R282

Directorate of Distance Education

130

M.Sc.

First Year

MATHEMATICS

Paper - IV

NUMERICAL METHODS and DIFFERENTIAL EQUATIONS

UNITS : 1 TO 10

MADURAI KAMARAJ UNIVERSITY
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PAPER IV

NUMERICAL ANALYSIS AND DIFFERENTIAL EQUATION

NUMERICAL ANALYSIS

Unit – I

Transcendental and polynomial equations : Introduction – Directive methods – Iterative methods – initial approximations – bisection method – iteration method (chord method) regula-falsi method Newton – Raphson method – iteration methods based on second degree equations – Muller method – Chebyshev method – multipoint iteration method – rate of convergence – iteration methods – first order method – second order method – high order methods – Aitken Δ^2 method – efficiency of a method – methods for multiple roots – methods for complex roots – polynomial equations – Descarte's Rule of signs – Iterative method – Graffe's root squaring method – model problems – choice of an iterative method and implementation.

Unit – II :

System of linear Algebraic equations and Eigen value problems – Introduction – Direct methods – forward substitution method – back substitution method – Cramer rule – Guass elimination method – Guass Jordan elimination method – triangularization method – Cholesky method – partition method – Error analysis – Iteration method – Jocobi Iteration method – Gauss seidal iteration method – convergence analysis – iterative method for A^{-1} – Eigen values, Eigen vectors – Faddeev-Leverrier method – Gerschgorin theorem – Brauer theorem – Jacobi method for Symmertic matrices – Given method for symmetric matrices – Householder's method for symmetric matrices – Rutishauser Method for Arbitrary Matrices – Power method – inverse power method – model problems – Choice of a method.

Unit – III

Interpolation and approximations : Introduction – Lagrange and Newton interpolations – Linear interpolation – truncation error bounds – higher order interpolation – iterated interpolation – Finite difference operators – interpolation polynomials using finite differences – Gregory – Newton forward difference interpolation – Striling and Bessel interpolation – Hermite interpolation – Piecewise and spline interpolation – Piecewise cubic interpolation – Bivariate interpolation – Lagrane and Newton approximation – Wierstrass theorem – least square approximation – Gram-Schmidt orthogonalisation process – Legendre polynomials – Chebyshev polynomials – model problems – Choice of a method.

Unit – IV :

Differentiation and integration : Numerical differentiation – methods based on iteration – non-uniform nodal points – uniform nodal points – methods based on finite differences – methods based on undetermined coefficients – optimum choice of step length – extrapolation methods – partial differentiation – numerical integration – methods based on integration – Newton – Cotes methods – Trapezoidal rule – Simpson's rule – 3/8th Simpson's rule – methods based on undetermined coefficients Guass–Legendre integration methods – Lobatto integration methods – Radan integration methods – Guass-Chebyshev integration method – Guass-Legendre integration method – Guass Hermite integration method – composite integration method – Trapesoidal rule – Simpon's rule – composite Simpson's rule – Romberg integration.

Unit – V :

Ordinary differential equations : Introduction - initial value problem – Boundary value problem – Reduction of higher order equations to the system of first order Differential Equations – Existence and uniquesness – Test equations – system of linear first order Differential Equations with constant coefficients – Numerical methods – Differential Equations Euler (Adam – Bashforth) method – Local Truncation error – Backward Euler method – Midpoint method – Single step method – Taylor series method – Runga Kutta methods – Second order method – minimization of local truncation error – Fourth order method – 2nd, 3rd, and 4th order Runga-Kutta methods – estimation of local truncation error – system of equations – implicit R.K. methods – multistep methods – Adams-Boshforth methods – Nyrtron method – Adams Moulton method – Milne Simpson method – Maximum order of K-step methods – bounds of the local truncation error – convergence of multi step methods – Predicator-Connector methods – PM_P, CM_C method.

DIFFERENTIAL EQUATIONS

Unit – VI :

Linear equations with variable coefficients : Introduction – Initial value problems for the homogeneous equations – Existence theorem – Uniqueness theorem – Solutions of the homogeneous equations – The Wronskian and linear independence reduction of the order of a homogeneous equation – The non-homogeneous equations – Homogeneous equations with analytic coefficients – Existence theorem for analytic coefficients – the Legendre equation – Justification of power series method.

Unit – VII :

Linear equations with regular singular points : Introduction – The Euler equations – 2^{nd} order equations with regular singular points an example – the general case. A convergence proof – the exceptional case – Frobenius method – The Bessel equation of order zero – Bessel equation of order α, $\alpha \neq 0$, Re $\alpha \geq 0$ - Regular singular points at infinity.

Unit – VIII :

Existence and Uniqueness of solution to first order equations – Introduction – equations with variable separated – exact equations – the method of successive approximation – Lipschitz condition of successive approximation – Existence theorem – non-local existence of solution – approximation to, and uniqueness of, solutions – uniqueness theorem – equations with complex valued functions.

Unit – IX :

Partial Differential Equations of the first order : Partial Differential Equations – origins of the first order Differential equations – Cauchy's problem for first order equations – Linear equations of the first order – Integral surfaces passing through a given curve.

Unit – X :

Non-linear Partial Differential equations : Surface orthogonal to a given system of surfaces – Non linear Partial Differential equations of the first order – Cauchy method for characteristics – compatible systems of first order equations – Charpit's method – Special types of first order equations.

TEXT BOOKS :

1. Lecture Material prepared by DDE, MKU.
2. Numerical methods for Engineers by Stephen Chopra and Raymond Canale – Tata McGraw Hill.
3. Numerical methods for Scientific and Engineering Computation by M.K.Jain, S.R.K.Iyengar and R.K.Jain.
4. An Introduction to Ordinary Differential Equations by Earl A Coddington.
5. Elements of Partial Differential Equations by Ian Sneddon.

SCHEME OF LESSONS

Unit – I

Transcental and Polynomial Equations

Unit – II

System of Linear Algebraic Equations and Eigen Value Problems

Unit – III

Interpolation and Approximation

Unit – IX

Partial Differential Equations of the first order

Unit – X

Non-Linear Partial Differential Equations

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UNIT – 1

TRANSCENTAL AND POLYNOMIAL EQUATIONS

1.1 Introduction

A problem of most importance in applied mathematics and engineering is that of finding the roots of an equation $f(x)=0$.

The function $f(x)$ may be given explicitly or implicitly. Example for explicit function is a polynomial $P_n(x)=x^n+a_1x^{n-1}+a_2x^{n-2}+......+a_n$ and the example for implicit function $f(x) = \cos x - xe^x$ and the implicit functions are called trascentendal function.

Definition D. 1.1 :

A number α is called a root or a zero of $f(x)=0$ if $f(\alpha) = 0$.

Definition D. 1.2 :

If $f(x) = (x-\alpha)^m\, g(x)$ where $g(x)$ is bounded and $g(\alpha)\neq 0$ then α is called a multiple root of multiplicity m to $f(x)=0$.

Note : If $m=1$ then α is said to be a simple root.

Generally two types of methods are used to find the roots of the equation $f(x)=0$. 1. Direct method & 2. Iterative method.

1.2.1. Direct Method :

This method gives the exact value of the roots in a finite number of steps and also gives all the roots of the function.

For example, the roots of the quadratic equation $ax^2+bx+c=0$ are given by $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

1.2.2. Iterative Method :

This method is based on the idea of successive approximation. That is starting with one or more initial approximations to the root, we obtain a sequence of approximations $\{x_n\}$, which converges to the root.

This method give only one root at a time.

Definition D. 1.3 :

A sequence of iterates $\{x_n\}$ is said to converge to the root α, if $\lim_{n\to\infty} |x_n - \alpha| = 0$.

1.3. Initial Approximations :

Initial approximations to the root are known from the physical considerations of the problem. Otherwise graphical methods are used to obtain initial approximation to the root.

We know that the value of x, at which the graph of the equation y=f(x) intersects the x-axis, gives the root of f(x)=0, and any value in the neighbourhood of this point be taken as an initial approximation to the root.

The initial approximation to the root be obtained from Intermediate Value Theorem. Statement of the aboe theorem (without proof) is *"If f(x) is a continuous function on some interval [a, b] and f(a).f(b)<0 then the equation f(x)=0 has at least one real root or an odd number of roots in the interval (a, b)."*

1.4. Bisection Method :

This method is based on the repeated application of the intermediate value theorem.

If we know that a root of f(x)=0 lies in the interval $I_0=(a_0, b_0)$, we bisect I_0 at the point $m_1=\frac{1}{2}(a_0+b_0)$.

Denote I_1 which is equal to (a_0, m_1) if $f(a_0).f(m_1)<0$ or the interval (m_1, b) if $f(m_1)f(b)<0$.

$\therefore I_1$ contains a root.

Bisect the interval I_1 and get a subinterval I_2 at whose end points f(x) takes the value of opposite signs and therefore contains the root.

After repeating the bisection process of times, then the root or find the interval I_q of length $\frac{b_0-a_0}{2^q}$ which contains the root. Take the midpoint of the last subinterval as the desired approximation to the root.

If the permissible error is $\in$, then the approximate number of iterations required be obtained from $\frac{b_0-a_0}{2^n} \leq \in$.

(i.e.,) $n \geq \frac{\log(b_0-a_0)-\log \in}{\log 2}$

Example E. 1.1 :

Find a root of the equation $x^3-4x-9=0$ correct to three decimal places.

Solution :

$$\text{Let } f(x) = x^3-4x-9$$

$$\text{Here } f(0) = -9 < 0$$

$$f(1) = 1-4-9 = -12 < 0$$

$$f(2) = -9 < 0$$

$$\text{and } f(3) = 6 > 0$$

$\therefore$ There is a root lies between 2 and 3.

$$\text{Take } x_0 = \frac{2+3}{2} = 2.5 \quad \text{------(1.1)}$$

be the first approximation to the root.

$$\therefore f(x_0) = f(2.5) = (2.5)^3-4(2.5)-9$$

$$= -3.375$$

$$< 0$$

Thus there is a root lies between x_0 and 3. (ie) 2.5 and 3.

$$\text{We take } x_1 = \frac{x_0+3}{2} = \frac{2.5+3}{2} = 2.75 \quad \text{------(1.2)}$$

$$f(x_1) = 0.796875 > 0$$

So a root lies between x_0 & x.

$$\therefore x_2 = \frac{x_0+x_1}{2} = \frac{2.5+2.75}{2} = 2.625 \quad \text{------(1.3)}$$

$$\text{Now } f(x_2) = (2.625)^3-4(2.625)-9$$

$$= -1.412109$$

$$< 0$$

So a root lies between x_1 & x_2

$$\text{we take } x_3 = \frac{x_1+x_2}{2} = \frac{2.75+2.625}{2}$$

$$= 2.6875 \quad \text{------(1.4)}$$

$$f(x_3) = (2.6875)^3 - 4(2.6875) - 9$$
$$= -0.339111 < 0$$
$$\therefore f(x_3) < 0$$

So a root lies between x_1 and x_3

$$\text{we take } x_4 = \frac{x_1 + x_3}{2} = \frac{2.75 + 2.6875}{2}$$
$$= 2.71875 \qquad \text{------(1.5)}$$
$$f(x_4) = (2.71875)^3 - 4(2.71875) - 9$$
$$= 220917 > 0$$
$$\therefore f(x_4) > 0$$

So a root lies between x_3 and x_4

$$\text{we take } x_5 = \frac{x_3 + x_4}{2} = \frac{2.6875 + 2.71875}{2}$$
$$= 2.703125 \qquad \text{------(1.6)}$$
$$f(x_5) = (2.703125)^3 - 4(2.703125) - 9$$
$$= -061077 < 0$$
$$\therefore f(x_5) < 0$$

So a root lies between x_4 and x_5

$$\text{we take } x_6 = \frac{x_4 + x_5}{2} = \frac{2.71875 + 2.703125}{2}$$
$$= 2.7109375 \qquad \text{------(1.7)}$$
$$f(x_6) = (2.7109375)^3 - 4(2.7109375) - 9$$
$$= 0.079423$$
$$f(x_6) > 0$$

So a root lies between x_5 and x_6

$$\text{we take } \quad x_7 = \frac{x_5 + x_6}{2} = \frac{2.703125 + 2.7109375}{2}$$
$$= 2.70703125 \qquad \text{------(1.8)}$$
$$f(x_7) = (2.70703125)^3 - 4(2.70703125) - 9$$
$$= 0.009048$$
$$f(x_7) > 0$$

So a root lies between x_5 and x_7

$$\text{we take } x_8 = \frac{x_5 + x_7}{2} = \frac{2.703125 + 2.70703125}{2}$$

$$= 2.705078 \qquad \text{------(1.9)}$$

$$f(x_8) = (2.705078)^3 - 4(2.705078) - 9$$

$$= -0.026047$$

$$f(x_8) < 0$$

So a root lies between x_7 and x_8

$$\text{we take } x_9 = \frac{x_7 + x_8}{2} = \frac{2.70703125 + 2.705078}{2}$$

$$x_9 = 2.706054 \qquad \text{------(1.10)}$$

$$f(x_9) = (2.706054)^3 - 4(2.706054) - 9$$

$$= -0.008518$$

$$f(x_9) < 0$$

So a root lies between x_7 and x_9

$$\text{we take } x_{10} = \frac{x_7 + x_9}{2} = \frac{2.70703125 + 2.706054}{2}$$

$$= 2.70654 \qquad \text{------(1.11)}$$

$$f(x_{10}) = (2.70654)^3 - 4(2.70654) - 9$$

$$= 0.000216$$

$$f(x_{10}) > 0$$

So a root lies between x_9 and x_{10}

$$\text{we take } x_{11} = \frac{x_9 + x_{10}}{2} = \frac{2.706054 + 2.70654}{2}$$

$$= 2.706297 \qquad \text{------(1.12)}$$

$$f(x_{11}) = (2.706297)^3 - 4(2.706297) - 9$$

$$= -0.004157 < 0$$

So a root lies between x_{10} and x_{11}

$$\text{we take } x_{12} = \frac{x_{10} + x_{11}}{2} = \frac{2.70654 + 2.706297}{2}$$

$$= 2.7064185 \qquad \text{------(1.13)}$$

$$f(x_{12}) = (2.7064185)^3 - 4(2.7064185) - 9$$
$$= -0.001967 < 0$$

So a root lies between x_{10} and x_{12}

$$\text{we take } x_{13} = \frac{x_{10}+x_{12}}{2} = \frac{2.70654+2.70642}{2}$$
$$= 2.70648 \qquad \text{------(1.14)}$$
$$f(x_{13}) = (2.70648)^3 - 4(2.70648) - 9$$
$$= -0.000862 < 0$$

So a root lies between x_{10} and x_{13}

$$\text{we take } x_{14} = \frac{x_{10}+x_{13}}{2} = \frac{2.70654+2.70648}{2}$$
$$= 2.70651 \qquad \text{------(1.15)}$$
$$f(x_{14}) = (2.70651)^3 - 4(2.70651) - 9$$
$$= -0.000323 < 0$$

So a root lies between x_{10} and x_{14}

$$\text{we take } x_{15} = \frac{x_{10}+x_{14}}{2} = \frac{2.70654+2.70651}{2}$$
$$= 2.706525$$

we find that $f(x_{10}) = 0.000216$ and $f(x_{14}) = -0.000323$ and these two approach the value zero. So they approximately satisfy the equation $f(x) = 0$.

So x_{15} which is the average of x_{10} & x_{14} is a better approximation. The values of x_{10}, x_{14}, x_{15} are respect 2.70654, 2.70651 and 2.706525.

We find that the value of the root has settled down to four places. Hence the root is 2.7065.

1.5. ITERATION METHODS BASED ON FIRST DEGREE EQUATION :

In the earlier section we have discussed bisection method. Now we shall discuss *Secant Method.*

1.5.1. Secant Method :

If x_{k-1} and x_k are two approximations to the roots, then we determine a_0 & a_1 in $f(x) = a_0x+a_1 = 0$, using the conditions

$$f_{k-1} = a_0 x_{k-1} + a_1 \qquad \text{------(1.16)}$$

$$f_k = a_0 x_k + a_1 \qquad \text{------(1.17)}$$

where $f_{k-1} = f(x_{k-1})$

and $f_k = f(x_k)$

Solving (1.16) & (1.17) we get,

$$a_0 = \frac{f_k - f_{k-1}}{x_k - x_{k-1}} \qquad \text{------(1.18)}$$

$$a_1 = \frac{x_k f_{k-1} - x_{k-1} f_k}{x_k - x_{k-1}} \qquad \text{------(1.19)}$$

The next approximation x_{k-1} to the root is given by

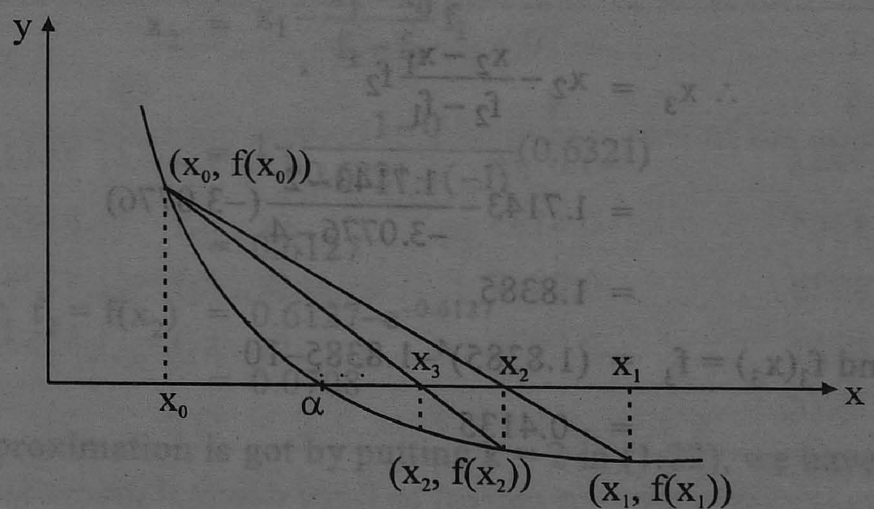
$$x_{k+1} = \frac{x_{k-1} f_k - x_k f_{k-1}}{f_k - f_{k-1}} \qquad \text{------(1.20)}$$

(because $f(x)=0 \Rightarrow a_0x+a1=0 \Rightarrow x = -\frac{a_1}{a_0}$)

$$\text{(i.e.,)} \quad x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}} . f_k \qquad \text{------(1.21)}$$

This method is called **secant** or the **chord method.**

Geometrically, in the secant method, we replace the function f(x) by a straight line or a chord passing through the points (x_{k-1}, f_{k-1}) and (x_k, f_k). Take the point of intersection of the striaght line with the x-axis as the next approximation to the root. If the approximations are such that $f_{k-1}f_k<0$ then the secant method is called **Refula-Falsi method**. This method be shown graphically as follows :



Example E. 1.2 :

Using secant method, find the roots of $x^4-x-10=0$ correct to three decimal places.

Solution :

$$\text{Let } f(x) = x^4-x-10$$

$$\text{Now } f(0) = -10 < 0$$

$$f(1) = 1-1-10 = -10 < 0$$

$$f(2) = 16-2-10 = 4 > 0$$

Hence the smallest positive root lies in the interval (1, 2)

The $(k+1)^{th}$ approximation of f(x) is given by

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}}.f_k \qquad \text{for } k = 1,2,3,....... \text{ \& } f_k=f(x_k)$$

Here $x_0 = 1, x_1 = 2$ and $f_0 = -10, f_1 = 4$

$$\therefore x_2 = x_1 - \frac{x_1 - x_0}{f_1 - f_0} f_1$$

$$= 2 - \frac{2-1}{4-(-10)}(4)$$

$$= 1.71429$$

$$\cong 1.7143$$

$$\text{and } f_2 = f(x_2) = (1.7143)^4 - 1.7143 - 10$$

$$= 8.6367 - 1.7143 - 10$$

$$= -3.0776$$

$$\therefore x_3 = x_2 - \frac{x_2 - x_1}{f_2 - f_1} f_2$$

$$= 1.7143 - \frac{1.7143-2}{-3.0776-4}(-3.0776)$$

$$= 1.8385$$

$$\text{and } f_3(x_3) = f_3 = (1.8385)^4 - 1.8385 - 10$$

$$= -0.4135$$

$$\therefore x_4 = x_3 - \frac{x_3 - x_2}{f_3 - f_2} f_3$$

$$= 1.8385 - \frac{1.8385 - 1.7143}{-0.4135 - (-3.0776)}(-0.4135)$$

$$= 1.8578.$$

Similarly proceding above, we get,

$$x_5 = 1.8556 \text{ and}$$

$$x_6 = 1.8556$$

Now $x_5 = x_6$ for correct to three decimal spaces and therefore the required three decimal root is 1.856.

Example E. 1.3 :

Using secant method, find the root of $x-e^{-x} = 0$ correct to three decimal places.

Solution :

Let $f(x) = x-e^{-x}$.

Now $f(0) = -1 < 0$ and $f(1) = 0.6321 > 0$

$\therefore$ There is a root lies between 0 and 1

Take $x_0 = 0, x_1 = 1$ & $f_0 = -1, f_1 = 0.6321$

The approximated root is given by

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}} f_k, \ k=1,2,3,\ldots\ldots \qquad \text{------(1.22)}$$

Put k=1 in (1), we get,

$$x_2 = x_1 - \frac{x_1 - x_0}{f_1 - f_0} f_1$$

$$= 1 - \frac{1-0}{0.6321-(-1)}(0.6321)$$

$$= 0.6127$$

$$\therefore f_2 = f(x_2) = 0.6127 - e^{-0.6127}$$

$$= 0.0708$$

The next approximation is got by putting k = 2 in (1.22), we have,

$$x_3 = x_2 - \frac{x_2 - x_1}{f_2 - f_1} f_2$$

$$= 0.6127 - \frac{0.6127 - 1}{0.0708 - 0.6321}(0.0708)$$

$$= 0.5638$$

Similarly, we get, $x_4 = 0.5671$ & $x_5 = 0.5671$

$\therefore$ The aproximated root of $f(x) = 0$ is 0.567.

Exercise :

Use the secant method, find a root of the equation $\cos x - xe^x = 0$.

Example E. 1.4 :

Find a root of the equation $x^4 - x - 10 = 0$ upto three decimal places using Regula-Falsi Method.

Solution :

We know that from example E.1.2 (page 8), there is a root between 1 and 2.

$\therefore x_0 = 1, x_1 = 2$ and $f_0 = -10, f_1 = 4$.

Using Regula-Falsi method the approximation to the root be obtained from

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}} f_k \text{ and } f_{k-1} f_k < 0.$$

We have the following table, and the last column of table shows the root α lies in the interval.

x	f(x)	α belongs to the interval.
$x_0=1$	-10	
$x_1=2$	4	
$x_2 = 1.7143$	-3.0776	$(x_2, x_1) = (1.7143, 2)$
$x_3 = 1.8385$	-0.4135	$(x_3, x_1) = (1.8385, 2)$
$x_4 = 1.8536$	-0.0487	$(x_4, x_1) = (1.8536, 2)$
$x_5 = 1.8553$	-0.0070	$(x_5, x_1) = (1.8553, 2)$
$x_6 = 1.8556$		

$\therefore$ From the above table the root correct to three decimal places is 1.856.

1.6 Newton – Raphson Method :

Given that $f(x) = a_0x+a_1$ ------(1.23)

and k^{th} approxmiated root is x_k.

Differentiate (1.23) with respect to x, we get,

$$f^1(x) = a_0 \quad \text{------(1.24)}$$

$$\text{If } f(x) = 0 \text{ then } x = -\frac{a_1}{a_0} \quad \text{------(1.25)}$$

Since x_k is a root of (1.23) then $a_0x_k+a_1 = f_k$

(i.e.) $$x_k.f^1(x_k)+a_1 = f_k$$

$$\Rightarrow a_1 = f_k-x_kf^1{}_k \quad \text{------(1.26)}$$

$$\text{where } f^1{}_k = f^1(x_k)$$

From (1.24), (1.25) & (1.26), we have, k+1 approximation is

$$x_{k+1} = -\frac{a_1}{a_0}$$

$$= \frac{-(f_k - x_kf^1{}_k)}{f^1{}_k}$$

$$= \frac{x_kf^1{}_k - f_k}{f^1{}_k}$$

$$= x_k - \frac{f_k}{f^1{}_k}$$

$$\text{Thus } x_{k+1} = x_k - \frac{f_k}{f^1{}_k}$$

This method is called **Newton-Raphson** method.

Example E. 1.5 :

Apply Newton-Raphson method to determine a root of the equation $3x-\cos x-1=0$.

Solution :

$$\text{Let } f(x) = 3x-\cos x-1 = 0$$

$$\text{Now } f(0) = -2<0$$

$$f(1) = 3-0.5403-1 = 1.4597 > 0$$

$\therefore$ a root of $f(x) = 0$ lies between 0 and 1.

Let us take $x_0 = 0.6$

(Note : One can select any value to x_0)

$$\text{Now } f(x) = 3x - \cos x - 1$$

$$\therefore f(x) = 3 + \sin x$$

By Newton-Raphson method, we have,

$$x_{k+1} = x_k - \frac{f_k}{f^1{}_k} \quad \text{for } k = 1, 2, 3\ldots\ldots$$

$$\begin{aligned}
\text{Now } f_0 = f(x_0) &= f(0.6) \\
&= 3(0.6) - \cos(0.6) - 1 \\
&= 1.8 - 0.825336 - 1 \\
&= -0.025336 \\
\text{and } f_0{}^1 = f^1(x_0) &= f^1(0.6) \\
&= 3 + \sin(0.6) \\
&= 3 + 0.564642 \\
&= 3.564642 \\
\therefore x_1 &= x_0 - \frac{f_0}{f^1{}_0} \\
&= 0.6 - \frac{(-0.025336)}{3.564642} \\
&= 0.607108 \\
\text{Now } f(x_1) = f_1 &= 3(0.607108) - \cos(0.607108) - 1 \\
&= 0.000023 \\
\text{and } f^1(x_1) &= 3 + \sin(0.607108) \\
&= 3.570495 \\
\text{Hence } x_2 &= x_1 - \frac{f_1}{f^1{}_1} \\
&= 0.607108 - \frac{0.000023}{3.570495} \\
&= 0.607102
\end{aligned}$$

Clearly $x_1 = x_2$ upto four decimal places and therefore the required root is 0.6071.

Example E. 1.6 :

Using Newton-Rephson method, establish the formula $x_{k+1} = \frac{1}{2}\left(x_k + \frac{N}{x_k}\right)$ to calculate the square root of N. Hence find the square root of 6 correct to four places of decimals.

