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PAPER - III

TOPOLOGY AND MEASURE THEORY

PART - II

UNITS - (6 to 10)

MEASURE THEORY

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UNIT - 6

MEASURE AND INTEGRATION

LEBESGUE MEASURE : (6.1)

Introduction

We are all familiar with lengths of line segments in one dimension, areas of plane regions in two dimensions and volumes of bodies in three dimension. They are all functions from intervals/ regions/ bodies to real numbers. That is, length is an example of a set function which associates an extended real number to each interval in a collection of interval. Each open set in R (the set of all reals) is composed of open interval and hence we can define the 'length' of an openset to be the sum of the lengths of the open intevals of which it is composed.

We wish to extend the defintion to each set in some collection $\mathcal{M}$ of sets of reals. Ideally, we would like to have a set function m which assigns to each set E in $\mathcal{M}$ a non-negative extended real number mE (called the measure of E) with the following properties.

(i) m is defined for each set E of real numbers.

i.e $\mathcal{M} = P(R)$

(ii) For an internal I, $mI = L(I)$ (the length of (I)

(iii) If E_n is a sequence of disjoint sets for which

m is defined, $[m(UE_n) = \Sigma mE_n$.

(iv) m is translation invariant; (i.e) if E is a set for which m is defined and if $E+Y = \{x+y;\ X \in E\}$ obtained by replacing each point $X \in E$ by the point $x+y$, then $m(E+y) = mE$.

Later we shall see that it is impossible to construct a set function having all four of these properties. If we assume the continum hypothesis (that every non-Countable set of real numbers can be put in one-to-one correspondence with the set of all real numbers) then it is not possible to construct a set function satisfying the first three properties. Consequently one of these properties must be weakned and it is most useful to retain the last three properties and to weaken the first condition so that mE need not be defined for all sets E of real numbers. We shall want mE to be defined for as many sets as possible and will find it convenient to require the family $\mathcal{M}$ of sets for which m is defined to be a $\sigma-$ algebra.

(A collection $\mathscr{M}$ of sets is called a $\sigma-$algebra or a Borel field if every field if every union of a countable collection of sets in $\mathscr{M}$ is again in $\mathscr{M}$ and if $A \in M$, then $\tilde{A} \in M$).

We can also weaken property (iii) i.e., we may replace the requirement of countable addivity by the weaker property of finite additivity (or) by countable sub additivity. The Lebesgue outer measure that is developed in the next section, satisfies the properties (i), (ii), (iv) and countable sub additivity we restrict this outer measure to a suitable collection of algebra of sets called measurable sets and get a lebesgue measure. We show that there exists a non-measurable set. We introduce measurable functions.

After studying in detail measurable sets, measurable functions and convergence you may be surprised to see that every measurable set is nearly a finite union of intervals every measurable function is nearly continuous; every convergent sequence of measurable functions is nearly uniformly convergent.

We start with the definition of outer measure.

Outer Measure :-

Definition :- (6.2)

Let A be a set of real numbers. The outer measure $m^*(A)$ of A is defined as

$$m^*(A) = \inf_{A \subset \cup I_n} \Sigma \ell (I_n) \text{ where } \ell (I_n) \text{ denotes the length of } I_n.$$

Note :- (1)

Given a set A of real numbers we can cover A by a countable collection of open intervals

since $R = \bigcup_{n=1}^{\infty} (-n, n)$. Since lengths of the intervals are positive numbers the sum $\Sigma \ell (I_n)$ is uniquely defined independently of the order of the terms.

Note :- (2)

m^* is a function $P(R) \to R \cup \{\infty\}$.

Note :- (3)

$m^* (A) \geq 0 \ \forall A$.

Since $m^* (A) = \inf_{A \subset \cup I_n} \Sigma \ell (I_n)$ given $\in > 0$, we get that there exist.

In such that $A \subset \cup I_n$ and $\Sigma l (I_n) < m^* (A) + \in$.

Note :- (4)

This outer measure is called Lebesgue outer measure after Henri: Lebesgue.

The following results are immediate from the definition of outer measure.

Result :

(i) If $A \subset (a,b)$ then $m^*(A) \leq (b-a)$.

(ii) $A \subset B \Rightarrow m^*(A) \leq m^*(B)$

(iii) $m^*\left[\{x\}\right] = 0$ where x is a real number.

(iv) $m^* \phi = 0$.

Theorem :- (i)

The outer measure of an interval is its length.

Proof :

Case (i)

Let I be a finite closed interval say [a, b]

Since $I \subset (a - \in, b + \in)$.

$m^*(I) \leq b\text{-}a + 2\in$.

This is true for every $\in$.

$\therefore m^*(I) \leq (b\text{-}a)$ .... (1)

Let $I \subset \bigcup\limits_{1}^{\infty} I_n$ by Heine - Borel thm, there exists a finite sub collection from $\{I_n\}$, say $I_1, I_2 \ldots\ldots I_m$ which covers I.

And $\sum\limits_{k=1}^{m} \ell(I_k) \leq \sum\limits_{n=1}^{\infty} \ell(I_n)$

Since $I = [a,b] \subset I_1 \cup \ldots\ldots \cup I_m$.
we get $a \in I_\ell$ for some l, $1 \leq l \leq m$.

Let I_ℓ be (a_1, b_1).

Then $a_1 < a < b_1$.

If $b < b_1$ then $a_1 < a \;\; b < b_1$.

$\therefore \ell(I_\ell) \geq (b\text{-}a)$.

$$\therefore \sum_{1}^{\infty} \ell(I_n) \geq \ell(I_\ell) \geq (b-a).$$

If $b_1 \leq b$ then $b_1 \in [a,b]$.

But $b_1 \notin I_\ell$.

$\therefore b_1 \in I_t$ for some t, $1 \leq t \leq m$.

Let $I_t = (a_2, b_2)$.

$\therefore a_2 < b_1 < b_2$.

If $b_2 \leq b_1$, continuing in this fashion we obtain a sequence (a_1, b_1) .... (a_k, b_k) from $\{I_n\}_{n=1}^{m}$ such that $a_i < b_{i-1}$ b_i.

Since $\{I_n\}_{n=1}^{m}$ is a finite collection, this process must terminate with some interval say (a_r, b_r).

But it terminates only if $b \in (a_r, b_r)$.

i.e. $a_r < b < b_r$.

Thus $\sum_{i=1}^{r} \ell\ (a_i, b_i) = (b_r - a_r) + ... + (b_1 - a_1)$

$$= b_r - (a_r - b_{r-1}) - (a_{r-1} - b_{r-2}) .... (a_2 - b_1) - a_1$$

$$> b_r - a_1 \quad \text{Since } a_i < b_{i-1}.$$

But $b_r > b$ and $a_1 < a$.

We have,

$$b_r - a_1 > b - a.$$

Hence $\sum_{i=1}^{r} \ell\ (a_i, b_i) > (b - a)$

$$\therefore \sum_{i=1}^{\infty} l\ (I) > (b - a)$$

$$\therefore \inf \sum_{i=1}^{\infty} l\ (I) \geq (b - a)$$

$\therefore m^* (I) \geq (b - a)$ .... (2)

From (1) & (2) we get,

$m^* (I) = (b - a)$.

Case (ii) :-

Let I be any finite interval.

$\therefore I = [a, b)$ or (a, b) or $(a, b]$

Then there exists $J = [a + \epsilon/4, \ b - \epsilon/4]$

Such that $J \subset I$.

But $\ell (J) = b - a \ \epsilon/2$ (by case (i))

$= \ell (I) - \epsilon/2$.

Then for each $\epsilon > 0$.

We have,

$$\ell (I) - \epsilon < \ell (J) = m^* (J) \leq m^*(I)$$

$$\leq m^* (\overline{I}) = \ell (\overline{I}) = \ell (I)$$

Where $\overline{I} = [a, b]$.

$$\therefore \ell (I) - \epsilon < m^* (I) \leq l (I)$$

This is true for every $\epsilon > 0$.

$\therefore m^* (I) = \ell (I)$.

Case (iii) :-

Let I be an infinite interval. Then given any real number Δ, there is a closed interval $J \subset I$ with

$\ell (J) = \Delta$.

Hence $m^* (I) \geq m^* (J) = \ell(J) = \Delta$.

Since $m^* (I) \geq \Delta$ for each Δ.

we get $m^* (I) = \infty = \ell(I)$.

Hence the theorem.

Theorem :- (ii)

Let $\left\{A_n\right\}_{n=1}^{\infty}$ be a countable collection of sets of real numbers.

Then $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(An).$

Proof :-

Case (i) :-

Suppose $m^*(A_n) = \infty$ for some n.

Then $\sum_{n=1}^{\infty} m^*(A_n) = \infty.$

And hence the inequality $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n)$ obviously follows.

Case (ii) :-

Let $m^*(A_n) < \infty$.

Given $\in > 0$, there exist countable collection $\left\{I_{n,i}\right\}_i$ of open intervals such that

$A_n \subset U_i \left(I_{n,i}\right)$ and $\Sigma \ell\, I_{n,i} \leq m^*\left(A_n\right) + 2^{-n} \in$

By the definition of m*

$m^*(A_n) = \inf_{A_n \subset UJ_K} \Sigma \ell(J_K)$ (by note :- (3))

Since union of a countable number of countable collections is countable.

We g . that $\left\{I_{n,i}\right\}_{n,i}$ is Countable and $UA_n \subset \bigcup_{n,i} I_{n,i}$.

Thus $m^*(U A_n) \leq \sum_{n,i} \ell(I_{n,i}) = \sum_n \sum_i \ell(I_{n,i}).$

$< \sum_n m^*(A_n) + 2^{-n} \in)$

$= \sum_n m^*(A_n) + \in \sum_n 2^{-n}$

$$= \sum_n m^*(A_n) + \epsilon \quad (\because \sum_n 2^{-n} = 1)$$

Since ϵ was an arbitrary positive number.

Hence $m^*(UA_n) \leq \Sigma m^*(A_n)$.

Corollary :-

If A is countable, $m^* A = 0$.

Proof :

Let $A = \{x_1, x_2, \ldots\ldots x_n, \ldots\ldots\}$

$\therefore A = \bigcup_i \{x_i\}$

$\therefore m^*A = m^* \bigcup_i \{x_i\} \leq \Sigma\, m^* \{x_i\} = 0$ by the above result.

$\therefore m^*A = 0$ by Result (iii)

Corollary :-

The set $[0, 1]$ is not countable.

Proof :

By thm (i)

$m^* [0, 1] = 1$

$\therefore [0,1]$ is uncountable [by cor -1].

But $m^* [0,1] = \ell[0,1) = 1$.

a contradiction

Hence the corollary.

Thm :- (iii)

Given any set A and $\epsilon > 0$, there is an open set U such that $A \subset U$ and $m^* U \leq m^* A + \epsilon$. There is a G_δ set G such that $A \subset G$ and $m^*A = m^* G$.

Proof :-

Let A be a set of real numbers.
Then there exist open intervals.

In such that $A \subset U\, I_n$ and $\Sigma \ell(I_n) \leq m^*(A) + \epsilon$.

Let $U = U I_n$.

Then U is an open set, $A \subset U$ and

$m^* (U) = m^* (UI_n)$.

$\leq \Sigma\ m^* (I_n) = \Sigma \ell (I_n) \leq m^*A + \in$.

$\therefore$ there exists an openset U such that $A \subset U$ and $m^* U \leq m^* A + \in$.

To find a G_δ set.

Let as take $\in = 1/n$.

Then from what we have proved above, we get for each n, an open set U_n such that $A \subset U_n$ and $m^* (U_n) \leq m^* (A) + 1/n$.

Let $G = \bigcap_n U_n$

Then G is a G_δ set and $A \subset G$.

Also $m^* (G) \leq m^* (U_n)$ $(\because G \subset U_n)$

$\leq m^* (A) + 1/n$ for each n.

$\therefore m^* (G) \leq m^* (A)$ .... (1)

But $A \subset G$ gives $m^* (A) \leq m^* (G)$ ....(2)

from (1) and (2)

$\therefore m^* (A) = m^* (G)$

Hence the thm.

Thm:- (iv)

m^* is translation invariant.

i.e $m^* (A+x) = m^*A\ \ \forall x \in R$.

Proof:-

Given $\in > 0$, Then $\exists \left\{ I_n \right\}_{n=1}^{\infty}$ such that

$A \subset UI_n$ and $\sum_{n=1}^{\infty} \ell (I_n) \leq m^* (A) + \in$

But

$A \subset UI_n \Rightarrow A + x \subset U (I_{n+x})$

But

$\ell (I_n) = \ell (I_n + x)$ $[\forall n$ and $\forall x)$

$\therefore m^* (A+x) \leq \Sigma \ell (I_n + x) = \Sigma \ell\ (I_n) \leq m^* (A) + \in$

$\therefore m^*(A+x) \le m^*(A)$ ($\therefore \in$ is arbitrary) .... (1)

But

$A = (A+x) - x = (A+x) + y$

Where $y = -x$.

$\therefore m^*\left[(A+x)+y\right] \le m^*(A+x)$ (using 1)

i.e $m^*A \le m^*(A+x)$ .... (2)

from (1) and (2)

we get

$m^*A = m^*(A+x)$

This proves the thm.

Problem : - (i)

Prove that if $m^*A = 0$ then $m^*(AUB) = m^*B$.

Solution :

$m^*(AUB) \le m^*(A) + m^*(B)$

$\Rightarrow m^*(AUB) \le m^*(B)$ .... (1) ($\therefore m^*$ is finitely subadditive)

But $B \subset A \cup B$ ($\therefore m^*A = 0$)

$\Rightarrow m^*(B) \le m^*(A \cup B)$ ....(2)

from (1) & (2) we get,

$m^*(A \cup B) = m^*(B)$

Problem :- (ii)

Let A be the set of rational numbers between 0 and 1. Let $\{I_n\}$ be a finite collection of open intervals covering A. Then $\Sigma \ell(I_n) \ge 1$.

Solution :-

Let $I_1, I_2, \ldots\ldots I_n$ be a finite collection of open intervals such that $A \le I_1 \cup I_2 \cup \ldots\ldots\ldots \cup I_n$.

$$\therefore \bar{A} \subset \bigcup_{j=1}^{n} I_j$$

$$\text{i.e., } [0,1] \subset \bigcup_{j=1}^{n} \bar{I}_j$$

($\therefore$ A is the set of rational numbers between 0 and 1 and hence $\bar{A} = [0,1]$)

$$\therefore m^*[0,1] \le \sum_{j=1}^{n} m^*\left(\overline{I}_j\right) = \sum_{j=1}^{n} m^*\left(\overline{I}_j\right)$$

$$= \sum_{j=1}^{n} \ell\left(\overline{I}_j\right)$$

$$\text{i.e, } 1 \le \sum_{j=1}^{n} \ell\left(\overline{I}_j\right)$$

Measurable sets And Lebesgue measure:-

While outer measure has the advantage that it is defined for all sets, is not countably additive. It becomes countably additive, however, it w suitably reduce the family of sets on which it is defined. Perhaps the bes way of doing this is to use the following definition due to caratheodory.

Definition :

A subset E of R is said to be measurable if for each subset A of R w have

$m^*A = m^*(A \cap \tilde{E}) + m^*(A \cap \tilde{E})$ where $E = R - E$.

(i.e, $\tilde{E}$ denotes the complement of E in R)

Note :

Since we are using Lebesgue outer measure m^* the measurable set are called Lebesgue measurable sets.

Remark :-

Since $A = (A \cap \tilde{E}) \cup (A \cap \tilde{E})$. We always have $m^*A \le m^*(A \cap E) + m^*(A \cap \tilde{E})$

Hence E is measurable if and only if for each set

$A \subset R$, $m^*A \ge m^*(A \cap E) + m^*(A \cap \tilde{E})$

Note :-

Since the definition of measurability is symmetric in E and $\tilde{E}$, we hav $\tilde{E}$ is measurable whenever $\overline{E}$ is.

Lemma : (i)

ϕ and R are measurable sets.

(i) Let $A \subset R$.

$A \cap R = A$ and $A \cap \tilde{R} = A \cap \phi = \phi$

$\therefore m^*(A) = m^*(A \cap \tilde{R}) + m^*(A \cap \tilde{R})$ holds trivially.

$\therefore$ R is measurable.

(ii) $A \cap \phi = \phi$; $A \cap \tilde{\phi} = A \cap R = A$.

$\therefore m^*(A) = m^*(A \cap \phi) + m^*(A \cap \tilde{\phi})$ holds trivially.

$\therefore \phi$ is measurable.

Lemma : - (ii)

If $m^* \in = 0$ then E is measurable.

Proof :-

Let A be any subset of R.

Then $A \cap E \subset E$ and so $m^*(A \cap E) \leq m^*(E) = 0$.

Thus $m^*(A \cap E) = 0$.

But $A \supset A \cap \tilde{E}$ gives $m^*(A) \geq m^*(A \cap \tilde{E})$

$$= m^*(A \cap \tilde{E}) + m^*(A \cap E)$$

(Since $m^*(A \cap E) = 0$)

Hence E is measurable.

Thm :-

If E_1 and E_2 are measurable, so is $E_1 \cup E_2$.

Proof :-

Let A be any subset of R.

Since E_2 is measurable.

$m^*(A \cap \tilde{E}_1) = m^*(A \cap \tilde{E}_1 \cap E_2) + m^*(A \cap \tilde{E}_1 \cap E_2)$ .... (1)

(By taking $A \cap \tilde{E}_1$ in the place of A)

now,

$$A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2)$$

$$= (A \cap E_1) \cup (A \cap E_2 \cap R)$$

$$= (A \cap E_1) \cup [A \cap E_2 \cap (\tilde{E}_1 \cup E_1)$$

$$= (A \cap E_1) \cup [A \cap E_2 \cap \tilde{E}_1) \cup A \cap E_2 \cap)E_1$$

$$= (A \cap E_1) \cup (A \cap E_2 \cap \tilde{E}_1)$$

[$\therefore A \cap E_2 \cap E_1 \subset A \cap E_1$ gives $(A \cap E_2 \cap E_1) \cup (A \cap E_1) = (A \cap E_1)$]

Hence $m^*(A \cap (E_1 \cup E_2) \leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap \tilde{E}_1)$

Adding

$m^*(A \cap \tilde{E}_1 \cap \tilde{E}_2)$ to both sides, we get

$m^*(A \cap (E_1 \cup E_2) + m^*(A \cap (\tilde{E}_1 \cap \tilde{E}_2) \leq m^*(A \cap E_1)$

$$+ m^*(A \cap E_2 \cap \tilde{E}_1) + m^*(A \cap \tilde{E}_1 \cap \tilde{E}_2)$$

$$= m^*(A \cap E_1) + m^*(A \cap \tilde{E}_1) \quad \text{using (1)}$$

$$= m^*(A)$$

(Since E_1 is measurable)

Thus $\quad m^*(A) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^{\sim})$

Hence $E_1 \cup E_2$ is measurable.

Note :

If E_1, E_2 are measurable, then $(E_1 \cap E_2) = (E_1 \cup E_2)^{\sim}$ and hence $E_1 \cap E_2$ is measurable.

Lemma :

Let A be any set and $E_1, \ldots\ldots E_n$ is a finite sequence of disjoint measurable sets.

$$\text{Then} \quad m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

Proof :

We prove this lemma by induction on n. When $n=1$, the result is obvious. Assume the result for n-1 sets ($n \geq 2$)

Let $E_1, E_2 \ldots\ldots E_n$ be n disjoint measurable sets.

$$\text{Then } A \cap \left(\bigcup_{i=1}^{n} E_i\right) \cap E_n = A \cap E_n \ldots\ldots\ldots (1)$$

$$\therefore \; E_n \subset \bigcup_{i=1}^{n} E_i$$

$$A \cap \left(\bigcup_{i=1}^{n} \tilde{E}_i\right) \cap \tilde{E}_n = A \cap \left((E_1 \cap E_n) \cup \ldots\ldots \cup (E_n \cap E_n)\right)$$

$$= A \cap \left(E_1 \cup \ldots\ldots \cup E_{n-1} \cup \phi\right)$$

$$= A \cap \left(\bigcup_{j=1}^{n-1} E_j\right) \text{----- (2)} \quad [\because \; E_i, E_n \text{ When if n are disjoint, } E_i \subset E_n]$$

Since E_n is measurable, we get

$$m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) = m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right) \cap E_n\right) + m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right) \cap E_n\right)$$

(using $A \cap \left(\bigcup_{i=1}^{n} E_i\right)$ in the place of A).

$$= m^*(A \cap E_n) + m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right)$$

(using (1) and (2))

$$= m^*(A \cap E_n) + \sum_{i=1}^{n-1} m^*(A \cap E_i)$$

(By induction hypothesis)

$$= \sum_{i=1}^{n} m^*(A \cap E_i)$$

$$\Rightarrow m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

This proves the lemma.

Remark :

Take A = R we get finite additivity of m*.

Theorem:

The collection $\mathscr{M}$ of measurable sets is a σ - algebra.

Proof :

We recall the definition of σ - algebra.

A collection T of subsets of a set x is called a σ - algebra if

(i) $A \in T \Rightarrow A \in T$.

(ii) If $A = \bigcup_{n=1}^{\infty} A_n$ and if $A_n \in T$ for every n then $A \in T$.

Let $E_n \in \mathcal{M}$, n = 1,2,3 .... and Let $E = \bigcup_{n=1}^{\infty} E_n$.

Claim:

There is a sequence $F_n \in \mathcal{M}$, n = 1,2....

Such that

$$F_i \cap F_j = \phi \quad \text{for } i \neq j \text{ and } \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$$

For : Set $F_1 = E_1$. For $n > 1$

define

$$F_n = E_n - \{E_1 \cup \ldots \cup E_{n-1}\}$$

$$= E_n \cap \tilde{E}_1 \cap \ldots \cap \tilde{E}_{n-1}$$

Since $\mathcal{M}$ is an algebra, $\tilde{E}_i \in \mathcal{M}$, $i \leq i \leq n-1$ and

$\tilde{E}_n \cap \tilde{E}_j \ldots \cap E_{n-1} \in \mathcal{M}$.

Hence $F_n \in \mathcal{M}$ for every n. Also $F_n \subset E_n$.

Let $m \neq n$

Suppose $m < n$ (Similar proof if $n < m$)

Then $F_m \subset E_m$, and so $F_m \cap F_n \subset E_m \cap F_n$

$$F_m \cap F_n = E_m \cap (E_n \cap \tilde{E}_1 \cap \ldots \tilde{E}_{n-1})$$

$$= E_n \cap \tilde{E}_1 \cap \ldots \cap E_m \cap \tilde{E}_m \cap \tilde{E}_{n-1} = f$$

$$= \phi \qquad (\because m < n \text{ and } E_m \cap \tilde{E}_m = \phi)$$

Thus

$$F_m \cap F_n = \phi$$

Since $F_i \subset E_j$, we have $\bigcup_{i=1}^{\infty} F_i \subset \bigcup_{i=1}^{\infty} E_j$ ------ (1)

Let $x \in \bigcup_{i=1}^{\infty} E_j$ Then $x \in E_n$ for some n.

Let m be the smallest value of n such that

$x \in E_m$.

Then $x \notin E_j$ for $j<m$.

$\therefore\ x \in \tilde{E}_j$ for $j< m$.

$\therefore\ x \in E_m \cap \tilde{E}_{m-1} \ldots\ldots \cap \tilde{E}_1$

$\therefore\ x \in F_m \quad \therefore\ x \in \bigcup_{i=1}^{\infty} F_i$

Hence

$$\bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} F_i \text{ ----------- (2)}$$

From (1) and (2) we get

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$$

Hence the claim.

Thus if E is the union of a countable collection of measurable sets, it must be the union of pairwise disjoint measurable sets.

Let

$$A_n = \bigcup_{i=1}^{n} E_i$$

Then A_i is measurable since $\mathcal{M}$ is an algebra.

$A_n \subset E$ gives $\tilde{A}_n \supset \tilde{E}$.

Hence

$$m^*(A) = m^*(A \cap A_n) + m^*(A \cap \tilde{A}_n)$$

$$\geq m^*(A \cap A_n) + m^*(A \cap \tilde{E}).$$

$$= m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) + m^*(A \cap \tilde{E}) \quad (\therefore A \cap \tilde{A}_n \supset A \cap \tilde{E})$$

$$= \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap \tilde{E})$$

(by the above thm)

Thus

$$m^*(A) \geq \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap E)$$

This is true for every n.

Hence

$$m^*(A) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap \tilde{E})$$

$$\geq m^*(A \cap E) + m^*(A \cap E)$$

(by the countable sub additivity of m^*)

$$= m^*\left(\bigcup_{i=1}^{\infty} A \cap E_i\right)$$

$$\leq \sum_{i=1}^{\infty} m^*(A \cap E_i)$$

Hence E is measurable.

Thus $\mathcal{M}$ is a σ - algebra.

Hence the theorem.

Note :

Since the complete of a measurable set is measurable, the intersection of a countable collection of measurable sets is measurable.

Theorem :

The interval (a, ∞) is measurable.

Proof :

Let A be any subset of R.

Let $A_1 = A \cap (a, \infty)$

$$A_2 = A \cap \widetilde{(a, \infty)} = A \cap (-\infty, a)$$

To show that (a, ∞) is measurable.

We have to prove that $m^*A \geq m^*A_1 + m^* A_2$.

If $m^*A = \infty$ Then there is nothing to prove.

If $m^*A < \infty$.

Then by **note : (3)** There exists open intervals $\{I_n\}_{n=1}^{\infty}$ such that $A \subset \bigcup_{n=1}^{\infty} I_n$

and $\sum \ell(I_n) \leq m^* A + E$ --- (1)

Let $I_n' = I_n \cap (a, \infty)$ and $I_n'' = I_n \cap [-\infty, a]$

Then I_n' and I_n'' are intervals.

$$\therefore \ell(I_n) = \ell(I_n') + \ell(I_n'')$$

$$= m^*(I_n') + m^*(I_n'') \text{ ---------- (2)}$$

$$\text{Since } A_1 = A \cap (a, \infty) \subset \left(\bigcup_{n=1}^{\infty} I_n\right) \cap (a, \infty)$$

$$= \bigcup_{n=1}^{\infty} I_n \cap (a, \infty)$$

$$= \bigcup_{n=1}^{\infty} I_n'$$

$$\therefore m^* A_1 \leq m^*\left(\bigcup_{n=1}^{\infty} I_n'\right) \leq \sum_{n=1}^{\infty} m^*(I_n')$$

(by countable sub additivity)

III^{ly}

$$m^* A_2 \leq m^*\left(\bigcup_{n=1}^{\infty} I_n''\right) \leq \sum_{n=1}^{\infty} m^*(I_n'')$$

Thus

$$m^*A_1 + m^*A_2 \le \sum_{n=1}^{\infty} (m^* I_n' + m^* I_n'')$$

$$\le \sum_{n=1}^{\infty} \ell(I_n) \qquad \text{(using (2))}$$

$$\le m^* A + E \text{ (using (1)}$$

Since $\in$ is arbitrary, we have

$$m^*A_1 + m^*A_2 \le m^* A$$

Hence (a, ∞) is measurable.

Theorem :

Every Borel Set is measurable.

Proof:

We recall the definition of a Borel Set in R. The smallest σ - algebra which contains all the open Sets of R is called the collection of all Borel sets.

Hence to show that every Borel set is measurable, it is enough if we showthat $\mathcal{M}$ contains all open sets.

Then $\mathcal{M}$ being a σ - algebra and $\mathcal{B}$ being the smallest σ - algebra containing all open sets of R, we get that $\mathcal{B}$ is contained in $\mathcal{M}$ and hence the result will follow.

Claim :

Every open set of R is measurable.

For : Consider (a, ∞) for any $a \in R$.

We have proved that (a, ∞) is measurable.

$\therefore (a, \infty) = (-\infty, a)$ is measurable.

For any $b \in R$ $(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b - 1/n)$

Since each of $(-\infty, b - 1/n)$ is measurable and since countable union of measurable sets is measurable.

We get that $(-\infty, b)$ is measurable.

consider any open interval (a, b).

Since both $(-\infty, b)$ and (a, ∞) are meaurable and since intersection of two measurable sets is measurable.

We get that (a, b) is measurable.

Each open set is the union of countable number of open intervals.

Hence each open set is measurable.

Hence the claim.

Note:

Since each open set, each closed set, any f_σ set and any G_δ set is Borel set. We get that these sets are also measurable.

Definition:-

Lebesgue measure:-

The Lebesgue measure m is the set function from the family $\mathcal{M}$ of Lebesgue measurable sets to $R \cup \{\infty\}$ defined by $m(E) = m^*(E)$ for any $E \in \mathcal{M}$.

ie) $m : \mathcal{M} \to R$ defined by $m(E) = m^*(E)$ for any $E \in \mathcal{M}$.

Theroem :

Let $\{E_i\}$ be a sequence of pairwise disjoint measurable sets.

$$\text{Then } m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i)$$

Proof:

Let $E_1, E_2 \ldots E_n$ be a finite sequence of measurable sets.

Already we proved the result. We get

$$m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) = \sum_{i=1}^{n} m^*(A \cap E_i) \text{ ---- (1)}$$

Where A is any subset of R.

Take $A = R$.

As R is measurable $R \cap E_i$, $R \cap \left(\bigcup_{i=1}^{n} E_i\right)$ are measurable.

Hence (1) gives.

$$m\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m(R \cap E_i)$$

[Since $m = m^* = m$ on $\mathscr{M}$]

$$m\left(\bigcup_{i=1}^{n} R_i\right) = \sum_{i=1}^{n} m(E_i)$$

$(\because \cup E_i \subset R,\ E_i \subset R)$

Thus m is finitely additive -------- (2)

Let $\{E_i\}$ be an infinite sequence of pairwise disjoint measurable sets.

Then $\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^{\infty} E_j$

And so,

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \geq m^*\left(\bigcup_{i=1}^{\infty} E_j\right)$$

$$\text{ie) } m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq m^*\left(\bigcup_{i=1}^{\infty} E_j\right)$$

$$= m\sum_{i=1}^{n} (E_j)$$

$(\therefore m^* = m$ on $\mathscr{M}$ and from 2)

Since this inequality is true for every n,

We get that $m\left(\bigcup_{i=1}^{\infty} E_j\right) \geq \sum_{i=1}^{\infty} m(E_i)$ --------- (3)

As m^* is countably sub additive, m is countably sub additive on $\mathscr{M}$.

ie) $m\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m(E_i)$ --------- (4)

From (3) and (4), we get

Hence $m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i)$

Thm:-

Let $\{E_n\}$ be an infinite decreasing sequence of measurable sets.

ie) $E_{n+1} \subset E_n$ for each n. Let $m E_1$ be finite.

Then $m\left(\bigcap_{i=1}^{\infty} E_j\right) = \lim_{n \to \infty} m E_n$

Proof:-

Let $E = \bigcap_{i=1}^{\infty} E_j$

Let $F_i = E_j - E_{i+1}$.

Then claim : $E_1 - E = \bigcup_{i=1}^{\infty} F_i$ and F_i^S are pairwise disjoint.

For: Let $x \in E_1 - E$. Then $x \in E_1$ and $x \notin E = \bigcap_{i=1}^{\infty} E_i$.

i.e) $x \in E_1$ and $x \notin E_i$ for some i.

now,

$E_1 \supset E_2 \supset \ldots \supset E_i \quad x \in E_i, x \notin E_j$.

Let j be the smallest suffix such that $x \notin E_j$.

Then

$x \in E_{j-1}, x \notin E_j$.

$\therefore x \in E_{j-1} - E_j = F_{j-1}$

$\therefore x \in \bigcup_{i=1}^{\infty} F_i$

$\therefore E_1 - E \subset \bigcup_{i=1}^{\infty} F_i$ ---- (1)

Let $y \in \bigcup_{i=1}^{\infty} F_i$

Then $y \in F_i$ for some i and hence $y \in E_i$, $y \notin E_{i+1}$

Since $E_1 \supset E_i$, $y \in E_1$ since $y \notin E_{i+1}$

$$y \notin \left(\bigcap_{i=1}^{\infty} E_i \right) = E$$

Hence $y \in E_1 - E$

Thus $\bigcup_{i=1}^{\infty} F_i \subset E_1 - E$ --------- (2)

From (1) and (2)

$$E_1 - E = \bigcup_{i=1}^{\infty} F_i$$

Let $i \neq j$, $F_i = E_i - E_{i+1}$, $F_j = E_j - E_{j+1}$

without loss of generality.

Let $i < j$

Then $E_i \supset E_j$

Hence $x \in F_i \Rightarrow x \in E_i$ and $x \notin E_{i+1}$

Since $E_{i+1} \supset E_j$ we get $x \notin E_j$

$\therefore x \notin F_j$

If $y \in F_j$ then $y \in E_j$ and $y \notin E_{i+1}$

$\therefore y \in E_i$, $y \in E_{i+1}$

since $E_i \supset E_{i+1} \geq E_j$

Thus $y \notin F_j$

Hence $F_j \cap F_j = \phi$

Hence the claim

Since $F_i = E_i - E_{j+1}$; F_i^s are measurable, Also $E_j - E = \bigcup_{i=1}^{\infty} F_i$, F_i^{s} disjoint

$$m(E_1 - E) = m\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} m(F_i)$$

$$= \sum_{i=1}^{\infty} m(E_i - E_{i+1}) \text{ ----------- } (3)$$

Since $E_1 \supset E_i$, $E_1 = E \cup (E_1 - E)$ a disjoint union.

And

$$E = \bigcup_{i=1}^{\infty} E_i \text{ is measurable}$$

Hence $E_1 - E$ is measurable

$$\therefore m(E_1) = m(E) + m(E_1 - E)$$

III ^{ly} $E_i \supset E_{i+1}$ gives $m(E_i) = m(E_{i+1}) + m(E_i - E_{i+1})$

Given

$$m E_i < \infty$$

$$\therefore m E_i \leq m E_1 < \infty \text{ for every } i \text{ (As } E_i \subset E_1)$$

And $m E < \infty$

$$\therefore m(E_1 - E) = m(E_1) - m(E) \text{ .... } (4) \text{ and}$$

$$m(E_i - E_{i+1}) = m(E_i) - m(E_{i+1}) \text{ ----------- } (5)$$

Thus

$$mE_1 - m E = m(E_1 - E) = \sum_{i=1}^{\infty} m(E_i - E_{i+j})$$ (Using (3)

$$= \sum_{i=1}^{\infty} (m E_i - m E_{i+1})$$ Using (5)

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (m E_i - m E_{i+1})$$

$$= \lim_{n\to\infty} (mE_1 - m E_n)$$

$$mE_1 - mE = mE_1 - \lim_{n\to\infty} mE_n.$$

Since $m E_1 < \infty$, We have

$$m E = \lim_{n\to\infty} m E_n.$$

$$\therefore\ m = \left(\bigcap_{i=1}^{\infty} E_i\right) \lim_{n\to\infty} m E_n$$ (Using (3)

This process the theorem.

Theorem :-

Let E be a given set. Then the following statements are equivalent.

(i) E is measurable

(ii) Given E > 0, there is an open set $U \supset E$ with $m^* (U - E) < \in$.

(iii) Given $\in > 0$, there is a closed set $F \subset E$ such that $m^* (E - F) < \in$.

(iv) There is a G_δ set G with $E \subset G$ such that $m^* (G - E) = 0$.

(v) There is an F_σ set F with $F \subset E$ such that $m^* (E - F) = 0$.

If $m^* E$ is finite, the above statements are equivalent to.

(vi) Given $\in > 0$, there is a finite union V of open intervals such that $m^* (V \Delta E) < \in$.

Proof :

(i) $\Rightarrow$ (ii)

Let E be measurable.

Case (i) :

Suppose $m^* E = mE < \infty$

Given $\in > 0$, by note : – 3 , There exist open intervals.

In, n = 1,2 ... such that $E \subset \bigcup_{n=1}^{\infty} I_n$ and

$m^* E + \in \geq \Sigma\ \ell\ (I_n)$ --------- (1)

Let $U = \bigcup_{n-1}^{\infty} I_n$. Then U is an open set and by countable sub additivity of m

we get

$$m(U) = m(\cup I_n) = \Sigma m(I_n) = \Sigma \ell(I_n) \quad ....(2)$$

(note that $I_n{}^s$ are measurable and U is measurable)

$\therefore$ From (1), (2) we get

$m^* E + E \geq m(U)$ ----- (3)

since E is measurable, and $E \subset U$ we get,

$m^* E = m E$

Also

$U = E \cup (U - E)$, a disjoint union we get,

$m(U) = m E + m(U - E)$

$\therefore m(U - E) = mU - m E = mU - m^* E \leq \epsilon$ (From (3))

Thus we get an open set U such that $E \subset U$ and

$m(U - E) \leq \epsilon$

$\therefore$ (ii) follows.

Case - 2 :

Suppose $m^* E = m E = \infty$

Let $x = \bigcup_{n=1}^{\infty} I_n$ where

$I_n = [n, n+1)$, $y = \bigcup_{n=1}^{\infty} I_n$ where $I_n = (-n-1, -n]$

Then $R = x \cup y$.

I_n, $I_n{}^1$ are countable collection of disjoint finite intervals.

Rename than as $\{J_n\}_{n=1}^{\infty}$

we have

$$E = E \cap R = E \cap \bigcup_{n=1}^{\infty} J_n = \bigcup_{n=1}^{\infty} \in \cap J_n.$$

$$= \bigcup_{n=1}^{\infty} E_n \text{ (say) where } E_n = E \cap J_n.$$

$E_n{}^s$ are measurable as E, J_n are measurable, Also $E_n \subset J_n$

$\therefore \; m(E_n) \le m(J_n) < \infty$

Thus, we can find open sets U_n.

(By case 1) such that $E_n \subset U_n$ and $m(U_n - E_n) < \dfrac{\in}{2^n}$.

Let $U = \cup U_n$.

Then $U - E = \cup U_n - \cup E_n \subset \cup (U_n - E_n)$

$$m(U - E) \le \sum_{n=1}^{\infty} m(U_n - E_n) < \sum_{n=1}^{\infty} \frac{\in}{2^n} = \in$$

$\therefore m^* (U - E) < \in$

Thus from case (1) and case (2) we get that (i) $\Rightarrow$ (ii)

To prove that (ii) $\Rightarrow$ (iv)

Given for every $\in > 0$, there exists an open set $U \supset E$ such that $m^* (U - E) < \in$.

