The coefficient matrix of the given system of equations

①, ② & ③ is

Space for Hints

Space for Hints

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} & |2| \geq |1| + |1| \\ \text{clearly } & |1| \not\geq |1| + |3| \\ & |1| \not\geq |1| + |1| \end{aligned}$$

$\therefore$ The given system of equations ①, ② & ③ is not diagonally dominant.

To make it diagonally dominant

$$\text{Now } A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix} R_2 \leftrightarrow R_3$$

$$\begin{aligned} & |2| \geq |-1| + |1| \\ \text{Again clearly } & |3| \geq |1| + |1| \\ \text{but } & |1| \not\geq |1| + |1| \end{aligned}$$

$\therefore$ The given system of equation is not diagonally

Problem 5 Solve the following system of linear equations using Gauss – Jacobi iteration method.

$$27x + 6y - z = 85 ; 6x + 15y + 2z = 72 ; x + y + 54z = 110$$

correct to 4 places of decimals.

$$\begin{aligned} \text{Solution Given } 27x + 6y - z &= 85 \longrightarrow \text{①} \\ 6x + 15y + 2z &= 72 \longrightarrow \text{②} \\ x + y + 54z &= 110 \longrightarrow \text{③} \end{aligned}$$

The coefficient matrix of the given system in

Space for Hints

$$A = \begin{pmatrix} 27 & 6 & -1 \\ 6 & 15 & 2 \\ 1 & 1 & 54 \end{pmatrix}$$

clearly $|27| \geq |6| + |-1|$
 $|15| \geq |6| + |2|$
 $|54| \geq |1| + |1|$

$\therefore$ The given system of equation is diagonally dominant

So we can apply Gauss – Jacobi iteration method.

From ①, ② & ③ is we can write

$$27x = 85 - 6y + z$$

$$\Rightarrow x = \frac{1}{27} [85 - 6y + z] \longrightarrow \textcircled{4}$$

$$15y = 72 - 6x - 2z \Rightarrow y = \frac{1}{15} [72 - 6x - 2z] \longrightarrow \textcircled{5}$$

$$54z = 110 - x - y \Rightarrow z = \frac{1}{54} [110 - x - y] \longrightarrow \textcircled{6}$$

We start the iteration by putting the initial

values $x^{(0)} = 0, y^{(0)} = 0, z^{(0)} = 0$ in RHS of ④, ⑤ & ⑥ we get

I-Iteration

$$x^{(1)} = \frac{1}{27} [85 - 6 \times 0 + 0] = \frac{85}{27} = 3.14815$$

$$y^{(1)} = \frac{1}{15} [72 - 2 \times 0 - 6 \times 0] = \frac{72}{15} = 4.8$$

$$z^{(1)} = \frac{1}{54} [110 - 0 - 0] = \frac{110}{54} = \frac{55}{27} = 2.03704$$

II -Iteration

Put $x^{(1)} = 3.14815, y^{(1)} = 4.8, z^{(1)} = 2.03704$ in RHS of ④, ⑤ & ⑥

we get

$$x^{(2)} = \frac{1}{27} [85 - 6 \times 4.8 + 2.03704] = 2.15693$$

$$y^{(2)} = \frac{1}{15} [72 - 2 \times 2.03704 - 6 \times 3.14815] = 3.26913$$

$$z^{(2)} = \frac{1}{54} [110 - 3.14815 - 4.8] = 1.88985$$

III - Iteration

Put $x^{(2)} = 2.15693$, $y^{(2)} = 3.26913$, $z^{(2)} = 1.88985$ in RHS of ④, ⑤ &

⑥ we get

$$x^{(3)} = \frac{1}{27} [85 - 6 \times 3.26913 + 1.88985] = 2.49167$$

$$y^{(3)} = \frac{1}{15} [72 - 2 \times 1.88985 - 6 \times 2.15693] = 3.68525$$

$$z^{(3)} = \frac{1}{54} [110 - 2.15693 - 3.26913] = 1.93655$$

Thus, the iteration process is continued.

The Results are tabulated as follows.

Iteration	x	y	z
0	0	0	0
1	3.14815	4.8	2.03704
2	2.15693	3.26913	1.88985
3	2.49167	3.68525	1.93655
4	2.40093	3.54513	1.92265
5	2.43155	3.58327	1.92692
6	2.42323	3.57046	1.92565
7	2.42603	3.57395	1.92604
8	2.42527	3.57278	1.92593
9	2.42552	3.57310	1.92596
10	2.42546	3.57300	1.92595

We find that the values in the 9th and 10th iterations are close to each other. They are practically the same, to 4 decimal places. So we can stop the iteration process.

Thus the value are $x = 2.4255$

$$y = 3.5730$$

$$z = 1.9260$$

Problem 6 Solve the system of equations

$$2x - 3y + 20z = 35; \quad 20x + y - 2z = 17; \quad 3x + 20y - z = -18$$

by Gauss – Jacobi method only two iterations

Solution Given $2x - 3y + 20z = 35 \longrightarrow \textcircled{1}$

$$20x + y - 2z = 17 \longrightarrow \textcircled{2}$$

$$3x + 20y - z = -18 \longrightarrow \textcircled{3}$$

The coefficient matrix of the given system is

$$A = \begin{pmatrix} 2 & -3 & 20 \\ 20 & 1 & -2 \\ 3 & 20 & -1 \end{pmatrix}$$

$$|2| \not\geq |-2| + |20|$$

Clearly $|1| \not\geq |20| + |-2|$

$$\& \quad |-1| \not\geq |3| + |20|$$

$\therefore A$ is not diagonally dominant.

To make it as A into diagonally dominant we have

$$A \sim \begin{pmatrix} 2 & 1 & -2 \\ 3 & 20 & -1 \\ 3 & -3 & 20 \end{pmatrix} \begin{array}{l} R_2 \Leftrightarrow R_1 \\ R_1 \Leftrightarrow R_3 \\ R_3 \Leftrightarrow R_2 \end{array}$$

$$|20| \geq |1| + |-21|$$

Clearly $|20| \geq |3| + |-1|$

$$|20| \geq |2| + |-3|$$

$\therefore A$ is diagonally dominant.

Now the system of equations now becomes

$$20x + y - 2z = 17 \longrightarrow \textcircled{\text{I}}$$

$$3x + 20y - z = -18 \longrightarrow \textcircled{\text{II}}$$

$$2x - 3y + 20z = 25 \longrightarrow \textcircled{\text{III}}$$

From (I), (II), & (III) we can write

$$20x = 17 - y + 2z$$

$$\Rightarrow x = \frac{1}{20} [17 - y + 2z] \longrightarrow \text{(IV)}$$

$$20y = -18 + z - 3x$$

$$\Rightarrow y = \frac{1}{20} [-18 + z - 3x] \longrightarrow \text{(V)}$$

$$20z = 25 - 2x + 3y$$

$$\Rightarrow z = \frac{1}{20} [25 - 2x + 3y] \longrightarrow \text{(VI)}$$

Let the initial solutions be $x = 0, y = 0, z = 0$

I-Iteration Put $x = 0, y = 0, z = 0$ in (IV), (V) & (VI) we get

$$x^{(1)} = \frac{1}{20} [17.0 + 2 \times 0] = \frac{17}{20} = 0.85$$

$$y^{(1)} = \frac{1}{20} [-18 + 0 - 3 \times 0] = \frac{-18}{20} = -0.9$$

$$z^{(1)} = \frac{1}{20} [25 - 2 \times 0 + 3 \times 0] = \frac{25}{20} = \frac{55}{27} = 1.25$$

II-Iteration Put $x^{(1)} = 0.85, y^{(1)} = -0.9$ & $z^{(1)} = 1.25$ in RHS

(IV), (V) & (VI) we get

$$x^{(2)} = \frac{1}{20} [17 - y^{(1)} + 2z^{(1)}] = \frac{1}{20} [17 - (-.9) + (1.25)] = 1.02$$

$$y^{(2)} = \frac{1}{20} [-18 - 3x^{(1)} + z^{(1)}] = \frac{1}{20} [-18 - 3(.85) + 1.25] = -0.965$$

$$z^{(2)} = \frac{1}{20} [25 - 2x^{(1)} + 3y^{(1)}] = \frac{1}{20} [25 - 2(.85) + 3(-.9)] = 1.03$$

$\therefore$ The solution is $x = 1.02, y = -.965, z = 1.03$

(only in 2 iterations)

Section 6.12

Gauss – Seidal Iteration Method

Gauss – Seidal iteration method is a refinement of Gauss – Jacobi method.

As in the Jacobi iteration method, let

$$x_1 = \frac{1}{a_{11}} [c_1 - a_{12}x_2 - a_{13}x_3 - \dots - a_{1n}x_n] \longrightarrow \textcircled{1}$$

$$x_2 = \frac{1}{a_{22}} [c_2 - a_{21}x_1 - a_{23}x_3 - \dots - a_{2n}x_n] \longrightarrow \textcircled{2}$$

$$x_n = \frac{1}{a_{nn}} [c_n - a_{n1}x_1 - a_{n2}x_2 - \dots - a_{nn}x_{n-1}] \longrightarrow \textcircled{3}$$

We start with the initial values $x_2^{(0)}, x_3^{(0)}, \dots, x_n^{(0)}$ and we get from $\textcircled{1}$

$$x_1^{(1)} = \frac{1}{a_{11}} [c_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - \dots - a_{1n}x_n^{(0)}]$$

In the 2nd equation we use $x_1^{(1)}$ for x_1 and $x_3^{(0)}$ for x_3 etc. & $x_n^{(0)}$ for x_n

(In the Jacobi method we use $x_1^{(1)}$ for x_1).

Thus we get

$$x_2^{(1)} = \frac{1}{a_{22}} [c_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)} - \dots - a_{2n}x_n^{(0)}]$$

Proceeding like this we find the 1st iteration values as $x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)}$.

The r^{th} iteration are $x_1^{(r)}, x_2^{(r)}, \dots, x_n^{(r)}$ then the values in the $(r+1)^{\text{th}}$

iteration are given by

$$x_1^{(r+1)} = \frac{1}{a_{11}} [c_1 - a_{12}x_2^{(r)} - a_{13}x_3^{(r)} - \dots - a_{1n}x_n^{(r)}]$$

$$x_2^{(r+1)} = \frac{1}{a_{22}} [c_2 - a_{21}x_1^{(r+1)} - a_{23}x_3^{(r)} - \dots - a_{2n}x_n^{(r)}]$$

$$x_n^{(r+1)} = \frac{1}{a_{nn}} [c_n - a_{n1}x_1^{(r+1)} - a_{n2}x_2^{(r)} - \dots - a_{nn}x_{n-1}^{(r+1)}]$$

Solved problems

Problem 1 Solve $2x + y = 3$; $2x + 3y = 5$ by Gauss seidal iteration method.

Solution Given $2x + y = 3 \longrightarrow \textcircled{1}$

$2x + 3y = 5 \longrightarrow \textcircled{2}$

The coefficient of the given system of equations $\textcircled{1}$ & $\textcircled{2}$ we get

Space for Hints

$$A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$$

Clearly $|2| \geq |1|$ & $|3| \geq |2|$

$\therefore$ The matrix A is diagonally dominant and hence Gauss – Seidal iteration method can be applied

The given equations can be written as

$$2x = 3 - y \Rightarrow x = \frac{1}{2}[3 - y] \longrightarrow \textcircled{3}$$

$$3y = 5 - 2x \Rightarrow y = \frac{1}{3}[5 - 2x] \longrightarrow \textcircled{4}$$

Put $y = 0$ in $\textcircled{3}$, we get $x = \frac{1}{2}[3 - 0] = \frac{3}{2} = 1.5$

using $x = 1.5$ in $\textcircled{4}$ we get $y = \frac{1}{3}[5 - 2 \times 1.5] = 0.6667$

Putting $y = 0.6667$ in $\textcircled{3}$ we get $x = 1.1667$

Putting $x = 1.1667$ in $\textcircled{4}$ we get $y = 0.8889$

Putting $y = 0.8889$ in $\textcircled{3}$ we get $x = 1.0556$

Putting $x = 1.0556$ in $\textcircled{4}$ we get $y = 0.9629$

Putting $y = 0.9629$ in $\textcircled{3}$ we get $x = 1.0186$

Putting $x = 1.0186$ in $\textcircled{4}$ we get $y = 0.9876$

Putting $y = 0.9876$ in $\textcircled{3}$ we get $x = 1.0062$

Putting $x = 1.0062$ in $\textcircled{4}$ we get $y = 0.9959$

using $y = 0.9959$ in $\textcircled{3}$ we get $x = 1.0021$

using $x = 1.0021$ in $\textcircled{4}$ we get $y = 0.9986$

using $y = 0.9986$ in $\textcircled{3}$ we get $x = 1.0007$

Putting $x = 1.0007$ in $\textcircled{4}$ we get $y = 0.9995$

Putting $y = 0.9995$ in $\textcircled{3}$ we get $x = 1.0001$

Putting $x = 1.0001$ in $\textcircled{4}$ we get $y = 0.9999$

using $y = 0.9999$ in $\textcircled{3}$ we get $x = 1$

Putting $x = 1$ in $\textcircled{4}$ we get $y = 1$

Hence $x = 1$, $y = 1$ is the solution of the given two equations.

Space for Hints

The above iterations can be simply carried out and exhibited in the following tabular form

Iterations	x	y
Start	-	0
1	1.5	0.6667
2	1.6667	0.8889
3	1.0556	0.9629
4	1.0186	0.9876
5	1.0062	0.9959
6	1.0021	0.9986
7	1.0007	0.9995
8	1.0001	0.9999
9	1	1
10	1	1

Space for Hints

Problem 2 Solve the following system of equations using Gauss – Seidal iteration method

$$6x + 15y + 2z = 72; \quad x + y + 54z = 110; \quad 27x + 6y - z = 85$$

Solution Given $6x + 15y + 2z = 72 \longrightarrow \textcircled{1}$

$$x + y + 54z = 110 \longrightarrow \textcircled{2}$$

$$27x + 6y - z = 85 \longrightarrow \textcircled{3}$$

The coefficient matrix of the given system of equations
①, ② & ③ is

$$A = \begin{pmatrix} 6 & 15 & 2 \\ 1 & 1 & 54 \\ 27 & 6 & -1 \end{pmatrix}$$

Clearly

$$\begin{array}{l} |6| \not\geq |15| + |2| \\ |1| \not\geq |1| + |54| \\ |-1| \not\geq |27| + |6| \end{array}$$

$\therefore A$ is not diagonally dominant.

However it can be made diagonally dominant by changing the rows as

$$A \sim \begin{pmatrix} 27 & 6 & -1 \\ 6 & 15 & 2 \\ 1 & 1 & 54 \end{pmatrix}$$

(changing 3rd row by 1st, 1st by 2nd, 2nd by 3rd)

Hence the corresponding system of equation is

$$27x + 6y - z = 85$$

$$6x + 15y + 2z = 72$$

$$x + y + 54z = 110$$

The above system of equations can be rewritten as

$$27x = 85 - 6y + z$$

$$\Rightarrow x = \frac{1}{27} [85 - 6y + z] \longrightarrow \textcircled{1}$$

$$15y = 72 - 6x - 2z$$

$$\Rightarrow y = \frac{1}{15} [72 - 6x - 2z] \longrightarrow \textcircled{2}$$

$$54z = 11 - 0 - x - y$$

$$\Rightarrow z = \frac{1}{54} [110 - x - y] \longrightarrow \textcircled{3}$$

First iteration

Putting $y = 0$ & $z = 0$ in $\textcircled{1}$ we get

$$x = \frac{1}{27} [85] = 3.1481$$

Putting $x = 3.1481$ & $z = 0$ in $\textcircled{2}$ we get

$$y = \frac{1}{15} [72 - 6 \times 3.1481 - 2 \times 0] = 3.5408$$

Putting $x = 3.1481$ & $y = 3.5408$ in $\textcircled{3}$ we get

$$z = \frac{1}{54} [110 - 3.1481 - 3.5408] = 1.9132$$

Second iteration

Putting $y = 3.5408$ & $z = 1.9132$ in $\textcircled{1}$ we get

$$x = \frac{1}{27} [85 - 6 \times 3.5408 + 1.9132] = 2.4322$$

Putting $x = 2.4322$ & $z = 1.9132$ in $\textcircled{2}$ we get

$$y = \frac{1}{15} [72 - 6 \times 2.4322 - 2 \times 1.9132] = 3.572$$

Putting $x = 2.4322$ & $y = 3.572$ in $\textcircled{3}$ we get

$$z = \frac{1}{54} [110 - 2.4322 - 3.572] = 1.9258$$

Third iteration

Putting $y = 3.572$ & $z = 1.9258$ in $\textcircled{1}$ we get

$$x = \frac{1}{27} [85 - 6 \times 3.572 + 1.9258] = 2.4257$$

Putting $x = 2.4257$ & $z = 1.9258$ in $\textcircled{2}$ we get

$$y = \frac{1}{15} [72 - 6 \times 2.457 - 2 \times 1.9258] = 3.5729$$

Space for Hints

Putting $x = 2.4257$ & $y = 3.5729$ in ③ we get

$$z = \frac{1}{54} [110 - 2.4257 - 3.5729] = 1.926$$

Fourth iteration

Putting $y = 3.5729$ & $z = 1.926$ in ① we get

$$x = \frac{1}{27} [85 - 6 \times 3.5729 + 1.926] = 2.4255$$

Putting $x = 2.4255$ & $z = 1.926$ in ② we get

$$y = \frac{1}{15} [72 - 6 \times 2.4255 - 2 \times 1.926] = 3.573$$

Putting $x = 2.4255$ & $y = 3.573$ in ③ we get

$$z = \frac{1}{54} [110 - 2.4255 - 3.573] = 1.926$$

The values of x, y, z in the 3rd and 4th iteration are almost equal.

∴ The solution of the given system of equations are

$x = 2.4255$
$y = 3.573$
$z = 1.926$

The above iterations can be simply carried out and shown in the following tabular form

Iteration	Initial value	1	2	3	4	5
x	-	3.1481	2.4322	2.4257	2.4255	2.4255
y	0	3.5408	3.5720	3.57299	3.5730	3.5730
z	0	1.9132	1.9258	1.9260	1.9260	1.9260

Problem 3 Solve the following system of equations using Gauss-Seidal iteration method.

$$10x + 2y + z = 9 \ ; \ x + 10y - z = -22 \ ; \ -2x + 3y + 10z = 22$$

Solution Given that $10x + 2y + z = 9$

$$x + 10y - z = -22$$

$$-2x + 3y + 10z = 22$$

The coefficient matrix of the given system of equations is

$$A = \begin{pmatrix} 10 & 2 & 1 \\ 1 & 10 & -1 \\ -2 & 3 & 10 \end{pmatrix}$$

$$|10| \geq |2| + |1|$$

Clearly $|10| \geq |1| + |-1|$

$$|10| \geq |-2| + |3|$$

$\therefore A$ is not diagonally dominant matrix.

Hence the given system of equations can be solved by Gauss – Seidal iteration method.

The given system of equations can be written as

$$10x = 9 - 2y - z$$

$$\Rightarrow x = \frac{1}{10} [9 - 2y - z] \longrightarrow \textcircled{1}$$

$$10y = -22 - x + z$$

$$\Rightarrow y = \frac{1}{10} [-22 - x + z] \longrightarrow \textcircled{2}$$

$$10z = 22 + 2x - 3y$$

$$\Rightarrow z = \frac{1}{10} [22 + 2x - 3y] \longrightarrow \textcircled{3}$$

First Iteration Putting $y = 0$ & $z = 0$ in $\textcircled{1}$ we get

$$x = \frac{1}{10} [9 - 0 - 0] = \frac{9}{10} = 0.9$$

Putting $x = 0.9$ & $z = 0$ in $\textcircled{2}$ we get

$$y = \frac{1}{10} [-22 - 0.9 + 0] = \frac{-22.9}{10} = -2.29$$

Putting $x = 0.9$ & $y = -2.29$ in $\textcircled{3}$ we get

Space for Hints

Space for Hints

$$z = \frac{1}{10} [22 + 2 \times .9 - 3 \times -2.29] = 3.067$$

Second Iteration Putting $y = -2.29$ & $z = 3.067$ in ① we get

$$x = \frac{1}{10} [9 - 2(-2.29) - 3.067] = 1.0513$$

Putting $x = 1.0513$ & $z = 3.067$ in ② we get

$$y = \frac{1}{10} [-22 - 1.0513 + 3.067] = -1.9984$$

Putting $x = 1.0513$ & $y = -1.9984$ in ③ we get

$$z = \frac{1}{10} [22 + 2 \times 1.0513 - 3(-1.9984)] = 3.0098$$

Third Iteration Putting $y = -1.9984$ & $z = 3.0098$ in ① we get

$$x = \frac{1}{10} [9 - 2(-1.9984) - 3.0098] = 0.9987$$

Putting $x = 0.9987$ & $z = 3.0098$ in ② we get

$$y = \frac{1}{10} [-22 - 0.9987 + 3.0098] = -1.9989$$

Putting $x = 0.9987$ & $y = -1.9989$ in ③ we get

$$z = \frac{1}{10} [22 + 2 \times 0.9987 - 3(-1.9989)] = 2.9989$$

Fourth Iteration Putting $y = -1.9989$ & $z = 2.9984$ in ① we get

$$x = \frac{1}{10} [9 - 2(-1.9989) - 2.9984] = 0.9998$$

Putting $x = 0.9998$ & $z = 2.9984$ in ② we get

$$y = \frac{1}{10} [-22 - 0.9998 + 2.9984] = -2$$

Putting $x = 0.9998$ & $y = -2$ in ③ we get

$$z = \frac{1}{10} [22 + 2 \times 0.9998 - 3(-2)] = 3$$

Proceeding like this in the next iteration we get $x = 1$, $y = -2$ & $z = 3$

$\therefore$ The solutions of the given system of equations are

$$x = 1, \quad y = -2 \quad \& \quad z = 3$$

Problem 4 Solve the following system of equations by (i) Gauss – seidal method (ii) Gauss – Jacobi method

Space for Hints

$$28x + 4y - z = 32 \quad ; \quad x + 3y + 10z = 24 \quad ; \quad 2x + 17y + 4z = 35$$

Solution Given $28x + 4y - z = 32$

$$x + 3y + 10z = 24$$

$$2x + 17y + 4z = 35$$

The coefficient matrix of the given system equations is

$$A = \begin{pmatrix} 28 & 4 & -1 \\ 1 & 3 & 10 \\ 2 & 17 & 4 \end{pmatrix}$$

Clearly $|28| \geq |4| + |-1| : |3| \geq |1| + |10|$
& $|4| \geq |2| + |17|$

$\therefore A$ is not diagonally dominant.

We can make it to diagonally dominant matrix

Now $A \sim \begin{pmatrix} 28 & 4 & -1 \\ 2 & 17 & 4 \\ 1 & 3 & 10 \end{pmatrix} R_2 \Leftrightarrow R_3$

Clearly $|28| \geq |4| + |-1|$
 $|17| \geq |2| + |4|$
 $|10| \geq |1| + |3|$

$\therefore A$ is diagonally dominant matrix.

Hence the given system of equation becomes

$$28x + 4y - z = 32 \longrightarrow \textcircled{1}$$

$$2x + 17y + 4z = 35 \longrightarrow \textcircled{2}$$

$$x + 3y + 10z = 24 \longrightarrow \textcircled{3}$$

From equations $\textcircled{1}$, $\textcircled{2}$ & $\textcircled{3}$ we have

$$28x = 32 - 4y + z$$

$$\Rightarrow x = \frac{1}{28} [32 - 4y + z] \longrightarrow \textcircled{4}$$

$$17y = 35 - 2x - 4z$$

$$\Rightarrow y = \frac{1}{17}[35 - 2x - 4z] \longrightarrow \textcircled{5}$$

$$10z = 24 - x - 3y$$

$$\Rightarrow z = \frac{1}{10}[24 - x - 3y] \longrightarrow \textcircled{6}$$

(i) Gauss – Seidel Method

First Iteration Putting $y = 0$ & $z = 0$ in $\textcircled{4}$ we get

$$x = \frac{1}{28}[32 - 0 + 0] = \frac{32}{28} = \frac{8}{7} = 1.1429$$

Putting $x = 1.1429$ & $z = 0$ in $\textcircled{5}$ we get

$$y = \frac{1}{17}[35 - 2 \times 1.1429 - 4 \times 0] = -1.9244$$

Putting $x = 1.1429$ & $y = -1.9244$ in $\textcircled{6}$ we get

$$z = \frac{1}{10}[24 - 1.1429 - 3 \times (-1.9244)] = 1.7084$$

Second Iteration Putting $y = 1.9244$ & $z = 1.7084$ in $\textcircled{4}$ we get

$$x = \frac{1}{28}[32 - 7.6976 + 1.7084] = 0.9290$$

Putting $x = 0.9290$ & $z = 1.7084$ in $\textcircled{5}$ we get

$$y = \frac{1}{17}[35 - 1.858 - 6.8336] = 1.5476$$

Putting $x = 0.929$ & $y = 1.5476$ in $\textcircled{6}$ we get

$$z = \frac{1}{10}[24 - 0.929 - 4.6428] = 1.8428$$

After second iteration we have

$$x = 0.929 ; y = 1.5476 ; z = 1.8428$$

Third Iteration Putting $y = 1.5476$ & $z = 1.8428$ in $\textcircled{4}$ we get

$$x = \frac{1}{28}[32 - 6.1904 + 1.8428] = 0.9876$$

Putting $x = 0.9876$ & $z = 1.8428$ in $\textcircled{5}$ we get

$$y = \frac{1}{17}[35 - 1.9752 - 7.3712] = 1.5090$$

Putting $x = 0.9876$ & $y = 1.5090$ in ⑥ we get

$$z = \frac{1}{10} [24 - 0.9876 - 4.527] = 1.8485$$

After 3rd iteration we have

$$x = 0.9876 ; y = 1.9426 ; z = 1.8485$$

Fourth Iteration Putting $y = 1.9426$ & $z = 1.8485$ in ④ we get

$$x = \frac{1}{28} [32 - 6.036 + 1.885] = 0.9933$$

Putting $x = 0.9933$ & $z = 1.8485$ in ⑤ we get

$$y = \frac{1}{17} [35 - 1.9866 - 7.394] = 1.5070$$

Putting $x = 0.9933$ & $y = 1.5070$ in ⑥ we get

$$z = \frac{1}{10} [24 - 0.9266 - 4.571] = 1.8486$$

Fourth Iteration Putting $y = 1.5070$ & $z = 1.8486$ in ④ we get

$$x = \frac{1}{28} [32 - 6.028 + 1.8486] = 0.9936$$

Putting $x = 0.9936$ & $z = 1.8486$ in ⑤ we get

$$y = \frac{1}{17} [35 - 1.9872 - 7.3944] = 1.5070$$

Putting $x = 0.9936$ & $y = 1.5070$ in ⑥ we get

$$z = \frac{1}{10} [24 - 1.9936 - 4.521] = 1.8485$$

Fourth and fifth iterations give almost the same values

Hence $x = 0.9936 ; y = 1.5070 ; z = 1.8485$

(ii) Gauss – Jacobi Method

Gauss Jacobi iteration formula is

$$x^{(r+1)} = \frac{1}{a_1} [d_1 - b_1 y^{(r)} - c_1 z^{(r)}]$$

$$y^{(r+1)} = \frac{1}{b_2} [d_2 - a_2 x^{(r)} - c_2 y^{(r)}]$$

Space for Hints

$$z^{(r+1)} = \frac{1}{c_3} [d_3 - a_3 x^{(r)} - c_3 y^{(r)}]$$

First Iteration Let the initial values be $x = y = z = 0$

put $x = y = 0$ in ④, ⑤ & ⑥ we get

$$x_1 = \frac{32}{28} = 1.1429 ; y_1 = \frac{35}{17} = 2.0588 ; z_1 = \frac{24}{10} = 2.4$$

Second Iteration

$$x_2 = \frac{1}{28} [32 - 4y_1 + z_1] = \frac{1}{28} [32 - 8.2352 + 2.4] = 0.9345$$

$$y_2 = \frac{1}{17} [35 - 2x_1 - 4z_1] = \frac{1}{17} [35 - 2.2858 + 9.6] = 1.3597$$

$$z_2 = \frac{1}{10} [24 - x_1 - 3y_1] = \frac{1}{10} [24 - 1.1429 - 6.1764] = 1.16681$$

Third Iteration

$$x_3 = \frac{1}{28} [32 - 4y_2 + z_2] = \frac{1}{28} [32 - 5.4388 + 1.6681] = 1.0082$$

$$y_3 = \frac{1}{17} [35 - 2x_2 - 4z_2] = \frac{1}{17} [35 - 1.869 + 6.6724] = 1.5564$$

$$z_3 = \frac{1}{10} [24 - x_2 - 3y_2] = \frac{1}{10} [24 - 0.9345 - 4.0791] = 1.8986$$

Fourth Iteration

$$x_4 = \frac{1}{28} [32 - 4y_3 + z_3] = \frac{1}{28} [32 - 6.2256 + 1.8986] = 0.9883$$

$$y_4 = \frac{1}{17} [35 - 2x_3 - 4z_3] = \frac{1}{17} [35 - 2.0164 - 7.5944] = 1.4935$$

$$z_4 = \frac{1}{10} [24 - x_3 - 3y_3] = \frac{1}{10} [24 - 1.0082 - 4.6692] = 1.8323$$

Fifth Iteration

$$x_5 = \frac{1}{28} [32 - 4y_4 + z_4] = \frac{1}{28} [32 - 5.974 + 1.8323] = 0.9949$$

$$y_5 = \frac{1}{17} [35 - 2x_4 - 4z_4] = \frac{1}{17} [35 - 1.9766 - 7.3292] = 1.5114$$

$$z_5 = \frac{1}{10} [24 - x_4 - 3y_4] = \frac{1}{10} [24 - 0.9883 - 4.4805] = 1.8531$$

Sixth Iteration

$$x_6 = \frac{1}{28} [32 - 4y_5 + z_5] = \frac{1}{28} [32 - 6.0456 + 1.8531] = 0.9931$$

$$y_6 = \frac{1}{17} [35 - 2x_5 - 4z_5] = \frac{1}{17} [32 - 1.9898 - 7.4124] = 1.5058$$

$$z_6 = \frac{1}{10} [24 - x_5 - 3y_5] = \frac{1}{10} [24 - 0.9949 - 4.5342] = 1.8471$$

Seventh Iteration

$$x_7 = \frac{1}{28} [32 - 4y_6 + z_6] = \frac{1}{28} [32 - 6.0232 + 1.8471] = 0.9937$$

$$y_7 = \frac{1}{17} [35 - 2x_6 - 4z_6] = \frac{1}{17} [35 - 1.9862 - 7.3884] = 1.5074$$

$$z_7 = \frac{1}{10} [24 - x_6 - 3y_6] = \frac{1}{10} [24 - 0.9931 - 4.5174] = 1.849$$

Eighth Iteration

$$x_8 = \frac{1}{28} [32 - 4y_7 + z_7] = \frac{1}{28} [32 - 6.0296 + 1.8490] = 0.9936$$

$$y_8 = \frac{1}{17} [35 - 2x_7 - 4z_7] = \frac{1}{17} [35 - 1.9874 - 7.396] = 1.5069$$

$$z_8 = \frac{1}{10} [24 - x_7 - 3y_7] = \frac{1}{10} [24 - 0.9937 - 4.5222] = 1.8484$$

The values in the seventh and eighth iteration are close to each other.

Hence the solution is given by

$$x = 0.9936 ; y = 1.5069 ; z = 1.8484$$

Problem 5 Solve the following system of equations by

- (i) Gauss – seidel method
- (ii) Gauss – Jacobi method

$$2x - y = 1.75 ; x - 3y = -1$$

Space for Hints

Soln Given $2x - y = 1.75$

$$x - 3y = -1$$

The coefficient matrix of one given system of equation is

$$A = \begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix}$$

$$\text{Clearly } |2| \geq |-1| \text{ \& } |-3| \geq |1|$$

$\therefore$ A is diagonally dominant

$$\text{Now } 2x = 1.75 + y$$

$$\Rightarrow x = \frac{1}{2}[1.75 + y] \longrightarrow \textcircled{1}$$

$$3y = x + 1$$

$$\Rightarrow y = \frac{1}{3}[x + 1] \longrightarrow \textcircled{2}$$

(i) Gauss – Seidel Method

First Iteration

put $y = 0$ & $x = 0$ in $\textcircled{1}$ & $\textcircled{2}$ respamly we get

$$x_1 = \frac{1}{2}[1.75] = 0.875$$

$$y_1 = \frac{1}{3}[0 + 1] = \frac{1}{3} = 0.333$$

Second Iteration

$$x_2 = \frac{1}{2}[1.75 + 0.333] = 1.0415$$

$$y_2 = \frac{1}{3}[1 + .875] = \frac{1.875}{3} = 0.625$$

Third Iteration

$$\begin{aligned} x_3 &= \frac{1}{2}[1.75 + y_2] = \frac{1}{2}[1.75 + .625] \\ &= \frac{1}{2}[1.750 + .625] = \frac{2.375}{2} = 1.1875 \end{aligned}$$

$$y_3 = \frac{1}{3}[x_2 + 1] = \frac{1.0415 + 1}{3} = \frac{2.0415}{3} = 0.6805$$

Fourth Iteration

$$x_4 = \frac{1}{2}[1.75 + y_3] = \frac{1}{2}[1.75 + 0.6805] = 1.21525$$

$$y_4 = \frac{1}{3}[x_3 + 1] = \frac{1.8515 + 1}{3} = 0.7291$$

Fifth Iteration

$$x_5 = \frac{1}{2}[1.75 + y_4] = \frac{1.75 + .7291}{2} = 1.23955$$

$$y_5 = \frac{1}{3}[x_4 + 1] = \frac{1.2152 + 1}{3} = 0.7384$$

Sixth Iteration

$$x_6 = \frac{1}{2}[1.75 + y_5] = \frac{1.75 + .7384}{2} = 1.2442$$

$$y_6 = \frac{1}{3}[x_5 + 1] = \frac{1.2395 + 1}{3} = \frac{2.2395}{3} = 0.7465$$

Seventh Iteration

$$x_7 = \frac{1}{2}[1.75 + y_6] = \frac{1.75 + .7465}{2} = 1.2483$$

$$y_7 = \frac{1}{3}[x_6 + 1] = \frac{1}{3}[1.2442 + 1] = \frac{2.2442}{3} = 0.7480$$

Eighth Iteration

$$x_8 = \frac{1}{2}[1.75 + y_7] = \frac{1.75 + .7480}{2} = 1.249$$

$$y_8 = \frac{1}{3}[x_7 + 1] = \frac{1}{3}[1 + 1.2483] = \frac{2.2483}{3} = 0.7494$$

Ninenth Iteration

$$x_9 = \frac{1}{2}[1.75 + y_8] = \frac{1}{2}[1.75 + .7494] = 1.2498$$

Space for Hints

$$y_9 = \frac{1}{3}[x_8 + 1] = \frac{1}{3}[1.249 + 1] = \frac{2.249}{3} = 0.7496$$

Tenth Iteration

$$x_{10} = \frac{1}{2}[1.75 + y_9] = \frac{1.75 + .7496}{2} = 1.2498$$

$$y_{10} = \frac{1}{3}[x_9 + 1] = \frac{1 + 1.2498}{3} = 0.7499$$

The values of x and y are almost same for 9th and 10th iterations.

∴ The solution is $x = 1.2498$; $y = 0.749$.

(ii) Gauss – Seidel Method

First Iteration put $y = 0$ in ① we get $x = \frac{1.75}{2} = 0.875$

put $x = .875$ in ② we get $y = \frac{x+1}{3} = \frac{0.875+1}{3} = \frac{1.875}{3} = 0.625$

Second Iteration put $y = 0.625$ in ① we get $x = \frac{1.75 + .625}{2} = 1.1875$

put $x = 1.1875$ in ② we get $y = \frac{1.1875+1}{3} = \frac{2.1875}{3} = 0.7291$

3rd Iteration put $y = .7291$ in ① we get $x = \frac{1}{2}[1.75 + .7291] = 1.2395$

put $x = 1.2395$ in ② we get $y = \frac{1}{3}[x+1] = \frac{2.2395}{3} = 0.7465$

4th Iteration put $y = .7465$ in ① we get $x = \frac{1.75 + .7465}{2} = 1.2483$

put $x = 1.2483$ in ② we get $y = \frac{1}{3}[x+1] = \frac{1}{3}[2.2483] = 0.7494$

5th Iteration put $y = 0.7494$ in ① we get $x = \frac{1.75 + .7494}{2} = 1.2497$

put $x = 1.2497$ in ② we get $y = \frac{1}{3}[x+1] = \frac{2.2497}{3} = 0.7499$

6th Iteration put $y = 0.7499$ in ① we get $x = \frac{1.75 + .7499}{2} = 1.2499$

put $x = 1.2499$ in ② we get $y = \frac{1 + x}{3} = \frac{2.2499}{3} = 0.7499$

Since the values of x & y from 5th and 6th iterations are almost equal.

∴ The solution is $x = 1.2499$
 $y = 0.7499$

Exercise Problems

1] Check whether Gauss – Seidel iteration method can be used to solve the following system of equations in less number of iterations? If possible solve them

(a) $x + y + z = 9$; $2x - 3y + 4z = 13$; $3x + 4y + 5z = 40$

(b) $2x - 3y + 20z = 25$; $20x + y - 2z = 17$; $3x + 20y - z = -18$

2] Solve the following system of equations using Gauss – Jacobi iteration method.