Solution :

$$\text{Let } x = \sqrt{N}$$

$$\therefore x^2 = N$$

$$\text{Let } f(x) = x^2 - N$$

$$\text{Thus } f^1(x) = 2x$$

By Newton-Raphson method, the (k+1) the iterate is given by

$$x_{k+1} = x_k - \frac{f_k}{f^1_k} \text{ where } f_k = f(x_k)$$

$$= x_k - \frac{x^2_k - N}{2x_k}$$

$$= \frac{x^2_k + N}{2x_k} = \frac{1}{2}\left(x_k + \frac{N}{x_k}\right),\ k = 0,1,2,3,\ldots..$$

Now N = 6. Take $x_0 = 2$

$$\therefore x_1 = \frac{1}{2}\left(x_0 + \frac{N}{x_0}\right)$$

$$= \frac{1}{2}\left(2 + \frac{6}{2}\right)$$

$$= \frac{1}{2}(2+3)$$

$$= 2.5$$

$$x_2 = \frac{1}{2}\left(2.5 + \frac{6}{2.5}\right)$$

$$= 2.45$$

$$x_3 = \frac{1}{2}\left(2.45 + \frac{6}{2.45}\right)$$

$$= 2.449490$$

$$x_4 = 2.449489743$$

Now x_3 & x_4 are approximately equal to 2.4495 upto four decimal places.

(Note that the correct value of square root of 6 upto four decimal places is 2.4495.)

Example E. 1.7 :

Show that the initial approximation x_0 for finding 1/N, where N is a positive integer, by the Newton-Raphson method must satisfy $0<x_0<2/N$, for convergence.

Solution :

$$\text{Let } x = \frac{1}{N}$$

$$\therefore 1/x = N$$

$$\text{Let } f(x) = \frac{1}{x} - N$$

$$\text{Thus } f^1(x) = \frac{-1}{x^2}$$

If x_k is the kth iterates then by Newton-Raphson method, we have,

$$x_{k+1} = x_k - \frac{f_k}{f^1_k} \quad \text{where } f_k = f(x_k)$$

$$\text{(i.e.,) } x_{k+1} = x_k - \frac{\frac{1}{x_k} - N}{\frac{-1}{x^2_k}}$$

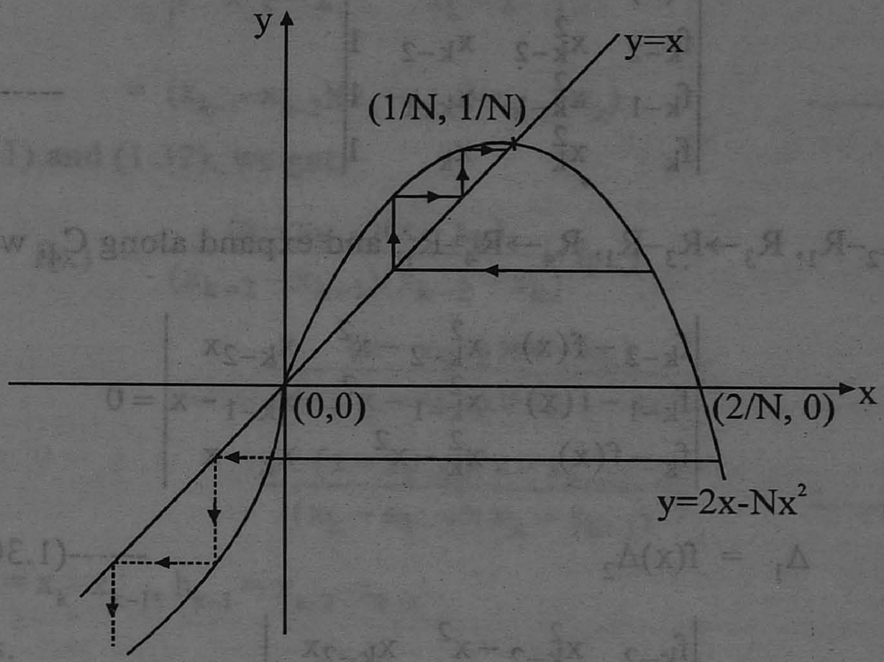
$$= x_k - \frac{\frac{1 - Nx_k}{x_k}}{\frac{-1}{x^2_k}}$$

$$= x_k + x_k(1 - Nx_k)$$

$$= 2x^2_k - Nx^2_k$$

Let us take y = x and $y = 2x - Nx^2$

Plot the above equations as curve in the graph, we have,



Clearly the point of intersection of the curves is (1/N, 1/N).

It is easy to find from the figure that any initial approximation outside the range $0<x_0<2/N$ diverges.

If $x_0 = 0$, the iteration does not converge to 1/N but remains zero always. This shows the importance of choosing a suitable initial approximation.

Iteration Methods Based on Second Degree Equation

In the previous section we assumed the linear equation and now we shall discuss the function f(x) is of second degree polynormial.

Let $f(x) = a_0x^2+a_1x+a_2$ and $a_0 \neq 0$. Where a_0, a_1 and a_2 are arbitrary parameters to be determined.

1.7 Muller Method :

If x_{k-2}, x_{k-1}, x_k be three approximations to the ξ of

$$f(x) = a_0x^2+a_1x+a_2 = 0 \qquad \text{------(1.27)}$$

Then

$$\left.\begin{aligned} f_{k-2} &= a_0x^2_{k-1}+a_1x_{k-2}+a_2 \\ f_{k-1} &= a_0x_{k-1}+a_1x_{k-1}+a_2 \\ f_k &= a_0x_k+a_1x_k+a_2 \end{aligned}\right\} \qquad \text{------(1.28)}$$

Eliminating a_0, a_1, a_2 from (1.27) and (1.28), we get

$$\begin{vmatrix} f(x) & x^2 & x & 1 \\ f_{k-2} & x_{k-2}^2 & x_{k-2} & 1 \\ f_{k-1} & x_{k-1}^2 & x_{k-1} & 1 \\ f_k & x_k^2 & x_k & 1 \end{vmatrix} \qquad \text{------(1.29)}$$

Applying $R_1 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$, and expand along C_4, we get

$$\begin{vmatrix} f_{k-2} - f(x) & x_{k-2}^2 - x^2 & x_{k-2}x \\ f_{k-1} - f(x) & x_{k-1}^2 - x^2 & x_{k-1} - x \\ f_k - f(x) & x_k^2 - x^2 & x_k - x \end{vmatrix} = 0$$

(i.e.,) $\qquad \Delta_1 = f(x)\Delta_2 \qquad$ ------(1.30)

$$\text{where } \Delta_1 = \begin{vmatrix} f_{k-2} & x_{k-2}^2 - x^2 & x_{k-2}x \\ f_{k-1} & x_{k-1}^2 - x^2 & x_{k-1} - x \\ f_x & x_k - x^2 & x_k - x \end{vmatrix}$$

$$\text{and } \Delta_2 = f(x)\begin{vmatrix} 1 & x_{k-1}^2 - x^2 & x_{k-2} - x \\ 1 & x_{k-1}^2 - x^2 & x_{k-1} - x \\ 1 & x_k^2 - x^2 & x_k - x \end{vmatrix}$$

$$\begin{aligned} \text{Now } \Delta_1 &= f_{k-2}[(x^2_{k-1}-x^2)(x_k-x)-(x^2_k-x^2)(x_{k-1}-x)] \\ &\quad -f_{k-1}[(x^2_{k-2}-x^2)(x_k-x)-(x^2_k-x^2)(x_{k-2}-x)] \\ &\quad +f_k[(x^2_{k-2}-x^2)(x_{k-1}-x)-(x^2_{k-1}-x^2)(x_{k-2}-x)] \\ &= f_{k-2}(x_{k-1}-x)(x_k-x)[x_{k-1}-x-x_k+x] \\ &\quad -f_{k-1}(x_{k-2}-x)(x_k-x)[x_{k-2}-x-x_k+x] \\ &\quad +f_k(x_{k-2}-x)(x_{k-1}-x)[x_{k-2}-x-x_{k-1}+x] \\ &= f_{k-2}(x_{k-1}-x)(x_k-x)(x_{k-1}-x_k) \\ &\quad -f_{k-1}(x_{k-2}-x)(x_k-x)(x_{k-2}-x_k) \\ &\quad +f_k(x_{k-2}-x)(x_{k-1}-x)(x_{k-2}-x_{k-1}) \qquad \text{-------(1.31)} \end{aligned}$$

$$\text{and } \Delta_2 = \begin{vmatrix} 1 & x^2_{k-2}-x^2 & x_{k-2}-x \\ 1 & x^2_{k-1}-x^2 & x_{k-1}-x \\ 1 & x^2_k - x^2 & x_k - x \end{vmatrix}$$

$$= (x_{k-1}-x_{k-2})(x_k-x_{k-2})(x_{k-1}-x_k) \qquad \text{------}(1.32)$$

From (1.30), (1.31) and (1.32), we get,

$$f(x) = \frac{(x-x_{k-1})(x-x_k)}{(x_{k-2}-x_{k-1})(x_{k-2}-x_k)} f_{k-2} + \frac{(x-x_{k-2})(x-x_k)}{(x_{k-1}-x_{k-2})(x_{k-1}-x_k)} f_k + \frac{(x-x_{k-2})(x-x_{k-1})}{(x_k-x_{k-2})(x_k-x_{k-1})} f_k \qquad \text{------}(1.33)$$

Take $h = x-x_k$, $h_k = x_k-x_{k-1}$, $h_{k-1} = x_{k-1}-x_{k-2}$

$\therefore$ (1.33) becomes,

$$f(x) = \frac{h(h+h_k)}{h_{k-1}(h_{k-1}+h_k)} f_{k-2} - \frac{h(h+h_k+h_{k-1})}{h_k h_{k-1}} f_{k-1} + \frac{(h+h_k)(h+h_k+h_{k-1})}{h_k(h_k+h_{k-1})} f_k = 0 \qquad \text{-----}(1.34)$$

Again, let $\lambda = h/h_k$, $\lambda_k = h_k/h_{k-1}$ and $\delta_k = 1+\lambda_k$

Hence (1.34) becomes,

$$\lambda^2 C_k + \lambda g_k + \delta_k f_k = 0 \qquad \text{-------}(1.35)$$

$$\text{where } g_k = \lambda^2_k f_{k-2} - \delta^2_k f_{k-1} + (\lambda_k+\delta_k) f_k$$

$$\text{and } C_k = \lambda_k(\lambda_k f_{k-2} - \delta_k f_{k-1} + f_k)$$

$$\text{Now (1.35)} \Rightarrow \delta_k f_k\left(\frac{1}{\lambda^2}\right) + \frac{g_k}{\lambda} + C_k = 0$$

$$\Rightarrow \delta_k f_k\left(\frac{1}{\lambda}\right)^2 + g_k\left(\frac{1}{\lambda}\right) + C_k = 0$$

which is quadratic in $1/\lambda$ and therefore we have,

$$\lambda = \frac{-2\delta_k f_k}{g_k \pm \sqrt{g_k^2 - 4\delta_k f_k C_k}} = \lambda_{k+1}(\text{say}) \qquad \text{------}(1.36)$$

The sign in the denominator in (10) is so chosen that λ_{k+1} has the smallest absolute value.

$$\therefore \lambda_{k+1} = \frac{x - x_k}{x_k - x_{k-1}}$$

$$\Rightarrow x = x_k + (x_k - x_{k-1})\lambda_{k+1} \qquad \text{------(1.37)}$$

Replacing x by x_{k+1}, we get,

$$x_{k+1} = x_k + (x_k - x_{k-1})\lambda_{k+1} \qquad \text{------(1.38)}$$

which is called the **MULLER** method.

Note :

1) Muller method converges for all initial approximations.

2) If no better approximations are known then take $x_0 = -1$, $x_1 = 0$, $x_2 = 1$.

Example E. 1.7 :

Perform 2 iterations with the MULLER method to the following equations.

$x^3 - ½ = 0$, $x_0 = 0$, $x_1 = 1$, $x^2 = ½$

Solution :

Let $f(x) = x^3 - ½$

Iteration 1 :

Given that $x_0 = 0$, $x_1 = 1$, $x_2 = ½$

$$\therefore f_0 = f(x_0) = f(0) = -½ = -0.5$$

$$\text{Similarly } f_1 = 0.5,$$

$$f_2 = -0.375$$

$$\therefore h_2 = x_2 - x_1 = -0.5,$$

$$h_1 = x_1 - x_0 = 1.0$$

$$\lambda_2 = h_2/h_1 = -0.5,$$

$$\delta_2 = 1 + \lambda_2 = 0.5$$

$$\text{and } g_2 = \lambda^2{}_2 f_0 - \lambda^2{}_2 f_1 + (\delta_2 + \lambda_2) f_2 = -0.25$$

$$C_2 = \lambda_2(\lambda_2 f_0 - \delta_2 f_1 + f_2) = 0.1875$$

$$\text{and } \lambda_3 = \frac{-2\delta_2 f_2}{g_2 \pm \sqrt{g_2^2 - 4\delta_2 f_2 C_2}}$$

Taking negative sign in the denominator, we get,

$$\lambda_3 = -0.5352$$

$$\therefore x_3 = x_2+(x_2-x_1)\lambda_3 = 0.7676.$$

Iteration 2 :

Now	$x_0 = 1,$	$x_1 = 0.5,$	$x_2=0.7676$
$\therefore$	$f_0 = 0.5,$	$f_1 = -0.375,$	$f_2 = 0.0477$
	$h_2 = 0.2676,$	$h_1 = -0.5,$	$\lambda_2 = -0.5352$
	$\delta_2 = 0.4647,$	$g_2 = 0.2276$	
	$C_2 = 0.0755,$	$\lambda_3 = 0.0945$	
and $\therefore$	$x_3 = 0.7929$		

1.8 Chebyshev Method :

$$\text{Let } f(x) = a_0x^2+a_1x+a_2 = 0 \text{ and } a_0 \neq 0 \quad \text{------(1.36)}$$

$$\therefore f_k = f(x_k) = a_0x^2_k+a_1x_k+a_2 \quad \text{------(1.37)}$$

$$f'_k = f'(x_k) = f'(x_k) = 2a_0x_k+a_1 \quad \text{------(1.38)}$$

$$f''_k = 2a_0 \quad \text{------(1.39)}$$

Eliminating a_0, a_1, a_2 from (1.36), (1.37), (1.38) & (1.39) we get,

$$f_k+(x-x_k)f'_k+½(x-x_k)f''_k = 0 \quad \text{------(1.40)}$$

Which is the Taylor series expansion of f(x) about $x=x_k$ such that the terms of order $(x-x_k)^3$ & higher powers are omitted.

Since (1.40) is an quadratic equation and the next to approximate root is given by

$$x_{k+1}-x_k = \frac{-f_k}{f'_k}-\frac{1}{2}(x_{k+1}-x_k)^2\frac{f''_k}{f'_k} \quad \text{------(1.41)}$$

We know that $x_{k+1} = x_k-\frac{f_k}{f^1_k}(\because f_k = a_0x_k+a_1 \;\&\; f'_k = a_1)$

$\therefore$ (1.41) becomes, $\boxed{x_{k+1} = x_k-\frac{f_k}{f^1_k}-\frac{1}{2}\;\frac{f^2_k}{f^{1^3}_1}f''_k}$ ------(1.42)

which is called the **CHEBYSHEV** method.

Example E. 1.8 :

Using Chebyshev method, find the root of $f(x) = \cos x - xe^x = 0$. Correct to four decimal places.

Solution :

$$\text{G.T. } f(x) = \cos x - xe^x$$

$$\therefore f'(x) = -\sin x - xe^x - e^x$$

$$\text{and } f''(x) = -\cos x - xe^x - e^x - e^x$$

$$= -\cos x - xe^x - 2e^x.$$

Now the $(k+1)^{th}$ approximated root of f(x) is given by

$$x_{k+1} = x_k - \frac{f_k}{f'_k} - \frac{1}{2}\frac{f^2_k}{f'^3_k} f''_k, \quad k = 0,1,2,.... \quad \text{------(1.42a)}$$

$$\text{Take } x_0 = 1.$$

$$\therefore f_0 = f(x_0) = f(1) = \cos 1 - 1e^1 = -2.177980$$

$$f'_0 = f^1(x_0) = -\sin(1) - e^1 - e^1 = -6.278035$$

$$f''_0 = -3.258584$$

Step 1 :

Put k = 0 in (1), we get,

$$x_1 = x_0 - \frac{f_0}{f'_0} - \frac{1}{2}\frac{f^2_0}{f'^3_0} f''_0$$

$$= 1 - \frac{(-2.177980)}{-6.278035} - \frac{1}{2}\frac{(-2.177980)^2}{(-6.278035)^3}(-3.258584)$$

$$= 1 - 0.346921 - 0.083346$$

$$= 0.569733$$

Step 2 :

$$\text{Now } x_1 = 0.569733$$

$$\therefore f_1 = f(x_1) = \cos x - xe^x$$

$$= -0.165126$$

$$f'_1 = -\sin x - xe^x - e^x$$

$$= -3.314373$$

$$f''_1 = -\cos x - xe^x - 2e^x$$
$$= -5.384806$$

Put x = 1 in (1), we get,

$$x_2 = x_1 - \frac{f_1}{f'_1} - \frac{1}{2}\frac{f_1^2}{f_1'^3} f''_1$$
$$= 0.569733 - 0.049821 - 0.002016$$
$$= 0.517896$$

Step 3 :

$$\text{Now } x_2 = 0.517896$$
$$\therefore f_2 = f(x_2) = 0.000421$$
$$f'_2 = -\sin x - xe^x - e^x$$
$$= -\sin x - (x+1)e^x$$
$$= -3.042828$$
$$f''_2 = -\cos x - xe^x - 2ex$$
$$= -\cos x - (x+2)e^x$$
$$= -5.095130$$

Put x = 2 in (1), we get,

$$x_3 = x_2 - \frac{f_2}{f'_2} - \frac{1}{2}\frac{f_2^2}{f_2'^3} f''_2$$
$$= 0.517896 - 0.000138 - 0.000000016$$
$$= 0.517758$$

Step 4 :

$$\text{Now } x_3 = 0.517758$$
$$\text{and } f_3 = \cos x - xe^x$$
$$= -0.000001935757$$
$$\text{and } f'_3 = -\sin x - (x+1)e^x$$
$$= -3.042126947$$
$$\text{and } f''_3 = -\cos x - (x+2)e^x$$
$$= -5.094385478$$

Put k = 3 in (1), we get,

$$x_4 = x_3 - \frac{f_3}{f'_3} - \frac{1}{2}\frac{f_3^2}{f_3'^3} f''_3$$

$$= 0.517758 - 0.0000006363169 - 0.3390248\times10^{-13}$$

$$= 0.517758636$$

Clearly from step 3 & step 4,

$$x_3 = x_4 \cong 0.5178$$

which is the required root upto four decimal spaces.

1.9 Multipoint Iteration Method :

From the derivation Chebyshev method, we have,

$$f_k+(x-x_k)f'_k+½(x-x_k)^2f''_k = 0$$

(i.e.,) $(x-x_k)[f'_k+½(x-x_k)f''_k] = -f_k$

$$\text{(i.e.,) } x-x_k = \frac{-f_k}{f'_k+\frac{1}{2}(x-x_k)f''_k}$$

$$\cong \frac{-f_k}{f'\left[x_k+\frac{1}{2}(x-x_k)\right]} \qquad \text{------(1.42b)}$$

Again replacing $x_{k+1}-x_k$ by in $-f_k/f'_k$ in (1), we get,

$$x_{k+1} = \frac{f_k}{f'\left(x_k-\frac{1}{2}\frac{f_k}{f'_k}\right)},\ k= 0,1,2,.......... \qquad \text{------(1.43)}$$

Here (1.43) is called the root of he equation $f(x) = a_0x^2+a_1x+a_2$ by MULTIPOINT ITERATION method.

Note :

Now $f_k+(x-x_k)f'_k+½(x-x_k)^2f''_k = 0$ can also be written as

$$x_{k+1}-x_k = \frac{1}{f'_k}(f(x_k+(x_{k+1}-x_k))-(x_{k+1}-x_k)f'_k) \qquad \text{------(1.44)}$$

Replacing $x_{k+1}-x_k$ by $-f_k/f'_k$ in (1.44), we get

$$x_{k+1} = x_k - \frac{f_k}{f'_k} - \frac{f\left(x_k - \frac{f_k}{f'_k}\right)}{f'_k}$$

$$\text{(i.e.) } x_{k+1} = x^*_{k+1} - \frac{f^*_{k+1}}{f'_k} \quad \text{------(1.45)}$$

$$\text{where } x^*_{k+1} = x_k - \frac{f_k}{f'_k} \text{ and } f^*_{k+1} = f(x^*_{k+1})$$

Here (1.45) can also be used as the approximation to the root of f(x) = 0 and it also called **multipoint iteration method.**

Example E. 1.9 :

Perform three iterations of the multipoint iteration method, to find the root of the equation $f(x) = \cos x - xe^x = 0$.

Solution :

Take $x_0 = 1$.

We shall tabulate the calculations as follows :

k	x_k	x^*_{k+1}	x_{k+1}	$f(x_{k+1})$
0	1.0	0.6530794	0.5797057	–0.1984
1	0.5797057	0.5207904	0.5180442	-0.8726×10^{-3}
2	0.5180442	0.5177574	0.5177573	-0.1006×10^{-9}
3	0.5177573	0.5177573	0.5177573	0.1388×10^{-16}

1.10 Rate of Convergence :

Definition D. 1.4 :

An iterative method is said to be of order p or has the rate of convergence p, if p is the largest positive real number for which there exists a finite constant $c(\neq 0)$ such that

$$|\epsilon_{k+1}| \leq C|\epsilon_k|^p \quad \text{------ (1.46)}$$

where $\epsilon_k = x_k - \xi$ is the error in the k^{th} iterate.

Here the constant C is called the asymmpptotic error.

Example E. 1.10 :

Find the rate of convergence of secant method.

Solution :

Let ξ be a simple root of $f(x) = 0$ and $x_k = \xi+\epsilon_k$ ------(1.47)

We know that the $(k+1)^{th}$ iterative value of the root of a function under secant method is

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}} f_k \quad \text{------(1.48)}$$

$\therefore$ From (1.47) and (1.48), we get,

$$\epsilon_{k+1} = \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1}) f(\xi+\epsilon_k)}{f(\xi+\epsilon_k) - f(\xi+\epsilon_{k-1})} \quad \text{------(1.49)}$$

Expanding $f(\xi+\epsilon_k)$ and $f(\xi+\epsilon_{k-1})$ in Taylor's series about the p+ξ, we get

$$\epsilon_{k+1} = \epsilon_k - \left[\epsilon_k + \frac{1}{2}\epsilon^2_k \frac{f''(\xi)}{f'(\xi)} + ...\right].$$

$$\left[1+\frac{1}{2}(\epsilon_{k-1}+\epsilon_k)\frac{f''(\xi)}{f'(\xi)}+......\right]^{-1} \quad (\because f(\zeta)=0)$$

$$\text{(i.e.)} \quad \epsilon_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\epsilon_k\epsilon_{k-1} + 0\left(\epsilon^2_k\epsilon_{k-1} + \epsilon_k\epsilon^2_{k-1}\right)$$

$$\text{(i.e)} \quad \epsilon_{k+1} = C\epsilon_k\epsilon_{k-1} \quad \text{------(1.49a)}$$

where $C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$ and higher powers of ϵ_k are neglected.

Now we shall find the values of A & p such that $\epsilon_{k+1} = A\epsilon^p_k$ ------(1.50)

From (1.50), $\epsilon_k = A\epsilon^p_{k-1}$

$$\Rightarrow \epsilon_{k-1} = A^{-1/p}\,\epsilon_k^{1/p}$$

$$\therefore \text{(1.49a)} \quad \Rightarrow \epsilon^p_k = CA^{-\left(1+\frac{1}{p}\right)}.\epsilon_k^{1+\frac{1}{P}} \quad \text{------(1.51)}$$

Comparing the powers of ϵ_k on both sides, we get, $p = 1 + \frac{1}{p}$

$$\Rightarrow p = \frac{1}{2}(1+\sqrt{5})$$

Omitting negative sign, the rate of convergence for the secant method is p = 1.618.

From (1.51), $A = \frac{p}{C^{p+1}}$

Example E. 1.11 :

Prove that Newton-Raphson method has second order convergence.

Proof :

Put $x_k = \xi+\epsilon_k$ in $x_{k+1} = x_k - \frac{f_k}{f''_k}$ and expanding $f(\xi+\epsilon_k)$ and $f'(\xi+\epsilon_k)$ in Taylor's series about the point the point ξ, we get

$$\epsilon_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}.\epsilon^2_k + O(\epsilon^3_k)$$

On neglecting ϵ^3_k and higher powers of ϵ_k, we get

$$\epsilon_{k+1} = C\epsilon_k \qquad \text{------(1.51a)}$$

$$\text{where } C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$$

Thus the Newton-Raphson method has second order convergence.

1.11 Iteration Methods :

We know that the iterative equation is given by $x_{k+1} = \phi(x_k)$ ------(1.52)

k=0,1,2,........

Here the function $\phi(x_k)$ is called **iteration function.**

If x_k converges to the value ξ then $\xi = \phi(\xi)$ ------(1.53)

$\therefore$ (1.53)–(1.52) $\Rightarrow x_{k+1}-\xi = \phi(x_k)-\phi(\xi)$ ------(1.54)

Put $x_k = \epsilon_k+\xi$, $a1 = \phi^1(\xi)$, $a_2 = ½\phi''(\xi)$

$\therefore$ (1.54) $\Rightarrow \epsilon_{k+1} = a_1\epsilon_k + a_2\epsilon^2_k + O(\epsilon^3_k)$ ------(1.55)

k = 0, 1, 2,......

We shall discuss different cases.

1.11.1 First Order Method :

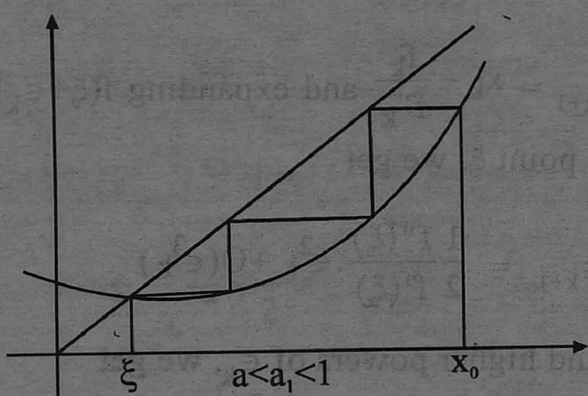
Let $a_1 \neq 0$ and (1.55) becomes $\epsilon_{k+1} = a_1 \epsilon_k$ ------(1.55a)

$k=0,1,2,.......$

Clearly from (1.55a), $\epsilon_k = a^k_1 \epsilon_0$

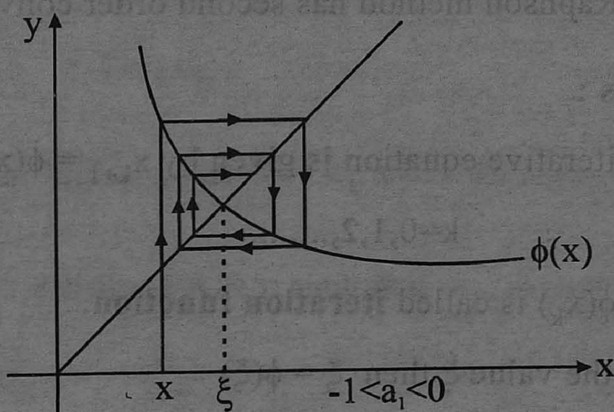
If $|a_1|<1$ and ϵ_0 is not very large then the iteration method (1.52) converges.