To prove that there exists a G_δ set G such that

$E \subset G$ and $m^* (G - E) = 0$

For $\in = 1/n$, $n = 1, 2 \ldots$

We have open sets U_n such that $U_n \supset E$ and

$m^* (U_n - E) < 1/n$.

$$\text{Let } G = \bigcap_{n=1}^{\infty} U_n$$

Then G is a G_δ set and $G \supset E$. ($\therefore U_n \supset E$ for every n)

Also

$G \subset U_n \Rightarrow G - E \subset U_n - E$ (for every n)

$\therefore m^* (G - E) \le m^* (U_n - E) < 1/n$ for every n.

$\therefore m^* (G - E) \le 0$

But $m^* (G - E) \ge 0$ ($\because$ m^* is a non- negative set function)

$\therefore m^* (G - E) = 0$.

Hence (iv) follows.

To prove (iv) $\Rightarrow$ (i)

$\therefore$ **Given :** There exists a G_δ set $G \supset E$.

with $m^*(G - E) = 0$.

To prove that E is measurable.

G is measurable since any G_δ set is measurable.

$\therefore m^*(G - E) = 0 \Rightarrow (G - E)$ is measurable.

(Since any set of outer measure 0 is measurable).

$\therefore E = G - (G - E)$ is measurable.

($\therefore$ difference of measurable set is measurable)

Hence (i) follows.

To prove (ii) $\Rightarrow$ (iii). Assume (ii) from what we have proved above, we have (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i)

Hence (ii) $\Rightarrow$(i)

$\therefore$ E is measurable.

$\therefore \tilde{E}$ is measurable.

Since (ii) is true. given an $\in > 0$ there exists an open set $U \supset \tilde{E}$ such that $m(U - \tilde{E}) < E$. As $U \supset \tilde{E}$ we get $\tilde{U} \subset E$ and

$$U - \tilde{E} = E - \tilde{U} = U \cap E$$

Since U is open. $\tilde{U}$ is closed.

Thus we have a closed set $\tilde{U}$ such that

$m^*(E - U)\, \tilde{U} = m^*(U - \tilde{E}) < E$ and $\tilde{U} \subset E$. Hence (iii) follows.

To Prove (iii) $\Rightarrow$ (v)

Assame (iii). For each positive integer n. There exists a closed set F_n such that $F_n \subset E$ and $m^*(E - F_n) < 1/n$.

Let $F = \bigcup_{n=1}^{\infty} F_n$ then F is a F_σ set.

Also $F_n \subset E$ for every $n \Rightarrow F \subset E$. And $E - F \subset E - F_n\ \forall n$.

$\therefore m^*(E - F) \leq m^*(E - F_n) < 1/n$ for every n.

$m^*(E - F) \leq 0$. But m^* is non-negative.

$\therefore$ $m^*(E-F) = 0$. Thus (v) follows.

To Prove (v) $\Rightarrow$ (i)

Given :

There exists an F_σ set F such that $F \subset E$ and $m^*(E-F) = 0$

Since each F_σ set is measurable, we get F is measurable.

Also $m^*(E-F) = 0$ gives $E - F$ is measurable.

$\therefore$ $E = F(E-F)$ is measurable. Hence (i) follows.

Thus (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i)

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (i) $\Rightarrow$ (ii)

Hence all the statements from (i) to (v) are equivalent.

To Prove (i) $\Rightarrow$ (vi) Suppose E is measurable.

Since (i) $\Rightarrow$ (ii) gives $\in > 0$, there exist an open set $U \supset E$ such that,

$m^*(U-E) < \in/2$ ----- (A)

As $m^*E < \infty$, $m^*(U) = m^*(E) + m^*(U-E) < \infty$.

Since $\cup$ is an open set, $\cup$ is a dis joint union open intervals say $\{I_n\}_{n=1}^{\infty}$

$$\therefore m^*(\cup) = m^*\left(\bigcup_{i=1}^{\infty} I_n\right)$$

Since $m = m^*$ on measurable sets we get,

$$m(\cup) = m\left(\bigcup_{i=1}^{\infty} I_n\right) = \sum_{i=1}^{\infty} \ell\left((I_n)\right) < \alpha \quad (\therefore m(U) = m^*U < \alpha)$$

$\therefore$ There exists $\cap$ such that

$$\sum_{n=1}^{\infty} L(I_n) < \in/2 \text{ ----- (B)}$$

$$\text{Let } V = \bigcup_{i=1}^{\infty} I_i$$

$E \Delta V = (E - V) \cup (V - E) \subset (\cup - V) \cup (\cup - E)$

$\therefore$ ($E \subset U$ and $V \subset U$)

$\therefore\ m^*(E \Delta V) \le m^*(U\text{-}V) + m^*(U\text{-}E)$

$< \in/2 + \in/2$ (from (A), B))

(Note : $\cup\text{-}V = \bigcup_{n=1}^{\infty} I_i$, and hence $m^*(U\text{-}V) < \in/2$)

$\therefore\ m^*(E \Delta V) < \in$

Thus, there is a finite Union V of open intervals such that $m^*(E\Delta V) < E$. Lastly assumed (vi). To prove (ii). Hence given $\in/3 > 0$, there exists intervals $I_1, I_2, \ldots I_n$ such that $m^*(E\Delta U) < \in/3$ .... (1)

Where $U = \bigcup_{i=1}^{n} I_i$ and $m^* E < \infty$.

To show that given $\in > 0$ there exists an open set W such that $E \subset W$ and $m^*(W\text{-}E) < \in$.

For $\in/3 > 0$, there exists open intervals $\{J_n\}$

Such that $E \subset \bigcup_{n=1}^{\infty} J_n$ and $m^*E + \in/3 \ge \sum_{n=1}^{\infty} L(J_n)$ .... (2)

Let $V = \bigcup_{n=1}^{\infty} J_n$. Then V is an open set.

$$m^*V = m^*\left(\bigcup_{n=1}^{\infty} J_n\right) \le \Sigma\, m^*(J_n) = \Sigma \ell(J_n) \quad \text{.............. (3)}$$

From (2), (3) we get $m^*V \le m^*E + \in/3$ .... (4)

Let $J = V \cap U$.

$\therefore\ J \subset V$ and J is an open set.

$\because$ (U and V are open sets)

Also

$J \Delta E = (J \cup E) - (J \cap E)$

But $J \cap E = V \cap U \cap E = U \cap E = V \cap E$.

$\therefore\ J \Delta E \subset (U \cup E) - (U \cap E)$ $\quad \because (E \subset V,\ \ E \subset V)$

$= U \Delta E$.

$\therefore$ From (1) We get,

$m^*(J \Delta E) < \in/3$ .... (5)

But for any three subsets A, B, C of a set x we have $A \Delta B \subset (A \Delta C) \cup (C \Delta B)$

$(\therefore (A-B) \subset (A-C) \cup (C-B)$ and $(B-A) \subset (B-C) \cup (C-A))$

$\therefore V \Delta E \subset (V \Delta J) \cup (J \Delta E)$

$\therefore m^*(V \Delta E) \leq m^*(V \Delta J) + m^*(J \Delta E)$ .... (6)

But

$V \Delta J = V - J \quad (\because J \subset V)$

$\therefore m^*(V \Delta J) = m^*(V-J) = m^*V - m^*J$ .... (7)

Since V and J are open sets and hence measurable.

Note :-

$V = (V-J) \cup J$ dis joint Union,

$m^* v = mv = m(v-J) + m(J) = m^*(v-J) + m^*(j)$

$\therefore m^* V - m^* J = m^*(V-J)$

Since $m^*(J) \leq m^* v \leq m^* E + \in/3 < \infty)$

But $E \subset J \cup (E-J)$

$\subset J \cup (E-J) \cup (J-E) \quad \left[\because E = E \cap (J \cup \tilde{J}) = (E \cap J) \cup (E \cap \tilde{J}) \subset J \cup (E-J)\right]$

$= J \cup (E \Delta J)$

$\therefore m^* E \leq m^* J + m^*(E \Delta J)$

$\leq m^* J + \in/3$ ----- (8) (by 5)

$\therefore m^*(V \Delta E) \leq m^*(V \Delta J) + m^*(J \Delta E)$ (by 6))

$\leq m^* V - m^* J + \in/3$ (by (5))

$\leq m^* E + \in/3 - m^* J + \in/3$ (by (4)

$\leq m^* J + \in/3 + \in/3 - m^* J + \in/3$ (by 8)

$= \in.$

Since $E \subset V$, $V \Delta E = V - E$

$\therefore m^*(V-E) \leq \in.$

Hence (ii) follows,

As (ii) $\Rightarrow$ (i) we get (i)

Hence (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (vi)

Thus all the six statements are equivalent.

Remark :-

The above theorem is known as Little wood's first principle.

Problems :-

1. Show that if E is a measurable set, them each translate $E+y$ of E is also measurable.

Solution :-

Since E is measurable, for each $\epsilon > 0$. There exists an open set $U \supset E$ such that $m^* (U-E) E < \epsilon$.

Since U is an open set. $U+y$ is an open set and $E+y \subset U+y$.

Also $(U+y)-(E+y) = (U-E)+y$.

And $m^* (U-E+y) = m^* (U-E) < \epsilon$

$\therefore m^* [(U+y)-(E+y)] < \epsilon$

Thus for each $\epsilon > 0$, there exists an open set $U+y$ such that $[m^* (U+y-(E+y)] < \epsilon$ and $E+y \in U+y$

Hence $E+y$ is measurable.

2) Show that if E_1 and E_2 are meaurable. Then $m(E_1 \cup E_2) + m(E_1 \cap E_2) = mE_1 + mE_2$.

Solution :-

$E_1 = (E_1 - E_2) \cup (E_1 \cap E_2)$ and this is a dis joint Union.

$mE_1 = m(E_1 - E_2) + m(E_1 \cap E_2)$. ----- (1)

$mE_2 = m(E_2 - E_1) + m(E_1 \cap E_2)$. ----- (2)

(Note that when E_1, E_2 are measurable, $E_1 - E_2$, $E_2 - E_1$, $E_1 \cap E_2$, $E_1 \cup E_2$ are all measurable)

from 1 and 2 adding, we get

$$mE_1 + mE_2 = m(E_1 - E_2) + m(E_2 - E_1) + m(E_1 \cap E_2) + m(E_1 \cap E_2)$$

$$= m(E_1 \cup E_2) + m(E_1 \cap E_2)$$

(Since $E_1 \cup E_2$ is the dis joint Union of $E_1 - E_2$, $E_2 - E_1$ and $E_1 \cap E_2$)

Hence the problem.

3) Let $\{E_i\}$ be a sequence of dis joint measurable set and A any set S.T

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$$

Solution :-

Already we have proved that,

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(A \cap E_i) \quad ----- \quad (1)$$

Let $E = \bigcup_{i=1}^{\infty} E_i$

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) = m^*\left(\bigcup_{i=1}^{\infty} A \cap E_i\right)$$

$$= \sum_{i=1}^{\infty} m^*(A \cap E_i) \text{ ------ } (2)$$

(By countable subaddivity of m^*)

$$m^*(A \cap E) = m^*\left(A \cap E \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) + m^*\left((A \cap E)\left(\cap \bigcup_{i=1}^{n} E_i\right)^{\sim}\right)$$

$$\left(\therefore \bigcup_{i=1}^{n} E_i \quad \text{is measurable}\right)$$

$$\therefore m^*(A \cap E) \geq m^*\left(A \cap E \cap \left(\bigcup_{i=1}^{n} E_i\right)\right)$$

$$= m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right)$$

$$\left(\therefore E = \bigcup_{i=1}^{\infty} E_i\right)$$

$$= \sum_{i=1}^{n} m^* (A \cap E_i) \text{ (by (1)]}$$

This is true for every n

$$\therefore m^* (A \cap E) \geq \sum_{i=1}^{n} m^* (A \cap E_i)$$

From (2) and (3) we get

$$m^* (A \cap E) = \sum_{i=1}^{\infty} m^* (A \cap E_i)$$

$$m^* \left(A \cap \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} m^* (A \cap E_i)$$

Hence the problem.

Problem :

Show that there exists a lebesgue measurable set which is not Borel set.

Solution :-

Let c be the cardinality of the continum. R has a countable base namely open intervals with rational end points.

The collection B of all Borel sets of R is the σ - algebra generated by this base.

Hence B has cardinality C.

The cantor set E has cardinality c and has measure O. Hence all subsets of E are Lebesgue measurable and of Lebesgue measure O. There are 2^c subsets of E. Since $2^c > c$, most subsets of E are not Borel sets this proves the solution.

UNIT - 7

TOPIC-1 :

A non- measurable set :-

In this section we show the existence of a non-measurable set.

Definition

Sum Module :-

Let x and y be real numbers in $(0,1)$. The sum module 1 of x and y is defined as $x+y<1$ and $x+y-1$ if $x+y \geq 1$. This is denoted by x ♀ y.

Remark :-

It is obvious from the definition that ♀ is a commutative and associative operation from $[0,1)$ x $[0,1)$ to $[0,1)$

Example :-

Let $f : [0,1) \rightarrow [0, 2\pi)$ by $f(x) = 2\pi x$.

Then

f (x ♀ y) $= 2\pi (x+y)$ if $x+y<1$

$= (x+y-1)\, 2\pi$ if $x+y \geq 1$

$= 2\pi (x+y) - 2\pi$

Hence addition module 1 corresponds to the addition of angles.

Definition :-

Translate module 1 :-

Let $E \subset (0,1)$. Then the translate module 1 of E is defined as

E♀ y = {Z : Z = x♀ y, $x \in E$}

Example :-

Let $E = [0,1)$

Let $y \in E$, E ♀ y = {x ♀ y : $x \in E$}

f (E ♀ y) = { x ♀ y : $x \in E$}

= $\{2\pi (x+y)$ or $2\pi (x+y) - 2\pi$ according ad $x+y<1$ or ≥ 1 where $x \in E\}$

i.e., the elements of f (E ♀ y) corresponds to rotations through an angle of $2\pi y$.

-emma .-

Let $E \subset (0,1)$ be a measurable set. Then for each $y \in [0,1)$ the set $E \dotplus y$ s measurable and $m(E \dotplus y) = mE$.

Proof:-

Let $E_1 = E \cap [0, 1-y)$ and $E_2 = E \cap [1-y, 0)$

Then E_1 and E_2 and disjoint measurable sets (Since any interval is measurable and intersection of two measurable sets is measurable).

Also $E = E_1 \cup E_2$

$\therefore mE = mE_1 + mE_2$.

ow

$E_1 \dotplus y = E_1 + y$ and so $E_1 \dotplus y$ is measurable

and $m(E_1 \dotplus y) = mE_1$.

Note that since every element of $[0, 1-y)$ added with y is less than 1)

Also $E_2 \dotplus y = E_2 + y - 1$

$(\because x \in [1-y, 1)$

$\Rightarrow x \geq 1 - y$

$\Rightarrow x + y \geq 1$

nd hence $x \dotplus y = x + y - 1)$

$\therefore m(E_2 \dotplus y) = m(E_2 + y - 1)$

$= mE_2$.

$(\because m$ is translation invariant)

ut

$E \dotplus y = (E_1 \dotplus y) \cup (E_2 \dotplus y)$ and the sets, $E_1 \dotplus y$, $E_2 \dotplus y$ are disjoir easurable sets.

ence

$E \dotplus y$ is measurable and $m(E \dotplus y) = m(E_1 \dotplus y) + m(E_2 \dotplus y) =$ $E_1 + mE_2$.

$\Rightarrow m(E \dotplus y) = mE$

Hence the lemma.

Construction of a non-measureable set :-

Define a relation $\sim$ on $(0,1)$ as follows. For all $x, y \in (0,1)$, $x \sim y$ if and only if $x-y$ is a rational number.

$\sim$ is an equivalence relation.

($\therefore$ $x-x$ is rational $\forall\, x \in (0,1)$ and so $x \sim x$.

If $x \sim y$, then $x-y$ is rational, hence $y-x$ is rational and so $y \sim x$.

If $x \sim y$, $y \sim z$ then $x-y$ and $y-z$ are rationals, hence $x-y+y-z$ is a rational.

i.e. $x-z$ is rational. $\therefore$ $x \sim z$)

This equivalent relation partitions $(0,1)$ in to equivalence classes such that any two elements of the same class differ by a rational number while any two elements of different classes differ by an irrational number.

By axiom choice, there is a set P which contains exactly one element from each equivalence class.

To show that P is non-measurable.

Let $\{r_i\}^{\infty}$ be an enumeration of the rational numbers in $(0,1)$ with $r_0 = 0$. (This is possible since the set of all rational numbers is countable).

Define $P_i = P ♀ r_i$ then $P_0 = P$

By the definition of ♀, each $P_i \subset (o,1)$ and hence $\bigcup_{i=1}^{\infty} P_i \; (0,1)$ .... (1)

Claim :

$P_i \cap P_j = \emptyset$ for $i \neq j$.

Suppose $x \in P_i \cap P_j$.

Then $x = P_i ♀ r_i = P_j ♀ P_j$ for some P_j, ♀ P_j for some $P_i P_j \in P$

$P_i ♀ r_i = P_i ♀ r_i$ or $P_i ♀ r_i - 1$

$$P_j \text{♀} r_j = P_j \text{♀} r_j \text{ or } P_j \text{♀} r_j - 1$$

As $P_i \text{♀} r_i = P_j \text{♀} r_j$ we get

$$P_i + r_i = P_j + r_j \text{ or } P_i + r_i = P_j + r_j - 1.$$

or $\quad P_i + r_i - 1 = P_j + r_j$ or $P_i + r_i - 1 = P_j + r_j - 1.$

i.e., $\quad P_i - P_j = r_j - r_i$ or $r_j - r_i - 1$ or $r_j - r_i + 1.$

Hence $P_i - P_j$ is rational and P_i, P_j belong to the same equivalence class.

Hence $P_i \sim P_j$

Since $P_i, P_j \in P$ and since P has only one element from each equivalence class we must have $i = j$.

$\Rightarrow$ If $i \neq j$. $P_i \cap P_j = \varnothing$.

i.e., $\{P_i\}$ is a pairwise disjoint sequence of sets ----- (2)

Let $x \in (0,1)$. This x is in some equivalence class. Let y be the element chosen for p from this equivalence, then $x \sim y$.

$\therefore$ x-y is rational.

Also if $x > y$, $x - y = r_i > 0$ and so $x = y + r_i \in P_i$.

If $x < y$, $y - x = r_i$, $y - r_i = x$.

$$y \text{♀} (1 - r_i) = y + 1 - r_i \bmod |$$

$$= (y - r_i + 1) \bmod |$$

$$= (x + 1) \bmod |$$

$$= x$$

$$\therefore x \in P_{1-ri}$$

If $x = y$ then $x \in P = P_0$

In any case $x \in UP_i$ ....(3)

From (1) and (3) we get

$UP_i = [0,1)$ .... (4)

From (2) and (4)

P_i is a collection of dis joint subsetswhose union is $[0,1)$

Since each P_i is a translate module 1 of P each P_i will be measurable if P is and will have the same measure. (by lemma if $m^*E=0$ than E is measurable). So, if P is measurable then each P_i is measurable and $m(P_i) = m(P)$.

$$m[0,1) = \sum_{i=1}^{\infty} mP_i = \sum_{i=1}^{\infty} mP.$$

The right side is either zero or infinite depending on mP is zero or positive.

Since $m[0,1)=1$

This is impossible. So P cannot be measurable. Thus P is non-measurable subset of R.

Remark:-

In the above proof, it should be noted that until the last sentence, we have made no use of properties of lebesgue meaure other than translation invariance and countable additivity.

Thm :-

If m is countably additive, translation invariant measure defined on a σ - algebra containing p then $m[0,1)$ is either zero or infinite.

Proof follows from the above remark:

Note :-

$m^*P \neq 0$. For if $m^*P = 0$ then P is measurable

Problems:-

1. Show that if E is measurable and $E \subset P$. Then $ME=0$

Solution :-

Let $\{r_i\}_{i=0}^{\infty}$ be an enumeration of rational numbers. Let $E_i = E \dotplus r_i$

Since $E \subset P$, $E_i = E \dotplus r_i \subset P \dotplus r_i = P_i$

Since P_i are disjoint, E_i are disjoint.

Also $\cup E_i \subset \cup P_i = [0,1)$

If E is measurable, each E_i is measurable and $mE = mE_i$.

$\therefore m(\cup E_i) = \Sigma mE_i \le m[0,1) = 1$

$\therefore \Sigma mE \le 1 \therefore mE = 0$

2) Show that if A is any set with $m^* A > 0$ then there is a non-measurable se $E \subset A$.

Solution :-

Case (i) :-

Let $A \subset [0,1)$

Let $E_i = A \cap P_i$

$A = A \cap [0,1) = A \cap \cup P_i = \cup A \cap P_i = \cup E_i$.

$\therefore m^*A > 0 \Rightarrow m^*E_i > 0$ for atleast one i.

But if E_i is measurable.

Its translate $E_i \dotplus (-r_i)$ is measurable.

$E_i \dotplus (-r_i) \subset (P_i + (-r_i))$

$= P \; (\because P_i = P \dotplus r_i)$

Hence by the previous problems

$m(E_i + (-r_i)) = 0$

$\therefore mE_i = 0$

$\therefore m^*E_i = 0$

$\Rightarrow\Leftarrow \; (\because mE_i > 0)$

Hence $E_i \subset A$ is non-measurable.

Case (ii)

Suppose $A \not\subset [0,1]$

For atlease one n, $A_n = A \cap [n, n+1]$ is non-empty and $m^* A_n > 0$.

(For : $A = \cup A_n$ if $m^* A_n = 0\ \forall n$ than $m^*A = m^*(\cup A_n) \le \Sigma m^* A_n = 0$ and hece $m^*A = 0$ a $\Rightarrow\Leftarrow$)

Let $B_n = A_n - A$. Then $B_n \subset (0,1)$ and $m^*B_n = m^*A_n > 0$.

By case 1, B_n contains a non-measurable set E. Then $E + n \subset A_n \subset A$ is the required non-measurable set.

Measurable functions

Since not all sets are measurable, it is of great importance to know that sets which arise naturally in certain constructions are measurable. It we start with a function f, the most important sets that arise from it are those listed in the following lemma.

Lemma :-

Let f be an extended real-valued function whose domain is measurable. Then the following statements are equivalent.

(i) For each real number α the set $\{ x : f(x) > \alpha \}$ is measurable.

(ii) For each real number α the set $\{ x : f(x) \ge \alpha \}$ is measurable.

(iii) For each real number α the set $\{x : f(x) < \alpha \}$ is measurable

(iv) For each real number α the set $\{ x : f(x) \le \alpha\}$ is measurable

These statements imply

(v) For each real number the set $\{x : f(x) = \alpha\}$ is measurable.

Proof :-

Let the domain of f b/D.

(i) $\Rightarrow$ (iv) For : $\{x : f(x) \le \alpha \} = D - \{x : f(x) > \alpha \}$ and the difference of two measurable sets is measurable.

IIIly (iv) $\Rightarrow$ (i)

Also (ii) $\Rightarrow$ (iii) ($\therefore$ The set in one is the complement of the set in order in D)

To prove (i) $\Rightarrow$ (ii)

$\{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - \frac{1}{n}\}$ and each of $\{x : f(x) > \alpha - \frac{1}{n}$ is

measurable by (i)

Also intersection of a sequence of measurable sets is measurable.

Hence $\{x : f(x) \geq \alpha\}$ is measurable.

Thus (ii) follows.

(ii) $\Rightarrow$ (i)

For : $\{x : f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \left\{x : f(x) \geq \alpha + \frac{1}{n}\right\}$ and the union of sequence of measurable sets is measurable.

Hence (i) $\Leftrightarrow$ (ii), (i) $\Leftrightarrow$ (iv), (ii) $\Leftrightarrow$ (iii)

Thus the first four statements are equivalent.

now

$\{x : f(x) = \alpha\} = \{x : f(x) \geq \alpha\}$ $\{x : f(x) \leq \alpha\}$ for any real number.

Thus (ii) and (iv) $\Rightarrow$ (v)

Suppose $\alpha = \infty$

Then $\{x : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) \geq n\}$.

(ii) $\Rightarrow$ (v) for $\alpha = \infty$

when $\alpha = -\infty$, $\{x : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) \leq -n\}$

$\therefore$ (iv) $\Rightarrow$ (v)

Thus (ii) and (iv) $\Rightarrow$ (v) in all cases.

Hence this theorem.

Definition :-

An extended real valued function f is said to lebesgue measurable if its domain is measurable and if it satisfies one of the following four statements.

(i) For each real number α the set $\{x : f(x) > \alpha\}$ is measurable.

(ii) For each real number α the set $\{x : f(x) \geq \alpha\}$ is measurable.

(iii) For each real number α the set $\{x : f(x) < \alpha\}$ is measurable.

(iv) For each real number α the set $\{x : f(x) \leq \alpha\}$ is measurable.

Example :-

(i) Constant functions are measurable.

Let $f : D \to R \cup \{\infty, -\infty\}$ be defined as $f(x) = c \ \forall \ x$ where D is a measurable subset of R.

$$\text{now } \{x : f(x) > \alpha\} = \begin{cases} D \text{ if } \alpha < c \\ \phi \text{ if } \alpha \geq c \end{cases}$$

Both D and ϕ are measurable and hence f is measurable.

(ii) If A is a measurable subset of R, then χ_A is a measurable function.

$$\text{For : } \chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

$$\therefore \{x : f(x) > \alpha\} = \begin{cases} R \text{ if } < 0 \\ A \text{ if } 0 \geq \alpha < 1 \\ \phi \text{ if } \alpha \geq 1 \end{cases}$$

R, A, ϕ are measurable. Hence χ_A is measurable.

Note that if A is not measurable, then clearly $\{x : f(x) > 0.5\} = A$ is not measurable and hence χ_A is not measurable.

(iii) Containuous functions from $R \to R$ are measurable.

For suppose f is continuous.

Then $\{x : f(x) > \alpha\} = f^{-1}(\alpha, \infty)$ is an open set.

Since f is continuous.

But any open set is measurable.

Hence f is measurable.

(iv) Any set function is measurable.

For : A real valued function ϕ defined on [a, b] is called a step function.

If there is a partition $a = x_0 < x_1 < .... < x_n = b$ such that for each i, the function ϕ assumes only one value in the interval (x_i, x_{i+1})

If ϕ is a step function on [a, b] and if $f(x_{i-1}, x_i) = \alpha_{i-1}$, $1 \le i \le n$

then $\{x : (x)\phi > \alpha\} = \begin{cases} \phi \text{ if } \alpha_{i-1} \le \alpha \le \forall i \quad 1 \le i \le n \\ [a, b] \text{ if } \alpha < \alpha_{i-1} \le \forall i \end{cases}$

{a finite union of open intervals if α lies between min and max of $\{\alpha_0, \alpha_{n-1}\}$

Hence ϕ is measurable.

Note :- (i)

If f is measurable we have proved that $\{x : f(x) = \alpha\}$ is measurable for any extended real number α.

The converse is not true.

Let R* be the extended real number system.

Let A be a non-measurable set contained in [0, 1]

$$\text{Define } f : R \to R^* \text{ as } f(x) = \begin{cases} x & \text{if } x \in A \\ -x & \text{if } x \notin A. \end{cases}$$

$$\text{Then } \{x : f(x) = \;=\begin{cases} \{\ \} & \text{if } x \in A \\ \{-\} & \text{if } x \in \tilde{A} \end{cases}$$

Since singleton sets are measurable.

$\{ x : f(x) = \alpha$ is measurable.

But f is not measurable.

Since $\{ x : f(x) \geq 0 \} = [-\infty, 0] \cup A$

Since A is not measurable, $[-\infty, 0] \cup A$ is not measurable.

$\therefore$ f is not measurable.

Note :- (ii)

If f is a measurable function and E is a measurable subset of the domain of f, then the function obtained by restricting f to E is also measurable.

For : $f : D \rightarrow R^*$ (entended real number system)

Let $E \subset D$ and E measurable.

Since t is measurable $\{ x : f(x) > \alpha\}$ is measurable.

$\therefore \{ x : f(x) > \alpha\} \cap E$ is measurable.

ii) f restricted to E is measurable.

Thm :-

Let c be a constant and f and g two measurable real valued functions defined on the same domain. Then the functions $f + c$, cf, $f + g$, $f - g$ and fg are also measurable.

Proof :-

To show that $f + c$ is measurable.

$\{x : f(x) + c < \alpha\} = \{x : f(x) < \alpha - c\}$

Since f is measurable, $\{x : f(x) < \alpha - c\}$ is measurable and hence $\{x$ $\{f(x) + c < \alpha\}$ is measurable.

$\therefore f(x) + c$ is measurable.

(ii) To show that cf is measurable.

If $c = 0$ then if $cf = 0$ a constant function and hence measurable.

$$\text{If } c \neq 0 \text{ then } \{x : cf(x) < \alpha\} = \begin{cases} \left\{x : f(x) < \dfrac{\alpha}{c}\right\} & \text{if } c > 0 \\ \left\{x : f(x) > \dfrac{\alpha}{c}\right\} & \text{if } c < 0 \end{cases}$$

In any case the R.H.S set is measurable and hence L.H.S.

$\therefore$ cf is measurable.

(iii) To show that f + g is measurable.

Consider

$$\{x : (f+g)(x) < \alpha\} = \{x : f(x) + g(x) < \alpha\}$$

$$= \{x : f(x) < \alpha - g(x)\}$$

Since between any two reals, there is a rational, there exists a rational say r such that

$$f(x) < r < \alpha - g(x)$$

$$\{x : (f+g)(x) < \alpha\} = \bigcup_r \left(\{x : f(x) < r\} \cap \{x : g(x) < \alpha - r\}\right)$$

Since rationals are countable and since each of $\{x : f(x) < r\}$, $\{x : g(x) < \alpha - r\}$ is measurable for every r (as f and g are measurable) we get that $\{x : (f+g)(x) < \alpha\}$ is measurable.

$\therefore$ f + g is measurable.

(iv) $f - g = f + (-1)g$ and hence by (ii) and

(iii) f - g is measurable when f and g are so.

(v) the function f^2 is measurable.

For : $\{x : f^2(x) > \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) < -\sqrt{\alpha}\}$

when $\alpha \geq 0$ and $\{x : f^2(x) > \alpha\} = D$

when $\alpha < 0$ where D is the domain of f.

In any case $\{x : f^2(x) > \alpha$ is measurable.

Since f is measurable and D is measurable.

$\therefore$ f^2 is measurable.

if f and g are measurable.

Then $fg = \frac{1}{2}\ [(f + g)^2 - (f - g)^2]$ and hence by (ii), (iii), (iv) and the above result, fg is measurable.

Remark:-

If f and g are extended real valued functions.

We define $(f + g)(x) = f(x) + g(x)$

If f(x) and g(x) are real or one of them $\pm\infty$ or both are $+\infty$ or both are $-\infty$

$\therefore$ $(f + g)(x) =$ a constant C

If $f(x) = \infty,\ g(x) = -\infty$

(or) $f(x) = -\infty,\ g(x) = \infty$

Then f + g is measurable.

IIIly we can define f - g. However f g is always measurable.

Problem :-

Suppose f and g are two real valued measurable functions on the same domain. Suppose $g(x) \neq 0\ \forall x$ of the domain. Then show that f / g is measurable.

Solution :-

Let $g(x) \neq 0\ \forall\ x \in D$ where D is the domain of f and g.

Let $E_1 = \{x : g(x) > 0\}$ and $E_2 = \{x : g(x) < 0\}$

Then E_1 and E_2 are measurable.

$$\therefore \text{ When } \alpha > 0, \left\{x = \frac{1}{g(x)} > \alpha\right\} \Rightarrow \left\{x : g(x) < \frac{1}{\alpha}\right\} \cap E_1$$

When $\alpha > 0, \dfrac{1}{g(x)} > \alpha \Rightarrow g(x)$ is $\left[+\text{ ve}\right]$

Which is measurable.

When $\alpha < 0, \left\{x : \dfrac{1}{g(x)} > \alpha\right\}$

$$\left\{x : \frac{1}{g(x)} > 0\right\} \cup \left\{x : \alpha < \frac{1}{g\,(x)} < 0\right\}$$

$$= E_1 \cup \left(\left\{x : g(x) < \frac{1}{\alpha}\right\} \cap E_2\right)$$

which is measurable.

Since E_1, E_2 and $\{x = g(x) < 1/\alpha\}$ are measurable.

So, for $\alpha > 0$ (or) $\alpha < 0, \left\{x : \dfrac{1}{g(x)} > \alpha\right\}$ is measurable.

When $\alpha = 0, \left\{x = \dfrac{1}{g(x)} > 0\right\} = \{x = g(x) > 0$

Then $\dfrac{1}{g(x)}$ is measurable.

Since the product of two measurable functions is measurable.

We get that

$$\frac{f}{g} = f\left(\frac{1}{g}\right) \text{ is measurable.}$$

Theorem :-

Let $f_1, f_2 \ldots\ldots\ldots f_n$ be measurable functions defined on the same domain D.

Then max $\{f_1, f_2 \ldots\ldots\ldots f_n\}$ and min $\{f_1, f_2 \ldots\ldots\ldots f_n)$ are measurable.

Proof :

Let $h = \max (f_1, f_2 \ldots\ldots f_n)$

i.e, $h(x) = \max (f, (x), f_2 (x) \ldots\ldots\ldots f_n (x) \quad \forall x \in D.$

$$\{x : h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x : f_i (x) > \alpha\}$$

For : Let $h(x) > \alpha$.

Then max of $f_i (x) > \alpha$.

i.e., $\exists$ i such that $f_i (x) > \alpha$.

$$\therefore x \in \bigcup_{i=1}^{n} \{x : f_i (x) > \alpha\}$$

Conversely,

$$\text{Let } x \in \bigcup_{i=1}^{n} \{x : f_i (x) > \alpha\}$$

Then $\exists$i such that $f_i (x) > (\alpha)$.

Since $h(x) \geq f_i (x) \ \forall i$.

$\therefore h(x) > \alpha$.

$\therefore x \in \{x : h(x) > \alpha\}$

$$\therefore \{x : h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x : f_i (x) > \alpha\}$$

Since each $\{x : f_i(x) > \alpha\}$ is measurable.

$\therefore \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$ is measurable.

$\therefore \{x : h(x) > \alpha\}$ is measurable.

Hence h (x) is measurable.

Let $h_1 = \min \{f_1, f_2 \ldots\ldots f_n\}$.

Then

$$\{x : h_1(x) < \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) < \alpha\}$$

For : Let $x : h_1(x) < \alpha$.

Then min of $f_i(x) < \alpha$.

i.e., $\exists$ i such that $f_i(x) < \alpha$.

$\therefore \{x \in \bigcup_{i=1}^{n} \{x : f_i < \alpha\}$

Conversely,

Let $x \in \bigcup_{i=1}^{n} \{x : f_i(x) < \alpha\}$

Then $\exists\, i \ni : f_i(x) < \alpha$.

Since $h(x) \leq f_i(x)\ \forall i$.

$\therefore h(x) < \alpha$.

$\therefore x \in \{x : h(x) < \alpha\}$.

$\therefore \{x : h(x)\} < \alpha = \bigcup_{i=1}^{n} \{x : f_i(x) < \alpha\}$

Since each $\{x : f_i(x) < \alpha\}$ is measurable, $\bigcup_{i=1}^{n} \{x : f_i(x) < \alpha\}$

and hence $\{x : h(x) < \alpha\}$ is measurable.

Hence, h(x) is measurable.

Remark :

If f and g are measurable.

Then $f \vee g = \max(f,g)$, $f \wedge g = \min(f,g)$ are measurable.

Remark :

If f is measurable.

$f^+ = f \vee o = \max(f, 0)$ is measurable.

$f^- = -(f \wedge 0) - \min(f,0)$ is measurable.

f^+ is called the positive part of f and f^- the negative part.

Note :- (i)

If f^+ and f^- are measurable, then $f = f^+ - f^-$ is measurable and $|f| = f^+ + f^-$ is measurable.

Note : (ii)

The Converse of the result, "$|f|$ is measurable $f \Rightarrow$ is measurable" is not true. Let A be a non-measurable subset of R.

Define :

$$f(\chi) = \begin{cases} 1 & \text{for } x \in A. \\ -1 & \text{for } x \notin A. \end{cases}$$

Then $\{x : f(x) > -1\}$ is A which is non-measurable.

Hence f is not measurable.

But $|f| = 1 \quad \forall x \in R$ is a constant function and hence measurable.

Theorem :-

Let $\{f_n\}$ be a sequence of measurable functions (with the same domain of definition). Then the functions $\sup f_n$, $\operatorname{Inf} f_n$, $\overline{\lim} f_n$, $\underline{\lim} f_n$ are all measurable

Proof :-

Let $g(x) = \sup f_n(x)$, $h(x) = \inf f_n(x)$.

Then

$$\{x : g(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > \alpha\}$$

So g is measurable.

$$\{x : h(x) < \alpha\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) < \alpha\}$$

So h is measurable.

Since

$$\overline{\lim} f_n = \operatorname*{Inf}_{n} \sup_{k \geq n} f_k.$$

we have $\overline{\lim} f_n$ is measurable.

III^{ly} $\underline{\lim} f_n = \sup_{n} \operatorname*{Inf}_{k \geq n} f_k$ and hence

$\underline{\lim} f_n$ is measurable.

Remark:

Let $\{f_n\}$ be a sequence of measurable functions on the same domain D.

Let $\lim_{n \to \infty} f_n = f$. Then f is measurable.

For: if $\lim_{n\to\infty} f_n$ exists then $f = \lim_{n\to\infty} f_n = \underline{\lim} f_n = \overline{\lim}_{n\to\infty} f_n$ and hence f is measurable.

Definition :-

A Property is said to hold almost every where (abbreviated a.e) if the set of points where is fails to hold is a set of measure zero. Thus, in particular we say f = g a .e if f and g have the same domain and $m\{x : f(x) \neq g(x)\} = 0$.

Definition :-

If $\{f_n\}$ is a sequence functions then $\{f_n\}$ is said to converge to g almost every where if there is a set E of measure zero such that $f_n(x)$ converges to $g(x)\ \forall x \notin E$.

Theorem :

If f is a measurable function and f = g a.e., then g is measurable.

Proof :-

Let $E = \{x : f(x) \neq g(x)$.

By hypothesis, $mE = 0$.