(a) $2x + y + z = 4$; $x + y + 2z = 4$; $x + 2y + z = 4$

(b) $4x + 2y + z = 14$; $x + 5y - z = 10$; $x + y + 8z = 20$

3] Solve the following system of equations using Gauss- Seidel iteration method

(a) $8x - y + z = 18$; $2x + 5y - 2z = 3$; $x + y - 3z = -6$

(b) $5x - 2y + z = -4$; $x + 6y - 2z = 1$; $3x + y + 5z = 13$

(c) $10x - 2y + z = 12$; $x + 9y - z = 10$; $2x - y + 11z = 20$.

4] Solve the following system of equations by

(i) Gauss Jacobi method (ii) Gauss – Seidel method upto 3 places of decimals.

(a) $x + y + 54z = 110$; $27x + 6y - z = 85$; $6x + 15y + 2z = 72$

(b) $10x - 2y - z - u = 3$; $-x - y + 10z - 2u = 27$;
 $-x - y - 2z + 10u = -9$; $-2x + 10y - z - u = 15$.

Space for Hints

Space for Hints

decimals.

- (i) Gauss-Jacobi method (ii) Gauss-Seidel method upto 3 places of decimals.

(f) Solve the following system of equations by

$$\begin{aligned} \text{(a)} \quad & 8x - y + z = 18; \quad 2x + 5y - 2z = 3; \quad x + y - 3z = -6 \\ \text{(b)} \quad & 2x - 2y + z = -4; \quad x + 6y - 2z = 1; \quad 3x + y + 2z = 13 \\ \text{(c)} \quad & 10x - 2y + z = 12; \quad x + 9y - z = 10; \quad 2x - y + 11z = 20 \end{aligned}$$

iteration method

- (g) Solve the following system of equations using Gauss-Seidel

$$\begin{aligned} \text{(a)} \quad & 2x + y + z = 4; \quad x + y + 2z = 4; \quad x + 2y + z = 4 \\ \text{(b)} \quad & 4x + 2y + z = 14; \quad x + 2y - z = 10; \quad x + y + 2z = 20 \end{aligned}$$

iteration method

- (h) Solve the following system of equations using Gauss-Jacobi

$$\text{(a)} \quad 2x - 3y + 20z = 22; \quad 20x + y - 2z = 17; \quad 3x + 20y - z = -18$$

$$\text{(b)} \quad x + y + z = 9; \quad 2x - 3y + 4z = 13; \quad 3x + 4y + 2z = 40$$

possible solve them

the following system of equations in less number of iterations? If

(i) Check whether Gauss-Seidel iteration method can be used to solve

Exercise Problems

The solution is
 $x = 1.2499$
 $y = 0.7499$

Since the values of x & y from 5th and 6th iterations are almost equal.

$$\text{From } x = 1.2499 \text{ in (i) we get } y = \frac{1+x}{3} = \frac{2.2499}{3} = 0.7499$$

Section 7.0 Introduction

Space for Hints

In this chapter we introduce the idea of finite differences and associated concepts, which have important applications in numerical analysis.

For example interpolation formulae are based on finite differences. Through finite differences we study the relation that exist between the values that are assumed by the functions whenever the independent variables change by finite jumps.

In this chapter we study the variation when the independent variables change by casual intervals.

Section 7.1

Difference Operators

In this section we introduce three difference operators namely forward, backward and central difference operators. Conscious the function $y = f(x)$ suppose we are given a table of values of the function at the points $x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \dots, x_n = x_0 + nh$

Let $f(x_0) = y_0, f(x_1) = y_1, f(x_2) = y_2, \dots, f(x_n) = y_n$

We define $\Delta[f(x)] = f(x+h) - f(x)$

Thus $\Delta y_0 = f(x_0 + h) - f(x_0)$

$$= f(x_1) - f(x_0) = y_1 - y_0$$

Similarly $\Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2$

$$\Delta y_{n-1} = y_n - y_{n-1}$$

is called the forward difference operator and $\Delta y_0, \Delta y_1, \Delta y_2, \dots, \Delta y_{n-1}$ are called the first forward differences in the function $y = f(x)$.

Space for Hints

The second order differences of the function $y = f(x)$ are defined by

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

$$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$$

$$\Delta^2 y_{n-1} = \Delta y_n - \Delta y_{n-1}$$

In a similar manner, higher order differences can be defined. In general the n^{th} order differences are defined by the equations.

$$\Delta^n y_i = \Delta^{n-1} y_{i+1} - \Delta^{n-1} y_i, \quad i = 0, 1, 2, 3, \dots, n$$

These differences of the function $y = f(x)$ can be systematically represented in the form of a table called forward differences table. We can construct the difference table for any number of arguments and a sample difference table is given for six consecutive arguments.

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0	Δy_0				
$x_1 = x_0 + h$	y_1	Δy_1	$\Delta^2 y_0$	$\Delta^3 y_0$		
$x_2 = x_0 + 2h$	y_2	Δy_2	$\Delta^2 y_1$	$\Delta^3 y_1$	$\Delta^4 y_0$	
$x_3 = x_0 + 3h$	y_3	Δy_3	$\Delta^2 y_2$	$\Delta^3 y_2$	$\Delta^4 y_1$	$\Delta^5 y_0$
$x_4 = x_0 + 4h$	y_4	Δy_4				
$x_5 = x_0 + 5h$	y_5					

Note 1 In this table y_0 is known as the First entry and $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0, \Delta^5 y_0$ are called the leading differences.

Note 2 Since each higher order difference is defined in terms of the previous lower differences by continuous substitution each higher order difference can be expressed in terms of the table values of the function.

$$\text{Thus } \Delta^2 y_0 = \Delta y_1 - \Delta y_0$$

$$= (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$

$$= [y_3 - 2y_2 + y_1] - [y_2 - 2y_1 + y_0]$$

$$= y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$$

$$= [y_4 - 3y_3 + 3y_2 - y_1] - [y_3 - 3y_2 + 3y_1 - y_0]$$

$$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

We observe that the coefficients occurring on the RHS are simply

Binomial coefficients in $(1-x)^n$. Hence in general we have

$$\Delta^n y_0 = y_n - nC_1 y_{n-1} + nC_2 y_{n-2} - \dots + (-1)^n y_0$$

Section 7.2

Properties of the Operation Δ (Properties for forward difference Operator)

1) Δ is Linear

ie, $\Delta[af(x) + bg(x)] = a\Delta[f(x)] + b\Delta[g(x)]$ Where a and b are constants.

$$\begin{aligned} \text{Proof: } \Delta[af(x) + bg(x)] &= [af(x+h) + bg(x+h)] - [af(x) + bg(x)] \\ &= a[f(x+h) - f(x)] + b[g(x+h) - g(x)] \\ &= a\Delta[f(x)] + b\Delta[g(x)] \end{aligned}$$

This shows that Δ is linear.

$$2) \Delta^m [\Delta^n [f(x)]] = \Delta^{m+n} [f(x)]$$

$$\begin{aligned} \text{Proof } \Delta^m [\Delta^n [f(x)]] &= [(\Delta \cdot \Delta \dots \Delta) \cdot (\Delta \cdot \Delta \dots \Delta)] f(x) \\ &= \underbrace{[\Delta \cdot \Delta \dots \Delta]}_{(m+n) \text{ times}} [f(x)] \\ &= \Delta^{(m+n)} [f(x)] \end{aligned}$$

$$\text{Thus } \Delta^n [\Delta^n [f(x)]] = \Delta^{m+n} [f(x)]$$

$$3) \Delta[f(x) \cdot g(x)] = f(x+h)\Delta[g(x)] + g(x)\Delta[f(x)]$$

Proof $\Delta[f(x) \cdot g(x)] = f(x+h)g(x+h) - f(x)g(x)$

$$= [f(x+h)g(x+h) - f(x+h)g(x)]$$

$$+ [f(x+h)g(x) - f(x)g(x)]$$

$$= f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]$$

$$= f(x+h)\Delta[g(x)] + g(x)\Delta[f(x)]$$

Thus $\Delta[f(x) \cdot g(x)] = f(x+h)\Delta[g(x)] + g(x)\Delta[f(x)]$

$$4) \Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta[f(x)] - f(x)\Delta[g(x)]}{g(x+h)g(x)}$$

Proof $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$

$$= \frac{g(x)f(x+h) - f(x)g(x+h)}{g(x+h)g(x)}$$

$$= \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)}$$

$$= \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{g(x+h)g(x)}$$

$$= \frac{g(x)\Delta[f(x)] - f(x)\Delta[g(x)]}{g(x+h)g(x)}$$

Thus $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta[f(x)] - f(x)\Delta[g(x)]}{g(x+h)g(x)}$

Section 7.3

Backward Differences

Consider the function $y = f(x)$. Suppose we are given a table of values of the function at the points

$$x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \dots, x_n = x_0 + nh$$

Let $f(x_0) = y_0, f(x_1) = y_1, f(x_2) = y_2, \dots, f(x_n) = y_n$

We define $\nabla[f(x)] = f(x) - f(x-h)$

Thus $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \nabla y_3 = y_3 - y_2, \dots, \nabla y_n = y_n - y_{n-1}$

∇ is called the Backward difference operator and

$\nabla y_1, \nabla y_2, \nabla y_3, \dots, \nabla y_n$ are called the First order backward differences of the function $y = f(x)$.

The second order differences of the function $y = f(x)$ are defined by

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$

$$\nabla^2 y_1 = \nabla y_3 - \nabla y_2$$

$$\nabla^2 y_4 = \nabla y_4 - \nabla y_2$$

.....

.....

$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$$

In a similar manner higher order differences can be defined.

In general the n^{th} order Backward differences of the function $y=f(x)$ are defined by

$$\nabla^n y_i = \nabla^{n-1} y_i - \nabla^{n-1} y_{i-1} \quad i = 1, 2, 3, \dots, n$$

These differences of the function $y = f(x)$ can be systematically represented in the form of a table called the Backward difference table.

Backward difference table for 6 arguments:

x	$y = f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
$x_1 = x_0 + h$	y_1	∇y_1				
$x_2 = x_0 + 2h$	y_2	∇y_2	$\nabla^2 y_2$			
$x_3 = x_0 + 3h$	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$		
$x_4 = x_0 + 4h$	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$	
$x_5 = x_0 + 5h$	y_5	∇y_5	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_5$	$\nabla^5 y_5$

Note The Relation between the two difference operators is given by

$$\nabla[f(x+h)] = \Delta[f(x)]$$

$$\text{For } \nabla[f(x+h)] = f(x+h) - f(x) = \Delta[f(x)]$$

$$\begin{aligned}\text{Similarly } \nabla^2[f(x+2h)] &= \nabla[f(x+2h) - f(x+h)] \\ &= \nabla f(x+2h) - \nabla[f(x+h)] \\ &= \Delta[f(x+h)] - \Delta[f(x)] \\ &= \Delta[f(x+h) - f(x)] \\ &= \Delta^2[f(x)]\end{aligned}$$

$$\text{In general } \nabla[f(x+nh)] = \Delta^n f(x)$$

Hence from the forward difference table of the function $y = f(x)$, we are obtain Backward difference of all orders.

Section 7.4

Central Difference Operator

Sometimes it is convenient to employ another system of differences known as central differences.

We define central difference operator δ as

$$\delta[f(x)] = f\left[x + \frac{h}{2}\right] - f\left[x - \frac{h}{2}\right]$$

Thus if $f(x_i) = y_i$, then we have

$$\delta y_{\frac{1}{2}} = y_1 - y_0; \quad \delta y_{\frac{3}{2}} = y_2 - y_1 \dots \delta y_{\frac{n-1}{2}} = y_n - y_{n-1}$$

Here the subscript of δy is the average of the subscripts of the two members of the difference. The higher order differences can be defined similar to forward and backward differences.

$$\delta^2 y_{\frac{1}{2}} = \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}}; \quad \delta^2 y_{\frac{3}{2}} = \delta y_{\frac{5}{2}} - \delta y_{\frac{3}{2}}.$$

$$\delta^3 y_{\frac{3}{2}} = \delta^2 y_{\frac{5}{2}} - \delta^2 y_{\frac{1}{2}} \text{ etc.,}$$

There differences of the function $y = f(x)$ can be systematically represented in the form of a table called the central difference table.

Central Difference Table

x	$y = f(x)$	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0	$\delta y_{\frac{1}{2}}$			
$x_1 = x_0 + h$	y_1	$\delta y_{\frac{3}{2}}$	$\delta^2 y_1$	$\delta^3 y_{\frac{3}{2}}$	
$x_2 = x_0 + 2h$	y_2	$\delta y_{\frac{5}{2}}$	$\delta^2 y_2$	$\delta^3 y_{\frac{5}{2}}$	$\delta^4 y_2$
$x_3 = x_0 + 3h$	y_3	$\delta y_{\frac{7}{2}}$	$\delta^2 y_3$		
$x_4 = x_0 + 4h$	y_4				

Theorem 1 (Fundamental theorem for finite differences)

Statement Let $f(x)$ be a polynomial of degree n .

Then the n^{th} difference of $f(x)$ is constant and all higher order differences are zero.

$$\Delta^r [f(x)] = \begin{cases} \text{Constant} & \text{if } r = n \\ 0 & \text{if } r > n \end{cases}$$

ie, $\Delta^r [f(x)] = 0$ if $r > n$

Proof Let $f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$ where $a_0 \neq 0$ be a polynomial of degree ' n '.

[Here a_0 is called the leading coefficient of $f(x)$]

$$\text{Now } \Delta[f(x)] = f(x+h) - f(x) =$$

$$[a_0(x+h)^n + a_1(x+h)^{n-1} + a_2(x+h)^{n-2} + \dots + a_{n-1}(x+h) + a_n]$$

$$- [a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n]$$

$$= \{a_0 [x^n + n c_1 x^{n-1} h + n c_2 x^{n-2} h^2 + \dots + h^n]$$

$$+ a_1 [x^{n-1} + (n-1) c_1 x^{n-2} h + \dots + h^{n-1}] + \dots + a_n\}$$

$$- \{a_0 x^n + a_1 x^{n-1} + a_2 x^{n-1} + a_2 x^{n-2} + \dots + a_n\}$$

$$= a_0 n h x^{n-1} + \text{terms involving lower power of } x.$$

Thus $\Delta^2[f(x)]$ is a polynomial of degree $(n-1)$ with leading coefficient $n h a_0$.

Similarly $\Delta^2[f(x)]$ is a polynomial of degree $(n-2)$ with leading coefficient $n(n-1)h^2 a_0$.

Proceeding this process we set $\Delta^n[f(x)]$ is a polynomial of degree zero with leading coefficient $n(n-1)(n-2)\dots 1 \cdot h^n a_0 = n! h^n a_0$

Since $\Delta^n[f(x)]$ is a constant polynomial it follows that

$$\Delta^n[f(x)] = n! h^n a_0$$

$\therefore$ The n^{th} difference of $f(x)$ is a constant and all the higher order differences of order greater than n are zero.

$$\text{ie, } \Delta^r[f(x)] = \begin{cases} \text{constant if } r = n \\ 0 \text{ if } r > n \end{cases}$$

Definition A product of the form $x(x-h)(x-2h)\dots[x-(n-1)h]$ is called a factorial function and is denoted by $x^{(n)}$

$$\text{ie, } x^{(n)} = x(x-h)(x-2h)\dots[x-(n-1)h]$$

If we take $h=1$, then $x^{(1)} = x$, $x^{(2)} = x(x-1)$ & $x^{(3)} = x(x-1)(x-2)$

We observe that $x^{(n)}$ is a polynomial of degree 'n' with leading coefficient 1.

Theorem 2 $\Delta x^{(n)} = n h x^{(n-1)}$. In particular when $h=1$, $\Delta x^{(n)} = n x^{(n-1)}$.

Proof $\Delta x^{(n)} = (x+h)^{(n)} - x^{(n)}$

$$= \{(x+h)x(x-h)\dots[x-(n-2)h]\} - \{x(x-h)(x-2h)\dots[x-(n-1)h]\}$$

$$\{(x+h) - [x-(n-1)h]\}$$

$$= x^{(n-1)} nh$$

Space for Hints

$$\therefore \Delta x^{(n)} = nhx^{(n-1)} \longrightarrow \textcircled{1}$$

put $h=1$ on both sides of $\textcircled{1}$ we get $\Delta x^{(n)} = nx^{(n-1)} \longrightarrow \textcircled{2}$

Note1 From $\textcircled{2}$ we get the formula for first order difference which is obtained by the simple differentiation Rule.

For example:
$$\begin{aligned} \Delta^2 x^{(n)} &= \Delta [\Delta x^{(n)}] \\ &= \Delta [nhx^{(n-1)}] \\ &= nh \Delta [x^{(n-1)}] \\ &= nh [(n-1)h \cdot x^{(n-2)}] \\ &= n(n-1)h^2 x^{(n-2)} \end{aligned}$$

Proceeding like this we get

$$\begin{aligned} \Delta^n [x^{(n)}] &= n(n-1)(n-2)\dots\dots 3.2.1 \cdot h^n x^{(n-n)} \\ &= n(n-1)(n-2)\dots\dots 3.2.1 \cdot h^n x^0 \\ &= \{1.2.3. \dots (n-1).n\} h^n = 1 \end{aligned}$$

$$\Delta^n [x^{(n)}] = n! h^n$$

Note 2 Any polynomial $f(x)$ of degree 'n' can be expressed in the form

$$f(x) = c_0 x^{(n)} + c_1 x^{(n-1)} + c_2 x^{(n-2)} + \dots + c_{n-1} x^{(1)} + c_n.$$

If $f(x)$ is divided successively by $x-0$, $x-1$, $x-2, \dots, x-(n-1)$, then the remainders gives the coefficient $C_n, C_{n-1}, C_{n-2}, \dots, C_0$.

Definition The Reciprocal factorial function $x^{(-n)}$

defined as

$$x^{(-n)} = \frac{1}{(x+h)(x+2h)\dots(x+nh)}$$

where n is a positive integer.

As in the case of factorial function, the formula for first order differences of $x^{(-n)}$ is similar to differentiation rule when $h=1$

Theorem 3

$$\Delta x^{(-n)} = (-n)hx^{[-(n+1)]}$$

In particular when $h=1$, $\Delta x^{(-n)} = -nx^{[-(n+1)]}$

Proof By defn. $\Delta x^{(-n)} = (x+n)^{(-n)} - x^{(-n)}$

$$= \frac{1}{(x+2h)(x+3h)\dots(x+(n+1)h)} - \frac{1}{(x+h)(x+2h)+\dots+(x+nh)}$$

$$= \frac{1}{(x+h)(x+3h)\dots(x+(n+1)h)} \left\{ (x+h) - (x+(n+1)h) \right\}$$

$$= \frac{x+h-x-nh-h}{(x+h)(x+2h)(x+3h)\dots(x+(n+1)h)}$$

$$= \frac{-nh}{(x+h)(x+2h)\dots(x+(n+1)h)}$$

$$\therefore \Delta x^{(-n)} = (-n)h x^{-(n+1)} \longrightarrow \textcircled{1}$$

put $h=1$ in $\textcircled{1}$ We get $\Delta x^{(-1)} = (-n) x^{-(n+1)} = -nx^{-(n+1)}$

Note $\Delta^2 [x^{(-n)}] = \Delta [\Delta x^{(-n)}]$

$$= \Delta [-nhx^{-(n+1)}]$$

$$= (-nh)\Delta [x^{-(n+1)}]$$

$$= (-nh)[- (n+1)h]x^{-(n+2)}$$

$$= (-1)^2 h^2 n(n+1)x^{-(n+2)}$$

$$\Delta^3 [x^{(-n)}] = (-1)^3 h^3 n(n+1)(n+2)x^{-(n+3)}$$

In general $\Delta^r [x^{(-n)}] = (-1)^r h^r n(n+1)(n+2)\dots(n+r-1)x^{-(n+r)}$

$$= (-1)^r h^r n(n+1)(n+2)\dots(n+r-1)x^{-(n+r)}$$

Example 1 Let $f(x) = \frac{1}{(x+1)(x+2)(x+3)} = x^{(-3)}$

$$\Delta[f(x)] = -3x^{(-4)}, \Delta^2[f(x)] = \Delta[\Delta f(x)]$$

$$= \Delta[-3x^{(-4)}]$$

$$= (-3)(-4)x^{-(4+1)}$$

$$\begin{aligned}
 &= 12 x^{[-(5)]} \\
 \Delta^3[f(x)] &= \Delta[\Delta^2 f(x)] = \Delta[12x^{-(5)}] \\
 &= 12 \Delta[x^{-(5)}] \\
 &= (12)(-5) x^{-(5+1)} \\
 &= -60x^{-(6)} \dots\dots\dots
 \end{aligned}$$

Example 2 Let $y = f(x) = \frac{1}{x(x+1)(x+2)}$

$$= \frac{1}{[(x-1)+1][(x-1)+2][(x-1)+3]} = (x-1)^{(-3)}$$

$$\begin{aligned}
 \Delta y &= \Delta[f(x)] = \Delta[(x-1)^{(-3)}] = (-3)(x-1)^{(-4)} \\
 &= -3(x-1)^{(-4)}
 \end{aligned}$$

$$\begin{aligned}
 \Delta^2 y &= \Delta[\Delta y] = \Delta[-3(x-1)^{(-4)}] = (-3)\Delta[(x-1)^{(-4)}] \\
 &= (-3)(-4)(x-1)^{(-5)} \\
 &= 12(x-1)^{(-5)}
 \end{aligned}$$

$$\begin{aligned}
 \Delta^3 y &= \Delta[\Delta^2 y] = \Delta[12(x-1)^{(-5)}] \\
 &= 12 \Delta[(x-1)^{(-5)}] \\
 &= (12)(-5)(x-1)^{(-6)} \\
 &= -60(x-1)^{(-6)} \text{ etc.}
 \end{aligned}$$

Solved Problems

Problem 1 Form the forward difference table for the following data.

x	0	1	2	3	4
y	8	11	9	15	6

Solution

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta_4 y$
0	8	3			
1	11	-2	-5	13	
2	9	6	8	-23	-36
3	15	-9	-15		
4	6				

Problem 2 Form the difference table for the following data (i)

x	0	1	2	3	4	5
y	1	5	19	55	125	241

and hence find $\Delta^5 y_0$

(ii)

x	0	1	2	3	4	5	6
y	2	5	8	20	38	10	3

and hence find $\Delta^6 y_0$

Solution

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	1	4				
1	5	4	0			
2	9	46	42	42	-60	
3	55	70	24	-18	-40	
4	125	116	46	22		20
5	244					

$\Delta^5 y_0$

Clearly from the table $\Delta^5 y_0 = 20$

Space for Hints

(ii)

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	2	3					
1	5	3	0				
2	8	12	9	-12			
3	20	18	6	-52	40		
4	38	-28	-46	-55	-122	70	
5	10	-7	21	67			30 $\rightarrow \Delta^6 y_0$
6	3						

Clearly from the table (last Column $\Delta^6 y$) the value of $\Delta^6 y_0 = 30$

Problem 3 Find $\Delta[2^x]$

Solution By defn. $\Delta[f(x)] = f(x+h) - f(x)$

$$\begin{aligned}\Delta[2^x] &= 2^{x+h} - 2^x \\ &= 2^x [2^h - 1]\end{aligned}$$

Problem 4 Find the n^{th} difference of e^x

Solution By definition, $\Delta[f(x)] = f(x+h) - f(x)$

$$\Delta[e^x] = e^{x+h} - e^x = e^x (e^h - 1)$$

$$\begin{aligned}\Delta^2[e^x] &= \Delta[\Delta e^x] = \Delta[e^x (e^h - 1)] = (e^h - 1) \Delta[e^x] = (e^h - 1)^2 e^x \\ &= (e^h - 1)^2 e^x\end{aligned}$$

$$\text{Similarly, } \Delta^3[e^x] = \Delta[\Delta^2(e^x)]$$

$$= \Delta[(e^h - 1)^2 e^x]$$

$$= (e^h - 1)^2 \Delta[e^x]$$

$$= (e^h - 1)^2 (e^h - 1) e^x$$

$= (e^h - 1)^3 \cdot e^x$

Space for Hints

Proceeding like this we get $\Delta^n[e^x] = e^x[e^h - 1]^n$

Problem 5

Form the difference table of the function

$y = x^3 + x^2 - 2x + 1$ for $x = -1, 0, 1, 2, 3, 4$.

Solution Given $y(x) = x^3 + x^2 - 2x + 1$ for $x = -1, 0, 1, 2, 3, 4$

$y(-) = (-1)^3 + (-1)^2 - 2(-1) + 1$ $y(0) = 0^3 + 0^2 - 2 \times 0 + 1 = 1$
 $= +1 + 1 - 2 + 1 = 3$ $y(1) = 1^3 + 1^2 - 2 \times 1 + 1$
 $= 2 - 2 + 1 = 1$

$y(2) = 2^3 + 2^2 - 2 \times 2 + 1$ $y(3) = 3^3 + 3^2 - 2 \times 3 + 1$
 $= 8 + 4 - 4 + 1 = 9$ $= 27 + 9 - 6 + 1 = 31$

$y(4) = 4^3 + 4^2 - 2 \times 4 + 1$
 $= 64 + 16 - 8 + 1 = 73$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
-1	3					
		-2				
0	1		2			
		0		6		
1	1		8		0	
		8		6		0
2	9		14		0	
		22		6		
3	31		20			
		42				
4	73					

Problem 6 Prove that $\Delta[\log(f(x))] = \log\left[1 + \frac{\Delta[f(x)]}{f(x)}\right]$

Solution By defn. $\Delta[f(x)] = f(x+h) - f(x)$

$$\therefore \Delta[\log f(x)] = \log[f(x+h)] - \log[f(x)]$$

$$= \log \left[\frac{f(x+h)}{f(x)} \right]$$

$$= \log \left[\frac{f(x) + f(x+h) - f(x)}{f(x)} \right]$$

$$= \log \left[\frac{f(x) + \Delta f(x)}{f(x)} \right]$$

$$= \log \left[\frac{f(x)}{f(x)} + \frac{\Delta[f(x)]}{f(x)} \right]$$

$$\text{Thus } \Delta[\log f(x)] = \log \left[1 + \frac{\Delta[f(x)]}{f(x)} \right]$$

Problem 7 Prove that $\Delta \left[\tan^{-1} \left(\frac{n-1}{n} \right) \right] = \tan^{-1} \left(\frac{1}{2n^2} \right)$

Solution Without loss of generality we take the interval of differencing as $h=1$

By defn. $\Delta[f(x)] = f(x+h) - f(x)$

$$\text{Now } \Delta \left[\tan^{-1} \left(\frac{n-1}{n} \right) \right] = \tan^{-1} \left[\frac{(n+1)-1}{n+1} \right] - \tan^{-1} \left(\frac{n-1}{n} \right)$$

$$= \tan^{-1} \left[\frac{n}{n+1} \right] - \tan^{-1} \left[\frac{n-1}{n} \right]$$

$$= \tan^{-1} \left[\frac{\left(\frac{n}{n+1} \right) - \left(\frac{n-1}{n} \right)}{1 + \left(\frac{n}{n+1} \right) \left(\frac{n-1}{n} \right)} \right]$$

By formula $\tan^{-1} A - \tan^{-1} B$

$$= \tan^{-1} \left[\frac{A-B}{1+AB} \right]$$

$$= \tan^{-1} \left[\frac{\left(\frac{n^2 - (n+1)(n-1)}{n(n+1)} \right)}{\frac{(n+1)n + n(n-1)}{(n+1)n}} \right]$$

$$= \tan^{-1} \left[\frac{n^2 - (n^2 - 1)}{n^2 + n + n^2 - n} \right] = \tan^{-1} \left(\frac{n^2 - n^2 + 1}{2n^2 + n - n} \right) = \tan^{-1} \left[\frac{1}{2n^2} \right]$$

$$\text{Thus } \Delta \left[\tan^{-1} \left(\frac{n-1}{n} \right) \right] = \tan^{-1} \left(\frac{1}{2n^2} \right)$$

Problem 8 If $f(x) = \frac{x}{x^2 + 7x + 12}$ find $\Delta(f(x))$ by taking the interval of differencing as unity.

Solution Given $f(x) = \frac{x}{x^2 + 7x + 12} = \frac{x}{(x+4)(x+3)}$ ①

Let us first split $f(x)$ as into its partial fraction conscious

$$\frac{x}{(x+4)(x+3)} = \frac{A}{(x+4)} + \frac{B}{(x+3)} \quad \text{②}$$

$$\frac{x}{(x+4)(x+3)} = \frac{A(x+3) + B(x+4)}{(x+4)(x+3)}$$

$$\Rightarrow x = A(x+3) + B(x+4) \quad \text{③}$$

put $x = -3$ in ③ we get $-3 = B(-3+4) \Rightarrow B = -3$

put $x = -4$ in ② we get $-4 = A(-3) = A(-1) \Rightarrow A = 4$

Substituting the values of A & B in ②

We set

$$\frac{x}{(x+4)(x+3)} = \frac{4}{x+4} - \frac{3}{x+3}$$

$$\text{ie, } f(x) = \frac{4}{x+4} - \frac{3}{x+3}$$

$$\text{Now } \Delta[f(x)] = f(x+h) - f(x)$$

$$= f(x+1) - f(x) (\because h=1)$$

$$\therefore \Delta[f(x)] = \left[\frac{4}{(x+1)+4} - \frac{3}{(x+1)+3} \right] - \left[\frac{4}{x+4} - \frac{3}{x+3} \right]$$

$$= \frac{4}{x+5} - \frac{3}{x+4} - \frac{4}{x+4} + \frac{3}{x+3}$$

$$\Delta[f(x)] = \frac{4}{x+5} - \frac{7}{x+4} + \frac{3}{x+3}$$

Problem 9 Evaluate the following taking the interval of differencing as 1

(a) Δe^x (b) $\Delta \tan^{-1} x$ (c) $\Delta 3^x$ (d) $\Delta \left[\frac{2^x}{x!} \right]$ (e) $\Delta(x + \cos x)$

(f) $\Delta[x(x+1)(x+2)(x+3)]$

Solution Given that $h=1$,

$$\therefore \text{By defn. } \Delta[f(x)] = f(x+1) - f(x) = f(x+1) - f(x)$$

(a) $\Delta e^x = e^{x+1} - e^x : e^h - e^x = e^x [e^h - 1]$

(b) $\Delta[\tan^{-1}] = \tan^{-1}(x+1) - \tan^{-1}(x)$

$$= \tan^{-1} \left[\frac{(x+1) - x}{1 + (x+1)x} \right] \text{ by formula } \tan^{-1} A \pm \tan^{-1} B = \tan^{-1} \left[\frac{A \pm B}{1 \mp AB} \right]$$

$$= \tan^{-1} \left[\frac{1}{x^2 + x + 1} \right]$$

(c) $\Delta[3^x] = 3^{x+1} - 3^x$

$$= 3^x \cdot 3 - 3^x = 3^x [3 - 1] = 3^x \times 2 = 2 \times 3^x$$

(d) $\Delta \left[\frac{2^x}{x!} \right] = \frac{2^{x+1}}{(x+1)!} - \frac{2^x}{x!}$

$$= \frac{2^{x+1}}{(x+1)x!} - \frac{2^x}{x!}$$

$$[\because (x+1)! = (x+1)x!]$$

$$= \frac{2^{x+1} - (x+1)2^x}{(x+1)x!}$$

$$\frac{2^x \times 2 - x2^x - 2^x}{(x+1)x!}$$

$$= \frac{2^x[2-x-1]}{(x+1)x!} = \frac{2^x[1-x]}{(x+1)x!} = \frac{2^x[1-x]}{(x+1)!}$$

$$\begin{aligned} \text{(e)} \quad \Delta[x + \cos x] &= \Delta[x] + \Delta[\cos x] \\ &= (x+1) - x + \cos(x+1) - \cos x \\ &= x + \cos(x+1) - \cos x \end{aligned}$$

$$\begin{aligned} \text{(f)} \quad \Delta[x(x+1)(x-12)(x+3)] \\ &= (x+1)(x+2)(x+3)(x+4) - x(x+1)(x+2)(x+3) \\ &= (x+1)(x+2)(x+3)[(x+1) - x] \\ &= (x+1)(x+2)(x+3) \end{aligned}$$

Problem 10 Find the 1st and 2nd order differences for $f(x) = ab^{cx}$

Solution By defn.

$$\begin{aligned} \Delta[f(x)] &= f(x+h) - f(x) \\ \Delta[ab^{cx}] &= ab^{c(x+h)} - ab^{cx} \\ &= ab^{cx}b^{ch} - ab^{cx} \\ &= ab^{cx}[b^{ch} - 1] \\ &= (b^{ch} - 1)(ab^{cx}) \end{aligned}$$

$$\begin{aligned} \Delta^2[f(x)] &= \Delta^2[ab^{cx}] \\ &= \Delta[\Delta[ab^{cx}]] \end{aligned}$$

$$\begin{aligned} &\Delta[(b^{ch} - 1)(ab^{cx})] \\ &= a(b^{ch} - 1)\Delta[b^{cx}] \\ &= a(b^{ch} - 1)[b^{c(x+h)}b^{ch} - b^{cx}] \\ &= a(b^{ch} - 1)[b^{cx}b^{ch} - b^{cx}] \\ &= a(b^{ch} - 1)b^{cx}[b^{ch} - 1] \\ &= a(b^{ch} - 1)^2 b^{cx} \end{aligned}$$

Problem 11 Evaluate $((\Delta - \nabla)x^2$ taking the interval of differencing as h .

Solution By defn.

$$\begin{aligned} \Delta[f(x)] &= f(x+h) - f(x) \\ \nabla[f(x)] &= f(x) - f(x-h) \end{aligned}$$

$$(\Delta - \nabla)(x^2) = \Delta(x^2) - \nabla(x^2)$$

$$\begin{aligned} \text{Now, } & \Delta(x^2) = [(x+h)^2 - x^2] - [x^2 - (x-h)^2] \\ &= [x^2 + h^2 + 2hx - x^2] - [x^2 - x^2 - h^2 + 2hx] \\ &= [2hx + h^2] - [2hx - h^2] \\ &= 2hx + h^2 - 2hx + h^2 \\ &= 2h^2 \end{aligned}$$

Problem 12 Find $\Delta^n [\text{Sin} x]$ by taking $h=1$

Solution By defn. $\Delta[f(x)] = f(x+h) - f(x)$

$$\Delta[\text{Sin} x] = \text{Sin}(x+h) - \text{Sin} x$$

$$= \text{Sin}(x+1) - \text{Sin} x$$

$$= 2\text{Cos}\left[x + \frac{1}{2}\right] \text{Sin}\left(\frac{1}{2}\right)$$

$$= 2\text{Sin}\left(\frac{1}{2}\right) \text{Cos}\left[x + \frac{1}{2}\right]$$

$$= 2\text{Sin}\left(\frac{1}{2}\right) \text{Sin}\left[\frac{\pi}{2} + x + \frac{1}{2}\right]$$

Formula

Sin C - Sin D

$$= 2\text{Cos}\left(\frac{C+D}{2}\right)$$

$$= \text{Sin}\left(\frac{C-D}{2}\right)$$

Formula

$$\text{Cos } \theta = \text{Sin}(90^\circ + \theta)$$

$$= \text{Sin}\left(\frac{\pi}{2} + 0\right)$$

$$\text{Thus } \Delta[\text{Sin} x] = 2\text{Sin}\left(\frac{1}{2}\right) \text{Sin}\left[\frac{\pi}{2} + \left(x + \frac{1}{2}\right)\right] \longrightarrow \textcircled{1}$$

$$\Delta^2[\text{Sin} x] = \Delta[\Delta(\text{Sin} x)]$$

$$\Delta\left[2\text{Sin}\left(\frac{1}{2}\right) \text{Sin}\left(\frac{\pi}{2} + \left(x + \frac{1}{2}\right)\right)\right] \quad (\text{Using } \textcircled{1})$$

$$= 2\text{Sin}\left(\frac{1}{2}\right) \Delta\left[\text{Sin}\left(\frac{\pi}{2} + \left(x + \frac{1}{2}\right)\right)\right]$$

$$= 2\text{Sin}\left(\frac{1}{2}\right) \left\{ \text{Sin}\left[\frac{\pi}{2} + x + 1 + \frac{1}{2}\right] - \text{Sin}\left[\frac{\pi}{2} + \left(x + \frac{1}{2}\right)\right] \right\}$$

$$= 2\text{Sin}\left(\frac{1}{2}\right) \left[2\text{Cos}\left(\frac{\pi}{2} + x + 1\right) \cdot \text{Sin}\left(\frac{1}{2}\right) \right]$$

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$$= \left[2\sin\left(\frac{1}{2}\right) \right] \left[2\sin\left(\frac{1}{2}\right) \cos\left[\frac{\pi}{2} + (x+1)\right] \right]$$

$$= \left[2\sin\left(\frac{1}{2}\right) \right]^2 \sin\left[\frac{\pi}{2} + \left(\frac{\pi}{2} + x + 1\right)\right]$$

$$= \left[2\sin\left(\frac{1}{2}\right) \right]^2 \sin\left[2\left(\frac{\pi}{2} + \frac{1}{2}\right) + x\right]$$

$$\text{Thus } \Delta[\sin x] = \left[2\sin\left(\frac{1}{2}\right) \right] \sin\left[\left(\frac{\pi}{2} + \frac{1}{2}\right) + x\right]$$

$$\Delta^2[\sin x] = \left[2\sin\left(\frac{1}{2}\right) \right]^2 \sin\left[2\left(\frac{\pi}{2} + \frac{1}{2}\right) + x\right]$$

$$\Delta^3[\sin x] = \left[2\sin\left(\frac{1}{2}\right) \right]^3 \sin\left[3\left(\frac{\pi}{2} + \frac{1}{2}\right) + x\right]$$

Problem 13 Evaluate $\Delta^{10}[(1-x) \cdot (1-2x^2) \cdot (1-3x^3)(1-4x^4)]$

Solution Let $y = f(x) = (1-x)(1-2x^2)(1-3x^3)(1-4x^4)$

This is a polynomial of degree 10 with leading coefficient 24

$$\therefore (1-x)(1-2x^2)(1-3x^3)(1-4x^4) = (24)x^4$$

$$\text{Since } \Delta^n y = \begin{cases} 0 & \text{if } n > 10 \text{ (by theorem ①)} \\ \text{constant if } n=10 \end{cases}$$

$$\text{We have } \Delta^{10} y = \Delta^{10} [24x^{10}] = 24\Delta^{10} [x^{10}] = 24 \cdot 10!$$

Problem 14 Find the Sixth term of the sequence

2, 6, 12, 20, 30,

Solution Let 'a' be the sixth term of one given sequence.