If x_0 is in the neighbourhood of the root ξ and $a_1>0$ then a convergent iteration method provides a staircase solution which is shown in the following figure.



(Stair case solution)

and if $a_1<0$ then convergent iteration method provides a spiral solution which is shown in the following figure.



The following theorem gives the sufficient condition for the convergence of $\phi(x)$.

Theorem :

If $\phi(x)$ is a continuous function in some interval [a, b] that contains the root and $|\phi'(x)| \leq C < 1$ in this interval, then for any choice of $x_0 \in [a, b]$, the sequence $\{x_k\}$ determined from $x_{k+1} = \phi(x_k)$, $k=0,1,2,3.........$ converges to the root ξ of $x = \phi(x)$.

Proof :

If ξ is a root of $x = \phi(x)$ then $\xi = \phi(\xi)$

$$\therefore \quad \xi - x_{k+1} = \phi(\xi) - \phi(x_k); \; k = 0,1,2,3,........$$

$\therefore$ By mean value theorem,

$$\xi - x_{k+1} = (\xi - x_k)\phi'(\xi_k); \; x_k < \xi_k < \xi$$

$$\begin{aligned} \Rightarrow \epsilon_{k+1} &= \epsilon_k \phi'(\xi_k) \text{ where } \epsilon_k = \xi - x_k \\ &= \epsilon_{k-1}\phi'(\xi_{k-1})\phi'(\xi_k) \\ &\ldots \\ &\ldots \\ &= \epsilon_0 \phi'(\xi_0)\phi'(\xi_1)........\phi'(\xi_k) \end{aligned}$$

where $x_i < \xi_i < \xi$, $i = 0,1,2,3,......k-1$.

If $|\phi'(\xi_r)| \leq C$, $r = 0,1,2,3,.......k$ then $|\epsilon_{k+1}| \leq |\epsilon_0| C^{k+1}$

If $C<1$ then R.H.S. tends to zero as $k \to \infty$.

$\therefore$ the iteration method converges to $|\phi'(x)| \leq C < 1$.

This proves the theorem.

1.11.2 Second Order Method :

Here $a_1 = 0$, $a_2 = 0$ then $\epsilon_{k+1} = a_1\epsilon_k + a_2\epsilon^2_k + O(\epsilon^3_k)$

becomes $\epsilon_{k+1} = a_2\epsilon^2_k$, $k = 0,1,2,.......$ ------(1.56)

The above equation (1.56) can be written as

$$\epsilon_k = a_2^{2^k - 1} . \epsilon_0^{2^k}$$

If $|a_2| \cong 1$, ϵ_0 is small, then the iteration method $x_{k+1} = \phi(x_k)$ converges and has second order convergence.

1.11.3 Higher Order Methods :

Definition D. 1.5 :

The iteration method $x_{k+1} = \phi(x_k)$ is said to be of the p^{th} order if

$\phi'(\xi) = \phi''(\xi) = ... = \phi^{(p-1)}(\xi) = 0$ and $\phi^{(p)}(\xi) \neq 0$ where ξ is the solution of $x = \phi(x)$.

1.12 Aitken Δ^2 Method :

The linear convergence of the iterative method $x_{k+1} = \phi(x_k)$ can be improved with the help of Aitken Δ^2 method.

The error in two successive approximation are $\epsilon_{k+1} = a_1 \epsilon_k$ & $\epsilon_{k+2} = a_1 \epsilon_{k+1}$

Eliminating a_1 from the above two equations, we get

$$\xi = \frac{x_k x_{k+2} - x_{k+1}^2}{x_{k+2} - 2x_{k+1} + x_k}$$

$$= x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k}$$

The above number ξ gives an improved value of the approximation x_{k+2}.

The computational procedure is given below :

Step 1 : Choose an initial approximation x_0

Step 2 : Calculate $x1 = \phi(x_0)$ and $x_2 = \phi(x_1)$

Step 3 : Determine the sequence $\{x_k\}$ as

$$x^*_{3k+2} = x_{3k} - \frac{(\phi(x_{3k}) - x_{3k})^2}{\phi(\phi(x_{3k})) - 2\phi(x_{3k}) + x_{3k}}, \quad k = 0,1,2,3,\ldots\ldots$$

Example E. 1.12 :

Find a real root of the equation $\cos x = 3x-1$ correct to 4 decimal places by iteration method.

Solution :

Let $f(x) = \cos x - 3x + 1$

$\therefore$ $f(0) = 2$

$f(1) = \cos 1 - 3 + 1 = -1.459698$

$\cong -1.45970$

$\therefore$ There is a root between 0 & 1.

Now the given equation can be written as

$$x = \frac{1}{3}(1 + \cos x)$$

$$= \phi(x) \text{ where } \phi(x) = \frac{1}{3}(1+\cos x)$$

Now. $$|\phi'(x)| = \left|\frac{1}{3}\sin x\right|$$

$$< 1 \text{ for all x.}$$

Hence the itertion method may be applied.

$$\text{Let } x_0 = 0.6$$

$$\therefore\ x_1 = \frac{1}{3}(\cos(0.6)+1) = 0.60845,$$

$$x_2 = 0.60684,$$

$$x_3 = 0.60715,$$

$$x_4 = 0.60709,$$

$$x_5 = 0.60710,$$

$$\text{and } x_6 = 0.60710$$

Thus required real root upto 4 decimal spaces is 0.60710.

Aitken's Δ^2 Method :

Use Aitken's Δ^2 method to solve the equation $2x = \cos x+3$ to three decimal places.

Solution :

$$\text{Let } f(x) = \cos x+3-2x = 0$$

The above function can be written as $x = ½(3+\cos x)$

$$= \phi(x) \text{ where } \phi(x) = ½(3+\cos x)$$

$$\therefore\ \phi'(x) = ½(-\sin x)$$

$$\&\ |\phi'(x)| = |½\sin x|<1$$

$\therefore$ we can apply Aitken's iteration method.

$$\text{Let } x_0 = 0,$$

$$x_1 = ½(3+\cos 0) = 2,$$

$$x_2 = ½(3+\cos 2) = 1.29193,$$

$$x_3 = ½(3+\cos(1.29193)) = 1.63763,$$

$$x_4 = \tfrac{1}{2}(3+\cos(1.63763)) = 1.46661,$$
$$x_5 = \tfrac{1}{2}(3+\cos(1.46661)) = 1.55200,$$
$$x_6 = \tfrac{1}{2}(3+\cos(1.55200)) = 1.50940,$$
$$x_7 = \tfrac{1}{2}(3+\cos(1.50940)) = 1.53068,$$
$$x_8 = \tfrac{1}{2}(3+\cos(1.53068)) = 1.52005,$$
$$x_9 = \tfrac{1}{2}(3+\cos(1.52005)) = 1.52536,$$
$$x_{10} = \tfrac{1}{2}(3+\cos(1.52536)) = 1.52271,$$
$$x_{11} = \tfrac{1}{2}(3+\cos(1.52271)) = 1.52403,$$
$$x_{12} = \tfrac{1}{2}(3+\cos(1.52403)) = 1.52337,$$
$$x_{13} = \tfrac{1}{2}(3+\cos(1.52337)) = 1.52370,$$
$$x_{14} = \tfrac{1}{2}(3+\cos(1.52370)) = 1.52354,$$
$$\text{and } x_{15} = \tfrac{1}{2}(3+\cos(1.52354)) = 1.52362$$

Clearly from above calculations.

$x_{13} = x_{14} \cong 1.524$ (approximate to 3 decimals)

$\therefore$ Required root upto 3 decimal places is 1.524.

1.13 Methods for Multiple Roots :

If ξ is a multiple root of multiplicty m of $f(x) = 0$ then

$$f(\xi) = f'(\xi) = \ldots\ldots = f^{(m-1)}(\xi) = 0 \text{ and } f^{(m)}(\xi) \neq 0 \qquad \text{------(1.56)}$$

If the multiplicity of the root is not known in advance, then we can apply the following procedure.

1.13.1 Using Newton-Raphson Method :

$$\text{Take } g(x) = \frac{f(x)}{f'(x)} \qquad \text{------(1.56a)}$$

Clearly $g(x)$ has a simple root ξ.

$\therefore$ Newton-Raphson method gives as

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)} \qquad \text{------(1.57)}$$

$$\text{From (1), } g'(x) = \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2}$$

$$\therefore (2) \Rightarrow x_{k+1} = x_k - \frac{\dfrac{f(x_k)}{f'(x_k)}}{\dfrac{(f'(x_k))^2 - f(x)f''(x_k)}{(f'(x_k))^2}}$$

$$= x_k - \frac{f(x_k)f'(x_k)}{(f'(x_k))^2 - f(x_k)f''(x_k)}$$

$$\text{(i.e.) } x_{k+1} = x_k - \frac{f_k f'_k}{(f'_k)^2 - f_k f''_k} \qquad \text{------(1.58)}$$

Similarly using secant method, we get

$$x_{k+1} = \frac{x_{k-1}f_k f'_{k-1} - x_k f_{k-1} f'_k}{f_k f'_{k-1} - f_{k-1} f'_k} \qquad \text{------(1.59)}$$

Note :

In (1.59), we can able to eliminate the derivative by assuming

$$g(x) \cong G(x) = \frac{-f^2(x)}{f(x - f(x)) - f(x)}$$

$$\text{Thus (1.59)} \Rightarrow x_2 = x_1 - (x_0 - x_1)\frac{G_1}{G_0 - G_1} \qquad \text{------(1.60)}$$

where $G_1 = G(x_1)$ & $G_0 = G(x_0)$

Example E. 1.13 :

Apply the Newton-Raphson method with $x_0 = 0.8$ to the equation $f(x) = x^3 - x^2 - x + 1 = 0$ and verify that the convergence is only of first order in each case. Then apply the Newton-Raphson method $x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$ with $m = 2$ and verify that the convergence is of second order.

Solution :

Given that $f(x) = x^3 - x^2 - x + 1$ is a given equation

$\therefore f'(x) = 3x^2 - 2x - 1$

Using Newton–Raghson method, we have,

$$x_{n+1} = x_n - \frac{f_n}{f'_n}, \quad n = 0,1,2,3,........$$

$$\text{(i.e.,) } x_{n+1} = x_n - \frac{x^3{}_n - x^2{}_n - x_n + 1}{3x^2{}_n - 2x_n - 1}; \; n = 0,1,2,..........$$

$$= \frac{3x^3{}_n - 2x^2{}_n - 1 - x^3{}_n - x^2{}_n + x_n - 1}{3x^2{}_n - 2x_n - 1}$$

$$= \frac{2x^3{}_n - x^2{}_n + x_n - 2}{3x^2{}_n - 2x_n - 1}; \; n=0,1,2,......$$

Given that $x_0 = 0.8$

$\therefore x_1 = 0.905882$

$x_2 = 0.954132$

$x_3 = 0.977338$

$x_4 = 0.988734$

Since exact root is 1 and therefore we have,

$|\epsilon_0| = |\xi - x_0| = 0.2 = 0.2 \times 10^0$

$|\epsilon_1| = |\xi - x_1| = 0.094118 = 0.94 \times 10^{-1}$

$|\epsilon_2| = |\xi - x_2| = 0.045868 = 0.46 \times 10^{-1}$

$|\epsilon_3| = |\xi - x_3| = 0.22 \times 10^{-1}$

$|\epsilon_4| = |\xi - x_4| = 0.11 \times 10^{-1}$

Hence the rate of convergence is linear.

Using the modified Newton-Raphson Method :

Given that $x_{n+1} = x_n - m\dfrac{f(x_n)}{f'(x_n)}, \quad n = 0,1,2,.......$

If m = 2 then $x_{n+1} = x_n - \dfrac{2f(x_n)}{f'(x_n)}; \quad n = 0,1,2,.......$

Given that $x_0 = 0.8$

$$\therefore x_1 = 1.011765$$

$$x_2 = 1.000034$$

$$x_3 = 1.000000074$$

Thus $|\epsilon_0| = |\xi - x_0| = 0.2 = 0.2\times10^0$

$$|\epsilon_1| = |\xi - x_1| = 0.12 \times 10^{-1}$$

$$|\epsilon_2| = |\xi - x_2| = 0.34 \times 10^{-4}$$

$$|\epsilon_3| = |\xi - x_3| = 0.74 \times 10^{-7}$$

which show second order convergence.

Exercise :

Find the multiple root of the equation $27x^5+27x^4+36x^3+28x^2+9x+1=0$ using (i) Newton-Raphson method & (ii) Secant method.

1.14 Methods for Complex Roots :

Consider $f(z) = 0$, $z = x+iy$ be a complex equation.

Let $f(z) = u(x, y)+iv(x, y)$ ------(1.61)

where $u(x, y)$, $v(x, y)$ are the real & imaginary parts of $f(z)$ respectively.

We know that C.R. equations for $f(z) = 0$ are

$$u_x = v_y \ \& \ v_x = -u_y \qquad \text{------(1.62)}$$

Using (1.62), the $(k+1)^{th}$ approximate values are given by

$$x_{k+1} = x_k - \frac{u(x_k,y_k)u_x(x_k,y_k)+v(x_k,y_k).v_x(x_k,y_k)}{u^2{}_x(x_k,y_k)+v^2{}_x(x_k,y_k)} \qquad \text{------(1.62)}$$

$$k=0,1,2,\ldots\ldots\ldots$$

$$\text{and } y_{k+1} = y_k - \frac{u(x_k,y_k)u_x(x_k,y_k)-v(x_k,y_k).v_x(x_k,y_k)}{u^2{}_x(x_k,y_k)+v^2{}_x(x_k,y_k)} \qquad \text{------(1.62)}$$

$$k=0,1,2,3,\ldots\ldots\ldots$$

(i.e.,) Newton-Raphson metod of $f(z) = 0$ is given by

$$z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)};\ k = 0,1,2,\ldots\ldots\ldots$$

Note : The initial approximation z_o **must be a complex number.**

Example E. 1.14 :

Perform three iterations of the Newton-Raphson method for solving the equation $1+z^2=0$ with $z_0 = \frac{1+i}{z}$.

Solution :

$$\text{Given that } f(z) = 1+z^2 \text{ with } z_0 = \frac{1+i}{z}$$

$$\text{Putting } f(z) = u(x,y)+iv(x, y) \text{ then}$$

$$u(x, y) = 1+x^2-y^2 \;\&$$

$$v(x, y) = 2xy$$

and $x_0 = ½$, $y_0 = ½$

Now $u_x = 2x$, $v_y = -2y$, $v_x = 2y$, $v_y = 2x$.

Thus using (1.62), we get the following approximations.

$u_0 = 1.0$	$v_0 = 0.5$
$x_1 = -0.25$	$y_1 = 0.75$
$u_1 = 0.5$	$v_1 = -0.375$
$x_2 = 0.075$	$y_2 = 0.975$
$u_2 = 0.055$	$v_2 = 0.14625$
$x_3 = 0.00172$	$y_3 = 0.9973$

1.15 Polynomial Equations :

In this section, for the polynomial $p_n(x)$, we want to know

(i) the exact number of real and complex roots along with there multiplicity and

(ii) the interval in which each root lies.

Definition D. 1.6 :

When in a polynomial (terms being written in order) a positive sign follows positive sign or a negative sign follows a negative sign, a continuation or a permanence of sign is said to occur. Otherwise a change in sign is said to occur.

1.16 Descartes' Rule of Signs :

The number of positive roots of a polynomial $P_n(x)=0$ cannot exceed the number of sign changes in $P_n(x)$ and the number of negative real roots of $P_n(x)=0$ cannot exceed the number of sign changes in $P_n(-x)$.

Note that desearte's rule of sign gives the upper bounds of positive or negative roots but not the exact number of real roots. In order get exact number of real roots we have to use sturm's theorem.

Definition D. 1.7 :

(Sturm functions or sturm sequence).

Lef f(x) be a given polynomial of degree n.

Let $f_1(x) = f'(x)$ and $f_2(x)$ be the remainder of f(x) divided by $f_1(x)$ taken with the reverse sign and $f_3(x)$ for the remainder of $f_1(x)$ divided by $f_2(x)$ with the reverse sign and so on, until a constant is arrived. We a sequence of function $f(x), f_1(x), f_2(x),.....f_n(x)$ called sturm functions or sturm sequence.

Sturm Theorem (without proof) :

The number of real roots of the equation $f(x) = 0$ on $[a, b]$ equals the difference between the number of changes of sign in the sturm sequence at x=a and x=b provided that $f(a) \neq 0$, $f(b) \neq 0$.

1.17 Iterative Methods :

We have Birge-Vieta method and Bairstow method to find a roots of a polynomial.

1.17.1 Birge-Vieta Method :

Let $P_n(x) = x_n+a_1x^{n-1}+.....+a_n$ ------(1.63)

be a polynomial of degree n.

Let $P_n(x) = (x-p)Q_{n-1}(x)+R$ ------(1.64)

where $Q_{n-1}(x)$, R are the quotient and remainder when $P_n(x)$ divided by x–p.

Let $Q_{n-1}(x) = x_{n-1}+b_1x_{n-2}+.........+b_{n-1}$ -------(1.65)

Clearly from (1.64), the value of R depends of p.

Let the initial approximation of p be p_0.

If we use Newton-Raphson method, we have

$$P_{k+1} = p_k - \frac{P_n(p_k)}{P^1_n(p_k)},\ k = 0, 1, 2, \ldots\ldots\ldots\ldots \qquad \text{------(1.66)}$$

On comparing coefficients of like powers of x on both sides of (1.63), we get,

$$\left.\begin{aligned} b_1 &= a_1+p \\ b_2 &= a_2+pb_1 \\ &\vdots \\ b_k &= a_k+pb_{k-1} \\ &\vdots \\ R &= a_n+pb_{n-1} \end{aligned}\right\} \qquad \text{------(1.67)}$$

Let $b_k = a_k+pb_{k-1}$, k=1,2,.......,n ------(1.68)
with $b_0 = 1$.

Again $P_n(p) = k$ (from 1.64)

$= b_n$ ------(1.69)

Differentiate (1.68) with respect to p, we get

$$\frac{db_k}{dp} = b_{k-1}+p\frac{db_{k-1}}{dp} \qquad \text{------(1.70)}$$

$$\text{Let } \frac{db_k}{dp} = C_{k-1} \qquad \text{------(1.71)}$$

$\therefore$ (1.70) becomes, $C_{k-1} = b_{k-1}+pC_{k-2}$

(i.e.) $C_k = b_k+pC_{k-1}$; k=1,2,3......n−1 ------(1.72)

Again differentiate (1.69) w.r.t. p, we get,

$$P'_n(p) = \frac{dR}{dp} = \frac{db_n}{dp} = C_{n-1} \qquad \text{------(1.73)}$$

$\therefore$ (1.66) changes as

$$\boxed{P_{k+1} = p_k - \frac{b_n}{C_{n-1}};\ k = 0,1,2,\ldots\ldots} \qquad \text{------(1.74)}$$

This method is called **Birge-Vieta method.**

Important Note :

The coefficients C_k, b_k are determined from b_k, a_k respectively and the calculations can be shown below.

p	1	a_1	a_2	a_3	...	...	a_{n-2}	a_{n-1}	a_n
	0	p	pb_1	pb_2	...	...	pb_{n-3}	pb_{n-2}	pb_{n-1}
p	1	b_1	b_2	b_3	...	...	b_{n-2}	b_{n-1}	b_n=R
	0	p	pC_1	pC_2	...	...	pC_{n-3}	pC_{n-2}	
	1	C_1	C_2	C_3	...	...	C_{n-2}	C_{n-2}=	dR/dp

Example E. 1.15 :

Use the Birge-Vieta method, to find a real root correct to three decimals of the following equations.

(i) $x^5 - x + 1 = 0$, $p = -1.5$

(ii) $x^3 - 11x^2 + 32x - 22 = 0$, $p = 0.5$

(iii) $x^6 - x^4 - x^3 - 1 = 0$, $p = 1.5$

Solution to (i) :

Given that $f(x) = x^5 - x + 1 = 0$, $p = -1.5$

Iteration 1 :

Here $p_0 = -1.5$

-1.5	1	0	0	0	-1	1
	0	-1.5	2.25	-3.375	5.0625	-6.0938
	1	-1.5	2.25	-3.375	4.0625	-5.0938
	0	-1.5	4.5	-10.125	20.25	
	1	-3	6.75	-13.5	24.3125	

We know that $p_{k+1} = p_k - \frac{b_k}{C_{k-1}}$; k=0,1,2,3,........

$$\therefore \; p_1 = p_0 - \frac{b_0}{C_{-1}}$$

$$= -1.5 - \frac{(-5.0938)}{24.3125}$$

$$= -1.2905$$

Iteration 2 :

-1.2905	1	0	0	0	-1	1
	0	-1.2905	1.6654	-2.1492	2.7735	-2.2887
	1	-1.2905	1.6654	-2.1492	1.7735	-1.2887
	0	-1.2905	3.3308	-6.4476	11.0941	
	1	-2.5810	4.9962	-8.5968	12.8676	

$$\therefore \; p_2 = p_1 - \frac{b_1}{C_0}$$

$$\text{(i.e.,)} \; p_2 = -1.2905 - \frac{(-1.2887)}{12.8676}$$

$$= -1.1903$$

Iteration 3 :

-1.1903	1	0	0	0	-1	1
	0	-1.1903	1.4168	-1.6864	2.0073	-1.1990
	1	-1.1903	1.4168	-1.6864	1.0073	-0.1990
	0	-1.1903	2.8336	-5.0593	8.0294	
	1	-2.3806	4.2504	-6.7457	9.0367	

$$\therefore \; p_3 = p_2 - \frac{b_2}{C_1}$$

$$= -1.1903 - \frac{(-0.1990)}{9.0367}$$

$$= -1.1683$$

Iteration 4 :

-1.1683	1	0	0	0	-1	1
	0	-1.1683	1.3649	-1.5946	1.8630	-1.0082
	1	-1.1683	1.3649	-1.5946	0.8630	-0.0082
	0	-1.1683	2.7298	-4.7838	7.4519	
	1	-2.3366	4.0947	-6.3784	8.3149	

$$\therefore \; p_4 = p_3 - \frac{b_2}{C_3}$$

$$= -1.1673 + \frac{0.0082}{8.3149}$$

$$= -1.1673$$

$\therefore$ The root correct to 3 decimal places is -1.167.

Qn (ii) & (iii) left as an exercise.

1.17.2 Bairstow Method :

The bairstow method extracts a quadratic factor of the form x^2+px+q from the polynomial $p_n(x)$, which may give a pair of complex or real roots.

Let $Q_{n-2}(x)$, R be the quotient and remainder when $p_n(x)$ is divided by x^2+px+q, where $Q_{n-2}(x)$ is a polynomial of order $n-2$.

$$\text{(i.e)}\; p_n(x) = (x^2+px+q)\, Q_{n-2}(x)+Rx+S \qquad \text{------(1.75)}$$

$$\text{where}\; Q_{n-2}(x) = x_{n-2}+b_1x^{n-3}+......+b_{n-3}x+b_{n-2}$$

Now we shall find the values of p and q such that

$$R(p, q) = 0 \;\&\; S(p, q) = 0 \qquad \text{------(1.76)}$$

Let (p_0, q_0) be an initial approximation and that $(p_0+\Delta p, q_0+\Delta q)$ be the true solution.

Using Newton-Raphson method, we have,

$$\Delta p = \frac{RSq - SRq}{RpSq - RqSp} \text{ and } \Delta q = \frac{RpS - RSp}{RpSq - RqSp} \qquad \text{------(1.77)}$$

where Rp, Rq, Sp, Sq are the partial derivatives of R and S with respect to p & q evaluated at p_0, q_0.

The coefficients b_i, R and S can be determined by comparing the like powers of x in $p_n(x)$.

$$\text{Thus} \left.\begin{array}{l|l} a_1 = p_1+p & b_1 = a_1-p \\ a_2 = b_2+pb_1+q & b_2 = a_2-pb_1-q \\ \vdots & \vdots \\ a_k = b_k+pb_{k-1}+qb_{k-2} & b_k = a_k-pb_{k-1}-qb_{k-2} \\ \vdots & \vdots \\ a_{n-1} = R+pb_{n-2}+qb_{n-3} & R = a_{n-1}-pb_{n-2}-qb_{n-3} \\ a_n = S+qb_{n-2} & S = a_n-qb_{n-2} \end{array}\right\} \qquad \text{------(1.78)}$$

Hence (1.78) can be put in a recurrence relation as

$$b_k = a_k-pb_{k-1}-qb_{k-2};\ k=1,2,\ldots\ldots,n \qquad \text{------(1.79)}$$

where $b_0 = 1$, $b_{-1} = 0$.

Comparing the last two equation of (1.78), we get

$$R = b_{n-1},\ S = b_n+pb_{n-3} \qquad \text{------(1.80)}$$

To find Rp, Sp, Rq, Sq :

Differentiate (1.80) partially with respect to p & q, we get

$$\left.\begin{aligned} -\frac{\partial b_k}{\partial p} &= b_{k-1} + p\frac{\partial b_{k-1}}{\partial p} + q\frac{\partial b_{k-2}}{\partial p};\ \frac{\partial b_0}{\partial p} = \frac{\partial b_{-1}}{\partial p} = 0 \\ -\frac{\partial b_k}{\partial p} &= b_{k-2} + p\frac{\partial b_{k-1}}{\partial q} + q\frac{\partial b_{k-2}}{\partial q};\ \frac{\partial b_0}{\partial q} = \frac{\partial b_{-1}}{\partial q} = 0 \end{aligned}\right\} \qquad \text{----(1.81)}$$

$$\text{Take } \frac{\partial b_k}{\partial p} = -C_{k-1};\ k = 1, 2, \ldots\ldots n$$

Thus first equation of (1.81) becomes

$$C_{k-1} = b_{k-1}-pC_{k-2}-qC_{k-3} \qquad \text{------(1.82)}$$

Again if we take $C_{k-2} = -\dfrac{\partial b_k}{\partial q}$ then the second equation of (1.81) becomes,

$$C_{k-2} = b_{k-2} - pC_{k-3} - qC_{k-4}$$

Hence we get a recurrence relation

$$C_k = b_k - pC_{k-1} - qC_{k-2}, \quad k = 1,2,.......,n-1$$

where $C_0 = 1$, $C_{-1} = 0$

$$\therefore Rp = -C_{n-2}$$

$$Sp = b_{n-1} - C_{n-1} - pC_{n-2}$$

$$Rq = -C_{n-3}$$

$$Sq = -(C_{n-2} + pC_{n-3})$$

Substituting the aboves values in (1.77), we get,

$$\Delta p = \frac{b_n C_{n-3} - b_{n-1} C_{n-2}}{C^2_{n-1} - C_{n-3}(C_{n-1} - b_{n-1})}$$

$$\text{and } \Delta q = \frac{b_{n-1}(C_{n-1} - b_{n-1}) - b_n C_{n-2}}{C^2_{n-1} - C_{n-3}(C_{n-1} - b_{n-1})}$$

The improved values of p_0 and q_0 are $p_1 = p_0 + \Delta p$, $q_1 = q_0 + \Delta q$.