$\{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \{x \in E : g(x) > \alpha\} - \{x \in E : g(x) \leq \alpha\}$

For :

$g(x) > \alpha$, if and only if

$x \in E$ and $g(x) > \alpha$ or $x \notin E$ and $g(x) > \alpha$.

i.e., $x \in E : g(x) > \alpha\} \cup \{x \notin E : g(x) > \alpha\}$

i.e., $\{x \in E : g(x) > \alpha\} \cup \{x \notin E : f(x) > \alpha\}$

$(\because f(x) = g(x)$ on $\tilde{E})$

i.e., $\{x \in E : g(x) > \alpha\} \cup \{x : f(x) > \alpha\} - \{x \in\in : g(x) \leq \alpha\}$

Hence

$\{x : g(x) > \alpha\} = \{x \in E : g(x) > \alpha\} \cup \{x : f(x) > \alpha\} - \{x \in E : g(x) \leq \alpha\}$

$\{x : f(x) > \alpha \}$ is measurable.

Since $mE = 0$, and since $\{x \in E : g(x) > \alpha\}$ and $\{x \in E : g(x) \le \alpha\}$ are subsets of E.

We get that they are of measure 0 and hence measurable.

$\therefore \{x : g(x) > \alpha\}$ is measurable.

i.e., g is measurable.

Definition :-

Simple function :-

A real valued function ϕ is called simple it is measurable and assumes only a finite number of values.

If ϕ is simple and has the values $\alpha_1, \ldots\ldots\alpha_n$ then $\phi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$

where $A_i = \{x : \phi(x) = \alpha_i\}$

Since f is measurable, each A_i is measurable.

Note :

The sum, product and the difference of two simple functions are simple. For the sum, product and difference of two simple functions are measurable and assume only a finite number of values.

Definition :-

A step function on a domain D is a simple function $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ are bounded intervals.

Problem :-

Let f be a non-negative measurable function on D. Then there exists a non-decreasing sequence f_n of non-negative simple functions converging to f.

Solution :

Given f is a non-negative measurable function.

i.e., $f(x) \ge 0.\ \forall x \in D$.

Define for each positive integer n.

$$f_n(\chi) = \begin{cases} \dfrac{i-1}{2^n} & \text{when } \dfrac{i-1}{2^n} \le f(\chi) \le \dfrac{1}{2^n},\ i = 1, 2, \ldots\ldots n. \\ n & \text{for } f(x) \ge n. \end{cases}$$

clearly f_n are simple, non-negative functions.

and $f_1 \le f_2 \le \ldots\ldots\ldots\ldots \le f_n \le \ldots\ldots\ldots$

Also if $f(x) < \infty$.

$$\text{then } 0 \le f(x) - f_n(x) \frac{1}{2^n} \le \in$$

for sufficiently large n.

$$\lim_{n\to\infty} f_n(x) = f(x)$$

If $f(x) = \infty$, $f_n = (x) = n \ \forall\ n$ and hence

$$\lim_{n\to\infty} f_n(x) = f(x)$$

Hence the Problem.

Note :

If f is any measurable function, f can be written as $f^+ - f^-$

Both f^+ and f^- are non-negative measurable functions.

Hence $f^+ = \lim\limits_n f_n$, $f^- = \lim\limits_n g_n$, where

f_n, g_n are simple functions.

$\therefore$ $f = \lim (f_n - g_n)$ and each $f_n - g_n$ is simple.

Problem:-

Let D be a dense set of real numbers. Let f be an extended real valued function on R such that $\{x: f(x) < \alpha\}$ is measurable for each $\alpha \in D$. Show that f is measurable.

Solution :-

Since D is dense set of real numbers every interval contains an element of D. Let a be any real number.

Then for every n, the interval $[a - 1/n, a]$ all contains an element $\alpha_n \in D$.

Then $\{x : f(x) < a\} = \bigcup_n \{x : f(x) < \alpha_n < a_n\}$

For : if for some x, $f(x) < a$, $\exists$ n such that

$f(x) < a - 1/n$ and so $f(x) < \alpha n$.

$\therefore x \in \{x : f(x) < \alpha_n\}$

i.e. $\{x : f(x) < a\} \subset \bigcup_n \{ : f(x) < \alpha_n < a\}$

The other inclusion is obvious.

Hence,

$$\{x : f(x) < a\} = \bigcup_n \{x : f(x) < \alpha_n < a\}$$

now

$\{x : f(x) < \alpha_n\}$ is measurable and countable union of measurable sets is measurable.

Hence f is measurable.

Definition :-

A function f is said to be Borel measurable if for each $\{x : f(x) > \alpha\}$ is a Borel set.

Problem :-

Show that there exists a measurable set which is not a Borel set.

Sol:-

For all $x \in [0,1]$

Let $x = \sum_{n=1}^{\infty} \frac{a_n(x)}{2^n}$ be the binary expansion

Where $a_n(x) = 0$ or 1.

(Choose a non-terminating expansion for $x > 0$ (i.e) $1/2 = .1 = .099999.....$)

Define the function f by

$$f(x) = \sum_{n=1}^{\infty} \frac{2 a_n}{3^n}$$

Since in the ternary expansion of numbers in the cantor ternary set C, a_n is) or 2. we get that

$f(x)$ lies in $C \; \forall x \in [0,1]$.

Each $a_n(x)$ is a measurable function.

For : $a_n(x)$ is either 0 or 1.

$a_n(x) = 0$ if either $0 \leq x < 1/2^n$.

$$\frac{1}{2^{n-1}} \leq x < \frac{1}{2^{n-1}} + \frac{1}{2^n}$$

$$\frac{1}{2^{n-2}} \leq x < \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} + \frac{1}{2^n}$$

and each is measurable set.

Thus $\{x = a_n(x) = 0\}$ is measurable and hence its complement is measurable.

Thus f is measurable.

$$\therefore f = \lim_{n \to \infty} \left(\frac{2a_1}{3} + \frac{2a_2}{3^2} + \ldots\ldots + \frac{2a_n}{3^n} \right) \text{ and}$$

$$\left(\frac{2a_1}{3} + \ldots\ldots + \frac{2a_n}{3}\right) \text{ is measurable}$$

f is 1-1 and on to its range.

Hence for any subset $V \subset [0,1]$

$f^{-1}(f(v)) = v.$

Let v be a non-measurable subset of $[0,1]$

[Take for example v=p, the non-measurable set that was constructed in $[0,1]$

Then $f(v) \subset C$.

But $m(C) = 0$.

Hence $m(f(v)) = 0$.

$\therefore$ f(v) is a measurable set.

But $f^{-1}(f(v)\_ = V$ is not measurable.

Hence $f(v) = V$ is not a Borel set.

Since if f(v) is a Borel set then $f^{-1}(f(v))$ is measurable.

Thus f(v) is a measurble set.

But not a Board set.

Remark :-

In this example we have proved that the pre-image of a measurable set under measurable function need not be measurable.

Little Wood's three Principles :-

J.E. Littlewood in his 'Lectures on the theory of functions' (Oxford 1994). The extent of knowledge required is nothing like so great as is sometimes supposed. There are three principles. roughly expressible in the following terms. Every measurable set is nearly a finite Union of intervals, every measurable function is nearly continuous, every convergent sequence of measurable functions is nearly uniformly convergent. Most of the results of the theory are fairly intuitive applications of these ideas, and the student armed with them should be equal to most occasions when real variable theory is called for. If one of the principles would be the obvious means to settle the problem if it were 'quite' true. It is naural to ask if the 'nearly' is near enough and for a problem that is actually solvable it generally is".

Theorem :- (i)

Let E be a measurable set of finite measure. Let {fn} be a sequence of measurable functions defined on E. Let f be a measurable real valued function such that for each $x \in E$. We have $f_n(x) - f(x)$. then given $\in > 0$, and $\delta > 0$. There is a measurable set $A \subset E$ with $mA < \delta$ and an integern such $\forall\, x \notin A$ and all $n \geq N$, $|f_n(x) - f(x)| < \in$.

Proof :-

Let $G_n = \{x \in E : |f_n(x) - f(x)| \geq \in\}$

Let $E_N = \bigcup\limits_{n=N}^{\infty} G_n = \{x \in E : |f_n(x) - f(x)| \geq \in \text{for some } n \geq N\}$

Since, $E_N = \bigcup\limits_{n=N}^{\infty} G_n$; $\quad E_{N+1} = \bigcup\limits_{n=N+1}^{\infty} G_n$.

We have

$E_{N+1} \subset E_N$

Since $f_n(x) \to f(x)$, $\forall\, x \in E$. We get that for each $x \in E$. There must be some E_N such that $x \notin E_N$.

Hence $\cap E_N = \phi$ (If $\cap E_N \neq \phi$ then $\exists\, x \in E$ such that $x \in E_N\ \forall\, N$, a contradiction).

Each G_n is a measurable set.

($\therefore f_n - f$ is meaurable $\forall n$).

$\therefore E_N$ is measurable $\forall$ N.

Since E is of finite measure and $E_N \subset E$,

$E_N{}^s$ are of finite measure.

Hence, $\lim_{n \to \infty} mE_n = m\left(\bigcap_{n=1}^{\infty} E_n\right) = m\phi = 0.$

i.e. $\lim_{n \to \infty} mE_n = 0$

Hence given $\delta > 0$, $\exists N$ such that $mE_N < \delta$.

i.e. $m\{x : |f_n(x) - f(x)| \geq \in \text{ for some } n \geq N\} < \delta$.

Let this E_N be denoted by A.

Then $mA < \delta$ and

$\tilde{A} = \{x \in E : |f_n(\chi) - f(x)| < \in \forall\, n \geq N.$

Hence the theorem.

Definition :-

Pointwise convergence:-

Let E be a set and let $\{f_n\}$ be a sequence of functions defined on E. We say that $\{f_n\}$ converges pointwise to f on W if $f_n(x) \to f(x)$ for each $x \in E$.

Definition :-

If there is a subset B of E such that $mB = 0$ and $f_n \to f$ pointwise on E–B. Then we say that $f_n \to f$ a.e on E.

Theorem :-

Let E be a measurable set of finite measure and $\{f_n\}$ a sequence of measurable functions which converge to a real valued function f a.e on E. Then given $\in > 0$ & $\delta > 0$ there is a set $A \subset E$ with $mA < \delta$ and an N such that $\forall\, x \notin A$ and all $n \geq N$, $|f_n(x) - f(x)| < \in$.

Proof :-

Let f_n converges to f on E - F.

Where m (F) = 0

Consider the sets G_n, E_n.

By thm (i) as sets contained in E-F.

Then $m(F_N) > \delta \Rightarrow m(E_N \cup F) \leq m(E_N) + m(F) < \delta$.

For $x \notin E_N \cup F \ \forall \ n \geq N$.

$\therefore |f_n(x) - f(x)| < \in$.

Example:-

Let $f_n : (0,1) \rightarrow R$ be defined by $fn(x) = \chi^n$.

$$\text{Then } \operatorname{Lim} f_n(x) = \begin{cases} 0 & \text{when } 0<x<1 \\ 1 & \text{when } x=1 \end{cases}$$

Let f be defined as $f(x) = 0 \ \forall x \in [0,1]$

Then f_n converges to f a.e. The one point at which f_n fails to coverges to f is 1 and its measure is 0. Note that f_n does not converge to f everywhere on $[0,1]$

Definition:-

Let $\{f_n\}$ be a sequence of function defined on E. Then f_n is said to converge to f uniformly on E if $\forall \in < 0$, there exists a '(+ve)' integer N depending on E only but not on $x \in E$ such that

$|f_n(x) - f(x)| < \in, \forall n \geq N, \forall x \in E$.

Definition:-

$|f_n|$ is said to converge almost uniformly on E if $\forall \in > 0$, there exists a measurable subset $B \subset E$ s.t $mB < \in$ and f_n converges to f uniformly on E-B.

Remark:-

When E is a measurable set $\{f_n\}$ are measurable on E, $f_n \rightarrow f$ almost uniformly on E. Then f is measurable. Note that almost uniform convergence $\Rightarrow$ convergence a.e.

Example:-

$f_n : [0,1] \to R,\ f_n(x) = x^n.$

Let $f(x) = 0 \quad \forall x \in [0,1]$

Then given $\in > 0$, f_n converges uniformly to f on $[0,1-\in]$

i.e. f_n converges to f a.u (almost uniformly)

Remark :-

From the above example we get that a.u. convergence neednot imply uniform convergence, a.u. convergence need not imply uniform convergence.

Example :-

This example shows that a.u. convergence need not imply a.u. convergence.

Let $f_n = \psi\ [n, n_{+1}]$ and $f=0$.

Then given $x \in R$, we can find a (+ve) integer N such that $N \le x < N+1$.

Then $f_n(x) = 0 \quad \forall\, n \ge N+1$ and hence $f_n(x) \to f(x)$.

Thus $f_n \to f$ every where and hence $f_n \to f$ a.e. But f_n does not converge to f a.u.

For : Suppose $fn \to f$ a.u.

Let $\in > 0$ be given then there is a measurable set $E \subset R$ such that $mE < \in$ and $f_n \to f$ uniformly on R-E.

i.e. Given $\in > 0$, there exists, a (+ve) integer N such that

$|f_n(x) - f(x) < \in \quad \forall\, x \ge N, \qquad \forall\, x \in R-E.$

i.e. $\quad |f_n(x)| < \in\ \forall\, x \ge N, \quad \forall\, x \in R-E.$

i.e. $\quad |\psi_{[n,\ n+1]}(x)| < \in \quad \forall\, x \ge N, \quad \forall\, x \in R-E.$

i.e. $\psi_{[n,\ n+1]}(x) = 0 \ \forall\ x \geq N, \quad \forall\ x \in R - \in$

$(\because \psi(x)$ is either 1 Or 0)

i.e. x is not in each of the intervals [N, N+1], [N+1, N+2] ..... for every $x \in R-E$.

$x < N \ \forall\ x \in R\text{-}E$.

i.e. $E = [N, \infty]$ m (E) is infinite a contradiction.

Since $mE < E$.

Hence f_n does not tend to f a.u.

Thm :-

If $f_n \to f$ a.u then $f_n \to f$ a.e

Proof :-

Let $f_n \to f$ a.u.

Then given $\in > 0$, $\exists$ a E such that $m E < \in$ and $f_n \to f$ uniformly on $\tilde{E}$

$\therefore$ For each n, $\exists\ E_n$ such that $mE_n < \frac{1}{n}$ and $f_n \to f$ uniformly on $\tilde{E}_n$.

Let $x \in \bigcup\limits_n \tilde{E}_n$

Then $x \in \tilde{E}_m$ for some m.

But f_n converges to f uniformly on $\tilde{E}_k$.

$$\therefore \lim_{n \to \infty} f_n(x) = f(x)$$

ie) $f_n(x) \to f(x) \ \forall\ x \in \bigcup \tilde{E}_n$.

$\therefore m\, (\widetilde{UEn}) = m\, (\bigcap\limits_n E_n) < m\,(E_n) = 1/n \ \forall n$.

$$m\widetilde{\left(\bigcup \tilde{E}_n\right)} = 0.$$

$\therefore$ $f_n \to f$ except on a set of measure zero.

ie) $f_n \to f$ a.e.

Hence the theorem.

Thm :- (EGOROFF'S theorem)

Let $\{f_n\}$ be a sequence of measurable functions which converges to a real valued function a.e. on a measurable set E of finite measure.

Then given $\eta > 0$, there is a set $A \subset E$ with $mA < \eta$ such that f_n converges to f uniformly on E – A.

Proof :-

By the above theorem,

take $\epsilon_n = \dfrac{1}{n}, \delta_n = \dfrac{\eta}{2^n}$

We get a set $A_n \subset E$ with $mA_n < \delta_n$ and n such that $\forall x \notin A_n$.

$| f_k(x) - f(x) | < 1/n \ \forall k \geq n.$

Let $A = \bigcup\limits_{n}^{\infty} A_n$

Then $m(A) = m \bigcup A_n \leq \sum\limits_{n=1}^{\infty} m(A_n) \leq \sum\limits_{n=1}^{\infty} \delta_n$

$= \eta \sum\limits_{n=1}^{\infty} \dfrac{1}{2^n} = \eta$

Let $\epsilon > 0$ be given

Then $\exists$ N such that $\dfrac{1}{N} < \epsilon$. And so $\forall n \geq N$, $\dfrac{1}{n} \leq \dfrac{1}{N} < \epsilon$

For $\chi \notin A$, and $n \geq N$ $\left| f_n(x) - f(x) \right| < \dfrac{1}{n} \epsilon$

(Since $x \notin A \Rightarrow x \notin A_n \ \forall n$)

Hence f_n converges to f uniformly on E – A and $mA < \delta$.

Note :-

If F is of finite measure, then F need not be bounded.

For example :- Q is of measure O but Q is not bounded.

But when F is closed, and F is of finite measure, then f is bounded.

For : if f is not bounded then given any $m > 0$ however large, $\exists\, a_n \in F$ such that $|a_n| > n$. Then $\{a_n\}$ diverges.

As F is closed, it contains the limit point.

ie) ∞ or $-\infty \in F$ a contraction to $mF < \infty$.

Theorem :- (LUSIN'S theorem)

Let f be a measurable real valued function on a closed interval [a, b]. Then given $\delta > 0$ there exists a continuous function ϕ on [a, b] such that $m\{\chi : (fx) \neq \phi(x)\} < \delta$.

Proof :-

Let $\delta > 0$ be given.

By Lemma, Let f be a finite valued and measurable on a set E of finite measure. Then given $\in > 0$, $\exists$ a closed set k such that $K \subset E$, $m(E - k) < \in$ and f is continuous on k.

$\therefore$ We can find a closed set k such that $K \subseteq \left[a + 8/4,\ b - 8/4\right]$ and f is continuous on k.

And

$$m\left(\left[a + \frac{\delta}{4},\ b - \frac{\delta}{4}\right] - k\right) < \frac{\delta}{2}$$

Hence

$$m\left(\left(\left[a + \frac{\delta}{4},\ b - \frac{\delta}{4}\right] - k\right) U \left(\left[a,\ a + \frac{\delta}{4}\right] U \left[b - \frac{\delta}{4},\ b\right]\right)\right)$$

$$< \frac{\delta}{2} + \frac{\delta}{4} + \frac{\delta}{4}$$

$$= \delta \text{ ---------- (1)}$$

Also the set $a + \frac{\delta}{4},\ b - \frac{\delta}{4} - k$ is open set in $\left[a + \frac{\delta}{4},\ b - \frac{\delta}{4}\right]$ and

so it is in the union of almost countable number of disjoint open intervals in

$\left(a + \frac{\delta}{\cdot}, b - \frac{\delta}{4}\right)$ now define g on [a. b] as follows.

If (x, y) is one such interval mentioned above define g on (x, y) as the linear function having $g(x) = f(x)$, $g(y) = f(y)$.

Define g = f on k.

on $\left(a, a + \frac{\delta}{4}\right)$ define $g(a) = 0$.

$g\left(a + \frac{\delta}{4}\right) = f\left(a + \frac{\delta}{4}\right)$ and g linear on $\left(a, a + \frac{\delta}{4}\right)$

III^{ly} on $\left(b - \frac{\delta}{4}, b\right)$

define $g\left(b - \frac{\delta}{4}\right) = f\left(b - \frac{\delta}{4}\right)$, $g(b) = 0$ and g is linear.

Then g is continuous and

$\{x : g(x) \neq f(x)\} = \left(a, a + \frac{\delta}{4}\right) \cup \left(b - \frac{\delta}{4}, b\right) \cup \left\{\left(a + \frac{\delta}{4}, b - \frac{\delta}{4}\right) - x\right\}$

$< \delta$ (Using (1))

∴ Hence the theorem.

Note :-

From the definition of g it is clear that if $m \leq f \leq M$. then $m \leq g \leq M$.

Corollary :-

Let f be measurable on R (= $(-\infty, \infty)$) and finite a.e. Thus given $\delta > 0$, ∃ a continuous function g on R such that

$m\{x : f(x) \neq g(x)\} < \delta$.

Proof :-

$\forall n \in Z$.

Define g on [n, n+1] as the function constructed in the above thm,

with δ replaced by $\frac{\delta}{2^{n+2}}$

Since $g(n) = 0 \forall n \in Z$.

and g is continuous on [n, n+1]

g is continuous on R.

$$\text{Also } m\{x : f(x) \neq g(x)\} < \frac{\delta}{2^2} + \frac{2\delta}{2^3} + \frac{2\delta}{2^4} + \ldots$$

$$= \frac{\delta}{2^2} + \frac{2\delta}{2^3}\left(1 + \tfrac{1}{2} + \ldots\right)$$

$$= \frac{\delta}{4} + \frac{\delta}{4} - 2$$

$$< \delta$$

Hence the corollary.

Corollary :-

Let f be measurable on R and finite a.e. Then $\exists$ a sequence f_n of continuous functions on R such that $f_n \rightarrow f$ a.e.

Proof :-

For each $x \in N$, let f_n be the function constructed as in the above corollary with $= \delta \frac{1}{2^n}$

Let $E_n = \{ x : f_n(x) \neq f(x) \}$

Let $E = \overline{\lim}\ En$

$$\text{Then } E = \bigcap_{n=1}^{\infty}\left(\bigcup_{n=k}^{\infty} E_n\right)$$

$$m\left(\bigcup_{n=k}^{\infty} E_n\right) \leq \sum_{K=N}^{\infty} m(E_k) \leq \sum_{k=n}^{\infty} \frac{1}{2^n}$$

$$= \frac{1}{2^{n-1}}$$

$$\therefore\ E \subset \bigcup_{n=k}^{\infty} E_k\ \forall n \Rightarrow m(E) \leq \frac{1}{2^{n-1}}\ \forall n.$$

$\therefore\ mE = 0.$

For $x \notin E, \chi \notin \bigcup_{k=n}^{\infty} E_k$ for some N.

$\therefore\ x \notin E_K,\ \forall K \geq N$

ie) $x \notin \tilde{E}_K,\ \forall K \geq N$

ie) $f_n(x) = f(x)\ \forall\ n \geq N$

$$\therefore \lim_{n\to\infty} f_n(x) = f(x)\ \forall \chi \notin E$$

So $f_n \to f$ a.e.

Hence the corollary.

Lemma :-

Let E be a set of finite measure.

$\forall \in > 0,\ \exists$ a step function s such that $m\{x : X_E(x) \neq s(x)\} < \in.$

Proof :-

We have proved that, given $\in > 0$, there exist finite intervals I_1, I_2. .... I_n such that

$m(E \Delta U) < \in$ where $U = \bigcup_{k=1}^{n} I_k$

(Since E is of finite measure)

Let $S = \chi_{I_1} + \chi_{I_2} + \ldots\ldots\ldots \chi_{I_n}$

$\{x : \psi_E(x) \neq s_{(x)}\} = E \Delta U$ and $m(E \Delta U) < \in$

Hence the Lemma.

Problem :-

Given a lebesgue measurable function f, show that $\exists$ a Borel measurable function g such that g = f a.e.

Proof :-

Let f be lebesgue measurable function.

Let $E_r = \{x : f(x) < r\}$ for every rational number r.

Then E_r is a measurable set.

We know there exists an F_r set $F_r \subset E_r$

and $m\left[(E_r - F_r)\right] = 0$

then $m \; U(Er - Fr) = 0$.

Since it is a countable Union.

Let N be a G_δ Set N such that

$N \supset \underset{r}{U}(E_r - F_r)$ and $m(N) = 0$.

Define

$g(x) = f(x)$ on $\tilde{N}$

$= 0$ on N.

Then g is a Borel measurable function and $g = f$ a.e.

Hence the Problem.

Topic :- 2

The Riemann Integral:-

Just as we defined the measure function on the collection of measurable sets, we will now define a function (called integration) from the collection of measurable functions to R^*. The enterded real number system.

We define this function in three steps.

Step -1:-

Integration of bounded measurable function on a set of finite measure.

Step - 2:-

Integration of non-negative measurable function.

Step - 3:-

Integration of any measurable function. To define the integration of bounded measurable function on a set of finite measure, we recall the definition of Riemann integral and try to generalize this definition.

Definition:- (The Riemann Integral)

Let f be a bounded real value function on $[a,b]$ (a,b finite real numbers)

Let $a = x_0 < x_1 < \dots\dots\dots < x_n = b$ be a subdivision of $[a, b]$

Since f is bounded on $[a,b]$ it is bounded on each subinterval (x_i, x_{i-1})

$M_i = \sup f(x)$ and $m_i = \inf f(\chi)$ exist. $\chi_{i-1} < \chi < M_i$.

Let $S = \sum_{i=1}^{n} (x_i - x_{i-1}) M_i$

and $S = \sum_{i=1}^{n} (x_i - x_{i-1}) M_i$

We define the upper Riemann integral of f by

$$R \int_{a}^{\bar{b}} f(x)\, d(x) = \inf S$$

and the lower Riemann integral of f by $R \int_{\bar{a}}^{b} f(x)\, dx = \text{Sup } S.$

Where the inf S and sup S are taken over all possible subdivision of (a,b).

It can be proved that

$$R \int_{\bar{a}}^{b} f(x)\, dx \le R \int_{a}^{\bar{b}} f(x)\, dx$$

If these two integrals concider, i.e. if

$$R \int_{\bar{a}}^{b} f(x)\, dx = R \int_{a}^{\bar{b}} f(x)\, dx$$

We say f is Riemann integrable and this common value is valled Riemann integral of f.

It is denoted as

$$R \int_{a}^{b} f(x)\, dx$$

Examples:-

1. Every continuous function on $[a,b]$ is Riemann integrable.
2. Every function on $[a,b]$ with finite numbr of points of discontinuity is Riemann integrable.
3. Every monotonic function on $[a,b]$ is Riemann integrable.

Note :- (i)

Let f be the Dirichlet's function defined on $[a,b]$ as

$f(x) = 1$ if x is rational, $a \le x \le b$

$= 0$ if x is irrational, $a \le x \le b$

i.e., $f = x_Q$ where Q is the set of rational numbers in $[a,b]$

Then f is not Riemann integrable.

For, if $a = x_0 < x_1, < \ldots < x_n = b$ is any subdivision. (x_{i-1}, x_i) contains both rational and irrational numbers.

Hence $M_i = 1$ and $m_i = 0$, whenever be the subdivision.

$$\therefore S = \sum_{i=1}^{n} (x_i - x_{i-1}) M_i = b-a$$

$$\text{and } S = \sum_{i=1} (x_i - x_{i-1}) M_i = 0$$

for all sub divisions.

Hence, the upper Riemann integral of f= (b-a) and the lower Riemann integral of f = 0.

$\therefore$ f is not Riemann integrable.

Note :- (ii)

Let $f : [a,b] \to R$ be the functions defined in Note (i)

(ii) for $a \leq x \leq b$, $f(x) = 1$ if x is rational

$= 0$ if x is irrational.

Since the set of rationals in countable, we can enumerate then $x_1, \ldots$ $x_n \ldots$ be the rational numbers in $[a,b]$

Define f_n : a,b $\to$ R as follows.

$f_n(x_i) = 1$ for $i = 1, 2 \ldots n$.

0 for all other $\chi \in (a, b)$. Then f_n are discontinuous at points $x_1, x_2 \ldots x_n$.

Hence f_n are Riemann integrable. So, $\{f_n\}$ is a monotonic increasing sequence of Riemann integrable functions converging to f, a function which is not Riemann integrable.

Topic - 3

The Lebesgue Integral of a Bounded function over a set of finite measure:-

In order to define the Lebesgue integral, we first recast the definition of the Riemann integral. For this, let us recall the definition of a step function.

Definition :-

The step function is a function from $D \subset R$ to R. S.t $D = \bigcup_{i=1}^{n} I_i$ and

$S(x) = C_i \ \forall \ x \in I_i$.

i.e. $S = \sum_{i=1}^{n} C_i \times I_i$

Let $a = x_0 < x_1 < x_2 < \dots x_n = b$ be a subdivision of (a,b)

Then $S = \sum_{i=1}^{n} C_i \chi (x_{i-1}, x_i)$ is a step function.

Note :-

Given a step function $S = \sum C_i \chi E_i$.

itsRiemann integral $R\int_a^b S(x)\, d(x) = \sum_{i=1}^{n} c_i \ell [I_i]$ (verify)

So, if f is a bounded function [a,b] and if $a = x_0 < x_1 < \dots\dots x_n = b$ is subdivision of [a,b] and if.

$M_i = \text{Sup } f(x)$ and $m_i = \inf f(x_i)$

$\chi_{i-1} \le \chi \le \chi_i \qquad \chi_{i-1} < \chi \le \chi_i$

Then $\psi = \sum_{i=1}^{n} M_i \chi (x_{i-1}, x_i)$ and

$\phi = \sum_{i=1}^{n} \mu_i \chi (x_{i-1}, x_i)$ are two step functions such that

$\phi \le f \le \psi$

Also,

$R\int_a^b \psi\, dx = \sum_i \mu (x_i - x_{i-1}) = S$

and $R\int_a^b \phi\, dx = \Sigma m_i (x_i - x_{i-1}) = s$

and $R\int_a^b \phi\, dx \le R\int_a^b f\, dx \le R\int_a^b \psi\, dx$

So, if we consider all step functions $g \ge f$, The above collection of ψ for various subdivisions of a,b will be a subcollection of the collection of all step function $g \ge f$.

$\therefore$ $\inf R\int_a^b g dx \le \inf R\int_a^b \psi\, dx$ ------------ (1)

III[ly], if we consider the collection of all step function $h \le f$ the above collection of ϕ will be its subcollection and hence.

$\text{Sup}\, R\int_a^b h dx \ge \text{Sup}\, R\int_a^b \phi\, dx$ ------------ (2)

If f is Riemann integrable.

$\text{Sup}\, R\int_a^b \phi\, dx = R\int_a^b f dx = \inf R\int_a^b \psi\, dx$ ------------ (3)

By (1), (2) We get

$\text{Sup}\, R\int_a^b \phi\, dx \le \text{Sup}\, R\int_a^b h dx \le \inf R\int_a^b g dx \le \inf \int_a^b \psi\, dx.$

By (3) all these inequalities will become equalities if f is Riemann integrable.

Also,

$\underset{h\le f}{\text{Sup}}\, R\int_a^b h dx = \underset{g\ge f}{\inf}\, R\int_a^b g dx.$

$\therefore$ g and h being step functions.

$$R\int_a^b f(x)\,dx = \sup_{h \le f} R\int_a^b h(x)\,dx = \inf_{g \ge f} R\int_a^b g(x)\,dx \quad \text{(Verify)}$$

We will new generalize this definition. First step towards this generalization is to consider simple function instead of step functions.

Let us recall the definition of on a simple function.

Defn:-

A simple function is a function assuming finite number of values, each on a measurable set.

Note (i)

Every step function is a simple function but χ_Q the set of rationals, is a simple function but not a step function.

Note (ii) :-

If f is measurable and s is a step function then fos is a simple function

Note (iii) :-

By note - (i), we see that the set of step functions on a given domain is a subcollection of the set of all simple function on the same domain.

We will first define the lebesque integral of a simple functions. If ϕ is a simple function. We use the notation $\int \phi$ to denote its Lebesgue integral. The simplest simple function is χ_A for any measurable set A.

(i) We define $\int \chi_A = m(A)$

(ii) Let ϕ be a simple function vanishing outside a set of finite measure and let

$\phi = \sum_{i=1}^{n} a_i \, \chi A_i$ be its canonical representation.

(i.e. a_i - distinct numbers, A_i - disjoint sets and $U_{A_i} = D$ the domain or ϕ)

Define $\int \phi(x)\,dx = \sum_{i=1}^{n} a_i\, mA_i$.

It is often convenient to use representations of simple function which are not canonical. The following Lemma gives the equivalent definition of the Lebesgue integral of a simple function.

Lemma :-

Let $\phi = \sum_{i=1}^{n} a_i \psi_{E_i}$ with $E_i \cap E_j = \phi$ for $i \neq j$.

Suppose each set E_i is a measurable set of finite measure.

Then $\int \phi = \sum_{i=1}^{n} a_i \; m_{E_i}$

Proof:-

Let $\{C_1, C_2 \ldots\ldots C_k\}$ be the distinct values assumed by ϕ.

Let $A_i = \{x : \phi(x) = C_i\}$

Clearly $A_i = \cup E_i$ ($\because$ E_i is a disjoint sets)

$$a_i = c_i$$

$\therefore$ A_i is measurable and it is finite Union of E_i.

$\therefore m(A_i) = \Sigma\, m(E_i)$

$$a_i = C_i$$

now, $\Sigma\, C_i\, X_{A_i}$ is the canonical representation of ϕ.

$$\therefore \int \phi = \sum_i C_i m(A_i)$$

$$= \left(\Sigma \;\; C \sum_{a_i = c_i} m(E_i) \right)$$

$$\therefore \int \phi = \Sigma\, a_i m(E_i)$$

Hence the lemma.

Lemma :-

Let ϕ and ψ be simple functions which vanish outside a set of finite measure. Then

$$\int (a\phi + b\psi) = a \int \phi + b \int \psi \text{ and if } \phi \geq \psi \text{ a.e., then } \int \phi \geq \int \psi$$

Proof:

Let A_i's be the set used in canonical representation in ϕ & B_j's be the sets used in canonical representation of ψ

Let A_0 be the set on which $\phi = 0$ and B_0 be the set on which $\phi = 0$.

Let E_k be measurable sets of the form $A_i' \cap B_j$. Then E'_k S are pairwise disjoint.

Lemma :

$$\phi = \sum_{k=1}^{n} a_k \ \psi \ E_K \text{ and } \psi = \sum_{k=1}^{n} b_K \psi_{E_K}$$

$$a\phi + b\psi = \sum_{k=1}^{n} (aa_k + bb_k)\, \psi_{E_K}$$

$$\int a\phi + b\psi = \sum_{k=1}^{n} (aa_k + bb_k)\, m(E_k)$$

$$= a \sum_{k=1}^{n} a_k \ m(E_k) + b_k \sum_{k=1}^{n} b_k \, m(E_k)$$

$$= a \int \phi + b \int \psi.$$

The integral of non-negative simple function vanishes outside the setof finite measure is non-negative.

$$\phi \geq \psi \Rightarrow \phi - \psi \geq 0$$

$$\Rightarrow \int (\phi - \psi) \geq 0$$

$$\Rightarrow \int \phi - \int \psi \geq 0$$

$$\Rightarrow \int \phi \geq \int \psi$$

Hence the lemma.

Definition :

Let ϕ be a simple function defined in D and $E \subset D$. Then we define

$$\int_E \phi = \int \phi \, X_E.$$

ie) If $\phi = \Sigma a_i \, X A_i$ than

$$\int_E \phi = \int (\Sigma a_i \chi_{A_i}) \chi_E$$

$$= \int \Sigma a_i \chi_{A_i \cap E}$$

$$\int_E \phi = \int \Sigma a_i m \ (A_i \cap E)$$

Lemma :

If $\phi = \sum_i a_i \chi_{A_i}$ be a simple function.

Then (i) If $m(\phi) = 0$, then $\int_E \phi = 0$

(ii) If $E_1, E_2 \ldots E_k$ are disjoint measurable set wtih $E = \bigcup_{=1}^{k} A_i$ then

$$\int_E \phi = \sum_{i=1}^{k} \int_{E_i} \phi.$$

Proof:

(i) $\phi = \Sigma a_i \chi_{A_i}$

$\phi \chi_E = \Sigma a_i \chi_{A_i \cap E}$

$\int_E \phi \, a_i m \ (A_i \cap E)^k$

Since $m(E) = 0$, $m(A_i \cap E) = 0 \ \forall \ i$.

$$\therefore \int_E \phi = 0$$

(ii) $\phi = \Sigma \chi_{A_i}$, $E = \bigcup_{j=1}^{k} E_j$

$$\phi \chi_{E_i} = \Sigma a_i \chi_{A_i \cap E_j}$$

$$\int_{E_i} \phi = \Sigma_j a_i m_i \ (A_i \cap E_i)$$

$$\therefore \sum_{j=1} \int_{Ei} \phi = \sum_j \sum_i a_i m\,(A_i \cap E_j)$$

$$= \sum_i a_i \left(\sum_j m(A_i \cap E_j) \right)$$

$$= \sum_i a_i\, m\,(\bigcup_j A_i \cap E_j) \quad (\because E_i \text{ are disjoint})$$

$$= \sum_i a_i m\,(A_i \cap (\bigcup_i E_i))$$

$$= \sum_i a_i m\,(A_i \cap E) = \int_E \phi$$

Hence the lemma.

Lemma :-

Let f be defined and bounded on a measurable set E with mE finite.

Then

$$\inf_{f \le \psi} \int_E \psi(x)\,dx = \sup_{f \ge \phi} \int_E \phi\,(x)\,dx$$

If and only if f is measurable.

Proof:

(i) Let f be measurable.

Let $|f(x)| \le M \ \forall\, x \in E$

Since f is a bounded function, such that Mexists]

Let

$$E_k = \left\{ x : \frac{(K-1)\,M}{n} < f\,(x) \le \frac{KM}{n} \right\}, \ -n \le K \le n$$

Since f is measurable, each E_k is measurable.

Also $\{E_k\}$ are disjoint and

$$\bigcup_{K=-n}^{n} m\,E_k = E$$

$$\therefore \sum_{K=-n}^{n} mE_k = mE$$

now, define the simple functions ψ_n, ϕ_n as

$$\phi_n(x) = \frac{M}{n} \sum_{K=-n}^{n} K\, X_{E_k}$$

$$\phi_n(x) = \frac{M}{n} \sum_{K=-n}^{n} (K-1)\, X_{E_k}$$

We get $\phi_n(x) \leq f(x) \leq \phi_n(x)$ by the definition of E_k.

$$\inf_{f \leq \psi} \int_E \psi(x)dx \leq \int_E \psi_n(x)\, dx = \frac{M}{n} \sum_{K=-n}^{n} K\, m\, E_k$$

$$[\because f \leq \psi_n]$$

and

$$\sup_{\phi \leq f} \int_E \phi(x)dx \geq \int_E \phi_n(x)\, dx = \frac{M}{n} \sum_{K=-n}^{n} (K-1)\, m\, E_k$$

$\therefore$ Since $\phi \leq f \leq \psi$

$$\therefore \sup \int_E \phi \leq \inf \int_E \psi.$$

$$\therefore 0 \leq \inf \int_E \psi(x)\, dx - \sup \int_E \phi(x)\, dx \leq \frac{M}{n} \sum_{k=-n}^{n} m\, E_k = \frac{M}{n}\, mE.$$

Since n is arbitrary, we get

$$\inf \int_E \psi(x)\, dx - \sup \int_E \phi(x)\, dx = 0$$

$$\therefore \inf \int_E \psi(x)\, dx = \sup \int_E \phi(x)\, dx.$$

(ii) Let f be a bounded function defined on a measurable set E of finite measure and let

$$\inf_{\phi \geq f} \int_E \psi (x)dx - \sup_{\phi \leq f} \int_E \phi (x) dx.$$

Then, given n, There are simple function ϕ_n and ψ_n such that $\phi_n \leq f \leq \psi_n$.

and $\int_E \psi_n (x)\, dx - \int_E \phi_n(x)\, dx < 1/n.$

Let $\psi^* = \inf \psi_n$

and $\phi^* = \sup \phi_n$.