∴ The difference table for the given data is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	2					
1	6	4				
2	12	6	2			
3	20	8	2	0		
4	30	10	2	0	0	
5	a	$a-30$	$a-40$	$a-42$	$a-42$	$a-42$

Since five values are given

$$\therefore \Delta^5 y = 0$$

$$\Rightarrow a - 42 = 0$$

Thus $a=42$

∴ The sixth term of the given sequence is 42.

Problem 15 If $U_1=1, U_3=17, U_4=43, U_5=89$
Find the value of U_2 .

Solution Given $U_1=1, U_3=17, U_4=43, U_5=89$

To find U_2

Let $U_2 = a$

We have the following difference table.

x	U	ΔU	$\Delta^2 U$	$\Delta^3 U$	$\Delta^4 U$
1	1				
2	a	$a-1$			
3	17	$17-a$	$18-2a$		
4	43	26	$a+9$	$3a-9$	
5	89	46	20	$11-a$	$29-4a$

Since only four values are given:

$$\therefore \Delta^4 y = 0$$

$$\therefore 20 - 4a = 0 \Rightarrow 4a = 20$$

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$$\Rightarrow a = 5$$

Hence the value of U_2 is 5 ie, $U_2=5$

Problem 16 Obtain the fifth term of the sequence 3,6,11,18

Solution Let a be one fifth term of the given sequence

The difference table for the given data is as follows.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	3	3			
1	6	5	2		
2	11	7	2	0	
3	18	$a-18$	$a-25$	$a-27$	$a-27$
4	a				

Since four values are given $\Delta^4 y = 0$

$$\text{Hence } a-27 = 0 \Rightarrow a=27$$

$\therefore$ The fifth value of the given sequence 27

Problem 17 Express $2x^3 - 3x^2 + 4x - 8$ as a factorial polynomial

Solution Let the factorial polynomial be $Ax^{(3)} + Bx^{(2)} + cx^{(1)} + D$

$$\therefore 2x^3 - 3x^2 + 4x - 8 = Ax^{(3)} + Bx^{(2)} + cx^{(1)} + D$$

Where A,B,C,D are to be determined by the Synthetic division.

0	2	-3	4	-8
		0	0	0
1	2	-3	+4	-8
		2	-1	
2	2	-1	3	
		4		
	2	3		

$\therefore$ The factorial polynomial is $2x^{(3)} + 3x^{(2)} + 3x^{(1)} - 8$

Problem 18*Space for Hints*

Express the following polynomials in terms of factorial polynomials

i) $x^3 - 2x^2 + x - 1$

ii) $x^4 - 12x^3 + 24x^2 - 30x + 9$

Solution Let the factorial polynomial be $Ax^{(3)} + Bx^{(2)} + cx^{(1)} + D$

$$\therefore x^3 - 2x^2 + x - 1 = Ax^{(3)} + Bx^{(2)} + cx^{(1)} + D$$

Where A, B, C, D are to be determined by synthetic division.

0	1	-2	1	-1
		0	0	0
1	1	-2	1	-1
		1	-1	
2	1	-1	0	
		2		
	1	1		

The factorial polynomial is $x^{(3)} + x^{(2)} + 0x^{(1)} - 1$ ii) Let the factorial polynomial be $Ax^{(4)} + Bx^{(3)} + Cx^{(2)} + Dx^{(1)} + E$

$$x^4 - 12x^3 + 24x^2 - 30x - 19 = Ax^{(4)} + Bx^{(3)} + Cx^{(2)} + Dx^{(1)} + E$$

Where A, B, C, D & E are to be determined by synthetic division.

0	1	-12	24	-30	9
		0	0	0	0
1	1	-12	24	-30	9
		1	-11	13	
2	1	-11	13	-17	
		2	-18		
3	1	-9	-5		
		3			
	1	-6			

The factorial Polynomial is $x^{(4)} - 6x^{(3)} - 5x^{(2)} - 17x^{(1)} + 9$.

Problem 19 Show that

$$\Delta(5x^4 + 6x^3 + x^2 - x + 7) = 20x^{(3)} + 108x^{(2)} + 108x^{(1)} + 11$$

Solution Let $y = 5x^{(4)} + Bx^{(3)} + cx^{(2)} + Dx^{(1)} + E$ be one factorial polynomial of y .

To find the values of $A, B, C, D,$ & E by synthetic division.

0	5	6	1	-1	+7
		0	0	0	0
1	5	6	1	-1	+7
		5	11	+12	
2	5	11	12	+11	
		10	42		
3	5	21	54		
		15			
	5	36			

$\therefore$ The factorial polynomial of y is $y = 5x^4 + 36x^{(3)} + 54x^{(2)} + 11x^{(1)} + 7$
 $\Delta y = 5x4x^{(3)} + 36x3x^{(2)} + 54x2x^{(1)} + 11x1x^{(0)}$
 $= 20x^{(3)} + 108x^{(2)} + 108x^{(1)} + 11$

Problem 20 Find the second difference of one polynomial

$$f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9 \text{ with } n=2$$

Solution First we shall express the given polynomial $f(x)$ in terms of factorial polynomial by synthetic division with $h=2$.

0	1	-12	42	-30	9
		0	0	0	0
2	1	-12	42	-30	9
		2	-20	44	
4	1	-10	22	14	
		4	-24		
6	1	-6	-2		
		6			
	1	0			

$$\therefore f(x) = x^{(4)} + 0x^{(3)} - 2x^{(2)} + 14x^{(1)} + 9$$

$$\begin{aligned}\Delta f(x) &= 8x^{(3)} + 0x^{(2)} - 8x^{(1)} + 28x^0 \quad (\because h = 2) \\ &= 8x^{(3)} - 8x^{(1)} + 28\end{aligned}$$

$$\Delta^2(f(x)) = 48x^{(2)} - 16 = 48x(x-2) - 16 = 48x^2 - 96x - 16$$

Problem 21 Find the function whose first difference is

$$x^3 + 3x^2 + 5x + 12$$

Solution Given $\Delta y = x^3 + 3x^2 + 5x + 12$ ①

To find y

we express ① in terms of function polynomial

0	1	3	5	12
		0	0	0
1	1	3	5	12
		1	4	
2	1	4	9	
		2		
	1	6		

$$\therefore \Delta y = x^{(3)} + 6x^{(2)} + 9x^{(1)} + 12$$

$$\begin{aligned}\therefore y &= \Delta^{-1}[x^{(3)} + 6x^{(2)} + 9x^{(1)} + 12] = 3\Delta^{-1}(x^{(3)}) + 6\Delta^{-1}(x^{(2)}) \\ &\quad + 9\Delta^{-1}(x^{(1)}) + \Delta^{-1}(12)\end{aligned}$$

$$= \frac{x^{(4)}}{4} + \frac{x^{(3)}}{3} + 9\frac{x^{(2)}}{2} + 12x^{(1)} + c$$

$$= \frac{x^{(4)}}{4} + 2x^{(3)} + \frac{9}{2}x^{(2)} + 12x^{(1)} + c$$

$$= \frac{1}{4}[x(x-1)(x-2)(x-3)] + [x(x-1)(x-2)]$$

$$+ \frac{9}{2}[x(x-1)] + 12x + c$$

Problem 22 If $\Delta y = 2x^3 + 3x^2 - 5x + 4$ Find y

Solution Given $\Delta y = 2x^3 + 3x^2 - 5x + 4 \longrightarrow \textcircled{1}$

To find y

we express $\textcircled{1}$ in terms of functional polynomial.

0	2	3	-5	4
		0	0	0
1	2	3	-5	4
		2	5	
2	2	5	0	
		4		
	2	9		

$$\therefore \Delta y = 2x^{(3)} + 9x^{(2)} + 0x^{(1)} + 4.$$

$$\therefore y = \Delta^{-1} [2x^{(3)} + 9x^{(2)} + 0x^{(1)} + 4] = 2\Delta^{-1}(x^{(3)}) + 9\Delta^{-1}(x^{(2)}) + 0\Delta^{-1}(x^{(1)}) + \Delta^{-1}(4)$$

$$= \frac{2x^{(4)}}{4} + 9\frac{x^{(3)}}{3} + 0\frac{x^{(2)}}{2} + 4x^{(1)} + c$$

$$= \frac{x^{(4)}}{2} + 3x^{(3)} + \frac{9}{2}x^{(2)} + 4x^{(1)} + c$$

$$= \frac{1}{2}[x(x-1)(x-2)(x-3)] + 3[x(x-1)(x-2)]$$

$$+ 4x + c$$

Problem 23 If $y = \frac{1}{(3x+1)(3x+4)(3x+7)}$ show that

$$\Delta^2 y = \frac{108}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}$$

Solution Given $y = \frac{1}{(3x+1)(3x+4)(3x+7)}$

$$= \frac{1}{3\left[x + \frac{1}{3}\right] \times 3\left[x + \frac{4}{3}\right] \times 3\left[x + \frac{7}{3}\right]}$$

$$= \frac{1}{27 \left(x + \frac{1}{3}\right) \left(x + \frac{4}{3}\right) \left(x + \frac{7}{3}\right)}$$

$$= \frac{1}{27 \left(x - \frac{2}{3} + 1\right) \left(x - \frac{2}{3} + 2\right) \left(x - \frac{2}{3} + 3\right)}$$

$$= \frac{1}{27} \left[x - \frac{2}{3} \right]^{(-3)}$$

$$\therefore \Delta y = \frac{1}{27} \times (-3) \left[x - \frac{2}{3} \right]^{-4} = \frac{-3}{27} \left[x - \frac{2}{3} \right]^{-4}$$

$$= \frac{-1}{9} \left[x - \frac{2}{3} \right]^{(-4)}$$

$$= \frac{-1}{9} \times (-4) \left[x - \frac{2}{3} \right]^{(-5)}$$

$$= \frac{4}{9} \left[x - \frac{2}{3} \right]^{(-5)}$$

$$= \frac{4}{9} \times \frac{1}{\left(x - \frac{2}{3} + 1\right) \left(x - \frac{2}{3} + 2\right) \left(x - \frac{2}{3} + 3\right) \left(x - \frac{2}{3} + 4\right) \left(x - \frac{2}{3} + 5\right)}$$

$$= \frac{4}{9} \times \frac{1}{\frac{(x-2+3)}{3} \frac{(x-2+6)}{3} \frac{(x-2+9)}{3} \frac{(x-2+12)}{3} \frac{(x-2+15)}{3}}$$

$$= \frac{4}{9} \times \frac{3^2 \times 3^3}{(x+1)(x+4)(x+7)(x+10)(x+13)}$$

$$= \frac{4 \times 27}{(x+1)(x+4)(x+7)(x+10)(x+13)}$$

$$\Delta^2 y = \frac{108}{(x+1)(x+4)(x+7)(x+10)(x+13)}$$

Problem 24 Prove that $\Delta \left[\frac{1}{f(x)} \right] = \frac{-\Delta[f(x)]}{f(x)f(x+1)}$

Proof Let $h = 1$,

By defn. $\Delta[f(x)] = f(x+1) - f(x)$

$$\begin{aligned}\therefore \Delta \left[\frac{1}{f(x)} \right] &= \frac{1}{f(x+1)} - \frac{1}{f(x)} \\ &= \frac{f(x) - f(x+1)}{f(x+1)f(x)} \\ &= \left[\frac{f(x+1) - f(x)}{f(x+1)f(x)} \right] = \frac{-\Delta[f(x)]}{f(x)f(x+1)}\end{aligned}$$

Exercises

1) Construct the difference table for the following data

(i)

x	0	1	2	3	4
y	8	11	9	15	6

(ii)

x	45	50	55	60	65	70	75
y	80	95	120	100	85	70	60

2) Obtain

(a) the difference table for $y = x^3 - 3x^2 - 5x - 7$

at $x = -1, 0, 1, 2, 3, 4, 5$

Hence find (i) Δy_{-1} , (ii) Δy_0 (iii) $\Delta^2 y_{-1}$ (iv) $\Delta^3 y_3$ (v) $\Delta^4 y_{-1}$

(b) Using the same table find

(i) $\Delta^2 y_0$, (ii) $\Delta^2 y_2$ (iii) $\Delta^3 y_3$ (iv) $\Delta^4 y_4$ (v) $\Delta^5 y_5$

3) If $y_0 = 2$, $y_1 = 7$, $y_2 = 27$, $y_3 = 49$, $y_4 = 32$, $y_5 = 0$.

Find (i) $\Delta^5 y_0$ (ii) $\Delta^4 y_1$ (iii) $\Delta^3 y_2$

4) If $y_0 = 3$, $y_1 = 12$, $y_2 = 81$, $y_3 = 200$, $y_4 = 100$, $y_5 = 8$.

Find (i) $\Delta^5 y_0$ (ii) $\Delta^4 y_1$ (iii) $\Delta^3 y_2$

5) Evaluate the following taking the interval of differencing as h .

(i) $\Delta[xe^x]$ (ii) $\Delta[\sin x]$ (iii) $\Delta[\tan x]$ (iv) $\Delta^2[e^{ax+b}]$ (v) $\Delta^2[x^3]$

6) Prove that $\Delta^2 \left[\frac{5x+12}{x^2+5x+6} \right] = \frac{2(5x+6)}{(x+2)(x+3)(x+4)(x+5)}$

7) Find the first order difference of the following table, taking $h = 1$.

(i) $\frac{x+1}{(x-1)(x-2)}$ (ii) $\frac{x-4}{(2-x)(3-x)}$

8) Evaluate the following

(i) $\Delta^3 [(1-x)(1-2x)(1-3x)]$

(ii) $\Delta^3 [(1-ax)(1-bx)(1-cx)]$

(iii) $\Delta^4 [(1-2x)(1-3x)(1-5x)(1-6x)]$

9) If $y = \frac{1}{x(x+3)(x+6)}$ Prove that

$$\Delta^2 y = \frac{108}{x(x+3)(x+6)(x+9)(x+12)}$$

10) Express $y = 2x^3 - 3x^2 + 3x - 10$ in factorial notation and have show that $\Delta^3 y = 12$.

Problem 25 Find the missing value of y if the 1st differences are given below

y	2	-	-	-	-	-
Δy	3	5	7	9	9	13

Solution

Let the unknowns be a,b,c,d,e. Then we have the following table.

y	2	a	b	c	D	e
Δy	3	5	7	9	9	13

By defn. we get

$$a - 2 = 5$$

$$\text{ie, } a = 5 + 2 = 7$$

$$b - a = 7$$

$$\Rightarrow b = 7 + a = 7 + 7 = 14$$

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$$c - b = 9 \Rightarrow c = 9 + 14 = 23$$

$$d - c = 9 \Rightarrow d = 9 + 23 = 32$$

$$e - d = 13 \Rightarrow e = 13 + a = 13 + 32 = 35$$

Section 7.5

Other Difference Operators

Definition The shift operator E is defined by

$$E[f(x)] = f(x + h)$$

Hence

$$E^2[f(x)] = E[E(f(x))] = E[f(x + h)] = f(x + 2h) - E^3[f(x)] = f(x + 3h)$$

In general for any +ve integer

$$n \quad E^n[f(x)] = f(x + nh)$$

In particular $E[y_0] = y_1$

$$E^2[y_0] = E[E(y_0)] = E(y_1) = y_2$$

$$E^3[y_0] = E^2(y_1) = E(y_2) = y_3$$

In general $E^n[y_0] = y_n$

The inverse operator E^{-1} is defined as $E^{-1}[f(x)] = f(x - h)$

$\therefore$ For any real number 'n' we have

$$E^n[f(x)] = f(x + nh)$$

Note $E^m E^n[f(x)] = E^{(m+n)}[f(x)]$

Definition The Averaging operator μ is defined by

$$\mu[f(x)] = \frac{f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)}{2}$$

There are several relations connecting Δ, ∇, E & μ and the differentiation operator D .

Section 7.6

Relations between the operators Δ, ∇, E, μ and D .

Theorem 4 $E = 1 + \Delta$

Proof By definition $\Delta[f(x)] = f(x+h) - f(x)$

$$\& \quad E[f(x)] = f(x+h) \longrightarrow \textcircled{1}$$

$$\begin{aligned} \text{now } \Delta[f(x)] &= f(x+h) - f(x) \\ &= E[f(x)] - f(x) \quad (\text{using } \textcircled{1}) \\ &= (E-1)f(x) \end{aligned}$$

$$\Rightarrow \Delta = E - 1$$

$$(\text{or}) \quad \boxed{E = 1 + \Delta}$$

Theorem 5 $\nabla = 1 - E^{-1}$

Proof By definition $\nabla[f(x)] = f(x) - f(x-h)$

$$\& \quad E^{-1}[f(x)] = f(x-h) \longrightarrow \textcircled{1}$$

$$\begin{aligned} \text{now } \nabla[f(x)] &= f(x) - f(x-h) \\ &= f(x) - E^{-1}[f(x)] \\ &= f(x)[1 - E^{-1}] \end{aligned}$$

$$(\text{or}) \quad \nabla[f(x)] = (1 - E^{-1})f(x)$$

$$\text{Thus } \nabla[f(x)] = [1 - E^{-1}]f(x)$$

$$(\text{or}) \quad \boxed{\nabla = 1 - E^{-1}}$$

Theorem 6 $\delta = E^2 - E^{-2}$

Proof By definition $E^n[f(x)] = f(x+nh)$ for any real number n .

$$\text{Now } \delta[f(x)] = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

(by defn. of central differences)

$$= E^{\frac{1}{2}}[f(x)] - E^{-\frac{1}{2}}[f(x)]$$

$$= \left[E^{\frac{1}{2}} - E^{-\frac{1}{2}} \right] [f(x)]$$

Thus $\delta[f(x)] = \left[E^{\frac{1}{2}} - E^{-\frac{1}{2}} \right] f(x)$

$$\therefore \boxed{\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}}$$

Theorem 7 $\mu = \frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}$

Proof By defn. $E^n[f(x)] = f(x + nh)$ for any real number n .

Also by definition of Averaging operator, μ , we have

$$\mu[f(x)] = \frac{f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)}{2}$$

$$= \frac{1}{2} \left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right) \right]$$

$$= \left[E^{\frac{1}{2}} [f(x)] + E^{-\frac{1}{2}} [f(x)] \right]$$

$$= \left[\frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2} \right] [f(x)]$$

$$\Rightarrow \boxed{\mu = \frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}}$$

Theorem 8 $\delta = E^{\frac{1}{2}} \nabla$

Proof By definition $E^n[f(x)] = f(x + nh)$ for any real number ' n '.

Also $\nabla[f(x)] = f(x) - f(x - h)$

Now By theorem ⑥

$$\delta[f(x)] = E^2[f(x)] - E^{-2}[f(x)]$$

$$= E^2 \left[1 - E^{-2} \right] [f(x)]$$

$$= E^2 \nabla[f(x)] \quad (\text{using theorem 5})$$

$$= \left[E^2 \nabla \right] [f(x)]$$

$$\therefore \delta = E^2 \nabla$$

Theorem 9 $E = e^{hD}$

Proof The Taylor's general expansion of $y = f(x)$ is given by

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \dots + \frac{h^n}{n!} f^{(n)}(x) + \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(x) + \dots$$

$$\text{ie, } E[f(x)] = f(x) + \frac{h}{1!} D[f(x)] + \frac{h^2}{2!} D^2[f(x)] + \dots + \frac{h^n}{n!} D^n[f(x)] + \dots$$

(Here D is the diffraction operator u, $D[f(x)] = \frac{d}{dx}[f(x)] = f'(x)$)

$$= \left[1 + \frac{hD}{1!} + \frac{h^2 D^2}{2!} + \dots + \frac{h^n D^n}{n!} + \frac{h^{n+1} D^{n+1}}{(n+1)!} + \dots \right] f(x)$$

$$\text{Hence } E = 1 + \frac{hD}{1!} + \frac{h^2 D^2}{2!} + \dots + \frac{h^n D^n}{n!} + \dots = e^{hD}$$

(by def. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$ where $|x| < D$)

Thus $E = e^{hD}$

Space for Hints

Theorem 10 $D = \frac{1}{h} \left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots \right]$

proof By theorem ⑨,

$$E = e^{hD}$$

$$\therefore \log E = \log e^{hD} = h D \therefore \log e^x = x$$

ie, $h D = \log e$

$$= \log[1 + \Delta] \quad (\text{using theorem ④})$$

$$= \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots \quad \text{By formula } \log(1+x)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad \text{where } |x| < 1$$

$$\Rightarrow D = \frac{1}{h} \left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \dots \right]$$

Solved Problems

Problem 1 Prove that $E\nabla = \nabla E = \Delta$

Solution WKT $\nabla = 1 - E^{-1}$ (using theorem ⑤)

$$\therefore E\Delta = E[1 - E^{-1}] = E - EE^{-1} = E - 1 = \Delta \quad (\text{Using theorem ④})$$

①

$$\text{Also } \nabla E = (1 - E^{-1})E = (E - E^{-1}E) = E - 1 = \Delta \quad (\text{Using theorem ④})$$

②

From ① & ②, we can write.

$$\nabla E = E\nabla = \Delta$$

Problem 2 $\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}} \right] [1 + \Delta]^2 = 2 + \Delta$

Solution By theorem ①

$$E = 1 + \Delta \Rightarrow E^{\frac{1}{2}} = (1 + \Delta)^{\frac{1}{2}}$$

$$\therefore \left(E^{\frac{1}{2}} + E^{-\frac{1}{2}} \right) (1 + \Delta)^2 = \left(E^{\frac{1}{2}} + E^{-\frac{1}{2}} \right) E^2$$

$$\begin{aligned}
&= E^2 - \frac{1}{E^2} + E - \frac{1}{E^2} \\
&= E' + E^0 \quad (\because E^m(E^n) = E^{m+n}) \\
&= E + 1 \\
&= (1 + \Delta) + 1 \quad (\text{Using theorem ④}) \\
&= 2 + \Delta
\end{aligned}$$

Problem 3 $\nabla\Delta = \Delta - \nabla = \delta^2$

Solution By theorem ④ & ⑤

$$E = 1 + \Delta \Rightarrow \Delta = E - 1$$

$$\& \Delta = 1 - E^{-1}$$

Now $\nabla\Delta = (1 - E^{-1})(E - 1)$

$$\begin{aligned}
&= E - 1 - E^{-1}E - E^{-1} \\
&= E - 1 - 1E^{-1} - 2 \quad (\because E^m E^n = E^{m+n} E^{-1} E^1 = E^0 = 1) \\
&= E + E^{-1} - 2 \\
&= \left[E^2 - E^{-2} \right]^2 \\
&= \delta^2 \quad (\text{by theorem ⑥}) \\
&\quad \longrightarrow \text{①}
\end{aligned}$$

Also $\Delta - \nabla = (E - 1) - (1 - E^{-1})$

$$\begin{aligned}
&= E - 1 - 1 + AE^{-1} = E + E^{-1} - 2 \\
&= \left(E^2 - E^{-2} \right)^2 = \delta^2 \longrightarrow \text{②}
\end{aligned}$$

From ① & ②, we can write.

$$\nabla\Delta = \Delta - \nabla = \delta^2$$

Problem 4 Prove that $E^2 = \mu + \frac{\delta}{2}$

Solution By theorem ⑦, $\mu = \frac{E^2 + E^{-2}}{2} \longrightarrow \textcircled{1}$

and by theorem ⑥, $\delta = E^2 - E^{-2} \longrightarrow \textcircled{2}$

$$\therefore \mu + \frac{\delta}{2} = \left[\frac{E^2 + E^{-2}}{2} \right] + \left[\frac{E^2 - E^{-2}}{2} \right]$$

$$= \frac{1}{2} \left[2E^2 + \cancel{E^{-2}} - \cancel{E^{-2}} \right]$$

$$= E^2$$

Problem 5 Prove that $\mu\delta = \frac{\Delta}{2} + \frac{\Delta E^{-1}}{2}$

Solution RHS = $\frac{\Delta}{2} + \frac{\Delta E^{-1}}{2} = \frac{\Delta}{2} [1 + E^{-1}]$

$$= \frac{1}{2} (E - 1)(1 + E^{-1}) \quad (\because \Delta = E - 1)$$

$$= \frac{1}{2} [E + EE^{-1} - 1 - E^{-1}]$$

$$= \frac{1}{2} [E + 1 - 1 - E^{-1}] = \frac{E - E^{-1}}{2}$$

$$= \frac{1}{2} \left\{ \left(E^2 + E^{-2} \right) \left(E^2 - E^{-2} \right) \right\}$$

$$= \mu\delta \quad (\text{by defn of } \mu \text{ \& } \delta)$$

$$= \text{LHS.}$$

Problem 6 Prove that $1 - e^{-hD} = \nabla$

Solution WKT $D = \frac{\log E}{h}$ (by theorem ⑨) $E = e^{hD} \Rightarrow hD = \log E$

$$\Rightarrow D \frac{1}{h} \log E$$

$$\Rightarrow hD = \log E$$

$$\Rightarrow e^{hD} = E$$

$$\Rightarrow \frac{1}{e^{hD}} = \frac{1}{E}$$

$$\Rightarrow e^{-hD} = E^{-1} = 1 - \nabla$$

$$\Rightarrow \boxed{\nabla = 1 - e^{-hD}}$$

Problem 7 Prove that $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

Solution WKT $\Delta = E - 1$ and $\nabla = -E^{-1}$

$$\begin{aligned} \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} &= \frac{E-1}{(1-E^{-1})} - \frac{1-E^{-1}}{(E-1)} \\ &= \frac{(E-1)(E-1) - (1-E^{-1})(1-E^{-1})}{(1-E^{-1})(E-1)} \\ &= \frac{(E-1)^2 - (1-E^{-1})}{(E-1-E^{-1}E-E^{-1})} \\ &= \frac{(E-1-1+E^{-1})(E-1+1-E^{-1})}{[E-1-1+E^{-1}]} \\ &= E-1+1-E^{-1} = \Delta + \nabla \end{aligned}$$

Problem 8 Prove that $\delta = \Delta E^{-\frac{1}{2}}$ and hence prove that

$$E = \left(\frac{\Delta}{\delta} \right)^2$$

Solution $\left[\Delta E^{-\frac{1}{2}} \right] (f(x))$

$$= \Delta \left[E^{-\frac{1}{2}} f(x) \right]$$

$$= \Delta \left[f\left(x - \frac{h}{2}\right) \right]$$

$$= f\left(x - \frac{h}{2} + h\right) - f\left(x - \frac{h}{2}\right)$$

$$= f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

$$= \delta[f(x)]$$

$$\left[\Delta E^{-\frac{1}{2}} \right] f(x) = \delta[f(x)]$$

$$\Rightarrow \Delta E^{-\frac{1}{2}} = \delta$$

$$\Rightarrow E^{-\frac{1}{2}} = \frac{\delta}{\Delta}$$

$$\Rightarrow E^{\frac{1}{2}} = \frac{1}{E^{-\frac{1}{2}}} = \frac{1}{\delta/\Delta} = \frac{\Delta}{\delta}$$

$$\Rightarrow E = \left(\frac{\Delta}{\delta}\right)^2$$

$$\text{Thus } \boxed{\delta = \Delta E^{-\frac{1}{2}}} \text{ \& } \boxed{E = \left(\frac{\Delta}{\delta}\right)^2}$$

Problem 9 Prove that $hD = \log(1 + \Delta) = -\log(1 - \nabla) = \text{Sinh}^{-1}(\mu\delta)$

Solution By theorem ⑨ we have $E = e^{hD}$

$$\Rightarrow e^{hD} = 1 + \Delta$$

Taking logarithm on both sides we have

$$-hD = \log(1 + \Delta) \longrightarrow \text{①}$$

Also WKT $\nabla = 1 - E^{-1}$

$$\Rightarrow E^{-1} = 1 - \nabla$$

$$\theta(e^{hD})^{-1} = 1 - \nabla$$

$$\Rightarrow e^{-hD} = 1 - \nabla$$

Taking logarithm on both sides we get

$$-hD = \log(1 - \nabla)$$

$$\therefore hD = -\log(1 - \nabla) \longrightarrow \text{②}$$

By definition of Sine hyperbolic function we have $\text{Sin h}[hD] = \frac{e^{hD} - e^{-hD}}{2}$

$$= \frac{E - E^{-1}}{2}$$

$$= \frac{1}{2} \left[\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}} \right) \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}} \right) \right]$$

$$= \mu\delta$$

$$\therefore hD = \text{Sinh}^{-1}(\mu\delta) \longrightarrow \textcircled{3}$$

From ①, ② & ③ we have

$$hD = \log(1 + \Delta) = -\log(1 - \nabla) = \text{Sinh}^{-1}(\mu\delta)$$

Problem 10 Prove that $\frac{\delta^2}{2} + 8\sqrt{1 + \frac{\delta^2}{4}} = \Delta$

LHS =

Solution

$$\begin{aligned} \frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}} &= \frac{1}{2}\delta \left[\delta + 2\sqrt{1 + \frac{\delta^2}{4}} \right] \\ &= \frac{1}{2}\delta \left[\delta + 2\sqrt{\frac{4 + \delta^2}{4}} \right] \\ &= \frac{1}{2}\delta \left[\delta + 2\sqrt{\frac{4 + \delta^2}{4}} \right] \\ &= \frac{1}{2}\delta \left[\left(\frac{1}{E^2} - E^2 \right) + \sqrt{4 + \left(\frac{1}{E^2} - E^2 \right)^2} \right] \text{ (Using theorem ③)} \\ &= \frac{\delta}{2} \left[\left(\frac{1}{E^2} - E^2 \right) + \sqrt{4 + \left(\frac{1}{E^2} \right)^2 - 2 + \left(E^2 \right)^2} \right] \\ &\quad (a - b)^2 = a^2 - 2ab + b^2 \\ &= \frac{\delta}{2} \left[\left(\frac{1}{E^2} - E^2 \right) + \sqrt{\left(\frac{1}{E^2} + E^2 \right)^2} \right] \\ &= \frac{\delta}{2} \left[\left(\frac{1}{E^2} - E^2 \right) + \left(\frac{1}{E^2} + E^2 \right) \right] \\ &= \frac{\delta}{2} \left[2E^2 \right] = \delta E^2 \\ &= \left(\frac{1}{E^2} - E^2 \right) E^2 \text{ (Using theorem ⑥)} \end{aligned}$$

Space for Hints

Space for Hints

$$\begin{aligned}
 &= E - 1 \\
 &= \Delta \quad (\text{Using theorem (4)}) \\
 &= \text{RHS.}
 \end{aligned}$$

Problem 11 Prove that $\Delta^\gamma [f(x)] = \Delta^\gamma [f(x - \gamma)]$ for any +ve integer γ .

Solution By theorems (4) & (5),

$$\Delta = E - 1 \text{ and } \nabla = 1 - E^{-1}$$

The proof is by induction on γ .

$$\text{when } \gamma = 1, \text{ LHS} = \nabla[f(x)] = \nabla[f(x)]$$

$$= (1 - E^{-1})f(x)$$

$$= f(x) - E^{-1}[f(x)]$$

$$= f(x) - f(x - 1)$$

$$= \Delta[f(x - 1)] \quad (\text{by defn. of } \Delta f(x))$$

$$= \text{RHS}$$

$\therefore$ The result is true for $\gamma = 1$

Let us assume that the result is true for $\gamma = k$.

To prove that the result is true for $\gamma = k + 1$

Since the result is true for $\gamma = k$.

$$\therefore \nabla^k [f(x)] = \Delta^k [f(x - k)] \longrightarrow \textcircled{1}$$

$$\text{Now } \nabla^k [f(x)] = \Delta[\Delta^k [f(x - k)]]$$

$$= \Delta[\Delta^k [f(x - k)]] \quad (\text{Using } \textcircled{1})$$

$$= (1 - E^{-1})\Delta^k [f(x - k)] \quad (\text{Using theorem (5)})$$

$$= \Delta^k [f(x - k)] - E^{-1}[\Delta^k [f(x - k)]]$$

$$= \Delta^k [f(x - k)] - \Delta^k [f(x - (k + 1))]$$

$$= \Delta^k [f(x - k) - f(x - (k + 1))]$$

$$= \Delta^k [\Delta[f(x - (k + 1))]]$$

$$= \Delta^k [\Delta(f(x - (k + 1)))]$$

$$= \Delta^{k+1} [(f(x - (k + 1)))]$$

Hence the result is true for $\gamma = k + 1$

Hence it is true for any +ve integer γ

$\therefore \Delta^\gamma [f(x)] = \Delta^\gamma [f(x - \gamma)]$ for all +ve integer γ .

Problem 12 Taking $h = 1$, find $(\Delta + \nabla)^2 [f(x)]$

where $f(x) = x^2 + x$.

Solution WKT $\nabla = 1 - E^{-1}$ and $\Delta = E - 1$

$$\therefore (\Delta + \nabla)^2 [f(x)] = (\Delta + \nabla)^2 [x^2 + x]$$

$$= (E - 1 + 1 - E^{-1})^2 [x^2 + x]$$

$$= (E - E^{-1})^2 (x^2 + x)$$

$$= [E + E^{-2} - 2] (x^2 + x)$$

$$= E^2 (x^2 + x) + E^{-2} (x^2 + x) - 2(x^2 + x)$$

$$= E^2 (x^2) + E^2 (x) + E^{-2} (x^2) + E^{-2} (x)$$

$$= (x + 2)^2 + (x + 2) + [(x - 2)^2 + (x - 2)] - 2x^2 - 2x$$

$$(\because E^k [f(x)] = f(x + kh) \therefore h = 2)$$

$$E^{-k} [f(x)] = f(x - kh) \therefore h = 2$$

$$= x^2 + 4x + 4 + x + x + x^2 - 4x + 4 + x - 2 - 2x^2 - 2x = 8$$

Thus $(\Delta + \nabla)^2 (x^2 + x) = 8$ by taking $h = 2$.

Problem 13 Prove that $y_4 = y_3 + \Delta y_2 + \Delta^2 y_1 + \Delta^3 y_1$

Solution RHS = $y_3 + \Delta y_2 + \Delta^2 y_1 + \Delta^3 y_1$

$$= y_3 + (E - 1)y_2 + (E - 1)^2 y_1 + (E - 1)^3 y_1$$

(Using theorem ④)

$$= y_3 + Ey_2 - y_2 + [E^2 + 2E + 1]y_1$$

$$+ (E^3 - 3E^2 + 3E - 1)y_1$$

$$= y_3 + y_3 - y_2 + E^2 y_1 - 2Ey_1 + E^3 y_1 + y_1$$

$$- 3E^2 y_1 + 3Ey_1 - y_1$$

$$= 2y_3 - y_2 + y_3 - 2y_2 + y_4 - 3y_3 + 3y_2 - y_1$$

Space for Hints

$$= 3y_3 - 3y_2 - 3y_3 + y_4 + 3y_2 - y_1 + -y_1 = y_4$$

Problem 14 Prove that $\Delta^2 y_2 = \nabla^2 y_4$

Solution $\Delta^2 y_2 = (E - 1)^2 y_2$

$$= (E^2 - 2E + 1)y_2 = E^2 y_2 - 2E y_2 + y_2$$

$$= y_4 - 2y_3 + y_2 \longrightarrow \textcircled{1}$$

$$\nabla^2 y_4 = (1 - E^{-1})^2 y_4$$

$$= [1 - 2E^{-1} + E^{-2}]y_4$$

$$= y_4 - 2E^{-1}(y_4) + E^{-2}(y_4)$$

$$= y_4 - 2y_3 + y_2 \longrightarrow \textcircled{2}$$

From $\textcircled{1}$ & $\textcircled{2}$ we can write $\Delta^2 y_2 = \nabla^2 y_4$

Problem 15 Given $u_0 = 2, u_1 = 11, u_2 = 80, u_3 = 200, u_4 = 200$

$$u_5 = 8. \text{ Find } \nabla^5 u_5$$

(i) without constructing the difference table.

(ii) by constructing the difference table.