Important Note 1 :

The computations of b_k, C_k can be done as follows :

	1	a_1	a_2	....	a_{n-2}	a_{n-1}	a_n
-p	0	-p	$-pb_1$	....	$-pb_{n-3}$	$-pb_{n-2}$	$-pb_{n-1}$
-q	0	-	-q	....	$-qb_{n-4}$	$-qb_{n-3}$	$-qb_{n-2}$
	1	b_1	b_2	....	b_{n-2}	b_{n-1}	b_n
-p	0	-p	$-pC_1$	....	$-pC_{n-3}$	$-pC_{n-2}$	
-q	0	0	-q	....	$-qC_{n-4}$	$-qC_{n-3}$	
	1	C_1	C_2	....	C_{n-2}	C_{n-3}	

Note 2 :

The polynomial $Q_{n-2}(x)$ is called deflated polynomial.

Note 3 :

Again applying same method to $Q_{n-2}(x)$, we get another set of quaduatic factor for $p_n(x)$. In this way we can able to find all the roots of $p_n(x)$.

Example E. 1.16 :

Perform two iteration of Bairstow's method to get the quadratic factor of the following equation

$x^4-x^3+6x^2+5x+10 = 0$ with (p, q) = (1.14, 1.42)

Solution :

We know that the recurrence relation of Bairstow's method are

$$p_{k+1} = p_k+\Delta p,\ q_{k+1} = q_k+\Delta q,\ R=0,1,2,......$$

$$\text{where } \Delta p = \frac{b_n C_{n-3} - b_{n-1} C_{n-2}}{C^2_{n-2} - C_{n-3}(C_{n-1} - b_{n-1})}$$

$$\text{and } \Delta q = \frac{b_{n-1}(C_{n-1} - b_{n-1}) - b_n C_{n-2}}{C^2_{n-2} - C_{n-3}(C_{n-1} - b_{n-1})}$$

Iteration 1 :

Here $p_0 = 1.14$, $q_0 = 1.42$.

	1	-1	6	5	10
-1.14	0	-1.14	2.4396	-8.0023	-0.0416
	0	0	-1.42	3.0388	-9.9678
	1	-2.14	7.0196	0.0365	-0.0094 (=bn)
-1.14	0	-1.14	3.7392	-10.6462	
-1.42	0	0	-1.14	4.6576	
		-3.28	9.3388	-5.9522 (=C_{n-1})	

$$\therefore\ \Delta p = 0.0046 \quad \& \quad \Delta q = 0.0019$$

$$\text{Thus } p_1 = 1.1446 \quad \& \quad q_1 = 1.4219$$

Iteration 2 :

	1	-1	6	5	10
-1.1446	0	-1.1446	2.4547	-8.0498	-0.0005
-1.4219	0	0	-1.4219	3.0494	-10
	1	-2.1446	7.0328	-0.0004 (= b_{n-1})	-0.0005 (= b_n)
-1.446	0	-1.446	3.7648	-10.7314	
-1.4219	0	0	-1.4219	4.6769	
	1	-3.2892	9.3757	-6.3561 (= C_{n-1})	

$$\therefore\ \Delta p = 0,\ \Delta q = 0$$

and hence $p_2 = 1.1446$, $q_2 \doteq 1.4219$.

1.18 Direct Method Graeffe's Root Squaring Method :

This method is used to find the roots of a polynomial with real coefficients

Let $P_n(a) = a^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n = 0$ ------(1.83)

Then $P_n(x)$ can be written as

$$(x^n + a_2 x^{n-2} + a_4 x^{n-4} + \ldots) = -(a_1 x^{n-1} + a_3 x^{n-3} + \ldots)$$

Squaring and simplifying, we get,

$$x^{2n} - (a_1^2 - 2a_2)x^{2n-2} + (a_2^2 - 2a_1 a_3 + 2a_4)x^{2n-4} - \ldots + (-1)^n a_n^2 = 0$$

put $x^2 = -z$ & the above equation changes as

$$z^n + b_1 z^{n-1} + \ldots + b_{n-1} z + b_n = 0 \qquad \text{------(1.84)}$$

$$\text{where } \left.\begin{aligned} b_1 &= a_{21} + 2a_2 \\ b_2 &= a_{22} - 2a_1 a_2 + 2a_4 \\ &\vdots \\ b_n &= a_{2n} \end{aligned}\right\} \qquad \text{------(1.85)}$$

Now the roots of (1.84) are $-\xi^2_1, -\xi^2_2, \ldots, \xi^2_n$ if $\xi_1, \xi_2, \ldots$n are the roots of (1.83).

The values of b_R can easily obtained as follows :

1	a_1	a_2	a_3	......	a_n
1	a^2_1	a^2_2	a^2_3	......	a^2_n
	$-2a_2$	$-2a_1a_3$	$-2a_2a_4$	......	
		$2a_4$	$2a_1a_5$	......	
			•	•	
			•	•	
			•	•	
1	b_1	b_2	b_3	......	b_n

The terms alternate in sign starting with a positive sign.

The first term is the square of the $(k+1)^{th}$ coefficient a_k.

The second term is twice the product of the nearest neighbouring coefficients a_{k-1} and a_{k+1}.

The third is twice the product of the next neighbouring coefficients a_{k-2} and a_{k+2}.

This produce is continued until there are no available coefficients to form the cross products.

This procedure can be repeated m times and we get the following equation

$$x^n+B_1x^{n-1}+B_2x^{n-2}+\ldots\ldots+B_{n-1}x+B_n = 0 \qquad \text{------(1.86)}$$

whose roots $R_1, R_2, \ldots\ldots R_n$ are the 2^m the power of the roots of the equation (1.83) with opposite signs, $R_i = -\xi^{2^m}_i$, $i = 1,2,\ldots\ldots.n$.

Assume that $|\xi_1| > |\xi_2| > \ldots\ldots\ldots > |\xi_n|$

Then $|R_1| >> |R_2| >> |R_3| >> \ldots\ldots\ldots >> |R_n|$

Also we have,

$$\left.\begin{aligned} -B_1 &= \sum R_i \cong R_1 \\ +B_2 &= \sum R_iR_j \cong R_1R_2 \\ -B_3 &= \sum R_iR_jR_R \cong R_1R_2R_3 \\ &\,.. \\ &\,.. \\ (-1)^n B_n &= R_1R_2\ldots\ldots R_n \end{aligned}\right\} \qquad \text{------(1.87)}$$

From (1.87), $$R_i = \frac{-B_i}{B_{i-1}}; i = 1,2,\ldots\ldots,n \quad \text{with } B_0 = 1$$

Now $$|R_i| = \frac{|B_i|}{|B_{i-1}|} = |\xi_i|^{2^m}$$

$$\Rightarrow \quad \log|\xi_i| = 2^{-m}(\log|B_i| - \log|B_{i-1}|, \quad i=1,2,\ldots\ldots,m.$$

This gives the absolute values of the roots and substituting in (1.83), we get the sign of the roots.

The squaring process is stopped when the cross product terms become negligible in comparison to square terms.

After few squaring, if the magnitude of the co-efficient B_k is about half the square of the magnitude of the corresponding coefficients a_n the previous equation, then it indicates ξ_k is a double root.

Now $$R_k \simeq \frac{-B_k}{B_{k-1}} \text{ and } R_{k+1} \simeq \frac{-B_{k+1}}{B_k}$$

$$\therefore R_k . R_{k+1} \simeq R^2_k \cong \left|\frac{B_{k+1}}{B_{k-1}}\right|$$

$$\text{(i.e.) } |R^2_k| = |\xi_k|^{2(2^m)} = \left|\frac{B_{k+1}}{B_{k-1}}\right|$$

which is the magnitude of the double root.

Substituting in (1.83) we get the sign of the root.

If ξ_k, ξ_{k+1} form a pair of complex roots, then

$$\xi_k . \xi_{k+1} = \beta_k e^{\pm i\phi_k}$$

and $$B_k^{2(2^m)} \simeq \left|\frac{B_{k+1}}{B_{k-1}}\right|$$

and ϕ is suitable determined from $2\beta_k^m \cos(m\phi_k) \cong \frac{B_{k+1}}{B_{k-1}}$.

Example E.1.17 :

Apply the Graeffe's root squaring method to find the roots of the following equations.

(a) $x^3-2x+2 = 0$ and (b) $x^3+3x^2-4 = 0$

Solution to (a) :

Given that $x^2-2x+2 = 0$

m	2^m				
0	1	1	0	-2	2
		1	0	4	4
			4	0	
1	2	1	4	4	4
		1	16	16	16
			-8	-32	
2	4	1	8	-16	16
		1	64	256	256
			32	-256	
3	8	1	96	0	256
		1	9216	0	65536
			0	-49152	
4	16	1	9216	-49152	65536
		$(=B_0)$	$(=B_1)$	$(=B_2)$	$(=B_3)$

Since B_2 is alternatively positive and negative, we have a pair of complex roots.

One real root is $|\xi_1| = 9216$

$\therefore\ |\xi_1| = 1.7692$

put $x = 1.7692$ in $P_n(x)$, we get $(1.7692)^3-2(1.7692)+2$

$= 3.9993$

$\neq 0.$

Thus $\xi_1 = -1.7692$

To find the pair of complex roots p+iq, we have

$$\beta^{32} = \left|\frac{B_3}{B_1}\right|$$

$$= \frac{65536}{9216}$$

$$= 7.1111$$

$$\therefore \ \beta = 1.0632$$

$$\text{Again } \ \xi_1 + 2p = 0$$

$$\Rightarrow \ 2p = -\xi_1 = +1.7692$$

$$\Rightarrow \ p = +0.8846$$

$$\text{and } \ p^2 + q^2 = \beta^2$$

$$\Rightarrow \ q^2 = \beta^2 - p^2$$

$$= 1.1304 - 0.7825$$

$$= 0.3479$$

$$\therefore \ \mathbf{q} = \mathbf{0.5898}$$

$$\text{Hence } \ p + iq = \mathbf{0.8846 + 0.5898i}$$

Solution (ii) :

Given that $x^3+3x^2-4 = 0$

m	2^m				
0	1	1	3	0	-4
		1	9	0	16
			0	24	
1	2	1	9	24	16
			81	576	256
			-48	-288	
2	4	1	33	288	256
		1	1089	82944	65536
			-576	-16896	
3	8	1	513	66048	65536
		1	263169	4362338304	16^8
			-132096	-67239936	
4	16	1	131073	4295098368	16^8
		$(=B_0)$	(almost half)		
			$(=B_1)$	$(=B_2)$	$(=B_3)$

Since B_1 is almost half of the corresponding value in the previous squaring, indicates that there may be a double root based on B_0, B_1 and B_2.

The doubel root is obtained from,

$$|\xi_1|^{32} = |\xi_2|^{32} = |B_2| = 4295098368$$

$$\therefore \quad |\xi_1| = |\xi_2| = 2.000$$

put x = 2 in the given polynomial, we get

$$2^3+3(2)^2-4 = 8+12-4 = 0$$

$\therefore$ The double root is 2.

& the simple root is obtained from

$$|\xi_3|^{16} = \left|\frac{B_3}{B_2}\right| = 1 \ \& \ \xi_3 = 1.000$$

Again putting x=1 in the given equation, we get $\xi_3 = 1.000$.

Hence the roots of the polynomial are 1, 2, 2.

UNIT – 2

SYSTEM OF LINEAR ALGEBRAIC EQUATIONS AND EIGEN VALVE PROBLEMS

Consider a system of n linear algebraic equations in n unknowns

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n &= b_2 \\ &\cdot\cdot \\ &\cdot\cdot \\ a_{n1}x_1 + a_{n2}x_2 + \dots\dots + a_{nn}x_n &= b_n \end{aligned}\right\} \quad \text{-------(2.1)}$$

where a_{ij} are the known coefficients,

b_i are the known valves and

x_i are the unknowns to be determined.

In this unit we use the following notations :

1) A – square matrix of order n.
2) a_{ij} – (i, j)th entry of A.
3) A^{-1} – inverse of A.
4) A^T – Transpose of A
5) |A| – determinent of A
6) M_{ij} – minor of a_{ij} in A
7) **O** – null matrix.
8) **I** – unit matrix of order n.
9) D – diagonal matrix of order n
10) L – Lower triangular matrix of order n
11) U – upper triangular matrix of order n
12) P – permutation matrix
13) **x** – column vector with elements x_i

14) $\mathbf{x}^T$ – row vector with elements x_i

15) $\rho(A)$ – spectral radius of A

16) $||A||$ – norm of A

17) $||x||$ – norm of x.

Consider a system of equations (2.1).

The characteristic equation of (2.1) is given by $\det(A-\lambda I)=0$ ------(2.2)

where λ is a parameter.

The n roots $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$ of (2.2) are called the **eigen valves** of A and the spectral radius of $A = \rho(A) = \max\{|\lambda_1|, |\lambda_2|, \ldots\ldots |\lambda_n|\}$.

Corresponding to each eigenvalve λ_i there exists an eigenvector $\mathbf{x}_i$ which is a nontrivial solution of $(A-\lambda_i I)\mathbf{x}_i = 0$ ------(2.3)

Using direct method or iterative method, we may find the solution of (2.1) and $(A-\lambda I)x = 0$ ------(2.4)

2.1 Direct Method

1) Using forward or backward substitution method we may find the solution of (2.1) when A = L or A = U.

2) **Cramer Rule :**

Using this rule valve of $x_i = \frac{|B_i|}{|A|}$, $i=1(1)n$ ------(2.5)

where $|B_i|$ is the determinant of the matrix obtained by replacing the i^{th} column of A by the right hand vector **b**.

Example 2.1 :

(1) Solve the equations

$2x_1+3x_2-x_3 = 5$

$4x_1+4x_2-3x_3 = 3$

$2x_1-3x_2+2x_3 = 2$

(i) by using Cramer's rule &

(ii) by determining the inverse of the coefficient matrix.

Solution :

The matrix form of the given equation is

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$$

$$\text{(i.e.) } A\mathbf{x} = \mathbf{b} \qquad \text{------(2.5a)}$$

$$\text{where } A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$$

$$\text{Thus } |A| = \begin{vmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 2 \end{vmatrix}$$

$$= 2(8-9)-3(8+6)-1(-12-8)$$

$$= -2-42+20$$

$$= -24$$

$$\text{and } |B_1| = \begin{vmatrix} 5 & 3 & -1 \\ 3 & 4 & -3 \\ 2 & -3 & 2 \end{vmatrix} = -24$$

$$|B_2| = \begin{vmatrix} 2 & 5 & -1 \\ 4 & 3 & -3 \\ 2 & 2 & 2 \end{vmatrix} = -48$$

$$\text{and } |B_3| = \begin{vmatrix} 2 & 3 & 5 \\ 4 & 4 & 3 \\ 2 & -3 & 2 \end{vmatrix} = -72.$$

$$\text{Then } x_1 = \frac{|B_1|}{|A|} = 1$$

$$x_2 = \frac{|B_2|}{|A|} = 2$$

$$x_3 = \frac{|B_3|}{|A|} = 3$$

(ii) Step 1 :

First we shall find A^{-1}.

$$\text{Now } |A| = -24,$$

$$\therefore A^T = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 4 & -3 \\ -1 & -3 & 2 \end{bmatrix}$$

$$\text{and } \operatorname{Adj}A = \begin{bmatrix} \begin{vmatrix} 4 & -3 \\ -3 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & -3 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ -1 & -3 \end{vmatrix} \\ -\begin{vmatrix} 4 & 2 \\ -3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ -1 & -3 \end{vmatrix} \\ \begin{vmatrix} 4 & 2 \\ 4 & -3 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & -3 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 3 & 4 \end{vmatrix} \end{bmatrix}$$

$$= \begin{pmatrix} -1 & 3 & -5 \\ -14 & 6 & 2 \\ -20 & 12 & -4 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\operatorname{Adj} = \frac{1}{-24}\begin{bmatrix} -1 & 3 & -5 \\ -14 & 6 & 2 \\ -20 & 12 & -4 \end{bmatrix}$$

Step 2 :

To find x.

$$\mathbf{x} = A^{-1}\mathbf{b} \text{ (from (2.5a)}$$

$$= \frac{1}{-24}\begin{pmatrix} -1 & 3 & -5 \\ -14 & 6 & 2 \\ -20 & 12 & -4 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$$

$$\text{(i.e.,) } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{-24}\begin{pmatrix} -24 \\ -48 \\ -72 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(i.e.,) $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

2.2 Gauss Elimination Method :

Consider a system of equation (2.1).

The Gauses elimination method gives

$$[A/b] \xrightarrow[\text{Elimination}]{\text{Gauss}} [U/C]$$

where [A/b] is the agumentend matrix.

Consider the agumented matrix [A/b] of (2.1).

Put $b_i^{(k)} = a^{(k)}_{i,n+1}$, i, k = 1(1) n ------(2.6)

The elements $a_{ij}(k)$ with i, j≥k are given by

$$a_{ij}^{(k)} = a_{ij}^{(k)} - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} a_{kj}^{(k)} \qquad \text{------(2.7)}$$

$i = k+1, k+2, \ldots\ldots, n; \quad j = k+1, k+2, \ldots\ldots, n, n+1$ and $a_{ij}^{(1)} = a_{ij}$.

Note :

$a_{ii}^{(i)}$ are called pivot element of i^{th} row.

The elimination is performed in (n–1) steps, k = 1,2,......n–1.

In the elimination process, if one of the pivot elements $a_{11}^{(1)}, a_{22}^{(2)}, \ldots\ldots, a_{nn}^{(n)}$ vanishes then rearrange the remaining rows so as to obtain a non-vanishing pivot. This method is called **pivoting.**

2.1.1 Partial Pivoting :

The pivot is chosen as follows :

Choose j, the smallest integer for which

$$\left|a_{jk}^{(k)}\right| = \max\left|a_{ik}^{(k)}\right|; \qquad k \le i \le n \qquad \text{------(2.8)}$$

and interchange rows k & j.

2.1.2 Complete Pivoting :

If at the k^{th} step, we interchange both the rows and column of the matrix so tha the largest number in magnitude in the remaining matrix is used as pivot

$$\text{(i.e.) } |a_{kk}| = \max\{|a_{ij}|\}, \; i, j = k, k+1, \ldots\ldots, n.$$

Example E. 2.2 :

Solve the following system of equations using Gauss–elimination mehtod :

$$4x_1+x_2+x_3 = 4$$
$$x_1+4x_2-2x_3 = 4$$
$$3x_1+2x_2-4x_3 = 6$$

Solution :

Given equations can be written as a matrix form as

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & -2 \\ 3 & 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

$\therefore$ The augmented matrix (A | b) is given by

$$(A/b) = \left[\begin{array}{ccc|c} 4 & 1 & 1 & 4 \\ 1 & 4 & -2 & 4 \\ 3 & 2 & -4 & 6 \end{array}\right]$$

$$R_2 \to R_2 - \tfrac{1}{4}R_1$$
$$R_3 \to R_3 - \tfrac{3}{4}R_1$$

$$\sim \left[\begin{array}{ccc|c} 4 & 1 & 1 & 4 \\ 0 & 15/4 & -9/4 & 3 \\ 0 & 5/4 & -19/4 & 3 \end{array}\right]$$

$$R_3 \to R_3 - \frac{1}{3}R_2$$

$$\sim \left[\begin{array}{ccc|c} 4 & 1 & 1 & 4 \\ 0 & 15/4 & -9/4 & 3 \\ 0 & 0 & -4 & 2 \end{array}\right]$$

Again writing in the matrix form, we have,

$$4x_1+x_2+x_3 = 4 \quad \text{------(1)}$$

$$\frac{15}{4}x_2 - \frac{9}{4}x_3 = 3 \quad \text{------(2)}$$

$$-4x_3 = 2 \quad \text{------(3)}$$

Using back-substitution method, we have, $x_3 = -1/2$, $x_2 = 1/2$, $x_1 = 1$.

2.3 Gauss-Jordan Elimination Method :

Here the coefficient matrix A is reduced to a diagonal matrix rather than a triangular matrix.

$$\text{(i.e.)} \quad [A/I] \xrightarrow[\text{Jordan}]{\text{Gauss}} [I/A^{-1}] \qquad \text{------(2.8)}$$

Example E. 2.3 :

Solve the following system of equations

$x_1+2x_2+x_3=5$,

$2x_1+3x_2-x_3=7$,

$2x_1-x_2+3x_3=12$ using Gauss-Jordan method.

Solution :

Given equation can be written as matrix form is

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix}$$

(i.e.,) $AX = \mathbf{b}$ where $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix}$

Step 1 : To find A^{-1}.

$$\text{Now} \quad [A/I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & -2 & 1 & 0 \\ 0 & -5 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & 16 & 8 & -5 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 7/16 & 5/16 \\ 0 & 1 & 0 & 1/2 & -1/16 & -3/16 \\ 0 & 0 & 1 & 1/2 & -5/16 & 1/16 \end{array}\right]$$

$$\text{Thus } A^{-1} = \begin{bmatrix} -1/2 & 7/16 & 5/16 \\ 1/2 & -1/16 & -3/16 \\ 1/2 & -5/16 & 1/16 \end{bmatrix}$$

Step 2 :

Now $A\mathbf{X} = \mathbf{b}$

$$\Rightarrow \mathbf{X} = A^{-1}\mathbf{b}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/2 & 7/16 & 5/16 \\ 1/2 & -1/16 & -3/16 \\ 1/2 & -5/16 & 1/16 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 69/16 \\ -27/16 \\ 17/16 \end{bmatrix}$$

$\therefore x_1 = 69/16, x_2 = -27/16, x_3 = 17/16.$

2.4 Triangulization Method :

This method is also called decomposition method. In this method the coefficient matrix A is written as A = L U where L, U are lower and upper triangula matrices.

$$\begin{aligned} \text{Thus } A\mathbf{X} &= \mathbf{b} \text{ be written as} \\ LU\mathbf{X} &= \mathbf{b} \\ \Rightarrow LZ &= \mathbf{b} \text{ where } Z = U\mathbf{X} \\ \Rightarrow Z &= L^{-1}\mathbf{b}. \\ \Rightarrow U\mathbf{X} &= L^{-1}\mathbf{b} \\ \Rightarrow \mathbf{X} &= U^{-1}L^{-1}\mathbf{b} \end{aligned}$$

Hence we can easily find A^{-1} is also.

Example 2.4 :

Solve the following system of equations :

$$\begin{aligned} 2x_1-3x_2+10x_3 &= 3 \\ -x_1+4x_2+2x_3 &= 20 \\ 5x_1+2x_2+x_3 &= -12 \end{aligned}$$

Solution :

The matrix form of the given equations is **AX** = **b**

where $A = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$

Let $A = LU$

(i.e.,) $\begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

(i.e.,) $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$

(i.e.,) $u_{11}=2$, $u_{12}=-3$, $u_{13}=10$, $l_{21}u_{11}=-1$,

$u_{21}u_{12}+u_{22}=4$, $l_{21}u_{13}+u_{23}=2$, $l_{31}u_{11}=5$,

$l_{31}u_{12}+l_{32}u_{22}=2$, $l_{31}u_{13}+l_{32}u_{23}+u_{33}=1$

On solving the above equations, we have,

$u_{11}=2$, $u_{12}=-3$, $u_{13}=10$, $u_{22}=5/2$, $u_{23}=7$,

$u_{33}=-253/6$, $l_{21}=-1/2$, $l_{31}=5/2$, $l_{32}=19/5$.

$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 5/2 & 19/5 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 5/2 & 7 \\ 0 & 0 & -253/6 \end{bmatrix}$

Now $AX = \mathbf{b}$

$\Rightarrow LUX = \mathbf{b}$

$\Rightarrow LZ = \mathbf{b}$ where $Z = UX = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ (say)

Thus $\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 5/2 & 19/5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$

$$\Rightarrow z_1 = 3$$

$$-\frac{1}{2}z_1 + z_2 = 20$$

$$\frac{5}{2}z_1 + \frac{19}{5}z_2 + z_3 = -12$$

Solving the above equations, we get,

$z_1 = 3$, $z_2 = 43/2$ & $z_3 = -506/5$

Again UX = Z

$$\Rightarrow \begin{bmatrix} 2 & -3/2 & 0 \\ 0 & 5/2 & 7 \\ 0 & 0 & -253/6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 43/2 \\ -506/5 \end{bmatrix}$$

(i.e.) $$2x_1 - \frac{3}{2}x_2 = 3$$

$$\frac{5}{2}x_2 + 7x_3 = 43/2$$

$$-\frac{253}{6}x_3 = -506/5$$

Solving the above equations, we get,

$x_1 = -4$, $x_2 = 3$, $x_3 = 2$

which is the required values of x_1, x_2 & x_3.

Exercise :

Try to solve the following equations using factorization method :

1) $x+2y+3z = 10$, $2x-3y+z = 1$, $3x+y-2z = 9$

[Answer : x=3, y=2, z=1]

2) $x_1+x_2+x_3 = 1$, $4x_1+3x_2-x_3 = 6$, $3x_1+5x_2+3x_3 = 4$

[Answer : x1 = 1, x2 = 1/2, x3 = –1/2].

2.5 Cholesky Method :

This method is also called square-root method and this method is used when the coefficient matrix A is real symmetric & positive definite.

If A is real symmetric & positive definite then $A = LL^T$ or $A = UU^T$ ------(2.9)

$$\text{Now } AX = b$$
$$\Rightarrow LL^TX = b$$
$$\Rightarrow LZ = b \text{ where } z = L^TX.$$

After finding z then we can able to find X.

Example E. 2.5 :

Solve the equations using Cholesky's method $x_1-x_2=1$, $-x_1+4x_2-x_3=0$, $-x_2+4x_3-x_4=0$, $-x_3+4x_4=0$. Also find inverse of the coefficient matrix.