Then ψ_n, ϕ_n measurable $\Rightarrow$ ϕ^* and ψ^* measurable.

Also

$\phi^*(x) \leq f(x) \leq \psi^*(x)$.

Let D $\{x:\phi^*(x) < \psi^*(x)\}$.

and $D_m = \left\{x : \phi^* (x) < \psi^* (x) - \frac{1}{m}\right\}$

Also $D_m \subset \left\{x : \phi_n{}^* (x) < \psi_n(x) - \frac{1}{m}\right\} V_n$ (verify)

Let $S_n \left\{x : \phi_n(x) < \psi_n(x) - \frac{1}{m}\right\}$

Then $\forall\, x \in S_n$

$$\frac{1}{m} < \psi_n (x) - \phi_n(x)$$

$$\therefore \frac{1}{m} X_s < \psi_n (x) - \phi_n(x)$$

$$\therefore \int_E \frac{1}{m} X_s < \int_E \psi_n - \phi_n < \frac{1}{n}$$

$$\therefore \frac{1}{m}\, m(S_n) < \frac{1}{n}$$

$$\therefore m(S_n) < \frac{m}{n}$$

$$D_m \subset S_n \Rightarrow m(D_m) < \frac{m}{n}$$

since n is arbitrary

$$\therefore m(D_m) = 0$$

$$\therefore m(D) = m(UD_m) \leq \Sigma\, m(D_m) = 0$$

Hence $\phi^* = \psi^*$ except on a set of measure zero.

$\therefore \phi^* = f$ a.e

Since ϕ^* is measurable.

Hence f is measurable.

Hence the lemma.

Definition:-

If f is a bounded measurable function defined on a measurable set E with mE finite, we define the (Lebesgue) integral of f over E by.

$$\int_E f(x)\, dx = \inf \int_E \psi(x)\, dx.$$

for all simple functions $\psi \geq f$.

Note:-

Because we are using the Lebesgue measure, we call the integral Lebesgue integral.

As soon as we define the Lebesgue integral of a bounded function, we verify that it is in fact a generalization of Riemann integral. ie) every Riemann integrable (R.I) function on (a,b) is Lebesgue integrable (L.I) Note that, when we say f is R.I it is a bounded function by the very definition of (R.I)

Lemma:-

Let f be a bounded function defined on [a, b]. If f is Riemann integrable on [a, b], then it is measurable and Lebesgue integrable and

$$R\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$$

Proof :

Since every step function is a simple function, the collection of step function $s \le f$ is a subcollection of simple functions $\phi \le f$ and the collection of step functions $s \ge f$ is a subcollection simple function $\phi \ge f$.

$$\therefore R\int_{-a}^{b} f(x)dx \le \sup_{\phi\le f}\int_a^b \phi(x)\,dx \le \inf_{\psi\ge f}\int_a^b \psi(x)dx \le R\int_a^{-b} f(x)\,dx$$

since f is Riemann integrable, we get equalities. Hence f is measurable and Lebesgue integrable.

Also $R\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$

Hence the Lemma.

Lemma :-

If f and g are bounded measurable functions defined on a set E of finite measure, then

(i) $\int_E (af+bg) = a\int_E f + b\int_E g$

(ii) If $f = g$ a.e , then

$$\int_E f = \int_E g$$

(iii) If $A \le f(x) \le B$, then

$$AmE \le \int_E f \le B\, m\, E.$$

(iv) If A and B are disjoint measurable sets of finite measure, then

$$\int_{A\cup B} f = \int_A f + \int_B f$$

Proof:

(i) Let f, g be bounded measurable functions defined on E, a set of finite measure.

a) ψ is a simple function if $a\psi$ is,

Let $a > 0$, Then $\psi \geq f \Leftrightarrow a\psi \geq af$.

$$\int_E af = \inf_{\psi \geq f} \int_E a\psi = a \inf_{\psi \geq f} \int_E \psi = a \int_E f$$

If $a < 0$.

$$\int_E af = \inf_{\phi \leq f} \int_E a\phi = a \sup_{\phi \leq f} \int_E \phi = a \inf_{\psi \geq f} \int_E \psi = a \int_E f$$

b) Let ψ_1, ψ_2 be simple functions such that

$\psi_1 \geq f$ and $\psi_2 \geq g$

Then $\psi_1 + \psi_2 \geq f + g$.

Hence

$$\int_E f+g \leq \int_E \psi_1 + \psi_2 = \int_E \psi_1 + \int_E \psi_2$$

$$\therefore \int_E f+g \leq \int_E f + \int_E g \text{ --------- (1)}$$

Also ϕ_1, ϕ_2 are simple functions such that

$\phi \leq f$ and $\phi_2 \leq g$

Then $\phi_1 + \phi_2 \leq f+g$

$$\int_E f+g \geq \int_E \phi_1 + \phi_2 \geq \int_E \phi_1 + \int_E \phi_2$$

$$\therefore \int_E f+g \geq \int_E f + \int_E g \text{ --------- (2)}$$

By (1) and (2), we have

$$\int_E f+g = \int_E f + \int_E g \text{ --------- (2)}$$

From (a) and (b) we get.

$$\int_E af+bg = a\int_E f+b\int_E g$$

Hence the proof. (i)

(ii) it now suffices to show that

$$\int_E f-g = 0$$

Since $f-g = 0$ a.e

it follows that if $\psi \geq f-g$, $\psi \geq 0$ a.e

From this it follows that

$$\int_E \psi \geq 0$$

When

$$\int_E f-g \geq 0$$

III^{ly}

$$\int_E f-g \leq 0$$

Hence $\int_E f = \int_E g$

Hence the proof (ii)

(iii) $A \leq f \Rightarrow A\, X_E \leq f.$

$A\, X_E \leq \int f.$

$\therefore Am(E) \leq \int f.$

IIIly $f \leq B \Rightarrow f \leq B\, X_E$

$\therefore \int f \leq \int B\, X_E = Bm(E)$

$\therefore$ we get $Am(E) \leq \int f \leq \int Bm(E)$

Hence the proof (iii)

(iv) Let A, B be two disjoint measurable sets mE.

$$\int_{AUB} f = \int f X_{AUB} = f\,(X_A + X_B)$$

$$= \int f X_A + fX_B$$

$$\Rightarrow \int_{AUB} f = \int_A f + \int_B f$$

Hence the proof (v)

Note:

Suppose f is bounded measurable on E. S.t. m (E) < ∞.

The $|f|$ is bounded measurable.

Also $f \leq |f| \Rightarrow \int f \leq \int |f|$

and $-f \leq |f| \Rightarrow -\int f = \int -f \leq \int |f|$

$$\therefore \int |f| \leq \int |f|$$

Lemma : (Bounded convergence theorem)

Let (f_n) be a sequence of measurable function defined on a set E of finite measure, and suppose that there is a real number M such that $|f_n(x)| \leq M$ $\forall n$ and all x. If $f(x) = \lim f_n(x)$ for each $x \in E$. Then $\int_E f = \lim \int_E f_n$.

Proof :

Let f_n be a sequence of bounded measurable functions on E S.t $m(E) \leq \infty$ and let $|f_n| \leq M$ $\forall n$.

Suppose f_n converges to f.

By Egoroff's theorem given $\epsilon > 0$ $\exists$ a measurable set $A \subset E$ s.t $m(A) < \frac{\epsilon}{4M}$

$$\therefore |f_n(x) - f(x)| < \frac{\epsilon}{2m(E)} \quad \forall x \in E-A$$
$$\forall x \in E-A$$
$$\&\ n \geq N$$

consider,

$$\therefore \left| \int_E f_n - \int_E f \right| = \left| \int_E (f_n - f) \right|$$

$$\leq \left| \int_{E-A} (f_n - f) \right| = \left| \int_{A} (f_n - f) \right|$$

$$\leq \left| \int_{E-A} |(f_n - f)| + \int_{A} |(f_n - f)| \right.$$

$$\therefore |f_n - f| < \frac{\epsilon}{2m(E)} \;\forall\; x \in E - A \;\&\; \forall\; n \geq N$$

$$\therefore \int_{E-A} |f_n - f| \leq \int_{E-A} \frac{\epsilon}{2m(E)} \leq \frac{\epsilon}{2m(E)} m(E-A) \leq \frac{\epsilon}{2m(E)} m(E)$$

$$\leq \frac{\epsilon}{2}$$

$$\therefore \int_{A} |f_n - f| \leq \int_{A} |f_n| + |f| \leq \int_{A} 2MX_A$$

$$(\therefore |f_n(x)| \leq M) \quad \forall x.$$

$\Rightarrow \lim |f_n(x)| \leq M \;\forall\; x.$

$\therefore |f(x)| \leq M \;\forall\; x.$

$$\therefore \int_{A} |f_n - f| \leq 2\,Mm\,(A).$$

$$\leq 2M \frac{\epsilon}{4M} = \frac{\epsilon}{2}$$

$$\therefore \left| \int_{E} f_n - \int_{E} f \right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\therefore \lim_{n} \int_{E} f_n = \int_{E} f.$$

Hence the lemma.

UNIT - 8

THE LEBESGUE INTEGRAL OF A NON-NEGATIVE MEASURABLE FUNCTION

Topic - I

Definition

Let f be a non-negative measurable function defined on a measurable set E.

Define $\int_E f = \sup_{h \leq f} \int_E h$

where h is a bounded measurable function such that m $\{x : h(x) \neq 0\}$ is finite, (is h vanishes outside a set of finite measure).

Definition:

A non-negative measurable function f is called integrable over the measurable set E if

$$\int_E f < \infty.$$

Note (i)

Every non-negative bounded function vanishing outside a set of finite measure is integrable. for, if $0 \leq f \leq D$ and $D = \{x : f(x) \neq 0\}$, $0 \leq f \leq M X_D$ and MX_D is a bounded measurable function and $h \leq f \leq MX_D \Rightarrow \int h \leq \int M X_D = M.m(D) < \infty$

$$\Rightarrow \int f \leq M.m(D) < \infty$$

Lemma:-

If f and g are non-negative measurable functions, Then

(i) $\int_E cf = c \int_E f$, $c > 0$

(ii) $\int_E f+g = f \int_E + \int_E g$

(iii) If $f \leq g$ a.e then $\int_E f \leq \int_E g$

Proof

(i) Let f be a non negative measurable function and $c > 0$

For every bounded measurable function h

$h \leq f \Rightarrow ch \leq cf$

and $h_1 \leq cf \Rightarrow \dfrac{h_1}{c} \leq f$

$$\therefore \int cf = \sup_{h_1 \leq cf} \int h_1 = \sup_{h_1/c < f} c\int \frac{h_1}{c}$$

$$= c \sup_{h < f} \int f$$

$$\Rightarrow \int cf = c \int f$$

Hence the proof (i)

(ii) Let h, k be bounded measurable functions vanishing outside sets of finite measures and let $h \leq f$ and $K \leq g$.

$\therefore \; h + K \leq f + g$

and $\int h + K \leq \int f + g$

$\therefore \int h + \int K \leq \int f + g$

Taking suprema,

$\therefore \int f + \int g \leq \int f + g$ ---- (1)

now, Let ℓ be a non-negative measurable function which vanishes outside a set of finite measure.

Let $\ell \leq f + g$

Define $h = \min (f, \ell)$

and $K = \ell - h$ (since ℓ is a bounded, his bounded hence $(\ell - h)$ is defined at all points of its domain.

Also $h \leq f$ [by defn]

$h + K \leq f + g$

$\therefore 0 \leq K \leq g$

For : $[f(x) \le \ell(x) \Rightarrow h(x) = f(x)]$

and

$$K(x) = \ell(x) - f(x) \le (f+g)x - f(x) = g(x)$$

and

$$f(x) \ge \ell(x) \Rightarrow h(x) = f(x) \text{ and } K(x) = g(x)$$

Moreover, $h, K \le \ell \Rightarrow h$ and K are bounded and they vanish outside a set of finite measure.

$\therefore h \le f$ and $k \le g$

$$\Rightarrow \int \ell = \int h + k = \int h + \int K \le \int f + \int g$$

$$\therefore \sup \int \ell \le \int f + \int g$$

$\ell \le f + g$

$$\therefore \int f+g \le \int f + \int g \qquad \text{------ (2)}$$

By (1) and (2), we get

$$\int f+g = \int f + \int g$$

Hence the proof (ii)

(iii) Let h be any bounded measurable.

$f_n \le f$

Let h vanish outside a set of finite measure

$\therefore f \le g,\ h \le g$

$$\therefore \int_E h \le \int_E g \quad (\because \text{ by defn. of } \int_E g)$$

Taking sup, on L.H.S we get

$$\int_E f \le \int_E g$$ Hence the proof (iii)

Lemma :

Let f be any non-negative measurable function. Then $f = 0$ a.e iff $\int f = 0$.

Proof:

Let $f = 0$ a.e

Let h be a bounded non-negative measurable function s.t $h \le f$.

Then $\{x : h(x) \neq 0\} \subset \{x : f(x) \neq 0\} \Rightarrow m\{x : h(x) \neq 0\} = 0$

$\therefore h = 0$ a.e

$\therefore \int h = 0$

$\therefore \int f = \sup_{h \le f} \int h \Rightarrow \int f = 0$

conversely,

Let $\int f = 0$

Let $E = \left\{x : f(x) \ge \frac{1}{n}\right\}$

Then $E = \{x : f(x) > 0\} = \cup E_n$.

Also $\frac{1}{n} X_E \le f \Rightarrow \frac{1}{n} X_E \le \int f$

$\Rightarrow m(E_n) = 0 \ \forall n$.

$m(E) = m(\cup E_n) \le \Sigma\, m(E_n) = 0$.

Hence $f = 0$ a.e

Hence the Lemma.

Lemma :-

Let f be a non-negative integrable function. Then $m\{x : f(x) = \infty\} = 0$ (i.e., f is integrable $\Rightarrow$ f is finite a.e)

Proof :-

Let $D = \{x : f(x) = \infty\}$

and $D_n = \{x : f(x) > n\}$

Then $D = \cap D_n$.

now $n\, XD_n \le f \Rightarrow \int n\, X_D \le \int f$.

$\therefore n.m(D_n) \le \int f = K$ (a finite number)

$\therefore m(D_n) \le \frac{1}{n} K$

$$D \subset D_n \ \forall n \Rightarrow m(D) \leq m(D_n) \leq \frac{k}{n} \ \forall n$$

$\therefore m(D) = 0$

$\therefore m\{x : f(x) = \infty\} = 0$

Hence the lemma.

Thm: (Fatou's Lemma)

If (f_n) is a sequence of non-negative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E. Then $\int f \leq \underline{\lim} \int_E f_n$

Proof :

Let (f_n) coverges to (f) a.e and Let $D = \{x : f_n(x)$ does not coverge to $f(x)\}$

Then $m(D) = 0$

and so $\int_D f_n = 0 \ \ \forall n$ and $\int_D f = 0$

So, we can assume, without loss of generality that $f_n(x)$ coverges to $f(x)$ $\forall x \in E$.

Let h be a bounded (non-negative) measurable function vanishing outside a set E′ of finite measure.

Let $h \leq f$.

Define $h_n = \min(h, f_n)$

Then $h_n \leq h \ \forall n$.

and if $h \leq M$, then $h_n \leq M \forall n$.

Also, $\forall x \in E'$

$\operatorname{Lim} h_n(x) = h(x)$.

$(\because \lim h_n = \min(h. \lim f_n) = \min(h.f))$

By Bounded convergence theorem.

$$\int_E h = \int_{E'} h = \lim \int_{E'} h_n \leq \underline{\lim} \int_E h_1 \text{ ---------- } (1)$$

Since $h_n = \min(h, f_n)$

$\therefore h_n \leq f_n$

and $\int h_n \le \int f_n$

$\therefore \lim \int_E h_n \le \underline{\lim} \int_E f_n$

$\therefore \int_E h \le \underline{\lim} \int_E f_n$ ($\because$ by (1))

Taking the supcemum over h, we get

$$\int_E f \le \sup_{n \le f} \int_E f \le \underline{\lim} \int_E fn$$

$$\therefore \int_E f \le \underline{\lim} \int_E f_n$$

Hence the theorem.

Note:

The above example in which $\int f = 0$ and $\underline{\lim} \int f_n = 1$ shows that the strict inequality may also hold.

Theorem : (Monotone convergence thm)

Let (f_n) be an increasing sequence of non-negative measurable functions, and let $f = \lim f_n$ a.e then $\int f = \lim \int f_n$.

Proof:

By Fatou's lemma.

$$\int f \le \underline{\lim} \int f_n.$$

now, $\forall$ n we have $f_n \le f$

$$\therefore \int f_n \le \int f.$$

$$\therefore \overline{\lim} \int f_n \le \int f$$

$$\therefore \int f \le \underline{\lim} \int f_n \le \overline{\lim} \int f_n \le \int f$$

$\therefore \lim \int f_n$ exists and

$$\int f = \lim \int f_n$$

Hence the theorem.

Remark:

Suppose f_n in monotone convergence theorem are integrable. Then f is integrable if $\lim \int f_n < \infty$.

Corollary :-

Let u_n be a sequence of non-negative measurable functions, and let

$$f = \sum_{n=1}^{\infty} u_n. \text{ Then}$$

$$\int f = \sum_{n=1}^{\infty} \int u_n$$

Proof :

$$\text{Let } S_n = \sum_{n=1}^{\infty} \int u_n$$

Since u_k are non-negative.

$\therefore$ S_n is an increasing sequence of measurable functions converging to f.

$$\int f = \lim \int S_n = \lim \int \sum_{K=1}^{n} u_k$$

$$= \lim \sum_{k=1}^{n} \int u_k$$

$$\int f = \sum_{n=1}^{\infty} \int u_n$$

Hence the corollary.

Lemma :-

Let f be a non-negative measurable function. Then f is integrable if there exists an increasing sequence of simple functions S_n such that $f = \lim S_n$ and $\lim \int S_n < \infty$.

Proof:

We have proved earlier that, given a non-negative measurable function f, there exists an increasing sequence of simple functions S_n such that

$$f = \lim S_n.$$

By (Monotone convergence thm)

$$\int f = \lim \int S_n$$

If f is integrable, $\int f < \infty$ and so $\lim S_n < \infty$

conversely it $S_n \to f$.

Then $\int f = \lim \int S_n < \infty$.

and hence f is integrable.

Hence the lemma.

Lemma :

Let f be a non-negative function and (E_i) a disjoint sequence of measurable sets.

Let $E = \cup E_i$. Then $\int_E f = \Sigma \int_{E_i} f$.

Proof :

Let $u_i = f X_{E_i}$

Then $X_{\overset{n}{\underset{i=1}{\cup}} E_i} = \sum_1^n f X_E = \sum_1^n u_i$

(since E_i are disjoint)

$$\therefore \int_{\overset{n}{\underset{1}{\cup}} E_i} f = \int \sum_1^n u_i = \sum_1^n u_i = \sum_1^n \int_{E_i} f.$$

And $f X_E = \sum_1^{\infty} f X_E = \sum_1^{\infty} u_i$

$$\therefore \int_E f = \int_E f X_E = \int \sum_1^{\infty} u_i = \sum_1^{\infty} \int u_i = \sum_1^{\infty} \int_{E_i} f$$

$$\Rightarrow \int_E f = \sum_{1}^{\infty} \int_{E_i} f.$$

Hence the lemma.

Lemma :

Let f be a non-negative function which is integrable over a set E. Then given $\in > 0$ there is $\delta > 0$ such that for every set $A \subset E$ with $mA < \delta$ we have

$$\int_A f < \in$$

Proof : case (i) Suppose f is bounded.

ie) $f \leq M$.

Let $\in > 0$ be given.

choose $\delta = \dfrac{\in}{M}$

If A is a subset of E s.t $m(A) < \delta$.

then $\int_A f = \int fX_A \leq M.m(A) < M.\delta = \in$.

Case (ii)

Given f define

$f_n = \min(f, n)$

Each f_n is bounded by n and

$\lim f_n = f$

Also, f_n is an increasing sequence of measurable functions.

Hence $\int f_n$ increase and $\lim \int f_n = \int f$

Let $\in > 0$ be given.

Then choose N.S.t $\int f_n > \int f - \dfrac{\in}{2}$

and $\int f - f_N = \int f - \int f_N < \dfrac{\in}{2}$

Let $\delta = \dfrac{\in}{2N}$

Then if $m(A) < 2.$

Then

$$\int_A f = \int_A f - f_N + \int_A f_N$$

$$< \int_E f - f_N + NM\,(A)$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in. \quad [\because f_N < N]$$

$$\therefore \int_A f < \in$$

Hence the lemma.

Lemma:-

Let f_n be a sequence of non-negative measurable functions which converges to f and suppose $f_n \le f$. Then $\int f = \lim \int f_n$.

Proof :

Let $\lim f_n = f$.

By Fatous lemma.

$\int f \le \underline{\lim} \int f_n$ ---- (1)

since $f_n \le f$

$\int f_n \le \int f \Rightarrow \overline{\lim} \int f_n \le \int f$------ (2)

By (1) & (2) we get,

$\text{Lim} \int f_n$ exists

$\therefore \int f = \text{Lim} \int f_n$.

Hence the Lemma.

Lemma:-

a) generalization of Fatou's Lemma :-

If f_n is a seauence of non-negative measurable functions then $\int \underline{\text{Lim}}\, f_n \le \underline{\text{Lim}} \int f_n$.

b) Using the functions f_n defined as.

$f_n(x) = 1$ if $n \leq x < n+1$

$= 0$ Otherwise.

Show that the strict inequality may hold in Fatou's lemma.

Proof :-

a) Let f_n be a sequence of non-negative measurable functions.

Let $g_n = \inf f_k = \inf (f_n\ f_{n+1}\ f_{n+2} \ldots) \quad k \geq n.$

Then $g_n \leq f_n$.

$\therefore \int g_n \leq \int f_n$.

Also $\underline{\text{Lim}} \int g_n \leq \underline{\text{Lim}} \int f_n$.

But $\underline{\text{Lim}}\ f_n = \text{Lim}\ g_n$.

Also g_n increase to $\underline{\text{Lim}}\ f_n$.

$\therefore$ By monotone convergence theorem.

$\text{Lim} \int g_n$ exists and

$\int \underline{\text{Lim}}\ f_n \leq \text{Lim} \int g_n$.

$\therefore \underline{\text{Lim}} \int g_n \leq \text{Lim} \int g_n$.

$\int \underline{\text{Lim}}\ f_n \leq \underline{\text{Lim}} \int f_n$.

Hence the proof (a)

b) f_n are defined as

$f_n(x) = 1$ if $n \leq x < n+1$

$= 0$ otherwise.

$\therefore \int f_n = \int (X_{n,\, n+1}) = m\,[\,n, n+1)] = 1$

$\therefore \text{Lim} \int f_n = 1$

$\therefore$ But $\text{Lim}\ f_n(x) = 0\ \forall\, x$

$\therefore \int f = \int \text{Lim}\ f_n = 0$

And $\int f = 0 < \text{Lim} \int f_n = \underline{\text{Lim}} \int f_n$.

Hence the proof (b)

Lemma :-

Let f be a non-negative, integrable function. show that the function F defined by.

$$F(x) = \int_{-\infty}^{x} f \text{ is continuous.}$$

Proof :-

$$F(\chi) = \int_{-\infty}^{x} f$$

$$F(x+\delta) = \int_{-\infty}^{x+\delta} f$$

$$\therefore F(x+\delta) - F(\chi) = \int_{-\infty}^{x+\delta} f - \int_{-\infty}^{x} f$$

$$= \int_{x}^{x+\delta} f$$

By Lemma,

Let f be a non-negative function which is integrable over a set E. Then given $\in > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $mA < \delta$ we have $\int_A f < \in$.

Given $\in > 0$, $\exists\ \delta > 0$ s.t. $\forall$ A, measurable and $m(A) > \delta$, $\int_A f < \in$.

for $y \in (x, x+\delta/2)$, $F(y) - F(x) = \int_x^y f < \in$.

and

for $y \in (x-\delta/2, x)$, $F(x) - F(y) = \int_y^x f < \in$.

$\therefore$ for $y \in (x-\delta/2, x+\delta/2)$

$\therefore |F(x) - F(y)| < \in$.

Hence F is continuous at x

$\therefore$ F is continuous function.

Hence the Lemma.

TOPIC - 2

The General Lebesgue Integral :-

In the previous two topies, we have defined. The integral of a bounded measurable function on a set of finite measure and of a non-negative measurable function. Now, we shall define. The integral of a general measurable function.

Given any function $f: R \to R$, we have defined f^+ and f^- as follows.

$f^+ = \max(f, 0)$ and $f^- = -\min(f, 0) = \max(-f, 0)$

Check $f = f^+ - f^-$ and

$|f| = f^+ + f^-$

Example (i)

Consider $f : (0, 2\pi) \to R$ defined as $f(x) = \text{Cos}\, x$.

$f^+(x) = \text{Max}(f, 0) = f(x)$ for $0 \leq x \leq \pi/2$.

$= 0$ for $\pi/2 \leq x \leq 3\pi/2$.

$= f(x)$ for $3\pi/2 \leq x \leq 2\pi$.

$f^-(x) = 0$ for $0 \leq x \leq \pi/2$

$= -f(x)$ for $\pi/2 \leq x \leq 3\pi/2$

$= 0$ for $3\pi/2 \leq x \leq 2\pi$

Example:- (ii)

Consider $f : [0,1] \to R$ defined as $f(x) = \frac{1}{2} + \sin 2\pi x$.

$f^+(x) = f(x)$ for $0 \leq x \leq 7/12$

$= 0$ for $7/12 \leq x \leq 11/12$

$= f(x)$ for $11/12 \leq x \leq 1$.

$f^-(x) = 0$ for $0 \le x \le 7/12$

$= -f(x)$ for $7/12 \in x \le 11/12$

$= 0$ for $11/12 \le x \le 1$.

Definition:-

(i) Given a measurable function f, if either f^+ or f^- is integrable over E, we define.

$$\int f = \int f^+ - \int f^-$$

(ii) If both f^+ and f^- are integrable $\int f$ in finite and so f is said to be integrable over E.

Lemma :-

A measurable function f is integrable over E if $|f|$ is integrable over E.

Proof :-

(i) Suppose f is integrable over E. Then f^+ and f^- are integrable over E.

and so $\int_E f^+$ and $\int_E f^-$ are finite.

$\therefore |f| = f^+ + f^- \Rightarrow \int_E |f| = \int_E f^+ + \int_E f^- < \infty.$

$\Rightarrow |f|$ is integrable over E.

(ii) Suppose $|f|$ is integrable over E. f is meaurable $\Rightarrow f^+$ and f^- are measurable.

Also,

$|f| = f^+ + f^- \Rightarrow f^+ \le |f|$ and $f^- \le |f|$

and so $\int_E f^+ < \infty$ and $\int_E f^- < \infty$.

$\therefore$ f is integrable over E.

Lemma:-

Let f be measurable and $f = f_1 - f_2$ where f_1 and f_2 are non-negative integrable function.

Then $$\int_E f = \int_E f_1 - \int_E f_2$$

Proof:-

now, $f = f^+ - f^- = f_1 - f_2$

$$\Rightarrow f^+ + f_2 = f_1 + f^-$$

$$\therefore \int_E f^+ + f_2 = \int_E f_1 + f^-$$

$$\therefore \int_E f^+ + \int_E f_2 = \int_E f_1 + \int_E f^-$$

Since all integrals are finite, we get

$$\int_E f^+ - \int_E f^- = \int_E f_1 + \int_E f_2$$

$$\Rightarrow \int_E f = \int_E f^+ - \int_E f^- = \int_E f_1 - \int_E f_2$$

Hence the Lemma.

Lemma :-

Let f and g be integrable over E. Then (i) the function cf is integrabl over E, and

$$\int_E cf = c\int_E f$$

(ii) The function f+g is integrable over E, and

$$\int_E f+g = \int_E f + \int_E g$$

(iii) If $f \le g$ a.e., then $\int_E f \le \int_E g$.

(iv) If A and B are disjoint measurable sets contained in E, then

$$\int_{A\cup B} f = \int_A f + \int_B f.$$

Proof:-

(i) Suppose $c \ge 0$.

$$c\,f = c\,(f^+ - f^-) = cf^+ - cf^-$$

($\therefore$ cf, cf^+, cf^- are non-negative and integrable)

By the above Lemma.

$$\int_E cf = \int_E cf^+ - \int_E cf^-$$

$$= c\int_E f^+ - c\int_E f^- = \left(c\int_E f^+ - \int_E f^-\right) = c\int_E f.$$

$$\Rightarrow \int_E cf = c\int_E f$$

Suppose $c < 0$.

Let $C = -d$, $d > 0$.

$cf = -d\,(f^+ - f^-) = df^- - df^+$ (by the above lemma)

$$\Rightarrow \int_E cf = \int_E df^- - \int_E df^+$$

$$= d\int_E f^- - d\int_E f^+$$

$$= -c\int_E f^- + c\int_E f^+ = c\int_E f^+ - c\int_E f^-$$

$$= \left(c \int_E f^+ - \int_E f^- \right)$$

$$= c \int_E f.$$

$$\Rightarrow \int_E cf = c \int_E f$$

Hence the proof (i)

(ii) $f + g = (f^+ - f^-) + (g^+ - g^-)$

$$= (f^+ + g^+) - (f^- + g^-)$$

(by the above lemma)

($\because$ both $(f^+ + g^+)$ and $(f^- + g^-)$ non negative and integrable)

$$\therefore \int_E f + g = \int_E (f^+ + g^+) - \int_E (f^- + g^-)$$

$$= \int_E f^+ + \int_E g^+ - \left(\int_E f^- + \int_E g^- \right)$$

$$= \int_E f^+ - \int_E f^- + \int_E g^+ - \int_E g^-$$

$$= \left(\int_E f^+ - \int_E f^- \right) + \left(\int_E g^+ - \int_E g^- \right) = \int_E f + \int_E g$$

$$\Rightarrow \int_E f + g = \int_E f + \int_E g$$

Hence the proof (ii)

(iii) Let $g = f + (g - f)$

So $\int g\,dx = \int f\,dx + \int (g\text{-}f)^+\,dx - \int (g - f)^-\,dx.$

But $(g-f)^+ = 0$ a.e. and $(g-f)^- = 0$ a.e.

$$\int gd\,x = \int fdx. \text{ If } f \le g \text{ a.e.}$$

$$\Rightarrow \int gd\,x \ge \int f\,dx$$

$$\therefore \int f\,dx \le \int g\,dx$$

Hence the proof (iii)

(iv) $$\int_{A\cup B} fd\,x = \int f\,\psi_{A\cup B}\,dx$$ (If A and B are disjoint - measurable sets)

$$= \int f\,(\psi_A + \psi_B)\,dx$$

$$= \int f\,\psi_A\,dx + \int f\,\psi_B\,dx$$

$$\Rightarrow \int_{A\cup B} fdx = \int_A fdx + \int_B fdx$$

Hence the Proof (iv)

Lemma:-

If $f=0$ a.u on E then $\int_E f = 0$

Proof:-

If $f = 0$ a.e. on E

Then $f^+ = 0$ a.e. on E

and $f^- = 0$ a.e. on E

$$\therefore \int_E f^+ = 0 \text{ and } \int_E f^- = 0$$

$$\therefore \int_E f = 0$$

Hence the Lemma.

Corollary:-

If f is integrable over E.

$$\left|\int_E f\right| \leq \int_E |f|$$

Proof :-

$$f \leq |f| \Rightarrow \int f \leq |f|$$

$$-f \leq |f| \Rightarrow \int -f \leq |f|$$

$$\therefore -\int f \leq \int |f|$$

$$\therefore \left|\int f\right| \leq \int |f|$$

Hence the corollary.

Corollary:-

If f=g a.l on E and if g is integrable on E, then f is integrable on E and

$$\int_E f = \int_E g$$

Proof:-

If f=g a.e $\Rightarrow f^+ = g^+$ a.e, $f^- = g^-$ a.e

$$\therefore \int_E f^+ = \int_E g^+ < \infty \text{ and } \int_E f^- = \int_E g^- < \infty$$

Since f is integrable and

$$\int_E f = \int_E f^+ - \int_E f^- = \int_E g^+ - \int_E g^- = \int_E g$$

$$\Rightarrow \int_E f = \int_E g$$

Hence the corollary.

Corollary:-

If f and g are both integrable on E and if $\int_A f = \int_A g$ for every measurable $A \subset E$,

then f = g a.e on E.

Proof:-

Let $E_1 = \{x \in E / f(x) > g(x)\}$

Then $f\text{-}g > 0$ on E_1.

Also $\int_{E_1} f = \int_{E_1} g \Rightarrow \int_E f\text{-}g = 0$

$\therefore f\text{-}g > 0$ on $E_1 \Rightarrow m(E_1) = 0$

III^{ly} if $E_2 = \{x \in E / f(x) < g(x)\}$

Then $\int_{E_2} g\text{-}f = 0$

$\therefore g - f > 0$ on $E_2 \Rightarrow m(E_2) = 0$

$\therefore$ If $E_3 = \{x \in E / f(x) = g(x)\}$ then

$E = E_1 \cup E_2 \cup E_3$ and so $f = g$ a.e on E.

Thm:-

Lebesgue convergence theorem:-
(Dominated convergence theorem)

Let g be integrable over E and let (f_n) be a sequence of measurable functions such that $|f_n| \leq g$ on E and for almost all $x \in E$ we have

$f(x) = \text{Lim}\, f_n(x)$. Then $\int_E f = \text{Lim} \int_E f_n$.

Proof:-

Since, for each n, $|f_n| \leq g$.

We have $|f| \leq g$ a.l.

So f_n and f are integrable.

Also $\{g + f_n\}$ is a sequence of non-negative meausrable functions.

By fatou's Lemma,

$$\text{Lim inf} \int (g + f_n)\, dx \geq \int \text{Lim inf}\, (g + f_n)\, dx$$

So,

$$\int g dx + \text{Lim inf} \int f_n dx \geq \int gd\, x + \int fd\, x$$

But $\int gd\, x$ is finite.

$$\therefore \text{Lim inf} \int f_n dx \geq \int fdx \text{ ----- (1)}$$

Again,

$\{g - f_n\}$ is also sequence of non-negative measurable functions.

$$\text{Lim inf} \int (g - f_n)\, dx \geq \int \text{Lim inf}\, (g - f_n)\, dx.$$

So,

$$\int gdx - \text{Lim sup} \int f_n dx \geq \int gdx - \int fdx.$$

$$\therefore \int fdx \geq \text{Lim sup} \int f_n dx \text{ ------ (2)}$$

(1) & (2) we get

$$\text{Lim sup} \int f_n d\, x \leq \int fdx \leq \text{Lim inf} \int f_n dx.$$

$\therefore \text{Lim} \int f_n$ exists.

$\therefore \int f\,dx = \text{Lim} \int f_n dx.$

Hence the thm.

Thm:-

Let (g_n) be a sequence of integrable functions. Which converges a.e to an integrable functioning. Let (f_n) be a sequence of measurable functions. Such that $|f_n| \leq g_n$ and (f_n) converges to f a.e. If $\int g = \text{Lim} \int g_n$, then $\int f = \text{Lim} \int f_n$

Proof:-

Consider $g_n - f_n$ and $g_n + f_n$ and prove.

Lemma:-

Let f_n be a sequence of integrable functions such that $f_n \to f$ a.e. with f integrable. Then $\int |f - f_n| \to 0$ iff $\int |f_n| \to \int |f|$.

Proof:-

(i) Let $\int |f - f_n| \to 0$

$0 \leq |f_n| - |f| \leq |f - f_n|$

$\therefore 0 \leq \int \big| |f_n| - |f| \big| \leq \int |f-f_n|$

$\therefore \text{Lim} \int |f - f_n| = 0 \Rightarrow \text{Lim} \int \big| |f_n| - |f| \big| = 0$

$\therefore \text{Lim} \int |f_n| \to \int |f|$

(ii) Let $\int |f_n| \to \int |f|$

$\big| |f_n| - |f| \big| = \big| (f_n^+ + f_n^-) - (f^+ + f^-) \big|$

$= \big| (f_n^+ - f^+) + (f_n^- - f^-) \big|$

$\therefore \big| f_n^+ - f^+ \big| \leq \big| |f_n| - |f| \big|$

$\therefore 0 \le \int |f_n^+ - f^+| \le \int \big|\, |f_n| - |f| \,\big|$

$\therefore \text{Lim} \int |f_n| = \int |f| \Rightarrow \text{Lim} \int |f_n^+ - f^-| = 0$

III^{ly} $\text{Lim} \int |f_n^- - f^-| = 0$

$\therefore 0 \le \int |f - f_n| = \int |(f^+ - f^-) - (f_n^+ - f_n^-)|$

$= \int \big| (f^+ - f_n^+) - (f^- - f_n^-) \big|$

$= \int |f^+ - f_n^+| + |f^- - f_n^-|$

$= \int |f^+ - f_n^+| + \int |f^- - f_n^-|$

$\therefore \text{Lim} \int |f - f_n| = 0$

Hence the Lemma.

Note:-

A measurable function f is integrable if ~~f~~ $|f|$ is integrable.

Lemma:-

Let g be an integrable function on E and let f_n be a sequence of measurable functions such that $|f_n| \le g$ a.e. on E.

Then $\int_E \underline{\text{Lim}}\, f_n \le \underline{\text{Lim}} \int_E f_n \le \overline{\text{Lim}} \int_E f_n \le \int_E \overline{\text{Lim}}\, f_n$.