Solution WKT $\nabla = 1 - E^{-1}$

$$\therefore \nabla^5 u_5 = (1 - E^{-1})^5 u_5$$

$$= [1 - 5E^{-1} + 10E^{-2} - 10E^{-3} + 5E^{-4} - E^{-5}](u_5)$$

$$= u_5 - 5E^{-1}(u_5) + 10E^{-2}(u_5) - 10E^{-3}(u_5)$$

$$+ 5E^{-4}(u_5) - E^{-5}(u_5)$$

$$= u_5 - 5u_4 + 10u_3 - 10u_2 + 5u_1 - u_0$$

$$= 8 - 500 + 2000 - 800 + 55 - 2$$

$$= 761.$$

(ii) We have $\nabla^n [u(x)] = \Delta^n u(x - n)$ (by problem (11))

$$\text{Hence } \nabla^5 [u_5] = \Delta^5 [u_6]$$

ie, $\nabla^5 u_5 = \Delta^5 u_0$

we construct the forward difference table is as follows.

x	u	Δu	$\Delta^2 u$	$\Delta^3 u$	$\Delta^4 u$	$\Delta^5 u$
0	2	9				
1	11	69	60			
2	80	+120	51	-9	-262	
3	200	-100	-220	-271	499	761
4	100	-92	8	228		
5	8					

$\therefore \nabla^5 u_5 = \Delta^5 u_0 = 761$

Problem 16 If $u_0 = 1, u_1 = 5, u_2 = 8, u_3 = 3, u_4 = 1, u_5 = 7, u_6 = 0$
find $\Delta^5 u_0$

Solution $\Delta^5 u_0 = (E - 1)^5 u_0$

$$\begin{aligned}
 &= (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)u_0 \\
 &= E^5(u_0) - 5E^4(u_0) + 10E^3(u_0) - 10E^2(u_0) + 5E(u_0) - u_0 \\
 &= u_5 - 5u_4 + 10u_3 - 10u_2 + 5u_1 - u_0 \\
 &= -35 + 30 - 80 + 25 - 1 = -61
 \end{aligned}$$

problem 17 Estimate the missing term in the following table.

x	0	1	2	3	4
$u(x)$	1	3	9	-	81

Explain why the resulting value difference from 3^3

Solution Let the missing form in $u(x)$ be a constants $\Delta^4 u_0 = 0$ (since 4 values are given) $\therefore (E - 1)^4 u_0 = 0$

Space for Hints

Space for Hints

$$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1)u_0 = 0$$

$$\Rightarrow E^4 u_0 - 4E^3 u_0 + 6E^2 u_0 - 4E u_0 + u_0 = 0$$

$$\Rightarrow u_4 - 4u_3 + 6u_2 - 4u_1 + u_0 = 0$$

$$\therefore 81 - 4a + 54 - 12 + 1 = 0$$

$$\therefore 124 - 4a = 0 \Rightarrow 4a = 124$$

$$\Rightarrow a = \frac{124}{4} = 31$$

We understand from the data that $u(x)$ satisfies the relation $u(x) = 3^x$.

While estimating for $u(3)$ the basic assumption is that $u(x)$ is a polynomial of degree 3. But 3^x is not a polynomial but an exponential function. Hence the assumption is violated in this case and so we are not getting $u(x) = 3^x = 27$.

Problem 18 Give an estimate of the population in 1971 from the following data from the table

year	1941	1951	1961	1971	1981	1991
Population in Lakhs	363	391	421	?	467	501

Solution Let the population in 1971 be 'a'.

Let $u_0 = 363$; $u_1 = 391$; $u_2 = 421$; $u_3 = a$;

$u_4 = 467$ and $u_5 = 501$.

Since only five values are given $\therefore \Delta^5 u_0 = 0$

$$\therefore (E - 1)^5 u_0 = 0 \quad (\because \Delta = 1 - E)$$

$$\Rightarrow (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)u_0 = 0$$

$$\Rightarrow E^5(u_0) - 5E^4(u_0) + 10E^3(u_0) - 10E^2(u_0) + 5E(u_0) - u_0 = 0$$

$$\Rightarrow u_5 - 5u_4 + 10u_3 - 10u_2 + 5u_1 - u_0 = 0$$

$$\Rightarrow 501 - 2335 + 10 \times a - 10 \times 421 + 5 \times 391 - 363 = 0$$

$$\Rightarrow 10a - 4452 = 0$$

$$\therefore a = 445.2 \text{ lakhs.}$$

Hence the estimated population in 1971 is 445.2 lakhs.

Problem 19 Given that $u_0 + u_8 = 80$; $u_1 + u_7 = 10$;

$$u_2 + u_6 = 5; \quad u_3 + u_5 = 10. \text{ Find } u_4$$

Solution Since 4 values are given we have $\Delta^n[u(x)] = 0$ for all $n \geq 4$

$$\text{In particular } \Delta^8 u_0 = 0$$

$$\Rightarrow (E - 1)^8 u_0 = 0 \quad (\because E - 1 = \Delta)$$

$$\Rightarrow \{E^8 - 8E^7 + 28E^6 - 56E^5 + 70E^4 - 56E^3 + 28E^2 - 8E + 1\}u_0 = 0$$

$$\begin{aligned} \Rightarrow E^8(u_0) - 8E^7(u_0) + 28E^6(u_0) - 56E^5(u_0) \\ + 70E^4(u_0) - 56E^3(u_0) + 28E^2(u_0) \\ - 8E(u_0) + u_0 = 0. \end{aligned}$$

$$\begin{aligned} \Rightarrow u_8 - 8u_7 + 28u_6 - 56u_5 + 70u_4 - 56u_3 \\ + 28u_2 - 8u_1 + u_0 = 0 \end{aligned}$$

$$(\because E^\gamma(u_0) = u_0 + \gamma)$$

$$\gamma = 1, 2, 3, \dots, n)$$

$$\Rightarrow (u_0 + u_8) - 8(u_1 + u_7) + 28(u_2 + u_6) - 56(u_3 + u_5) + 70u_4 = 0$$

$$\Rightarrow 80 - 80 + 140 - 560 + 7044 = 0$$

$$\Rightarrow 7044 = 420$$

$$\Rightarrow 44 = \frac{420}{70} = 6$$

$$\therefore u_4 = 6.$$

Problem 20 Given that $y_0 + y_6 = 2.37$, $y_1 + y_5 = 2.292$,

$$y_2 + y_4 = 3.01$$

Find y_3

Solution Given $y_0 + y_6 = 2.37$, $\longrightarrow (1)$

$$y_1 + y_5 = 2.292, \longrightarrow (2)$$

Space for Hints

$$y_2 + y_4 = 3.01 \longrightarrow (3)$$

Since only 3 values are given

$$\therefore \Delta^n [y(x)] = 0 \text{ for all } n \geq 3$$

In particular $\Delta^6 y_0 = 0$

$$\Rightarrow (E - 1)^6 y_0 = 0$$

$$\Rightarrow [E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1]y_0 = 0$$

$$\Rightarrow E^6(y_0) - 6E^5(y_0) + 15E^4(y_0) - 20E^3(y_0)$$

$$+ 15E^2(y_0) - 6E(y_0) + y_0 = 0$$

$$\Rightarrow y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0 = 0$$

$$\Rightarrow (y_0 + y_6) - 6(y_1 + y_5) + 15(y_2 + y_4) - 20y_3 = 0$$

$$\Rightarrow (2.37) - 6 \times 2.292 + 15(3.01) - 20y_3 = 0$$

$$\Rightarrow 2.37 - 13.752 + 45.15 - 20y_3 = 0$$

$$\Rightarrow (2.37 + 45.15) - 13.752 = 20y_3 = 0$$

$$\Rightarrow 47.52 - 13.752 = 20y_3$$

$$\Rightarrow 33.768 = 20y_3$$

$$\Rightarrow y_3 = \frac{33.768}{20} = \frac{16.884}{10} = 1.6884$$

$\therefore$ The value of y_3 is 1.6884.

Problem 21 If $u_1 = (12 - x)(4 + x)$; $u_2 = (5 - x)(4 - x)$;

$$u_3 = (x + 18)(x + 16) \quad \text{and} \quad u_4 = 94.$$

obtain a value of x assuming second differences constants

Solution Since the 2nd order differences are assumed to be constants, the third order differences of one function u will all be zero.

$$\therefore \Delta^3 u_1 = 0$$

$$\Rightarrow (E - 1)^3 u_1 = 0 \quad (\because \Delta = E - 1)$$

$$\Rightarrow [E^3 - 3E^2 + 3E - 1]u_1 = 0$$

$$\Rightarrow E^3 u_1 - 3E^2 u_1 + 3E u_1 - u_1 - u_1 = 0$$

$$\Rightarrow u_4 - 3u_3 + 3u_2 - u_1 = 0$$

$$\Rightarrow 94 - 3(x+18)(x+6) + 3(5-x)(4-x) - (12-x)(4+x) = 0$$

$$\Rightarrow 94 - 3[x^2 + 24x + 108] + 3[20 - 9x + x^2]$$

$$\Rightarrow x^2[-3 + 3 + 12] + x[-72 - 27 - 96] + [94 - 324 + 60 - 576] = 0$$

$$\Rightarrow 12x^2 - 195x + [154 - 900] = 0$$

$$\Rightarrow 12x^2 - 195x - 756 = 0$$

$$\Rightarrow 4x^2 - 65x - 252 = 0$$

$$\Rightarrow x = \frac{65 \pm \sqrt{4225 + 4032}}{8}$$

$$= \frac{65 \pm \sqrt{8259}}{8}$$

$$= \frac{65 \pm 90.86}{8} = \frac{65 + 90.86}{8}, \frac{65 - 90.86}{8}$$

$$= 19.48 \quad \& \quad -3.23$$

$$\therefore x = -3.23 \quad (\text{or}) \quad x = 19.48$$

Problem 22 Prove that $\left(\frac{\Delta^2}{E}\right)x^3 = 6x$

Solution $\left(\frac{\Delta^2}{E}\right)x^3 = \Delta^2[E^{-1}(x^3)]$

$$= \Delta^2(x-1)^3$$

$$= \Delta^2[x^3 - 3x^2 + 3x - 1]$$

Let $y = x^3 - 3x^2 + 3x - 1$, we now express y as factorial polynomial

0	1	-3	3	-1
	0	0	0	0
1	1	-3	3	-1
	0	1	-2	
2	1	-2	1	
	0	2		
	1	0		

$$y(x) = x^{(3)} + 0x^{(2)} + x^{(1)} - 1$$

$$\therefore \Delta y = \Delta [x^{(3)} + x^{(1)} - 1]$$

$$= 3x^{(2)} - 1$$

$$\Delta^2 y = 2 \times 3x^{(1)} = 6x$$

$$\text{Hence } \left(\frac{\Delta^2}{E} \right) (x^3) = 6x$$

Problem 23 Find $\left(\frac{\Delta}{E} \right)^2 f(x)$

$$\text{Solution } \left(\frac{\Delta}{E} \right)^2 [f(x)] = (\Delta E^{-1})^2 [f(x)]$$

$$= ([E - 1]E^{-1}) [f(x)]$$

$$= [(1 - E^{-1})^2] [f(x)]$$

$$= [(1 - E^{-1})^{-1} (1 - E^{-1})] f(x)$$

$$= (1 - E^{-1}) [f(x) - f(x - h)]$$

$$= f(x) - f(x - h)$$

$$- f(x - h) + f(x - 2h)$$

$$= f(x) - 2f(x - h)$$

$$+ f(x - 2h)$$

Problem 24 Prove that (i) $\nabla^2 y_8 = y_8 - 2y_7 + y_6$

(ii) $\delta^2 y_5 = y_6 - 2y_5 + y_4$

Solution (i) $\nabla^2 [y_8] = \nabla [\nabla (y_8)]$

$$= \nabla (y_8 - y_7)$$

$$= \nabla y_8 - \nabla y_7 \quad (\because \nabla \text{ is linear})$$

$$= [y_8 - y_7] - [y_7 - y_6]$$

$$= y_8 - 2y_7 + y_6$$

$$(i) \delta^2 y_5 = \delta[\delta(y_8)]$$

$$= \delta \left[\begin{matrix} y_{11} & y_9 \\ 2 & 2 \end{matrix} \right]$$

$$= \delta y_{11} - \delta y_9 \quad (\because \delta \text{ is linear})$$

$$= \begin{matrix} y_6 & y_5 \\ 2 & 2 \end{matrix} - (y_5 - y_4)$$

$$= y_6 - 2y_5 + y_4$$

Problem 25 Explain the difference between $\left(\frac{\Delta^2}{E}\right) f(x)$ and $\frac{\Delta^2 f(x)}{E f(x)}$

and find the values of these when $f(x) = x^2$

Solution $\left[\frac{\Delta^2}{E}\right] f(x) = [(E-1)^2 E^{-1}] f(x)$

$$= [(E^2 - 2E + 1)E^{-1}] f(x)$$

$$= [E^{-1} - 2 + E^{-1}] f(x)$$

$$= E[f(x)] - 2f(x) + f(x-h) \longrightarrow \textcircled{1}$$

Also $\frac{\Delta^2[f(x)]}{f(x)} = \frac{(E-1)^2 f(x)}{E[f(x)]}$

$$= \frac{(E^2 - 2E + 1)f(x)}{f(x+h)}$$

$$= \frac{E^2[f(x)] - 2E[f(x)] + f(x)}{f(x+h)}$$

$$= \frac{f(x+2h) - 2f(x+h) + f(x)}{f(x+h)} \longrightarrow \textcircled{2}$$

From $\textcircled{1}$ & $\textcircled{2}$ we can write

$$\left(\frac{\Delta^2}{E}\right) f(x) \neq \frac{\Delta^2 f(x)}{E f(x)}$$

Space for Hints

Taking $f(x) = x^2$ in ① & ② we have

$$\begin{aligned} \left(\frac{\Delta^2}{E} \right) f(x) &= \left(\frac{\Delta^2}{E} \right) (x^2) = (x+h)^2 - 2x^2 + (x-h)^2 \\ &= x^2 + 2hx + h^2 - 2x^2 + x^2 - 2hx + h^2 \\ &= 2h^2 \end{aligned}$$

$$\begin{aligned} \frac{\Delta^2(x^2)}{Ex^2} &= \frac{(x+2h)^2 - 2(x+h)^2 + x^2}{(x+h)^2} \\ &= \frac{x^2 + 4hx + 4h^2 - 2x^2 - 4hx - 2h^2 + x^2}{(x+h)^2} \\ &= \frac{2h^2}{(x+h)^2} \end{aligned}$$

Problem 26 Prove that $\left(\frac{\Delta^2}{E} \right) e^x \frac{E(e^x)}{\Delta^2(e^x)} = e^x$

$$\begin{aligned} \text{Solution } \left(\frac{\Delta^2}{E} \right) (e^x) \frac{E(e^x)}{\Delta^2(e^x)} &= (\Delta^2 E^{-1})(e^x) \frac{E(e^x)}{\Delta^2(e^x)} \\ &= \Delta^2 [E^{-1}(e^x)] \frac{E(e^x)}{\Delta^2(e^x)} \\ &= \Delta^2 [e^{(x-h)}] \frac{E(e^x)}{\Delta^2(e^x)} \\ &= \Delta^2 (e^{x-h} c) \frac{E(e^x)}{\Delta^2(e^x)} \\ &= \Delta^2 (e^x) e^{-h} \frac{E(e^x)}{\Delta^2(e^x)} \\ &= e^{-h} \times e^{x+h} = e^x \cdot e^{-h} \cdot e^h = e^x \end{aligned}$$

$$\text{Thus } \left(\frac{\Delta^2}{E} \right) e^x \frac{E(e^x)}{\Delta^2(e^x)} = e^x$$

Problem 27 Find $y_0 = 9, y_1 = 18, y_2 = 20, y_3 = 24$, given that the 3rd differences are constants

Solution Since the third differences are constants.

we have $\Delta^4 y_0 = \Delta^5 y_0 = \Delta^6 y_0 = 0$

To find y_6

Now $y_6 = E^6 y_0 = (1 + \Delta)^6 y_0$

$$\begin{aligned} &= [1 + 6c_1\Delta + 6c_2\Delta^2 + 6c_4\Delta^4 + 6c_5\Delta^5 + 6c_6\Delta^6] y_0 \\ &= [1 + 6(E-1) + 15(E-1)^2 + 20(E^3 - 3E^2 + 3E - 1)] y_0 \\ &= [1 - 24E + 15E^2 + 9 + 20E^3 - 60E^2 + 60E - 20] y_0 \\ &= [-10 + 36E - 45E^2 + 20E^3] y_0 \\ &= -10y_0 + 36Ey_0 - 45E^2 y_0 + 20y_3 \\ &= -10y_0 + 36y_1 - 45y_2 + 20y_3 \\ &= -10(9) + 36(18) - 45(20) + 20(24) \\ &= -10 + 648 - 900 + 480 = 138. \end{aligned}$$

Problem 28 Prove that $\Delta^n e^x = e^{x+h} - nc_1 e^{x+n-1} + nc_2 e^{x+n-2} + (-1)^n e^x$

Solution WKT $\Delta = (E - 1)$

$$\therefore \Delta^n e^x = (E - 1)^n e^x$$

$$\begin{aligned} &= [E^n - nc_1 E^{n-1} + nc_2 E^{n-2} \dots - 1(-1)^n - 1] e^x \\ &= E^n(e^x) - nc_1 E^{n-1}(e^x) + nc_2 E^{n-2}(e^x) \dots + (-1)^n e^x \end{aligned}$$

$$\Delta^n [e^x] = e^{x+n} - nc_1 e^{x+n-1} + nc_2 e^{x+n-2} \dots (-1)^n e^x$$

(by taking $h = 1$)

Problem 29 Show that $\frac{\Delta^2 x^2}{E(x + \log x)} = \frac{2}{(x+1) + \log(x+h)}$

Solution Take $h = 1$.

By defn. $E[f(x)] = f(x+h) = f(x+1)$

$$\therefore E[x + \log x] = E(x) + E[\log x]$$

Space for Hints

Space for Hints

$$= (x+1) + \log(x+1)$$

$$\Delta^2 x^2 = (E-1)^2 (x^2)$$

$$= (E^2 - 2E + 1)(x^2)$$

$$= E^2(x^2) - 2E(x^2) + x^2$$

$$= (x+2)^2 - 2(x+1)^2 + x^2$$

$$= [x^2 + 4x + 4] - 2x^2 - 4x - 2 + x^2$$

$$= x^2 + 4x + 4 - 2x^2 - 4x - 2 + x^2$$

$$= 2x^2 - 2x^2 + 4 - 2$$

$$= 2$$

$$\therefore \frac{\Delta^2 [x^2]}{E[x + \log x]} = \frac{2}{(x+1) + \log(x+1)}$$

Problem 30 Prove that

$$(i) \Delta^2 = (1 + \Delta)\delta^2$$

$$(ii) \delta = \Delta(1 + \Delta)^{-\frac{1}{2}} = \nabla[1 - \nabla]^{-\frac{1}{2}}$$

$$(iii) \Delta^n = \delta^n E^2$$

$$(iv) \nabla = 1 - (1 + \Delta)^{-1}$$

Solution (i) WKT $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$ & $\Delta = E - 1$

$$RHS = (1 + \Delta)\delta^2$$

$$= (1 + E - 1) \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}} \right)^2$$

$$= E \{ E - 2 + E^{-1} \}$$

$$= \{ E^2 - 2E + 1 \}$$

$$= (E - 1)^2$$

$$= \Delta^2 = LHS$$

$$(ii) \Delta(1+\Delta)^{-\frac{1}{2}} = \Delta \left(E^{-\frac{1}{2}} \right) (\because E = 1 + \Delta)$$

$$= (E-1) \left(E^{-\frac{1}{2}} \right)$$

$$= E^{\frac{1}{2}} - E^{-\frac{1}{2}} = \delta \rightarrow \textcircled{1}$$

$$\text{Also } \Delta(1+\Delta)^{-\frac{1}{2}} = \text{WKT } \nabla = 1 - E^{-1}$$

$$= \Delta(E^{-1})^{\frac{1}{2}} = \nabla E^{\frac{1}{2}}$$

$$= \left(1 - E^{-\frac{1}{2}} \right) E^{\frac{1}{2}}$$

$$= E^{\frac{1}{2}} - E^{-\frac{1}{2}} = \delta \rightarrow \textcircled{2}$$

$$\text{From } \textcircled{1} \text{ \& } \textcircled{2}, \text{ we using } \delta = \Delta(1+\Delta)^{-\frac{1}{2}} = \nabla(1-\nabla)^{-\frac{1}{2}}$$

$$(iii) \text{WKT } \delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

$$\Rightarrow \delta^n = E \left[E^{\frac{1}{2}} - E^{-\frac{1}{2}} \right]^n$$

$$E^2 \delta^n = E^2 \left\{ E^{\frac{1}{2}} - E^{-\frac{1}{2}} \right\}^n$$

$$= \left\{ E^{\frac{1}{2}} \left[E^{\frac{1}{2}} - E^{-\frac{1}{2}} \right] \right\}^n$$

$$= \left\{ E^{\frac{1}{2}} E^{\frac{1}{2}} - E^{\frac{1}{2}} E^{-\frac{1}{2}} \right\}^n$$

$$= \{E-1\}^n = \Delta^n \quad (\because \Delta = (E-1))$$

$$\text{Thus } \delta^n E^2 = \Delta^n \rightarrow \textcircled{1}$$

Put $n = 2$ in $\textcircled{1}$ we get

Space for Hints

$$\Delta^2 = \delta^2 E^2 \quad (1^{\text{st}} \text{ subdues on})$$

(iv) WKT $E = 1 + \Delta$

now RHS = $1 - [1 + \Delta]^{-1}$

$$= 1 - E^{-1}$$

$$= \nabla \quad (\text{by formula } \nabla = 1 - E^{-1})$$

$$= \text{LHS}$$

Problem 31 Estimate the missing term from the following.

x	1	2	3	4	5
y	2	5	7	-	32

Solution

Let the missing value be a .

The forward difference table for one given data is as shown below.

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	2				
2	5	3			
3	7	2	-1		
4	a	$a-7$	$a-9$	$a-8$	
5	32	$32-a$	$39-2a$	$48-3a$	$56-4a$

Since only 4 values are given $\Delta^4 y_0 = 0$

$$\Rightarrow 56 - 4a = 0$$

$$\Rightarrow 56 - 4a$$

$$\Rightarrow a = 14$$

The missing value is 14.

Problem 32 Estimate the missing term from the following

Year	1994	1995	1996	1997	1998	1999	2000
Population	200	220	260	-	350	390	430

Solution Let the population in 1997 be 'a'.

Let $u_0 = 200, u_1 = 220, u_2 = 260, u_3 = a, u_4 = 350, u_5 = 390, u_6 = 430$

Since 6 values are given $\Delta^6 u_0 = 0$

$$\therefore (E-1)^6 u_0 = 0$$

$$\therefore [E^6 - 6C_1 E^5 + 6C_2 E^4 - 6C_3 E^3 + 6C_4 E^2 - 6C_5 E + 6C_6 E^0] u_0 = 0$$

$$\Rightarrow E^6(u_0) - 6E^5 u_0 + 15E^4 u_0 - 20E^3 u_0 + 15E^2 u_0 - 6E u_0 + u_0 = 0.$$

$$\Rightarrow u_6 - 6u_5 + 15u_4 - 20u_3 + 15u_2 - 6u_1 + u_0 = 0$$

$$\therefore 430 - 6 \times 390 + 15 \times 350 - 20a + 15 \times 260 - 6 \times 220 + 200 = 0$$

$$\Rightarrow 430 - 2340 + 5250 - 20a + 3900 - 1320 + 200 = 0$$

$$\Rightarrow 6300 - 20a = 0$$

$$\Rightarrow 20a = 6300$$

$$\Rightarrow a = \frac{6300}{20} = 315$$

$\therefore$ The missing terms $a = 315$

Exercise Problems

1] Taking $h = 1$, prove that

$$(i) \nabla^2 y_6 = y_6 - 2y_5 + y_4 \quad (ii) \nabla^4 y_6 = y_6 - 4y_5 + 6y_4 - 4y_3 + y_2$$

2] Prove the following

$$(i) \delta^2 y_7 = y_8 - 2y_7 + y_6 \quad (ii) \delta^2 y_4 = y_5 - 2y_4 + y_3$$

$$(iii) \delta^2 y_4 = \frac{y_{11}}{2} - 3\frac{y_9}{2} + 3\frac{y_7}{2} - \frac{y_5}{2}$$

$$(iv) \delta^{12} y_4 = y_6 - 4y_5 + 6y_4 - 4y_3 + y_2$$

$$3] \text{ Prove that } \mu = \frac{2\Delta}{2\sqrt{1+\Delta}}$$

Space for Hints

Space for Hints

4] Estimate the missing term from the following

(i)

x	1	2	3	4	5
y	7	-	13	21	37

(ii)

x	1	2	3	4	5	6	7
y	2	4	8	-	32	64	128

(iii)

x	0	5	10	15	20	25
y	6	10	-	17	-	31

5] Given $y_6 = 2$, $y_7 = 6$, $y_8 = 8$, $y_9 = 9$, and $y_{10} = 17$

Calculate $\Delta^4 y_6$

6] Given that $y_0 + y_8 = 1.924$, $y_1 + y_7 = 1.959$

$y_2 + y_6 = 1.982$, $y_3 + y_5 = 1.996$ find y_4 .

UNIT : 8

NUMERICAL METHODS

Section 8.0 Introduction

Space for Hints

Interpolation is the process of finding the most appropriate estimate for missing data. It is the “art of reading between the lines of a table”. For making the most probable estimate it requires the following assumptions.

(i) The frequency distribution is normal and not marked by sudden ups and downs.

(ii) The changes in the series are uniform within a period.

Interpolation technique is used in various disciplines like economics, business, population studies, price determination etc. It is used to fill in the gaps in the statistical data for the sake of continuity of information. For example, if we know the records for the past five years except the third year which is not available due to unforeseen conditions the interpolation technique helps to estimate the record for that year too under the assumption that the changes in the records over these five years have been uniform.

It is also possible that we may require information for future in which case the process of estimating the most appropriate value is known as extrapolation. Given a set of tabular values of a function $y = f(x)$ where the explicit nature of the function is not known, then $f(x)$ is replaced by a simpler function $\phi(x)$ such that $f(x)$ and $\phi(x)$ agree with the set of tabulated points. Any other value may be calculated from $\phi(x)$. This function $\phi(x)$ is known as an interpolating

function. In particular if $\phi(x)$ is a polynomial then the process is called polynomial interpolation and $\phi(x)$ is called an interpolating polynomial. The existence of an interpolating polynomial is supported by weierstrass approximation theorem which asserts that any continuous function on a closed interval can be approximated by a polynomial. In this chapter we introduce various interpolation polynomials using the concepts of forward difference, backward difference and central difference.

Section 8.1

Newton's Interpolation Formula

Theorem 8.1.

(Newton's Forward interpolation formula for equal intervals).

Let the function $y = f(x)$ take the values $y_0, y_1, \dots, y_n$ at the points $x_0, x_1, \dots, x_n$ where $x_i = x_0 + ih$. Then Newton's forward interpolation polynomial is given by

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots + \frac{p(p-1)\dots(p-\bar{n}-1)}{n!}\Delta^n y$$

where $x = x_0 + ph$.

Proof

Let $\phi(x)$ be an interpolating polynomial of degree n which represents $y = f(x)$ in $x_0 \leq x \leq x_0 + nh$.

Then $\phi(x)$ can be written in the form

$$\begin{aligned} \phi(x) = & a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_0 - h) \\ & + a_3(x - x_0)(x - x_0 - h)(x - x_0 - 2h) + \dots + \\ & a_n(x - x_0)(x - x_0 - h)\dots(x - x_0 - (n-1)h) \rightarrow \textcircled{1} \end{aligned}$$

By definition of interpolating polynomial we have $y_0 = \phi(x_0)$.

Putting $x = x_0$ in $\textcircled{1}$, we get $a_0 = y_0$.

Putting $x = x_0 + h$ in ①, we get $a_0 = y_0$

$$a_1 = \frac{1}{h}(y_1 - a_0)$$

$$= \frac{1}{h}(y_1 - y_0)$$

$$\therefore a_1 = \frac{\Delta y_0}{h}$$

Putting $x = x_0 + 2h$ in ①, we get $a_0 = y_0$

$$a_2 = \frac{1}{2h^2}(y_2 - a_0 - 2ha_1)$$

$$= \frac{1}{2h^2}(y_2 - y_0 - 2\Delta y_0)$$

$$= \frac{1}{2h^2}(y_2 - 2y_1 + y_0)$$

$$a_2 = \frac{\Delta^2 y_0}{2!h^2}$$

Similarly substituting $x = x_0 + 3h, \dots, x_0 + n-1h$ we get

$$a_3 = \frac{\Delta^3 y_0}{3!h^3}, \dots, a_n = \frac{\Delta^n y_0}{n!h^n}$$

substituting these values in ① we get

$$\phi(x) = y_0 + (x - x_0) \frac{\Delta y_0}{1!h} + (x - x_0)(x - x_0 - h) \frac{\Delta^2 y_0}{2!h^2} + \dots +$$

$$(x - x_0)(x - x_0 - h) \dots (x - x_0 - n-1h) \frac{\Delta^n y_0}{n!h^n}$$

This gives the Newton's forward interpolation polynomial

Since $\phi(x)$ is the interpolating polynomial which represents

$y = f(x)$ then function $\phi(x)$ can be written as y .

$$\therefore y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots +$$

$$\frac{p(p-1)\dots(p-n+1)}{n!}\Delta^n y_0 \text{ where } x = x_0 + ph.$$

$$\text{ie, } p = \frac{x - x_0}{h}$$

Space for Hints

This formula is known as Newton's forward interpolation formula (or) Gregory – Newton's formula for forward interpolation.

Theorem 8.2 (Newton's backward interpolation formula).

Let the function $y = f(x)$ take the values $y_0, y_1, \dots, y_n$ at the points $x_0, x_1, \dots, x_n$ where $x_i = x_0 + ih$. Then Newton's backward interpolation polynomial is given by

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \dots + \frac{p(p+1)\dots(p+n-1)}{n!} \nabla^n y_n$$

where $x = x_n + ph$.

Proof

Let $\phi(x)$ be an interpolating polynomial of degree n which represents $y = f(x)$ in $x_0 \leq x \leq x_0 + nh$.

Then $\phi(x)$ can be written in the form

$$\phi(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \dots + a_n(x - x_n)(x - x_{n-1})\dots(x - x_1).$$

As in the proof of theorem 8.1 we get

$$a_n = \frac{\nabla^n y_n}{n!h^n} a ; n = 0, 1, 2, \dots, n.$$

$$\text{Thus } y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \dots + \frac{p(p-1)\dots(p+n-1)}{n!} \nabla^n y_n.$$

$$\text{where } p = \frac{x - x_n}{h}.$$

This is known as Newton's formula for backward interpolation.

Remark

Newton's forward interpolation formula is used to interpolate the values of y near the beginning of the set of tabulated values or for extrapolating values of y to the left of the beginning. Newton's backward

interpolation formula is used to interpolate the values of y near the end of the set of tabulated values or for extrapolating values of y to the right of the last tabulated value y_n .

Solved Problems

Problem 1

— If $y(75) = 246$, $y(80) = 202$, $y(85) = 118$, $y(90) = 40$, find $y(79)$.

Solution Hence $x_0 = 75$, $h = 5$ and we have to find the value of y at $x = 79$.

Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \dots + \frac{p(p-1)\dots(p-(n-1))}{n!}\Delta^n y_0$$

where $p = \frac{x - x_0}{h}$

$$\therefore p = \frac{79 - 75}{5} = 0.8.$$

we now form the forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
75	246			
		-44		
80	202		-40	
		-84		46
85	118		6	
		-78		
90	40			

using the values of Δy_0 , $\Delta^2 y_0$, $\Delta^3 y_0$ in ① we get,

$$y_{0.8} = 246 + 0.8(-44) + \frac{(0.8)(0.8-1)}{2!}(-40) + \frac{(0.8)(0.8-1)(0.8-2)}{3!}(46)$$

$$= 215.472$$

Problem 2 Find a cubic polynomial which takes the following Values

x	0	1	2	3
$f(x)$	1	2	1	10

Solution Let us form the difference table first

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
0	1	1		
1	2	-1	-2	
2	1	9	10	12
3	10			

By Newton's - Gregory formula,

$$f(x) = f(x_0) + (x - x_0) \frac{\Delta f(x_0)}{1!h} + (x - x_0)(x - x_0 - h) \frac{\Delta^2 f(x_0)}{2!h^2} + \dots$$

here $x_0 = 0$ and $h = 1$.

$$\begin{aligned} \therefore f(x) &= 1 + (x - 0) \frac{1}{1!} + (x - 0)(x - 1) \frac{(-2)}{2!} + (x - 0)(x - 1)(x - 2) \frac{12}{3!} \\ &= 1 + x - x(x - 1) + 2x(x - 1)(x - 2) \\ &= 1 + x - x^2 + x + 2x^3 - 6x^2 + 4x \\ &= 2x^3 - 7x^2 + 6x + 1. \end{aligned}$$

$$\therefore f(x) = 2x^3 - 7x^2 + 6x + 1$$

Prob 3 A function $y = f(x)$ is given by the following table.

Find $f(0.2)$ by a suitable formula.

x	0	1	2	3	4	5	6
$y = f(x)$	176	185	194	203	212	220	229

Solution Since the value $x=0.2$ is near the beginning of the table we use Newton's forward interpolation formula.

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots + \frac{p(p-1)\dots p(p-(n-1))}{n!}\Delta^n y_0 \quad \text{①}$$

where $p = \frac{x - x_0}{h}$.

Here $x_0 = 0$, $h = 1$ and we want to find the value of $f(x)$ at $x = 0.2$

$$\therefore p = \frac{0.2 - 0}{1} = 0.2$$

Now we form the forward difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
0	176	9					
1	185	9	0				
2	194	9	0	0			
3	203	9	0	0	-1		
4	212	8	-1	-1	3	4	5
5	220	9	1	2			
6	229						

$\therefore$ ① becomes,

$$y_{0.2} = 176 + 0.2 \times 9 + \frac{(0.2)(0.2-1)(0.2-3)(0.2-4)}{5!}(-1) + \frac{(0.2)(0.2-1)(0.2-2)(0.2-3)(0.2-4)(0.2-5)}{6!}(5)$$

$$= 177.67232.$$

Hence $f(0.2) = 177.67$.

Problem 4 Construct Newton's forward interpolation polynomial for the following data

x	4	6	8	10
y	1	3	8	16

use it to find the value of y for $x = 5$.

Solution Here $x_0 = 4$ and $h = 2$. The Newton's forward interpolation formula is

$$y = y_0 + \frac{(x - x_0)}{1!h} \Delta y_0 + (x - x_0)(x - x_0 - h) \frac{\Delta^2 y_0}{2!h^2} + \dots \rightarrow \textcircled{1}$$

we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

$$\begin{aligned} \therefore y &= 1 + \frac{(x-4) \times 2}{1!2} + \frac{(x-4)(x-6) \times 3}{2!2^2} + \frac{(x-4)(x-6)(x-8) \times 0}{3!2^3} \\ &= 1 + (x-4) + \frac{3(x-4)(x-6)}{8} \end{aligned}$$

$\therefore y = 1 + (x-4) + \frac{3}{8}(x^2 - 10x + 24)$ is the required interpolating

polynomial, when $x = 5$, $y_5 = 1 + 1 + \frac{3}{8}(5^2 - 50 + 24) = 1.625$.

Problem 5 Find the value of y from the following data at $x = 2.65$

Space for Hints

x	-1	0	1	2	3
y	-21	6	15	12	3

Solution Since the value of $x = 2.65$ is near the end of the table we use Newton's backward interpolation formula.

The formula is $y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \dots$

where $p = \frac{x - x_n}{h}$

Here $x = 2.65$, $x_n = 3$ and $h = 1$.

$$\therefore p = \frac{2.65 - 3}{1} = -0.35$$

$$\therefore p = -0.35$$

To find ∇y_n , $\nabla^2 y_n$ etc we form the backward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-1	-21				
		27			
0	6		-18		
		9		6	
1	15		-12		0
		-3		6	
2	12		-6		
		-9			
3	3				

$$\begin{aligned} \therefore y_{-0.35} &= 3 + (-0.35)(-9) + \frac{(-0.35)(-0.35+1)}{2!}(-6) \\ &\quad + \frac{(-0.35)(-0.35+1)(-0.35+2)}{3!} \times 6 \\ &= 6.4571. \end{aligned}$$

Space for Hints

Problem 6 The following data gives the melting point of an alloy of zinc and lead, θ is the temperature and x is the percentage of lead. Using Newton's interpolation formula find (i) θ when $x = 48$
(ii) θ when $x = 84$

x	40	50	60	70	80	90
θ	184	204	226	250	276	304

Solution (i) Since $x = 48$ is near the beginning of the table we use Newton's forward interpolation formula.