Solution :

The matrix form of the given system of equations is AX = **b** where,

$$A = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Let } A = LL^T. \text{ where } L = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix}$$

Now $LL^T = A$

$$\Rightarrow \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} & l_{41} \\ 0 & l_{22} & l_{32} & l_{42} \\ 0 & 0 & l_{33} & l_{43} \\ 0 & 0 & 0 & l_{44} \end{bmatrix} = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

Multiplying LHS matrices and equating with the RHS matrix, we get,

$$l_{11} = 2$$

$$l_{21} = -1/2,\ l_{22} = \sqrt{\frac{15}{4}}$$

$$l_{31} = 0,\ l_{32} = -\sqrt{\frac{4}{15}},\ l_{33} = \sqrt{\frac{56}{15}}$$

$$l_{41} = 0,\ l_{42} = 0,\ l_{43} = -\sqrt{\frac{15}{56}},\ l_{44} = \sqrt{\frac{209}{56}}$$

$$\text{Now } AX = \mathbf{b}$$

$$\Rightarrow LL^TX = \mathbf{b}$$

$$\Rightarrow LZ = \mathbf{b} \text{ where } Z = L^TX = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \text{(say)}$$

$$\text{(i.e.)} \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1/2 & \sqrt{15/2} & 0 & 0 \\ 0 & -\sqrt{4/15} & \sqrt{56/15} & 0 \\ 0 & 0 & \sqrt{15/56} & \sqrt{209/56} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(i.e.,)
$$2z_1 = 1$$

$$-\frac{1}{2}z_1 + \sqrt{\frac{56}{15}}z_3 = 0$$

$$\sqrt{\frac{4}{15}}z_2 + \sqrt{\frac{56}{15}}z_3 = 0$$

$$\sqrt{\frac{15}{56}}z_3 + \sqrt{\frac{209}{56}} = 0$$

Solving the above equations, we get,

$$z_1 = 1/2,\ z_2 = \frac{1}{\sqrt{60}},\ z_3 = \sqrt{\frac{1}{840}},\ z_4 = \frac{1}{\sqrt{11704}}$$

$$\text{Again } L^TX = Z$$

$$\Rightarrow \begin{bmatrix} 2 & -1/2 & 0 & 0 \\ 0 & \sqrt{\frac{15}{4}} & -\sqrt{\frac{4}{15}} & 0 \\ 0 & 0 & \sqrt{\frac{56}{15}} & -\sqrt{\frac{15}{56}} \\ 0 & 0 & 0 & \sqrt{\frac{209}{56}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/\sqrt{60} \\ 1/\sqrt{840} \\ 1/\sqrt{11704} \end{bmatrix}$$

$$\Rightarrow x_1 = \frac{1}{209},\ x_2 = \frac{15}{209},\ x_3 = \frac{4}{209},\ x_4 = \frac{56}{209}$$

To find A^{-1}

Since L is a lower triangular matrix then L^{-1} is also lower triangular matrix.

$$\text{Let } L^{-1} = \begin{bmatrix} l^1_{11} & 0 & 0 & 0 \\ l^1_{21} & l^1_{22} & 0 & 0 \\ l^1_{31} & l^1_{32} & l^1_{33} & 0 \\ l^1_{41} & l^1_{42} & l^1_{43} & l^1_{44} \end{bmatrix}$$

$$\text{Now } LL^{-1} = I$$

$$\Rightarrow l^1_{11} = \frac{1}{2}, l^1_{21} = \frac{1}{\sqrt{60}}, l^1_{22}\frac{2}{\sqrt{15}}$$

$$l^1_{31} = \frac{1}{\sqrt{840}},\ l^1_{32} = \sqrt{\frac{2}{105}}, l^1_{33} = \sqrt{\frac{15}{56}}$$

$$l^1_{41} = \frac{1}{\sqrt{11704}}, l^1_{42} = \sqrt{\frac{2}{1463}}, l^1_{43} = \sqrt{\frac{15}{11704}}, l^1_{44} = \sqrt{\frac{56}{209}}$$

$$\text{Hence } L^{-1} = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0.1291 & 0.5164 & 0 & 0 \\ 0.0345 & 0.1380 & 0.5176 & 0 \\ 0.0092 & 0.0370 & 0.1387 & 0.5176 \end{bmatrix}$$

$$\text{Now } A = LL^T$$

$$\therefore A^{-1} = (L^T)^{-1}L^{-1}$$

$$= (L^{-1})^T L^{-1}$$

$$= \begin{bmatrix} 0.2679 & 0.0718 & 0.0191 & 0.0048 \\ 0.0718 & 0.2871 & 0.0766 & 0.0192 \\ 0.0191 & 0.0766 & 0.2871 & 0.0718 \\ 0.0048 & 0.0192 & 0.0718 & 0.2679 \end{bmatrix}$$

Example E. 2.6 :

Find the inverse of the matrix $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 1 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ by the Cholesky method.

Solution :

$$\text{Let } A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 1 & -1 \\ 2 & -1 & 3 \end{bmatrix} \text{ and } A = LL^T$$

$$\text{where } L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$\text{Now } LL^T = A$$

$$\Rightarrow l_{11} = \sqrt{2}, l_{21} = \frac{-1}{\sqrt{2}}, l_{22} = \frac{1}{\sqrt{2}},$$

$$l_{31} = \sqrt{2}, l_{32} = 0, l_{33} = 1$$

$$\therefore L = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \sqrt{2} & 0 & 1 \end{pmatrix}$$

Since L is a lower triangular matrix and therefore L^{-1} is also lower triangular matrix.

$$\text{Now } LL^{-1} = I$$

$$\Rightarrow \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} l^1_{11} & 0 & 0 \\ l^1_{21} & l^1_{22} & 0 \\ l^1_{31} & l^1_{32} & l_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiplying the LHS matrices and equating it with I, we get,

$$L^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\text{Thus } A = LL^T$$

$$\Rightarrow A^{-1} = (L^{-1})^T L^{-1}$$

$$= \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

which is the required inverse of A.

2.6 Partition Method :

This method is used to find the inverse of a matrix.

$$\text{Let } A = \left[\begin{array}{c|c} B & C \\ \hline E & D \end{array}\right]$$

where B, C, D, E are r×r, r×s, s×s, s×r matrices respectively and r+s=n, (r, s are positive integers).

$$\text{Let } A^{-1} = \left[\begin{array}{c|c} X & Y \\ \hline Z & V \end{array}\right]$$

where X, Y, Z and V are matrices of the same orders as B, C, E, D respectively.

$$\therefore\ AA^{-1} = \left[\begin{array}{c|c} B & C \\ \hline E & D \end{array}\right]\left[\begin{array}{c|c} X & Y \\ \hline Z & V \end{array}\right]$$

$$= \left[\begin{array}{c|c} I_r & O \\ \hline O & I_s \end{array}\right] \qquad \text{------(2.11)}$$

where I_r, I_s are identity matrices of orders r & s respective.

From (2.11), we have,

$$\begin{aligned} BX+CZ &= Ir \\ BY+CV &= O \\ EX+DZ &= O \\ EY+DV &= I_s \end{aligned}$$

Solving the above equations, we get,

$$\begin{aligned} V &= (D-EB^{-1}C)^{-1}, \\ Y &= -B^{-1}CV \\ Z &= -VEB^{-1} = -(D-EB^{-1}C)^{-1}EB^{-1} \\ X &= B^{-1}(I_r-CZ) = B^{-1}-B^{-1}CZ. \end{aligned}$$

Thus we can able to find the inverse of the given matrix.

Example E. 2.7 :

Find the inverse of the matrix $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ by the partition method.

Solution :

$$\text{Let } A = \left[\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ \hline 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$= \left[\begin{array}{c|c} B & C \\ \hline E & D \end{array}\right]$$

where $B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ & $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$\text{Let } A^{-1} = \begin{pmatrix} X & Y \\ Z & V \end{pmatrix}$$

$$\text{We know that } AA^{-1} = I$$

$$\therefore \begin{pmatrix} B & C \\ E & D \end{pmatrix}\begin{pmatrix} X & Y \\ Z & V \end{pmatrix} = I$$

$$\text{(i.e.)} \begin{pmatrix} BX+CZ & BY+CV \\ EX+DZ & EY+DV \end{pmatrix} = \left(\begin{array}{c|c} I_2 & 0 \\ \hline 0 & I_2 \end{array}\right)$$

$$\text{(i.e.)} \left.\begin{array}{l} BX+CZ = I_2 \\ BY+CV = O \\ EX+DZ = O \\ EY+DV = I_2 \end{array}\right\} \qquad \text{------(2.12)}$$

Solving the equations in (2.12), we get,

$$V = (D-EB^{-1}C)^{-1}$$

$$Y = -B^{-1}CV,$$

$$Z = -VEB^{-1}$$

$$\text{and } X = B^{-1}-B^{-1}CZ$$

$$\text{Now } B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\therefore |B| = 4-1 = 3$$

$$\text{Thus } B^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\therefore\ V = (D-EB^{-1}C)^{-1} = \frac{3}{5}\begin{pmatrix} 2 & -1 \\ -1 & \frac{4}{3} \end{pmatrix}$$

$$Y = -B^{-1}CV = -\frac{1}{5}\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}$$

$$Z = -VEB^{-1} = -\frac{1}{5}\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}$$

$$\text{and } X = B^{-1}-B^{-1}CZ = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -3 & 6 \end{pmatrix}$$

$$\text{Thus } A^{-1} = \frac{1}{5}\begin{pmatrix} 4 & -3 & 2 & -1 \\ -3 & 6 & -4 & 2 \\ 2 & -4 & 6 & -3 \\ -1 & 2 & -3 & 4 \end{pmatrix}$$

Example E.2.8 :

Using partition method, find the inverse of $\begin{bmatrix} 13 & 14 & 6 \\ 8 & -1 & 13 \\ 6 & 7 & 3 \end{bmatrix}$

Solution :

$$\text{Let } A = \left[\begin{array}{cc|c} 13 & 14 & 6 \\ 8 & -1 & 13 \\ \hline 6 & 7 & 3 \end{array}\right] = \left[\begin{array}{c|c} B & C \\ \hline E & D \end{array}\right]$$

where, $B = \begin{pmatrix} 13 & 14 \\ 8 & -1 \end{pmatrix}$, $C = \begin{bmatrix} 6 \\ 13 \end{bmatrix}$, $D = (3)$, $E = (6\ \ 7)$.

$$\text{Now } B^{-1} = -\frac{1}{125}\begin{pmatrix} -1 & -14 \\ -8 & 13 \end{pmatrix}$$

$$\text{Let } A^{-1} = \left[\begin{array}{c|c} X & Y \\ \hline Z & V \end{array}\right]$$

Then $AA^{-1} = I$ gives us

$$X = B^{-1} - B^{-1}CZ,$$

$$Y = -B^{-1}CV,$$

$$Z = -VEB^{-1}$$

$$\text{and } V = (D - EB^{-1}C)^{-1}$$

$$\text{Now } EB^{-1}C = -\frac{1}{125}(6 \quad 7)\begin{pmatrix} -1 & -14 \\ -8 & 13 \end{pmatrix}\begin{pmatrix} 6 \\ 13 \end{pmatrix}$$

$$= -\frac{1}{125}(6 \quad 7)\begin{pmatrix} -188 \\ 121 \end{pmatrix}$$

$$= \frac{-1}{125}(-281)$$

$$= \frac{281}{125}$$

$$\therefore D - EB^{-1}C = (3) - \left(\frac{281}{125}\right) = \frac{94}{125}$$

$$\text{Thus } V = (D - EB^{-1}C)^{-1}$$

$$= \left(\frac{94}{125}\right)^{-1}$$

$$= \left(\frac{125}{94}\right)$$

$$\text{and } Z = -VEB^{-1}$$

$$= -\left\{\frac{125}{94}(6 \quad 7)\right\}\left\{\frac{-1}{125}\begin{pmatrix} -1 & -14 \\ -8 & 13 \end{pmatrix}\right\}$$

$$= \frac{1}{94}(-62 \quad 7)$$

$$= \frac{1}{94}(-62 \quad +7)$$

$$\text{and } Y = -B^{-1}CV$$

$$= \frac{1}{125}\begin{pmatrix} -1 & -14 \\ -8 & 13 \end{pmatrix}\begin{pmatrix} 6 \\ 13 \end{pmatrix}\begin{pmatrix} 125 \\ 94 \end{pmatrix}$$

$$= \frac{1}{94}\begin{pmatrix} -188 \\ 121 \end{pmatrix}$$

$$\text{and } X = B^{-1}-B^{-1}CZ$$

$$= B^{-1}(I_2-CZ)$$

$$= \frac{-1}{125}\begin{pmatrix}-1 & -14\\ -8 & 13\end{pmatrix}\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\frac{1}{94}\begin{pmatrix}6\\ 13\end{pmatrix}\begin{pmatrix}-62 & +7\end{pmatrix}\right]$$

$$= \frac{-1}{125}\begin{pmatrix}-1 & -14\\ -8 & 13\end{pmatrix}\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\frac{1}{94}\begin{pmatrix}-372 & -42\\ -806 & +91\end{pmatrix}\right]$$

$$= \frac{-1}{125}\begin{pmatrix}-1 & -14\\ -8 & 13\end{pmatrix}\cdot\frac{1}{94}\begin{pmatrix}466 & -42\\ 806 & 3\end{pmatrix}$$

$$= \frac{-1}{125\times 94}\begin{pmatrix}-11750 & 0\\ 6750 & 375\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\ -\frac{27}{47} & -\frac{3}{94}\end{pmatrix}$$

$$\therefore\ A^{-1} = \begin{pmatrix}1 & 0 & 2\\ \frac{27}{47} & \frac{-3}{94} & \frac{121}{94}\\ \frac{-31}{47} & \frac{7}{94} & \frac{125}{94}\end{pmatrix}$$

which is the required inverse of A.

2.7 Error Analysis :

In the earlier sections we have discussed some methods. In these methods some operations like division and multiplication are involved & the number of appearance is called operational count for that method.

Operational count for Gauss Elimination method is $\frac{n}{3}(n^2+3n-1)$ and if n is large then operational count is approximately equal to $\frac{n^3}{3}$.

The number of addition and subtractions is equal to $\frac{n(n-1)(2n+5)}{6}$

Smilarly the operational court for the Cholesky method is $\frac{n^3+9n^2+2n}{6}$

2.8 Iteration Methods :

A general linear iterative metod for the solution of the system of equations AX=**b** may be defined in the form.

$x^{(k+1)} = Hx^{(k)}+C$, k=0,1,2,...... ------(2.12)

where $x^{(k+1)}$ & $x^{(k)}$ are the approximations for x at the $(k+1)^{th}$ and k^{th} iterations respectively. Here H is called iteration matrix depending on A and C, the column vector.

When $k \to \infty$ then $x^{(k)}$ converges to the exact solution $\mathbf{x} = A^{-1}\mathbf{b}$.

$\therefore$ (2.12) becomes,

$$A^{-1}\mathbf{b} = HA^{-1}\mathbf{b}+C$$

$$\Rightarrow \quad C = (I-H)A^{-1}\mathbf{b} \qquad \text{------(2.13)}$$

2.9 Jacobi Iteration Method :

Consider the system equations (2.1), in which a_{ii} are pivot elements. Thus the equations (2.1) can be rewritten as:

$$\left.\begin{aligned} a_{11}x_1 &= -(a_{12}x_2 + a_{13}x_3 + ...a_{1n}x_n) + b_1 \\ a_{22}x_2 &= -(a_{11}x_2 + a_{23}x_3 + ...a_{2n}x_n) + b_2 \\ &\vdots \\ a_{nn}x_n &= -(a_{n1}x_1 + a_{n2}x_2 + ...a_{n,n-1}x_{n-1}) + b_{n-1} \end{aligned}\right\} \qquad \text{------(2.14)}$$

Hence Jacobi iteration method may be defined at (k+1)th iterated valves of $x_1, x_2, ...x_n$ are

$$\left.\begin{aligned} x_1^{(k+1)} &= \frac{-1}{a_{11}}\left(a_{12}x_2^{(k)} + a_{13}x_3^{(k)} + ...+a_{1n}x_n^{(k)} - b_1\right) \\ x_2^{(k+1)} &= \frac{-1}{a_{22}}\left(a_{21}x_1^{(k)} + a_{23}x_2^{(k)} + ...+a_{2n}x_n^{(k)} - b_2\right) \\ &\vdots \\ x_n^{(k+1)} &= \frac{-1}{a_{nn}}\left(a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + ...+a_{n,n-1}x_{n-1}^{(k)} - b_n\right) \end{aligned}\right\} \qquad \text{------(2.15)}$$

k = 0,1,2,.........

The equations in (2.15) can be written in the matrix form as

$$
\begin{aligned}
X^{(k+1)} &= -D^{-1}(L+U)X^{(k)}+D^{-1}b \\
&= HX^{(k)}+C \qquad \text{------(2.16)} \\
& \quad k = 0,1,2,\ldots\ldots\ldots
\end{aligned}
$$

where $H = -D^{-1}(L+U)$ & $C = D^{-1}\mathbf{b}$ and L, U are respectively lower and upper triangular matrices with zero diagonal entries and D is the diagonal matrix such that $A = L+D+U$.

Now (2.16) is also written as

$$
\begin{aligned}
x^{(k+1)} &= x^{(k)}-[I+D^{-1}(L+U)x^{(k)}+D^{-1}\mathbf{b}] \\
&= x^{(k)}-D^{-1}[D+L+U]x^{(k)}+D^{-1}\mathbf{b} \\
&= x^{(k)}+D^{-1}[\mathbf{b}-Ax^{(k)}] \\
\text{(ie)}\ x^{(k+1)}-x^{(k)} &= D^{-1}(\mathbf{b}-Ax^{(k)}) \\
v^{(k)} &= D^{-1}r^{(k)} \\
\text{where}\ v^{(k)} &= x^{(k+1)}-x^{(k)}\ \&\ r^{(k)} = \mathbf{b}-Ax^{(k)}
\end{aligned}
$$

Here $v^{(k)}$ is the error in the approximation and $r^{(k)}$ is the residual.

Hence we have $Dv^{(k)} = r^{(k)}$ and therefore we solve for $v^{(k)}$ and find $x^{(k+1)} = x^{(k)}+v^{(k)}$

2.10 Gauss-Seidel Iteration Method :

For the equations (2.1), using Gauss-Jeidel iteration method, the (k+1)th iterated value be written as

$$
\left.
\begin{aligned}
x_1^{(k+1)} &= \frac{-1}{a_{11}}\left(a_{11}x_2^{(k)}+a_{13}x_3^{(k)}+\ldots\ldots+a_{1n}x_n^{(k)}\right)+\frac{b_1}{a_{11}} \\
x_2^{(k+1)} &= \frac{-1}{a_{22}}\left(a_{21}x_1^{(k+1)}+a_{23}x_3^{(k)}+\ldots\ldots+a_{2n}x_n^{(k)}\right)+\frac{b_2}{a_{22}} \\
&\vdots \\
x_n^{(k+1)} &= \frac{-1}{a_{nn}}\left(a_{n1}x_1^{(k+1)}+a_{n2}x_2^{(k+1)}+\ldots\ldots+a_{n,n-1}x_{n-1}^{(k+1)}\right)+\frac{b_n}{a_{nn}}
\end{aligned}
\right\} \qquad \text{-----(2.17)}
$$

The above equations (2.17) can be written as

$$\left.\begin{aligned}
a_{11}x_1^{(k+1)} &= -\sum_{i=2}^{n} a_{1i}x_i^{(k)} + b_1 \\
a_{21}x_1^{(k+1)} + a_{22}x_2^{(k+2)} &= -\sum_{i=3}^{n} a_{2i}x_i^{(k)} + b_2 \\
&\;\vdots \\
a_{n1}x_1^{(k+1)} + a_{n2}x_2^{(k+1)} + \ldots\ldots + a_{nn}x_n^{(k+1)} &= b_n
\end{aligned}\right\} \quad \text{------}(2.18)$$

Now (2.18) can be written in the matrix form as

$$\begin{aligned}
(D+L)x^{(k+1)} &= -Ux^{(k)}+\mathbf{b} \\
\text{(i.e.,)}\ x(k+1) &= -(D+L)^{-1}Ux^{(k)}+(D+L)^{-1}\mathbf{b} \\
\text{(i.e.)}\ x^{(k+1)} &= Hx^{(k)}+C \qquad \text{------}(2.19) \\
&\qquad k = 0,1,2,\ldots\ldots \\
\text{where}\ H &= -(D+L)^{-1}U\ \&\ C=(D+L)^{-1}\mathbf{b}. \\
\text{(i.e.,)}\ x^{(k+1)} &= x^{(k)}-[I+(D+L)^{-1}U]x^{(k)}+(D+L)^{-1}\mathbf{b} \\
\text{(i.e.,)}\ x^{(k+1)} &= x^{(k)}-(D+L)^{-1}Ax^{(k)}+(D+L)^{-1}\mathbf{b} \\
\text{(i.e.,)}\ x^{(k+1)}-x^{(k)} &= (D+L)-1(\mathbf{b}-Ax(k)) \\
\text{(i.e.,)}\ v^{(k)} &= (D+L)^{-1}r^{(k)} \\
\text{where}\ v^{(k)} &= x^{(k+1)}-x^{(k)}\ \&\ r^{(k)} = \mathbf{b}-Ax^{(k)} \\
\text{(i.e.,)}\ (D+L)v^{(k)} &= r^{(k)}
\end{aligned}$$

Solve $v^{(k)}$ by forward substitution.

The solution is then found from $x^{(k+1)} = x^{(k)}+v^{(k)}$.

Example E. 2.9 :

Consider the system of equations

$2x_1-x_2 = 7,$

$-x_1+2x_2-x_3 = 1,$

$-x_2+2x_3 = 1$

Use the Gauss-Seidel iterative method and perform three iterations.

Solution :

The matrix form of the given system of equations is AX = b where

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Now } A = L+D+U \text{ where } L = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$\text{and } U = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Using Gauss-Seidel iterative method, the approximated values of x_1, x_2, x_3 are given by

$$x^{(k+1)} = -(D+L)^{-1}Ux^{(k)}+(D+L)^{-1}\mathbf{b} \quad \text{-------(2.19a)}$$

$$\text{Now } D+L = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\therefore (D+L)^{-1}U = \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & -\frac{1}{8} & -\frac{1}{4} \end{bmatrix}$$

$$\text{and } (D+L)^{-1}\mathbf{b} = \begin{bmatrix} 7/2 \\ 9/4 \\ 11/8 \end{bmatrix}$$

$$\therefore (2.19a) \Rightarrow x^{(k+1)} = \begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{bmatrix} x^{(k)} + \begin{bmatrix} 7/2 \\ 9/4 \\ 13/8 \end{bmatrix}, \quad k = 0, 1, 2,,......$$

Assuming $x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, we get

$$x^{(1)} = \begin{pmatrix} 3.5 \\ 2.25 \\ 1.625 \end{pmatrix}, \; x^{(2)} = \begin{pmatrix} 4.625 \\ 3.625 \\ 2.3125 \end{pmatrix} \; \& \; x^{(3)} = \begin{pmatrix} 5.3115 \\ 4.3125 \\ 2.6563 \end{pmatrix}.$$

2.11 Matrix Norm :

The matrix norm, $||A||$, is a non-negative number which satisfies the following properties :

(i) $||A||>0$; if $A \neq 0$ and $||0|| = 0$

(ii) $||cA|| = |c| \, ||A||$, for $c \in \boldsymbol{C}$.

(iii) $||A+B|| \leq ||A||+||B||$

(iv) $||AB|| \leq ||A|| \, ||B||$

2.12 Convergence Analysis :

Using this analysis, we study the behaviour of the difference between the exact solution x and the approximation $x^{(k)}$.

If x is the exact solution, then, we have $x = Hx+C$ ------(2.20)

we know that $x^{(k+1)} = Hx^{(k)}+C$ ------(2.21)

Subtraction (2.20) from (2.21), we get,

$$x^{(k+1)}-x = H(x^{(k)}-x)$$

$$\text{(i.e.,)} \; \in^{(k+1)} = H\in^{(k)} \text{ where } \in^{(k)} = x^{(k)}-x, \qquad k = 0, 1, 2, \ldots\ldots\ldots$$

$$\text{Thus } \in^{(k)} = H^k \in^{(0)}, \; k = 0, 1, 2,,,,\ldots\ldots$$

(Here we have the assumption that the itertion matrix H remains constant for each iteration).

Theorem T. 2.1 :

Let A be a square matrix. Then $\lim\limits_{m \to \infty} A^m = 0$ if $||A||<1$ or if and only if $\rho(A)<1$.

Proof :

$$\text{If } \|A\|<1 \Rightarrow \|A^m\| \le \|A\|^m$$

$$\Rightarrow \|\lim A^m\| \le \lim\|A\|^m = 0$$

For our convenience, assume that all the eigen values of A are distinct. Then there exists a similarity transformation S, such that $A = S^{-1}DS$ where D is the diagonal matrix having the eigen values of A on the diagonal.

$$\therefore A^m = S^{-1}D^mS$$

$$\text{where } D^m = \begin{bmatrix} \lambda_1^m & & O \\ & \lambda_2^m \ddots & \\ O & & \lambda_n^m \end{bmatrix}$$

Now $\lim\limits_{m\to\infty} A^m = 0$ iff all the eigen values satisfy $|\lambda_i|<1$; that is $\rho(A)<1$.

Theorem T. 2.2 :

The infinite series $I+A+A^2+.....$ converges if $\lim\limits_{m\to\infty} A^m = 0$. The series equations to $(I-A)^{-1}$.

Proof :

If $\lim\limits_{m\to\infty} A^m = 0$, then by theorem (T.2.1), $\rho(A)<1$.

$\because |I-A|\neq 0$ & hence $(I-A)^{-1}$ exists

We know that $(I+A+A^2+.......+A^m)(I-A) = I-A^{m+1}$

$$\Rightarrow I+A+A^2+.....+A^m = (I-A^{m+1})(I-A)^{-1}$$

$$\Rightarrow \lim_{m\to\infty}(I+A+A^2+......A^m) = \lim_{m\to\infty}\left[(I-A^{m+1})(I-A)^{-1}\right]$$

$$\Rightarrow I+A+A^2+........ = (I-A)^{-1}.$$

This proves the theroem.

Theorem T. 2.3 :

No eigenvalues of a matrix A exceeds the norm of the matrix.

Proof :

$$
\begin{aligned}
\text{We know that } AX &= \lambda X \\
\Rightarrow \|AX\| &= \|\lambda x\| \\
\Rightarrow \|\lambda X\| &= \|AX\| \le \|A\|\,\|X\|. \\
\Rightarrow |\lambda|\,\|X\| &= \|A\|\,\|X\| \\
\Rightarrow |\lambda| &\le \|A\|
\end{aligned}
$$

This proves the theorem.

Theorem T. 2.4 :

The iteration method of the form $x^{(k+1)} = Hx^{(k)}+C$ for the solution of $AX=b$ converges to the exact solution for any initial vector; if $\|A^{-1}\|<1$.

Proof :

Without loss of generality, assume that $x^{(0)} = 0$

$$
\begin{aligned}
\therefore x^{(k+1)} &= Hx^{(k)}+C \\
&= H(1+x^{(k-1)}+C)+C \\
&= H^2x^{k-1}+HC+C \\
&= H^3x^{k-2}+H^2C+HC+C \\
&\bullet \\
&\bullet \\
&= (H^k+H^{k-1}+H^{k-2}+.......+I)C
\end{aligned}
$$

$$
\therefore \lim_{k\to\infty} x^{(k+1)} = (I-H)^{-1}C \quad \text{if } \|H\|<1. \qquad \text{------(2.22)}
$$

This proves the result.