Proof:-

Already we have proved $\int_E \underline{\text{Lim}}\, f_n \le \underline{\text{Lim}} \int_E f_n$.

Also,

$\underline{\text{Lim}} \int f_n \le \overline{\text{Lim}} \int f_n$ ----- (1)

now $f_n \le |f_n| \le g \Rightarrow g - f_n \ge 0$

$\therefore f_n$ is finite a.e.

$$\int \underline{\text{Lim}}\,(g - f) \leq \underline{\text{Lim}} \int (g-f_n)$$

$$\int g - \int \overline{\text{Lim}}\, f_n \leq \underline{\text{Lim}}\, \left(\int g - \int f_n\right)$$

$$\therefore \int g - \int \overline{\text{Lim}}\, f_n \leq \int g - \overline{\text{Lim}} \int f_n$$

$$\therefore \overline{\text{Lim}} \int f_n \leq \int \overline{\text{Lim}}\, f_n \quad \text{----- (2)}$$

From (1) and (2) we get,

$$\int_E \underline{\text{Lim}}\, f_n \leq \underline{\text{Lim}} \int_E f_n \leq \overline{\text{Lim}} \int_E f_n \leq \int_E \overline{\text{Lim}}\, f_n$$

Hence the Lemma.

TOPIC - 3

Convergence in Measure:-

Let us first recall the definitions of convergence of measurable functions.

(i) Pointwise (ii) a.e. (iii) uniformly (iv) a.u

(v) in measure and (vi) in mean.

Let $\{f_n\}$ be a sequence of measurable functions on E.

Definition:- (i)

f_n is said to converge pointwise to f if $\text{Lim}_n f_n(x) = f(x), \forall x \in E.$

if $\forall \in > 0$ and $x \in E, \exists N$ S.t. $\forall n \geq N$,

$|f_n(x) - f(x)| < \in.$

We write $\text{Lim}_n f_n = f$

Definition:- (ii)

f_n is said to converge to f almost every where if $\exists A \subset E.\ m(A) = 0$ and if $\underset{n}{\text{Lim}}\, f_n(x) = f(x)\ \forall\, x \in E|A$.

We write Lim $f_n = f$ a.e.

(We also say that f_n converges to f for all almost all $x \in E$, i.e.) Lim $f_n(x) = f(x)$ for a.ex $\in E$).

Definition:- (iii)

f_n is said to converge to f uniformly in E. if $\forall \in > 0, \exists N$ s.t. $\forall\, n \geq N$.

$|f_n(x) - f(x)| < \in \forall\, x \in E$.

We write Lim $f_n = f$ (uni f)

Definition:- (iv)

f_n is said to converge to f almost uniformly, if $\forall\, \delta > 0, \exists A \subset E$ s.t. $m(A) > \delta$ and f_n converges to f uniformly in E/A we write Lim $f_n = f$ a.u.

Definition:- (v)

f_n is said to converge to f in measure. if $\forall \in > 0$, Lim $m\{x : |f(x) - f_n(x)| \geq \in\} < 0$

ii) $\forall \in > 0$, & $\delta > 0\ \exists N$ s.t $\forall\, n \geq N$.

$m\{x : |f(x) - f_n(x)| \geq \in\} < \delta$

We write Lim $f_n = f$ (mse)

Definition:- (vi)

f_n is said to converge to f in the mean if Lim $\int |f_n - f| = 0$ we write.

Lim $f_n = f$ (mean)

note :- (i)

Uniform convergence $\Rightarrow$ pointwise convergence (follows from the definition)

note :- (ii)

If $m(E) < \infty$ a.e convergence $\Rightarrow$ a.u convergence (Egoroff's theorem).

Lemma:

Convergence a.u $\Rightarrow$ convergence in measure. ie) if f_n is sequence of measurable functions converging almost uniformly to f, then f_n will converge in measure to f.

Proof:

Let $\in > 0$, $\exists\ \delta > 0$ be given.

Then $\exists$ E s.t $m(E_\delta) < \delta$ and

f_n converges uniformly to f on E_δ

$\therefore \exists$ N s.t

$|f_n(x) - f(x)| < \in \ \forall\ n \geq N$ and $\forall x \in E_\delta$

$\therefore \{x : |f_n(x) - f(x)| \geq \in\} \subset E_\delta \ \forall\ n \geq N$

and

$m\{x : |f(x) - f_n(x)| \geq \in\} \leq m(E_\delta) < \delta \ \forall\ n \geq N$

$\therefore \lim\limits_{n} m\{x : |f_n(x) - f(x)| \geq \in\} = 0$

$\therefore \lim f_n = f$ (mse)

Hence the lemma.

Lemma:

Convergence in mean $\Rightarrow$ convergence in measure (i.e) If f_n is a sequence of measurable function and f is a measurable function.

s.t

$\lim \int |f_n - f| = 0$ then $\lim f_n = f$ (mse)

Proof:-

Suppose f_n doesnot converge in measure to f.

Then $\exists \in > 0$ & $\delta > 0$ s.t

$m\{x : |f_n(x) - f(x)| > \in\} > \delta$

for an infinite number of n.

Let $D = \{x : |f_n(x) - f(x)| \geq \in\}$ then

$m(D) > \delta$

$\therefore \int |f_n - f| \geq \delta$ then $|f_n - f| \geq \in \delta$

for an infinite number of n.

- a contradiction to meanage.

Hence the lemma.

Lemma

If $m(E) < \infty$, unif convergence $\Rightarrow$convergent in mean.

Proof:

Let $\{f_n\}$ converge uniformly to f and let $m(E) = K\ (\neq 0)$

Note : if $m(E) = 0$, $\int |f_n - f| = 0$ & hence the result]

Let $\in > 0$ be given

Then $\exists$ N.S.t $\forall n \geq N$.

$$|f_n(x) - f(x)| \leq \frac{\in}{K} \ \forall x \in E.$$

$$\therefore \int_E |f_n - f| \leq \frac{\in}{K} m(E) = \in$$

Hence the result.

To prove some other implications, we need the following lemma.

Lemma

Let $\{f_n\}$ be a sequence of measurable functions that converges in measure to f. Then there is a subsequence $\{f_{nk}\}$ that converges to f almost everywhere.

Proof:

Since $(f_n) \to f$ in measurable $\forall$ each (+ve) integer n then $\exists$ a (+ve) integer n_m such that

$$m\left\{x : |f_n(x) - f(x)| \geq \frac{1}{2^m}\right\} < \frac{1}{2^m}$$

$$\text{Let } E_m = \left\{x : |f_{n_m}(x) - f(x)| \geq \frac{1}{2^m}\right\}$$

$$\text{then } \forall x \notin \bigcup_{m=k}^{\infty} E_m \ |f_{n_m}(x) - f(x)| < \frac{1}{2^m} \ \forall\, m \geq k.$$

$$\left(f_{n_m}(x)\right) \to f \text{ on } E-A$$

$$\text{where } A = \bigcap_{k=1}^{\infty}\left(\bigcup_{k}^{\infty} E_m\right)$$

$$\therefore m(A) \leq m\left(\bigcup_{k}^{\infty} E_m\right) \leq \sum_{m=k}^{\infty} m(E_m) < \sum_{m=k}^{\infty} \frac{1}{2^m} = \frac{1}{2^{k-1}} \ \forall k$$

$$m(A) = 0$$

Thus $\left(f_{n_k}\right) \to f$ a.e on E.

Hence the lemma.

Thm:

Let $\{f_n\}$ be a sequence of non-negative measurable function and f be a measurable function such that $f_n \to f$ in measure then

$$\int f \leq \lim\ \inf \int f_n.$$

Proof :

Case (i)

Let $\int f < \infty$

Suppose $\int f > \lim\inf \int f_n$.

Then $\exists\ \delta > 0$ a sequence (n_k) s.t

$\forall$ K $\int f_{n_k} < (\int f) < \delta$.

f_n converges in measure to f.

$\Rightarrow \left(f_{n_k}\right)$ converges in measure to f.

(by lemma,

If (f_n) is a sequence which converges to f in measure then subsequence (f_{n_k}) will converge to f in measure).

$\Rightarrow \left(f_{n_k}\right)$ a subsequences of $\left(f_{n_k}\right)$ a.e. to f

$\therefore$ By Fatou's lemma,

$\int f \leq \lim\inf \int f_{n_{k_i}} \leq (\int f) < \delta$ a contradiction.

$\therefore \int f \leq \lim\inf \int f_n$.

Case (ii) :

Suppose $\int f = \infty$ and $\liminf \int f_n < \infty$ we can find $K > 0$ and a subsequence $\left(f_{n_k}\right)$ s.t

$\int\left(f_{n_k}\right) < K \ \forall K.$

f_n converges in measure to $f \Rightarrow \left(f_{n_k}\right)$ converges in measure to $f \Rightarrow \left(f_{n_{k_i}}\right)$

a subsequence of $\left(f_{n_k}\right)$ converges a.e to f

$\therefore$ By Fatou's lemma.

$\int f \leq \liminf \left(f_{n_{k_i}}\right) \leq k$ a contradiction to $\int f = \infty$

Hence the thm.

Thm :-

Let f_n be an increasing sequence of non-negative measurable functions and let f_n converges in measure to f. Then

$\int f = \lim f_n.$

Proof:-

Bv lemma

convergence a.u $\Rightarrow$ convergence measure.

$\int f \leq \liminf \int f_n$

since $f_n \leq f \ \forall n.$

$\int f_n \leq \int f \ \ \forall n$

and

$\limsup \int f_n \leq \int f.$

Hence $\liminf \int f_n = \limsup \int f_n = \lim \int f_n = \int f$

Hence the theorem.

Thm:-

Let (f_n) be a sequence of measurable functions such that $|f_n| \le g$, an integrable function and let f be a measurable function s.t $f_n \to f$ in measure.

Then f is integrable and

(i) $\lim \int f_n = f$ and $\lim \int |f_n - f| = 0$.

Proof :

By lemma, let (f_n) be a sequence of measurable functions which converges in measure to f. Then there is a subsequence (f_{nk}) which converges to f almost every where.

$\exists$ a subsequence $\left(f_{n_k}\right)$ of (f_n) s.t $\lim\limits_{k} f_{n_k} = f$ a.e

$\therefore |f_n| \le g \ \forall n \Rightarrow |f| \le g.$

(i) For each n, $g + f_n \ge 0$ and $(g+f_n)$ converges in measure to $(g+f)$ By thm,

Let $\{f_n\}$ be a sequence of non-negative measurable function and f be a measurable function such that $f_n \to f$ in measure.

Then $\int f \le \liminf \int f_n$.

$\therefore \int g+f \le \liminf \int g+f_n$

$\therefore \int f \le \liminf \int f_n$ ---- (1)

III^{ly}

$g - f_n \geq 0$ and $(g - f_n)$ converges in measure to $(g - f)$

By thm,

Let $\{f_n\}$ be a sequence of non-negative measurable function and f be a measurable function such that $f_n \to f$ in measure.

Then $\int f \leq \liminf \int f_n$.

$\therefore \int g - f \leq \liminf \int g - f_n$

$\therefore \int f \geq \limsup \int f_n$.

$\therefore \int f \leq \liminf \int f_n \leq \limsup \int f_n \leq \int f$

$\therefore \int f$ exists, $\int f = \lim \int f_n$.

$\therefore \int f = \lim \int f_n$

Hence the proof (i)

(ii) $|f_n| \leq g$ and $|f| \leq g \Rightarrow |f_n - f| \leq 2g$.

Also $\lim_n |f_n - f| = 0$ in measure.

$\therefore$ By (i) $\lim_n \int |f_n - f| = 0$

Hence the proof (ii)

corollary :-

Let (f_n) be a sequence of measurable functions defined on a measurable set E of finite measure. Then (f_n) converges to f in measure iff every subsequence of (f_n) has in turn a subsequence that converges almost everywhere to f.

Proof:

(i) Suppose f_n converges to f in measure.

Then by thm,

Let (f_n) be a sequence of measurable functions which converges in measure to f, then there is a subsequence (f_{nk}) which converges to f almost everywhere.

Then $\exists$ a subsequence (f_{nk}) converging to f a.e and so every subsequence of (f_{nk}) will converge to f a.e.

(ii) Let $m(E) < \infty$ and every subsequence of (f_n) has a subsequence converging to f a.e

Let $\in > 0$ be given and

$$\left.\begin{array}{l} E_n = \{ x / |f_n x) - f(x) \} > \in \\ \text{we want to prove that } \lim m(E_n) = 0 \end{array}\right\}$$

Suppose not:

$\exists\ \delta > 0$ and a sequence (f_{nk}) s.t

$$m\left(E_{n_k}\right) > \delta$$

consider the subsequence $\left(f_{n_k}\right)$.

This has a subsequence $\left(f_{n_k}\right)$ converging to f are

$$E_{n_{k_i}} = \{x : |f_{n_{k_i}}(x) - f(x)| > \epsilon\}$$

$$\lim E_{n_{k_i}} = \bigcap_m \bigcup_{i \geq m} E_{n_{k_i}}$$

For $x \notin \lim E_{n_{k_i}}$, $\lim f_{n_{k_i}}(x) = f(x)$

$$m(\lim E_{n_{k_i}}) = 0$$

$$m(E) < \infty \Rightarrow$$

$\therefore 0 + m \lim E_{n_{k_i}} \geq \lim m \, E_{n_{k_i}} \geq \delta > 0$ a contradiction

$$\therefore \lim m(E_n) = 0$$

$\therefore f_n$ converges to f in measure.

Hence the corollary.

UNIT - 9

MEASURE AND INTEGRATION

Measure spaces 9-1

The purpose of the present chapter is to abstract the most important properties of Lebesgue measure and Lebesgue integration. We shall do this by giving certain axioms which Lebesgue measure satisfies and base our integration theory on these axioms. As a consequence our theory will be valid for every system satisfying the given axioms.

We begin by recalling that a σ - algebra β is a family of subsets of a given set x which contains ϕ and is closed with respect to complements and with respect to countable unions. By a set function μ we mean a function which assigns an extended real numbers to certain sets. With this is mind we make the following definitions.

Definition : 9.1.1

A pair (X, β) consisting of a set X and σ- algebra β of subsets of X is called a measurable space.

A subset A of X is called a measurable set if $A \in \beta$.

Definition : 9.1.2

A subset C of X is called a non-measure set if $C \not\subset \beta$.

Definition : 9.1.3

A triple (X, β, μ) is called a measure space if (X, β) is a measurable space and μ is a measure on β.

Definition : 9.1.4

A function $\mu : \beta \rightarrow R^*$ satisfying the following properties.

i) $\mu(A) > 0 \ \forall A \in \beta$

ii) $\mu(\phi) = 0$

iii) $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu E_i$ for any sequence E_i of disjoint measurable sets

is called a measure on (X, β)

Example i) $(R, \mathscr{M}, m)$ where R is the set of real numbers $\mathscr{M}$, the Lebesgue measurable sets of real numbers and m- Lebesgue measure.

ii) (R, β, m) where β is the class of Borel sets and m- is again Lebesgue measure.

Definition: 9.1.5

Let (X, β, μ) be a measure space.

i) A measure μ is called finite measure if $\mu(X) < \infty$

ii) μ is called σ - finite measure, if there is a sequence $X_n, \in \beta$ such that

$$X = \bigcup_{n=1}^{\infty} X_n \text{ and } \mu(X_n) < \infty$$

iii) A measure μ is said to be semi-finite measure if each measurable set of infinite measure contains measurable sets of arbitrary **large finite** measure.

Definition : 9.1.6

A measure space (X, β, μ) is said to be complete if β contains all subsets of sets of measure zero.

(ie) if $B \in \beta$ $\mu\beta = 0$ and $A \subset B \Rightarrow A \in \beta$.

Example :

i) Lebesgue measure is complete.

Proposition : 9.1.1

If (X, β, μ) is a measure space, then we can find a complete measure space (X, β_0, μ_0) such that

i) $\beta \subset \beta_0$

ii) $E \in \beta \Rightarrow \mu E = \mu_0 E$

iii) $E \in \beta_0 \Leftrightarrow E = A \cup B$ where $B \in \beta$ and $A \subset C$, $C \in \beta$, $\mu C = 0$

Proof :

Let (X, β, μ) be the given measure space and $\beta = \{E \Delta N / E \in \beta \,.\, N \subset M$ $M \in \beta$ and $\mu(M) = 0\}$

i) we first observe that

$E \Delta N = (E - M) \cup (M \cap E \Delta N)$ ----- (1) For any sets E, M, N such that $M \supseteq N$.

Let $X \in E \Delta N$.
Then if $x \in M$
We have $X \in M_n \cap (E \Delta N)$

and if $X \notin M$, then $x \notin N$ since $N \subset M$.

And so $X \in E - N$

Hence $X \in E - M$ ($\because E \Delta N = (E - N) \cup (N - E)$

$\therefore X \in (E - M) \cup (M \cap (E \Delta N)$

And if $X \in M \cap (E \Delta N)$

Then $x \in E \Delta N$ and if $x \in E - M$.

We have $x \in E - N \subseteq E \Delta N$.

Let $D \in \beta$, $D = E \Delta N$, as above with $N \subseteq M \in \beta$ where $\mu(M) = 0$. Then by (1) and $D = F \cup A$ where $F \cap A = 0$ and $F \in \beta$ and $A \subseteq M \in \beta$ with $\mu(E) = 0$ and since for F, A disjoint.

We have $F \cup A = F \Delta A$ the two characterizations of the sets of β, are equivalent.

i) Now if $D_i \in \beta$ $i = 1, 2 \ldots$

$D_i = F_i \cup A_i = F_i \Delta A_i$.

We see that $\bigcup_{i=1}^{\infty} D_i \in \beta$ and $\cup A_i \subset \cup F_i$ and $\mu_i(\cup M_i) \leq \mu(M_i) = 0$

$\therefore \cup D_i \in \beta_0$. .... (i)

If $D_1 = E_1 \Delta N_1$ and $D_2 = E_2 \Delta N_2$

$D_1 \Delta D_2 = (E_1 \Delta E_2) \Delta (N_1 \Delta N_2)$

(Since Δ is commutative and associative)

$E_1 \Delta E_2 \in \beta$ and $N_1 \Delta N_2 \subset M_1 \cup M_2$ and $\mu(M_1 \cup M_2) = 0$

$\therefore D_1 \Delta D_2 \in \beta_0$ .... (ii)

$D_1 - D_2 = (D_1 \cup D_2) \Delta D_2 \in \beta_0$ (by (i) and (ii))

$\therefore$ Clearly $x \in \beta_0$.

$\therefore \overline{D}_1 \in \beta_0$ for $D_1 \in \beta$.

$\therefore \beta_0$ is a $\sigma -$ algebra.

clearly $\beta_0 \supset \beta$

ii) Define $\mu_0 : \beta_0 \to R^*$ as follows.

$\mu_0(D) = \mu_0(E \cup N) = \mu_0(E \Delta N) = \mu(E)$

[For $N \subset M_i$ $\mu(M) = 0$ and $E \cap N = \phi$)

Suppose $D_1 = D_2$ then $D_1 \Delta D_2 = \phi$

conversely $D_1 = E_1 \Delta N_1$, $D_2 = E_2 \Delta N_2$

$D_1 \Delta D_2 = (E_1 \Delta E_2) \Delta (N_1 \Delta N_2)$

If $D_1 \Delta D_2 = \phi$ then $E_1 \Delta E_2 = N_1 \Delta N_2$

[Since $A \Delta B = \phi \Rightarrow (A-B) \cup (B-A) = 0$

$\Rightarrow A - B = \phi$ and $B - A = \phi$

$\Rightarrow A \subset B$ and $B \subset A$ and so $A = B$]

Now, we have

$\mu(\tilde{E_1} \Delta E_2) = 0 = \mu(N_1 \Delta N_2)$

[$\because N_1 \Delta N_2 \in \beta$ and $N_1 \Delta N_2 \subset M_1 \cup M_2$ and $\mu(M_1 \cup M_2) = 0$]

since $E_1, E_2 \in \beta$, $E_1 - E_2$ and $E_2 - E_1 \in \beta$.

And so $\mu(E_1 - E_2) = 0$ $\mu(E_2 - E_1) = 0$

$\mu(E_1) = \mu(E_1 - E_2) \cup (E_1 \cap E_2)$

$= \mu(E_1 \cap E_2)$

$\mu(E_1) = \mu(E_2)$ ....(2)

$\mu_0(D_1) = \mu_0(E_1 \Delta N_1) = \mu(E_1)$

$\therefore D_1 = D_2 = \mu_0(D_1) = \mu_0(D_2)$ by (2)

$\therefore \mu_0$ is well- defined

μ_0 is a measure on β_0.

For clearly $\mu_0(\phi) = \mu(\phi) = 0$

Let $D_i \in \beta_i$ be a disjoint sequence.

Let $D_i = F_i \cup A_i$

So that $F_i \cap A_i = \phi$ for all i and j

(Since D_i are disjoint)

$\therefore \mu_0(\cup D_0) = \mu_0(\cup F_i \cup \cup A_i)$

$= \mu_0(\cup F_i \Delta \cup A_i)$

$= \Sigma \mu_0 (U F_i)$

$= \Sigma \mu_0 (F_i \cup A_i)$

$\mu_0(UD_i) = \Sigma_i \mu_0(D_i)$

(μ_0 is countable additive)

iii) Finally μ_0 is complete.

For let $D \subset D_0$, $D_0 \in \beta$ and where $\mu_0 (D_0)$

$\mu_0 (D_0) = 0$

So $D_0 = E_0 \Delta N$, where $N_0 \subseteq M_0, E_0, M_0 \in \beta$

$\mu_0(E_0) = \mu_0 (M_0) = 0$

And so $D_0 \subseteq M'_0 = E_0 \cup M_0 \in \beta$

And $\mu_0 (M'_0) = 0$.

Then $D = E \Delta N$ with $E = \phi$

$N = D \subseteq E_0 \cup M_0$

$\therefore D \in \beta_0$

Hence the result.

Proposition : 9.1.2

Show that the extension $\overline{\mu}$ of By the previous theorem in unique in the sense that if μ' is a complete measure on a σ- ring $\beta' \supseteq \beta$ and $\mu' = \mu$ on β then $\mu' = \overline{\mu}$ or β_0.

Proof :

Since $\overline{\mu}'$ is complete on $\beta' \supset \beta_0$.

And $\beta_0 = \{ E \Delta N / E \in \beta, N \subset M, \mu (M) = 0\}$

$\beta_0 \subset \beta'$

Also if $D \in \beta$.

We have $D = FUA$, F, A disjoint set with $F \in \beta$, $A \subseteq M \in \beta$ with $\mu(M) = 0$

So

$\mu'(D) = \mu'(FUA) = \mu'(F) + \mu'(A)$

$\mu'(D) = \mu(F) = \overline{\mu}_0 (D)$.

Note:

The completion of a σ- finite measure is σ- finite.

Proof :

Let $D \in \beta_0$

As in Theorem : 1

$D = FUA$ where $F \in \beta$

and $\mu_0(A) = 0$

So $F = \bigcup_{i=1}^{\infty} F_i$ where $\mu(F_i) < \infty$

Hence $D = A \cup \bigcup_{i=1}^{\infty} F_i$ is a countable union of sets of finite μ_0 measure.

Definition : 9.1.7

Let (X, β, μ) be a measure space A subset E of X is said to be locally measurable if $E \cap B \in \beta$ for each $B \in \beta$ with $\mu(B) < \infty$.

Proposition : 9.1.3

Let (X, β, μ) be a measure space and γ the algebra of locally measurable sets.

For $E \in \gamma$ define $\bar{\mu}(E) = \mu(E)$ if $E \in \beta$.

$= \infty$ if $E \notin \beta$.

Then $(x, \gamma, \bar{\mu})$ is a saturated measure space.

Proof:

i) Clearly $\bar{\mu}(\phi) = \mu(\phi) = 0$

ii) Let $E_n \in \gamma$

a) Suppose $E_n \in \beta \; \forall n$ then $UE_n \in \beta$

And $\mu E_n = E_n \; \bar{\mu}(UE_n) = \mu(UE_n)$

$\bar{\mu}(UE_n) = \mu(UE_n) = \Sigma \mu(E_n) = \Sigma \bar{\mu}(E_n)$

b) Suppose $E_k \in \gamma$ for some k and $E_k \notin \beta$

If $U E_n \in \gamma / \beta$, $\bar{\mu}(UE_n) = \infty$

If $UE_n \in \beta$ and $\mu(UE_n) < \infty$ Then

$E_k = E_k \cap (UE_0) \in \beta \Rightarrow \Leftarrow$

$\therefore \mu(UE_n) = \infty = \bar{\mu}(UE_n)$

In any case $\bar{\mu}(UE_n) = \infty$

Since $E_k \in \gamma \cap \bar{\beta}$

$\mu(E_k) = \infty$.

$$\therefore \bar{\mu}\left[(UE_n)\right] = \sum_1^{\infty} \bar{\mu}(E_n)$$

Hence μ is a measure on γ.

ii) Let γ be the set of all locally measurable sets w.r. to μ.

Let $A \in \bar{\gamma}$

Then $A \in \bar{\gamma} \Rightarrow A \cap B = \gamma$ for $\bar{\mu}(B) < \infty$ and $B \in \gamma$.

So, if $B \in \beta$ and $\mu(B) < \infty$

$$\bar{\mu}(B) = \mu(B) < \infty$$

and so $A \cap B \in \gamma$

But $\bar{\mu}(A \cap A) \leq \bar{\mu}(B) < \infty \Rightarrow (A \cap B) \in \beta$

By the definition of $\bar{\mu}$

$\therefore \forall A \in \bar{\gamma}, A \cap B \in \beta$ if $\mu(B) < \infty$

and $B \in \beta$

$\therefore A \in \gamma$

(ie) $\bar{\gamma} \subset \gamma$

$\therefore \bar{\mu}$ is a saturated measure.

Note :-

The extension of μ to μ is not unique.

Example : 1

If (x, β, μ) is a complete measure space show that $E_1 \in \beta$ and $\mu(E_1 \Delta E_2) = 0 \Rightarrow E_2 \in \beta$

Proof :

$\mu(E_1 \Delta E_2) = \mu(E_1 \sim E_2) \cup (E_2 \sim E_1) = 0$

$\therefore \mu(E_1 \sim E_2) = 0$ and $\mu(E_2 \sim E_1) = 0$

Since μ is complete $E_1 \sim E_2$

and $E_2 \quad E_1 \in \beta$

$\therefore E_1 \cap E_2 = E_1 \sim (E_1 \sim E_2) \in \beta$

and $E_2 = (E_1 U E_2) \cup (E_2 \sim E_1) \in \beta$

Example : 2

Suppose (x, β, μ) is a measure space and (x, β_0, μ_0) it completion show that if $A, B \in \beta$ with $A \subset E \subset B$, $\overline{\mu}(B-A) = 0$. Then $E \in \beta_0$ and $\mu(E) = \mu(A) = \mu(B)$.

proof :

Let $A \subset E \subset B$ and $\mu(B-A) = 0$

$\therefore \mu(B) = \mu(A) + \mu(B-A) = \mu(B) = \mu(A)$

Also $(E-A) \subset (B-A)$

$A, B, \in \beta$ (since it is a subset of measure zero)

$\therefore E = A \cap (E - A) \in \beta_0$

And $\mu(E) = \mu(A) = \mu(B)$

$\therefore \mu(E) = \mu(A) = \mu(B)$

Hence the prove.

Topic - 9.2 : General Measurable Function

Since not all sets are measurable it is of great importance to know that sets which arise naturally in certain construction are measurable. If we start with a function f. The most important sets that arise from it are those listed in the following.

Proposition : 9.2.1

If f is an extended real valued function f defined on x. Then the following statements are equivalent. i) f is a measurable function.

ii) $\{x : f(x) \geq \alpha\}$ is measurable $\forall \alpha$.

iii) $\{x : f(x) < \alpha\}$ is measurable $\forall \alpha$

iv) $\{x : f(x) \leq \alpha\}$ is measurable $\forall \alpha$

Proof :

To prove (i)$\Rightarrow$(ii)

Let f be measurable for each $\alpha \in R$.

$$\{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \left\{\{x: f(x) > \alpha - \frac{1}{n}\right\} \text{ is measurable.}$$

To prove (ii)⇒(iii)

For each $\alpha \in R$ $\{x : f(x) \geq \alpha\}$ be measurable.

Then $\{x \in X : f(x) < \alpha\} = \{x : f(x) \geq \alpha\}$ is measurable.

To prove (iii) ⇒ (iv)

For each $\alpha \in R$

$\{x : f(x) < \alpha\}$ is measurable.

$$\{x : f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} x : f(x) < + \alpha \frac{1}{n} \quad \text{is measurable.}$$

To prove (iv) ⇒ (i)

For each $\alpha \in R$ $\{x \in X : f(x) \leq \alpha\}$ is measurable.

Then its complement $\{x : f(x) > \alpha\}$

∴ f is measurable.

Definition : 9.2.1

Let (x, β, μ) be a measure space. A function $f : x \to R^*$ is called a measurable function (with respect to β) . If any one of the above statements hold.

Example : 1

Prove that if f is measurable. Then $\{x : f(x) = \alpha\}$ in measurable for each extended real number α.

Solution :

For finite α

$\{x: f(x) = \alpha\} = \{x : f(x) \geq \alpha\}$ $\{x: f(x) \leq \alpha\}$ is measurable.

For : $\alpha = -\alpha$

$$\{x : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \in X : f(x) > n\}$$

$$\{x : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \in X : f(x) < -n\}$$

is measurable set

Example : 2

The constant functions are measurable.

Solution :

Suppose f in a measurable on X. such that

$g(x) = f(x)$ if $x \in A$

$= c$ if $x \notin A$.

Let $a \in A$

if $a \leq C$

$\{x : f(x) < a\} = \{x : f(x) < a\} \cap A$

If $a > c$

$\{x : g(x) < a\} = [\{x : f(x) < a\} \cap A] \cap (x - A)$

Since f is measurable and $A \in \beta$

$\{x : g(x) < a\}$ $\forall$ a are measurable.

Hence g is measurable.

Example 2.1

The characteristic function χ_A of the set A is measurable if A is measurable.

Solution :

Suppose $\chi_A = 1 \quad x \in A$

$= 0 \quad x \notin A$

$$\therefore \{x : f(x) > \alpha\} = \begin{cases} R & \text{if } \alpha < 0 \\ A & \text{if } 0 \leq \alpha < 1 \\ \phi & \text{if } \alpha \geq 1 \end{cases}$$

$\therefore$ R, A, ϕ is measurable.

$\therefore \chi_A$ is measurable.

$\Rightarrow A \in \beta$

$\therefore \chi_A$ is measurable iff A is measurable.

Example : 3

Continuous functions from $f : R \to R$ are measurable.

For: suppose f is continuous.

Then $\{x : f(x) > \alpha\}$ is open set.

$\therefore$ But any open set is measurable .

$\therefore$ f is measurable.

Theorem : 9.2.1

Let c be any real number and let f and g be real valued measurable functions defined on the same measurable set E. Then $f + c$, cf, $f+g$, $f-g$ and fg on also measurable.

Proof :

To show that $f + c$ is measurable.

For each α

$$\{x: f(x) + c < \alpha\} = \{x : f(x) < \alpha - c\}$$

since f is measurable.

$\{x : f(x) < \alpha - c\}$ is measurable.

$\therefore \{x : f(x) + c < \alpha\}$ is measurable.

$\therefore$ $f + c$ is measurable.

ii) To show that cf is measurable.

If $c = 0$ then if $cf = 0$ a constant function

$\therefore$ cf is measurable.

$$\text{If } c \neq 0 \text{ then } \{x : c\,f(x) < \alpha\} = \begin{cases} x : f(x) < \alpha/c & \text{if } c > 0 \\ x : f(x) > \alpha/c & \text{if } c < 0 \end{cases}$$

$\therefore$ R.H.S set is measurable.

$\therefore$ L.H.S $\Rightarrow$ cf is measurable.

(iii) To show that $f + g$ is measurable consider

$$\text{consider } x \in A = \{x : (f+g)(x) < \alpha\}$$

$$= \{x : f(x) < \alpha - g(x)\}$$

(ie) There exists a rational r_i such that

$$f(x) < r_i < \alpha - g(x) \text{ where } \{r_i, i = 1,2....\}$$

$\{x : (f+g)(x) < \alpha\} = \bigcup_r \{x : f(x) < r_i\} \cap \{x : g(x) < \alpha - r\}$

Since rationals are countable and since each of $\{x : f(x) < r_i\}$, $\{x : g(x) < \alpha - r_i\}$ is measurable for every r. (as f and g are measurable) we get $\{x : (f+g)x < \alpha\}$ is measurable.

(ie) $f + g$ is measurable.

(iv) $f - g = f + (-g)$

Hence by (ii) and (iii)

$f - g$ is measurable when f and g are so.

v) The function f^2 is measurable.

For : $\{x : f^2(x) > \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) < -\sqrt{\alpha}\}$

For $\alpha \geq 0$ and

$\{x : f^2(x) > \alpha\} = D$. when $\alpha < 0$ wher D is the domain of f.

In any case

$\{x : f^2(x) > \alpha\}$ is measurable.

Since f is measurable and D is measurable.

$\because f^2$ is measurable.

If f and g are measurable.

Then $fg = \frac{1}{2}\left[(f+g)^2 - (f-g)^2\right]$

$\therefore$ (ii), (iii), (iv) and the above result.

$\therefore$ fg is measurable.

Corollary : 1

If f and g are extended real valued functions. We define $(f+g)(x) = f(x) + g(x)$ if $f(x)$ and $g(x)$ are real or one of them $\pm\alpha$ or both are $+\alpha$ or both $-\alpha$.

If $(f+g)x =$ a constant C.

If $f(x) = \alpha$, $g(x) = -\alpha$ or $f(x) = -\alpha$, $g(x) = \alpha$

Then $f+g$ in measurable.

Similarly, Define $f - g$

However fg is always measurable.

Theorem : 9.2.2

Let $\{f_n\}$ be a sequence of measurable functions (with the same domain of definition). Then the functions $\sup\{f_1, f_2 \ldots f_n\}$, $\operatorname{Inf}\{f_1, f_2 \ldots f_n\}$

$\sup\limits_n f_n$, $\operatorname{Inf}\limits_n f_n$, $\overline{\operatorname{Lim}}\, f_n$, $\underline{\operatorname{Lim}}\, f_n$. are all measurable.

Proof :-

i) Let h is defined by $h(x) = \sup\{f_1(x) \ldots f_n(x)\}$

Then

$$\{x : h(x) > \alpha\} = \bigcup_{1}^{n} \{x : f_i(x) > \alpha\}$$

Hence $\sup\limits_{1 \le i \le n} f_i$ is measurable.

ii) $\operatorname{Inf}\limits_{1 \le i \le n} f_i = -\sup\limits_{1 \le i \le n} (-f_i)$

$\therefore \operatorname{Inf}\limits_{1 \le i \le n} f_i$ is measurable.

iii) Let $g(x) = \sup f_n(x)$

$$\{x : \sup f_n(x) > \alpha\} = \bigcup_{1}^{n} \{x : f_n(x) > \alpha\}$$

$$\{x : g(x) > \alpha\} = \bigcup_{1}^{\infty} \{x : f_n(x) > \alpha\}$$

$\therefore \sup f_n(x)$ is measurable.

iv) Let $h(x) = \operatorname{Inf} f_n(x)$

$$\{x : h(x) < \alpha\} = \bigcup_{1}^{\alpha} \{x : f_n(x) < \alpha\}$$

$\therefore$ h is measurable.

v) Since $\overline{\operatorname{Lim}}\, f_n = \operatorname{Inf}\limits_n \sup\limits_{k \ge n} f_k$ By (iii) and (iv)

$\therefore \overline{\operatorname{Lim}}\, f_n$ is measurable.

vi) Since $\underline{\text{Lim}}\, f_n = \text{Sup}_n \text{ In f}_{zk\geq n} f_k$

By iii and iv

$\therefore \underline{\text{Lim}}\, f_n$ is measurable.

Remark :

i) Let $\{f_n\}$ be a sequence of measurable functions on the same domain D.

Let $\text{Lim}_{n\to\infty} f_n = f$. Then f is measurable.

For : If $\text{Lim}_{n\to\infty} f_n$ exists. Then

$f = \text{Lim}_{n\to\infty} f_n = \overline{\text{Lim}}\, f_n = \underline{\text{Lim}}\, f_n$

$\therefore$ f is measurable .

Note :-

1) f is the measurable if f f^+ and f^- are measurable.

$(\because f = f^+ - f^-)$

2) f is measurable $\Rightarrow |f|$ measurable. Converge not true.

Definition :- 9.2.2.

A simple function is a finite linear combination of characteristic functions of Subsets in β.

(i.e) $\sum_{i=1}^{n} a_i \chi_{Ei}$ is a simple function.

Proposition :- 9.2.2

Let f be a non-negative measurable function. Then there is a sequence (ϕ_n) of simple functions with $\phi_{n+1} \geq \phi_n$ such that $f = \text{Lim}\, \phi_n$ at each point of X. It f is defined on a σ - finite measure space then we may choose the functions ϕ_n. So that each varnishes outside a set of finite measure.

Proposition :- 9.2.3

If μ is a complete measure and f is a measurable function. Then f = g a.e implies g is measurable.

Proof :-

Let $E = \{x : f(x) \neq g(x)\}$

Then $E \subset F$. $\mu(F) = 0$.

Since μ is complete. E is measurable and $\mu(E) = 0$.

Given f is measurable.

$\{x : g(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \{x \in E : g(x) > \alpha\} \sim \{x \in E : g(x) \leq \alpha\}$

For : $g(x) > \alpha$ iff.

$x \in E$ and $g(x) > \alpha$ or $x \notin E$ and $g(x) > \alpha$

ie) $\{x \in E : g(x) > \alpha\} \cup \{x \notin E : g(x) > \alpha\}$

ie) $\{x \in E : g(x) > \alpha\} \cup \{x \notin E : f(x) > \alpha\}$ $\{\therefore f(x) = g(x)$ on $\bar{E})$

ie) $\{x \in E : g(x) > \alpha\} \cup \{x : f(x) > \alpha\} - \{x \in E : g(x) \leq \alpha)$

Hence $\{x : g(x) > \alpha\} = \{x \in E : g(x) > \alpha\} \cup \{x : f(x) > \alpha\} - \{x \in E : g(x) \leq \alpha)$

$\{x : f(x) > \alpha\}$ is measurable.