The formula can be written as

$$\theta_p = \theta_0 + p\Delta\theta_0 + \frac{p(p-1)}{2!}\Delta^2\theta_0 + \dots$$

$$\text{where } p = \frac{x - x_0}{h} = \frac{48 - 40}{10} = 0.8$$

To find $\Delta\theta_0$, $\Delta^2\theta_0$ etc we form forward difference table.

x	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$	$\Delta^5\theta$
40	184	20				
50	204	22	2			
60	226	24	2	0		
70	250	26	2	0	0	
80	276	28	2	0	0	0
90	304					

$$\begin{aligned}\theta_p &= 184 + 0.8 \times 20 + \frac{(0.8)(0.8-1)}{2!} \times 2 \\ &= 199.84 \approx 200.\end{aligned}$$

$$\therefore \theta_p = 199.84 \approx 200.$$

(ii) Since $x = 84$ is nearer to the end of the table we use Newton's backward interpolation formula

$$\theta_p = \theta_n + p\nabla\theta_n + \frac{p(p+1)}{2!}\nabla^2\theta_n + \dots$$

$$\text{where } p = \frac{x - x_n}{h} = \frac{84 - 90}{10} = -0.6$$

The values of $\theta_n, \nabla\theta_n, \nabla^2\theta_n$ etc. Can be obtained from the forward difference table by sloping backwards with respect to the increasing direction of x .

$$\therefore \theta_n = 304, \nabla\theta_n = 28, \nabla^2\theta_n = 2, \nabla^3\theta_n = \nabla^4\theta_n = \nabla^5\theta_n = 0.$$

Hence we have

$$\theta - 0.6 = 304 + (-0.6) \times 28 + \frac{(-0.6)(-0.6-1)}{2!}(2)$$

$$= 286.96 \approx 287.$$

$$\therefore \theta - 0.6 = 286.96 \approx 287.$$

Problem 7 From the data given below, find the number of student whose weight is between 60 and 70.

weight	0-40	40-60	60-80	80-100	100-120
No.of students	250	120	100	70	50

Solution The less than cumulative frequency table of the given data is as shown below.

weight less than x	40	60	80	100	120
No.of students	250	370	470	540	590

Space for Hints

we now form the difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
40	250				
60	370	120	-20	-10	
80	470	100	-30	10	20
100	540	70	-20		
120	590	50			

Number of students whose weight is between 60 and 70 is got from

$y_{70} - y_{60}.$

we have $y_{60} = 370.$

Now we shall find y_{70} by Newton's forward interpolation formula.

$$y_p = y_0 + p \nabla y_0 + \frac{p(p-1)}{2!} \nabla^2 y_0 + \dots + \frac{p(p-1) \dots (p-(n-1))}{n!} \nabla^n y_0$$

where $p = \frac{x - x_0}{h}.$

Here $x_0 = 40, h = 20$ and $x = 70.$

Hence $p = \frac{70 - 40}{20} = 1.5$

Now,

$$y_{70} = y_{1.5} = 250 + 1.5 \times 120 + \frac{(1.5)(0.5)}{2!} (-10) + \frac{(1.5)(0.5)(-0.5)(-1.5)}{4!} (20)$$

$$= 423.59375$$

$$y_{70} = 424 \text{ (approximately).}$$

$\therefore$ Number of persons whose weight is between 60 and 70 is $424 - 370 = 54.$

Exercise Problems

(1) For the following data find $f(9)$ using Newton's forward interpolation formula

x	8	10	12	14	16
$f(x)$	1000	1900	3250	5400	8950

(2) Find $f(2.5)$ using Newton's forward difference formula for the given data

x	1	2	3	4	5	6
y	0	1	8	27	64	125

(3) From the following table find the value of $\tan 45^\circ 15'$

x°	45	46	47	48	49	50
$\tan x^\circ$	1.00000	1.03553	1.07237	1.11061	1.15037	1.19175

(4) From the table given below find $\sin 52^\circ$ by using Newton's forward interpolation formula.

x	45	50	55	60
$\sin x$	0.7071	0.7660	0.8192	0.8660

(5) The following data is taken from the steam table. Find the pressure at temperature $t = 142^\circ$ and $t = 175^\circ$

Temperature $^\circ\text{C}$	140	150	160	170	180
Pressure kg/cm^2	3.685	4.854	6.302	8.075	10.225

(6) The following table gives the census population of a town for the years 1931-1991. Estimate the population (i) for the year 1965 (ii) for the year 1933 by using appropriate interpolation formula.

Year	1931	1941	1951	1961	1971
Population in lakhs	36	66	81	93	101

(7) Estimate the value of $f(22)$ and $f(42)$ from the following data

x	20	25	30	35	40	45
$f(x)$	354	332	291	260	231	204

Space for Hints

(8) Using Newton's forward interpolation formula find a polynomial of degree four which takes the values

(i)

x	2	4	6	8	10
y	0	0	1	0	0

(ii)

x	4	6	8	10
y	1	3	8	16

Hence find $f(5)$.

(9) Using Newton's backward formula find the polynomial of degree three passing through (3,6) (4,24) (5,60) and (6,120).

Hence find the ordinate when the abscissa is 5.5.

10) Given the following data, express y as a function of x .

Hence find the value of y at $x=0.5$.

x	0	1	2	3	4
y	3	6	11	18	27

Section 8.2

Central Difference Interpolation Formulae

Newton's forwards interpolation formula and Newton's backward interpolation formula are applicable for interpolating near the beginning and end of the tabulated values. However, they are not applicable to interpolate near the central value of the table because they contain forward and backward differences which are far away from the central line. To obtain more accurate estimation near the middle of a table we

introduce central difference interpolation formula such as

- (i) Gauss forward interpolation formula,
- (ii) Gauss backward interpolation formula
- (iii) Sterling's formula
- (iv) Bessel's formula

(v) Laplace Everett's formula

Consider the function $y = f(x)$ whose values at a collection of equally spaced points are given. Denote the middle point as x_0 so that the set of equally spaced points are given by

$$\dots, x_0 - 3h, x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h, x_0 + 3h, \dots$$

Let $f(x_0 + ih) = y_i$;

with this notation the table giving the values of the function takes the following form

x	...	$x_0 - 3h$	$x_0 - 2h$	$x_0 - h$	x_0	$x_0 + h$	$x_0 + 2h$	...
$f(x)$	...	y_{-3}	y_{-2}	y_{-1}	y_0	y_1	y_2	...

we now form a difference table in the case of seven values

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
$x_0 - 3h$	y_{-3}	Δy_{-3}	$\Delta^2 y_{-3}$				
$x_0 - 2h$	y_{-2}	Δy_{-2}		$\Delta^3 y_{-3}$	$\Delta^4 y_{-3}$		
$x_0 - h$	y_{-1}	Δy_{-1}	$\Delta^2 y_{-2}$	$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0	Δy_0	$\Delta^2 y_{-1}$	$\Delta^3 y_{-1}$	$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
$x_0 + h$	y_1	Δy_1	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_{-1}$	$\Delta^5 y_{-2}$	
$x_0 + 2h$	y_2	Δy_2	$\Delta^2 y_1$				
$x_0 + 3h$	y_3						

Space for Hints

The above table can also be written in terms of central differences using the operator δ as follows

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$	$\delta^6 y$
$x_0 - 3h$	y_{-3}	$\delta y - \frac{5}{2}$					
$x_0 - 2h$	y_{-2}	$\delta y - \frac{3}{2}$	$\delta^2 y - 2$				
$x_0 - h$	y_{-1}		$\delta^2 y - 1$	$\delta^3 y - \frac{3}{2}$	$\delta^4 y - 1$		
x_0	y_0	$\delta y - \frac{1}{2}$		$\delta^3 y - \frac{1}{2}$		$\delta^5 y - \frac{1}{2}$	
			$\delta^2 y_0$	$\delta^3 y \frac{1}{2}$	$\delta^4 y_0$		$\delta^6 y_0$
$x_0 + h$	y_1	$\delta y \frac{1}{2}$	$\delta^2 y_1$		$\delta^4 y_1$	$\delta^5 y \frac{1}{2}$	
				$\delta^3 y \frac{3}{2}$			
$x_0 + 2h$	y_2	$\delta y \frac{3}{2}$	$\delta^2 y_2$				
$x_0 + 3h$	y_3	$\delta y \frac{5}{2}$					

The entries in this table can be related to the entries of the previous table by using the operator relation $\delta = \Delta E^{-1/2}$. For example,

$$\delta^3 y_{1/2} = \left(\Delta E^{-1/2} \right)^3 y_{1/2} = \Delta^3 E^{-3/2} y_{1/2} = \Delta^3 y_{-1}.$$

Hence the difference given in both the tables all the same in corresponding positions.

Example

Form the central difference table for the following data choosing $x = 35$ as origin.

x	20	25	30	35	40	45
y	12	15	20	27	38	52

x	$p = \frac{x-35}{5}$	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
20	-3	12					
25	-2	15	3				
30	-1	20	5	2	0		
35	0	27	7	2	3	3	
40	1	39	12	5	-4	-7	-10
45	2	52	13	1			

Here $\Delta y_{-3} = 3$, $\Delta^2 y_{-3} = 2$, $\Delta^3 y_{-3} = 0$, $\Delta^4 y_{-3} = 3$, $\Delta^5 y_{-3} = -10$
 $\Delta y_{-2} = 5$, $\Delta^2 y_{-2} = 2$, $\Delta^3 y_{-2} = 3$, $\Delta^4 y_{-2} = -7$,
 $\Delta y_{-1} = 7$, $\Delta^2 y_{-1} = 5$, $\Delta^3 y_{-1} = -4$,
 $\Delta y_0 = 12$, $\Delta^2 y_1 = 1$,
 $\Delta y_1 = 13$,

Section 8.2.1

(i) Gauss Forward interpolation formula

we know that Newton's forward interpolation formula is given by,

$$y_p = y_0 + p\nabla y_0 + \frac{p(p-1)}{2!}\nabla^2 y_0 + \frac{p(p-1)(p-2)}{3!}\nabla^3 y_0 + \dots + \frac{p(p-1)\dots(p-(n-1))}{n!}\nabla^n y_0$$

①

where $x = x_0 + ph$.

We have $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

Hence $\Delta^2 y_0 - \Delta^2 y_{-1} + \Delta^3 y_{-1}$

Similarly $\Delta^3 y_0 - \Delta^3 y_{-1} + \Delta^4 y_{-1}$, etc.

Also, $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2} \therefore \Delta^3 y_{-1} - \Delta^3 y_{-2} + \Delta^4 y_{-2}$

Space for Hints

Similarly $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^3 y_{-2}$, etc.

Substituting these values in ① we get,

$$\begin{aligned} y_p &= y_0 + p \nabla y_0 + \frac{p(p-1)}{2!} (\nabla^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!} \\ &\quad (\nabla^3 y_{-1} + \Delta^4 y_{-1}) + \frac{p(p-1)(p-2)(p-3)}{4!} (\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \dots \\ &= y_0 + p \Delta y_0 + \binom{p}{2} \Delta^2 y_{-1} + \left[\binom{p}{2} + \binom{p}{3} \right] \Delta^3 y_{-1} + \left[\binom{p}{3} + \binom{p}{4} \right] \Delta^4 y_{-1} \\ &\quad + \left[\binom{p}{4} + \binom{p}{5} \right] \Delta^5 y_{-1} + \dots \\ &= y_0 + \binom{p}{1} \Delta y_0 + \binom{p}{2} \Delta^2 y_{-1} + \binom{p+1}{3} \Delta^3 y_{-1} + \binom{p+1}{4} [\Delta^4 y_{-2} + \Delta^5 y_{-2}] \\ &\quad + \binom{p+1}{5} [\Delta^5 y_{-2} + \Delta^6 y_{-2}] + \dots \\ &= y_0 + \binom{p}{1} \Delta y_0 + \binom{p}{2} \Delta^2 y_{-1} + \binom{p+1}{3} \Delta^3 y_{-1} + \binom{p+1}{4} \Delta^4 y_{-2} \\ &\quad + \binom{p+1}{4} \binom{p+1}{5} \Delta^5 y_{-2} + \dots \\ \therefore y_p &= y_0 + \binom{p}{1} \Delta y_0 + \binom{p}{2} \Delta^2 y_{-1} + \binom{p+1}{3} \Delta^3 y_{-1} \\ &\quad + \binom{p+1}{4} \Delta^4 y_{-2} + \binom{p+1}{5} \Delta^5 y_{-2} + \dots \end{aligned}$$

The above formula is known as Gauss – Forward interpolation formula.

Remark 1

Gauss forward interpolation formula involves odd differences below the central line and even differences on the central line

Central Line :
$$\begin{array}{ccccccc} \dots y_0 & \dots & \Delta^2 y_{-1} & \dots & \Delta^4 y_{-2} & \dots \\ & \swarrow & & \swarrow & & \swarrow \\ & \Delta y_0 & \dots & \Delta^3 y_{-1} & \dots & \Delta^5 y_{-2} & \dots \end{array}$$

Remark 2

This formula is used to interpolate the values of y for $0 < p < 1$.

Section 8.2.2**(ii) Gauss backward interpolation formula**

The Newton's forward interpolation formula is

$$y_p = y_0 + p \nabla y_0 + \frac{p(p-1)}{2!} \nabla^2 y_0 + \frac{p(p-1)(p-2)}{3!} \nabla^3 y_0 + \dots$$

①

where $p = \frac{x - x_0}{h}$.

We have $\Delta y_0 - \Delta y_{-1} = \Delta^2 y_{-1}$

ie, $\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$

Similarly $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2} \text{ etc.}$$

Substituting these values in ①, we get

$$\begin{aligned} y_p = y_0 + p[\Delta y_{-1} + \Delta^2 y_{-1}] + \frac{p(p-1)}{2!}[\Delta^2 y_{-1} + \Delta^3 y_{-1}] + \frac{p(p-1)(p-2)}{3!} \\ [\Delta^3 y_{-1} + \Delta^4 y_{-1}] + \frac{p(p-1)(p-2)(p-3)}{4!}[\Delta^4 y_{-1} + \Delta^5 y_{-1}] + \dots \end{aligned}$$

Simplifying we get,

$$y_p = y_0 + \binom{p}{1} \Delta y_{-1} + \binom{p+1}{2} \Delta^2 y_{-1} + \binom{p+1}{3} \Delta^3 y_{-2} + \binom{p+1}{4} \Delta^4 y_{-2} + \dots$$

The above formula is known as Gauss backward interpolation formula.

Remark 1

Gauss backward interpolation formula contains odd difference above the central line and even difference on the central line.

Central Line :

$$\begin{array}{ccccccc} \dots \Delta y_{-1} & \dots & \Delta^3 y_{-2} & \dots & \Delta^5 y_{-3} & \dots \\ \nearrow & \searrow & \nearrow & \searrow & \nearrow & \\ \dots y_0 & \dots & \Delta^2 y_{-1} & \dots & \Delta^4 y_{-2} & \dots \end{array}$$

Remark 2

This is used to interpolate the values of y for values of p lying between -1 and 0.

Section 8.2.3

(iii) Stirling's Formula

Gauss forward interpolation formula is

$$y_p = y_0 + p \nabla y_0 + \frac{p(p-1)}{2!} \nabla^2 y_{-1} + \frac{p(p-1)(p-2)}{3!} \nabla^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!} \Delta^4 y_{-2} + \dots$$

①

Gauss backward interpolation formula is

$$y_p = y_0 + p \nabla y_{-1} + \frac{(p+1)p}{2!} \nabla^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \nabla^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!} \Delta^4 y_{-2} + \dots$$

②

Taking the mean of ① and ② we get,

$$y_p = y_0 + p \left[\frac{\Delta y_0 + \Delta y_{-1}}{2} \right] + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2-1)}{3!} \left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right] + \frac{p^2(p^2-1)}{4!} \Delta^4 y_{-2} + \dots$$

This is called Stirling's formula.

Space for Hints

Remark 1

Stirling's formula is the average of Gauss forward and Gauss backward interpolation formula.

Remark 2

This formula involves the average of the odd differences just above and just below the central line and even differences on the central line.

$$\text{Central line } y_0 \left[\begin{array}{c} \Delta y_{-1} \\ \Delta y_0 \end{array} \right] \Delta^2 y_{-1} \left[\begin{array}{c} \Delta^3 y_{-2} \\ \Delta^3 y_{-1} \end{array} \right] \Delta^4 y_{-2}$$

$$\text{Co-efficients } 1 \quad p \quad \frac{p^2}{2} \quad \frac{p(p^2 - 1^2)}{3!} \quad \frac{p^2(p^2 - 1^2)}{4!}$$

Averages should be taken for the values within the square brackets.

Remark 3

We use this formula for the values of p lying between $-\frac{1}{2}$ and $\frac{1}{2}$. Good estimate will be obtained for $-\frac{1}{4} < p < \frac{1}{4}$.

Section 8.2.4

(iv) Bessel's Formula

Gauss forward interpolation formula is

$$y_p = y_0 + p \nabla y_0 + \frac{p(p-1)}{2!} \nabla^2 y_{-1} + \frac{p(p-1)p(p-2)}{3!} \nabla^3 y_{-1} \\ + \frac{(p+1)p(p-1)(p-2)}{4!} \Delta^4 y_{-2} + \dots$$

①

we know that $\Delta y_0 = y_1 - y_0$

$$\therefore y_0 = y_1 - \Delta y_0 \longrightarrow \text{②}$$

$$y_{-1} = y_0 - \Delta y_{-1}$$

$$\therefore \Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1} \longrightarrow \text{③}$$

Similarly, $\Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^5 y_{-2}$ etc.

$\therefore$ ① can be written as

Space for Hints

$$\begin{aligned}
 y_p &= \left(\frac{y_0}{2} + \frac{y_1}{2} \right) + p\Delta y_0 + \left[\frac{1}{2} \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{1}{2} \frac{p(p-1)}{2!} \Delta^2 y_0 \right] \\
 &\quad + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-1} + \dots \\
 &= \frac{y_0}{2} + \frac{1}{2} (y_1 - \Delta y_0) + p\Delta y_0 + \frac{1}{2} \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{1}{2} \frac{p(p-1)}{2!} \Delta^2 y_0 \\
 &\quad + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-1} + \dots \\
 &= \left(\frac{y_0 + y_1}{2} \right) + \left(p - \frac{1}{2} \right) \Delta y_0 + \frac{1}{2} \frac{p(p-1)}{2!} (\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{p(p-1)}{2!} \Delta^3 y_{-1} \\
 &\quad + \left(\frac{-1}{2} + \frac{p+1}{3} \right) \Delta^3 y_{-1} + \dots \\
 y_p &= \left(\frac{y_0 + y_1}{2} \right) + \left(p - \frac{1}{2} \right) \Delta y_0 + \frac{p(p-1)}{2!} \left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} \right) + \\
 &\quad \frac{\left(p - \frac{1}{2} \right) p(p-1)}{3!} \Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!} \left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} \right) + \dots
 \end{aligned}$$

This is called Bessel's formula.

Remark 1

Bessel's formula involves odd differences below the central line and averages of the even differences on and below the central line.

$$\text{Central line} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \Delta y_0 \begin{bmatrix} \Delta^2 y_{-1} \\ \Delta^2 y_0 \end{bmatrix} \Delta^3 y_{-1} \begin{bmatrix} \Delta^4 y_{-2} \\ \Delta^4 y_{-1} \end{bmatrix} \dots$$

Remark 2

If $p = \frac{1}{2}$ the coefficients of all the odd order differences are zero. In this case Bessel's formula becomes

$$y_{\frac{1}{2}} = \left(\frac{y_0 + y_1}{2} \right) - \frac{1}{8} \left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} \right) + \frac{3}{128} \left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} \right) \dots$$

and this formula enables us for interpolating to halves.

Hence it can be used to compute the values of the function midway

between any two given values.

Space for Hints

Section 8.2.5

(v) Laplace – Everett's formula

Gauss forward interpolation formula is

$$y_p = y_0 + \binom{p}{1} \Delta y_0 + \binom{p}{2} \Delta^2 y_{-1} + \binom{p+1}{3} \Delta^3 y_{-1} + \binom{p+1}{4} \Delta^4 y_{-2} + \binom{p+1}{5} \Delta^5 y_{-2} + \dots$$

①

we have $\Delta y_0 = y_1 - y_0$

$$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$$

$$\Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2} \text{ etc.}$$

Substituting these odd differences in ① we get

$$\begin{aligned} y_p &= y_0 + \binom{p}{1} (y_1 - y_0) + \binom{p}{2} \Delta^2 y_{-1} + \binom{p+1}{3} (\Delta^2 y_0 - \Delta^2 y_{-1}) + \binom{p+1}{4} \Delta^4 y_{-2} \\ &\quad + \binom{p+2}{5} (\Delta^4 y_{-1} - \Delta^4 y_{-2}) + \dots \\ &= (1-p)y_0 + py_1 + \left[\binom{p}{2} - \binom{p+1}{3} \right] \Delta^2 y_{-1} + \binom{p+1}{3} \Delta^2 y_0 + \\ &\quad \left[\binom{p+1}{4} - \binom{p+2}{5} \right] \Delta^4 y_{-2} + \binom{p+2}{5} \Delta^4 y_{-1} + \dots \\ &= (1-p)y_0 + py_1 + \frac{p(p-1)(p-2)}{3!} \Delta^2 y_{-1} + \frac{(p+1)(p-1)}{3!} \Delta^2 y_0 \\ &\quad - \frac{(p+1)p(p-1)(p-2)(p-3)}{5!} \Delta^4 y_{-2} \\ &\quad + \frac{(p+2)(p+1)p(p-2)}{5!} \Delta^4 y_{-1} + \dots \end{aligned}$$

To change the terms with negative signs, put $p = 1 - q$.

$\therefore$ we have

$$y_p = qy_0 + \frac{q(q^2 - 1^2)}{3!} \Delta^2 y_{-1} + \frac{q(q^2 - 1^2)}{5!} \Delta^4 y_{-2} + \dots$$

$$+ py_1 + \frac{p(p^2 - 1^2)}{3!} \Delta^2 y_0 + \frac{p(p^2 - 1^2)(p^2 - 2^2)}{5!} \Delta^4 y_{-1} + \dots$$

This is known as Laplace – Everett's formula.

Remark 1

Laplace – Everett's formula involves only even differences on and below the central line,

$$\begin{array}{ccccccc} \text{Central line} & \dots & \dots & y_0 & \dots & \dots & \Delta^2 y_{-1} & \dots & \dots & \Delta^4 y_{-2} & \dots \\ & & & y_0 & & & \Delta^2 y_0 & & & \Delta^4 y_{-1} & \end{array}$$

Remark 2

There is a close relationship between Bessel's formula and Everett's formula and one can be deduced from the other by suitable rearrangements.

Remark 3

Bessel's formula truncated after third differences is equivalent to Everett's formula truncated after second differences.

Remark 4

Choice of an interpolation formula.

In the proceeding sections we have desired several interpolation formulae for calculating y_p from a set of equispaced values.

The following is a natural question which formula gives the most accurate result in a particular problem?

The following rules will be found useful.

1. If interpolation is desired near the beginning of the table we use Newton's forward interpolation formula since higher order central differences do not exist at the beginning of the table.
2. If interpolation is desired near the end of the table we use Newton's backward interpolation formula.

3. Gauss's forward interpolation formula gives best result for $0 < p < 1$ and Gauss's backward formula gives best result for $-1 < p < 0$.

4. To find an interpolated value near the middle of the table

sterling's formula gives most accurate result for $-\frac{1}{4} \leq p \leq \frac{1}{4}$ and

Bessel's formula and Everett's formula give the most accurate

result $\frac{1}{4} \leq p \leq \frac{3}{4}$.

Solved Problems

Problem 1

Apply Gauss forward interpolation formula to find $y(25)$ for the following data.

x	20	24	28	32
y	2854	3162	3544	3992

Solution

Here $h=4$

To find $y(25)$ take the origin at $x_0 = 24$.

$$\text{At } x = 25, p = \frac{x - x_0}{h} = \frac{25 - 24}{4} = 0.25$$

we apply Gauss forward formula

$$y_p = y_0 + \binom{p}{1} \Delta y_0 + \binom{p}{2} \Delta^2 y_{-1} + \binom{p+1}{3} \Delta^3 y_{-1} + \dots$$

Central difference table

x	$p = \frac{x - 24}{4}$	y	∇y	$\nabla^2 y$	$\nabla^3 y$
20	-1	2854	308		
24	0	3162	382	74	
28	1	3544	448	66	-8
32	2	3992			

Space for Hints

Space for Hints

$$\begin{aligned}y_{0.25} &= 3162 + 0.25 \times 382 + \frac{0.25 \times (0.25 - 1)}{2!} \times 74 \\&\quad + \frac{(0.25 + 1)(0.25)(0.25 - 1)}{3!} \times (-8) \\&= 3162 + 95.5 - 6.9375 + .3125 \\&= 3250.875 \\&\therefore y(25) \approx 3251.\end{aligned}$$

Problem 2

Using Gauss forward formula find $f(30)$ from the following data

x	21	25	29	33	37
$f(x)$	18.4708	17.8144	17.1070	16.3432	15.5154

Solution

Here $h = 4$. To find $f(30)$ we take 29 as origin.

$$\therefore p = \frac{x - x_0}{h} = \frac{30 - 29}{4} = 0.25$$

Gauss forward interpolation formula is

$$f_p = f(0) + \binom{p}{1} \Delta f(0) + \binom{p}{2} \Delta^2 f(-1) + \binom{p+1}{3} \Delta^3 f(-1) + \binom{p+1}{4} \Delta^4 f(-2) + \dots$$

The central difference table

x	$p = \frac{x - 29}{4}$	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
21	-2	18.4708				
25	-1	17.8144	-0.6564			
29	0	17.1070	-0.7074	-0.0510		
33	1	16.3432	-0.7638	-0.0564	-0.0054	
37	2	15.5154	-0.8272	-0.0640	-0.0076	-0.0022

$$\begin{aligned}
 f(0.25) &= 17.1070 + (0.25)(-0.7638) + \frac{(0.25)(-0.75)}{2}(-0.0564) \\
 &\quad + \frac{(1.25)(0.25)(-0.75)}{6}(-0.0076) \\
 &\quad + \frac{(1.25)(0.25)(-0.75)(-1.75)}{24}(-0.0022) \\
 &= 17.1070 - 0.19095 + 0.00529 + 0.003 - 0.00004 \\
 &= 16.9216
 \end{aligned}$$

$$\therefore f(30) \approx 16.9216$$

Problem 3

Using the following table, apply Gauss's forward formula to

$$f(3.75)$$

x	2.5	3.0	3.5	4.0	4.5	5.0
$f(x)$	24.145	22.043	20.225	18.644	17.262	16.047

Solution

Here $h = 0.5$. To find $f(3.75)$ we take 3.5 as origin.

Gauss forward interpolation formula is

$$f(p) = f(0) + p\Delta f(0) + \frac{p(p-1)}{2!}\Delta^2 f(-1) + \frac{(p+1)p(p-1)}{3!}\Delta^3 f(-1) + \dots$$

$$\text{where } p = \frac{x - x_0}{h} = \frac{3.75 - 3.5}{0.5} = 0.5$$

we now form the difference table

x	$p = \frac{x-3.5}{0.5}$	$y = f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
2.5	-2	24.145					
3.0	-1	22.043	-2.102				
3.5	0	20.225	-1.818	0.284			
4.0	1	18.644	-1.581	0.237	-0.047		
4.5	2	17.262	-1.382	0.199	-0.038	0.009	
5.0	3	16.047	-1.215	0.167	-0.032	0.006	-0.003

Space for Hints

$$\begin{aligned}\therefore f(0.5) &= 20.225 + 0.5(-1.581) + \frac{(0.5)(-0.5)}{2!}(0.237) \\ &\quad + \frac{(1.5)(0.5)(-0.5)}{3!}(-0.038) \\ &\quad + \frac{(1.5)(0.5)(-0.5)(-1.5)}{4!}(0.009) \\ &\quad + \frac{(2.5)(1.5)(0.5)(-0.5)(-1.5)}{5!}(-0.003) \\ &= 19.4074\end{aligned}$$

$$\therefore f(3.75) \approx 19.4074$$

Problem 4

Apply Gauss backward interpolation formula to find $y(25)$ for the following data.

x	20	24	28	32
y	2854	3162	3544	3992

Solution

Here $h = 0.4$. To find $y(25)$ take the origin at $x_0 = 28$.

$$\text{At } x = 25, \quad p = \frac{x - x_0}{h} = \frac{25 - 28}{4} = -0.75$$

Gauss backward interpolation formula is

$$y_p = y_0 + \binom{p}{1} \Delta y_{-1} + \binom{p+1}{2} \Delta^2 y_{-1} + \binom{p+1}{3} \Delta^3 y_{-2} + \dots$$

The Central difference table is

x	$p = \frac{x - 28}{4}$	y	∇y	$\nabla^2 y$	$\nabla^3 y$
20	-2	2854			
			308		
24	-1	3162		74	
			382		-8
28	0	3544		66	
			448		
32	1	3992			

$$\begin{aligned}
 y_{-0.75} &= 3544 + (-0.75) \times 382 + \frac{(-0.75+1)(-0.75)}{2} \times 66 \\
 &\quad + \frac{-(0.75+1)(-0.75)(0.75-1)}{6} \times -8 \\
 &= 3544 - 286.5 - 6.1875 - 0.0625 \\
 &= 3251.25
 \end{aligned}$$

$$\therefore y(25) = 3251.$$

Problem 5

Using Gauss's backward interpolation formula find $\sin 45^\circ$ from the following table.

x°	20	30	40	50	60	70
$\sin x^\circ$	0.342	0.502	0.642	0.766	0.866	0.939

Solution Gauss's backward interpolation formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-2} + \dots$$

where $p = \frac{x - x_0}{h}$.

To find the value of $\sin 45^\circ$ we choose 40 as the origin and the interval of differencing is 10.

$$\therefore p = \frac{45 - 40}{10} = 0.5$$

Central difference table is

x	p	$y = \sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
20	-2	0.342	0.16				
30	-1	0.502	0.14	-0.02			
40	0	0.642	0.124	-0.016	0.004	-0.012	
50	1	0.766	0.1	-0.024	-0.008	0.005	0.017
60	2	0.866	0.073	-0.027	-0.003		
70	3	0.939					

Space for Hints

$$\begin{aligned}
 y_p &= 0.642 + 0.5 \times 0.14 + \frac{(1.5)(0.5)}{2!}(-0.016) + \frac{(1.5)(0.5)(-0.5)}{3!}(0.004) \\
 &\quad + \frac{(2.5)(1.5)(0.5)(-0.5)}{4!}(-0.012) \\
 &= 0.642 + 0.07 - 0.006 - 0.00025 + 0.000469 \\
 &= 0.7062.
 \end{aligned}$$

Problem 6 The following table gives the specific heat of ethyl alcohol at different temperatures. Estimate the specific heat corresponding to 15°C and 45°C .

Temperature ($x^\circ\text{C}$)	0	10	20	30	40	50
Specific heat (y)	0.51	0.55	0.57	0.59	0.62	0.67

Solution Here $h = 10$.

(i) To find y_{15} we take the origin at $x_0 = 20$ and apply Gauss backward interpolation formula.

Here $p = \frac{x - x_0}{h} = \frac{15 - 20}{10} = -0.5$ we have to find $y - 0.5$.

The central difference table is

x	$p = \frac{x - 20}{10}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	-2	0.51					
10	-1	0.55	0.04				
20	0	0.57	0.02	-0.02			
30	1	0.59	0.02	0	0.02		
40	2	0.62	0.03	0.01	0.01	-0.01	
50	3	0.67	0.05	0.02	0.01	0	0.01

By Gauss backward formula

$$y_p = y_0 + \binom{p}{1} \Delta y_{-1} + \binom{p+1}{2} \Delta^2 y_{-1} + \binom{p+1}{3} \Delta^3 y_{-2} + \binom{p+2}{4} \Delta^4 y_{-2} + \binom{p+2}{5} \Delta^5 y_{-3} + \dots$$

At $p = -0.5$

$$\begin{aligned} y_{-0.5} &= 0.57 + \frac{(-0.5)(0.2)}{1} + \frac{(-0.5+1)(-0.5)}{2} \times 0 \\ &+ \frac{(-0.5+1)(-0.5)(-0.5-1)}{6} \times (0.02) + \frac{(-0.5+2)(-0.5+1)(-0.5)(-0.5-1)}{24} \\ &+ \frac{(-0.5+2)(-0.5+1)(-0.5)(-0.5-1)(-0.5-2)}{120} \times (0.01) \\ &= 0.47 + 0.00125 - 0.00023 - 0.00012 \\ &= 0.4709 \end{aligned}$$

$\therefore$ The specific heat corresponding to $15^\circ \text{C} \approx 0.4709$.

(ii) To find the value of y at $x = 45$ we use Newton's backward interpolation formula, since $x = 45$ is near the end of the table.

$$y_p = y_n + p \Delta y_n + \frac{p(p+1)}{2!} \Delta^2 y_n + \dots$$

$$\text{where } p = \frac{x - x_n}{h} = \frac{45 - 50}{10} = -0.5.$$

we now form the backward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	0.51	0.04				
10	0.55	0.02	-0.02			
20	0.57	0.02	0	0.02		
30	0.59	0.03	0.01	0.01	-0.01	
40	0.62	0.05	0.02	0.01	0	0.01
50	0.67					

Space for Hints

$$y_{-0.5} = 0.67 + (-0.5) \times 0.05 + \frac{(-0.5)(-0.5+1)}{2!}(0.02) \\ + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(0.01) \\ + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+4)}{5!}(0.01)$$

$\therefore$ The specific heat corresponding to 45°C is 0.6416.

Problem 7 Apply Stirling's formula to find $y(25)$ for the following data

x	20	24	28	32
y	2854	3162	3544	3992

Solution

Here $h = 4$. To find $y(25)$ take the origin at $x_0 = 24$.

At $x = 25$, $p = \frac{x - x_0}{h} = \frac{25 - 24}{4} = -0.25$

Stirling's formula is

$$y_p = y_0 + \left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2 - 1^2)}{3!} \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right) + \dots$$

The Central difference table is

x	$p = \frac{x - 24}{4}$	y	∇y	$\nabla^2 y$	$\nabla^3 y$
20	-1	2854	308		
24	0	3162		74	
28	1	3544	382	66	
32	2	3992	448		

$$\therefore y_{-0.25} = 3162 + 0.25 \left(\frac{882 + 308}{2} \right) + \frac{(0.25)^2}{2!} \times 74 \\ = 3250.5625 \\ y(25) \approx 3251.$$

Problem 8

Space for Hints

Using stirling's formula compute y_{35} given that $y_{10} = 600$; $y_{20} = 512$; $y_{30} = 439$; $y_{40} = 346$; $y_{50} = 243$

Solution Here $h=10$. To find y_{35} we take origin at $x_0 = 30$.

At $x = 35$, $p = \frac{x - x_0}{h} = \frac{35 - 30}{10} = 0.05$.

stirling formula is

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2 - 1^2)}{3!} \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2 - 1^2)}{4!} \Delta^4 y_{-2} + \dots$$

The central difference table

x	$p = \frac{x - 30}{10}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	-2	600				
20	-1	512	-88	15		
30	0	439	-20	-20	-35	45
40	1	346	-10	-10	1	
50	2	243				

By stirling's formula,

$$y_{0.5} = 439 + 0.5\left(\frac{-93 + (-73)}{2}\right) + \frac{0.5^2}{2!} (-20) + \frac{0.5(0.5^2 - 1^2)}{3!} \left(\frac{10 + (-35)}{2}\right) + \frac{(0.5)^2(5^2 - 1^2)}{4!} \times 45$$

$= 439 - 41.5 - 2.5 + 0.7813 - 0.3516$
 $= 395 - 4297$
 $\therefore y(35) \approx 395.$

Space for Hints

Problem 9 The following are the number of deaths in four successive ten year age groups. Calculate the no. of deaths at 45-50.

Age group	25-35	35-45	45-55	55-65
No. of deaths	13229	18139	24225	31496

Solution The frequency table can be written as follows, taking the mid values of class intervals as x and the corresponding frequency as y .

Age x	30	40	50	60
Death y	13229	18139	24225	31496

Here $h = 10$: we take the origin at $x_0 = 50$. The age in the required class interval 45-50 corresponds to 47.5.

$$\text{Here } p = \frac{x - x_0}{h} = \frac{47.5 - 50}{10} = \frac{-2.5}{10} = -0.25.$$

Since $p = -0.25$ we apply stirling formula for good accuracy of the result.

The stirling formula is

$$y_p = y_0 + p \left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2 - 1^2)}{3!} \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right) + \dots$$

The central difference table.

x	$p = \frac{x - 50}{10}$	y	∇y	$\nabla^2 y$	$\nabla^3 y$
30	-2	13229	4910		
40	-1	18139	6086	1176	9
50	-- 0 --	24225	- -	1185	--
60	1	31496	7271		

$$\therefore y_{-0.25} = 24225 + (-0.25) \left(\frac{727 + 6086}{2} \right) + \frac{(-0.25)^2}{2!} \times (1185) = 2259.406.$$

$\therefore$ The number of deaths in the age group 45-50 is approximately 22592.

Problem 10 Apply central difference formula to obtain $f(32)$

that $f(25) = 0.2707$; $f(30) = 0.3027$; $f(35) = 0.3386$; $f(40) = 0.3794$.

Solution Here $h = 5$. to find $f(32)$ take the origin at $x_0 = 30$.

$$\text{At } x = 32, \quad p = \frac{x - x_0}{h} = \frac{32 - 30}{5} = 0.4.$$

Here we can apply Gauss forward interpolation formula.

However, since $p = 0.4$ which lies in the range $(0.25, 0.75)$ we can use

Bessel formula to find $f(32) (= f_{0.4})$

Bessel formula is

$$f_p = \left(\frac{f_0 + f_1}{2} \right) + \left(p - \frac{1}{2} \right) \Delta f_0 + \frac{p(p-1)}{2!} \left(\frac{\Delta^2 f_{-1} + \Delta^2 f_0}{2} \right) + \frac{\left(p - \frac{1}{2} \right) p(p-1)}{3!} \Delta^3 f_{-1} + \dots$$

The central difference table

x	$p = \frac{x-30}{5}$	f	∇f	$\nabla^2 f$	$\nabla^3 f$
25	-1	0.2707	0.0320		
30	0	0.3027		0.0039	
35	1	0.3386	0.0359	0.0049	0.0010
40	2	0.3794	0.0408		

$$\begin{aligned} \therefore f_{0.4} &= \left(\frac{0.327 + 0.3386}{2} \right) + \left(0.4 - \frac{1}{2} \right) (0.0359) + \frac{0.4(0.4-1)}{2!} \\ &\quad \left(\frac{0.0039 + 0.0049}{2} \right) + \frac{\left(0.4 - \frac{1}{2} \right) (0.4)(0.4-1)}{3!} \times (0.001) \\ &= 0.3207 - 0.00359 - 0.000528 + 0.000004 \\ &= 0.316586 \\ \therefore f(32) &\approx 0.3166. \end{aligned}$$

Problem 11 Apply Laplace Everett's formula to find $y(25)$ for the following data.

x	20	24	28	32
y	2854	3162	3544	3992

Solution Here $h = 4$. To find $y(25)$ take the origin at $x_0 = 24$.