Note : The above theorem can be verified for Jacobi method.

$$
\begin{aligned}
(I-H)^{-1}C &= [I+D^{-1}(L+U)]^{-1}D^{-1}\mathbf{b} \\
&= [D^{-1}D+D^{-1}(L+U)]^{-1}D^{-1}\mathbf{b} \\
&= [D^{-1}(D+L+U)]^{-1}D^{-1}\mathbf{b} \\
&= (D^{-1}A)D^{-1}\mathbf{b} \\
&= (A^{-1}D)(D^{-1}\mathbf{b}) \\
&= A^{-1}\mathbf{b} \quad = X
\end{aligned}
$$

Thus the theorem be verified for Jocabi method.

Theorem T. 2.5 :

A necessary and sufficient condition for convergence of an iterative method of the form $x^{(k+1)} = Hx^{(k)}+C$ is that the eigenvalues of the iteration matrix satisfy $|\lambda_i(H)|<1$; $i = 1,2,3,.......n$.

Proof :

We prove the theorem for the case when the iteration matrix H has n independent eigen-vectors $x_1, x_2,......,x_n$ with eigenvalues $\lambda_1, \lambda_2,.....\lambda_n$ respectively.

The error vector $\in^{(0)} = C_1x_1+C_2x_2+......+C_nx_n$ -------(2.23)

Using the relation $\in^{(k)} = H^k\in^{(0)}$, (2.23) be rewritten as

$$\in^{(k)} = C_1\lambda_1^kx_1+C_2\lambda_2^kx_2+........+C_n\lambda_n^kx_n \quad \text{-------(2.24)}$$

(i) Necessary Condition :

If $\lim\limits_{k\to\infty} \in^{(k)} = 0$ for any arbitrary initial vector $x^{(0)}$ and thus for any arbitrary error vector $\in(0)$, then by (2.24), $|\lambda_i|<1$, $i = 1(1)n$.

(ii) Sufficient Condition :

For $|\lambda_i|<1$, $i = 1(1)n$, then convergence of $\in^{(k)}$ towards the zero vector follows from (2.24).

This proves the theorem.

Definition D. 2.1 :

The rate of convergence of an iterative method is $v = -\log \rho(H)$ where $\rho(H)$ is the spectral radius of H.

2.13 Iterative Method for A^{-1} :

Let A be a non-singular square matrix and B be the approximate inverse of A.

$\therefore AB \neq I.$

The error matrix is $E = AB-I$

(i.e.,) $AB = E+I$ ------(2.25)

From (2.25), we have,

$$A^{-1} = B(I+E)^{-1}$$

$$= B(I-E+E^2-E^3+......) \text{ if } ||E||<1 \quad \text{-------(2.26)}$$

$$\text{Again } A^{-1} \cong B(I-E)$$

$$A^{-1} = B(2I-AB)$$

which gives an iterative method

$$B^{(k+1)} = B^{(k)}(2I-AB^{(k)}) \qquad \text{-------(2.27)}$$

$$k = 0, 1, 2, \ldots\ldots$$

Note : The convergence of the iterative method is guadratic, because

$$AB^{(k+1)} = AB^{(k)}(2I-AB^{(k)})$$

$$= AB^{(k)}-(AB^{(k)})^2$$

$$\text{Hence } I-AB^{(k+1)} = (I-AB^{(k)})^2$$

Thus the convergence of the iterative.

Example E. 2.10 :

Find the inverse of the matrix $\begin{pmatrix} 5 & 2 \\ 3 & -1 \end{pmatrix}$ using the iterative method, given that its approximated inverse is $\begin{pmatrix} 0.1 & 0.2 \\ 0.3 & -0.4 \end{pmatrix}$. Perform two iterations.

Solution :

Given that $A = \begin{pmatrix} 5 & 2 \\ 3 & -1 \end{pmatrix}$ & $B^{(0)} = \begin{pmatrix} 0.1 & 0.2 \\ 0.3 & -0.4 \end{pmatrix}$.

We know that for iterative method,

$B^{(k+1)} = B^{(k)}(I-AB^{(k)})$; $k = 0,1,2,3,\ldots\ldots$

Step 1 :

$$\text{Now } AB^{(0)} = \begin{pmatrix} 5 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 0.1 & 0.2 \\ 0.3 & -0.4 \end{pmatrix}$$

$$= \begin{pmatrix} 1.1 & 0.2 \\ 0 & 1 \end{pmatrix}$$

$$\therefore I-AB^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1.1 & 0.2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -0.1 & -0.2 \\ 0 & 0 \end{pmatrix}$$

$$\therefore\; B^{(1)} = B^{(0)}(I-AB^{(0)})$$

$$= \begin{pmatrix} 0.1 & 0.2 \\ 0.3 & -0.4 \end{pmatrix}\begin{pmatrix} -0.1 & -0.2 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.1 & -0.2 \\ -0.03 & -0.06 \end{pmatrix}$$

Step 2 :

$$AB^{-1} = \begin{pmatrix} 5 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 0.1 & -0.2 \\ -0.03 & -0.06 \end{pmatrix}$$

$$= \begin{pmatrix} 0.44 & -1.12 \\ 0.33 & -0.54 \end{pmatrix}$$

$$\therefore\; I-AB^{(1)} = \begin{pmatrix} 0.56 & 1.12 \\ -0.33 & 1.54 \end{pmatrix}$$

$$\therefore\; B^{(2)} = B^{(1)}(I-AB^{(1)})$$

$$= \begin{pmatrix} 0.1 & -0.2 \\ -0.03 & -0.06 \end{pmatrix}\begin{pmatrix} 0.56 & 1.12 \\ -0.33 & 1.54 \end{pmatrix}$$

$$= \begin{pmatrix} 0.122 & -0.196 \\ 0.003 & -0.126 \end{pmatrix}$$

2.14 Eigenvalues and Eigenvectors :

The eigenvalues of a matrix A are given by the roots of the characteristic equations

$$\det\;(A-\lambda I) = 0 \qquad \text{-------(2.28)}$$

On expanding the determinant in LHS of (2.28), we get a polynomial

$$p(\lambda) = (-1)^n\lambda^n + a_1\lambda^{n-1} + \ldots\ldots + a_n = 0 \qquad \text{--------(2.29)}$$

where $(-1)^n$ is used to give the sign of the terms of the polynomial.

2.14.1 Faddeev - Leverrier Method :

This method is used to find the coefficients of the characteristic polynomial of A given in (2.29).

Let $B_1 = A$ and $a_1 = tr(B_1)$

$B_2 = A\,(B_1 - a_1 I)$ and $a_2 = \frac{1}{2}\text{tr}(B_2)$

$B_3 = A\,(B_2 - a_2 I)$ and $a_3 = \frac{1}{3}\text{tr}(B_3)$

•

•

•

$B_n = A\,(B_{n-1} - a_{n-1} I)$ and $a_n = \frac{1}{n}\text{tr}(B_n)$

Here tr A = sum of the diagonal entries of A.

Theorem T. 2.6 : Gerschgorin's Theorem :

Statement : The largest eigenvalue in modulus of the square matrix A cannot exceed the largest sum of the module of the elements along any row or any column.

Solution :

Let λ_i be an eigenvalue of A and x_i be the corresponding eigenvector.

Then $Ax_i = \lambda_i, x_i$

On expanding, we get,

$$\left.\begin{array}{l} a_{11}x_{i1} + a_{12}x_{i2} + \dots\dots\dots + a_{1n}x_{in} = \lambda_i x_{i1} \\ a_{21}x_{i1} + a_{22}x_{i2} + \dots\dots\dots + a_{2n}x_{in} = \lambda_i x_{i2} \\ \vdots \\ a_{n1}x_{i1} + a_{n2}x_{i2} + \dots\dots\dots + a_{nn}x_{in} = \lambda_i x_{in} \end{array}\right\} \qquad \text{------(2.30)}$$

Let $|x_{ik}| = \max\limits_{r}\{|x_{ir}|\}$

Select k^{th} equation in (2.30) and divide it by x_{ik}, we get,

$$\lambda_i = a_{k1}\left(\frac{x_{i1}}{x_{ik}}\right) + a_{k2}\left(\frac{x_{i2}}{x_{in}}\right) + \dots\dots + a_{kk} + \dots\dots + a_{kn}\left(\frac{x_{in}}{x_{ik}}\right) \text{----(2.31)}$$

$$\text{and } |\lambda_i| \le |a_{k1}| + |a_{k2}| + \dots\dots + |a_{kk}| + \dots\dots + |a_{kn}| \qquad \text{------(2.32)}$$

$$\left(\because \left|\frac{x_{ij}}{x_{ik}}\right| \le 1;\ j = 1(1)n\right)$$

Since k is unknown, we have from (2.32)

$$|\lambda| \le \max_i \left[\sum_{j=1}^{n} |a_{ij}| \right]$$

$\therefore$ This theroem is true for any row.

Since the eigenvalues of A^T are same as those of A. & $\therefore$ the theorem is truefor any column.

This proves the theorem.

2.7 BRAUER Theorem :

Let P_k be the sum of the modulii of the elements along the k^{th} row excluding the diagonal element a_{kk}. Then every eigenvalue of λ lies inside or on the boundary of atleast one of the circles $|\lambda - a_{kk}| = P_k$; $k = 1(1)n$.

Proof :

From (2.31),

$$\lambda_i - a_{kk} = a_{k1}\left(\frac{x_{i1}}{x_{ik}}\right) + a_{k2}\left(\frac{x_{i2}}{x_{ik}}\right) + \ldots + a_{kn}\left(\frac{a_{in}}{a_{ik}}\right)$$

$$\therefore \ |\lambda_i - a_{kk}| \le \sum_{\substack{j=1 \\ i \ne k}}^{n} \left|a_{jk}\right| = P_k \qquad \text{------(2.33)}$$

(i.e.,) all the eigenvalues of A lie inside or on the union of the circle (2.33).

This proves Brauer's theorem.

2.15 Jacobi Method for Symmetric Matrices :

Let A be a real symmetric matrix.

The eigenvalues of A are real and there exists a real orthogonal matrix S such tht $S^{-1}AS$ is a diagonal matrix D. The diagonalization is done by applying a series of orthogonal transformations $S_1, S_2, \ldots S_n \ldots$ as follows :

Choose $|a_{ik}|$ be the numerically largest element among the off-diagonal elements of A.

Then the elements $\begin{pmatrix} a_{ii} & a_{ik} \\ a_{ki} & a_{kk} \end{pmatrix}$ form a 2×2 sub-matrix A_1 which can be transformed to a diagonal form.

$$\text{Let } S_1^* = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{-------}(2.34)$$

and find θ such that 2×2 submatrix A_1 be diagonalised.

$$\therefore S_1^{*-1}A_1S_1^* = \begin{bmatrix} a_{ii}\cos2\theta + a_{ik}\sin2\theta + a_{kk}\sin2\theta & (a_{kk}-a_{ii})\sin\theta\cos\theta + a_{ik}\cos2\theta \\ (a_{kk}-a_{ii})\sin\theta\cos\theta + a_{ik}\cos2\theta & a_{ii}\sin2\theta - a_{ik}\sin2\theta + a_{kk}\cos2\theta \end{bmatrix} \text{---}(2.35)$$

Choose θ such that $S_1^{*-1}A_1S_1^*$ becomes a diagonal matrix.

$\therefore (a_{kk}-a_{ii})\sin\theta\cos\theta + a_{ik}\cos2\theta = 0.$

$$\Rightarrow \tan2\theta = \frac{2a_{ik}}{a_{ii}-a_{kk}} \quad \text{-------}(2.36)$$

This equation produces four values of θ and in order that we may get smallest rotation, we require $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.

Now (2.36) becomes $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{ik}}{a_{ii}-a_{kk}}\right)$ if $a_{ii} \ne a_{kk}$ --------(2.37)

and if $a_{ii} = a_{kk}$ then (2.36) $\Rightarrow \theta = \begin{cases} \frac{\pi}{4}; & a_{ik} > 0 \\ -\frac{\pi}{4} & a_{ik} < 0 \end{cases}$ -------(2.38)

With the value of θ, the off-diagonal elements in (2.35) varish and the diagonal elements are simplifed. Thus the rotation of $S_1^{-1}AS_1$ is completed.

Next step is to find new rotation matrix by choosing the largest off-digonal element in magnitude in $S_1^{-1}AS_1$. Perform a series of such two-dimensional rotations. After finding θ at each step, the rotation is performed with the corresponding orthogonal matrix.

After finding θ at each step, the rotation is performed with the corresponding orthogonal matrix. For example, if $|a_{ik}|$ is the largest off-diagonal element then S_1 is given by

$$S_1 = \begin{bmatrix} 1 & & & & & & \\ & 1 \ddots & & & & & \\ & & \cos\theta & \cdots & -\sin\theta & & \\ & & \vdots & 1 \ddots & \vdots & & \\ & & \vdots & 1 & \vdots & & \\ & & \sin\theta & \cdots & \cos\theta \ddots & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix}$$

where $\cos\theta$, $-\sin\theta$, $\sin\theta$, $\cos\theta$ are located at (i, i), (i, k), (k, i) and (k, k) positions respectively. After making r transformations, we get,

$$B_r = S_r^{-1} S_{r-1}^{-1} \ldots\ldots S_1^{-1} A S_1 S_2 \ldots\ldots S_{r-1} S_r$$

$$= S^{-1} AS \qquad \text{-------(2.39)}$$

$$\text{where } S = S_1 S_2 \ldots\ldots S_r$$

As $r \to \infty$, B_r approches a diagonal matrix with the eigen values on the leading diagonal and corresponding columns of S gives the eigenvector of A.

Example E. 2.11 :

Apply Jacobi's method to find the eigen values of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

Solution :

$$\text{Given that } A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

The largest off-diagonal element is 1.

First Rotation :

Consider $a_{23} = a_{32} = 1$ and $\therefore\ a_{22} = a_{33} = 4$.

$$\therefore\ \tan 2\theta = \frac{2\times 1}{4-4} = \infty$$

$$\Rightarrow\ 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

$$\text{Thus } S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ 0 & \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7071 & -0.7071 \\ 0 & 0.7071 & 0.7071 \end{bmatrix}$$

$$\text{and } S_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7071 & 0.7071 \\ 0 & -0.7071 & 0.7071 \end{bmatrix}$$

$$\therefore\ B_1 = S_1^{-1} A S_1$$

$$= \begin{bmatrix} 2 & 0.7071 & -0.7071 \\ 0.7071 & 4.9999 & 0 \\ -0.7071 & 0 & 2.9999 \end{bmatrix}$$

Second Iteration :

In B_1 the largest off-diagonal value is $a_{12} = a_{21} = 0.7071$

$$\therefore\ \tan 2\theta = \frac{2\times 0.7071}{2-4.9999}$$

$$\Rightarrow\ \theta = -27^{o}.22' \text{ (using trigonometric tables)}$$

$$\Rightarrow\ \cos\theta = 0.8881\ \&\ \sin\theta = -0.4596$$

$$\Rightarrow S_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8881 & -0.4596 & 0 \\ 0.4596 & 0.8881 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

giving new value of A, say B_2.

$$\therefore B_2 = S_2^{-1} B_1 S_2$$

$$= \begin{bmatrix} 2.0562 & -0.8161 & -0.6279 \\ -0.8160 & 4.9431 & -0.3249 \\ -0.6279 & -0.3249 & 0 \end{bmatrix}$$

continuing this process, after 12 iterations,

$$\text{we get, } B_{12} = \begin{bmatrix} 1.52 & 0 & 0 \\ 0 & 5.17 & 0 \\ 0 & 0 & 3.31 \end{bmatrix}$$

$\therefore$ The eigenvalues of A are 1.52, 5.17, 3.31.

Example E. 2.12 :

Find all the eigenvalues and eigen vectors of the matrix.

$$\begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

Solution :

$$\text{Let } A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

The largest off-diagonal element is $a_{13}=a_{31}=2$. The other two elts in this (2, 2) submatrix are $a_{11}=1$ & $a_{33}=1$.

$$\therefore \quad \frac{1}{2}\tan^{-1}\left(\frac{4}{0}\right) = \frac{\pi}{4}$$

$$\therefore \; S_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\text{Hence } B_1 = S_1^{-1} A S_1 = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Again, the largest off-diagonal element in B_1 is $a_{12}=a_{21}=2$. The other elements are $a_{11}=a_{22}=3$.

$$\text{Thus } \theta = \pi/4$$

$$\text{and } S_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore \; B_2 = S_2^{-1} B_1 S_2 = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\text{and } S = S_1 S_2 = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Thus the eigenvalues are 5, 1 and –1 and the corresponding eigenvectors are the columns of S.

2.17 GIVENS METHOD FOR SYMMETRIC MATRICES :

Let A he a real, symmetic matrix. The Given's method has the following steps.

1) reduce A to a tridiagonal form using plane rotation.
2) form a Sturm sequence, study the changes in sign in the sequence and find the eigenvalues.
3) find the signvectors.

The reduction to a tridiagonal form is achieved by using the orthoyonal transformations as in the Jacobimethod. However, in this case we start with the subspace containing the elements a_{22}, a_{23}, a_{32}, a_{33}.

Perform the plane rotation $S_1^{-1}AS_1$, using the orthogonal matrix

$$S_1^* = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \text{-------(2.40)}$$

Let the new matrix be $A^1 = (a^1_{ij})$ and the angle θ be obtained by putting $a^1_{13} = a^1_{31} = 0$ and not be putting $a^1_{23} = 0 = a^1_{32}$ as in the Jacobi method.

Thus we find $a^1_{13} = -a_{12}\sin\theta + a_{13}\cos\theta = 0$

$$\Rightarrow \tan\theta = \frac{a_{13}}{a_{12}} \quad \text{-------(2.41)}$$

With this θ and make the plane rotation, we produce zeros in (3, 1) and (1, 3) positions. Then we perform rotations in (2, 4) subspace and put $a^1_{14} = a^1_{41} = 0$. This would not affect zeros that have been obtained earlier. Proceeding in this way we put $a^1_{15} = a^1_{51} = 0$ etc. by performing rotations in (2, 5),.......(2, n) subspaces. Then we pass on to the elements a^1_{24}, a^1_{25},......,a^1_{2n} and make them zero by performing rotations in (3, 4),......(3, n) subspaces. Hence we have a matrix of the form.

$$B = \begin{bmatrix} b_1 & c_1 & & & & \\ c_1 & b_2 & c_2 & & & \\ & c_2 & b_3 & c_3 & & O \\ & & & \ddots & & \\ & O & & c_{n-2} & b_{n-1} & c_{n-1} \\ & & & & c_{n-1} & b_n \end{bmatrix} \quad \text{------(2.42)}$$

Note that A and B have the same eigenvalues.

Let $f_n = |\lambda I - B|$

$$= \begin{vmatrix} \lambda - b_1 & -c_1 & & & & \\ -c_1 & \lambda - b_2 & c_2 & & & O \\ & & & \ddots & & \\ & O & & -c_{n-2} & \lambda - b_{n-1} & -c_{n-1} \\ & & & & -c_{n-1} & \lambda - b_n \end{vmatrix}$$

Expanding by minors, the sequence $\{f_n\}$ satisfies $f_0 = 1$, $f_1=\lambda-b_1$ and

$$fr = (\lambda-b_r)f_{r-1}-c^2_{r-1}f_{r-2};\ 2 \le r \le n \qquad \text{-------(2.43)}$$

Note that f_n is the characteristic equation. If any $c_i = 0$, then the system degenerate as

$$B = \begin{bmatrix} P & O \\ O & Q \end{bmatrix}$$

$\therefore$ $f_n = (\lambda I-B)$ = (characteristic equation of P) (characteristic equation of Q)

If none of the $c_1, c_2, \ldots c_{n-1}$ vanish, then $\{f_n\}$ is a sturm sequence.

(i.e.,) if V(x) is the number of changes in sign in the sequence for a given number x, then the number of zeros of f_n in [a, b] is V(a)–V(b) provided a or b is not a zero of f_n. In this way we may approximately compute the eigenvalues and by repeated bisections, we can improve these estimeters.

If v and u are the eigenvectors of B and A respectively there u = Sv where $S = S_1S_2 \ldots\ldots S_j$ is the product of the orthogonal matrixes need in the plane rotations. The eigenvectors of B may be found as follows :

Neglect a particular equation say i^{th} equation and then solve the remaining equations. Then v is the eigenvector determined from these solutions and by putting a zero in the i^{th} position.

Example E. 2.13 :

Transform the matrix to $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ to tridiagonal form using Givens method. Find the eigenvector corresponding to the largest eigenvalue from the eigenvectors of the tridiagonal matrix.

Solution :

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\therefore\ \tan\theta = \frac{a_{13}}{a_{12}} = 1$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

$$\text{Hence } S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ 0 & \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\therefore B_1 = S^{-1}AS = \begin{bmatrix} 1 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

which is the required tridiagonal form.

Again the characteristic equation of B_1 is

$$f_n = |\lambda I - B| = \begin{vmatrix} \lambda-1 & -2\sqrt{2} & 0 \\ -2\sqrt{2} & \lambda-3 & 0 \\ 0 & 0 & \lambda+1 \end{vmatrix}$$

The Sturm sequence $\{f_n\}$ is given by

$$f_0 = 1$$

$$f_1 = \lambda-1$$

$$f_2 = (\lambda-3)f_1-(-2\sqrt{2})^2 f_0 = \lambda^2-4\lambda-5$$

$$f_3 = (\lambda+1)(\lambda+1)(\lambda-5)$$

Now $f_3(-1) = 0$ & $f_3(5) = 0$

and eigenvalues of A are $-1, -1, 5$ and the largest eigenvalue in magnitude is 5.

The eigenvector corresponding to B_1 is $v_1 = (1\ \sqrt{2}\ 0)^T$ and the corresponding eigenvector of A is $v = 5v_1 = (1\ 1\ 1)^T$.

Example E. 2.14 :

Use the Givens method to find the eigenvalues of the tridiagonal matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Solution :

$$\text{Let } A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\begin{aligned} \text{Thus } f_n &= |\lambda I - B| \\ &= |\lambda I - A| \ (\because \text{ here } B = A) \\ &= \begin{vmatrix} \lambda-2 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-2 \end{vmatrix} \end{aligned}$$

Hence the Sturm sequence is

$$\begin{aligned} f_0 &= 1, \\ f_1 &= \lambda-2, \\ f_2 &= (\lambda-2)f_1 - f_0 = (\lambda-2)^2-1 \\ f_3 &= (\lambda-2)f_2 - f_1 = (\lambda-2)^3-2(\lambda-2). \end{aligned}$$

Let $V(x)$ denote the number of changes in sign in the sequence, for a given number x.

λ	f_0	f_1	f_2	f_3	$V(\lambda)$
–1	+	–	+	–	3
0	+	–	+	–	3
1	+	–	0	+	2
2	+	0	–	0	–
3	+	+	0	–	1
4	+	+	+	+	0

Here $f_3(2)=0$ and therefore $\lambda=2$ is an eigenvalue and there is an eigenvalue in (0, 1) and (3, 4).

To find an approximate eigenvalue in (0, 1) we use bisection method.

λ	f_0	f_1	f_2	f_3	$V(\lambda)$
1/2	+	–	+	–	3

The eigenvalue is located in (0.5, 1)

Again 0.75	+	–	+	+	2

The eigenvalue is located in (0.5, 0.75)

$\therefore$ 0.625	+	–	+	+	2

The eigenvalue is located in (0.5, 0.625)

$\therefore$ 0.5625	+	–	+	–	3

The eigenvalue is located in (0.5625, 0.625)

0.59375	+	–	+	+	2

The eigenvalue is located in (0.5625, 0.59375)

We repeat this procedure until the required accuracy.

Similary we can find required approximated eigenvalue which lies in (3, 4).

2.18 HOUSEHOLDER'S METHOD FOR SYMMETRIC MATRICES :

In this method, the given matrix A is reduced to tridiagonal form by orthogonal transformations representing reflexions. The orthogonal transformations are of the form

$$P = I-2WW^T \qquad \text{------(2.43)}$$

where W is a column vector, $W \in \mathbf{R^n}$ such that

$$W^TW = x_1^2+x_2^2+\dots\dots+x_n^2 = 1 \qquad \text{-------(2.44)}$$

Clearly P is symmetric and orthogonal.

The vectors W are construted with the first (r–1) components as zeros,

$$\text{(i.e.,)} \; W_r^T = (0,0,\dots\dots\dots,0, x_r, x_{r+1},\dots\dots\dots,x_n)$$

$$\text{Since } W_r^TW_r = 1$$

$$\Rightarrow \quad x_r^2+x_{r+1}^2+\dots\dots x_n^2 = 1$$

with this choice of W_r form the matrices $P_r = I-2W_rW_r^T$

The similarity transformation is given by

$$P_r^{-1}AP_r = P_r^TAP_r = P_rAP_r \qquad \text{------(2.45)}$$

since P_r is symmetric & orthogonal.

Put A = A1 and form successively

$$Ar = P_r A_{r-1} P_r, \quad r = 2,3,......n-1 \qquad \text{------(2.46)}$$

At the first transformation, we find x_r^1s such that we get zeros in the positions (1, 3), (1, 4),.......(1, n) and in the corresponding positions in the first column.

∴ one rotation brings n–2 zeros in the first row and column.

In the second rotation, we find x_r^1s such that we have zeros in (2, 4),(2, 5),...,(z, n) positions. The final matrix is a tridiagonal matrix as in the Givens method.

After getting tridiagonal matrix, then the procedure of finding eigenalue and eigenvector is same as in the Givens method.

2.19 RUTISHASTER METHOD FOR ARBITRARY MATRICES :

Rutishaster proposed the LU transformation.

In the limit, the upper triangular matrix gives the eigenvalues of A on the leading diagonal.

Starting with the matrix $A=A_1$, split it into two triangular matrices.

$$A_1 = L_1U_1 \text{ with } l_{ii} = 1$$

$$\text{Then form } A_2 = U_1L_1$$

$$\text{Since } A_2 = U_1A_1U_1^{-1}, A_1 \ \& \ A_2 \text{ have the same eigenvalues.}$$

$$\text{Again } A_2 = L_2U_2 \text{ with } l_{ii} = 1.$$

$$\text{Form } A_3 = U_2L_2$$

Proceeding in this way, we get a sequence of matrices A_1, A_2, A_3,.. which reduces to an upper triangular matrix.

Note : There are difficulties associated with the practical application of the method.

Example E. 2.15 :

Find all the eigenvalues of the matrix $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}$ using the Hauseholder method.

Solution :

Choose $W_2^T = (0 \ x_2 \ x_3)$ such that $x_2^2+x_3^2=1$.