Since $mE = 0$ and Since $\{x \in E : g(x) > \alpha\}$ and $\{x \in E : g(x) \leq \alpha\}$ are subset of E. We get that they are of measure of 0.

Hence measurable.

$\therefore \{x : g(x) > \alpha\}$ is measurable. (i.e) g is measurable.

Note :-

If μ is not a complete measure for each α Hence g is not a measurable function.

Example :-

Let E be a Lebesgue measurable set of measure zero which is not Borel measurable.

Then χ_E is Lebesgue measurable but not borel measurable and $\chi_E = 0$ a.e. with reference to borel measure also (For $m(E) = 0 \Rightarrow \exists\ G_\delta$ set (which in Borel set) $G \ni : E$ (G and $m(G) = 0$)

Proposition :- 9.2.4

Suppose that to each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \mathcal{B}$ such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$. Then there is a unique measurable extended real-valued function f on x such that $f \leq \alpha$ on B_α and $f \geq \alpha$ and on $x \sim B_\alpha$.

Proof :-

For each $x \in X$.

Define $f(x) = \inf\{\alpha \in D : x \in B_\alpha\}$. Where the usual $\inf \phi = \infty$

If $x \in B_\alpha$. Then $f(x) \leq \alpha$.

If $x \notin B_\alpha$. Then $x \notin B_\beta$ for each $\beta < \alpha$

And so $\beta < \alpha$, $f(x) \geq \alpha$.

To show that f is measurable.

We take $\lambda \in R$ and choose a sequence $<\alpha_n>$ from D with $\alpha_n < \lambda$ and $\lambda = \text{Lim}\ \alpha_n$.

$$\text{Then } \{x : f(x) < \lambda\} = \bigcup_{n=1}^{\infty} B_{\alpha n}$$

For if $f(x) < \lambda$, then $f(x) < \alpha_n$ for some n, And so $x \in B_{\alpha n}$

If $x \in \beta\alpha_n$ for only n.

Then $f(x) \leq \alpha_n < \lambda$.

Thus the sets $\{x : f(x) < \lambda\}$ are all measurable and

$\therefore$ f is measurable.

To prove the unicity of f.

Let g be any extended real valued function with $g \leq \alpha$ on B_α and $g \geq \alpha$ on $\overline{B}_\alpha$.

Then $x \in B_\alpha$ imples $g(x) \leq \alpha$.

And so $\{\alpha \in D : x \in B_\alpha\} \subset \{\alpha \in D : \alpha \geq g(x)\}$

Since $g(x) < \alpha$ implies that $x \in B_\alpha$.

We have $\{\alpha \in D : \alpha > g(x)\} \subset \{\alpha \in D : x \in B_\alpha\}$

Because of the density of D.

We have

$$\begin{aligned} g(x) &= \inf \{\alpha \in D : \alpha > g(x)\} \\ &= \inf \{\alpha \in D : \alpha \geq g(x)\} \\ &= \inf \{\alpha \in D : x \in B_\alpha\} \end{aligned}$$

$g(x) = f(x)$ Hence the result

Note :-

The preecending proposition shows that a function is uniquely detemined by its ordinate sets and that these ordinate sets may be taken to be an arbitrary increasing family $\{B_\alpha\}$ provided we are flexible about whether or not an x with $f(x) = \alpha \in B_\alpha$.

Remark :-

If f and g are finite a.e. and measurable then $f \pm g$ are measurable.

For Let $f_1(x) = f(x)$ when $f(x)$ is a real number and $f_1(x) = 0$ when $f(x) = \pm\alpha$

Let $g_1(x) = g(x)$ when $g(x)$ is finite and $g_1(x) = 0$ when $g(x) = \pm\alpha$,

Then $f_1 = f$ a.e.

$g_1 = g$ a.e.

Hence f_1, g are measurable. By 9.2.3

$f_1 + g_1$ is measurable

But $f + g = f_1 + g$ a.e

Hence $f + g$ is measurable.

Proposition 9.2.5 :-

Supppose that for each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \beta$. Such that $\mu(B_\alpha \cup B_\beta) = 0$ for $\alpha < \beta$. Then there is a measurable function f such that $f \leq \alpha$ a.e. on B_α and $f \geq \alpha$ a.e. on $X \sim B_\alpha$. If g in any other function with this property then $g = f$ a.e.

Proof :-

Let C be a countable dense subset of D.

And $N = \cup(B_\alpha \cup B_\beta)$ for α and β in C with $\alpha < \beta$.

$$C = (\cup B_\alpha) \cap \overline{B}_\beta = \underset{\alpha<\beta}{\cup}(B_\alpha - B_\beta)$$

$$\mu/c = (\mu(\underset{\alpha<\beta}{\cup}(B_\alpha \cap \overline{B}_\beta) = \mu(\underset{\alpha<\beta}{\cup}(B_\alpha - B_\beta))$$

$\mu(c) = 0$

Since $\mu(B_\alpha - B_\beta) = 0$ for each $\alpha < \beta$.

Let $B_\alpha^1 = B_\alpha \cup N$ For α and β in C

With $\alpha < \beta$.

We have $B_\alpha^1 \sim B_\beta^1 = (B_\alpha \sim B_\beta) \sim N = \phi$.

$\therefore$ By the theorem.

Suppose that to each α in a dense set D of real numbers there in assigned a set $B_\alpha \in \beta$ such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$. Then there in a unique measurable extented real valued function f on x. Such that $f \leq \alpha$ on B_α and $f \geq \alpha$ on $X \sim B_x$.

There is a measurable function f such that $f \leq \gamma$ on $B^1\gamma$ and $f \geq \gamma$ on $\overline{B}^1\gamma$.

Let $\alpha \in D$ and choose a sequence (γ_n) from C with $\alpha < \gamma_n$ and $\alpha = \text{Lim}\ \gamma_n$.

Then $B_\alpha \cup B^1\gamma_n \subset (B_\alpha \cup B_\gamma)$.

Thus $P = \underset{n}{\cup} (B_\alpha \sim B^1_\gamma)$ is countable union of null sets and so a null sets.

Let $A = \cap B^1_\gamma$

Then $f \le \ln f\gamma = \alpha$ on A.

And $A \sim B_\alpha \subset P$.

Thus $f \le \alpha$ almost every where on B_α.

A similar argument shows that $f \ge \alpha$ almost everywhere on $\overline{B}_\alpha$.

Let g be an extended real-valued function with $g \le \gamma$ a.e. on B_γ and $g \ge \gamma$ on $\overline{B}_\gamma$ for each $\gamma \in C$.

Then $g \le \gamma$ on B^1_γ and $g \ge \gamma$ on $\overline{B}_\gamma$ except for x in a null set ϕ_γ.

Thus $Q = \cup Q_\gamma$ is a null set.

$\therefore$ f = g on $x \sim Q$.

Example :1

Let (X, β, μ) be a measure space and (X, β_0, μ_0) be its completion. It a function f is measurable w.r. to β_0 then there is a function g measurable w.r. to β. Such that f = g a.e. is the sense that there is a set $E \in \beta$ with $\mu(E) = 0$ and f = g on X - E.

Proof:-

Let for each rational number.

$$E_r = \{x / f(x) < r\}$$

Then $E_r = F_r \cup N_r$ where $F_r \in \beta$ $N_r \subset M_r$.

$\mu(M_r) = 0$ and $M_r \in \beta$

Let $M = \underset{n}{\cup} M_n$ then $\mu(M) \le \Sigma \mu(M_r) = 0$

And $\cup N \subset M$

Define g (x) = f (x) on M^-

= 0 on M

Then g is a measurable w.r. to β because for $r \le 0$, rational $\{x / g(x) < r\}$.

$\{x / g(x) < r\} = M \sim \cup F_i \in \beta$ and for $r > 0$ rational $\{x / g(x) < r\} = M \cup (F_1 \cap M\sim) \in \beta$.

Example :- 2

A function f on a measure space (X, β, μ) is called locally measurable if $\forall E \in \beta$ with $\mu(E) < \infty$, f/E in measurable, (ie) $f\chi_E$ is measurable show that f in locally measurable if it is measurable w.r. to the algebra of locally measurable sets.

Proof :-

i) Let f be locally measurable.

(ie) $\forall E \in \beta$ such that $\mu(E) < \alpha f \chi_E$ is a measurable set.

Let β be that σ - algebra or locally measurable.

(i.e.) $\beta = \{B / \text{ for } E \in \beta \text{ with } \mu(E) < \infty, E \cap B \in \beta\}$

Let $E \in \beta$ with $\mu(E) < \infty$, Let $a \leq 0$, $\{x / f \chi_E(x) < a\} = \{x / f(x) < a\} n\, E \in \beta$.

For $a > 0$

$\{x / f \chi_E(x) < a\} = E \sim \cup \{x / f(x) < a\} \cap E\} \in \beta$.

Since $E \in \beta$.

$\therefore \{x / f x < a\} \cap E \in \beta \; \forall a$

$\therefore \{x / f x < a\} \in \beta$

$\therefore$ f is measurable w.r. to β.

ii) Retracting the steps, we get the result.

INTEGRATION

Topic - 3

Definition: 9.3.1

A measurable simple function ϕ is one taking a finite number of non-negative vlaues, each on a measurable set. So if $a_1, a_2 \ldots a_n$ are the distinct values of ϕ.

We have $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ where $A_i = \{x : \phi(x) = a_i\}$

Then the integral of ϕ, $\left\{\text{We have } \phi = \sum_{i=1}^{n} a_i \chi_{A_i}\right\}$ with respect to μ is given by

$$\int \phi \, d\mu = \sum_{i=1}^{n} a_i \, \mu(A_i)$$

Definition : 9.3.2

Let f be measurable $f : x \to [0, \alpha]$ Then the integral of f is $\int f \, d\mu = \text{Sup} \left(\int \phi \, d\mu : \phi \le f \; \phi \text{ a measurable simple function} \right)$

Definition : 9.3.3

Let $E \in \beta$ and let f be a measurable function $f : E \to [0, \alpha]$. Then the integral of f over E is $\int_E f d\mu = \int f \chi_E \, d\mu$.

Theorem :-

Fatou's Lemma :- 9.3.1

If $\{f_n\}$ be a sequence of non-negative measurable function which converge almost everywhere on a set E to the functions f. Then

$$\text{Lim inf} \int f_n \, dx \ge \int \text{Lim inf} \, f_n \, dx.$$

Proof :-

Let $f = \text{Lim inf} \, f_n$.

Then f is a non-negative measurable function. By the definition.

For any non-negative measurable function f, the integral of f. $\int f \, dx$ is given by

$\int f \, dx = \sup \int \phi \, dx$ where the supremum is taken over all measurable simple functions ϕ, $\phi \le f$.

For each measurable simple function ϕ with $\phi \le f$.

We have $\int \phi \, dx \le \text{Lim inf} \int f_n \, dx$ ----- (1)

Case i) If $\int \phi \, dx = \infty$

Since ϕ is a simple function, there is a measurable set $A \subset E$ with $\mu(A) > \infty$ and $a > 0$ such that

$\phi(x) > a$ for all $x \in A$

Let $g_k(x) = \inf\limits_{i \ge k} f_i(x)$

And $A_n = \{x : g_k(x) > a\} \; \forall \; K \ge n$.

i) Clearly A_n is an increasing sequence of measurable set.

ii) Since $\phi \le f = \text{Lim} \, f_n$.

$A \subset \bigcup\limits_1^\infty A_n$ ($\because$ if $x \notin \bigcup\limits_1^\infty A_n$, Then $f_k(x) < a$ for infintely many K and so lim $f_k(x) \le a$ and so $\phi(x) \le a$)

Hence $\text{Lim } m(A_n) = \infty$

$\int f_n \, dx \geq \int g_n \, dx > am(A_n)$

We have lim in f $\int f_n \, dx = \alpha$ and (1) holds

Caes ii) $\int \phi(x) \, dx < \infty$

If $B = \{x : \phi(x) > 0\}$ Then $m(B) < \infty$

Let M be the largest value of ϕ and if $0 < \in < 1$

Define $B_n = \{x : g_k(x) > (1-\in)\phi(x), k \geq n\}$

Where g_k is as defined

Then i) A_n is an increasing sequence of measurables sets.

ii) $A_n \subseteq A_{n+1}$ for each n.

And $\bigcup_{n=1}^{\infty} A_n \supset A$

Consider $A - A_n$

Then $A - A_n$ is a decrease sequence of sets, $\cap(A - A_n) = \phi$ As $m(B) < \infty$

By the theorem.

Let $\{E_i\}$ be a sequence of mesurable sets. Then

i) If $E_1, \subseteq E_2 \subseteq \ldots\ldots$ we have $m(\text{Lim } E_i) = \text{Lim } m(E_i)$

ii) If $E_1, \supseteq E_2 \supseteq \ldots\ldots$ and $m(E_i) < \alpha$ for each i then we have $m(\text{Lim } E_i) = \text{Lim } m(E_i)$

There exists N such that $m(B - B_n) \leq \in \forall n \geq N$

So if $n \geq N$

$$\int g_n \, dx \geq \int_{B_n} g_n \, dx \geq (1-\in) \int_{B_n} \phi \, dx$$

$$= (1-\in)\left(\int_B \phi \, dx - \int_{B-B_n} \phi \, dx\right)$$

$$\geq (1-\in) \int \phi \, dx - \int_{B-B_n} \phi \, dx$$

$$\geq \int \phi \, dx - \in \int \phi \, dx - \in M.$$

Since $\in$ is aibitrary.

Let M be the largest value assued by the simple function ϕ.

$\therefore \int f_n(x)\,dx \geq (1-\epsilon)\,(\int \phi(x)\,dx - M.m(B-B_n)$

$\int f_n(x)\,dx \geq (1-\epsilon)\,(\int \phi(x)\,dx - M.m(B-B_n)\ \forall n \in Z_+$.

$\therefore$ Lim In f $\int f_n(x)\,dx \geq (1-\epsilon)\,\phi(x)\,dx$ – Lim in f $M.m(B-B_n)$

Lim In f $\int f_n(x)\,dx \geq (1-\epsilon)\,\phi(x)\ \forall \epsilon > 0$

$\therefore$ Lim in f $\int f_n(x)\,dx \geq \int \phi(x)\,dx.\ \forall \phi \leq f$.

$\therefore$ Lim in f $\int f_n(x)\,dx \geq \sup_{\phi \leq f} \int \phi(x)\,dx = \int f(x)\,d\,x$.

$\therefore$ Lim in f $\int f_n(x)\,dx \geq \int$ Lim in f $f_n(x)\,d\,x$.

Theorem:- 9.3.2

(Monotone Convergence Theorem)

Let $\{f_n\}$ be a sequence of non-negative measurable functions which converge almost every where to a function f and suppose that $f_n \leq f$ for all n. Then $\int f = \lim \int f_n$.

Proof :-

Since $f_n \leq f$ for all n.

We have $\int f_n \leq \int f$ for all n $\Rightarrow \overline{\text{Lim}} \int f_n \leq f$ ------ (1)

By Fatou's lemma

Let $\{f_n\}$ be a sequence of non-negative measurable functions which converge. Almost everywhere on a set E to the function of.

Lim in f $\int f_n\,dx \geq \int \lim\inf f_n\,dx$.

Let f = Lim in f f_n.

$\int f\,dx = \int \lim\inf f_n\,dx \leq \lim\inf \int f_n\,dx$.

$\therefore \int f\,dx \leq \underline{\text{Lim}} \int f_n$ ----- (2)

By (1) and (2)

We get $\int f \leq \underline{\text{Lim}} \int f_n \leq \overline{\text{Lim}} \int f_n \leq \int f$

$\therefore \overline{\text{Lim}} \int f_n$ exist.

$\therefore \int f = \lim \int f_n$.

Corollary:- 1

Let f be a non-negative measurable funtion. Then a non decreasing sequence of simple function S_n. Such that $\lim S_n = f$ and $\int f = \lim \int S_n$.

Proof:-

Since $\{S_n\}$ is a non-decreasing sequence of simple functions. Suc that $\lim S_n = f$. By the Monotone convergence Theorem.

$\int f = \operatorname{Lim} \int S_n$.

Corollary:- 2

Let f_n be a sequence of non-negative mesurable functions whicl convergence almost every where to a functions f and suppose that $f_n \le f_{n+1}$ n. Then $\int f = \operatorname{Lim} \int f_n$.

Proposition : 9.3.1

If f and g are non-negative measurable functions and a and b be non negative constants. Then $\int af + bg = a \int f + b \int g$ We have $\int f \geq 0$ witl equality only if $f = 0$ a.e.

Proof:-

To prove the first statement.

Let $\{\phi_n\}$ and $\{\psi_n\}$ be increasing sequences of simple functions whic converge to f and g

Then $(a\phi_n + b\psi_n\}$ is an increasing sequence of simple functions converge to $(a.f + bg)$

By Monotone convergence theorem.

$$\int af + bg = \lim \int a\,\phi_n + b\,\psi_n$$

$$= \lim (a \int \phi_n + b \int \int \psi_n)$$

$$\int af + bg = a \int f + b \int g.$$

Clearly $\int f \geq 0$ If $\int f = 0$

Let $A_n = \{x : f(x) \geq 1/n\}$

Then we have $f \geq (1/n)\,\chi_A$

And so $\mu A_n = \int \chi_A = 0$

Since the set where $f > 0$ is the union of the sets An. It has measure zer

Proposition : 9.3.2

Let $\{f_n\}$be a sequence of non-negative measuable functions. Then

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n.$$

Proof:

Proof follows from Monotone convergence theorem.

Definition : 9.3.4

An arbitrary function f is said to be integrable if both f^+ and f^{-1} are integrable in this case we define.

$$\int_E f = \int_E f^+ - \int_E f^-$$

Note: 1

$\int f$ is a measurable function and if either

$\int_E f^+$ or $\int_E f^-$ is finite

we can define

$$\int_E f = \int_E f^+ - \int_E f^-$$

$$\int f = \pm\infty$$

as $\int_E f^+$ or $\int_E f^- = \infty$

If $\int f < \infty$ we say f is integrable.

Note : 2

If $f = f^+ - f^- = f_1 - f_2$ non-negative measurable function then $\int f = \int f_1 - f_2$.

Proof:

$$f^+ - f^- = f_1 - f_2 \Rightarrow f^+ + f_2 = f_1 + f^-$$

(ie) $\int f^+ + \int f_2 = \int f_1 + \int f$

Since all integrals are finite.

We get $\int f = \int f^+ - \int f^- = \int f_1 - \int f_2$.

Proposition 9.3.3

If f is integrable, then $|f|$ is integrable and $|\int f| \leq \int |f|$

Proof :

Suppose f is integrable.

$\Rightarrow \int f^+ \, dx < \infty$ and $\int f^- \, dx < \infty$

$\therefore |f| = f^+ + f^- \Rightarrow \int |f| = \int f^+ + f^- < \infty$

$\Rightarrow$ $|f|$ is integrable.

Also $f = f^+ - f^-$

$$f \le f^+ \text{ and so } \int f \le \int f^+ - \int f^-$$

$$\le \int f^+ + \int f^- = \int |f|$$

$$-f = f^- - f^+$$

$$\Rightarrow -\int f = \int -f \le \int f^- \le \int f + \int f^+ = \int |f|$$

$$\therefore \quad \left|\int f\right| \le \int |f|$$

Note:

If f is measurable and $|f|$ integrable. Then f is integrable.

Proposition: 9.3.4

If f and g are integrable functions.

i) a f is integrable , and $\int afdx = a \int fdx$.

ii) $f+g$ is integrable and $\int (f+g)dx = \int fdx + \int gdx$

iii) If $f \le g$ a.e then $\int fdx \le \int gdx$

Proof:

Suppose that $a \ge 0$

Then $(af)^+ = af^+$

$(af)^- = af^-$

So $\int (af)^+ dx < \infty$ and $\int (af)^- dx < \infty$

so af is integrable and

$\int af\,dx = \int af^+ dx - \int af^- dx = a \int fdx$

Suppose that $a = -1$

Then $(-f)^+ = f^-$

$(-f)^- = f^+$

So $-f$ is integrable.

And $\int (-f)\, dx = \int f^- dx - \int f^+ dx = -\int fdx$

But for $a < 0$ $af = -|a|\, f$

$$\text{So } \int afdx = -\int |a|\, fdx = -|a| \int fdx$$

$$= a \int fdx.$$

ii) $(f+g)^+ \le f^+ + g^+$

$(f+g)^- \le f^- + g^-$

So $f+g$ is integrable.

Also $(f+g)^+ - (f+g)^- = f+g = f^+ + g^+ - f^- - g^-$

so $(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$ ---- (1)

By the theorem

Let f and g be non-negative measurable functions. Then $\int fdx + \int gdx = (f+g)\, dx$.

Using Theorem

To both sides of (1) and rearrange the terms.

$\therefore \int (f+g)\, dx = \int fdx + \int gdx.$

iii) let $g = f + (g - f)$

so $\int gdx = \int fdx + \int (g-f)^+ dx - \int (g-f)^- dx$

But $(g-f)^- = 0$.a.e

$\therefore$ $fdx \leq gdx$.

Proposition : 9.3.5

Let f be integrable then i) $\mu\{(x/f(x) = \pm\infty\} = 0$ (ie) f in finite a.e. ii) $\{x/f(x) \neq 0\}$ is a σ- finite measure.

Proof:

Let $\{x/f(x) = \pm\infty\} = \{x / |f(x)| = \infty\}$

Let $g = |f|$

Then g is non-negative integrable function.

Let $A_n = \{x / g(x) > n\}$

Then $A = \{x / g(x) = \infty\} = \cap A_n$

Now,

$n\chi_{A_n} \leq g$

$\int n\chi_{A_n} \leq \int g$

$n\mu(A_n) \leq \int g$

$n\mu(A_n) \leq \int g$

$\mu(A_n) \leq 1/n \int g$

$\mu(A) \leq \mu(A_n)\ \forall\ n \leq 1/n \int g\ \forall\ n$

Since $\int g$ is a finite number.

$\mu(A) = 0$

(ie) $\mu\{x/f(x) = \pm\infty\} = 0$

ii) $\{x: f(x) \neq 0\} = \{x : |f(x)| \neq 0\}$

Let $g = |f|$ g is a non-negative integrable function .

Let $B_n = \{x \mid g(x) \geq 1/n\}$ Then $B = \{x / g(x) \geq 0\} = \cup B_n$

And $1/n\, X_{B_n} \leq g \Rightarrow 1/n\, \mu(B_n) \leq \int g$

$\therefore \mu(B_n) \leq n \int g < g.$

$\therefore B = \cup B_n \ni : \mu(B_n) < \infty\ \forall n$

(ie) $\{x . f(x) \neq 0\}$ is the countable union of sets of finites.

Theorem : 9.3.3

Lebesgue's (Demionted) convergence Theorem.

Let g be integrable over E, and suppose that $\{f_n\}$ is a sequence of measurable functions. Such that on E $|f_n(x)| \leq g(x)$ and such that almost everywhere on E.

$f_n(x) \to f(x)$

Then $\int_E f = \lim \int_E f_n$

Proof:

i) Consider $g + f_n$

Since $f_n \leq |f_n| \leq g.$

$g + f_n \geq 0$

And $\lim (g + f_n) = g + f$ a.e

By Fatou's Lemma.

$\int g + f \leq \underline{\lim} \int g + f_n$

$\therefore \int g + \int f \leq \int g + \underline{\lim} \int f_n$

since g is integrable is finite.

So $\underline{\lim} \int f_n dx \geq \int f dx$ ------- (1)

ii) Consider $g - f_n$

$f_n \leq |f_n| \leq g.$

$g - f_n \geq 0$

And $\lim (g - f_n) = g - f$ a.e

$\therefore$ By Fatou's Lemma.

$\int g - f \leq \underline{\lim} \int g - f_n$

(ie) $\int g - \int f \leq a - \underline{\lim} (f_n)$

since $\int g < \infty$.

We get

$\int f \geq \overline{\lim} \int f_n$ ------- (2)

By (1) and (2)

$\int f \leq \underline{\lim} \int f_n \leq \overline{\lim} \int f_n \leq \int f$

$\therefore \lim \int f_n$ exists

$\therefore \int f = \lim \int f_n$

Hnece the result.

Example : (Tchebychev's inequality)

Let f be measurable function and let $A = \{ x / f(x) \geq 0 \}$

Then for $c > 0$, $\mu \{ x / f(x) > c \} \leq c^{-1} \int_A f d\mu$

Let $c > 0$ and $B = \{ x / f(x) > c \}$

Then $B \subset A$ and

$c \chi_B \leq f \chi_A$

$\therefore \int c \chi_B \leq \int f \chi_A$

(ie) $C \mu (B) \leq \int_A f$

(ie) $\mu \{ x / f(x) > C \} \leq 1/c \int_A f$

$\therefore \mu \{ x/f(x) > c \} \leq C^{-1} \int_A f d\mu$

Example : 2

Let f_n be a sequence of integrable functions such that $\sum_n |f_n| < \infty$.

Then $\sum_1^\infty f_n$ converges a.e. and its sum f is integrable and $\int f = \sum_1^\infty \int f$

Proof

Let $g = \sum |f_n|$

By Prop : Let f_n be a sequence of non-negative function Then $\int \sum_1^\infty f_n = \sum_1^\infty \int f_n$.

$\int g = \Sigma |f_n|$

since $\Sigma|f_n| < \infty$ g is integrable.

$\{x/g(x) = \infty\}$ is of measure zero.

(ie) $\{x / \Sigma f_n(x) = \infty\}$ is of measure zero.

(ie) $\Sigma|f_n|$ converges a.e.

(ie) Σf_n converges absolutely a.e

$\therefore \Sigma f_n$ converges a.e

Define $f(x) = \Sigma f_n(x)$ wherever $\Sigma f_n(x)$ converges

$= 0$ otherwise

Then $|f| \leq \Sigma |f_n| = g$

$\therefore$ f is integrable.

Also if $h_n = \sum_1^n f_k$ then

$$|h_n| \leq \sum_1^n |f_k| \leq g$$

And $\lim h_n = f$ a.e

By Dominated Convergence Theorem.

$$\int f = \lim \int h_n = \text{Lim} \sum_1^n \int f_k = \sum_1^n \int f_k.$$

CONVERGENCE THEOREM

Topic : 9.4

Definition 9.4.1

Let (x, β) be a measurable space and (μ_n) a sequence of set function defined on β.

We say μ_n converges setwise to the set function μ if for each $E \in \beta$

We have $\mu E = \lim \mu_n E$

Proposition : 9.4.1

Let (x, β) be a measurable space (μ_n) a sequence of measures that converge setwise to a measure μ and (f_n) a sequence of non negative measurable functions that converge pointwise to the function f then

$\int f d\mu \leq \underline{\lim} \int f_n d\mu_n$

Proof:

Setwise convergence of μ_n to μ

$\Rightarrow \int \phi d\mu = \lim \int \phi\, d\mu_n$

for any simple function ϕ

From the definition of $\int f d\mu$ it suffices to prove that $\int \phi d\mu \leq \underline{\lim} \int f_n f d\mu_n$ for any simple function $\phi \leq f$.

Case (i)

Suppose $\int \phi d\mu < \infty$

Then ϕ vanishes outside a set E of finite measure.

Let $\in > 0$

$E_n = \{x : f_k(x) \geq (1-\in)\phi(x) \ \forall \ k \geq n\}$

Then (E_n) is an increasing sequence of sets whose union contains E.

And so $(E \sim E_n)$ is decreasing sequence of measurable sets whose intersection is empty.

Thus there is an m such that

$\mu(E \sim E_n) < \in$

Since $\mu(E \sim E_n) = \lim \mu_k (E - E_n)$

we may choose $n \geq m$

So that $\mu_k (E \sim E_m) < E_k$ for $k \geq n$

Since $E \sim E_k \subset E \sim E_m$

We have $\mu_k (E \sim E_k) < E$ for $k \geq n$

Thus

$$\int_{E_k} f_k d\mu \geq \int_{E_k} f_k d\mu_k \geq (1-E) \int_{E_k} \phi\, d\mu_k$$

$$\geq (1-\in) \int_E \phi\, d\mu_k - \int_{E\sim E_x} \phi\, d\mu_k$$

$$\geq (1-\in) \int_E \phi d\mu_k - M\in$$

where M is the maximum of ϕ.

Thus

$$\underline{\lim} \int f_k \, d\mu_k \geq \int_E \phi d\mu - \epsilon \quad [M + \int \phi d\mu]$$

Since ϵ was arbitrary

we get

$$\int_E \phi \, d\mu \leq \underline{\lim} \int_E f_k d\mu_k$$

Hence $\int f d\mu \leq \lim \int f_k d\, k\mu$

Similarly $\int \phi d\mu = \infty$

$$\int f d\mu \leq \lim \int f_k \, d\, \mu$$

Hence the result.

Proposition 9.4.2

Let (x, β) be a measurable space and (μ_n) a sequence of measures on β that converge setwise to a measure μ. Let (f_n) and (g_n) be two sequences of measurable functions that converge pointwise to f and g. Suppose that $|f_n| \leq g_n$ and that

$$\lim \int g_n \, d\mu_n = \int g d\mu < \infty$$

Then $\lim \int f_n d\mu_n = \int f d\, \mu$.

Proof:

i) Consider $g_n + f_n$

since $|f_n| \leq g_n \quad g_n + f_n \geq 0$

$\therefore \lim (g_n + f_n) = g + f$

By the proposition.

Let (x, β) be a measurable space (μ_n) a sequence of measure that converge setwise to a measure μ and (f_n) a sequence of non-negative measurable function that converge pointwise to be function f. Then $\int f d\mu \leq \underline{\lim} \int f_n d\mu_n$ ----- (1)

$$\int (g + f) \, d\mu \leq \underline{\lim} \, (g_n + f_n) \, d\mu_n$$

$$\int g d\mu + \int f d\mu \leq \underline{\lim} \int g_n \, d\mu_n + \underline{\lim} \int f_n \, d\, \mu_n$$

$$\leq \int g d\, \mu + \underline{\lim} \int f_n \, d\mu_n$$

Since $\int g d\mu < \infty$

we get $\int f d\mu \leq \lim \int f_n \, d\, \mu_n$ ---- (2)

ii) consider $g_n - f_n$

since $|f_n| \le g_n$ $\quad g_n - f_n \ge 0$

$\lim (g_n - f_n) = g - f$

By (1)

$\int (g-f)d\mu \quad \le \underline{\lim} \left(\int g_n d\mu - \int f_n \, d\mu \right)$

$\int g d\mu - \int f d\mu \le \underline{\lim} \int g_n d\mu - \overline{\lim} \int f_n d\mu_n$

$\le \int g d\mu - \overline{\lim} \int f_n d\mu_n.$

Since $\int g d\mu < \infty$ $\quad \int f d\mu \ge \lim f_n \, d\, \mu_n$ ----------- (2)

From (2) and (3)

We get $\quad \int f d\mu = \lim \int f_n \, d\, \mu_n$

Example :

Let (x, β) be a measurable space and (μ_n) a sequence of measures on β. Since that for each $E \in \beta$, $\mu_{n+1}(E) \ge \mu_n(E)$. Let $\mu E = \lim \mu_n E$. Then μ is a measure on β.

Proof:

Let $\mu(E) = \lim \mu_o (E) \ \forall E \in \beta$

clearly $\mu(E) \ge 0, \ \forall E \in \beta$ and $\mu(\phi) = 0$

Also $\mu_n(E) \le \mu(E) \ \forall E \in \beta$

(Since $\mu_n(E) \le \mu_{n+1}(E) \ \forall_n$

Let E_i be a disjoined sequence of measurable sets such that $E = \bigcup_1^{\infty} E_i$

$$\mu_n(E) = \sum_1^{\infty} \mu_n (E_i) \le \sum_1^{\infty} \mu \ (E_i)$$

$$\therefore \mu(E) = \lim \mu_n (E) \le \Sigma \mu(E_i) \text{ ----- (1)}$$

Conversely

$$\sum_1^{K} \mu_n(E_i) \le \mu_n(E) \le \mu(E) \ \forall K$$

$$\text{Lim} \sum_1^{K} \mu_n (E_i) \le \mu (E)$$

(ie) $\sum_{1}^{K} \mu_o(E_i) \leq \mu(E)$

$\sum_{1}^{\infty} \mu(E_i) \leq \mu(E)$ ------ (2)

$\therefore \mu(E) = \sum_{1}^{\infty} \mu_n(E)$

Example - 2:

Given a.. example of a decreasing sequence (μ_n) of measures on a measurable space such that the set function μ defined by $\mu(E) = \lim \mu_n(E)$ is not a measure.

Solution :-

Let $x = \{1,2,3 \ldots.\}$ and $\beta = P(x)$

Let $\sum_{1}^{\infty} P_n$ be a convergent series of number P satisfying $0 \leq P_i \leq 1$

Define $P_{n_i} = p_i$ for $i \leq n$
$= 1$ for $i > n$

(ie) $P_{11} = P_1, P_1K = 1$ for $K>1$, $P_{21} = P_1, P_{22}=P_2\ P_{2k}= 1$ for $K>2$

$P_{31} = P_1\ P_{32} = P_2, P_{33} = P_3\ P_{3k} = 1$ for $K>3$

Define $\mu_n(E) \sum_{i \in E} P_{n_i}$

We can prove that $\{\mu_n\}$ are measure

let $\mu(E) \lim \mu_n(E)$

$\mu_n(i) = P_i\ \forall\ i \leq n$ and $\mu_n(i) = 1$ for $i > n$

$\mu(i) = P_i$

$\therefore \Sigma\mu(i) = \Sigma p_i < \infty$

But $\mu_n(x) = \infty\ (\mu_n(x) = P_1 + P_2 + P_{(n+1)} + \ldots..$

$$\mu(x) = \lim_{n} \mu_n(x) = \sum_{i=1}^{\infty} \mu(i)$$

$\therefore \mu$ is not a measure.

SIGNED MEASURES

Topic 9.5

Definition : 9.5.1

Let (x, β) be a measurable space. By a signed measures on (x, β) we mean extended real-valued set function V defined for the sets of B and satisfying the following conditions:

i) V assumes at most one of the values $+\infty$, $-\infty$

ii) $V(\phi) = 0$

iii) $V\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} VE_i$ for any sequence E_i

of disjoint measurable sets.

The equality taken to mean that the servies on the right converges absolutely if $V(\cup E_i)$ in finite and that it properly diverges otherwise.

Example :1

Show that if $\phi(E) = \int f d\mu$ where $\int f d\mu$ isdefined then ϕ is a signed measure.

Solution :

We have $\int f^+ d\mu < \infty$ or $\int f^- d\mu < \infty$

By the Definition signed Measurer.

So (i) is true.

(ii) is trival.

Let $\{E_i\}$ be a sequence of disjointsets of s and for $E \in S$.

$\varnothing^+(E) = \int_E f^+ d\mu$, $\varnothing^-(E) = \int_E f^- d\mu$

By the theorem

Let (X,β, μ) be a measure space and f a non-negative measurable function. Then $\varnothing(E) = \int f d\mu$ is a measure on the measureable space (x^E, δ). If in addition $\int f d\mu < \infty$ then $\forall \in > 0 \ni \delta > 0 \to$ If $A \in \beta$ and $\mu(A)$ Then $\varnothing(A) < \in$.

$\therefore \varnothing^+$ and $\varnothing^-$ are measures.

Then

$$\varnothing\left(\bigcup_{i=1}^{\infty} E_i\right) = \varnothing^+\left(\bigcup_{i=1}^{\infty} E_i\right) - \varnothing^-\left(\bigcup_{i=1} E_i\right)$$

$$= \sum_{i=1}^{\infty} \varnothing^{+} (E_i) - \sum_{i=1}^{\infty} \varnothing^{-} (E_i)$$

$$\varnothing (\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \varnothing (E_i)$$

as we cannot get $\infty - \infty$ at any stage.

Definition : 9.5.2

i) A is a positive set w.r. to the signed measure ν on (X, β) if $A \in B$ and $V(E) \geq 0$ for each measurable subset E of A. We will omit 'with respect to v' if the signed measure is obvious from the context.

Clearly $\varnothing$ is a positive set with respect to every signed measure. Also $VA \geq 0$ is necessary but not in general sufficient for A to be a positive set w.r. to v.

ii) A is negative set w.r. to v if its a positive set w.r. to - ν.

Example:

If A is a positive set w.r. to ν and if for $E \in B$, $\mu (E) = \nu (E \cap A)$. Then μ is a measure.

Dfinition 9.5.3.

A is a null set w.r. to ν or a ν - null set if it is both a positive and negative set w.r. to ν. Equivalently, A is a ν - null set if $A \in \beta$ and $V(E) = 0 \ \forall \ E \in \beta, E \subseteq A$

Example :

If A is a positive set w.r. to ν, then every measurable subset of A is a positive set.

Proposition : 9.5.1

Suppose E, F are measurable sets and μ is a signed measure such that $E \subset F$ and $/ \mu (F) / < \infty$

Proof :

Let $E = E \cup (F - E)$ - disjoint union.

$\therefore \mu (F) = \mu (E) + \mu (F-E)$

Since μ assumes atmost one of the values $\pm \infty$ if one or both of μ (E) and μ (F-E) are infinite, μ (F) will be infinte.

$\therefore$ Since $| \mu (F) | < \infty$, both $| (\mu (E) |$ and $| \mu (F-E)$ must be finite.

$\therefore | \mu (E) \backslash < \alpha$

Proposition : 9.5.2

If μ is a signed measure E_n, a monotone sequence of measurable sets and if in case E_n is a decreasing sequence $/ \mu(E_n) / < \infty$ for atleast one n then $\mu(\lim E_n) = \lim \mu(E_n)$

Proof :

i) Let E_n be increasing sequence.

Then let $F_1 = E_1, F_0 = \varnothing$

$$F_2 = E_2 - E_1$$

$$F_n = E_n - E_{n-1}$$

Then F_n are disjoint sets and also

$$\bigcup_1^\infty E_n = \bigcup_1^\infty F_n$$

$$\therefore \mu\left(\bigcup_1^\infty E_n\right) = \mu\left(\bigcup_1^\infty F_n\right) = \sum_1^\infty \mu(F_n) = \text{Lim} \sum_1^n \mu(F_n)$$

$$= \text{Lim}\, \mu\left(\bigcup_1^\infty F_n\right) = \text{Lim}\, \mu(E_n) \quad \left(\therefore E_n = \bigcup_1^\infty F_n\right)$$

ii) E_n be a decreasing sequence

And let $\mu(E_n) < \infty$ for some n.