At $x = 25$, $p = \frac{x - x_0}{h} = \frac{25 - 24}{4} = 0.25$

Laplace – Everett's formula is

$$y_p = \left[qy_0 + \frac{q(q^2 - 1^2)}{3!} \Delta^2 y_{-1} + \frac{q(q^2 - 1^2)}{5!} \Delta^4 y_{-2} + \dots \right] + \left[py_1 + \frac{p(p^2 - 1^2)}{3!} \Delta^2 y_{-1} + \frac{q(q^2 - 1^2)(q^{-2})}{5!} \Delta^4 y_{-1} + \dots \right]$$

where $q = 1 - p = 1 - 0.25 = 0.75$.

The difference table

x	$p = \frac{x - 24}{4}$	y	∇y	$\nabla^2 y$	$\nabla^3 y$
20	-1	2854	308		
24	0	3162	382	74	
28	1	3544	448	68	-8
32	2	3992			

$$\begin{aligned} y_{0.25} &= \left[0.75 \times 3162 + \frac{0.75(0.75^2 - 1^2)}{3!} \times 74 + \dots \right] \\ &\quad + \left[0.25 \times 3544 + \frac{0.25(0.25^2 - 1^2)}{3!} \times 66 + \dots \right] \\ &= [2371.5 - 4.0469] + [886 - 2.5781] \\ &= 3250.875. \\ \therefore y(25) &\approx 3251. \end{aligned}$$

Problem 12 Using Laplace – Everett's formula find $\log 337.5$ given that

x	310	320	330	340	350	360
$\log x$	2.491	2.505	2.518	2.531	2.544	2.556

Solution Here $h = 10$. To find $\log 337.5 = y(337.5)$ we take the origin at $x_0 = 330$.

$$\text{At } x = 337.5, \quad p = \frac{x - x_0}{h} = \frac{337.5 - 330}{10} = 0.75$$

Since $p = 0.75$ lies close to the range $(0.25, 0.75)$ we apply Laplace – Everett's formula to get good accuracy of the result.

Laplace – Everett's formula is

$$y_p = \left[qy_0 + \frac{q(q^2 - 1^2)}{3!} \Delta^2 y_{-1} + \frac{q(q^2 - 1^2)}{5!} \Delta^4 y_{-2} + \dots \right] \\ + \left[py_1 + \frac{p(p^2 - 1^2)}{3!} \Delta^2 y_0 + \frac{q(q^2 - 1^2)(q^{-2})}{5!} \Delta^4 y_{-1} + \dots \right]$$

where $q = 1 - p = 1 - 0.75 = 0.25$.

The central difference table is

x	$p = \frac{x - 330}{10}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
310	-2	2.491					
320	-1	2.505	0.019				
330	0	2.518	0.013	-0.001			
340	1	2.531	0.013	0	0.001		
350	2	2.544	0.013	0	0	-0.001	
360	3	2.556	0.012	-0.001	-0.001	-0.001	0

$$y_{0.75} = \left[0.25 \times 2.518 + \frac{0.25(0.25^2 - 1^2)}{3!} \times 0 + \frac{0.25(0.25^2 - 1)(0.25^2 - 2^2)}{5!} \times (-0.001) \right]$$

Space for Hints

$$\begin{aligned}
 &= \left[0.75 \times 2.531 + \frac{0.75(0.75^2 - 1^2)}{3!} \times 0 + \frac{0.75(0.75^2 - 1)(0.75^2 - 2^2)}{5!} \times (-0.001) \right] \\
 &= \left[0.6295 + \frac{0.25(-0.9375)(-3.9375)}{120} \times (-0.001) \right] \\
 &= \left[1.8983 + \frac{0.75(-0.4375)(-3.4375)}{120} \times (-0.001) \right] \\
 &= [0.6295 - 0.0000077] + [1.8983 - 0.000009] \\
 &= 2.5278
 \end{aligned}$$

Here $\log 337.5 \approx 2.5278$.

Exercises

- (1) Apply Gauss forward formula to obtain $f(x)$ at $x = 3.5$ from the table below

x	2	3	4	5
$f(x)$	2.626	3.454	4.784	6.986

- (2) Use Gauss forward formula to get use given that $u_{21} = 18.4708$,
 $u_{25} = 17.8144$, $u_{29} = 17.1070$, $u_{33} = 16.3432$, $u_{37} = 15.5154$.

- (3) Using Gauss forward formula find the population in 1935

Year	1930	1932	1934	1936	1938	1940
Population in crores	12	16	21	27	32	40

- (4) Find $\log_{10} 337.5$ using Gauss backward difference formula

x	310	320	330	340	350	360
$\log x$	2.4914	2.5052	2.5185	2.5315	2.5441	2.5563

- (5) Find the value of $\cos 51^\circ 42'$ by using Gauss backwards interpolation formula from the table given below

x	50°	51°	52°	53°	54°
$\cos x$	0.6428	0.6293	0.6157	0.6018	0.5878

(6) Find y_{35} using stirling's formula from the following data

x	20	30	40	50
y	512	439	346	243

(7) Use stirling formula to find $\tan 16^\circ$ given that

θ°	0	5	10	15	20	25	30
$\tan \theta^\circ$	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

(8) Estimate $\sqrt{1.12}$ using stirling's formula from the following table

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
$\sqrt{x}$	1.0000	1.0247	1.0488	1.0724	1.0954	1.1180	1.1402

(9) Apply Bessel's formula to find $y(25)$ given that

x	20	24	28	32
y	2854	3162	3544	3992

(10) Employ Bessel's formula to find the value of $f(1.95)$ given that

x	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$f(x)$	2.979	3.144	3.283	3.391	3.463	3.997	4.491

(11) Using Bessel's formula find $y(62.5)$ given

x	60	61	62	63	64	65
y	7782	7853	7924	7993	8062	8129

(12) Using Everett's formula find $f(80)$ given that

x	20	28	36	44
$f(x)$	2854	3162	7088	7984

(13) Using Everett's formula find $\log 23.75$ from the following data

x	21	22	23	24	25	26
$\log x$	1.3222	1.3424	1.3617	1.3802	1.3979	1.4150

Space for Hints

(14) Find y at $x = 34$ using Laplace Everett's formula for the following data

x	20	25	30	35	40
y	11.4699	12.7834	13.7648	14.4982	15.0463

Section 8.3

Interpolation with unequal intervals

The various interpolation formulae we have derived so far are applicable only when the values of the function are given at equally spaced points. We now proceed to develop interpolation formulae for unequally spaced points. In this section we study Lagrange's interpolation formula and in the next section Newton's divided difference formula which are applicable for unequally spaced values of x .

Section 8.3.1

Lagrange's Interpolation formula

Theorem 8.3 (Lagrange's Interpolation formula).

Let $y_0, y_1, \dots, y_n$ be the values of $f(x)$ at $x_0, x_1, \dots, x_n$ (not necessarily at equally interval) then an interpolating polynomial $\phi(x)$ for $f(x)$ is given by

$$\phi(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1 \\ + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}y_n.$$

Proof

Since n values for $f(x)$ are given we can assume $f(x)$ to be a polynomial of degree $n-1$.

$$\text{Let } \phi(x) = A_1(x-x_1)(x-x_2)\cdots(x-x_n) + A_2(x-x_0)(x-x_2)\cdots(x-x_n) \\ + \cdots + A_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

└───────────┐ ①

when $x = x_0$, we get $y_0 = A_1(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)$

$$\therefore A_1 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)}$$

Similarly when $x = x_1, x_2, \dots, x_n$ we get

$$A_2 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)}$$

$$A_n = \frac{y_n}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})}$$

Substituting these values in ① we get Lagrange's formula.

Note Since $\phi(x)$ is the interpolation polynomial which represents $f(x)$, the function $\phi(x)$ can be replaced by $f(x)$ in ①.

Hence Lagrange's formula becomes

$$y = f(x) = \frac{(x - x_1)(x - x_2) \cdots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2) \cdots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)} y_1 + \cdots + \frac{(x - x_0)(x - x_1) \cdots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})} y_n.$$

Solved Problems

Problem 1 Use Lagrange's formula to find the value of y at $x = 6$ from the following data.

x	3	7	9	10
y	168	120	72	63

Solution

The Lagrange's formula for four set of data is

$$y = \left[\frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \right] y_0 + \left[\frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \right] y_1 + \left[\frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \right] y_2 + \left[\frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \right] y_3 \rightarrow \text{①}$$

Here $x_0 = 3, x_1 = 7, x_2 = 9, x_3 = 10$

$y_0 = 168, y_1 = 120, y_2 = 72, y_3 = 63$

Space for Hints

To find $y(6)$ put $x = 6$ in ①

$$\begin{aligned}
 y(6) &= \left[\frac{(6-7)(6-9)(6-10)}{(3-7)(3-9)(3-10)} \right] \times 168 + \left[\frac{(6-3)(6-9)(6-10)}{(7-3)(7-9)(7-10)} \right] \times 120 \\
 &+ \left[\frac{(6-3)(6-7)(6-10)}{(9-3)(9-7)(9-10)} \right] \times 72 + \left[\frac{(6-3)(6-7)(6-9)}{(10-3)(10-7)(10-9)} \right] \times 63 \\
 &= 12 + 180 - 72 + 27 \\
 &= 147.
 \end{aligned}$$

Problem 2 The Lagrange's formula to fit a polynomial to the data.

x	0	1	3	4
y	-12	0	6	12

find the value of y when $x=2$.

Solution

The Lagrange's formula for four set of data is

$$\begin{aligned}
 y &= \left[\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \right] y_0 + \left[\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \right] y_1 \\
 &+ \left[\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \right] y_2 + \left[\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \right] y_3
 \end{aligned}$$

Here $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 4$ and

$$y_1 = 12, y_2 = 0, y_3 = 6, y_4 = 12$$

$$\begin{aligned}
 \therefore y &= \left[\frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)} \right] \times (-12) + \left[\frac{x(x-3)(x-4)}{1(-2)(-3)} \right] \times 0 + \left[\frac{x(x-1)(x-4)}{3 \times 2 \times (-1)} \right] \times 6 \\
 &+ \left[\frac{x(x-1)(x-3)}{4 \times 3 \times 1} \right] \times 12 \\
 &= (x+1)(x-3)(x-4) - x(x-1)(x-4) + x(x-1)(x-3) \\
 &= (x-1)[(x-3)(x-4) - x(x-4) + x(x-3)] \\
 &= (x-1)(x^2 - 6x + 12) \\
 &= x^3 - 7x^2 + 18x - 12
 \end{aligned}$$

when $x = 2$, $y = y(2) = 2^3 - 7x2^2 + 18 \times x2 - 12 = 4$.

Space for Hints

Problem 3 Find the form of the function y for the following data.

Hence find y_3

x	0	1	2	5
y	2	3	12	147

Solution

Here $x_0 = 0$; $x_1 = 1$; $x_2 = 2$; $x_3 = 5$

$y_0 = 2$; $y_1 = 3$; $y_2 = 12$; $y_3 = 147$

Applying Lagrange's formula we get

$$\begin{aligned}
 y &= \left[\frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} \right] \times 2 + \left[\frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} \right] \times 3 + \\
 &\quad + \left[\frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} \right] \times 147 \\
 &= \frac{(x^3 - 8x^2 + 17x - 10)}{5} + \frac{3(x^3 - 7x^2 + 10x)}{4} - 2(x^3 - 6x^2 + 5x) \\
 &\quad + \frac{x^3 - 3x^2 + 2x}{60} \times 127 \\
 &= \frac{1}{60} [60x^3 + 60x^2 - 60x + 120]
 \end{aligned}$$

$$\therefore y = x^3 + x^2 - x + 2$$

$$\begin{aligned}
 \text{At } x = 3, \quad y(3) &= 3^3 + 3^2 - 3 + 2 \\
 &= 35.
 \end{aligned}$$

Problem 4

The values of $U(x)$ are known at a, b, c . Show that maximum (or) minimum of Lagrange's interpolation polynomial is attained at

$$x = \frac{\sum Ua(b^2 - c^2)}{2 \sum Ua(b - c)}$$

Solution

By Lagrange's formula for 3 set of values we have

$$U(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)} \times Ua + \frac{(x-a)(x-c)}{(b-a)(b-c)} \times Ub + \frac{(x-a)(x-b)}{(c-a)(c-b)} \times Uc$$

$$= \sum \left[\frac{x^2 - (b+c)x + bc}{(a-b)(a-c)} \right] Ua$$

$U(x)$ attains its maximum (or) minimum when $\frac{dU}{dx} = 0$.

$$\text{Now, } \frac{dU}{dx} = 0 \Rightarrow \sum \left[\frac{2x - (b+c)}{(a-b)(a-c)} \right] U = 0$$

$$\Rightarrow \left[\frac{2x - (b+c)}{(a-b)(a-c)} \right] Ua + \left[\frac{2x - (c+a)}{(b-c)(b-a)} \right] Ub + \left[\frac{2x - (a+b)}{(c-a)(c-b)} \right] Uc = 0$$

$$\Rightarrow (b-c)[2x - (b+c)]Ua + (c-a)[2x - (c+a)]Ub + (a-b)[2x - (a+b)]Uc = 0$$

$$\Rightarrow 2x[Ua(b-c) + Ub(c-a) + Uc(a-b)] = (b^2 - c^2)Ua + (c^2 - a^2)Ub + (a^2 - b^2)Uc$$

$$\therefore x = \frac{(b^2 - c^2)Ua + (c^2 - a^2)Ub + (a^2 - b^2)Uc}{2[Ua(b-c) + Ub(c-a) + Uc(a-b)]}$$

$$= \frac{\sum Ua(b^2 - c^2)}{2 \sum Ua(b-c)}$$

Exercises

- (1) Use Lagrange's interpolation formula to find the value of y when $x = 10$ if the following values of x and y are given.

x	5	6	9	11
y	12	13	14	16

- (2) Use Lagrange's formula to find $f(x)$ when $x = 0$, given the following data.

x	-1	-2	2	4
$f(x)$	-1	-9	11	69

- (3) Find U_5 given that $U_1 = 4$; $U_2 = 7$; $U_4 = 13$ and $U_7 = 30$.

- (4) The following table gives the normal weight of a baby during first 6 months of life.

Age of month	0	2	3	5	6
Weight in lbs	5	7	8	10	12

- (5) The amount A of a substance remaining in a reacting system after an interval of time t in a certain chemical experiment is given in the following table.

t	2	5	8	14
A	94.8	87.9	81.3	68.7

- (6) Determine by Lagrange's formula the percentage of number of criminals under 35 years

Age	% No. of Criminals
Under 25 years	52.0
Under 30 years	67.3
Under 40 years	84.1
Under 50 years	94.4

- (7) Use Lagrange's formula find a polynomial to the data.

x	0	1	3	4
$U(x)$	-12	0	6	12

- (8) Use Lagrange's formula to find a polynomial to the following data.

x	-1	0	2	3
$U(x)$	-8	3	1	12

Space for Hints

(9) Find the form of $U(x)$ given that $U_0 = 8$; $U_1 = 11$; $U_4 = 78$;
 $U_5 = 123$.

(10) Apply Lagrange's formula to find $f(x)$ for the following data.

x	1	2	-4
$f(x)$	3	-5	4

(11) From the following table find the value of y when $x = 10$.

x	5	6	9	11
y	12	13	14	16

Hence find $f(2)$.

Section 8.4

Divided Differences

Lagrange's interpolation formula has the disadvantage that if another interpolation value were inserted then we have to recompute the interpolation coefficients. Hence it is desirable to have an interpolating formula which has the property that a polynomial of higher degree may be derived from it by simply adding new terms. Newton's divided difference formula is one such formula and this formula employs the concept of divided differences, which we now introduce.

Definition

Let $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ be a given set of $n + 1$ points.

The first divided differences are defined by the following relations.

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

$$[x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1}$$

...

...

$$[x_{n-1}, x_n] = \frac{y_n - y_{n-1}}{x_n - x_{n-1}}$$

The second divided differences are defined by

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

and so on.

The third divided differences are defined by

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$$

and so on.

The divided differences are also denoted by $\hat{u}, \hat{u}^2, \hat{u}^3$ etc

The divided difference table is given below

x	y	$\hat{u}$	$\hat{u}^2$	$\hat{u}^3$	$\hat{u}^4$
x_0	y_0	$[x_0, x_1]$	$[x_0, x_1, x_2]$	$[x_0, x_1, x_2, x_3]$	$[x_0, x_1, x_2, x_3, x_4]$
x_1	y_1	$[x_1, x_2]$	$[x_1, x_2, x_3]$	$[x_1, x_2, x_3, x_4]$	
x_2	y_2	$[x_2, x_3]$	$[x_2, x_3, x_4]$		
x_3	y_3	$[x_3, x_4]$			
x_4	y_4				

Note The divided differences are independent of the order of arguments.

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_0 - y_1}{x_0 - x_1} = [x_1, x_0]$$

$$\text{Similarly } [x_0, x_1, x_2] = [x_1, x_2, x_0] = [x_2, x_0, x_1]$$

Example

The divided differences table for the following table

x	-1	0	2	4	5
y	0	1	9	65	126

is given below.

x	y	first div diff	Second div. diff	Third div. diff.	Fourth div. diff.
-1	0	$\frac{1-0}{0-(-1)} = 1$	$\frac{4-0}{2-(-1)} = 1$	$\frac{6-1}{4-(-1)} = 1$	$\frac{1-1}{5-(-1)} = 0$
0	1	$\frac{9-0}{1-0} = 9$	$\frac{28-4}{4-0} = 6$	$\frac{11-6}{5-0} = 1$	
2	9	$\frac{65-9}{4-2} = 28$	$\frac{61-28}{5-2} = 11$		
4	65	$\frac{126-65}{5-4} = 61$			
5	126				

Result If $f(x)$ and $g(x)$ are two function a and b are constants then $\hat{\cup}$
 $[af(x)+bg(x)] = a \hat{\cup} f(x) + b \hat{\cup} g(x)$ (i.e) The operator $\hat{\cup}$ is linear.

$$\text{Proof } \hat{\cup} [af(x)+bg(x)] = \frac{[af(x_1)+bg(x_1)] - [af(x_0)+bg(x_0)]}{x_1 - x_0}$$

$$= \frac{a[f(x_1)-f(x_0)]}{x_1 - x_0} + \frac{b[g(x_1)-g(x_0)]}{x_1 - x_0}$$

$$a \hat{\cup} f(x) + b \hat{\cup} g(x).$$

Section 8.4.1

Space for Hints

Relation between divided differences and forward differences

Let the arguments of the function $U(x)$ be $x_0, x_1, \dots, x_n$ such that

$$x_1 - x_0 = x_2 - x_1 = \dots = x_n - x_{n-1} = h.$$

Then $\hat{U}^n U = \frac{\Delta^n U_0}{n! h^n}$

$$\hat{U}^1 U = [x_0, x_1] = \frac{U_1 - U_0}{x_1 - x_0} = \frac{\Delta U_0}{h}$$

$$\hat{U}^2 U = [x_0, x_1, x_2] = \frac{\hat{U}^1 U_1 - \hat{U}^1 U_0}{x_2 - x_0} = \frac{\frac{1}{h} \Delta U_1 - \frac{1}{h} \Delta U_0}{2h}$$

$$= \frac{1}{2h^2} (\Delta U_1 - \Delta U_0)$$

$$= \frac{1}{2h^2} \Delta(U_1 - \Delta U_0)$$

$$= \frac{1}{2h^2} \Delta(\Delta U_0)$$

$$= \frac{\Delta^2 U_0}{2h^2}$$

Proceeding like this we obtain $\hat{U}^n U = \frac{\Delta^n U}{n! h^n}$

Corollary The n^{th} divided differences of a polynomial of degree n are constants.

Solved Problems

Problem 1

If $f(x) = \frac{1}{x^2}$ find the first divided differences (i) $[a, b]$ (ii) $[a, b, c]$.

Solution Here a, b, c are arguments for $f(x) = \frac{1}{x^2}$

$$\therefore f(a) = \frac{1}{a^2}; f(b) = \frac{1}{b^2} \text{ and } f(c) = \frac{1}{c^2}$$

Space for Hints

$$(i) \therefore [a, b] = \frac{f(b) - f(a)}{b - a} = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{b - a}$$

$$= \frac{-(a + b)}{a^2 b^2}$$

$$(ii) [a, b, c] = \frac{[b, c] - [a, b]}{c - a}$$

$$= \frac{\left[-\frac{(b + c)}{b^2 c^2} \right] - \left[-\frac{(a + b)}{a^2 b^2} \right]}{c - a}$$

$$= \frac{1}{c - a} \left[\frac{c^2(a + b) - a^2(b + c)}{a^2 b^2 c^2} \right]$$

$$= \frac{c^2 a + c^2 b - a^2 b - a^2 c}{a^2 b^2 c^2 (c - a)}$$

$$= \frac{1}{c - a} \left[\frac{ac(c - a) + b(c^2 - a^2)}{a^2 b^2 c^2} \right]$$

$$= \frac{ab + bc + ca}{a^2 b^2 c^2}$$

Problem 2 Find the divided difference table for the function

$f(x) = x^2 + 2x + 2$ whose arguments are 1, 2, 4, 7, 10.

Solution When $x = 1, 2, 4, 7, 10$ the values of the function are

5, 10, 26, 65, 122 respectively.

The divided difference table is as shown below.

x	y	first div diff	Second div. diff	Third div. diff.	Fourth div. diff.
1	5	$\frac{10-5}{2-1} = 5$	$\frac{8-5}{4-1} = 1$		
2	10	$\frac{26-10}{4-2} = 8$		0	
4	26	$\frac{65-26}{7-4} = 13$	$\frac{13-8}{7-2} = 1$	0	0
7	65		$\frac{19-13}{10-4} = 1$		
10	122	$\frac{122-65}{10-7} = 19$			

Exercises

1) Form the divided differences table for the following data.

(i)

x	0	1	2	5
$f(x)$	2	3	12	147

(ii)

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

(iii)

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

(iv)

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1260	2028

2) Find the following divided differences of $f(x) = \frac{1}{x^2}$ whose arguments are 1,2,3,4,5 in this order.

(i) [1,2] (ii) [1,2,4] (iii) [1,2,4,5] (iv) [2,4,5].

3) If $f(x) = \frac{1}{x}$ whose arguments are a,b,c,d in this order prove that

$$[a,b,c,d] = -\frac{1}{abcd}.$$

Section 8.5

Newton's Divided differences formula

Theorem 8.4 (Newton's divided difference formula).

Let $f(x)$ be a function taking values $y_0, y_1, y_2, \dots, y_n$ corresponding to the values $x_0, x_1, \dots, x_n$. Then the Newton's divided difference formula is given by

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \dots + (x - x_0)(x - x_1) \dots (x - x_{n-1})[x_0, x_1, \dots, x_n]$$

Proof $[x_1, x_0] = \frac{f(x) - f(x_0)}{x - x_0}$

$$\therefore f(x) = f(x_0) + (x - x_0)[x_1, x_0] \longrightarrow \textcircled{1}$$

Also, $[x_1, x_0, x_1] = \frac{[x_1, x_0] - [x_0, x_1]}{x - x_1}$

$$\therefore [x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1] \longrightarrow \textcircled{2}$$

Substituting $\textcircled{2}$ in $\textcircled{1}$ we have,

$$f(x) = f(x_0) + (x - x_0) \{ [x_0, x_1] + (x - x_1)[x_1, x_0, x_1] \}$$

Proceeding like this we get the result.

Solved Problems

Problem 1 Using Newton's divided difference formula evaluate $f(8)$ given that

x	4	5	7	10	11	13
$f(x)$	-48	100	294	900	1210	2028

Solution

Space for Hints

Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \dots$$

At $x = 8$ we have,

$$\begin{aligned} f(8) &= 48 + (8 - 4)[4, 5] + (8 - 4)(8 - 5)[4, 5, 7] + (8 - 4)(8 - 5)(8 - 7) \\ &\quad [4, 5, 7, 10] + (8 - 4)(8 - 5)(8 - 7)(8 - 10)[4, 5, 7, 10, 11] \\ &\quad + (8 - 4)(8 - 5)(8 - 7)(8 - 10)[4, 5, 7, 10, 11, 13] \end{aligned}$$

The divided difference table is given below

x	$f(x)$	1 st div. diff.	2 nd div. diff.	3 rd div. diff.	4 th div. diff.	5 th div. diff.
4	48					
		52				
5	100		15			
		97		1		
7	294		21		0	
		202		1		0
10	900		27		0	
		310		1		
11	1210		33			
		409				
13	2028					

Substituting the values of all divided differences from the divided difference table we have,

$$\begin{aligned} f(8) &= 48 + (4 \times 52) + (4 \times 3 \times 15) + (4 \times 3 \times 1 \times 1) \\ &= 48 + 208 + 180 + 12 \\ &= 448. \end{aligned}$$

Problem 2

Find the equation of the cubic curve which passes through the points (4, -43) (7, 83) (9, 327) and (12, 1053).

Space for HintsHence find $f(10)$.

Solution We have to find the equation of the cubic curve from the following data

x	4	7	9	12
$f(x)$	-43	83	327	1053

we apply Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \dots$$

The divided difference table is

x	$f(x)$	1 st div. diff.	2 nd div. diff.	3 rd div. diff.
4	-43			
		42		
7	83		16	
		122		1
9	327		24	
		242		
12	1053			

$$\begin{aligned} \therefore f(x) &= -43 + (x - 4) \div 42 + (x - 4)(x - 7) \times 16 + (x - 4)(x - 7)(x - 9) \times 1 \\ &= -43 + x^3 - 7x - 4x^2 + 28 \end{aligned}$$

$$\therefore f(x) = x^3 - 4x^2 - 7x - 15$$

$$f(10) = 1000 - 400 - 70 - 15 = 515.$$

Problem 3 Given the values

x	5	7	11	13	17
y	150	392	1452	2366	5202

Evaluate y_9 using (i) Lagrange's formula (ii) Newton's divided differences formula.

Solution

Space for Hints

(i) Lagrange's formula

Given $x_0 = 5, x_1 = 7, x_2 = 11, x_3 = 13, x_4 = 17$

$$y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366, y_4 = 5202$$

Putting $x = 9$ in the Lagrange's formula

$$\begin{aligned} y_9 = & \left[\frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \right] \times 150 + \left[\frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \right] \times 392 \\ & + \left[\frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-13)(11-17)} \right] \times 1452 + \left[\frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \right] \times 2366 \\ & + \left[\frac{(9-5)(9-7)(9-13)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \right] \times 5202. \end{aligned}$$

$$= 1615.3334 - 805.3334.$$

$$\therefore y_9 = 810.$$

(ii) The Newton's divided difference formula

$$y = f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \dots$$

Taking $x = 9$ in the formula

$$\begin{aligned} f(9) &= 150 + (9-5) \times 121 + (9-5)(9-7) \times 24 + (9-5)(9-7)(9-11) \times 1 \\ &= 150 + 484 + 192 - 16 \\ &= 810. \end{aligned}$$

The values 121, 24 and 1 in the above are obtained from the following divided difference table

x	$f(x)$	1 st div. diff.	2 nd div. diff.	3 rd div. diff.	4 th div. diff
5	150				
7	392	121			
11	1452	265	24	1	
13	2366	457	32	1	0
17	5202	709	42		

- (1) By Newton's divided difference formula find the value of $f(8)$ given that

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

- (2) Find $f(19)$ using Newton's divided difference formula given that

x	11	17	21	23
$f(x)$	14646	83526	19448	27984
			6	6

- (3) Find U_2 given that

U_0	U_1	U_3	U_4
-12	0	6	12

- (4) Find the cubic function from the following table using Newton's divided difference formula

x	0	1	2	5
$f(x)$	2	3	12	147

- (5) Find the function $f(x)$ from the following table

x	0	1	4	5
$f(x)$	8	11	78	123

- (6) Given the following data

x	654	658	659	661
$\log_{10} x$	2	3	12	147

Find $\log_{10} 656$ using (i) Lagrange's interpolation formula

(ii) Newton's divided difference formula.

Section 8.6**Inverse Interpolation**

So far given a table of values of x and y , using one of the interpolation formula we find the value of y corresponding to some value of x which is not in the table. On the other hand the process of estimating the value of x for some value of y which is not in the table is called **inverse interpolation**. There are two methods for inverse interpolation, one when the values are unequally spaced and another when the values are equally spaced.

Section 8.6.1**Lagrange's method**

This method is used when the values of x are unequally spaced.

Lagrange's interpolation formula can be simply viewed as a relation between two variables and any one of the variable can be taken as an independent variable. Therefore interchanging the variables x and y in Lagrange's formula we get

$$x = \frac{(y - y_1)(y - y_2) \cdots (y - y_n)}{(y_0 - y_1)(y_0 - y_2) \cdots (y_0 - y_n)} x_0 + \frac{(y - y_0)(y - y_2) \cdots (y - y_n)}{(y_1 - y_0)(y_1 - y_2) \cdots (y_1 - y_n)} x_1 + \cdots + \frac{(y - y_0)(y - y_1) \cdots (y - y_{n-1})}{(y_n - y_0)(y_n - y_1) \cdots (y_n - y_{n-1})} x_n.$$

Which is used for inverse interpolation.

Section 8.6.2**Iterative method**

Newton's forward difference formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \cdots + \frac{p(p-1) \cdots (p-n+1)}{n!} \Delta^n y_0.$$

$$\therefore p = \frac{1}{\Delta y_0} \left[y_p - y_0 - \frac{p(p-1)}{2!} \Delta^2 y_0 - \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 \cdots \right]$$

Space for Hints

Neglecting the second and higher order differences we obtain the first approximation to p given by

$$\therefore p_1 = \frac{1}{\Delta y_0} (y_p - y_0).$$

To find the second approximation to p we retain the term with second difference and replace p by p_1

$$\therefore p_2 = \frac{1}{\Delta y_0} \left[y_p - y_0 - \frac{p_1(p_1 - 1)}{2!} \Delta^2 y_0 \right]$$

To find the third approximation we retain the forms upto third order difference and replace p by p_2

$$\therefore p_3 = \frac{1}{\Delta y_0} \left[y_p - y_0 - \frac{p_2(p_2 - 1)}{2!} \Delta^2 y_0 - \frac{p_2(p_2 - 2)}{3!} \Delta^3 y_0 \right]$$

Continue this process till the successive values of p are approximately equal.

Solved Problems

problem 1 Find the value of x correct to one decimal place for which $y = 7$ given

x	1	3	4
y	4	12	19

Solution we have to find x when $y = 7$.

we need to use Lagrange's inverse interpolation formula.

Here $x_0 = 1$; $x_1 = 3$; $x_2 = 4$

$y_0 = 4$; $y_1 = 12$; $y_2 = 19$

$$x = \frac{(y - y_1)(y - y_2)}{(y_0 - y_1)(y_0 - y_2)} x_0 + \frac{(y - y_0)(y - y_2)}{(y_1 - y_0)(y_1 - y_2)} x_1 + \frac{(y - y_0)(y - y_1)}{(y_2 - y_0)(y_2 - y_1)} x_2$$

$$= \frac{(7 - 12)(7 - 19)}{(4 - 12)(4 - 19)} \times 1 + \frac{(7 - 4)(7 - 19)}{(12 - 4)(12 - 19)} \times 3 + \frac{(7 - 4)(7 - 12)}{(19 - 4)(19 - 12)} \times 4$$

$$= 0.5 + 1.9286 - 0.5714$$

$$= 1.8572.$$

Problem 2

Space for Hints

Tabulate $y = x^3$ for $x = 2, 3, 4, 5$ and calculate the cube root of 10 correct to three decimal places.

Solution For $x = 2, 3, 4, 5$ the values of y are 8, 27, 64, 125 respectively.

Here $h = 1$. we form the forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2	8	19		
3	27	37	18	
4	64	61	24	6
5	125			

The first approximation is given by

$$\begin{aligned} p_1 &= \frac{1}{\Delta y_0} (y_p - y_0) \\ &= \frac{1}{19} (10 - 8) \\ &= 0.1 \end{aligned}$$

The second approximation is

$$\begin{aligned} p_2 &= \frac{1}{\Delta y_0} \left[y_p - y_0 - \frac{p_1(p_1 - 1)}{2!} \Delta^2 y_0 \right] \\ &= \frac{1}{19} \left[10 - 8 - \frac{(0.1)(0.1 - 1)}{2!} (18) \right] \\ &= 0.15 \end{aligned}$$

The third approximation is

$$\begin{aligned} p_3 &= \frac{1}{\Delta y_0} \left[y_p - y_0 - \frac{p_2(p_2 - 1)}{2!} \Delta^2 y_0 - \frac{(p_2 - 1)(p_2 - 2)}{3!} \Delta^3 y_0 \right] \\ &= \frac{1}{19} \left[10 - 8 - \frac{(0.15)(0.15 - 1)}{2!} \times 18 - \frac{(0.15)(0.15 - 1)(0.15 - 2)}{3!} \times 6 \right] \\ &= 0.1532 \end{aligned}$$

Space for Hints

The third approximation is

$$\begin{aligned} p_3 &= \frac{1}{\Delta y_0} \left[y_p - y_0 - \frac{p_2(p_2 - 1)}{2!} \Delta^2 y_0 - \frac{(p_2 - 1)(p_2 - 2)}{3!} \Delta^3 y_0 \right] \\ &= \frac{1}{19} \left[10 - 8 - \frac{(0.15)(0.15 - 1)}{2!} \times 18 - \frac{(0.15)(0.15 - 1)(0.15 - 2)}{3!} \times 6 \right] \\ &= 0.1532 \end{aligned}$$

The fourth approximation is

$$\begin{aligned} p_4 &= \frac{1}{\Delta y_0} \left[y_p - y_0 - \frac{p_3(p_3 - 1)}{2!} \Delta^2 y_0 - \frac{p_3(p_3 - 1)(p_3 - 2)}{3!} \Delta^3 y_0 \right] \\ &= \frac{1}{19} \left[10 - 8 - \frac{0.1532(0.1532 - 1)}{2!} \times 18 - \frac{0.1532(0.1532 - 1)(0.1532 - 2)}{3!} \times 6 \right] \\ &= 0.1541. \end{aligned}$$

The fifth approximation is

$$\begin{aligned} p_5 &= \frac{1}{\Delta y_0} \left[y_p - y_0 - \frac{p_4(p_4 - 1)}{2!} \Delta^2 y_0 - \frac{p_4(p_4 - 1)(p_4 - 2)}{3!} \Delta^3 y_0 \right] \\ &= \frac{1}{19} \left[10 - 8 - \frac{0.1541(0.1541 - 1)}{2!} \times 18 - \frac{0.1541(0.1541 - 1)(0.1541 - 2)}{3!} \times 6 \right] \\ &= 0.1542. \end{aligned}$$

Hence $p_4 \approx p_5 = 0.154$ (correct to 3 places of decimals)

we have to find $3\sqrt{10}$. Since 10 lies between the values of y

corresponding to $x = 3$, the required value of $3\sqrt{10}$ is

$$2 + \text{hps} = 2 + 0.154 = 2.154.$$

Exercises

(1) The values of x and U_x are given below

x	2	3	5
U_x	113	286	613

Find correct to one decimal place the value of x for which $U_x = 1001$.

(2) The values of x and U_x are given below

x	5	6	9	11
U_x	12	13	14	16

Find the value of x when $U_x = 15$.

(3) Find x when $f(x) = 14$ for the following data using Lagrange's inverse interpolation formula.

x	0	5	10	15
$f(x)$	16.35	14.88	13.59	12.46

(4) Find the value of x corresponding to $y = 100$ from the table using lagrange inverse formula.

x	3	5	7	9	11
y	6	24	58	108	174

(5) Find the value of θ if $f(\theta) = 0.3887$ from the table given below

θ	21°	23°	25°
$f(\theta)$	0.3706	0.4068	0.4433

(6) Find the values of x when $y = 85$ given that

x	2	5	8	14
y	94.8	87.9	81.3	68.7

(7) The following values of $y = f(x)$ are given

x	10	15	20
y	1754	2648	3564

Find the value of x for $y = 3000$ by iterative method.

Space for Hints

Find the value of x for $y = 3000$ by iterative method.

x	10	15	20
y	1754	2648	3564

(7) The following values of $y = f(x)$ are given

x	2	5	8	14
y	94.8	87.9	81.3	68.7

(6) Find the values of x when $y = 85$ given that

θ	21°	23°	25°
$\lambda(\theta)$	0.3706	0.4068	0.4433

(5) Find the value of θ if $\lambda(\theta) = 0.3887$ from the table given below

x	3	5	7	9	11
y	6	24	58	108	174

Lagrange inverse formula.

(4) Find the value of x corresponding to $y = 100$ from the table using

x	0	2	5	10	15
$\lambda(x)$	16.32	14.88	13.39	12.46	

inverse interpolation formula.

(3) Find x when $\lambda(x) = 14$ for the following data using Lagrange's

Find the value of x when $U_x = 15$.

x	2	6	9	11
U_x	12	13	14	16

(2) The value of x and U_x are given below

Space for Hint

UNIT : 9

NUMERICAL DIFFERENTIATION

Section 9.0 Introduction

Space for Hints

Suppose a function $y = f(x)$ is given by a table values of (x_i, y_i) .

The process of computing the derivative $\frac{dy}{dx}$ for some particular value of x is called the Numerical Differentiation.