The parameters in the first Householder transformation are obtained as follows :

$$s_1 = \sqrt{a_{12}^2 + a_{13}^2} = \sqrt{5}$$

$$x_2^2 = \frac{1}{2}\left[1 + \frac{a_{12}}{s_1}\,\mathrm{sign}(a_{12})\right]$$

$$= \frac{1}{2}\left(1 + \frac{2}{\sqrt{5}}\right)$$

$$= \frac{\sqrt{5}+2}{2\sqrt{5}}$$

$$x_3 = \frac{a_{13}\,\mathrm{sign}(a_{12})}{2s_1x_2}$$

$$= \frac{-1}{2s_1x_2}$$

$$x_2x_3 = \frac{-1}{2\sqrt{5}}$$

$$\therefore\ P_2 = I - 2W_2W_2^T$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

The required Householder transformation is

$$A_2 = P_2A_1P_2 = \begin{pmatrix} 1 & -\sqrt{5} & 0 \\ -\sqrt{5} & \frac{-3}{5} & \frac{-6}{5} \\ 0 & \frac{6}{5} & \frac{13}{5} \end{pmatrix}$$

Using Givens method, Sturm's sequence are given by

$$f_0 = 1$$

$$f_1 = \lambda - 1$$

$$f_2 = \lambda^2 - (2/5)\lambda - (28/5)$$

$$f_3 = \lambda^3 - 3\lambda^2 - 6\lambda + 16$$

Let $V(\lambda)$ denote the number of changes in sign in the sturm sequence. We have the following table giving $V(\lambda)$.

λ	f_0	f_1	f_2	f_3	$V(\lambda)$
–3	+	–	+	–	3
–2	+	–	–	+	2
–1	+	–	–	+	2
0	+	–	–	+	2
1	+	+	–	+	2
2	+	+	–	0	1
3	+	+	+	–	1
4	+	+	+	+	0

Since $f_3 = 0$ for $\lambda=2$, $\lambda=2$ is an eigenvalue.

The remaining two eigenvalues lie in the intervals $(-3, -2)$ and $(3, 4)$.

Using repeated bisection and application of Sturm's sequence gives the eigenvalues as $\lambda_2 = -2.372$ and $\lambda_3 = 3.372$.

Example E. 2.16 :

Find approximately the eigenvalue of the matrix $\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$. Using Rutishauser method. Apply the procedure until the elements of the lower triangular part are less than 0.005 in magnitude.

Solution :

$$A_1 = A = L_1U_1$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & \frac{2}{3} \end{pmatrix}$$

$$A_2 = U_1L_1$$

$$= \begin{pmatrix} \frac{10}{3} & 1 \\ \frac{2}{9} & \frac{2}{3} \end{pmatrix}$$

$$= L_2U_2$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{1}{15} & 1 \end{pmatrix}\begin{pmatrix} \frac{10}{3} & 1 \\ 0 & \frac{3}{5} \end{pmatrix},$$

$$\text{and } A_3 = U_2L_2$$

$$= \begin{pmatrix} \frac{17}{5} & 1 \\ \frac{1}{25} & \frac{3}{5} \end{pmatrix}$$

$$= L_3U_3$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{1}{85} & 1 \end{pmatrix}\begin{pmatrix} \frac{17}{5} & 1 \\ 0 & \frac{10}{17} \end{pmatrix},$$

$$\text{and } A_4 = U_3L_3$$

$$= \begin{pmatrix} \frac{58}{17} & 1 \\ \frac{2}{289} & \frac{10}{17} \end{pmatrix}$$

$$= L_4U_4$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{1}{493} & 1 \end{pmatrix}\begin{pmatrix} \frac{58}{17} & 1 \\ 0 & \frac{289}{498} \end{pmatrix},$$

$$\text{and } A_5 = U_4L_4$$

$$= \begin{pmatrix} 3.4138 & 1 \\ 0.0012 & 0.5862 \end{pmatrix}$$

Thus the required accuracy the eigenvalues are 3.4138 and 0.5862.

2.20 POWER METHOD :

This method is used to determine the largest eigenvalue in magnitude and the corresponding eigenvector of the system $AX=\lambda X$.

Let $\lambda_1, \lambda_2, \ldots\ldots\lambda_n$ be the distinct eigenvalues such that

$$|\lambda_1|>|\lambda_2|>\ldots\ldots\ldots>|\lambda_n| \qquad \text{-----(2.46)}$$

and $v_1, v_2, \ldots\ldots v_n$ be the corresponding eigenvectors.

Any eigenvector v in the space of eigenvectors $v_1, v_2, \ldots\ldots v_n$ can be written as

$$v = c_1v_1+c_2v_2+\ldots\ldots+c_nv_n \qquad \text{------(2.47)}$$

$$\therefore\ Av = c_1Av_1+c_2Av_2+\ldots\ldots\ldots+c_nAv_n$$

$$= c_1\lambda_1v_1+c_2\lambda_2v_2+\ldots\ldots+c_n\lambda_nv_n$$

$$= \lambda_1\left(c_1v_1+c_2\left(\frac{\lambda_2}{\lambda_1}\right)v_2+\ldots\ldots+c_n\left(\frac{\lambda_n}{\lambda_1}\right)v_n\right)$$

$$\text{Similarly } A^2v = \lambda_1^2\left(c_1v_1+c_2\left(\frac{\lambda_2}{\lambda_1}\right)^2v_2+\ldots\ldots+c_n\left(\frac{\lambda_n}{\lambda_1}\right)^2v_n\right)$$

$$\vdots$$

$$A^kv = \lambda_1^k\left(c_1v_1+c_2\left(\frac{\lambda_2}{\lambda_1}\right)^kv_2+\ldots\ldots+c_n\left(\frac{\lambda_n}{\lambda_1}\right)^3v_n\right) \qquad \text{------(2.48)}$$

$$\text{and } A^{k+1}v = \lambda_1^{k+1}\left[c_1v_1+c_2\left(\frac{\lambda_2}{\lambda_1}\right)^{k+1}v_2+\ldots+c_n\left(\frac{\lambda_n}{\lambda_1}\right)^{k+1}v_n\right] \text{---(2.49)}$$

As $k\rightarrow\infty$ then R.H.S. of (2.48) & (2.49) tends to $\lambda_1^kc_1v_1$ and $\lambda_1^{k+1}c_1v_1$ respecitvely... because

$$\left|\frac{\lambda_i}{\lambda_1}\right|<1, \quad i = 1,2,3,\ldots\ldots,n$$

$\therefore$ The vector $c_1v_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k v_2 + \ldots\ldots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k v_n$ tends to c_1v_1 which is the eigenvector corresponding to λ_1.

$$\therefore \lambda_1 = \lim_{k\to\infty} \frac{\left(A^{k+1}v\right)_r}{\left(A^k v\right)_r}; r = 1(1)\, n \qquad \text{------(2.50)}$$

where the suffix r denotes the rth component of the vector.

Note : If v_0 is an initial non-null arbitrarly vector then

$$\left.\begin{aligned} y_{k+1} &= Av_k \\ v_{k+1} &= \frac{y_{k+1}}{m_{k+1}} \end{aligned}\right\} \qquad \text{------(2.51)}$$

where m_{k+1} is the largest element in magnitude of y_{k+1} and v_{k+1} is the required eigenvector.

Example E. 2.17 :

Find the dominant eigenvalue and corresponding eigen vector of the matrix

$$\begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Solution :

$$\text{Let } A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\text{Let } v_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore Y_1 = Av_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1.\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= m_1v_1 \text{ where } m_1 = 1, v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{Now } Y_2 = Av_1$$

$$= \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix}$$

$$= 7\begin{pmatrix} 1 \\ 0.4 \\ 0 \end{pmatrix}$$

$$= m_2 v_2 \text{ where } m_2 = 7,\ v_2 = \begin{pmatrix} 1 \\ 0.4 \\ 0 \end{pmatrix}$$

$$\text{and } Y_3 = Av_2$$

$$= \begin{pmatrix} 3.4 \\ 1.8 \\ 0 \end{pmatrix}$$

$$= 3.4\begin{pmatrix} 1 \\ 0.52 \\ 0 \end{pmatrix}$$

$$= m_3 v_3 \quad \text{where} \quad m_3 = 3.4,\ v_3 = \begin{pmatrix} 1 \\ 0.52 \\ 0 \end{pmatrix}$$

$$\text{and } Y_4 = Av_3$$

$$= \begin{pmatrix} 4.12 \\ 2.04 \\ 0 \end{pmatrix}$$

$$= 4.12\begin{pmatrix} 1 \\ 0.49 \\ 0 \end{pmatrix}$$

$$= m_4 v_4 \quad \text{where} \quad m_4 = 4.12,\ v_4 = \begin{pmatrix} 1 \\ 0.49 \\ 0 \end{pmatrix}$$

$$\text{and } Y_5 = Av_4 = \begin{pmatrix} 3.94 \\ 1.98 \\ 0 \end{pmatrix} = 3.94 \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$$

$$= m_5 v_5 \text{ where } m_5 = 3.94 \;\&\; v_5 = \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$$

$$\text{and } Y_6 = Av_5 = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$$

$$= m_6 v_6 \text{ where } m_6 = 4,\; v_6 = \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$$

$\therefore$ Dominant latent root is 4 and the corresponding eigenvector is $\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$.

2.21 Inverse Power Method :

This method is more powerful that the power method. This method has an advantage that it can give approximation to any eigenvalue rather than to λ_1 or λ_n.

If λ is an eigenvalue of A and v is the corresponding eigenvector, then $1/\lambda$ is an eigenvalue of A^{-1} corresponding to the same eigenvector.

Choose any non-zero eigenvector $y_0 \in \mathbf{R}^n$ and express it as a linear combination of $v_1, v_2, \ldots\ldots, v_n$. Applying power method to A^{-1}, we have

$$\left.\begin{aligned} z_{k+1} &= A^{-1} y_k \\ y_{k+1} &= \frac{1}{m_{k+1}} z_{k+1} \end{aligned}\right\} \qquad \text{------(2.52)}$$

This gives an approximation to the dominant eigenvalue of A^{-1}, that is, the smallest eigenvalve of A in modulus.

One need not find A^{-1} to find the smallest eigenvalue (in modulus) of A.

$\therefore$ From (2.52), $\quad Az_{k+1} \;=\; y_k$ ------(2.53)

We find z_{k+1} by solving the linear system of algebraic equation (2.53).

The ratio of the corresponding components tends to $1/\lambda_i$ where λ_i are the eigenvalues of A.

$$\text{(i.e.)} \quad \frac{1}{\lambda_i} \;=\; \lim_{k \to \infty} \frac{(z_{k+1})_r}{(y_k)_r} \qquad \text{------(2.54)}$$

We normalize the vectors at each stage of iteration. It is known that this inverse iteration is the most powerful and accurate of all methods for computing eigenvectors.

UNIT – 3

INTERPOLATION AND APPROXIMATION

3.1 Introduction :

In this unit we consider the problem of approximating a given function to a polynomial called interpolating polynomial. There are two advantages in interpolating polynomial. First advantage is in reconstructing the function f(x) when it is not given explicitly and only the values of f(x) and / or its certain order derivatives at a set of points, called nodes, tabular points or arguments are known. The second advantage is to replace the function f(x) by the interpolating polynomial P(x) so that many common operations such as determination of roots, differentiation and integration etc., which are intended for the function f(x) may be performed using P(x). In approximation, we measure the deviation of the given function f(x) form the approximating polynomial P(x) for all values of x over a given interval [a, b].

Definition D. 3.1 :

A polynomial P(x) is called an interpolating polynomial if the values of P(x) and / or its certain order derivatives concide with those of f(x) and / or its same order derivatives at one or more nodes.

Definition D. 3.2 :

If the polynomial P(x) is written as the Taylor's expansion for the function f(x) at a point x_0, $x_0 \in [a, b]$, as

$$P(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \ldots\ldots + \frac{1}{n!}(x - x_0)^n f^{(n)}(x_0) \quad \text{------(3.1)}$$

then P(x) is an interpolating polynomial of degree n, satisfying the conditions

$$P^{(k)}(x_o) = f^{(k)}(x_0); \qquad k = 0, 1, 2, 3, \ldots\ldots, n \quad \text{-------(3.2)}$$

$$\text{The term } R_n = \frac{1}{(n+1)!}(x - x_0)^{n+1} f^{(n+1)}(\xi),\ x_0 < \xi < x \quad \text{-------(3.3)}$$

has been neglected in (3.1), is called the remainder or the truncation error.

3.2 LAGRANGE AND NEWTON INTERPOLATIONS :

Assume that f(x) is continuous in the given closed interval [a, b]. Also assume that $a=x_0<x_1<x_2<........<x_n = b$ be n+1 distinct points and the corresponding functional values $f_0, f_1,.......f_n$ of f(x) are known. (Here $f_i = f(x_i)$). We have to find a polynomial.

$$P(x) = a_0+a_1x+a_2x^2+.......+a_nx^n \qquad ------(3.4)$$

satisfying the condition $P(x_i) = f(x_i)$ for $i = 0, 1, 2,........,n$ ------(3.5)

Now P(x) exists if the Vandermonde's determinant.

$$V(x_0, x_1,.........x_n) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdot & \cdot & \cdot & x_0^n \\ 1 & x_1 & x_1^2 & \cdot & \cdot & \cdot & x_1^n \\ \cdot \\ \cdot \\ \cdot \\ 1 & x_n & x_n^2 & \cdot & \cdot & \cdot & x_n^n \end{vmatrix}$$

$$\text{Let} \quad V(x_0, x_1,..........x_{n-1}, x) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdot & \cdot & \cdot & x_0^n \\ 1 & x_1 & x_1^2 & \cdot & \cdot & \cdot & x_1^n \\ \cdot \\ \cdot \\ \cdot \\ 1 & x_{n-1} & x_{n-1}^2 & \cdot & \cdot & \cdot & x_{n-1}^n \\ 1 & x & x^2 & \cdot & \cdot & \cdot & x^n \end{vmatrix}$$

Clearly $V(x_0, x_1,......x_{n-1}, x) = (x-x_0)(x-x_1).......(x-x_{n-1})A$ ------(3.6)

where A is a constant to be determined.

Comparing the coefficient of x^n of (3.6) on both sides, we get,

$$A = V(x_0, x_1,........,x_{n-1})$$

$$\therefore \; V(x_0, x_1, x_2,......,x_n) = V(x_0, x_1,.......x_{n-1}) \prod_{i=0}^{k-1} (x_n - x_i)$$

$$= \prod_{\substack{i,j=0 \\ i>j}}^{n} (x_i - x_j) \neq 0$$

Claim :

The polynomial P(x) so obtained is unique.

Proof of the Claim :

Assume that there is another polynomial $P^*(x)$ which also satisfies

$$P^*(x_i) = f(x_i);\ i = 0, 1, 2, 3, \ldots\ldots n \qquad \text{------(3.7)}$$

$$\text{Let } Q(x) = P(x) - P^*(x)$$

$$\text{Now } \deg Q(x) = P(x_i) - P^*(x_i) = 0;\ i = 0(1)\, n \qquad \text{------(3.8)}$$

$\therefore$ Q(x) is a polynomial of degree $\leq n$ which has n+1 distinct roots $x_0, x_1, \ldots\ldots x_n$.

This implies Q(x) = 0, because a polynomial. Q(x) of degree n has exactly n roots.

$$\therefore \qquad P^*(x) = P(x)$$

This proves the claim.

We discuss now interpolations of various degrees.

3.3 Linear Interpolation :

$$\text{Let } P(x) = a_0 + a_1 x \qquad \text{------(3.9)}$$

where a_0, a_1 are constants to be determined and f(x), $x \in [x_0, x_1]$ be the given function such that

$$f(x_0) = P(x_0) = a_1 x_0 + a_0 \qquad \text{-------(3.10)}$$

$$f(x_1) = P(x_1) = a_1 x_1 + a_0 \qquad \text{-------(3.10)}$$

Eliminating a_0 and a_1 from (3.9) & (3.10),

we get,
$$\begin{vmatrix} P(x) & x & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0 \qquad \text{------(3.11)}$$

Expanding the LHS determinant of (3.11) and after simplification, we get,

$$P(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$= l_0(x) f(x_0) + l_1(x) f(x_1) \qquad \text{------(3.12)}$$

where
$$l_0(x) = \frac{x - x_1}{x_0 - x_1}, \quad l_1(x) = \frac{x - x_0}{x_1 - x_0}$$

The functions $l_0(x)$ & $l_1(x)$ are called the **Lagrange fundamental polynomials.** and satisfy the conditions $l_0(x)+l_1(x) = 1$

$$
\begin{aligned}
\Rightarrow l_0(x_0) &= 1,\ l_0(x_1) = 0 \\
l_1(x_0) &= 0,\ l_1(x_1) = 1 \\
\Rightarrow l_i(x_j) &= \delta_{ij} \\
&= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \text{------(3.13)}
\end{aligned}
$$

Note : The equations (3.12) is called the linear Lagrange interpolating polynomial.

3.4 Iterated Linear Interpolation :

We can write (3.12) as

$$
\begin{aligned}
P(x) &= \frac{1}{x_1 - x_0}[(x_1 - x)f(x_0) - (x_0 - x)f(x_1)] \\
&= \frac{1}{x_1 - x_0}\begin{vmatrix} I_0(x) & x_0 - x \\ I_1(x) & x_1 - x \end{vmatrix} \qquad \text{------(3.14)} \\
&= I_{0,1}
\end{aligned}
$$

where $\quad I_0(x) = f(x_0),\ I_1(x) = f(x_1)$

Clearly $\quad I_{0,1}(x_0) = f(x_0)$ and $I_{0,1}(x_1) = f(x_1)$

The equation (3.14) is called Aitken's or iterated linear interpolating polynomial.

3.5 Newton's Divided difference Interpolation :

Expand the determinant (3.11) along first-row we get,

$$
\begin{aligned}
P(x) &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \\
&= f(x_0) + f[x_0, x_1](x - x_0) \qquad \text{------(3.15)}
\end{aligned}
$$

$$
\text{where } f[x_0, x_1] = \frac{x(x_1) - f(x_0)}{x_1 - x_0} \qquad \text{------(3.16)}
$$

This ratio $f[x_0, x_1]$ is called first divided difference of $f(x)$ relative to x_0 and x_1.

Further (3.15) can be written as

$$\frac{P(x)-f(x_0)}{x-x_0} = f[x_0, x_1] \qquad \text{------(3.17)}$$

The equation (3.15) or (3.17) is called linear Newton interpolating polynomial with divided differences.

3.6 Truncation Error Bounds :

Let P(x) be the interpolating polynomial for the given function f(x).

$$\text{Then} \quad E_1(f; x) = f(x)-P(x) \qquad \text{------(3.18)}$$

for $x \in [x_0, x_1]$ is called the truncation error.

Now we shall derive an expression for $E_1(f; x)$, $x \in [x_0, x_1]$.

Clearly $E_1(f; x_0) = 0$ & $E_1(f; x_1) = 0$

Fix $x \in [a, b]$ and $x \neq x_0$, $x \neq x_1$.

Define a function g(t) corresponding to this x as

$$g(t) = f(t)-P(t)-[f(x)-P(x)]\frac{(t-x_0)(t-x_1)}{(x-x_0)(x-x_1)} \qquad \text{------(3.19)}$$

Clearly g(t) = 0 for $t = x_0, x_1, x$.

Differentiating (3.19) twice with respect to t, we get,

$$g''(t) = f''(t)-\frac{2(f(x)-P(x))}{(x-x_0)(x-x_1)} \qquad \text{------(3.20)}$$

Then by ROLL's theorem, there exist $\xi \in (a, b)$ such that $g''(\xi) = 0$

$\therefore$ (3.20) becomes $f''(x)-\dfrac{2(f(x)-P(x))}{(x-x_0)(x-x_1)} = 0$

$$\text{(i.e.)} \quad f''(x) = \frac{2(f(x)-P(x))}{(x-x_0)(x-x_0)}$$

$$\text{(i.e.)} \quad f(x) = P(x)+\tfrac{1}{2}(x-x_0)(x-x_1)f''(x) \qquad \text{------(3.21)}$$

where $\min(x_0, x_1, x) < \xi < \max(x_0, x_1, x)$

$$\therefore \quad E_1(f; x) = \tfrac{1}{2}(x-x_0)(x-x_1)f''(\xi) \qquad \text{------(3.22)}$$

If $f''(x) \leq M_2$ for $x_0 \leq x \leq x_1$ then $|f(x)-p(x)| = |\frac{1}{2}(x-x_0)(x-x_1)f''(\xi)|$

$$\leq \frac{1}{2} \max_{x_0 \leq x \leq x} \{|(x-x_0)(x-x_1)f''(\xi)|\}$$

$$\leq \frac{1}{2} \max_{x_0 \leq x \leq x} |(x-x_0)(x-x_1)| M_2$$

$$\leq \frac{1}{8}(x_1-x_0)^2 M_2 \qquad \text{------(3.23)}$$

(because max $|(x-x_0)(x-x_1)|$ occurs at $x=\frac{1}{2}(x_0+x_1)$)

Note : If x are equally spaced with interval h, then the maximum truncation error using the linear interpolating polynomial P(x) is less than given $\in(>0)$.

$$\therefore \text{ we have } \frac{h^2}{8} . \max_{a \leq x \leq b} |f''(x)| < \in \qquad \text{------(3.24)}$$

Example E. 3.1 :

Using sin(0.1) = 0.09983 and sin(0.2) = 0.19867, find an approximate value of sin (0.15) by Lagrange interpolation. Also find a bound on the truncation error.

Solution :

Here $x_0 = 0.1$, $x_1 = 0.2$, $x = 1.5$

$f(x_0) = 0.09983$, $f(x_1) = 0.19867$

$\therefore$ The interpolating polynomial is

$$P(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$$

$$= \frac{1.5-0.2}{0.1-0.2}(0.09983) + \frac{1.5-0.1}{0.2-0.1}(0.19867)$$

$$= 0.14925$$

and the truncation error is given by

$$E_1(f; x) = \frac{(x-x_0)(x-x_1)}{2} f''(\zeta)$$

$$= \frac{1}{2}(x-0.1)(x-0.2)(-\sin\xi)$$

where $0.1<\xi<0.2$

$$\text{But } |-\sin\xi| \leq \max|-\sin\xi|$$

$$= \sin(0.2) = 0.19867$$

$$\therefore \ |E_1(f; x)| \le \frac{1}{2}|(1.5-0.1)(1.5-0.2)|0.19867$$

$$\cong 0.00025$$

which is the required bound for truncation error.

3.7 Higher Order Interpolation :

The Lagrange fundamental polynomial of degree n based on n+1 distinct points $a \le x_0 < x_1 < \ldots\ldots\ldots < x_n \le b$ and which satisfy $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$ can be written as

$$l_i(x) = \frac{(x-x_0)(x-x_1)\ldots.(x-x_{i-1})(x-x_{i+1})\ldots.(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots.(x_i-x_{i-1})(x_i-x_{i+1})\ldots..(x_i-x_n)}$$

$$\text{for } i = 0,1,2,\ldots\ldots..,n \qquad \text{------(3.25)}$$

$$\text{(i.e.,)} \ l_i(x) = \frac{w(x)}{(x-x_i)w^1(x_i)} \text{ where } w(x)=(x-x_0)(x-x_1)\ldots(x-x_n)$$

$$\therefore \text{ The polynomial} \quad P(x) = \sum_{i=0}^{n} l_i(x)f(x_i) \qquad \text{------(3.26)}$$

where $l_i(x)$ are given by (3.25) is the Lagrange interpolating polynomial of degree n.

The truncation error in the Lagrange interpolation is given by

$$E_n(f; x) = f(x)-P(x)$$

$$= \frac{w(x)}{(n+1)!}f^{(n+1)}(\xi)$$

where ξ is some point in $(x_0, x_1, \ldots\ldots.. x_n, x)$.

3.8 Iterated Interpolation :

The iterated form of the Lagrange interpolation is given as

$$I_{0,1,2,\ldots\ldots,n}{}^{(x)} = \frac{1}{x_n - x_{n-1}}\begin{vmatrix} I_{0,1,2,\ldots\ldots,n-1}{}^{(x)} & x_{n-1}-x \\ I_{0,1,2,\ldots\ldots,n-2}{}^{(x)} & x_n - x \end{vmatrix} \qquad \text{------(3.27)}$$

where $I_i(x) = f(x_i)$

3.9 Newton's Divided Difference Interpolation :

We can generatise the linear Newton divided difference interpolation (3.15). The higher order divided difference is given by

$$f(x_0, x_1, x_2) = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$$

$$f[x_0, x_1, \ldots\ldots x_{k-1}, x_k] = \frac{f[x_1,x_2,\ldots\ldots x_k]-f[x_0,x_1,\ldots\ldots x_{k-1}]}{x_k-x_0}$$

$$k=3,4,5, \ldots\ldots\ldots n$$

Thus the n^{th} divided difference is written as

$$f[x_0, x_1, x_2,\ldots\ldots\ldots,x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\sum\limits_{\substack{j=0\\ i\neq j}}^{n} (x_i-x_j)} \quad \text{--------(3.28)}$$

The divider differences can be calculated from the following divided difference table.

x_0	$f[x_0]$				
x_1	$f[x_1]$	$f[x_0,x_1]$			
x_2	$f[x_2]$	$f[x_1,x_2]$	$f[x_0,x_1,x_2]$		
x_3	$f[x_3]$	$f[x_2,x_3]$	$f[x_1,x_2,x_3]$		
.	.	.	.		
.	.	.	.		
.	.	.	.	....	
x_n	$f[x_n]$	$f[x_{n-1},x_n]$	$f[x_{n-2},x_{n-1},x_n]$		$f[x_0,x_1,\ldots\ldots,x_n]$

Note that $f[x_0, x_1,\ldots\ldots\ldots,x_n] = f[x_n, x_{n-1},\ldots\ldots\ldots x_0]$

Now the interpolating polynomial $P_n(x)$, interpolating at the n+1 distinct points $x_0, x_1,\ldots\ldots\ldots,x_n$ can be written as

$$P_n(x) = a_0+(x-x_0)a_1+(x-x_0)(x-x_1)a_2$$

$$+\ldots\ldots+(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})a_n \quad \text{------(3.29)}$$

If we substitude $x = x_0$ in (3.29), then we get $a_0 = f[x_0]$.

If we substitude $x = x_1$ in (3.29), then we get $a_1 = f[x_0, x_1]$

If we substitude $x = x_2$ in (3.29), then we have, $a_2 = f[x_0, x_1, x_2]$,

•

•

and if we put $x = x_n$ in (3.29), we get, $a_n = f[x_0, x_1, x_2, \ldots\ldots, x_n]$ ------(3.30)

$$\therefore \text{(3.29) becomes, } P_n(x) = f[x_0]+(x-x_0)f[x_0, x_1]+\ldots\ldots$$
$$+(x-x_0)(x-x_1)\ldots(x-x_{n-1})f[x_0,\ldots x_n] \quad \text{------(3.31)}$$

Example E. 3.2 :

Find the unique polynomial P(x) of degree 2 or less such that P(1)=1, P(3)=27, P(4)=64 using each of the following methods :

(i) Lagrange interpolation method

(ii) Newton-divided difference method.

(iii) and Aitkten's iterated formula.

Also evaluate P(1.5).