Without loss generality,

We assume $|\mu(E_1)| < \infty$.

Hence By Prop. 9.5.1

$\mu(E_n) < \infty \; \forall_n$

consider $F_i' = E_1 - E_i$

Then F_i and increasing sequence and so

$$\mu(\lim_n F_n) = \lim \mu(F_n)$$

$$\mu(\lim_n F_n) = \lim [\mu(E_1 - E_n]$$

$$= \lim [\mu(E_1) - \mu(E_n)]$$

(Since $\mu(E_0) < \infty \; \forall n$)

$$\mu(\lim_n F_n) = \mu(E_1) - \lim \mu(E_n)$$

(i.e) $\mu(\cup F_n) = \mu(E_1) - \lim \mu(E_n)$

(i.e) $\mu(\cup E_1 - E_n) = \mu(E_1) - \lim \mu(E_n)$

(i.e) $\mu(E_1 - \cup E_n) = \mu(E_1) - \lim \mu(E_n)$

(i.e) $\mu(E_1) - \mu(\cup E_n) = \mu(E_1) - \lim \mu(E_n)$

$\therefore \mu(E_n) = \lim \mu(E_n)$ [Since $|\mu(E_1)| < \infty$]

Hence the result.

Theorem : 9.5.1

A Countable Union of sets positive w.r. to a signed measure V is a positive set.

Proof :

Let $\{A_n\}$ be a sequence of positive sets.

Then, as in Theorem.

Let $\{A_i\}$ be a sequence in a ring R, then there is a sequence $\{B_i\}$ of disjoint sets of R such that

$B_i \subseteq A_i$ for each i and $\bigcup_{i=1}^{N} A_i = \bigcup_{i=1}^{N} B_i$ for each N, so that $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$

We have $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$ where the sets $B_n \in \beta$

$$B_n \subseteq A_n$$

And $B_n \cap B_m = \phi$ if $n \neq m$.

Now let $E \subseteq \bigcup_{n=1}^{\infty} A_n$

Then $E = \bigcup_{n=1}^{\infty} (E \cap B_n)$

So $V(E) = \sum_{n=1}^{\infty} V(E \cap B_n) \geq 0$ as $E \cap B_n$ is a positive set for each n.

So $\bigcup_{n=1}^{\infty} A_n$ is a positive set.

Example :

Let β be a set of Lebesgue measurable sets of [0,1].

Define $f : [0,1] \to R$ as

$$f(x) = 1 \text{ if } 0 \le x \le 1/4$$
$$= 0 \text{ if } 1/4 < x < 3/4$$
$$= -1 \text{ if } 3/4 \le x \le 1$$

And define

$$V(E) = \int_E f d\mu \ \forall E \in \beta.$$

Proposition : 9.5.3

Let υ be a signed measure. Let E be a measurable set such that $0 < V(E) < \alpha$. Then there is a positive set $A \subset E$ with $V(A) > 0$.

Proof :

i) Suppose E is a positive set. Then E is the required set.

ii) Suppose E is not a positive set. Then it contains set of negative measure.

Let n_1 be the smallest positive integer. Such that there is a measurable set $E_1 \subset E$ with $V(E_1) < -1/n_1$

Consider $E - E_1$

If $(E - E_1)$ contains no sets of negative measure then $(E - E_1)$ in required positive set.

For this we have only to show that

$V(E - E_1) > 0$

Suppose $V(E - E_1) = 0$

$\therefore V(E) - V(E_1) = 0$

$\therefore V(E) - V(E_1) < -1/n_1$

$\therefore V(E) < 0$ – a contradition

$\therefore V(E - E_1) > 0$

Suppose $E - E_1$ contains sets of negative measure. Let n_2 be the smallest positive integer such that

$E_2 \subset E - E_1$ and $V(E_2) < -1/n_2$

By the definition of n_1 $n_2 \leq n_2$.

Consider $E - (E_1 \cup E_2)$

If this is not a positive set proceed as above. Thus proceeding inductively.

Let n_k be the smallest positive integer such that

$$E_k \subset E \sim \bigcup_{i=1}^{k-1} E_i$$

and $V(E_k) < -1/n_k$

Also by the choice of n_i

$n_1 \leq n_2 \leq n_3 \leq \ldots\ldots\ldots \leq n_k$

If $(E - \bigcup_1^K E_i)$

Contains no sets of negative measure at any stage then it in the required positive set,

$$\text{Since } V(E - \bigcup_1^K E_i) = V(E) - \sum_1^K (V(E_i) = 0$$

$$\Rightarrow V(E_i) < 0.$$

(Since E_i are disjoint sets)

If the process does not stop at any finite stage.

$$\text{Since } A \subset B - \bigcup_1^\infty E_i \subset (E - \bigcup_1^K E_i)$$

If $B \subset A$.

(i.e) A can contain no measurable sets with measure less than $-(n_k-1)^{-1} \geq -\epsilon$

Thus A Contains no measurable sets of measure less than $-\epsilon$

Since ϵ in arbirary positive number.

$\therefore$ A Can contain no sets of negative measure and so must be a positive set.

Theorem: 9.5.2

(Hahn Decomposition Theorem)

Let υ be a signed measure on the measurable space (x, β). Then there is a positive set A and a negative set B such that $X = A \cup B$ and $A \cap B = \phi$.

Proof :

Without loss of generality.

We may assume V does not take he value of $+\infty$.

Let $\lambda = \text{Sup}\,\{VA \,/\, A \text{ - Positive set w.r. to } \upsilon\}$

Since the empty set ϕ is (VA) is non - empty.

Also $\lambda \geq 0$.

We get $E_k \subset E \sim \bigcup_{j=1}^{k-1} Ej$

$$V(E_k) < -1 / n_k$$

Set $A = E \sim (\bigcup_1^{\infty} E_k)$

Then $E = A \cup (\bigcup_1^{\infty} E_k)$

$\therefore V(E) = V(A) + \sum_1^{\infty} V E_k$

(Since E_k are disjoint)

Since VE is finite.

$A \subset E \Rightarrow |V(A)| < \infty \;\&\; \sum_1^{\infty} V(E_k)$

Converges absolutely.

$\therefore \Sigma\, 1/n_k$ converges and $n_k \to \infty$

Also $V(E) > 0$ and $V(E_k) \leq 0 \ \forall K$

$\Rightarrow V(A) > 0$

To show that A is a Positive set.

Let $\in > 0$ be given

Since $n_k \to \infty$

We may choose k. $\Rightarrow \therefore \dfrac{1}{(n_{k-1})} < \in$

There exists (A_i) is a sequence of Positive sets

Such that $\lambda = \lim\limits_{i \to \infty} V A_i$

and set $A = \bigcup\limits_{i=1}^{\infty} A_i$

By the Proposition.

Every measurable subset of a positive set is itself positive. The union of a countable collection of positive sets is positive.

A is a Positive set.

And so $\lambda \geq VA$.

But $A \sim A_i \subset A$ and so $V(A \sim A_i) \geq 0$

Thus $VA = VA_i + V(A \sim A_i) \geq A_i$

$\therefore VA \geq \lim V A \geq \lambda$

$\therefore VA = \lambda$ and $\lambda < \infty$

Let $\beta = \sim A$

Suppose E is a positive subset of B. Then E and A are disjoint and $E \cup A$ is a positive set.

$\therefore \lambda \geq V(E \cup A) = VE + VA = VE + \lambda$

$\therefore V(E) = 0$ since $0 \leq \lambda < \infty$

$\therefore$ B Contains no Positive subset of Positive measure.

By the theorem.

Let E be a measurable set such that $0 < VE < \infty$. Then there is a Positive set A contained in E with $VA > 0$.

$\therefore$ B is a negative set.

Note :

A decomposition of x into two disjoint sets A and B such that A is Positive for v and B negative is called a Hahn decomposition for v and negative sets need not be unique.

Consider $V(E) = \int_E fd\,\mu$ where $f = 1$ on $[0, \frac{1}{4}]$

$= 0$ on $[\frac{1}{4}, \frac{3}{4}]$

$= -1$ on $[\frac{3}{4}, 1]$

For every $K \in (\frac{1}{4}, 2/4)$, $(0,1) = (0,1) = (0,k) \cup (k,1)$ is a Hahn Decomposition.

If $\{A, B\}$ is a Hahn decomposition for v, then we may define two measures v^+ and v^- with

$V = V^+ - V^-$ by setting

$V^+(E) = V(E \cap A)$

and $V^-(E) = -V(E \cap B)$

Two measures v_1 and v_2 on (X, β) are said to be mutually singular if there are disjoint measurable sets A and B with $x = A \cup B$ such that $V_1(A) = V_2(B) = 0$.

Thus the measures V^+ and V^- defined are above are mutually singular.

Proposition: 9.5.4

Let v be a signed measure on (X, β). Suppose of $X = A_1 \cup B_1 = A_2 \cup B_2$ are two Hahn decomposition of X. Then $A_1 \Delta A_2$ and $B_1 \Delta B_2$ are null sets.

Proof :

Let $X = A_1 \cup B_1 = A_2 \cup B_2$

Where A_1, A_2 are positive sets and B_1, B_2 are negative sets.

$A_1 \Delta A_2 = (A_1 \sim A_2) \cup (A_2 \sim A_1)$

Now $A_1 \sim A_2 = A_1 \cap A_2 = A_1 \cap B_2$

And $A_1 \cap B_2 \subset A_1 \Rightarrow V(A_1 \cap B_2)$ is a Positive Set.

$A_1 \cap B_2 \subset B_2 \Rightarrow V(A_1 \cap B_2)$ is a negative Set.

$\therefore (A_1 \cap B_2)$ is a null set.

(i.e) $A_1 \sim A_2$ is a null set.

Similarly $A_2 \sim A_1$ is a null set.

$\therefore (A_1 \Delta A_2) = (A_1 - A_2) \cap (A_2 \sim A_1)$ is a null set.

Similarly, we can show that $B_1 \Delta B_2$ is a null set.

Proposition : 9.5.6

Let $x = A_1 \cup B_1 = A_2 \cup B_2$ be two Hahn decomposition of V.

Then $V(E \cap A_i) = V(E \cap A_2)$ and

$V(E \cap B_1) = V(E \cap B_2)$ for $E \in \beta$.

Proof :

Let $E \in \beta$. $E \cap (A_1 \sim A_2) = E \cap (A_1 \cap B_2) \subset A_1$

$\therefore V \; (E \cap (A_1 \sim A_2) \geq 0$

And $E \cap (A_1 \sim A_2) \subset B_2$

$\therefore V \; (E \cap (A_1 \sim A_2) \leq 0$

$\therefore V \; (E \cap (A_1 \sim A_2) = 0$

Similarly $V(E \cap (A_2 \sim A_1) = 0$

Now, $V(E \cap (A_1 \cup A_2) = V(E \cap (A_1 \cup A_2 \sim A_1))$

$= V(E \cap A_1) \cup (E \cap (A_2 \sim A_1)$ disjoint union

$= V(E \cap A_1) + V(E \cap (A_2 \sim A_1)$

$\therefore V(E \cap (A_1 \cup A_2) = V(E \cap A_1)$

Also $V(E \cap (A_1 \cup A_2)) = V(E \cap (A_1 \cup (A_2 \sim A_1)$

$= V(E \cap A_2)$

$\therefore V(E \cap A_1) = V(E \cap A_2)$

Similarly,

$V(E \cap (B_1 \cup B_2) = V(E \cap B_1) = -V(E \cap B_2)$ for $E \in \beta$.

Proposition : (Jordan decomposition of v) 9.5.7

Let V be a signed measure on the measurable space (X, β). Then there are two mutually singular measures V^+ and V^- on (X, β) such that $V = V^+ - V^-$. Moreover, there is only one such pair of mutually singular measures.

Proof :

Let A, B be a Hahn decomposition of X with respect to v and define V^+ and V^- by

$V^+(E) = V(E \cap A)$. $V^-(E) = -V(E \cap B)$ ....(1)

for $E \in \beta$.

Then V^+ and V^- are measures.

If A is a Positive set with respect to v and if for $E \in \beta$, $\mu(E) = V(E \cap A)$. Then μ is a measure.

And $V^+(B) = V^-(A) = 0$

So $V^+ \perp V^-$

Also for $E \in \beta$.

$V.(E) = V(E \cap A) + V(E \cap B) = V^+(E) - V^-(E)$

So $V = V^+ - V^-$ and the proof will be complete when we show that the decomposition is unique.

Let $V = V_1 - v_2$ be any decomposition of v into mutually singular measures.

Then we have $X = A \cup B$ where there exist two disjoint set A, B.

$V_1(B) = 0 = V_2(A)$

Let $D \subseteq A$.

Then $V(D) = V_1(D) - V_2(D) = V_1(D) \geq 0$.

$\therefore$ A is a Positive set W.r. to v.

Similarly B is a negative set.

For each $E \in \beta$.

We have $V_1(E) = V_1(E \cap (A \cup B))$

$= V_1[(E \cap A) \cup (E \cap B)]$

$= V_1(E \cap A) + V_1(E \cap B)$

$= V_1(E \cap A)$

[Since $V_1(B) = 0 \Rightarrow V_1(E \cap B) = 0$

$= V_1(E \cap A) - V_2(E \cap A)$

[$\because V_2(A) = 0 \Rightarrow V_2(E \cap B) = 0$]

$= (V_1 - V_2)(E \cap A)$

$V_1 - (E) = -V(E \cap B)$

Similarly $V_2(E) = -\overline{V}(E \cap B)$

So every such decomposition of v is obtained from a Hahn decomposition of x as in ....(1)

So it is enough to show that if A, B and A^1, B^1 are two Hahn decomposition then the measures obtained as in (1) are the same.

We have

$$v(A \cup A^1) = v(A \cap A^1) + v(A \Delta A^1) = v(A \cap A^1)$$

By the Hahn Decomposition Theorem.

For each $E \in \beta$ as $(A \cup A^1)$ is a Positive set,

we have

$$v(E \cap A \cap A^1) \le v(E \cap A) \le v(E \cap (A \cup A^1))$$

and $v(E \cap A \cap A^1) \le v(E \cap A^1) \le v(E \cap (A \cup A^1))$

But the first and last terms in each of these inequalities are the same.

So $V(E \cap A) = V(E \cap A^1)$

And V^+ defined in (1) is unique.

But then $v^- = v^+ - v$ is also unique.

Hence the theorem.

Note :

The Hahn decomposition in of the set x and is not unique where as the Jordon decomposition is of the signed measure and is unique.

Example :

Let (X, β, μ) be a measure space and let f be an integrable function w.r. to this space. Then $V(E) = \int_E f \, d\mu$ for $E \in \beta$.

Find a Hahn decomposition w.r. to v and the Jordon decomposition of v.

Solution :

V is signed measure.

Let $A = \{x : f(x) \ge 0\}$ and $B = \{x : f(x) < 0\}$

Then A, B from a Hahn decomposition while V^+, V^- given by

$V^+(E) = \int_E f^+ \, d\mu$, $V^-(E) = \int_E f^- \, d\mu$ form the Jordan decomposition.

Example : 1

Show that signed measure v is finite or $\sigma-$ finite respectively iff $|v|$ is or iff both V^+ and V^-

Proof :

i) Suppose V is finite.

(ie) $|V(E)| < \infty$

(ie) $|V^+(E) - V^-(E)| < \infty$

Since both V^+ and V^- cannot be infinite.

We have $V^+(E) < \infty$

and $V^-(E) < \infty$ and so $|V|(E) < \infty$

Convesely, if $|V|(E) < \alpha$ Then

$|V(E)| \leq |V|(E) \Rightarrow |V(E)| < \infty$

Also $V^+(E)$ and $V^-(E)$ are finite.

ii) σ– finite $\Rightarrow |V|$ σ– finite and converse of proof (exercise).

Example : 2

If V_1 and V_2 are two finite signed measures show that $|V_1 + V_2| \leq |V_1| + |V_2|$

Proof :

Since V_1 and V_2 are signed measures.

We have $V_1 = V_1^+ - V_1^-$ and $V_2 = V_2^+ - V_2$

$\therefore V_1 + V_2 = (V_1^+ + V_2^+) - (V_1^- + V_2^-)$ is one decomposition.

$|V_1 + V_2| \leq |V_1^+ + V_2^+| + |V_1^- + V_2^-|$

$\leq V_1^+ + V_2^+ + V_1^- + V_2^-$

$|V_1 + V_2| \leq |V_1| + |V_2|$

Example : 3

Define integration w.r. to a signed measure V as follows $\int f dV = \int f dv^+ - \int f dv$.

If $|f| \leq M$ show that

$\int_E f dv = |v|(E)$.

Proof :

i) If $|f| \leq M$

$$\left|\int f\,d v\right| = \left|\int_E f\,d v^+\right| - \left|\int_E f\,d v^-\right|$$

$$\leq \left| \int_E f\, d\, v^+ \right| - \left| \int_E f\, d\, v^- \right|$$

$$\leq MV^+ (E) + MV^- (E)$$

$$\leq M\, |V|\, (E).$$

ii) Let (A, B) be a Han decomposition of V.

Define f as

$$f(x) = 1 \quad \text{if } x \in A.$$

$$= 1 \quad \text{if } x \in B.$$

$$\int f\, d\, v = \int_{E \cap A} f\, d\, v + \int_{E \cap B} f\, d\, v$$

$$= \int_{E \cap A} dv + \int_{E \cap B} (-1)\, d\, v$$

$$= V(E \cap A) - V(E \cap B)$$

$$= V^+ (E) + V^- (E)$$

$$= |V|\, (E)$$

Hence the result.

Topic : 9.6

RADON - NIKODYM THEOREM

Definition : 9.6.1

i) If μ, ν are signed measures on $[X, \beta]$ and $V(E) = 0$ whenever $\mu(E) = 0$. Then we say that V is absolutely continous with respect to μ and write $V << \mu$.

ii) If μ, ν are signed measures on $[X, \beta]$ and $V(E) = 0$ whenever $|\mu(E)| = 0$ then ν is absolutely continous with respect to μ, $\nu << \mu$.

Example : 1

Show that the following conditions on the signed measures μ and ν on $[X, \beta]$ are eqivalent i) $V << \mu$, ii) $|V| << |\mu|$ iii) $V^+ << \mu$ and $V^- << \mu$.

Solution :

From Definition absolutely continuous

We see that $V << \mu$ iff $V << |\mu|$

So we may assume that $\mu \geq 0$

As $|V| = V^+ + V^-$

We see that $|V| << \mu \Rightarrow V^+ << \mu$.

And $V^- << \mu$.

So $\nu << \mu$.

Suppose that $\nu = \nu^+ - \nu^-$ with a Hahn decomposition A, B. Then if $V << \mu$ and $\mu(E) = 0$.

We have $\mu(E \cap A) = 0$

So $V^+(E) = 0$

Similarly $V^-(E) = 0$

So $|V|(E) = 0$.

Definition: 9.6.2

If μ, ν are two signed Measures on (X, β) such that $\mu << V$ and $V << \mu$. Then μ and ν are said to be equivalent. We denote by $\mu \equiv \nu$.

Note :

i) Whenever we are dealing with more than one measure on a measurable space (X, β). The term almost everywhere becomes ambiguous, and we must specify almost everywhere with respect to μ or almost everywhere with respect to ν etc. These are usually abbreciated a.e $[\mu]$ and a.e $[\nu]$.

ii) If $V << \mu$ and a property holds a.e $[\mu]$. Then it holds a.e $[v]$.

Theorem: 9.6.1 (The Radon Nikodym Theorem)

Let (X, β, μ) be a σ-finite measure space and let υ be a measure space and let υ be a measure defined on β which is absolutely Continuous with respect to μ. Then there is a non-negative measurable function f such that for each set E in β. We have

$$VE = \int_E f \, d\mu.$$

The function f is unique in the sense that if g is any measurable function with this property. Then $g = f$ a.e $[\mu]$

Proof :

Suppose that the result has been proved for finite measures.

We have $X = \bigcup_{n=1}^{\infty} A_n$

$\mu(A_n) < \alpha$ and $X = \bigcup_{m=1}^{\infty} \beta_m$

$V(B_m) < \alpha$ and $\{A_n\}$, $\{B_m\}$ may be supposed to be sequences of disjoint sets.

So set $X = \bigcup_{n,m=1}^{\infty} (A_n \cap \beta_n)$

We obtain X as the union of disjoint sets on which both μ and υ are finite.

$$X = \bigcup_{n=1}^{\alpha} x_n$$

Let $\beta_n = [E_n n X_n, E \in \beta]$ σ-algebra over X_n.

And Considering μ and υ restricted to β_n.

(a σ-algebra over X_n)

We obtain

Since μ and υ are finite, a non-negative function f_n such that if $E \in \beta_n$.

$$v(E) = \int_E f_n \, d^{\mu}.$$

So if $A \in \beta$.

$$A = \bigcup_{n=1}^{\infty} A_n \text{ where } A_n \in \beta_n.$$

Then define $f = f_n$ on X_n gives a measurable function on X.

$$\text{And } V(A) = \sum_{n=1}^{\infty} \int_{A_n} f_n \, d\mu = \int_A f \, d\mu.$$

So the general case.

We need to prove that for finite measures such a function f exists. Let k be the class of non-negative functions measurable with respect to μ and satisfying

$$\int_E f \, d\mu \le v(E) \quad \forall E \in \beta.$$

But then by the inductive hypothesis.

$$\text{We have } F_n = \bigcup_{i=1}^{n} B_i$$

And $g_{n+1}^{(x)} = f_i(x)$ for $X \in B_i$, $i = 1 \ldots\ldots\ldots\ldots n+1$.

Now, Since each $f_i \in K$.

$$\int_B g_n \, d\mu = \sum_{i=1}^{n} \int_{B_i} f_i \, d\mu \le \sum_{i=1}^{n} V(B_i) = V(B) \ldots\ldots\ldots\ldots (1)$$

Also, we have $g_n \uparrow$

So $f_o = \lim g_n$

Then (1) and the Lebesgue Monotone Convergence Theorem

$$\Rightarrow \int_E f_o \, d\mu = \lim \int_E g_n \, d\mu \le V(E)$$

So $f_o \in K$.

Hence $\alpha \ge \int f_o \, d\mu \ge \int g_n \, d\mu \ge \int f_n \, d\mu$

So $\alpha = \int f_o \, d\mu$

Since $\int f_o \, d\mu \leq V(x) < \alpha$

There exists a finite - valued measurable function f, also non-negative, such that $f = f_o$ a.e (μ)

Then K is non-empty as $O \in K$.

Let $\alpha = \text{Sup} [\int f \, d^{\mu} : f \in K]$.

And let $\{f_n\}$ be a sequence in K such that $\text{Lim} \int f_n \, d^{\mu} = \alpha$

If B is any fixed measurable set, n a fixed positive integer.

And $g_m = \max \{f_1 \ldots\ldots\ldots\ldots f_m\}$

Then we can prove by induction that B in the Union of disjoint measurable sets B_i, $i = 1, 2 \ldots\ldots n$ Such that $g_n = f_i$ on B_i, $i = 1, 2 \ldots\ldots n$

For, Let $n = 2$ and Let $B_1 = [x : x \in B, \ f_1(x) \geq f_2(x)]$

$B_2 = B - B_1$

Then $B = B_1 \cup B_2$ has the desired property.

Supposing decomposition possible for n,

Let $g_{n+1} = \text{Max} (f_1 \ldots\ldots f_{g+1}) = \max (g_n, f_1$

So $B = F_n \cup B_{n+1}$ where $g_{n+1} = f_{n+1}$ on B_{n+1}

$g_{n+1} = g_n$ on F_n

And $F_n \cap B_{n+1} = \phi$.

We will show that if $V_0(E) = V(E) - \int_E f d\mu$

Then $V_0(E) = 0$ for each $E \in \beta$.

By the construction of f, V_o is non-negative.

If V_0 is not identically zero no β.

Let $C \in \beta$ and $V_0(C) > 0$.

Then for a suitable $\in$, $0 < \in < 1$.

$(V_0 - \in^{\mu})(C) > 0$

But then By Theorem.

Let V be a signed measure on (X, β) Let $E \in \beta$ and $V(E) > 0$. Then there exists A, a set positive w.r. to v such that $A \subseteq E$ and $V(A) > 0$.

We can find A such that $(V_0 - \epsilon\mu)$ (A) > 0 Where A is a positive set w.r. to $v_0 - \epsilon\mu$.

Also μ (A) > 0 for otherwise as $V << \mu$.

We have V (A) = 0.

Hence $(V_0 - \epsilon\mu)$ (A) = 0

So for $E \in B$

$$\epsilon\mu\ (E \cap A) \leq V_0\ (E \cap A) = V\ (E \cap A) - \int_{E \cap A} f\, d\mu$$

Hence if $g = f + \epsilon_{\chi_A}$ for each $\epsilon \in \beta$.

We have

$$\int_E g\, d\mu = \int_E f\, d\mu + \epsilon\mu\ (E \cap A) \leq \int_{E-A} f\, d\mu + V\ (E \cap A) \leq V\ (E)$$

And so $g \in K$.

But $\int g\, d\mu = \int f\, d\mu + \epsilon\mu\ (A) > \alpha$.

Contradicting the maximality of a.

So $V_0 = 0$ on β.

(ie) $\int_E f\, d\mu = V\ (E)$

So f has the desired properties.

Let g also have these properties. Then for $E \in \beta$.

$$\int_E (f - g)\, d\mu = 0$$

And taking $E = [x: f(x) > g(x)]$

We get $f \leq g$ a.e

Similarly $f \geq g$ a.e

$\therefore f = g$ a.e

Corollary : 1

Suppose v is σ-finite. Then f is finite valued a.e.

Proof :

Suppose v is σ-finite and $X = \cup B_n$.

$V(B_n) < \infty$ and B_n disjoint. Then

$X = \cup A_m = \cup B_n = \cup (A_m \cap B_n)$

$\mu (A_m \cap B_n) < \infty$ and $V (A_m \cap B_n) < \infty$.

(Since μ in $\sigma-$finite with $\mu (A_m) < \infty : A_m$ disjoint).

Let $C = A_m \cap B_n$

Then both μ and v are finite

So if we proced as above.

$\mu(E_\infty) > 0 \Rightarrow V (\mu_\infty) = \infty$, show that $\mu(E_\infty) = 0$ and so $V (E_\infty) = 0$

So f is finite except on E_∞.

(ie) f is finite a.e (μ) and also a.e (v) on C.

Complete the proof for X.

Corollary : Theorem 9.6.1 can be extended to the case when V is a $\sigma-$finite signed measure.

Proof :

The Jordon decomposition of v given $V = V^+ - V^-$

Then $V^+ (E) = \int_E f_1 \, d\mu$., $V=(E) = \int_E f_2 \, d\mu$

where f_1, f_2 are finite valued (a.e μ) non-negative measurable functions.

$\therefore V(E) = V^+ (E) - V=(E)$

$V(E) = \int_E f_1 \, d\mu - \int_E f_2 \, d\mu$

$= \int_E f d\mu$ where $f = f_1 - f_2$

Since V is a signed measure, either V^+ or V^- is finite.

Hence $\int_E (f_1 - f_2) d\mu$ is well defined.

Hence the result.

Corollary : 3

This theorem can be further extended to the case where μ is a $\sigma-$finite signed measure and v, $\sigma-$finite signed measure.

Proof :

Let $\mu = \mu^+ - \mu^-$ be its Jordon decomposition with (A,B) Hahn decomposition.

Now, $V < < \mu^+$ and μ^+ is $\sigma-$finite.

Applying Theroem 9.6.1 set

We get $V(E \cap A) = \int_{E\cap A} f_1 \, d\mu^+$ for measurable function f_1 defined on A.

Similarly, we get

$V(E \cap B) = \int_{E \cup B} f_2 \, d\mu-$ for some measurable function f_2 defined on B.

Define $f = f_1$ on A.

$= -f_2$ on B.

Then f is measurable and

$$V(E) = V(E \cup A) + V(E \cup B)$$

$$= \int_{E\cap A} f_1 \, d\mu^+ + \int_{E\cap B} f_2 \, d\mu^- = \int_{E\cap A} f d\mu^+ - \int_{E\cap B} f_2 \, d\mu^-$$

$$= \int_E f \, d\mu^+ - \int_E f \, d\mu^-$$

$$= \int_E f \, d\mu$$ (Since V is signed measure $(\infty, -\infty)$ will not occur at only stage)

Hence the result.

Note :

The hypothesis that μ is $\sigma-$finite cannot be omitted.

Let X = [0,1]. B-Lebesgue measurable subsets of [0,1]. Let μ- Counting measure and V-Lebesgue measure.

$V < < \mu$ and μ, not $\sigma-$finite.

If $f = 0$ then $V(E) = \int_E f \, d\mu = 0 \; \forall E$

If $f = 0$ $x \in [0,1] : \rightarrow f(x) \neq 0$

$\therefore \int_{\{x\}} f \, d\mu = f(x) \neq 0$

But $V(\{x\}) = 0$

So, for no f, we get $V(E) = \int_E f\, d\mu$

Definition :

Suppose μ, v are two measuers on (x, β) and $V << \mu$ with $V(E) = \int_E f\, d\mu$

Then f is called the Radon Nikodym derivative of v w.r. to μ and it is denoted as $[dv / d\mu]$.

Definition :

i) Two measures v_1 and v_2 on (X, β) are said to be mutually singular (in symbols $v_1 \perp v_2$) If there disjoint measurable sets A and B with $X = A \cup B$ ∋: $V_1(A) = V_2(B) = 0$

ii) Two signed measures v_1 and v_2 are said to be mutually singular ($v_1 \perp v_2$) if there disjoint measurable sets A and B with $X = A \cup B$ such that

$|V_1|(A) = |V_2|(B) = 0$

Proposition : 9.6.1. (Lebesgue Decomposition)

Let (X, β, μ) be a σ−finite measure space and v a σ-finite measure defined on β. Then we can find a measure V_0, singular w.r. to μ, and a measure v_1, absolutely continuous w.r. to μ, such that $V = V_0 + V_1$. The measures V_0 and V_1 unique.

Proot :

Since μ and v are σ-finite measures, So in the measure $\lambda = \mu + v$

Since both μ and v are absolutely Continuous w.r. to λ.

The Radon - Nikodym Theorem asserts the existence of non-negative measurable functions f and g such that for each $E \in \beta$.

$\mu E = \int_E f d\lambda, \quad VE = \int_E g d\lambda.$

Let $A = \{x : f(x) > 0\}$ and $B = \{x : f(x) = 0\}$

Then X in the disjoint Union of A and B.

$\mu B = 0$

If we define V_0 by

$V_0 E = V(E \cap B)$

We have $V_0(A) = 0$ and So $V_0 \perp \mu$.

Let $V_1(E) = V(E \cap A) = \int_{(E \cap A)} g d\lambda$

Then $V = V_0 + V_1$

And we have only to show that $V_1 << \mu$.

Let E be a set of μ measure zero.

Then $0 = \mu E = \int_{(E)} f d\lambda$.

And $f = 0$ a.e $[\lambda]$ on E.

Since $f > 0$ on $A \cap E$.

We must have $\lambda(A \cap E) = 0$

Hence $V(A \cap E) = 0$

And so $V_1(E) = V(A \cap E) = 0$

$\therefore V_1 << \mu$

Thus we nave the decomposition of v into V_0 and v_1.

Next, to prove the uniqueness part.

Suppose $V_0 = V_0 + V_1 = V_0^1 + V_1^1$

Such that $V_0 \perp \mu$, $V_0^1 \perp \mu$

$V_1 << \mu$, $V_1^1 << \mu$.

Let A_1, B_1, A^1, B^1 be subsets of X such that $X = A \cup B = A^1 \cup B^1$ and $A \cap B = \phi$, $A^1 \cap B^1 = \phi$ and $V_0(B) = V_0^1(B) = \mu(A) = \mu(A^1) = 0$. Let $E \in \beta$.

$$E = E \cap X = (E \cap A) \cup (E \cap B^1$$

$$= (E \cap A) \cap (A^1 \cap B^1) \cap (E \cap B) \cap (A^1 \cap B^1)$$

$$= (E \cap A \cap A^1) \cup (E \cap A \cap B^1) \cup (E \cap B \cap A^1) \cup (E \cap B \cap B^1)$$

Clearly μ is zero on the first three subsets of R.H.S.

And Hence V_1 and V_1^1 are zero on these sets since they are absolutely Continuous w.r. to μ.

Now, $v = V_0 + V_1 = V_0^1 + V_1^1 \Rightarrow V_0 - V_0^1 = V_1^1 - V_1$

$\therefore (V_1^1 - V_1^1)\ E = (V_1^1 - V_1)\ (E \cap (A \cup B^1)$

$$= (V_0 - V_0^1) \quad (E \cap (B \cup B)$$

$$= 0 \quad \text{Since } V_0^1 \ (B) = 0.$$

$$\therefore \ V_1^1 \ (E) = V_1 \ (E)$$

$$\therefore \ V_1^1 \ (E) = V_0 \ (E)$$

This is true for each E in β.

$$\therefore \ V_0 = V_0^1 \text{ and } V_1 = V_1^1.$$

Hence, the decomposition is unique.

Example : 1

If $\mu, v \in > o$, there exists $\delta > 0$ such that when every $\{\mu\}$ (E) $< \delta$ we have $|v|$ $(\in) < \in$, Then $V < < \mu$.

Proof :

Let $|\mu|$ (E) $= 0$

Let $\in > 1/n$. Then for any $\delta < 0$, $|\mu|$ (E) $= 0 < \delta$.

(ie) $|v|$ (E) $< 1/n$ for each n.

$\therefore |V|$ (E) $= 0$

$\therefore V < < \mu$.

Example : 2

Let μ be a signed measure on (X, β) and let υ be a finite valued signed measure on (X, β) such that $V < < \mu$. Then given $\in > 0$ $\exists \delta \geq 0 \ni : |V|$ (E) $<$ $\in$ whenever $|\mu|$ (E) $< \delta$.

Proof :

By definition $v < < \mu \Rightarrow |v| < < |\mu|$

Also v-infinite valued $\Rightarrow$ $|V|$ - finite valued.

So, we may suppose that μ and v are measures and v is finite valued.

Suppose the given result is not true.

Then $\exists$ a positive $\in$ and a sequence E_n of sets in β, $\ni$:

μ $(E_n) < 1/2''$ but V $(E_n) \geq \in$.

$$\text{Let } E = \lim \text{Sup } E = \bigcap_{k=1}^{\infty} \ \bigcap_{m=k}^{\infty} E_m$$

Let $F = \bigcap_{m=k}^{\infty} E_m$ Then $E = \bigcap_{k=1}^{\infty} F_k \ \forall K$

$$\mu(\text{Lim Sup } E_n) = \mu(E) \leq \mu(E_k) = \mu\left(\bigcup_{m=k}^{\infty} Em\right)$$

$$\leq \sum_{m=k}^{\infty} \frac{1}{2^m} = \frac{1}{2^{k-1}}$$

$\therefore \mu$ (lim Sup E_n) = 0

(ie) $\mu(E) = 0$

But, for each k, $V(F_k) = V\left(\bigcup_{k}^{\infty} E_m\right) \geq \in$

Since V is finite

$V(E) = \lim v(F_k) \geq \in$

$\Rightarrow \Leftarrow$ to

$V < < \mu$.

Hence the result.

Example : 3

If μ, v_1, v_2 are measures on (X, β) and $v_1 < < \mu$ and $V_1 < < \mu$. Then

$$\frac{d}{d\mu}[v_1 + v_2] = \frac{dv_1}{d\mu} + \frac{dv_2}{d\mu}$$

Proof :

Since $V_1 < < \mu$ and $V_2 < < \mu$

$V_1 \perp V_2 < < \mu$

Also $V_1 + V_2(E) = V_1(E) + V_2(E)$

$$= \int \frac{dv_1}{d\mu}.\, d\mu + \frac{dv_2}{d\mu} d\mu$$

$$= \int \left(\frac{dv_1}{d\mu}.\, d\mu + \frac{dv_2}{d\mu}\right) d\mu = \int \left(\frac{dv_1}{d\mu} + \frac{dv_2}{d\mu}\right) d\mu$$

From the uniqueness of the Radon Nikodym derivative.

We get the result.

Example :

If V_1 V_2 and μ are finite signed measures such that both V_1 and V_2 are singular with respect to μ. Then $V_1 + V_2$ is singular w.r. to μ.

Proof :

Let $X = A_1 \cup A_2$ and $A_2 U B_2$ be decompositions such that $|\mu|(A_1) = 0 = |\mu|(A_2)$

And $|V_1|(B_1) = 0$ and $|V_2|(B_2) = 0$

$$X = A_1 \cup B_1 = A_2 \cup B_2$$

$$= [A_1 \cap (A_2 \cup B_2)] \cup (B, \cap A_2 \cup B_2)$$

$$= [(A_1 \cap A_2) \cup (A_1 \cap B_2) \cup (B_1 \cap A_2) \cup (B_1 \cap B_2)$$

$$X = |\mu|[(A_1 \cap A_2) \cup (A_1 \cap B_2) \cup (B_1 \cap A_2) = 0$$

$$(v_1 + v_2) \leq |v_1 + v_2|$$

And $|v_1|(B_1 \cap B_2) = 0 = |v_2|(B_1 \cap B_2)$

$$\Rightarrow |v_1 + v_2| (B_1 \cap B_2) = 0$$

Let $A = (A_1 \cap A_2) \cup (A_1 \cap B_2) \cup (B_1 \cap A_2)$

$B = (B_1 \cap B_2)$

Then $A \cap B = \phi$ and $A \cup B = x$

and $|\mu|(A) = 0 = |v_1 + v_2|(B)$

Hence the result.

Example :

A set function a which assigns a complex number VE to each E in a σ-algebra β is called a complex measure if

i) $V(\phi) = 0$ and ii) $V(\cup E_i) = \sum_{i=1}^{\infty} V E_i$

with absolute convergence on the right.

Show that each complex measure v may be expressed a, $v = \mu_1 - \mu_2 + i\,\mu_3 - i\mu_4$ where μ_i (i = 1 to 4) are finite measure.

Proof :

Let v be a complex measure on (X, β).

a) Let VE to each E.