In unit 8 we have developed several interpolating polynomials which agree at the set of tabulated points. The general method of Numerical differentiation is to differentiate the interpolating polynomial. Hence corresponding to each of the interpolating polynomial derived in unit 8, We can derive a corresponding formula for derivative. The choice of the interpolating formula depends as usual on the particular value of x at which $\frac{dy}{dx}$ is desired.

Section 9.1

Derivatives using Newton's Forward difference formula

Newton's interpolation formula for equal interval

$$\text{is } y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots \longrightarrow \textcircled{1}$$

$$\text{Where } p = \frac{x - x_0}{h} \longrightarrow \textcircled{2}$$

Now Differentiating $\textcircled{1}$ with respect to p We get

$$\begin{aligned} \frac{dy}{dp} &= \Delta y_0 + \left(\frac{2p-1}{2!} \right) \Delta^2 y_0 + \left(\frac{3p^2 - 6p + 2}{3!} \right) \Delta^3 y_0 \\ &+ \left(\frac{4p^3 - 18p^2 + 22p - 6}{4!} \right) \Delta^4 y_0 + \dots \longrightarrow \textcircled{3} \end{aligned}$$

Space for Hints

Differentiating equation ② with respect to x we have $\frac{dp}{dx} = \frac{1}{h}$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx}$$

$$\therefore \left(\frac{dy}{dx} \right)_{x=x_0+ph} = \frac{1}{h} \left[\Delta y_0 + \left(\frac{2p-1}{2!} \right) \Delta^2 y_0 + \left(\frac{3p^2-6p+2}{3!} \right) \Delta^3 y_0 + \left(\frac{4p^3-18p^2+22p-6}{4!} \right) \Delta^4 y_0 + \dots \right]$$

At $x = x_0$ We have $p = 0$

$$\therefore \left(\frac{dy}{dx} \right)_{x=x_0} = \frac{1}{h} \left(\Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 - \frac{1}{4} \Delta^4 y_0 + \dots \right)$$

$$\begin{aligned} \text{Now } \frac{d^2 y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) \\ &= \frac{d}{dp} \left(\frac{dy}{dx} \right) \frac{dp}{dx} \end{aligned}$$

$$= \frac{d}{dp} \left[\frac{dy}{dx} \right] \times \frac{1}{h} \longrightarrow \text{④}$$

$$\text{But } \frac{d}{dp} \left(\frac{dy}{dx} \right) = \frac{d}{dp} \left\{ \frac{1}{4} \left[\Delta y_0 + \left(\frac{2p-1}{2!} \right) \Delta^2 y_0 + \left(\frac{3p^2-6p-12}{3!} \right) \Delta^3 y_0 + \left(\frac{4p^3-18p^2+22p-6}{4!} \right) \Delta^4 y_0 + \dots \right] \right\}$$

$$= \frac{1}{h} \left\{ \Delta^2 y_0 + (p-1) \Delta^3 y_0 + \left(\frac{6p^2-18p+11}{12} \right) \Delta^4 y_0 + \dots \right\} \longrightarrow \text{⑤}$$

Substituting ⑤ in ④ we get

$$\frac{d^2 y}{dx^2} = \frac{1}{h^2} \left\{ \Delta^2 y_0 + (p-1) \Delta^3 y_0 + \left(\frac{6p^2-18p+11}{12} \right) \Delta^4 y_0 + \dots \right\}$$

At $x = x_0$, we have $p = 0$

$$\therefore \left(\frac{d^2 y}{dx^2} \right)_{x=x_0} = \frac{1}{h^2} \left\{ \Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 + \dots \right\}$$

Similarly we can find the higher order derivatives.

Section 9.2

Derivatives using Newton's Backward difference formula

We know that the Newton's Interpolation formula for Backward difference is

$$y_p = y_n + p \nabla y_n + \frac{p(p+1)}{2!} \Delta^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \dots \rightarrow \textcircled{1}$$

$$\text{where } p = \frac{x - x_n}{h} \rightarrow \textcircled{2}$$

Differentiating $\textcircled{1}$ with respect to p we get

$$\begin{aligned} \frac{dy}{dp} = & \nabla y_n + \left(\frac{2p+1}{2!} \right) \Delta^2 y_n + \left(\frac{3p^2+6p+2}{3!} \right) \nabla^3 y_n \\ & + \left(\frac{4p^3+18p^2+22p+6}{4!} \right) \nabla^4 y_n + \dots \rightarrow \textcircled{3} \end{aligned}$$

Differentiating equation $\textcircled{2}$ with respect to x we get $\frac{dp}{dx} = \frac{1}{h}$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx}$$

$$\begin{aligned} &= \left[\nabla y_n + \left(\frac{2p+1}{2} \right) \nabla^2 y_n + \left(\frac{3p^2+6p+2}{6} \right) \nabla^3 y_n \right. \\ &\quad \left. + \left(\frac{4p^3+18p^2+22p+6}{24} \right) \nabla^4 y_n + \dots \right] \frac{1}{h} \\ &= \frac{1}{h} \left\{ \nabla y_n + \left(\frac{2p+1}{2} \right) \nabla^2 y_n + \left(\frac{3p^2+6p+2}{6} \right) \nabla^3 y_n \right. \\ &\quad \left. + \left(\frac{4p^3+18p^2+22p+6}{24} \right) \nabla^4 y_n + \dots \right\} \end{aligned}$$

Space for Hints

At $x = x_n$, $p = 0$

$$\begin{aligned}\text{Now } \frac{d^2 y}{dx^2} &= \frac{d}{dx} \left[\frac{dy}{dx} \right] \\ &= \frac{d}{dx} \left[\frac{dy}{dx} \right] \times \frac{dp}{dx} \\ &= \frac{1}{h^2} \frac{d}{dp} = \left\{ \nabla y_n + \left(\frac{2p+1}{2} \right) \nabla^2 y_n + \left(\frac{3p^2+6p+2}{6} \right) \nabla^3 y_n \right. \\ &\quad \left. + \left(\frac{4p^3+18p^2+22p+6}{24} \right) \nabla^4 y_n + \dots \right\} \\ &= \frac{1}{h^2} \left[\nabla^2 y_n + (p+1) \nabla^3 y_n + \left(\frac{6p^2+18p+11}{12} \right) \nabla^4 y_n + \dots \right]\end{aligned}$$

At $x = x_n$, $p = 0$

$$\therefore \left(\frac{d^2 y}{dx^2} \right)_{x=x_n} = \frac{1}{h^2} \left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \dots \right]$$

Similarly we can find the Higher order derivatives.

Section 9.3

Derivatives Using Central difference formulae

When we want to interpolate near the central value we use any of the central difference interpolation formulae, namely Gauss forward and Gauss Backward interpolation formulae, Stirling's formulae, Bessel's formula Bessel's formula and Laplace. Everett's formula. We can use these formula to get the derivatives of the polynomials.

Section 9.3.1.**Derivatives Using Stirlings Formula**

The Stirling's formula is

$$y_p = y_0 + \frac{p}{1!} \left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2-1^2)}{3!} \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right) + \frac{p^2(p^2-1)}{4!} \Delta^4 y_{-2} + \frac{p(p^2-1^2)(p^2-2^2)}{5!} \left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} \right) + \dots \quad \textcircled{1}$$

Where $p = \frac{x - x_0}{h} \longrightarrow \textcircled{2}$

Differentiating equation $\textcircled{1}$ with respect to p we get

$$\frac{dy}{dp} = \left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + p \Delta^2 y_{-1} + \left(\frac{3p^2 - 1}{3!} \right) \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right) + \frac{4p^3 - 2p}{4!} \Delta^4 y_{-2} + \dots \right]$$

Differentiate equation $\textcircled{2}$ with respect to x we get

$$\frac{dp}{dx} = \frac{1}{h}$$

Now $\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx}$

$$= \frac{1}{h} \left\{ \left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + p \Delta^2 y_{-1} + \left(\frac{3p^2 - 1}{3!} \right) \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right) + \left(\frac{4p^3 - 2p}{4!} \right) \Delta^4 y_{-2} + \dots \right\}$$

At $x = x_0, p = 0$

$$\therefore \left(\frac{dy}{dx} \right)_{x=x_0} = \frac{1}{h} \left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) - \frac{1}{6} \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right) + \frac{1}{30} \left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} \right) + \dots \right]$$

Similarly we can derive

$$\left(\frac{d^2 y}{dx^2} \right)_{x=x_0} = \frac{1}{h^2} \left[\Delta^2 y_{-1} - \frac{1}{12} \Delta^4 y_{-2} + \dots \right]$$

In a Similar manner we can find the higher derivatives.

Note We can also use the other central difference formulae to find the derivative.

Solved Problems

Problem 1 Find $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$ at $x = 51$ from the following data

x	50	60	70	80	90
y	19.96	36.65	58.81	77.21	94.61

Solution Here $h = 10$.

To find the derivatives of y at $x = 51$, we use Newton's Forward difference formula, by taking the origin at $a = 50$.

$$\text{we know that } p = \frac{x - x_0}{h} = \frac{51 - 50}{10} = \frac{1}{10} = 0.1$$

$$\therefore \text{At } x = 51, \quad p = 0.1$$

$$\therefore \left(\frac{dy}{dx} \right)_{x=51} = \left(\frac{dy}{dx} \right)_{p=0.1}$$

$$= \frac{1}{h} \left\{ \Delta y_0 + \frac{(2p-1)}{2!} \Delta^2 y_0 + \frac{(3p^2 - 6p + 2)}{3!} \Delta^3 y_0 + \frac{(4p^3 - 18p^2 + 22p - 6)}{4!} \Delta^4 y_0 + \dots \right\}$$

The forward difference table is given by

x	$p = \frac{x-50}{10}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
50	0	19.96				
60	1	36.65	16.69	5.47		
70	2	58.81	22.16	-3.76	-9.23	
80	3	77.21	18.40	-1.00	2.76	11.99
90	4	94.61	17.40			

$$\therefore \left(\frac{dy}{dx} \right)_{p=0.1} = \frac{1}{10} \left\{ 16.69 + \frac{(0.2-1)}{2} (5.47) + \left[\frac{3(1)^2 + 6(1) + 2}{6} \right] (-9.23) \right. \\ \left. + \left[\frac{4(1)^3 + 18(1)^2 + 22(1) - 6}{24} \right] \times 11.99 \right\}$$

$$= \frac{1}{10} [16.69 - 2.188 - 2.1998 - 1.9863] = 1.0316$$

$$\left(\frac{d^2 y}{dx^2} \right)_{p=0.1} = \frac{1}{h^2} \left[\Delta^2 y_0 + (p-1) \Delta^3 y_0 + \frac{(6p^2 - 8p + 11)}{12} \Delta^4 y_0 + \dots \right]$$

$$= \frac{1}{100} \left[5.47 + (0.1-1)(-9.23) + \left[\frac{6(1)^2 - 18(1) + 11}{12} \right] \times 11.99 \right]$$

$$= \frac{1}{100} [5.47 + 8.307 + 9.2523] = 0.2303$$

Thus $\left(\frac{dy}{dx} \right)_{p=0.1} = 1.0316$

$$\left(\frac{d^2 y}{dx^2} \right)_{p=0.1} = 0.2303$$

Problem 2 Find $y'(x)$ given

x	0	1	2	3	4
$y(x)$	1	1	15	40	85

Solution Here $h = 1$. we apply Newton's forward Interpolation formula for derivative

$$y' = \frac{1}{h} \left\{ \Delta y_0 + \frac{(2p-1)}{2!} \Delta^2 y_0 + \frac{(3p^2 - 6p + 2)}{3!} \Delta^3 y_0 + \frac{(4p^3 - 18p^2 + 22p - 6)}{4!} \Delta^4 y_0 + \dots \right\}$$

where $p = \frac{x - x_0}{h}$. Choose $x_0 = 0$, so that $p = x$

The forward difference table is now below

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1	0			
1	1	14	14		
2	15	25	11	-3	12
3	40	45	20	9	
4	85				

$$\therefore y' = \Delta y_0 + \frac{(2x-1)}{2} \Delta^2 y_0 + \frac{(3x^2 - 6x + 2)}{6} \Delta^3 y_0 + \frac{(4x^3 - 18x^2 + 22x - 6)}{24} \Delta^4 y_0 + \dots$$

$$= 0 + \left(\frac{2x-1}{2} \right) \times 14 + \left(\frac{3x^2 - 6x + 2}{6} \right) (-3) + \frac{(4x^3 - 18x^2 + 22x - 6)}{24} \times 12$$

$$= 7(2x-1) - \frac{(3x^2 - 6x + 2)}{2} + 2x^3 - 9x^2 + 11x - 3$$

$$y'(x) = 2x^3 - \frac{21}{2}x^2 + 28x - 11$$

Now $y'(x)$ at $x = 0.5$ is

$$y'(.5) = 2(.5)^3 - 10.5(.5)^2 + 28(.5) - 11 = 0.625$$

$\therefore y'(x)$ at $x = 0.5$ is

$$y' = 0.625$$

Problem 3 Given $u_0 = 5$; $u_1 = 15$; $u_2 = 57$ and

$$\frac{du}{dx} = 4 \text{ at } x = 0 \text{ and } 72 \text{ at } x = 2.$$

Find $\Delta^3 u_0$ and $\Delta^4 u_0$.

Solution Here $h = 1$

$\therefore$ By Newton's forward formula for differentiation, we have

$$\frac{du}{dx} = \frac{1}{h} \left\{ \Delta u_0 + \frac{(2p-1)}{2} \Delta^2 u_0 + \frac{(3p^2 - 6p + 2)}{6} \Delta^3 u_0 + \frac{(4p^3 - 18p^2 + 22p - 6)}{24} \Delta^4 u_0 + \dots \right\}$$

$$\text{where } p = \frac{x-a}{h} \quad \text{--- (1)}$$

$$\text{Choosing origin at } x = 0 \text{ we have } p = \frac{x-a}{h} = \frac{x-0}{1} = x$$

The forward difference table is

x	u	Δu	$\Delta^2 u$
0	5	10	32
1	15	42	
5	57		

Space for Hints

Given that $\left(\frac{du}{dx}\right)_{x=0} = 4$ and $\longrightarrow (2)$

$$\left(\frac{du}{dx}\right)_{x=2} = 72 \longrightarrow (3)$$

Now put $x = 0$ in (1) and using (2) we get & $h = 1$

$$\left(\frac{du}{dx}\right)_{x=0} = 4 = 10 - \frac{1}{2} \times 32 + \frac{2}{6} \Delta^3 u_0 - \frac{6}{24} \Delta^4 u_0$$

$$\text{ie, } 4 = 10 - 16 + \frac{1}{3} \Delta^3 u_0 - \frac{1}{4} \Delta^4 u_0$$

$$\text{ie, } \frac{1}{3} \Delta^3 u_0 - \frac{1}{4} \Delta^4 u_0 = 10$$

$$\Rightarrow \frac{4\Delta^3 u_0 - 3\Delta^4 u_0}{12} = 10$$

$$\Rightarrow 4\Delta^3 u_0 - 3\Delta^4 u_0 = 120 \longrightarrow (4)$$

Now putting $x = 2$ and $h = 1$ in (1) and using (3) we get

$$72 = 10 + \frac{3}{2} \times 32 + \frac{1}{3} \Delta^3 u_0 - \frac{1}{12} \Delta^4 u_0$$

$$\text{ie, } 10 + 48 + \frac{1}{3} \Delta^3 u_0 - \frac{1}{12} \Delta^4 u_0 = 72$$

$$\Rightarrow \frac{1}{3} \Delta^3 u_0 - \frac{1}{12} \Delta^4 u_0 = 72 - 58 = 14$$

$$\Rightarrow \frac{4\Delta^3 u_0 - \Delta^4 u_0}{12} = 14$$

$$\Rightarrow 4\Delta^3 u_0 - \Delta^4 u_0 = 14 \times 12 = 168 \longrightarrow (5)$$

Now Let us solve equations (4) and (5) we get

$$4\Delta^3 u_0 - 3\Delta^4 u_0 = 120$$

$$\begin{array}{r} (-) 4\Delta^3 u_0 \quad (+) -\Delta^4 u_0 \quad (-) = 168 \\ \hline -2\Delta^4 u_0 = -48 \end{array}$$

$$\Rightarrow \Delta^4 u_0 = \frac{48}{2} = 24$$

Substituting the value of $\Delta^4 u_0 = 24$ in equation (5) we get

$$4\Delta^3 u_0 = 168 + \Delta^4 u_0$$

$$\text{ie, } 4\Delta^3 u_0 = 168 + 24 = 192$$

$$\Rightarrow \Delta^3 u_0 = \frac{192}{4} = 48$$

$$\text{Thus } \Delta^3 u_0 = 48$$

$$\text{and } \Delta^4 u_0 = 24$$

Problem 4

Find the first and 2nd order derivatives of the function tabulated below the point $x = 1.5$

x	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	3.375	7.0	13.625	24.0	38.875	59.0

Solution The table of difference is as follows

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.5	3.375					
2.0	7.0	3.625				
2.5	13.625	6.625	3.0			
3.0	24.0	10.375	3.75	0.75		
3.5	38.875	14.875	4.50	0.75	0	
4.0	59.0	20.125	5.25	0.75	0	0

Here we have to find the derivative at the point $x = 1.5$ which is the starting value of the table.

$\therefore$ By Newton's forward difference formula for derivatives at $x = x_0$ we have

Space for Hints

$$f'(x_0) = \frac{1}{h} \left[\Delta y_0 - 2 \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} \dots \right]$$

Here $x_0 = 1.5$, $h = 0.5$

$$\therefore f'(1.5) = \frac{1}{.5} \left[3.625 - \frac{3.0}{2} + \frac{0.75}{3} - \frac{0}{4} \dots \right]$$

ie, $f'(1.5) = 4.75$

At the point $x = x_0$,

$$f''(x_0) = \frac{1}{h^2} \left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 \dots \right]$$

Here $x_0 = 1.5$, $h = 0.5$

$$\begin{aligned} \therefore f''(1.5) &= \frac{1}{(.5)^2} [3 - 0.75 + 0 + 0] \\ &= \frac{2.25}{.25} = 9.0 \end{aligned}$$

Thus $f'(1.5) = 4.75$

$f''(1.5) = 9.0$

Problem 5 The population in millions of a certain town is shown in the following table. Find the rate of growth of the population in 1961.

Year x	1931	1941	1951	1961	1971
Population y	40.62	60.80	79.95	103.56	132.65

Solution Here $h = 10$. Since the rate of growth of population is $\frac{dy}{dx}$, we

have to find $\frac{dy}{dx}$ at $x = 1961$, which is nearer to the end value of the table.

Hence we choose the origin at $x = 1971$, and we use Newton's backward interpolation formula for derivative.

$$\frac{dy}{dx} = \frac{1}{h} \left\{ \nabla y_4 + \left(\frac{2p+1}{2} \right) \nabla^2 y_4 + \left(\frac{3p^2+6p+2}{6} \right) \nabla^3 y_4 + \left(\frac{2p^3+9p^2+11p+3}{12} \right) \nabla^4 y_4 + \dots \right\}$$

where $p = \frac{x - x_0}{h} = \frac{1961 - 1971}{10} = -1$.

The Backward difference table is as follows:

Year x	Population y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1931	40.62				
1941	60.80	20.18			
1951	79.95	19.15	-1.03		
1961	103.56	23.61	4.46	5.49	
1971	132.65	29.09	5.48	1.02	-4.47

$$\therefore \left(\frac{dy}{dx} \right)_{p=1} = \frac{1}{10} \left\{ 29.09 + \left(\frac{-1}{2} \right) (5.48) + \left(\frac{3(-1)^2 + 6(-1) + 2}{6} \right) \times (1.02) + \left(\frac{2(-1)^3 + 9(-1)^2 + 1(-1) + 3}{12} \right) (-4.47) \right\}$$

$$= \frac{1}{10} \{ 29.09 - 2.74 - 0.17 + 0.3725 \}$$

$$= \frac{1}{10} [26.5525] = 2.6553$$

$\therefore$ The rate of growth of the population in the year 1961 is 2.6553

Problem 6 A rod is rotating in a plane. The following table given the angular θ (radians) through which the rod has turned for various values of time t (Seconds)

t	0	0.2	0.4	0.6	0.8	1.0
θ	0	0.12	0.49	1.12	2.02	3.20

Calculate the angular velocity and the acceleration of the rod when $t=0.6$ seconds.

Solution Here $h=0.2$. To find the angular velocity and acceleration of the

rod at $t = 0.6$ seconds, we have to find $\left(\frac{d\theta}{dt}\right)_{t=0.6}$ and $\left(\frac{d^2\theta}{dt^2}\right)_{t=0.6}$

Since $t = 0.6$ is nearer to the middle value of t , we are using Stirling's formula

Choosing the origin at $t_0 = 0.6$, we have $p = \frac{t - t_0}{h} = \frac{0.6 - 0.6}{0.2} = 0$

Now Stirling's formula for first order derivative is $\left(\frac{d\theta}{dt}\right)_{t=0.6} = \left(\frac{d\theta}{dt}\right)_{p=0}$

$$= \frac{1}{h} \left\{ \frac{(\Delta\theta_0 + \Delta\theta_{-1})}{2} - \frac{1}{6} \left(\frac{\Delta^3\theta_{-p} + \Delta^3\theta_{-2}}{2} \right) + \frac{1}{30} \left(\frac{\Delta^5\theta - 2 + \Delta^5\theta - 3}{2} \right) + \dots \right\}$$

The central difference table is given below

t	p	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$	$\Delta^5\theta$
0	-3	0					
0.2	-2	0.12	0.12	0.25			
0.4	-1	0.49	.37	0.26	0.01		
0.6	0	1.12	0.63	0.27	0.01	0	
0.8	1	2.02	0.90	0.28	0.01	0	
1.0	2	3.20	1.18				

$$\therefore \left(\frac{d\theta}{dt}\right)_{p=0} = \frac{1}{0.2} \left\{ \left(\frac{0.9 + 0.63}{2} \right) - \frac{1}{6} \left(\frac{0.01 - 0.01}{2} \right) \right\} = 5 \left[\frac{1.53}{2} \right]$$

$$\left(\frac{d\theta}{dt}\right)_{t=0.6} = 3.825 \text{ radians / Second}$$

$$\begin{aligned} \left(\frac{d^2\theta}{dt^2}\right)_{t=0.6} &= \left(\frac{d^2\theta}{dt^2}\right)_{p=0} = \frac{1}{h^2} \left\{ \Delta^2\theta_{-1} - \frac{1}{12}\Delta^4\theta_{-2} + \dots \right\} \\ &= 25 \left[0.27 - \frac{1}{2}(0) \right] = \frac{.27}{25} \end{aligned}$$

$$\left(\frac{d^2\theta}{dt^2}\right)_{t=0.6} = 6.75 \text{ radian/second}^2$$

Problem 7 Find the gradient of the rod at the middle point of the elevation above a datum line of seven points of a rod which are given below the following table.

x	0	300	600	900	1200	1500	1800
y	135	149	157	183	201	205	193

Solution Since we have to find the gradient of the rod at the middle point of the elevation we have to find $\left(\frac{dy}{dx}\right)_{x=x_0}$ by using stirlings formula.

Here $h=300$. Choose the origin at $x = x_0 = 900$

$$\text{We get } p = \frac{x - x_0}{h} = \frac{900 - 900}{300} = 0$$

Stirlings formula for first order derivative is

$$\begin{aligned} \left(\frac{dy}{dx}\right)_{x=x_0} &= \frac{1}{h} \left\{ \left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) - \frac{1}{6} \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right) \right. \\ &\quad \left. + \frac{1}{30} \left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} \right) + \dots \right\} \end{aligned}$$

We now form the central difference table and it is given below.

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	-3	135						
			14					
300	-2	149		-6				
			8		24			
600	-1	157		18		-50		
			26		-26		+20	
900	0	183		-8		4		-86
			18		-6		-16	
1200	1	201		-14		-2		
			4		-2			
1500	2	205		-16				
			-12					
1800	3	193						

$$\left(\frac{dy}{dx}\right)_{x=900} = \frac{1}{300} \left\{ \left(\frac{18+26}{2} \right) - \frac{1}{6} \left(\frac{-6+(-20)}{2} \right) + \frac{1}{30} \left(\frac{-16+70}{2} \right) \right\}$$

$$= 0.0852$$

Problem 8

From the following data obtain the first and second derivatives of

$y = \log_e^x$ (i) at $x = 500$

(ii) at $x = 550$

(iii) at $x = 520$

x	500	510	520	530	540	550
$y = \log_e^x$	6.2146	6.2344	6.2538	6.2729	6.2916	6.3099

Also calculate the actual values of the derivatives at these points.

Solution Here $h=10$

(i) Choosing the origin at $x_0 = 500$, we apply Newton's Forward difference formula for derivatives.

$$\text{At } x = 500, \quad p = \frac{x - x_0}{h} = \frac{500 - 500}{10} = 0$$

Now the forward difference table is given below.

x	$y = \log_e x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
500	6.2146					
510	6.2344	0.00198				
520	6.2358	0.0194	-0.0004			
530	6.2729	0.0191	-0.0003	0.0001	-0.0002	
540	6.2916	0.0187	-0.0004	-0.0001	0.0001	0.0003
550	6.3099	0.0183	-0.0004	0		

$$\therefore \left(\frac{dy}{dx} \right)_{x=500} = \left(\frac{dy}{dx} \right)_{p=0} = \frac{1}{h} \left\{ \Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 - \frac{1}{4} \Delta^4 y_0 + \dots \right\}$$

$$= \frac{1}{10} \left\{ 0.0198 - \frac{1}{2} (-0.0004) + \frac{1}{3} (0.0001) - \frac{1}{4} (-0.0002) + \frac{1}{5} (0.0003) \right\}$$

$$= \frac{1}{10} \{ 0.0198 + 0.0002 + 0.00003 + 0.00005 + 0.00006 \} = 0.002014$$

$$\left(\frac{d^2 y}{dx^2} \right)_{x=500} = \left(\frac{d^2 y}{dx^2} \right)_{p=0} = \frac{1}{h^2} \left\{ \Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 \right\}$$

$$= \frac{1}{100} \left[-0.0004 - 0.0001 - \frac{11}{12} (0.0002) \right]$$

$$= \frac{1}{100} \{ -0.0004 - 0.0001 - 0.00018 \}$$

$$= -0.00000518$$

(ii) Choosing the origin at $x_n = 550$, we apply Newton's Backward difference formula for derivatives.

$$\text{At } x = 550, \quad p = \frac{x - x_n}{h} = \frac{550 - 550}{10} = 0$$

$$\left(\frac{d\theta}{dt}\right)_{t=0.6} = 3.825 \text{ radians / Second}$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=500} = \left(\frac{dy}{dx}\right)_{p=0}$$

$$= \frac{1}{h} \left\{ \nabla y_5 + \frac{1}{2} \nabla^2 y_5 + \frac{1}{3} \nabla^3 y_5 + \frac{1}{4} \nabla^4 y_5 + \frac{1}{5} \nabla^5 y_5 \right\}$$

$$= \frac{1}{10} \left\{ 0.0183 + \frac{1}{2}(-0.0004) + \frac{1}{4}(0.0001) + \frac{1}{5}(0.0003) \right\}$$

$$= \frac{1}{10} \{ 0.0183 - 0.0002 + 0.000025 + 0.00006 \}$$

$$= \frac{1}{10} \{ 0.018185 \} = \frac{1}{10} \{ 0.01819 \}$$

$$= 0.001819$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x=500} = \left(\frac{d^2 y}{dx^2}\right)_{p=0} = \frac{1}{h^2} \left\{ \nabla^2 y_5 - \nabla^3 y_5 + \frac{11}{12} \nabla^4 y_5 \right\}$$

$$= \frac{1}{100} \left[-0.0004 + -\frac{11}{12}(0.0001) \right]$$

$$= \frac{1}{100} \{ -0.0004 + 0.000092 \}$$

$$= -0.00000308$$

(iii) To find the values of the derivatives at $x=520$, choose the origin at $x_0 = 520$. Since $x = 520$ is nearer to the mid value of x we can use Stirling's formula for derivatives.

By Stirlings formula for derivatives;

$$\left(\frac{dy}{dx}\right)_{p=0} = \frac{1}{h} \left\{ \left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) - \frac{1}{6} \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right) + \frac{1}{30} \left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} \right) \right\}$$

$$= \frac{1}{10} \left\{ \left(\frac{0.0191 + 0.0194}{2} \right) - \frac{1}{6} \left(\frac{-0.0001 + 0.0001}{2} \right) \right\}$$

$$= \frac{1}{10} [0.01925] = 0.001925$$

The central difference table is given below

x	$p = \frac{x-520}{h}$	$y = \log_e x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
500	-2	6.2146	0.0198				
510	-1	6.2344	0.0194	-0.0004	0.0001	-0.0002	
520	0	6.2358	0.0191	-0.0003	0.0001	0.0001	0.0003
530	1	6.2729	0.0187	-0.0004	0		
540	2	6.2916	0.0183				
550	3	6.3099					

$$\left(\frac{d^2 y}{dx^2} \right)_{p=0} = \frac{1}{h^2} \left\{ \Delta^2 y_{-1} - \frac{1}{12} \Delta^4 y_{-2} + \dots \right\}$$

$$= \frac{1}{100} \{-0.0003\}$$

$$= -0.000003$$

Now we calculate the actual values of the derivatives at these points

Since $y = \log_e x$, $\frac{dy}{dx} = \frac{1}{x}$ & $\frac{d^2 y}{dx^2} = \frac{-1}{x^2}$

$$\therefore \left(\frac{dy}{dx} \right)_{x=500} = \frac{1}{500} = 0.002; \left(\frac{d^2 y}{dx^2} \right)_{x=500} = \frac{-1}{500^2}$$

$$= -0.000004$$

$$\left(\frac{dy}{dx} \right)_{x=520} = \frac{1}{520} = 0.001923; \left(\frac{d^2 y}{dx^2} \right)_{x=520} = \frac{-1}{(520)^2} = 0.0000037$$

From the above results we have the following table for comparison of actual and derived values.

x	First Derivative		Second Derivative	
	Actual Value	Calculated Value	Actual Value	Calculated Value
500	.00200	0.002014	-0.000004	-0.00000518
550	0.001818	0.001819	-0.000003	-0.00000308
520	0.001923	0.001925	-0.0000037	-0.00000300

The calculated values of the first derivatives agree very well with the actual values. Hence the error is not significant.

However, the calculated values of the second derivatives differ much from their actual values. Hence the error is significant.

Section 9.4

Maxima and Minima of the Interpolating Polynomials

Since the derivative of a function $y = f(x)$ given by the table values is defined to be the derivative of the interpolating polynomial, the maxima and minima of $f(x)$ can be obtained by equating the first derivatives to zero.

Newton's forward interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots + \dots \rightarrow \textcircled{1}$$

$$\text{where } p = \frac{x-x_0}{h}$$

$$\therefore \frac{dy}{dp} = \Delta y_0 + \frac{(2p-1)}{2} \Delta^2 y_0 + \frac{3p^2 - 6p + 2}{6} \Delta^3 y_0 + \dots$$

For y to be a maximum (or) minimum, $\frac{dy}{dp} = 0$

$$\therefore \Delta y_0 + \left(\frac{2p-1}{2} \right) \Delta^2 y_0 + \left(\frac{3p^2 - 6p + 2}{6} \right) \Delta^3 y_0 = 0 \rightarrow \textcircled{2}$$

(neglecting the higher order differences like

$$\Delta^4 y_0, \Delta^5 y_0, \dots)$$

Substituting in ②, the known values of Δy_0 , $\Delta^2 y_0$ and $\Delta^3 y_0$ from the difference table we get a quadratic equation in p , which can be solved for p .

The corresponding value of x at which $y(x)$ has maximum (or) minimum is given by $x = x_0 + ph$.

Problem 9 Find the maximum and minimum value of y from the following table.

x	0	1	2	3	4	5
y	0	$\frac{1}{4}$	0	$\frac{9}{4}$	16	$\frac{225}{4}$

Solution Here $h = 1$

Newton's forward difference formula is

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0 + \dots$$

$$\left(\frac{dy}{dx}\right)_{x=x_0+h} = \frac{1}{h} \left\{ \Delta y_0 + \left(\frac{2p-1}{2!}\right) \Delta^2 y_0 + \left(\frac{3p^2-6p+2}{3!}\right) \Delta^3 y_0 + \left(\frac{4p^3-18p^2+22p-6}{4!}\right) \Delta^4 y_0 + \dots \right\}$$

Space for Hints

Space for Hints

The forward difference table is given below

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	0	0.25				
1	0.25	-0.25	-0.50	3		
2	0	2.25	2.50	9	6	
3	2.25	13.75	11.50	15	6	0
4	16	40.25	26.50			
5	56.25					

Choosing the origin $x_0 = 0$

$$\therefore p = \frac{x-x_0}{h} = \frac{x-0}{1} = x, \text{ we have}$$

$$\frac{dy}{dx} = 0.25 + \frac{(2p-1)}{2}(-0.5) + \left(\frac{3p^2-6p+2}{6} \right) \times 3$$

$$+ \left(\frac{4p^3-18p^2+22p-6}{24} \right) \times 6 + \dots$$

$$= 4p^3 - 12p^2 + 8p$$

$$\frac{dy}{dx} = 0 \Rightarrow 4p^3 - 12p^2 + 8p = 0$$

$$\Rightarrow p\{4p^2 - 12p + 8\} = 0$$

$$\Rightarrow 4p\{p^2 - 3p + 2\} = 0$$

$$\Rightarrow 4p\{p-1\}(p-2) = 0 \quad (\because 4 \neq 0)$$

$$\Rightarrow p = 0 \quad (\text{or}) \quad p-1=0 \quad (\text{or}) \quad p-2=0$$

$$\Rightarrow p = 0, \quad p = 1, \quad p = 2.$$

$$\text{Also } p = 0, \quad \frac{d^2y}{dx^2} = 8 > 0$$

$$\text{At } p = 1, \quad \frac{d^2y}{dx^2} = 12(1)^2 - 24(1) + 8$$

$$= 20 - 24$$

$$= -4 < 0$$

$$\begin{aligned} \text{At } p = 2, \quad \frac{d^2 y}{dx^2} &= 12(2)^2 - 24 \times 2 + 8 \\ &= 48 - 48 + 8 \\ &= 8 > 0 \end{aligned}$$

Hence y is maximum at $p = 1$ and minimum at $p = 0$ and $p = 2$.

$\therefore$ The Maximum value of y at $p=1$ [ie, $x=1$]

$$\begin{aligned} \text{is } y(1) &= y_0 + \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 \\ &= 0 + 1(0.25) = 0.25 \end{aligned}$$

Minimum value of y at $p=0$ and $p=2$ [ie, $x=0$ and $x=2$] are $y(0)=0$ and $y(2)=0$ respectively

Exercise Problems

1] Obtain an approximate value of $u'(0)$ where $u(x)$ is given by the following table

x	0	1	2	3	4	5
$u(x)$	43	40	38	42	45	50

2] Find y'_0 from the following table.

x	0	1	2	3	4
y	768	913	1039	1143	1225

3] Find the value of $\cos(1.74)$ from the following table.

x	1.70	1.74	1.78	1.82	1.86
$\sin x$	0.9917	0.9857	0.9782	0.9691	0.9585

4] Find $y'(0)$ and $y''(0)$ from the following table

x	0	1	2	3	4	5
y	4	8	15	7	6	2

5] The following table gives the velocity v of a body at time t . Find its acceleration at $t = 1.1$

t	1.0	1.1	1.2	1.3	1.4
v	43.1	47.7	52.1	56.4	60.8

6] Given that

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (i) $x = 1.1$ (ii) $x = 1.6$

7] From the following table find the minimum value of $f(x)$

x	0	1	2	3	4	5
$f(x)$	58	43	40	45	52	60

8] Find the minimum value of $f(x)$ which has the value

x	0	2	4	6
$f(x)$	3	3	11	27

9] The population of a certain town in lakhs is shown in the following table.

Year	1951	1961	1971	1981	1991
Population	19.96	38.65	58.81	77.21	94.61

Find the rate of growth of the population in 1971.

10] The table given below the results of an observation : ϕ is the observed temperature, in Celsius of a vessel of cooling water and t is the time in minutes from the beginning of observation

t	1	3	5	7	9
ϕ	85.3	74.5	67.0	60.5	54.3

Find an approximation of the rate of cooling at $t = 3$ and $t = 5.5$.

UNIT : 10

NUMERICAL INTEGRATION

Section 10.0

Introduction

Space for Hints

We frequently come across integrals whose integrand is an empirical function given by table values. In such cases we can use a numerical method of approximate integration. The problem of numerical integration is solved by representing the integrand by an interpolation formula between the given limits.

Section 10.1

NUMERICAL INTEGRATION

The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$. Which is not known explicitly is called numerical integration. In this chapter we derive the following formula for numerical integration for which the basic formula is Newton-cote's quadrature formula which uses Newton formula for forward interpolation.

- (i) Trapezoidal rule
- (ii) Simpson's one-third rule
- (iii) Simpson's three – eight rule
- (iv) Weddle's rule.

Taking $\frac{x - x_0}{h} = p$ we know that the Newton's forward interpolation formula is

$$f(x) = y = y_0 + \frac{p}{1!} \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \dots + \frac{p(p-1) \dots (p-(n-1))}{n!} \Delta^n$$

[Refer theorem 8.1]

NEWTON – COTE'S QUADRATURE FORMULA

Let $I = \int_a^b f(x)dx$ where $f(x)$ takes the values $y_0, y_1, \dots, y_n$ for

$$x = x_0, x_1, \dots, x_n.$$

Let us divide the interval (a, b) into n subintervals of width h so that

$$x_0 = a, x_1 = x_0 + h, x_2 = x_0 + 2h, \dots, x_n = x_0 + nh = b.$$

Consider $I = \int_a^b f(x)dx$

$$= \int_{x_0}^{x_0+nh} f(x)dx$$

Take $p = \frac{x - x_0}{h}$. Hence $x = x_0 + ph$ and $dx = hdp$.

$$\therefore I = h \int_a^b f(x_0 + ph)dp$$

$$= h \int_0^1 \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \dots \right] dp.$$

(by Newton's forward difference formula)

$$= h \left[py_0 + \frac{p^2}{2} \Delta y_0 + \frac{1}{2} \left(\frac{p^3}{3} - \frac{p^2}{2} \right) \Delta^2 y_0 + \frac{1}{6} \left(\frac{p^4}{4} - p^3 + p^2 \right) \Delta^3 y_0 + \dots \right]_0^1$$

$$= h \left[ny_0 + \frac{n^2}{2} \Delta y_0 + \frac{1}{2} \left(\frac{n^3}{3} - \frac{n^2}{2} \right) \Delta^2 y_0 + \dots \right]$$

$$\therefore \int_{x_0}^{x_0+nh} f(x)dx = h \left[ny_0 + \frac{n^2}{2} \Delta y_0 + \frac{1}{2} \left(\frac{n^3}{3} - \frac{n^2}{2} \right) \Delta^2 y_0 + \dots \right]$$

This is called Newton – cote's quadrature formula. It is a general quadrature formula from which we can derive various special formulae giving different values for n .

Section 10.3

TRAPEZOIDAL RULE

Put $n = 1$ in the quadrature formula. Then all the differences except the first order differences Δy_0 will become zero.

$$\begin{aligned}\int_{x_0}^{x_0+h} f(x)dx &= h \left[y_0 + \frac{1}{2} \Delta y_0 \right] \\ &= h \left[y_0 + \frac{1}{2} (y_1 - y_0) \right] \\ &= \frac{h}{2} (y_0 + y_1)\end{aligned}$$

Similarly, $\int_{x_0}^{x_0+2h} f(x)dx = \frac{h}{2} (y_1 + y_2)$

...

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2} (y_{n-1} + y_n)$$

Adding these n integrals we obtain.

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2} (y_0 + y_1) + \frac{h}{2} (y_1 + y_2) + \cdots + \frac{h}{2} (y_{n-1} + y_n)$$

$$\therefore \int_{x_0}^{x_n} f(x)dx = \frac{h}{2} [(y_0 + y_1) + 2(y_1 + y_2 + \cdots + y_{n-1})]$$

Space for Hints

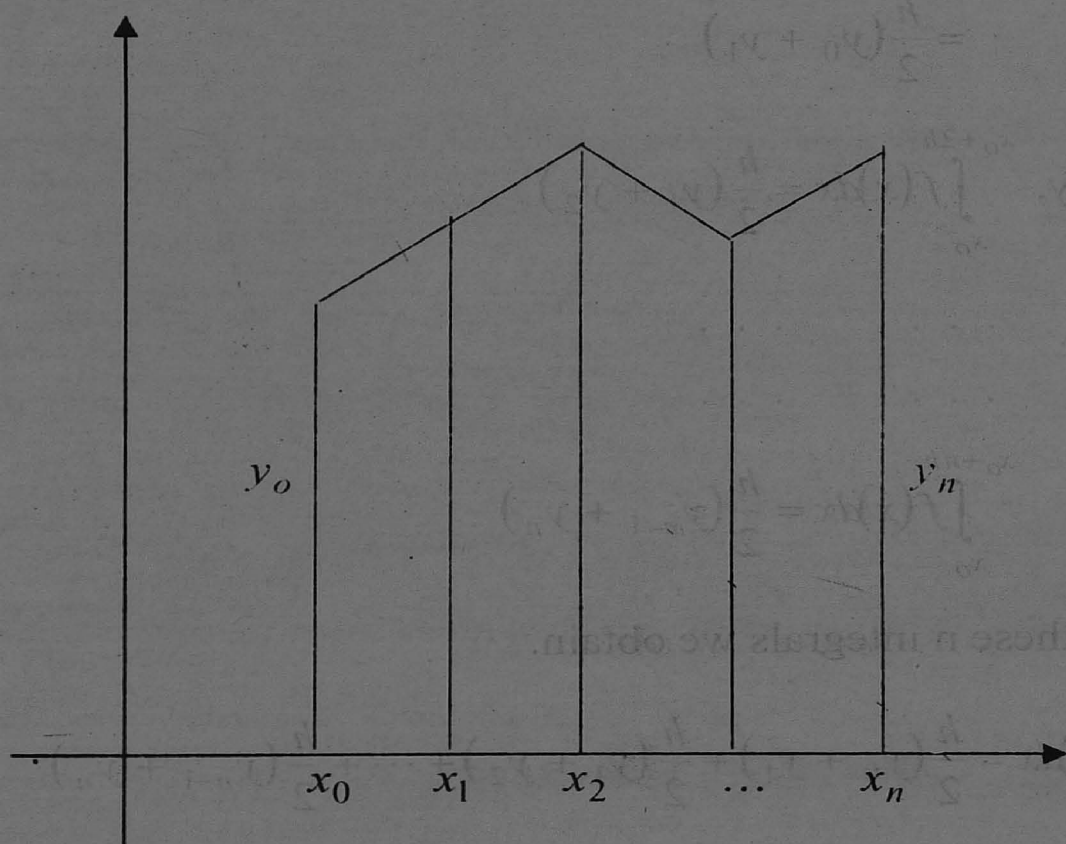
$\therefore \int_{x_0}^{x_n} f(x) dx = \frac{h}{2} \times$ [Sum of the first and last ordinates + 2 (sum of the intermediate ordinates.)]

This is known as Trapezoidal rule.

Section 10.3.1

Geometrical interpretation of Trapezoidal rule

The definite integral $\int_{x_0}^{x_n} f(x) dx$ represents the area bounded by the curve $y = f(x)$, the ordinates $x = x_0$, $x = x_n$ and the x -axis. In Trapezoidal rule the curve $y = f(x)$ is replaced by n straight line segments joining the points (x_0, y_0) and (x_1, y_1) ; (x_1, y_1) and (x_2, y_2) ; $\dots$; (x_{n-1}, y_{n-1}) and (x_n, y_n) as shown in the figure.



Area $\int_{x_0}^{x_n} f(x) dx$ is approximately given by the sum of the areas of the n trapeziums obtained. Hence this method is known as Trapezoidal rule

Note The trapezoidal rule is a very simple formula from calculation point of view of numerical integration. But it is least accurate. The accuracy can be improved by increasing the number of intervals or equivalently by decreasing the value of h .

Section 10.4

SIMPSON'S ONE - THIRD RULE

Put $n = 2$ in Newton Cote's quadrature formula.

Since x takes one of the three values x_0, x_1 , or x_2 all the differences of third and higher order become zero.

$$\int_{x_0}^{x_2} f(x) dx \approx h \left[2y_0 + \frac{1}{2} \Delta y_0 + \frac{1}{2} \left(\frac{8}{3} - \frac{4}{2} \right) \Delta^2 y_0 \right]$$

$$= h \left[2y_0 + 2(y_1 - y_0) + \frac{1}{3} \Delta^2 y_0 \right]$$

$$= h \left[2y_0 + 2(y_1 - y_0) + \frac{1}{3} (E - 1)^2 y_0 \right]$$

$$= h \left[2y_0 + 2y_1 - 2y_0 + \frac{1}{3} (E^2 - 2E + 1) y_0 \right]$$

$$= h \left[2y_1 + \frac{1}{3} (y_2 - 2y_1 + y_0) \right]$$

$$= h \left[\frac{1}{3} y_2 + \frac{4}{3} y_1 + \frac{1}{3} y_0 \right]$$

$$\text{Thus } \int_{x_0}^{x_2} f(x) dx = \frac{h}{3} [y_0 + 4y_1 + y_2]$$

$$\text{Similarly, } \int_{x_2}^{x_4} f(x) dx = \frac{h}{3} [y_2 + 4y_3 + y_4]$$

$$\text{Finally } \int_{x_{n-2}}^{x_n} f(x) dx = \frac{h}{3} [y_{n-2} + 4y_{n-1} + y_n] \text{ (assuming } n \text{ to be even)}$$

Space for Hints

Adding all these integrals, we have

$$\int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)dx = \frac{h}{3} [y_0 + 4y_1 + y_2] \\ + [(y_2 + 4y_3 + y_4) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)]$$

$$\therefore \int_{x_0}^{x_n} f(x)dx = \frac{h}{3} [(y_0 + y_n) + 2(y_2 + y_4 + \cdots) + 4(y_1 + y_3 + \cdots)]$$

This is called Simpson's one third rule (or) simply Simpson's rule.

Section 10.5

SIMPSON'S THREE – EIGHTH RULE

Put $n = 3$ in Newton – Cote's quadrature formula.

Since x takes one of the four values x_0, x_1, x_2 or x_3 all differences of fourth and higher order become zero.

We can prove $\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8} [y_0 + 3y_1 + 3y_2 + y_3]$

Similarly, $\int_{x_3}^{x_6} f(x)dx = \frac{3h}{8} [y_3 + 3y_4 + 3y_5 + y_6]$ etc.

Assuming that n is a multiple of three we get.

$$\int_{x_0}^{x_0+nh} f(x)dx = \int_{x_0}^{x_3} f(x)dx + \int_{x_3}^{x_6} f(x)dx + \cdots + \int_{x_0+(n-3h)}^{x_0+nh} f(x)dx \\ = \frac{3h}{8} [(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \cdots + (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)]$$

$$\therefore \int_{x_0}^{x_n} f(x)dx = \frac{3h}{8} [(y_0 + y_n) + 3(y_1 + y_2 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots)]$$

The above equation is called Simpson's three eight rule.

Section 10.5.1**Error in Trapezoidal rule**

The Taylor's series expansion of $y = f(x)$ about x_0 is given by

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \dots \longrightarrow \textcircled{1}$$

$$\begin{aligned} \therefore \int_{x_0}^{x_1} f(x) dx &= \int_{x_0}^{x_1} \left[f(x) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \dots \right] dx \\ &= \left[x f(x) + \frac{(x-x_0)^2}{2!} f'(x_0) + \frac{(x-x_0)^3}{3!} f''(x_0) + \dots \right]_{x_0}^{x_1} \\ &= (x_1 - x_0) f(x_0) + \frac{(x-x_0)^2}{2!} f'(x_0) + \frac{(x-x_0)^3}{3!} f''(x_0) + \dots \\ &= h y_0 + \frac{h^2}{2!} y'_0 + \frac{h^3}{3!} y''_0 + \dots \longrightarrow \textcircled{2} \end{aligned}$$

where $h = x_1 - x_0$, $f(x) = y_0$, $f'(x_0) = y'_0, \dots$

Now, the area of the trapezium in the first interval (x_0, x_1) is given by

$$A_0 = \int_{x_0}^{x_1} y dx = \frac{h}{2} (y_0 + y_1) \longrightarrow \textcircled{3}$$

Putting $x = x_1$ in $\textcircled{2}$ we get,

$$y_1 = y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \dots \longrightarrow \textcircled{4}$$

Substituting this value of y_1 in $\textcircled{3}$ we get

$$A_0 \approx \frac{h}{2} \left[y_0 + y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \dots \right]$$

$$\therefore A_0 \approx h y_0 + \frac{h^2}{2!} y'_0 + \frac{h^3}{2! \cdot 2} y''_0 + \dots \longrightarrow \textcircled{5}$$

$$\therefore \int_{x_0}^{x_1} y dx - A_0 = h^3 y''_0 \left(\frac{1}{3!} - \frac{1}{2! \cdot 2} \right) + \dots = \frac{-h^3}{12} y''_0 + \dots$$

Space for Hints

$\therefore$ The principal part of error in the interval of $(x_0, x_1) = \frac{-h^3}{12} y_0''$.

Similarly, the principal error in the interval of $(x_1, x_2) = \frac{-h^3}{12} y_1''$.

$$\begin{array}{ccccccc} \dots & & \dots & & \dots & & \dots \\ \dots & & \dots & & \dots & & \dots \end{array}$$

The principal error in the interval of $(x_{n-1}, x_n) = \frac{-h^3}{12} y_{n-1}''$

$\therefore$ The total error E is given by $E \approx \frac{-h^3}{12} (y_0'' + y_1'' + \dots + y_{n-1}'')$

$\therefore |E| < \frac{nh^3}{12} M$ where $M = \max \left\{ |y_0''|, |y_1''|, \dots, |y_{n-1}''| \right\}$

$\therefore |E| < \frac{(b-a)h^2}{12} M \quad \left(\because h = \frac{b-a}{n} \right)$

Hence the error in the Trapezoidal rule is of order h^2 .

Section 10.5.2

Truncation error in Simpson's formula

The Taylor's series expansion of $y = f(x)$ about x_0 is given by

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \dots \rightarrow \textcircled{1}$$

$$\therefore \int_{x_0}^{x_2} f(x) dx = \int_{x_0}^{x_2} \left[f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \dots \right] dx$$

$$= \left[x f(x_0) + \frac{(x-x_0)^2}{2!} f'(x_0) + \frac{(x-x_0)^3}{3!} f''(x_0) + \dots \right]_{x_0}^{x_2}$$

$$= (x_2 - x_0) f(x_0) + \frac{(x_2 - x_0)^2}{2!} f'(x_0) + \frac{(x_2 - x_0)^3}{3!} f''(x_0) + \dots$$

$$= 2hf(x_0) + \frac{4h^2}{2!} f'(x_0) + \frac{8h^3}{3!} f''(x_0) + \frac{16h^4}{4!} f'''(x_0)$$

$$\therefore \int_{x_0}^{x_2} f(x) dx = 2hy_0 + 2h^2 y_0' + \frac{4h^3}{3} y_0'' + \frac{2h^4}{3} y_0''' + \dots \rightarrow \textcircled{2}$$

Also by Simpson's rule

$$\text{Area } A_0 = \int_{x_0}^{x_2} f(x) dx = \frac{h}{3!} [y_0 + 4y_1 + y_2] \rightarrow \textcircled{3}$$

Putting $x = x_1$ in $\textcircled{1}$ we have

$$\begin{aligned} y_1 &= y_0 + \frac{(x_1 - x_0)}{1!} y_0' + \frac{(x_1 - x_0)^2}{2!} y_0'' + \dots \\ &= y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \dots \rightarrow \textcircled{4} \end{aligned}$$

Putting $x = x_2$ in $\textcircled{1}$ we have

$$y_2 = y_0 + \frac{2h}{1!} y_0' + \frac{4h^2}{2!} y_0'' + \dots \rightarrow \textcircled{5}$$

$$A \approx \frac{h}{3} \left[y_0 + 4 \left(y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \dots \right) + \left(y_0 + \frac{2h}{1!} y_0' + \frac{4h^2}{2!} y_0'' + \dots \right) \right]$$

$$\therefore A_0 \approx 2hy_0 + 2h^2 y_0' + \frac{4h^3}{3} y_0'' + \frac{2h^4}{3} y_0''' + \frac{5h^5}{18} y_0'''' + \dots \rightarrow \textcircled{6}$$

From $\textcircled{2}$ and $\textcircled{6}$ we get,

$$\therefore \int_{x_0}^{x_2} y dx - A_0 = \left(\frac{4}{15} - \frac{5}{18} \right) h^5 y_0'''' + \dots$$

$$= \frac{-h^5}{90} y_0'''' + \dots$$

$$\therefore \text{The principal part of error in the interval of } (x_0, x_2) = \frac{-h^5}{90} y_0''''$$

$$\text{Similarly, the principal error in the interval of } (x_2, x_4) = \frac{-h^5}{90} y_2''''$$

$\therefore$ The total error in all the intervals is given by

$$E \approx \frac{-h^5}{90} (y_0'''' + y_2'''' + \dots)$$

Space for Hints

$$\therefore |E| < \frac{nh^5}{90} M \text{ where } M = \max \{y_0''', y_2''', y_4''', \dots\}$$

$$\therefore |E| < \frac{(b-a)h^4}{180} M \quad (\because 2nh = b-a)$$

Hence the error in Simpson's one third rule is of order h^4 .

Note

Proceeding as Simpson's one – third rule, it can be proved that the

principal part of the error in (x_0, x_3) is $\frac{-3}{80} h^5 y''''$

Solved Problems

Problem 1

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule with $h = 0.2$. Hence

determine the value of π .

Solution

Here $h = 0.2$.

$$\text{Let } y = \frac{1}{1+x^2}.$$

The interval is $[0,1]$. The values of y for each pint of subdivision $h = 0.2$ are got and tabulated as in the following table.

For example, when $x = 0.2$ we get $y = \frac{1}{1+(0.2)^2} = \frac{1}{1.04} = 0.9615$

x	0	0.2	0.4	0.6	0.8	1.0
$y = f(x)$	1	0.9615	0.8621	0.7353	0.6098	0.5

By Trapezoidal rule

$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{2!} [(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)]$$

$$\therefore \int_0^1 \frac{dx}{1+x^2} = \frac{0.2}{2} [(1+0.5) + 2(0.9615+0.8621) + 0.7353 + 0.6098] \\ = 0.7837 \longrightarrow \textcircled{1}$$

To find the value of $\bar{\Delta}$

By actual integration

$$\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \bar{\Delta}/4 \longrightarrow \textcircled{2}$$

from $\textcircled{1}$ & $\textcircled{2}$ we get

$$\bar{\Delta}/4 = 3.1348.$$

$$\therefore \bar{\Delta} = 3.1348.$$

Problem 2

Evaluate $\int_0^5 \frac{dx}{4x+5}$ by Trapezoidal rule using 11 coordinates.

Solution

Here the length of interval is 5. Take $h = 0.5$.

Let $y = \frac{1}{4x+5}$. The values of y for each point of subdivision are as follows.

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
y	.20	.1429	.1111	.0909	.0769	.0667	.0588	.0526	.0476	.0435	.04

$$\text{By Trapezoidal rule : } \int_0^5 \frac{dx}{4x+5} = \frac{h}{2} [(y_0 + y_{10}) + 2(y_1 + y_2 + \dots + y_9)]$$

$$\int_0^5 \frac{dx}{4x+5} = \frac{0.5}{2} [(0.2 + 0.04) + 2(.1429 + .1111 + .0909 + .0769 + 0.0667 \\ + 0.0588 + .0526 + .0476 + .0435)]$$

Space for Hints

$$= 0.25[0.24 + 2(0.691)]$$

$$= 0.25 (1.622)$$

$$= 0.4055$$

Problem 3

Evaluate $\int_0^1 e^{-x^2} dx$ by dividing the range into four equal parts using

Trapezoidal rule

Solution

Let $y = e^{-x^2}$. Take $h = 0.25$.

The values of y for each point of subdivision are as follows.

x	0	0.25	0.5	0.75	1.0
y	1	.9394	.7788	.5698	.3679

By Trapezoidal rule

$$\int_0^1 e^{-x^2} dx = \frac{h}{2} [(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.25}{2} [1.3679 + 2(2.288)]$$

$$= 0.7430.$$

Problem 4

Evaluate $\int_0^{\frac{\pi}{2}} \sin x dx$ by Simpson's $\frac{1}{3}$ rule dividing the range into six equal parts.

Solution

Here the length of interval is $\frac{\pi}{2}$. We subdivide this interval into Six

equal parts with $h = \frac{\frac{\pi}{2}}{6}$.

Let $y = \sin x$. The values of y at each point of subdivision with

$h = \frac{\pi}{12}$ are as follows:

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$y = \sin x$	0	.2588	.500	.7071	.8660	.9659	1.00

By Simpson's one third rule

$$\int_0^{\pi/2} \sin x \, dx = \frac{h}{3} [(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$= \frac{\pi}{36} [(0 + 1) + 2(0.5 + .866) + 4(.2588 + .7071 + .9659)]$$

$$= 0.0873 [1 + 2(1.366) + 4(1.9318)]$$

$$= 1.0004 \text{ (approximately.)}$$

Problem 5

Calculate $\int_{0.5}^{0.7} e^{-x} x^2 \, dx$ taking 5 ordinates by Simpson's $\frac{1}{3}$ rule.

Solution

Let $y = e^{-x} x^2$. Length of the interval is 0.2.

Take $h = 0.04$ the values of y for each point of subdivision are tabulated as follows:

x	0.5	0.54	0.58	0.62	0.66	0.7
y	.4289	.4282	.4264	.4236	.4199	.4155

Simpson's one third rule is

$$\int_{0.5}^{0.7} e^{-x} x^2 \, dx = \frac{h}{3} [(y_0 + y_5) + 2(y_2 + y_4) + 4(y_1 + y_3)]$$

$$= \frac{0.04}{3} [(0.4289 + 0.4155) + 2(0.4264 + 0.4199) + 4(0.4282 + 0.4236)]$$

Space for Hints

$= 0.0793.$

Problem 6

Find the value of $\log 2^3$ from $\int_0^1 \frac{x^2}{1+x^3} dx$ using Simpson's $\frac{1}{3}$ rule with

$h = 0.25.$

Solution

Let $y = \frac{x^2}{1+x^3}$. The values of y corresponding to the subdivision point are as follows.

x	0	0.25	0.5	0.75	1
y	0	0.0615	0.2222	0.3956	0.5

By Simpson's one third rule,

$$\begin{aligned} \int_0^1 \frac{x^2}{1+x^3} dx &= \frac{h}{3} [(y_0 + y_4) + 2y_2 + 4(y_1 + y_3)] \\ &= \frac{0.25}{3} [0.5 + 2(0.2222) + 4(0.0615 + 0.3956)] \\ &= 0.2311 \longrightarrow \textcircled{1} \end{aligned}$$

By actual integration,

$$\begin{aligned} \int_0^1 \frac{x^2}{1+x^3} dx &= \frac{1}{3} \int_0^1 \frac{3x^2}{1+x^3} dx \\ &= \frac{1}{3} [\log(1+x^3)]_0^1 \end{aligned}$$

$$= \log 2^{\frac{1}{3}} \longrightarrow \textcircled{2}$$

$\therefore$ By $\textcircled{1}$ and $\textcircled{2}$ $\log 2^{\frac{1}{3}} = 0.2311.$

Problem 7

Evaluate $\int_0^{10} \frac{dx}{1+x^2}$ by using

(i) Trapezoidal rule (ii) Simpson one third rule.

Solution

Here length of interval is 10. Take $h=1$.

Let $y = \frac{1}{1+x^2}$. The values of y for each point of subdivision with $h=1$ are as follows.

x	0	1	2	3	4	5	6	7	8	9	10
y	1	0.5	0.2	0.1	0.0588	0.0385	0.0270	0.2	.0154	.0122	.0099

(i) Trapezoidal rule

$$\begin{aligned} \int_0^{10} \frac{dx}{1+x^2} &= \frac{1}{2} [(y_0 + y_{10}) + 2(y_1 + y_2 + \dots + y_9)] \\ &= \frac{1}{2} [(1 + 0.0099) + 2(.5 + .2 + .1 + .0588 + .0385 + .027 + .02 + .0154 + .0122)] \end{aligned}$$

$$= \frac{1}{2} [1.0099 + 2(0.9719)]$$

$$= \frac{1}{2} [2.9537]$$

$$= 1.4769$$

(ii) Simpson one third rule

$$\int_0^{10} \frac{dx}{1+x^2} = \frac{h}{3} [(y_0 + y_{10}) + 2(y_2 + y_4 + y_6 + y_8) + 4(y_1 + y_3 + y_5 + y_7 + y_9)]$$

$$= \frac{1}{3} [(1 + 0.0099) + 2(.2 + .0588 + .027 + .0154) + 4(.5 + .1 + .0385 + .02 + .0122)]$$

$$= \frac{1}{3} [4.2951]$$

$$= 1.4317.$$

Space for Hints

Problem 8

The velocity v of a particle at distance s from a point on its path is given by the table below.

S in metres	0	10	20	30	40	50	60
V in mlsec	47	58	64	65	61	52	38

Estimate the time taken to travel 60 metres by using Simpson's $\frac{1}{3}$ rule.

Solution

Here $h = 10$. We know that $v = \frac{ds}{dt}$ Hence $dt = \frac{ds}{v}$

To find the time taken to travel 60 metres we have to evaluate

$$\int_0^{60} dt = \int_0^{60} \frac{ds}{v}$$

Let $y = \frac{1}{v}$. The table values for y for different values of s are given

below.

S	0	10	20	30	40	50	60
$y = \frac{1}{v}$	.0213	.0172	.0156	.0154	.0164	.0192	.0263

By Simpson's one third rule,

$$\int_0^{60} y ds = \frac{h}{3} [(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

ie.

$$\begin{aligned} \int_0^{60} ds &= \frac{10}{3} [(0.0213 + 0.263) + 2(.0156 + .0164) + 4(.0172 + .0154 + .0192)] \\ &= 1.0627. \end{aligned}$$

$\therefore$ Time taken to travel 60 metres = 1.0627 seconds.

Problem 9

Space for Hints

A curve passes through the points as given in the table.

Find (i) the area bounded by the curve, the x -axis and $x = 1$ to 9.

(ii) the volume of the solid generated by revolving this area about the x -axis.

x	1	2	3	4	5	6	7	8	9
y	.2	.7	1	1.3	1.5	1.7	1.9	2.1	2.3

Solution

(i) Here $h = 1$. Area $A = \int_1^9 y dx$

We use Simpson's one third rule. (Since the number of ordinates is 9 (odd))

$$\begin{aligned}\int_1^9 y dx &= \frac{h}{3} [(y_0 + y_8) + 2(y_2 + y_4 + y_6) + 4(y_1 + y_3 + y_5 + y_7)] \\ &= \frac{h}{3} [(0.2 + 2.3) + 2(1 + 1.5 + 1.9) + 4(0.7 + 1.3 + 1.7 + 2.1)] \\ &= 11.5 \text{ Sq. units}\end{aligned}$$

(ii) Volume $v = \int_1^9 y^2 dx$

we find $\int_1^9 y^2 dx$ using Simpson's one third rule.

$$\therefore \int_1^9 y^2 dx =$$

$$\frac{1}{3} [(0.2^2 + 2.3^2) + 2(1^2 + 1.5^2 + 1.9^2) + 4(0.7^2 + 1.3^2 + 1.7^2 + 2.1^2)]$$

$$= \frac{1}{3} [5.33 + 13.72 + 37.92]$$

$$= \frac{1}{3} (56.97)$$

$$= 18.99.$$

$\therefore$ the required volume $v = \bar{\Delta}(18.99)$

$$= 59.6588 \text{ cubic units.}$$

Problem 10

Evaluate $\int_0^1 \frac{dx}{1+x}$ using (i) Trapezoidal rule (ii) Simpson's one third rule.

(iii) Simpson's three eight rule (iv) Find the error in each method by comparing with the actual integration upto 4 places of decimals. Take

$$h = \frac{1}{6} \text{ for all cases.}$$

Solution

Here $h = \frac{1}{6}$. Let $y = f(x) = \frac{1}{1+x}$. The values of $f(x)$ for the points of

subdivisions are as follows:

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y = \frac{1}{1+x}$	1	.8571	.75	.6667	.6	.5455	.5

(i) Trapezoidal rule

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{3} [(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$\approx \frac{1}{12} [(1 + 0.5) + 2(.8571 + .75 + .6667 + .6 + .5455)]$$

$$= 0.6949.$$

(ii) Simpson's one third rule

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{2} [(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$\approx \frac{1}{18} [(1 + 0.5) + 2(.75 + .6) + 4(.8571 + .6667 + .5455)]$$

$$= 0.6932.$$

(iii) Simpson's three eight rule

$$\int_0^1 \frac{dx}{1+x} = \frac{3h}{8} [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$\approx \frac{1}{16} [(1 + .5) + 3(.8571 + .75 + .6 + .5455) + 2(.6667)]$$

$$= 0.6932.$$

(iv) By actual integration

$$\int_0^1 \frac{dx}{1+x} = [\log_e (1+x)]_0^1 = \log_e 2 = 0.6931 \text{ (upto 4 decimal places).}$$

Comparing (i) and (v) error in Trapezoidal rule is

$$0.6931 - 0.6949 = -0.0018.$$

Comparing (iii) and (v) error in Simpson's $\frac{3}{8}$ rule is

$$0.6931 - 0.6935 = -0.0001.$$

Exercises

(1) Using Trapezoidal rule evaluate $\int_{0.6}^2 f(x)dx$ from the following tables

x	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$	1.23	1.58	2.03	4.32	6.25	8.36	10.23	12.45

2. A river is 80 metres wide. The depth d in metres at a distance x from one bank is given by the following table. Calculate the area of cross section of the river using Simpson's rule.

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3

3. Evaluate numerically $\int_0^1 e^{-x} dx$ in steps of 0.1 by trapezoidal rule.

4. Evaluate $\int_{1.0}^{1.3} \sqrt{x} dx$ taking $h = 0.05$ by Trapezoidal rule.

5. Using Trapezoidal rule evaluate $\int_0^3 \tan x \, dx$ taking 8 intervals.
6. Apply the trapezoidal rule to obtain a value of the integral $\int_4^{6.2} \log x \, dx$ taking $h = 0.2$.
7. The following table gives the values of $f(x)$ at equal intervals of x

x	0	0.5	1.0	1.5	2.0
$f(x)$	0.399	0.352	0.242	0.129	0.054

Evaluate $\int_0^2 f(x) dx$ using Simpson's rule.

8. Apply Simpson's rule to find the value of $\int_0^2 \frac{dx}{1+x^2}$ dividing the interval $[0,2]$ into 4 equal parts.

9. Evaluate $\int_0^2 \frac{dx}{1+x^2}$ to three decimals dividing the range of integration into 8 equal parts using Simpson's rule.

10. A curve is drawn to pass through the points given by the following table

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	2	2.4	2.7	2.8	3	2.6	2.1

using Simpson's rule find (i) the area bounded by the curve, the x -axis and the lines $x = 1$, $x = 4$ (ii) the volume of revaluation of the area about the x -axis.

11. The following table gives the velocity v of a particle at time t .

t (seconds)	0	2	4	6	8	10	12
V metres per second	4	6	16	34	60	94	136

Find (i) the distance moved by the particle in 12 seconds.

12. Find the value of $\log 2^3$ from $\int_0^1 \frac{x^2}{1+x^3} dx$ using Simpson's $\frac{1}{3}$ rule taking $h = 0.25$.

13. Evaluate $\int_0^1 e^{-x^2} dx$ dividing the range into 4 equal parts by

(i) Trapezoidal rule (ii) Simpson's rule.

14. The following table gives the values of $f(x)$ at equal intervals of x .

x	0.0	0.5	1.0	1.5	2.0
$f(x)$	0.399	0.352	0.242	0.129	0.054

Evaluate $\int_0^2 f(x) dx$ using (i) Simpson's rule (ii) Trapezoidal rule and

Compare the two results.

15. The velocity v of a particle at distance s from a point on its path is given by the table below

S in metre	0	10	20	30	40	50	60
V in metre per sec	47	58	64	65	61	52	38

Estimate the time taken to travel 60 metres by using Simpson's $\frac{3}{8}$ rule.

$$\text{Hint: } v = \frac{ds}{dt} \Rightarrow t = \int_0^{60} \frac{ds}{v}$$

SECTION 10.6

WEDDLCY'S RULE

Put $n = 6$ in Newton - Cote's quadrature formula Neglecting all differences of order higher than Sixth, we get

Space for Hints

$$\int_{x_0}^{x_0+6h} f(x)dx = h \left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0 \right]$$

We now replace the term $\frac{41}{140}\Delta^6 y_0 + \frac{41}{140}\Delta^6 y_0 = \frac{3}{10}\Delta^6 y_0$.

As a result of this adjustment the error in this $\frac{h\Delta^6 y_0}{140}$ which is very

small and negligible when h and $\Delta^6 y_0$ are small.

Using $E - 1 = \Delta$ and replacing all differences in terms of y' we get

$$\int_{x_0}^{x_0+6h} f(x)dx = \frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + y_5 + y_6]$$

$$\text{Similarly, } \int_{x_0+6h}^{x_0+12h} f(x)dx = \frac{3h}{10} [y_6 + 5y_7 + y_8 + y_9 + y_{10} + 5y_{11} + y_{12}]$$

$\vdots$

$\vdots$

$\vdots$

$$\int_{x_0+(n-6)h}^{x_0+6h} f(x)dx = \frac{3h}{10} [y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n]$$

(assuming n is a multiple of 6)

Adding all these integrals we get,

$$\int_{x_0}^{x_0+6h} f(x)dx = \frac{3h}{10} \left[(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5) + (2y_6 + 5y_7 + y_8 + y_9 + y_{10} + 5y_{11}) \right] + \dots$$

This equation is called Weddle's rule.

Solved Problems

Problem 1

Evaluate $\int_0^1 \frac{dx}{1+x}$ using weddle's rule.

Solution

Here $h = \frac{1}{6}$. Let $y = f(x) = \frac{1}{1+x}$.

The values of $f(x)$ for the points of subdivision are as follows.

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y = \frac{1}{1+x}$	1	.8571	.75	.6667	.6	.5455	.5

Space for Hints

By Weddle's rule.

$$\int_0^1 \frac{1}{1+x} dx = \frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$\approx \frac{1}{20} [1 + 5(.8571) + .75 + 6(.6667) + 0.6 + 5(.5455) + 0.5]$$

$$= 0.69320.$$

Problem 2

Evaluate $I = \int_0^6 \frac{dx}{1+x}$ by taking $h = 1$, by using

- Trapezoidal Rule
- Simpson's Rule
- Weddle's Rule. Also check up by direct integration.

Solution Take the no. of intervals as 6

$$\therefore h = \frac{6-0}{6} = \frac{6}{6} = 1$$

x	0	1	2	3	4	5	6
$y = \frac{1}{1+x}$	1	0.5	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$

(i) By Trapezoidal Rule,

$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$$

Here $x_0 = 0$, $x_n = x_6$ ($n = 6$)

$$\int_0^6 \frac{dx}{x+1} = \frac{1}{2} \{ (y_0 + y_6) + 2[y_1 + y_2 + y_3 + y_4 + y_5] \}$$

$$= \frac{1}{2} \left\{ \left(1 + \frac{1}{7} \right) + 2 \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \right) \right\}$$

$$= 2.02142857$$

(ii) By Simpson's one – third Rule,

$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{3} [(y_0 + y_n) + 2(y_2 + y_4 + \dots) + 4(y_1 + y_3 + \dots)]$$

$$\therefore \int_0^6 \frac{dx}{x+1} = \frac{1}{3} [(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$= \frac{1}{3} \left[\left(1 + \frac{1}{7} \right) + 2 \left(\frac{1}{3} + \frac{1}{5} \right) + 4 \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} \right) \right]$$

$$= \frac{1}{3} \left[1 + \frac{1}{7} + \frac{16}{15} + \frac{22}{6} \right] = 1.95873016$$

(iii) By Simpson's three – eight's Rule:

$$\int_{x_0}^{x_n} f(x) dx = \frac{3h}{8} \left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \dots) \right]$$

$$\therefore \int_0^6 \frac{dx}{x+1} = \frac{3 \times 1}{8} \{ (y_0 + y_6) + 3[y_1 + y_2 + y_4 + y_5] + 2(y_3) \}$$

$$= \frac{3}{8} \left\{ \left(1 + \frac{1}{7} \right) + 3 \left[\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} \right] + 2 \times \frac{1}{4} \right\}$$

$$= 1.96607143$$

(iv) By Weddle's Rule

$$\int_{x_0}^{x_0+nh} f(x) dx = \frac{3h}{10} \left\{ (y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5) + 2(2y_6 + 5y_7 + y_8 + y_9 + \dots) + 2y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n \right\}$$

$$\therefore \int_0^6 \frac{dx}{x+1} = \frac{3 \times 1}{10} [(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)]$$

$$= \frac{3}{10} \left[1 + 5(.5) + \frac{1}{3} + 6\left(\frac{1}{4}\right) + \frac{1}{5} + 5\left(\frac{1}{6}\right) + \frac{1}{7} \right]$$

$$= 1.95285714.$$

(v) By Actual Integration

$$\begin{aligned}\therefore \int_0^6 \frac{dx}{1+x} &= [\log(1+x)]_0^6 = \log_e^7 - \log_e^1 \\ &= \log_e^7 \\ &= 1.94591015\end{aligned}$$

Problem 3 Using Weddle's Rule find $\int_1^7 y dx$ for the function tabulated

below

x	1	2	3	4	5	6	7
y	3.95	4.07	4.18	4.30	4.42	4.54	4.67

Solution Here $b - 1 = 7 - 1 = 6$

$$\therefore h = \frac{7-1}{6} = 1$$

$\therefore$ By Weddle's Rule

$$\int_{x_0}^{x_0+nh} y dx = \frac{3h}{10} \left\{ [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5] + (2y_6 + 5y_7 + y_8 + y_9 + y_{10} + y_{11} + \dots + y_n) \right\}$$

Here $x_0 = 1$, $n = 6$, $h = 1 \therefore x_0 + nh = 1 + 6 = 7$

$$\begin{aligned}\int_1^7 y dx &= \frac{3 \times 1}{10} \{y_1 + 5y_2 + y_3 + 6y_4 + y_5 + 5y_6 + y_7\} \\ &= \frac{3}{10} \{3.95 + 5(4.07) + 4.18 + 6(4.3) + 4.42 + 5(4.54) + 4.67\} \\ &= (.30) \{86.07\} \\ &= 25.821\end{aligned}$$

Exercise Problems

1] Using Weddle's rule find $\int_1^2 \frac{dx}{\sqrt{1-x^2}}$ taking $h = .1$ Compare with the actual value and get one numeric difference between them.

Space for Hints