$V(E) = V_1(E_i) + V_2(E_i)$

$V_1(E) = \text{Re}(V(E))$ and $V_2(E) = \text{Im}(V(E))$

Let $E \subset \bigcup_{1}^{\infty} E_i$, E_i - disjoint

Then $V(\bigcup E_i) = \bigcup_{1}^{\infty} V(E_i)$ where $\sum_{1}^{\infty} |V(E_i)| < \alpha$

$\therefore \sum V_1 |(E_i)| \leq \sum |V(E)|$

$\Rightarrow \sum V_1(E_i)$ Converges absolutely

(Since $|RZ| \leq |Z|$)

$\therefore V_i$ is a finite valued signed measure. So III^{ly} V_2 is a finite values signed measure using Jordan decomposition of v_1, v_2 we get

$V = v_1 + iv_2 = \mu_1 \mu_2 + i(\mu_3 - \mu_4)$

Where $v_1 = \mu_1 - \mu_2$ and $v_2 = \mu_3 - \mu_4$ $\mu_1 = V_1$ $\mu_2 = V^-_1$, $\mu_3 = V_2^+$, $\mu_4 = V^-_2$

Since v_1, v_2 are finite valued.

μ are finite measure for $i = 1$ to 4

Example :

Show that for each complex measure v, there is a measure μ and a complex valued
function ϕ such that $\forall E \in \beta$. $V(E) = \int_E \phi \, d\mu$

The measure μ is called the total variation or absolute variation of v and is denote as $|V|$.

Solution :

Define $\forall E \in \beta$.

$$|V|(E) = \text{Sup} \sum_{1}^{n} |V(E_k)| \quad E_1, E_2 \ldots\ldots E_n \text{ disjoint sets of } \beta \ni : UE_i = E$$

clearly $|V|(\phi) = 0$

Let E_i be a pairwise disjoint sequence of sets in $E \in \beta$.

Let $E = U E_i$

Let $\beta < |V|(E)$

Then there exists $A_1, A_2 \ldots A_n$ disjoint and $UA_1 = E$ such that $\sum_{1}^{n} |V(A_k)| > \beta$

$$\text{Now } V(A_k) = V(A_k \cap E) = V(A_k \cap (UE_i))$$

$$= V(U(A_k \cap E_i))$$

$$= \sum_{i=1}^{\infty} V(A_k \cap E_i) \quad \text{(Since V in measure)}$$

$$\therefore \sum_{1}^{n} |V(A_k)| \leq \sum_{k=1}^{n} \sum_{k=1}^{\infty} |V(A_k) \cap E_i)|$$

$$\leq \sum_{i=1}^{\infty} \sum_{K=1}^{n} |V(A_k) \cap E_i)|$$

$$\leq \sum_{i=1}^{\infty} |V|(E_i)$$

[Since [$(A_k \cap E$] for $k = 1, \ldots\ldots\ldots n$ is a partition for E.]

$$\text{ie. } \beta < \sum_{1}^{n} |V(A_k)| \leq \sum_{1}^{\infty} |V|(E_i)$$

$$\therefore |V|(E) \leq \sum_{1}^{\infty} |V|(E_i) \ldots\ldots\ldots (1)$$

Now $|V|(E)$ cannot be infinite.

$|V_i(E)| < \infty$

$\forall i$ choose $(E_{i,1}{}^1 \ldots\ldots\ldots E_{1n_i})$ of E_i

such that $\sum_{k=1}^{n} (E_{i,k}) > V(E_i) - \epsilon/2^i \ \forall m$

$$\sum_{i=1}^{n} |V|(E_i) < \sum_{i=1}^{n} \left(\frac{\epsilon}{2^i} + \sum_{1}^{n} |V(E_{i,k})| \right)$$

$$\leq \epsilon + \sum_{j=1}^{n} \sum_{k=1}^{n} |V(E_{j,k})| + \left| V\left(\bigcup_{m+1}^{\infty} E_i \right) \right|$$

$$\leq \epsilon + v(E)$$

$\therefore$ Since ϵ in arbitary $\sum_{1}^{\infty} |V|(E_i) \leq |V|(E)$ ....(2)

From 1 & 2 we get $|V|$ to be countably additive & so $|V|$ is measure.

$|V|$ is called the total variation of v.

Also $v = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$

Let (A_1, B_1) (A_2, B_2) be the Hahn decomposition of V_1 and V_2 resecuvely. Then for any $E_1 E_2 \ldots\ldots\ldots E_k$ of E.

$V(E_i) = |(\mu_1 - \mu_2 + i\mu_3 - i\mu_4)(E_i| \leq E_i(E_1) + E_2(E) + E_3(E_1) + E_4(E)$

$$\sum_{1}^{K} |V(E_i)| \leq \sum_{j=1}^{4} \mu_i(E)$$

$\therefore V(E)| \leq \mu_1(E) + \mu_2(E) + \mu_3(E) + \mu_4(E)$ ....(1)

Also $\mu_1(E) - V_1(E \cap A_1) \leq V(E)$

Similarly $\mu_i \leq |V| \ \forall i = 2.3.4$

Also so each $\mu_i \leq <|V|$

Let $\mu = |V|$.

μ is a finite measure.

Since all μ are finite.

Also each μ_i is abs.cont w.r. to μ.

$\therefore$ There exists $f_i \ni : \mu_i (E) = \int_E f_i \, d\mu$

$\therefore V(E) = (\mu_1 - \mu_2 + i\mu_3 - i\mu_4)(E)$

$= \int_E (f_1 - f_2 + i(f_3 - f_4)) \, d^{\mu}$

$= \int_E \phi \, d^{\mu}$ where $\phi = f_1 - f_2 + i(f_3 - f_4)$

Example :

If μ, υ are signed measures $\ni : V \perp \mu$ and $V << \mu$ Then $v = 0$

Proof .

$V \perp \mu \Rightarrow$ There exists A, B such that $X = A \cup B$, $A \cap B = \phi$ and $\mu(A) = 0 = V(B)$

Since $V << \mu$.

$\mu(A) = 0$

$\Rightarrow V(A) = 0$

$\therefore V(X) = 0$

(ie) $V = 0$

Exercise :

i) If v, λ, μ are $V << \mu << \lambda$

$$\text{Then} \left(\frac{dv}{d\lambda}\right) = \frac{dv}{d\mu} \cdot \frac{d\mu}{dv}$$

ii) If $v << \mu$ and $\mu << V$.

$$\text{Then} \left(\frac{dv}{d\mu}\right) = \left(\frac{d\mu}{dv}\right)^{-1}$$

UNIT - 10

MEASURE AND OUTER MEASURE

Outer Measure and Measulability

Let A be a non-negative extended real value set function defined on all subsets of a space x and having the following properties.

i) $\mu^* \phi = 0$

ii) $A \subset B \Rightarrow \mu^* A \leq \mu^* B$

iii) $E \subset \bigcup_{i=1}^{\infty} E_i \Rightarrow \mu E^* \leq \sum_{i=1}^{\infty} \mu^* E_i$

The second property is called monotonicity and the third countable subaddititivity.

i) Finite subadditivity follow from (iii) Because of (ii) property (iii) can be replaced by

$$E = \bigcup_{i=1}^{\infty} E_i,\ E_i \text{ disjoint} \Rightarrow \mu^* E \leq \sum_{i=1}^{\infty} \mu^* E_i$$

The outer measure μ^* is called finite if $\mu^* x < \infty$.

Definition:-

A subset E of R is said to be measurable if for each subset A of R. We have

$$m^* A = m^* (A \cap E) + m^* (A \cap \tilde{E}) \text{ Where } \bar{E} = R - E.$$

Theorem: 10.1.1.

The class β of μ^* - measurable sets is a σ - algebra. If $\bar{\mu}$ is μ^* restricted to β Then $\bar{\mu}$ is a comnlete measure on β.

Proof:-

Trivially, the empty set is measurable

The symmetry of the definition of measurability in E and $\tilde{E}$ show that $\tilde{E}$ is measurable whenever E is.

Let E_1 and E_2 be measurable sets.

From the measurability of E_2

$$\mu^*(A) = \mu^*(A \cap E_2) + \mu^*(A \cap \bar{E}_2)$$

And $\mu^*(A) = \mu^*(A \cap E_2) + \mu^*(A \cap \bar{E}_2 \cap E_1) + \mu^*(A \cap \bar{E}_1 \cap \bar{E}_2)$

By the measurability of E_1.

Since $A \cap (E_1 \cup E_2) = (A \cap E_2) \cup (A \cap E_1 \cap \bar{E}_2)$

We have

$$\mu^* (A \cap (E_1 \cup E_2) \leq \mu^* (A \cap E_2) + \mu^* (A \cap \bar{E}_2 \cap E_1)$$

by subadditivity and so

$$\mu^* A \geq \mu^* (A \cap (E_1 \cup E_2) + \mu^* (A \cap \bar{E}_1 \cap \bar{E}_2)$$

This means that $E_1 \cup E_2$ is measurable.

And so

$$\mu^* (A) \geq \mu^* (A \cap \tilde{E}) + \sum_{i=1}^{\infty} \mu^* (A \cap E_i)$$

$$\geq \mu^* (A \cap \tilde{E}) + \mu^* (A \cap E)$$

$$\therefore A \cap E \subset \bigcup_{1}^{\infty} (A \cap E_i)$$

Thus E is measurable.

Since the union of any sequence of sets in an algebra can be replaced by a disjoint union of sets in the algebra. It follows that β is a σ - algebra.

We next demonstrate the finite additivity of $\bar{\mu}$.

Let F_1 and E_2 be disjoint measurable sets.

Then the measurabaility of E_2.

$$\Rightarrow \bar{\mu} (E_1 \cup E_2) = \mu^* (E_1 \cup E_2)$$

$$\Rightarrow \bar{\mu} (E_1 \cup E_2) = \mu^* (E_1 \cup E_2) \cap E_2 + \mu^* (E_1 \cup E_2) \cap \tilde{E}_2$$

$$= \mu^* E_2 + \mu^* E_1$$

Finite additivity follows by induction.

Since $\sim (E_1 \cup E_2) = \tilde{E}_1 \cap \tilde{E}_2$

Thus the Union of two measurable sets in measurable and by induction the union of any finite number of mesurable sets is measurable. Showing that β is an algebra of sets.

Assume that $E = \cup E_i$ where (E_i) is a disjoint sequence of measurable sets.

And set $G_n = \bigcup_{i=1}^{n} E_i$

Then G_n is measurable.

And $\mu^*(A) = \mu^*(A \cap G_n) + \mu^*(A \cap \overline{G}_n)$

$$\geq \mu^*(A \cap G_n) + \mu^*(A \cap \overline{E})$$

Since $\widetilde{E} \subset \widetilde{G}_r$

Now $G_n \cap E_n = E_n$ and $G_n \cap \overline{E}_n = G_{n-1}$

And by the measurability of E_n

We have

$$\mu^*(A \cap G_n) = \mu^*(A \cap E_n) + \mu^*(A \cap G_{n-1})$$

By induction

$$\mu^*(A \cap G_n) = \sum_{i=1}^{n} \mu^*(A \cap E_i)$$

If E is the disjoint Union of the measurable sets $\{E_i\}$, Then

$$\overline{\mu}(E) \geq \overline{\mu}\left(\bigcup_1^n E_i\right) = \sum_i^n \overline{\mu}(E_i)$$

And so $\sum_{i=1}^{\infty} \overline{\mu}(E_i) \geq \overline{\mu}(E)$

$$\overline{\mu}(E) \leq \sum_{i=1}^{\infty} \overline{\mu}(E_i)$$

But $\overline{\mu}(E) \leq \sum_{i=1}^{\infty} \overline{\mu}(E_i)$

By the subadditivity of μ^*.

Hence $\overline{\mu}$ is countably additive and thus a measure since it in non-negative and $\overline{\mu}\phi = \mu^*\phi = 0$

Problems :1

Assume that (E_i) is a sequence of disjoint measurable sets and $E = \cup E_i$. Then for any set A.

We have $\mu^*(A \cap E) = \Sigma \mu^*(A \cap E_i)$ (Exercise)

THE EXTENSION THEOREM

Topic : 10.2

Definition : 10.2.1

A non-negative extended real valued set function μ defined on an algebra R of sets such that,

i) $\mu(\phi) = 0$

ii) If $<A_i>$ is a disjoint sequence of sets in R. Whose union is also is R. Then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu A_i$$

Thus a measure on an algebra R is a measure iff R is a σ - algebra.

The purpose of this section is to show that if we start with a measure on an algebra R of a sets, we may extend it to a measure defined on a σ - algebra β containing R. We shall do this by using the mesure on the algebra to constract an outer measure $\mu*$ and show that the measure μ inducted by $\mu*$ is the desired extension of μ. The process by which we construct $\mu*$ from μ is amalogous to that by which we constructed lebegue outer measure from the length of intervals

We define

$$\mu^* E = \inf \sum_{i=1}^{\infty} \mu A_i \text{ ----- (1)}$$

where (A_i) ranges over all sequences from R. Such that $E \subset \bigcup_{i=1}^{\infty} A_i$.

Lemma : 10.2.1

If $A \in R$ and if $<Ai>$ is any sequence of sets in R such that $A \subset \bigcup_{i=1}^{\infty} Ai$. Then

$$\mu A \leq \sum_{i=1}^{\infty} \mu A_i$$

Proof : Set

$B_n = A \cap A_n \cap \tilde{A}_{n-1} \cap \ldots\ldots \cap \tilde{A}_1$

Then $B_n \in R$ and $B_n \subset A_n$

But A is the disioint union of the sequence (B_n) and so by countable additivity.

$$\mu A = \sum_{n=1}^{\infty} \mu B_n \leq \sum_{n=1}^{\infty} \mu A_n$$

Corollary :

If $A \in R$, $\mu^* A = \mu A$

Lemma : 10.2.2.

The set function μ^* is an outer measure

Proof :

Since μ^* is clearly a monotone non-negative set function defined for all sets and $\mu^* \phi = 0$.

We have only to show that it is countably subadditive.

Let $E \subset \bigcup_{i=1}^{\infty} E_i$

If $\mu^* E_i = \infty$ for any i.

We have $\mu^* E \leq \Sigma \mu^* E_i = \infty$

If not, given $\in > 0$. There is for each i a sequence $(A_{ij})^{\infty}_{j=1}$ of sets in R such that

$$E_i \subset \bigcup_{j=1}^{\infty} A_{ij} \text{ and } \sum_{j=1}^{\infty} \mu A_{ij} < \mu^* E_i + \in/2^i$$

$$\text{Then } \mu^* E \leq \sum_{ij}^{\infty} \mu A_{ij} < \sum_{i=1}^{\infty} \mu E_i + E$$

Since $\in$ was an arbitrary positive number. $\mu^* E \leq \sum_{i=1}^{\infty} \mu^* E_i$

and μ^* is subadditive.

Lemma : 10.2.3.

If $A \in R$, Then A is measurable with respect to μ^*.

Proof :

Let E be an arbitrary set of finite outer measure and $\in$ is a positive number.

Then there is a sequence (A_i) from R such that $E \subset \cup A_i$.

And $\Sigma \mu A_i < \mu^* E + \in$.

By the additivity of μ on R.

We have,

$$\mu(A_i) = \mu(A_i \cap A) + \mu(A_i \cap \bar{A})$$

Hence

$$\mu^* E + \in > \sum_{i=1}^{\infty} \mu(A_i \cap A) + \sum_{i=1}^{\infty} \mu(A_i \cap \bar{A})$$

$$\geq \mu^*(E \cap A) + \mu^*(E \cap \bar{A})$$

Since $E \cap A \subset \cup (A_i \cap A)$

And $E \cap \bar{A} \subset \cup (A_i \cap \bar{A})$

Since $\in$ was an arbitrary positive number,

$\mu^* E \geq \mu^*(E \cap A) + \mu^*(E \cap \bar{A})$

And A is measurable.

Note :

The outer measure μ^* be a measure on an algebra R, μ^* the outer measure induced by μ, and E any set. For a given algebra R of sets we use R_σ to denote those sets that are countable unions of sets of R and use $R\sigma\delta$ to those sets are countable intersections of sets in $R\sigma$.

Proposition : 10.2.4

Let μ be a measure on an algebra R, μ^* the outer measure induced by μ, and E any set, Then for $\in > 0$, There is a set $A \in R_\sigma$ with $E \subset A$ and

$\mu^* A \leq \mu^* E + \in$

There is also a set $B \in R_{\sigma\delta}$ with $E \subset B$ and $\mu^* E = \mu^* \beta$.

Proof :

By the definition of μ^*

There is a sequence (A_i) from R such that $E \subset U A_i$ and

$$\sum_{i=1}^{\infty} \mu A_i \leq \mu^* E + \epsilon$$

Set $A = \cup A_i$

Then $\mu^* A \leq \mu^* A_i = \Sigma \mu A_i$

To prove the second statement.

We note that for each positive integer n there is a set A_n in R_σ with $E \subset A_n$ and

$\mu^* A_n < \mu^* E + 1/n$

Let $B = \cap A_n$.

Then $B \in R_{\sigma\delta}$ and $E \subset B$.

Since $B \subset A_n$, $\mu^* B \leq \mu^* A_n < \mu^* E + 1/n$

Since n is abitiary.

$\mu^* B \leq \mu^* E$

But $E \subset B$

And so $\mu^* B \geq \mu^* E$

by monotonicitv

Hence $\mu^* B = \mu^* E$

Definition : 10.2.2

An outer measure μ^* is said to be regular if given any subset E of x and any $\epsilon > 0$ there is a μ^* measurable set A with $E \subset A$ and $\mu^* A \leq \mu^* E + \epsilon$.

If follows from lemma and proposition that every outer measure induced by a measure on an algebra is a regular outer measure.

If we apply this proposition in the case that E is a measurable set of finite measure. We see that E must be the difference of a set B in $R_{\sigma\delta}$ and a set of measure zero. This gives us the structure of the measurable sets of finite measure.

Proposition 10.2.5

Let μ be a σ - finite measure on algebra R and let μ^* be the outer measure generated by μ. A set E is μ^* measurable iff E is the proper difference $A \sim B$ of a set A in $R_{\sigma\delta}$ and a set B with $\mu^*B = 0$ is contained in a set C is $R_{\sigma\delta}$ with $\mu^* C = 0$.

Proof:

"If" part of the proposition follows from the fact that each set in $R_{\sigma\delta}$ must be measurable.

Since the measurable sets form a $\sigma-$ algebra while each set of μ^*- measure zero must be measurable.

Since μ is complete.

To prove the 'only if' part of the proposition.

Let $\{x_i\}$ be a countable disjoint collection of sets in R with μx_i finite and $x = Ux_i$

If E is measurable.

Then E is the disjoint union of the measurable sets $E_i = x_i \cap E$.

Ry proposition 10.2.4

Let μ be a measure on a algebra R. μ^* the outer, measure induced by μ and E any set. Then for $\in > 0$. There is a set $A \in R_\sigma$ with $E \subset A$ and

$$\mu^*A \leq \mu^* E + \in$$

There is also a set $B \in R_{\sigma\delta}$ with $E \subset B$ and $\mu^*E = \mu^*B$.

We can find for each positive integer n, a set A_{ni} is R_σ such that $E_i \subset A_{ni}$ and $\bar{\mu} A_{ni} \leq \bar{\mu} E_i + 1/n2^i$

set

$$A_n = \bigcup_{i=1}^{\infty} A_{ni}$$

$$\text{Then } E \subset A_n \text{ and } A_n \sim E \subset \bigcup_{i=1}^{\infty} (A_{ni} \sim E)$$

$$\text{Hence } \bar{\mu}(A_n \sim E) \leq \sum_{i=1}^{\infty} \mu(A_{ni} \sim E_i)$$

$$\leq \sum_{i=1} \frac{1}{n2^i} = \frac{1}{n}$$

Since $A_n \in R_\sigma$ the set $A = \bigcap_{n=1}^{\infty} A_n$ is in $R_{\sigma\delta}$ and for each n.

$$A \sim E \subset A_n \sim E.$$

Hence $\overline{\mu}\,(A \sim E) \leq \overline{\mu}\,(A_n \sim E) \leq 1/n$

Since this holds for each positive integer n,

we must have

$$\overline{\mu}\,(A \sim E) = 0$$

$$\therefore \overline{\mu}\,(A \sim E) = 0$$

Hence the Theorem

Theorem (cara theodory) : 10.2.1

Let μ be a measure on an algebra R and μ^* The outer measure induced by μ. Then the restriction μ of μ^* to the μ^* measurable sets is an extension of μ to a $\sigma-$ algebra containing R. If μ is finite (or $\sigma-$ finite) so is μ if μ is $\sigma-$ finite then μ is the only measure on the smallest $\sigma-$ algebra containing R which is an extension of μ.

Proof

The fact that μ is an extension of μ from R to be a measure on a $\sigma-$ algebra containing R follows directly from lemma 10.2.1. Theorem 10.1.1 and Lemma 10.2.3 it is readly verified that μ is finite or $\sigma-$ finite whenever μ is.

To show the unicity of μ when μ is $\sigma-$ finite.

Let β be the smallest $\sigma-$ algebra containing R and μ some measure on β that agrees with μ on R.

Since each set in R_σ can be a expressed as a disjoint countable union of sets in R the measure $\overline{\mu}$ must agree with $\overline{\mu}$ on R_σ.

Let B be any set in β with finite outer measure. Then there is an A in R_σ

such that $B \subset A$ and

$$\mu^* A \leq \mu^* B + \epsilon$$

Since $B \subset A$

$$\overline{\mu} B \leq \overline{\mu} A = \mu^* A \leq \mu^* B + \epsilon$$

Since ϵ is an arbitrary positive number β

We have $\bar{\mu}\ B \le \mu^* B$ for each $B \in \beta$.

Since the class of self measurable w.r to μ^* in a σ–algebra containing R each B in β must be measurable.

If B in measurable and A is in R_σ with $B \subset A$ and $\mu^* A \le \mu^* B + \in$

Then $\mu^* A = \mu^* B + \mu^* (A \sim B)$

And so $\bar{\mu} (A \sim B) \le \mu^* (A \sim B) \le \in$

f $\mu^* B < \infty$

Hence $\mu^* B \le \mu^* A = \bar{\mu} A$

$$= \bar{\mu} B + \bar{\mu} (A \sim B)$$

$$\le \bar{\mu} B + \in$$

Since $\in$ is arbitrary

we have $\mu^* B \le \bar{\mu} B$

ıd so $\mu^* B = \bar{\mu} B$

If μ is a σ–finite measure.

Let $\{x_i\}$ be a countable disjoint collection of sets in R with

$X = Ux_i$ and μx_i finite.

If B is any set in β.

Then $B = U (x_i \cap B)$ and this is a countable disjoint union of sets in β

So we have.

$\bar{\mu} B = \Sigma \bar{\mu} (x_i \cap B)$

and $\bar{\mu} B = \Sigma \bar{\mu} (x_i \cap B)$

Since $\mu^* (x_i \cap B) < \infty$

We have $\bar{\mu} (x_i \cap B) = \bar{\mu} (x_i \cap B)$

Topic : 10.3 Product Measures

Definition : 10.3.1

Let (x, R, μ) and (y, β, ν) be two complete measure spaces, and consider the direct product X xY of X and Y if $A \subset x$ and $B \subset y$ we call A x B a rectangle.

If, $A \in R$ and $B \in \beta$ we call $A \times B$ a measurable rectangle.

The collection of R measurable rectangler is a semi algebra.

Since $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ and

$\sim (A \times B) = (\tilde{A} \times B) \cup (A \times \overline{B}) \cup (\bar{A} \times \overline{B})$

If $A \times B$ is a measurable rectangle.

We get $\lambda(A \times B) = \mu A \, \nu B$

Lemma : 5.3.1

Let $\{(A_i \times B_i)\}$ be a countable disjoint collection of measurable rectangles whose union is a measurable rectangle $A \times B$ Then

$\lambda(A \times B) = \Sigma \lambda (A_i \times B_i)$

Proof :

Fix a point $x \in A$.

Then for each $y \in B$, the point $<x,y>$ belongs to exactly one rectangle $A_i \times B_i$. Thus B in the disjoint union of those B_i. Such that x is in the corresponding A_i.

Hence $\Sigma \nu B_i \, \chi_{A_i}(x) = \nu B . \chi_A(x)$

Since v is countably additive.

Thus by the Monotone convergence Theorem.

We have

$\Sigma \int \nu B_i \chi A_i \, d\mu = \int \nu(B) : \chi_A \, d\mu$

(or) $\Sigma \nu B_i . \mu A_i = \nu B . \mu A$.

Note :

The lemma imples that λ satisfies the condition of proposition.

Hence has a unique extension to a measure on the algebra R consisting of all finite disjoint unions of sets in R Theorem. Carathodory allows us to extend λ to be a complete measure on a σ – algebra S Containing R. This extended measure is called the product measure of μ and υ and is denote by $\mu \times \nu$.

If μ and ν one finite (or σ finite) so is $\mu \times \nu$. If x and y are the real line and μ and ν are both lebesgue measure. Then $\mu \times \nu$ is called two-dimensional lebesgue measure for the plane.

The purpose of the next few lemmas is to describe the structure of the sets which are measurable with r to the product measure $\mu \times \nu$. If E is any subset of X x Y and x a point of x. we define the x cross section E_x by

$E_x = \{y : (x, y) \in E\}$ and similarly for the y cross section for y in Y.

The characteristic function of E_x is related to that of E by

$\chi_{E_x}(y) = \chi_E(x, y)$

We have $(\bar{E})_x = \sim (E_x)$

and $(\cup E_\alpha)_x = \cup (E_\alpha)_x$ for any collection $\{E_\alpha\}$

Lemma : 10.3.2

Let x be a point of X and E a set in $R_{\sigma\delta}$. Then E_x is a measurable subset of y.

Proof :

The Lemma is trivially true. If E is in the class R of measurable rectangles.

We next show it to be true for $E \in R_\sigma$.

Let $E = \bigcup_{i=1}^{\infty} E_i$ where each E_i is a measurable rectangle. Then

$$\chi_{E_x}(y) = \chi_E(x,y)$$
$$= \operatorname{Sup}_i \chi_{E_i}(x,y)$$
$$= \operatorname{Sup}_i \chi_{(E_i)_x}(y)$$

Since each E_i is a measurable rectangle $\chi_{(E_i)_x}(y)$ is a measurable function of y and so χ_{E_x} must also be measurble.

Hence E_x is measurable.

Suppose that $E = \bigcap_{i=1}^{\infty} E_i$ with $E_i \in R_\sigma$.

Then $\chi_{E_x} = \chi_E(x,y) = \inf_i \chi_E(x,y)$

$$= \inf_i \chi\left[E_i\right]_x (y)$$

And we see that χ_{E_x} is measurable.

Thus E_x is measurable for any $E \in R_{\sigma\delta}$

Lemma : 10.3.3.

Let E be a set in $R_{\sigma\delta}$ with $\mu \times \nu$ (E) less than ∞. Then the function g defined by $g(x) = \nu E_x$ is a measurable function of x and $\int g\, d\mu = \mu \times \nu\, (E)$.

Proof :

The lemma is trivially true. If E is a Measurable rectangle.

We first note that any set in R_σ is a disjoint Union of measurable rectangles.

Let (E_i) be a disjoint sequence of measurable rectangles and

Let $E = \cup E_i$

Set $g_i(x) = \left(\nu\, (E_i)_x\right)$

Then each g_i is a non-negative measurable function and $g = \Sigma g_i$

Thus g is measurable.

And by the Monotone convergence theorem

we have $\int g\, d\mu = \Sigma \int g_i\, d\mu$

$$= \Sigma\, \mu \times \nu\, (E_i)$$

$$= \mu \times \nu\, (E)$$

Consequently the lemma holds for $E \in R_\sigma$.

Let E be a set of finite measure in $R_{\sigma\delta}$. Then there is a sequence (E_i) of sets in R_σ such that $E_{i+1} \subset E_i$ and $E = \cap E_i$.

It follows from proposition. 10.2.4.

That we may take $\mu \times v\,(E_i) < \infty$

Let $g_i\,(x) = v\left[(E_1)_x\right]$

Since $\int g_1\, d\mu = \mu \times v\,(E_i) < \infty$

We have $g_1\,(x) < \infty$ for almost all x.

For an x with $g_1\,(x) < \infty$.

We have $\langle (E_i)_x \rangle$ a decreasing sequence of measurable sets of finite measure whose intersection is E_x.

Thus, we have

$$g\,(x) = v\,(E_x) = \lim v\left[(E_1)_x\right]$$

$$= \lim g_i\,(x)$$

Hence $g_i \to g$ a.e.

and so g is measurable.

Since $0 \le g_i \le g$

The lebesgue convergence Theorem

$$\Rightarrow \int g\, d\mu = \lim \int g_i\, d\mu.$$

$$= \lim \mu \times v\,(E_i)$$

$$\therefore \int g\, d\mu = \mu \times v\,(E)$$

Hence the theorem.

Lemma : 10.3.4

Let E be a set for which $\mu \times v\,(E) = 0$

Then for almost all x. we have $v\,(E_x) = 0$.

Proof :

By the proposition 10.2.4.

There is a set F in $R_{\sigma\delta}$ suchthat $E \subset F$ and $\mu \times v\,(F) = 0$

It follows from lemma that for almosts all x we have $v\,(F_x) = 0$

But $E_v \subset F_x$ And so $VE_x = 0$ for almost all x. Since V is complete.

Proposition : 10.3.1

Let E be a measurable subset of X x Y such that $\mu \times V$ (E) is finite. Then for almost all x the set E_x is a measurable subset of y. The function g defined by

$$g(x) = v\,(E_x)_X$$

is a measurable function defined for almost all x and $\int g\, d\mu = \mu \times V$ (E).

Proof :

By proposition 10.2.4

There is a set F in $R_{\sigma\delta}$ such that $E \subset F$ and $\mu \times v$ (F) = $\mu \times v$ (E).

Let G = F ~ E

Since E and F are measurable so is G.

And $\mu \times v$ (F) = $\mu \times v$ (E) + $\mu \times v$ (G)

Since $\mu \times v$ (E) is finite and equal to $\mu \times v$ (F)

We have $\mu \times v$ (G) = 0

Thus ɒy Lemma 10.3.4.

We have V G_x = 0 for almost all x.

Hence g (x) = VE_x = VF_x a.e.

So g is a measurable function by Lemma 10.3.3

Again by Lemma

$\int g\, d\mu = \mu \times v$ (F)

$\int g\, d\mu = \mu \times v$ (E)

Hence the theorem.

Theorem : (Fubini) 10.3.1

Let (X, R, μ) and (Y, β, υ) be two complete measure spaces and f on integrable function on X x Y. Then

i) For almost all x the function f_x defined by f_x (y) = f (x, y) is an integrable function on Y.

ii) For almost all y the function f_y defined by f_y (x) = f (x, y) is an integrable function on x.

iii) $\int_y \left[f(x,y) \right] d\,v\,(y)$ is an integrable function on X.

iv) $\int_x f(x,y)\, d\,\mu\,(x)$ is an integrable function on y.

v) $\int_x \left(\int_y f dv\right) d\,\mu = \int_{x \times y} f d\,(\mu \times v) = \int_y \left(\int_x f d\mu\right) dv.$

Proof :

If the results holds for two functions f_1 and f_2 they will obviously hold for $f_1 + f_2$ also and hence.

We may take f to be non-negative.

Case : 1

Let $f = \chi_E$ and f is integrable.

Then $\int f\, d\,(\mu \times v) = \mu \times v\,(E) < \alpha.$

And hence by prop. Let (x, R, μ) and (γ, β, v) be two complete measure space.

Let T be a measurable subset of X x Y. Such that $\mu \times v\,(E) < \infty$. Then for almost all x. The set E_x is a measurable subset of y.

And the function g defined by $g(x) = V\,(E_x)$ is a measurable function defined for almost all x and $\int g\, d\mu = \mu \times v\,(E)$.

The set E_x is measurable and

$f_x = (\chi_E)_x = \chi_{E_x}$

$\int f_x\, dv = \int_x (\chi_E\, dv = V\,(E_x)$

The function $g(x) = V\,(E_x) = \int f_x\, dv = \int f\,(x, y)\, d\,vy$ is a measurable function defined for a.a.x and $\int g\, d\mu = \mu \times v\,(E) < \infty.$

$\Rightarrow \int f_x\, dv$ is an integrable function.

Also $\int_x \left(\int_y f_x\, dv\right) d\mu = \int (g\, d\mu = \mu \times v\,(E) = \int f\, d\,(\mu \times v)$

Case : 2

Let f be $\sum_{1}^{n} a_i \chi_{E_i} \mu \times v(E_i) < \infty$

(i.e.) f is a simple function which vanishes outside a set of finite measure. By the observation in the beginning, we see that the above results hold or f.

Case : 3

Let f be a non-negative integrable function. Then there exists on increasing sequence ϕ_n of non-negative simple functions such that $f = \lim \phi_n$.

Also each ϕ_n is integrable and so it vanishes outside a set of finite measure.

By the observation in the beginning, we see that the above results hold for all x.

Case : 4

Let f be a non-negative integral function. Then there exists on increasing sequence ϕ_n of non-negative simple function such that $f = \lim \phi_n$.

Also each ϕ_n is integrable and so it vanishes outside a set of finite measure.

$\therefore f = \lim \phi_n - \phi_n$ increasing sequence of simple function in Y.

$\therefore f_x$ is the limit of the increasing sequence $(\phi_n)_x$) and is measurable.

By monotone convergence Theorem.

$$\int f(x,y)\, dv(y) = \lim \int_Y \phi_n (x,y)\, dv(y)$$

And so this integral is a measurable function of x. Again by the Monotone convergence Theorem.

$$\int_x \left(\int_y f dv \right) = \lim \int_x \left(\int_y \phi_n\, dv \right) d\mu$$

$$= \lim \int_{X \times Y} \phi_n\, (d(\mu \times v)$$

$$= \int_{X \times Y} f\, d(\mu \times v)$$

By symmetry between x and y the other result follow.

Theorem : (Tonelli) 10.3.2

Let (X, a, μ) and (Y, β, υ) be two σ-finite measure spaces, and let f be a non-negative measurable function on $X \times Y$ Then

i) For almost all x the function f_x, defined by $f_x(y) = f(x, y)$ is a measurable function on Y.

i)' For almost all y the function f_y defined by $f_y(x) = f(x, y)$ is a measurable function on X.

ii) $\int_Y f(x, y)\, dv(y)$ is a measurable function on X.

ii) $\int_X f(x, y)\, d^{\mu}(x)$ is a measurable function on Y.

iii) $\int_x \left(\int_y f dv\, d^{\mu} \right) = \int_{X \times Y} f d^{(\mu \times \upsilon)} = \int_y \left(\int_x f d^{\mu} dv \right)$

Proof :

For a non-negative measurable function f the only point in the proof of Fubini Theorem where the integrability of f was used was to infer the existence of an increasing sequence (ϕ_n) of simple functions each vanishing outside a set of finite measure such that

$$f = \lim \phi_n.$$

But if μ and υ are σ-finite.

Then, so is $\mu \times \upsilon$ and any non-negative measurable function on $X \times Y$ can be so apporoximate by proposion 9.2.2.

Example :

Let $X = [0,1] = Y$, $\mu =$ Lebesgue measure on $[0,1]$ $\lambda =$ counting measure on Y.

Let $f(x,y) = 1$ if $x = y$, $f(x,y) = 0$ if $x \neq y$.

$\int_X f(x,y)\, d^{\mu}(x) = 0$

Since for a given y $f(x, y) = 1$ when $x = y$ and υ at all other x.

Also Lebesgue measure of a single point is 0.

(ie) $\int_{x} f(x, y)\, d^{\mu}(x) = \int_{(y)} d^{\mu} = 0$

$\int_{y} f(x, y)\, d\lambda(x) = \int_{(x)} (d\lambda) = 1$

($\because$ λ is the counting measure)

Hence $\int_{y} (d\lambda(y) \int_{x} f(x, y)\, d\mu(x) = 0$

$$\int_{x} d^{\mu}(x) \int_{y} f(x, y)\, d\lambda(y) = \int_{0}^{1} d^{\mu}(x)$$

$$= \mu\ 0,1 = 1$$

Hence $\int_{x} d^{\mu}(y) \int_{y} f(x, y)\, d\lambda(y) \neq \int_{y} d\lambda(y) \int_{x} f(x, y)\, d\mu(x)$

To show that f is$(R \times \beta)$ measurable where R is the class of all Lebesgue measurable sets in 0,1 β consists of all subset of 0,1

Since $f(x,y) = 1$ if $x = y$ $f(x,y) = 0$ if $x \neq y$

We see that $f = \chi_D$ where D is the diagonal of the unit square.

Given a '+ve' integer n.

Let $I_j = \left(\dfrac{j-1}{n}, \dfrac{j}{n} \right) \quad i \leq j \leq n$

Let $Q_n = (I_1 \times I_1) \cup (I_2 \times I_2) \cup \dots\dots\dots \cup (I_n \times I_n)$

when $n = 1$

$I_1 = [0,1], \quad Q_1 = I_1 \times I_1$ is the unit square when $n = 2$.

$I_1 = \left[0, \tfrac{1}{2}\right] \qquad I_2 = \left[\tfrac{1}{2}, 1\right]$

$$Q_2 = \left[0, \tfrac{1}{2}\right] \times \left[0, \tfrac{1}{2}\right] \cup \left[\tfrac{1}{2}, 1\right] \times \left[\tfrac{1}{2}, 1\right]$$

$$Q_3 = \left[0, \tfrac{1}{3}\right] \times \left[0, \tfrac{1}{3}\right] \cup \left[\tfrac{1}{3}, \tfrac{2}{3}\right] \times \left[\tfrac{2}{3}, 1\right] \times \left[\tfrac{2}{3}, 1\right] \text{ etc.}$$

Thus Q_n is a finite union of measurable rectangles and $D = \cap Q_n$

Hence $D \in R \times \beta$

$\therefore f = \chi_D$ is $R \times \beta$ measurable.

In this case, the theroem fails due to the fact that λ is not σ–finite.

Since λ is the Counting measure.

If $Y = \bigcup_{n=1}^{\infty} Y_n$ a disjoint Union such that $\lambda(y_n) < \infty \; \forall n$. Then every Y_n is a finite set.

Hence Y is Countable a contradiction

Since $Y = [0,1]$

Thus λ is not σ-finite

$\therefore$ Thus the σ–finiteness of λ, μ cannot be omitted.

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