Solution :

We know that Lagrange interpolation formula is

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}P(x_1)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}P(x_1)$$
$$+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}P(x_2)$$

$$\text{(i.e.) } P_2(x) = \frac{(x-4)(x-3)}{(1-4)(1-3)}(1)+\frac{(x-1)(x-4)}{(3-1)(3-4)}(27)$$
$$+\frac{(x-1)(x-3)}{(4-1)(4-3)}(64)$$

$$= 8x^2-19x+12$$

Using Newton's divided difference formula,

$$P_2(x) = f[x_0]+(x-x_0)f[x_0, x_1]$$
$$+(x-x_0)(x-x_1)f[x_0, x_1, x_2] \quad \text{-----(3.31a)}$$

x	f(x)	Δ	Δ^2
1	1	$\frac{27-1}{3-1}=13$	$\frac{37-13}{4-1}=8$
3	27	$\frac{64-27}{4-3}=37$	
4	64		

$$\therefore (3.31a) \Rightarrow P_2(x) = 1+(x-1)(13)+(x-1)(x-3)8$$
$$= 8x^2-19x+12$$

(iii) Using iteration formula, we get,

$$I_{0,1}(x) = \frac{1}{x_1-x_0}\begin{vmatrix} I_0(x) & x_0-x \\ I_1(x) & x_1-x \end{vmatrix}$$
$$= \frac{1}{3-1}\begin{vmatrix} 1 & 1-x \\ 27 & 3-x \end{vmatrix}$$
$$= 13x-12$$
$$I_{0,2}(x) = \frac{1}{x_2-x_0}\begin{vmatrix} I_0(x) & x_0-x \\ I_2(x) & x_2-x \end{vmatrix}$$
$$= \frac{1}{3}\begin{vmatrix} 1 & 1-x \\ 64 & 4-x \end{vmatrix}$$
$$= 21x-20$$
$$\text{and } I_{0,1,2}(x) = \frac{1}{x_2-x_1}\begin{vmatrix} I_{0,1}(x) & x_1-x \\ I_{0,2}(x) & x_2-x \end{vmatrix}$$
$$= \begin{vmatrix} 13x-12 & 3-x \\ 21x-20 & 4-x \end{vmatrix}$$
$$= 8x^2-19x+12$$
$$\text{Hence } P_2(x) = 8x^2-19x+12$$
$$\text{and } P_2(1.5) = 8(1.5)^2-19(1.5)+12 = 1.5$$

Example E. 3.3 :

Calculate the n^{th} divided difference of $1/x$.

Solution :

$$\text{Let } f(x) = 1/x$$

$$\therefore \quad f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$$= \frac{\frac{1}{x}-\frac{1}{x_0}}{x_1-x_0}$$

$$= \frac{-1}{x_0x_1}$$

$$\text{and } f[x_0, x_1, x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$$

$$= \frac{1}{x_2-x_0}\left[\frac{-1}{x_1x_2}-\left(\frac{-1}{x_0x_1}\right)\right]$$

$$= \frac{1}{x_0x_1x_2}$$

Assume that result is true for k.

$$\text{(i.e.,)} \quad f[x_0, x_1, \ldots\ldots\ldots, x_k] = \frac{(-1)^k}{x_0x_1\ldots\ldots x_k}$$

$$\text{Hence} \quad f[x_0,x_1,\ldots\ldots,x_{k+1}] = \frac{f[x_1,x_2,\ldots\ldots,x_{n+1}]-f[x_0,x_1,\ldots\ldots\ldots x_k]}{x_{k+1}-x_0}$$

$$= \frac{1}{x_{k+1}-x_0}\left[\frac{(-1)^k}{x_1x_2\ldots\ldots x_{k+1}}-\frac{(-1)^k}{x_0x_1x_2\ldots\ldots x_k}\right]$$

$$= \frac{(-1)^{k+1}}{x_0x_1\ldots\ldots\ldots x_{k+1}}$$

$\therefore$ By mathematical induction, $f[x_0, x_1, \ldots\ldots, x_n] = \dfrac{(-1)^n}{x_0, x_1, x_2, \ldots\ldots x_n}$ for $n \in N$.

Practice Problem :

Find the unique polynomial of degree 2 or less such that f(0)=1, f(1)=3, f(3)=55.

Using (i) the Lagrange interpolation

(ii) the interated interpolation and

(iii) the Newton divided difference interpolation

[Ans. : $P^2(x) = 8x^2-6x+1$]

3.10 Finite Difference Operators :

Let the tabular points $x_0, x_1, x_2, \ldots\ldots, x_n$ be equally spaced (i.e.,) $x_i = x_0+ih$; $i=0(1)h$, and the corresponding functional values be $f(x_0), f(x_1), \ldots\ldots f(x_n)$.

Denote $f_i = f(x_i)$

We define the following operators.

(1) The shift operator E be defined as $Ef_i = f_{i+1}$

(i.e.,) $\mathbf{Ef(x_i) = f(x_i+h)}$

(2) The forward-difference operator Δ is defined as $\Delta f_i = f_{i+1}-f_i$

(i.e.,) $\mathbf{\Delta f(x_i) = f(x_i+h)-f(x_i)}$

(3) The backward-difference operator ∇ is defined as $\nabla f_i = f_i-f_{i-1}$

(i.e.,) $\mathbf{\nabla f(x_i) = f(x_i)-f(x_i-h)}$

(4) The central-difference operator δ is defined as

$$\delta f(x_i) = f(x_i+\frac{h}{2})-f(x_i-\frac{h}{2})$$

(5) The averaging operator μ is defined as

(i.e.,) $\mu f(x_i) = \frac{1}{2}[f(x_i+\frac{h}{2})+f(x_i-\frac{h}{2})]$

Example 3.4 :

Using the definitions of difference operators, we can easily verify the following :

$$E^n f(x_i) = f(x_i+nh)$$

$$\Delta^n f(x_i) = \Delta^{n-1} f_{i+1} - \Delta^{n-1} f_i$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!}$$

$$\nabla^n f(x_i) = \nabla^{n-1} f_i - \nabla^{n-1} f_{i-1}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f_{i-k}$$

$$\delta f(x_i) = \delta^{n-1} f_{i+\frac{1}{2}} - \delta^{n-1} f_{i-\frac{1}{2}}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f_{i+\frac{n}{2}-k}$$

Further

$$\Delta f_i = \nabla fi+1 = f_{i+½}$$

$$\Delta = E-1$$

$$\nabla = 1-E^{-1}$$

$$\delta = E^{½}-E^{-½}$$

The following table shows the relations among various operators.

Operator	E	Δ	∇	δ
E	E	$\Delta+1$	$(1-\nabla)^{-1}$	$1+\frac{1}{2}\delta^2+\sqrt{1+\frac{1}{4}\delta^2}$
Δ	E^{-1}	Δ	$(1-\nabla)^{-1}-1$	$\frac{1}{2}\delta^2+\delta.\sqrt{1+\frac{1}{4}\delta^2}$
∇	$1-E^{-1}$	$1-(1+\Delta)^{-1}$	∇	$-\frac{1}{2}\delta^2+\delta.\sqrt{1+\frac{1}{4}\delta^2}$
δ	$E^{\frac{1}{2}}-E^{-\frac{1}{2}}$	$\Delta(1+\Delta)^{-½}$	$\nabla(1-\nabla)^{-½}$	δ
μ	$\frac{1}{2}\left(E^{\frac{1}{2}}+E^{-\frac{1}{2}}\right)$	$(1+½\Delta)(1+\Delta)^{½}$	$(1-½\nabla)(1-\nabla)^{-½}$	$\sqrt{1+\frac{1}{4}\delta^2}$

Now we shall find the Newton divided difference in terms of forward, backward and central differences.

We know that $f[x_0, x_1] = \dfrac{f(x_1)-f(x_0)}{x_1-x_0} = \dfrac{1}{h}\Delta f_0$

$$f[x_0, x_1, x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$$

$$= \frac{\frac{1}{h}\Delta f_1-\frac{1}{h}\Delta f_0}{2h}$$

$$= \frac{1}{2!h^2}\Delta^2 f_0$$

By induction, we have, $f[x_0, x_1,\ldots\ldots,x_n] = \dfrac{1}{n!h^n}\Delta^n f_0$ ------(3.32)

$$\text{Again } f[x_0, x_1] = \frac{1}{h}\nabla f_1$$

$$f[x0, f_1, f_2] = \frac{1}{2!h^2}\nabla^2 f_2$$

$$\boxed{f[x_0, x_1, \ldots\ldots, x_n] = \frac{1}{n!h^n}\nabla^n f_n} \quad \text{------(3.33)}$$

$$\text{Again } f[x_0, x_1] = \frac{1}{h}\delta f_{\frac{1}{2}}$$

$$f[x_0, x_1, x_2] = \frac{1}{2!h^2}\delta^2 f_1$$

$$\vdots$$

$$\left.\begin{aligned} f[x_0, x_1, \ldots\ldots x_{2m}] &= \frac{1}{(2m)!h^{2m}}\delta^{2m} f_m \\ \& \ f[x_0, x_1, \ldots\ldots, x_{2m+1}] &= \frac{1}{(2m+1)!h^{2m+1}}\delta^{2m+1} f_{m+\frac{1}{2}} \end{aligned}\right\} \quad \text{------(3.34)}$$

3.11 Interpolating Polynomials Using Finite Differences :

Substituting (3.32) in (3.31), we get,

$$P(x) = f_0 + \frac{(x-x_0)}{h}\Delta f_0 + \frac{(x-x_0)(x-x_1)}{2!h^2}\Delta^2 f_0 + \ldots\ldots$$

$$+ \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})}{n!h^n}\Delta^n f_0 \quad \text{----(3.35)}$$

$$\text{put } u = \frac{x-x_0}{h} \text{ in (3.35), we get}$$

$$P(x) = f_0 + uC_1\Delta f_0 + uC_2\Delta^2 f_0 + \ldots\ldots + uC_n\Delta^n f_0$$

$$= \sum_{i=0}^{n} uC_i\Delta^i f_0 \quad \text{------(3.36)}$$

This formula (3.36) is called Gregory-Newton forward difference formula & its truncation error is given by

$$E_n(f; x) = \frac{u(u-1)(u-2)\ldots\ldots(u-n)}{(n+1)!}h^{n+1}f^{(n+1)}(\xi)$$

Derive Gregory-Newton backward difference interpolation.

Derivation :

$$f(x) = f\left(x_n + \frac{x - x_n}{h}.h\right)$$

$$= f(x_n + uh) \text{ where } u = \frac{x - x_n}{h}$$

$$= E^u f(x_n)$$

$$= (1-\nabla)^{-u} f(x_n)$$

$$= \left[1 + u\nabla + \frac{u(u+1)}{2!}\nabla^2 + \frac{u(u+1)(u+2)}{3!}\nabla^3 + \ldots\right] f(x_n)$$

$$= f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(x_n) + \ldots\ldots\ldots \qquad \text{-----(3.37)}$$

The polynomial $P(x_n+h_u) = f_n + u\nabla f_n + \frac{u(u+1)}{2!}\nabla^2 f_n + \ldots\ldots + \frac{u(u+1)(u+2)\ldots\ldots(u+n-1)}{n!}\nabla^n f_n$

$$= \sum_{i=0}^{n} (-i)^i \binom{-u}{i} \nabla^i f_n \qquad \text{------(3.38)}$$

is called the Gregory-Newton backward difference interpolation and the truncation error is given by

$$E_n(f; x) = \frac{u(u+1)\ldots\ldots(u+n)}{(n+1)} h^{n+1} f^{(n+1)}(\xi) \qquad \text{-------(3.39)}$$

3.12 Stirling and Bessel Interpolations :

Using the following central difference table, we can form formulae called Stirling & Bessel inpolation formulae.

x	f(x)	δf	$\delta^2 f$	$\delta^3 f$	$\delta^4 f$
x_0	f_0				
		$\delta f_{1/2}$			
x_1	f_1		$\delta^2 f_1$		
		$\delta f_{3/2}$		$\delta^3 f_{3/2}$	
x_2	f_2		$\delta^2 f_2$		$\delta^4 f_2$
		$\delta f_{5/2}$		$\delta^3 f_{5/2}$	
x_3	f_3		$\delta^2 f_3$		
		$\delta f_{7/2}$			
x_4	f_4				

For n even, denote the modal points as $x_{-p}, x_{-(p-1)}, \ldots\ldots x_{-1}, x_0, x_1, x_2, \ldots\ldots x_{p-1}, x_p$

The **Stirling** interpolation is given by

$$\begin{aligned} P(x) &= f(x_0)+\frac{u}{2}[\delta f_{1/2}+\delta f_{-1/2}]+\frac{u^2}{2!}\delta^2 f_0 \\ &\quad +\frac{u(u^2-1^2)}{3!}\frac{1}{2}\left(\delta^3 f_{1/2}+\delta^3 f_{-1/2}\right)+\ldots\ldots \\ &\quad +\frac{u(u^2-1^2)(u^2-2^2)\ldots.(u^2-(p-1)^2)}{(2p\text{-}1)!}.\frac{1}{2}\left[\delta^{2p-1}f_{\frac{1}{2}}+\delta^{2p-1}f_{-\frac{1}{2}}\right] \\ &\quad +\frac{u^2(u^2-1^2)\ldots\ldots(u^2-(p-1)^2)}{(2p)!}\delta^{2p}f_0 \end{aligned} \qquad \text{-----(3.40)}$$

$$\text{where } u = \frac{x-x_0}{h}$$

For n odd, denote the nodal points as $x_{-p}, x_{-(p-1)}, \ldots\ldots x_{-1}, x_0, x_1, \ldots\ldots x_p, x_{p+1}$ then Bessel interpolation formula is given by

$$\begin{aligned} P(x) &= \frac{1}{2}[f_0+f_1]+v\delta f_{1/2}+\frac{v^2-\frac{1}{4}}{2!}\frac{1}{2}.\left(\delta^2 f_0+\delta^2 f_1\right) \\ &\quad +\frac{v(v^2-1/4)}{3!}\delta^3 f_{1/2}+\ldots\ldots\ldots \\ &\quad +\frac{\left(v^2-\frac{1}{4}\right)\left(v^2-\frac{9}{4}\right)\ldots\ldots\left(v^2-\frac{(2p-1)^2}{4}\right)}{(2p)!}\frac{1}{2}\left[\delta^{2p}f_0+\delta^{2p}f_1\right] \\ &\quad +\frac{v\left(v^2-\frac{1}{4}\right)\left(v^2-\frac{9}{4}\right)\ldots\ldots\left(v^2-\frac{(2p-1)^2}{4}\right)}{(2p+1)!}\delta^{2p+1}f_{1/2} \end{aligned} \qquad \text{------(3.41)}$$

$$\text{where } v = u-\frac{1}{2}.$$

Note :

1) The Newton-Gregory interpolations with forward and backward differences are used if the interpolation is desired near the beginning and end respectively of the table values.

2) The Stirling interpolation is used for calculations between x_0–h/4 and x_0+h/4 and also what the first neglected highest difference is of odd order.

3) The Bessel interpolation is used if the calculations are performed between $x_0+h/4$ and $x_1-h/4$ and also when the first neglected highest difference is of even order.

4) If the entire data is not to be used, we can choose a suitable initial point, so that u is small and terms in the interpolation formulae reduce fast.

Example E. 3.5 :

The following table gives the sales of a concern for the last five years. Estimate the sales for the year 1997.

Years	:	1992	1994	1996	1998	2000
Sales (Rs. '000)	:	40	43	48	52	57

Solution :

Here $$u = \frac{x-x_0}{h} = \frac{x-1992}{2}$$

& the Newton-Gregory forward interpolation formula is

$$P(x) = f(x_0)+uC_1\Delta f_0+uC_2\Delta^2 f_0+\ldots\ldots\ldots \quad \text{------(3.41a)}$$

First we shall find difference table.

years x	sales f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$	Δ^4
1992	40				
		3			
1994	43		2		
		5		−3	
1996	48		−1		5
		4		2	
1998	52		1		
		5			
2000	57				

When $x = 1997$, then $u = \frac{1997-1992}{2} = 2.5$

Thus (3.41a) $\Rightarrow$ $$P(2.5) = 40+2.5\times 3+\frac{2.5(2.5-1)}{2}(2)+\frac{2.5(2.5-1)(2.5-2)}{6}(-3)$$
$$+\frac{2.5(2.5-1)(2.5-2)(2.5-3)}{24}(5)$$
$$= 50.21$$

Hence the sales for the year 1997 is Rs.50.21 thousand.

Example E. 3.6 :

For linear interpolation in the case of equispaced tabular data, show that the error does not exists 1/8 of the second difference.

Proof :

The error in the linear interpolation is

$$|E_1| \cong \left|\frac{(x-x_0)(x-x_1)}{2!}\right|\left|\frac{\Delta^2 f_0}{h^2}\right|$$

$$\leq \max_{x_0 \leq x \leq x_1}\{|(x-x_0)(x-x_1)|\}\left|\frac{\Delta^2 f_0}{2h^2}\right|$$

$$= \left[\frac{1}{2}(x_0+x_1)-x_0\right]\left[\frac{1}{2}[x_0+x_1]-x_1\right]\left|\frac{\Delta^2 f_0}{2h^2}\right|$$

$$= \left|\left(\frac{x_1-x_0}{2}\right)\left(\frac{x_0-x_1}{2}\right)\right|\left|\frac{\Delta^2 f_0}{2h^2}\right|$$

$$= \left|\left(\frac{h}{2}\right)\left(-\frac{h}{2}\right)\right|\left|\frac{\Delta^2 f_0}{2h^2}\right| \qquad (\because x_1 = x_0 + h)$$

$$= \frac{|h^2|}{4}\left|\frac{\Delta^2 f_0}{2h^2}\right| = \frac{1}{8}|\Delta^2 f_0|$$

This proves the problem.

Example E. 3.7 :

Find the number of men getting wages between Rs. 10 and Rs. 15 from the following data :

Wages (in Rs.)	:	0–10	10–20	20–30	30–40
Frequency	:	9	30	35	42

Solution :

To the given data, first we shall find the difference table.

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$
under 10	9			
		30		
under 20	39		5	
		35		2
under 30	74		7	
		42		
under 40	116			

$$\text{Here } u = \frac{x-x_0}{h}$$

$$= \frac{15-10}{10}$$

$$= 0.5 \qquad (\because h = 10, x_0 = 10)$$

Thus from Newton forward formula, we have,

$$f(15) = f_0 + u\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 + \ldots\ldots$$

$$= 9 + 0.5 \times 30 + \frac{0.5(0.5-1)}{2}(5) + \frac{0.5(0.5-1)(0.5-2)}{6}(2)$$

$$= 23.5$$

$$\cong 24$$

Hence 24 men were getting wages between Rs. 10 and 15.

Example E. 3.8 :

Using divided difference formula, find the value of f(15) from the following table :

x	:	4	5	7	10	11	13
f(x)	:	48	100	294	900	1210	2028

Solution :

x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$	Δ^4
4	48	$\frac{100-48}{5-4}=52$	$\frac{97-52}{7-4}=15$	1	0
5	100	97	21	1	0
7	294	202	27	1	
10	900	310	33		
11	1210	409			
13	2028				

$$\therefore f(15) = 48+(15-4)52+(15-4)(15-5)(15)+(15-4)(15-5)(15-7)(1)$$

$$= 3150$$

Example E. 3.9 :

Use Stirling formula to find y_{35}, given y_{20}=512, y_{30}=439, y_{40}=346, y_{50}=243 where y_x represents the number of persons at age x years in the life table.

Solution :

Taking 30 as origin & 10 years as unit, we get, the difference table as

u	f(u)	$\Delta f(u)$	$\Delta^2 f(u)$	$\Delta^3 f(u)$
-1	512			
		-73		
0	439		−20	
		-93		10
1	346		−10	
		-103		
2	243			

We know that the stirling formula as

$$f(u) = f(0) + uC_1\left(\frac{\Delta f_0 + \Delta f_{-1}}{2}\right) + \frac{u^2}{2}\Delta^2 f_{-1} + \frac{u(u^2-1^2)}{3!}\left(\frac{\Delta^3 f_{-1} + \Delta^3 f_{-2}}{2}\right) + \ldots..$$

$$\therefore\ f(0.5) = 439 + 0.5\left(\frac{-93-73}{2}\right) + \frac{(0.5)^2}{2!}(-20)$$

$$= 395$$

Example E. 3.10 :

For the following data, calculate the differences and obtain the forward and backward difference polynomials. Interpolate at x=0.25 and x=0.35.

x	:	0.1	0.2	0.3	0.4	0.5
f(x)	:	1.4	1.56	1.76	2	2.28

Solution :

The difference table is given by

x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$	Δ^4
0.1	1.40				
		0.16			
0.2	1.56		0.04		
		0.20		0	
0.3	1.76		0.04		0
		0.24		0	
0.4	2.00		0.04		
		0.28			
0.5	2.28				

Using forward difference polynomial is given by

$$P(x) = 1.4 + \frac{(x-0.1)}{1!}.\frac{0.16}{0.1} + \frac{(x-0.1)(x-0.2)}{2!}.\frac{0.04}{0.01}$$

$$= 2x^2 + x + 1.28$$

Using backward difference polynomial, we have,

$$P(x) = 2.28+(x-0.5)\left(\frac{0.28}{0.1}\right)+\frac{(x-0.5)(x-0.4)}{2!}\left(\frac{0.04}{0.1}\right)$$

$$= 2x^2+x+1.28$$

$$\text{Thus } f(0.25) = 2(0.25)^2+(0.25)+1.28$$

$$= 1.655$$

$$\text{and } f(0.35) = 2(0.35)^2+(0.35)+1.28$$

$$= 1.875$$

Example E. 3.11 :

Determine the step size that can be used in the tabulation of $f(x) = \sin x$ in the interval $[0, \pi/4]$ at equally spaced nodal points so that the truncation error of the quadratic interpolation is less than 5×10^{-8}.

Proof :

Le x_{i-1}, x_i, x_{i+1} denote three equispaced points with step size h.

Since $|f^{111}(x)| = |-\sin x|<1$ for $x\in[0, \pi/4]$

Hence the truncation error of the quadratic Lagrange interpolation is bounded by

$$|E_2(f; x)| \leq \frac{1}{6}|(x-x_{i-1})(x-x_i)(x-x_{i+1})| \qquad \text{------(3.42)}$$

where $x_{i-1}\leq x\leq x_{i+1}$

put $t=\frac{x-x_i}{h}$ in the right hand expression of (3.42), we have,

$$\max_{-1\leq t\leq 1}\frac{|(t-1)t(t+1)|}{6}h^3 = \frac{h^3}{9\sqrt{3}}$$

$$\text{Hence } |E_2(f; x)| < \frac{h^3}{9\sqrt{3}}$$

Thus the step size h be obtained from $\frac{h^3}{9\sqrt{3}}<5\times10^{-8}$

$$\Rightarrow h^3\leq5\times9\times\sqrt{3}\times10^{-8}$$

$$\Rightarrow h \cong 0.009$$

3.13 Hermite Interpolations :

The Hermite interpolating polynomial interpolates not only the function f(x) but also its derivatives (certain order) at a given set of tabular points.

Since the interpolating polynomial P(x) satisfying

$$P(x_i) = f(x_i)$$

$$P'(x_i) = f'(x_i),\ i = 0(1)n$$

then P(x) must be a polynomial of degree ≤2n+1.

Thus the required polynomial be written as

$$P(x) = \sum_{i=0}^{n} A_i(x)f(x) + \sum_{i=0}^{n} B_i(x)f'(x_i) \qquad \text{------(3.43)}$$

where $A_i(x)$ and $B_i(x)$ are polynomials of degree ≤2n+1 and satisfy.

$$\left.\begin{array}{l} \text{(a) } A_i(x_j) = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} \\ \text{(b) } A_i^1(x_j) = 0 \,\forall\, i \,\&\, j \\ \text{(c) } B_i(x_j) = 0 \,\forall\, i \,\&\, j \\ \text{(d) } B_i^1(x_j) = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} \end{array}\right\} \qquad \text{------(3.44)}$$

Using the Lagrange fundamental polynomials $l_i(x)$, we have,

$$\left.\begin{array}{l} A_i(x) = \lambda_i(x) l_i^2(x) \\ B_i(x) = \delta_i(x) l_i^2(x) \end{array}\right\} \qquad \text{------(3.45)}$$

Since $l_i(x)$ is a polynomial of degree 2n then $\gamma_i(x)$, $\delta_i(x)$ must be linear polynomials.

$$\left.\begin{array}{r} \text{Let } \gamma_i(x) = a_i x + b_i \\ \delta_i(x) = c_i x + d_i \end{array}\right\} \qquad \text{-------(3.46)}$$

Because of the conditions (3.44), we get,

$$\left.\begin{array}{l} a_i = -2l_i^1(x_i) \\ b_i = 1 + 2x_i l_i^1(x_i) \\ c_i = 1 \\ \&\, d_i = -x_i \end{array}\right\} \qquad \text{-----(3.47)}$$

From (3.43), (3.45), (3.46) & (3.47), we get

$$P(x) = \sum_{i=0}^{n}[1-2(x-x_i)l_i^1(x_i)]\, l_i^2(x)f(x_i)$$

$$+\sum_{i=0}^{n}(x-x_i)l_i^2(x)f^1(x_i) \qquad \text{------(3.48)}$$

Here (3.48) is called the Hermite interpolating polynomial.

Note : We can verify that $l_i^1(x_i) = \dfrac{w''(x_i)}{2w'(x_i)}$ and the truncation error.

Example E. 3.12 :

Given the following values of f(x) & f '(x). Estimate the values of the f(–0.5) & f(0.5) using Hermite interpolation.

x	f(x)	f'(x)
-1	1	-5
0	1	1
1	3	7

Solution :

Here $n = 2$, $x_0 = -1$, $x_1 = 0$, $x_2 = 1$.

$$\text{Now } P(x) = \sum_{i=0}^{n}A_i(x)f(x_i)+\sum_{i=0}^{n}B_i(x)f'(x_i) \qquad \text{------(3.48a)}$$

$$\text{where } A_0(x) = [1-2(x-x_0)l_0^1(x_0)]l_0^2(x)$$

$$A_1(x) = [1-2(x-x_1)l_1^1(x_1)]l_1^2(x)$$

$$A_2(x) = [1-2(x-x_0)l_2^1(x_2)]l_2^1(x)$$

$$B_0(x) = (x-x_0)l_0^2(x),$$

$$B_1(x) = (x-x_1)l_1^2(x)$$

$$\text{and } B_2(x) = (x-x_2)l_2^2(x)$$

$$\text{Now } l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$= \frac{x(x-1)}{2}$$

$$\therefore l_0^1(x) = \frac{2x-1}{2}$$

$$\Rightarrow l_0^1(-1) = -\frac{3}{2}$$

$$\text{Similarly } l_1(x) = -(x^2-1),\ l_1^1(0)$$

$$= 0$$

$$l_2(x) = \frac{x(x+1)}{2},\ l_2^1(1)$$

$$= \frac{3}{2}$$

$$\text{Hence } A_0(x) = [1+3(x+1)]\frac{x^2(x-1)^2}{4}$$

$$= \frac{1}{4}[3x^5-2x^4-5x^3+4x^2],$$

$$A_1(x) = x^4-2x^2+1,$$

$$\&\ A_2(x) = \frac{1}{4}[-3x^5-2x^4+5x^3+4x^2]$$

$$\text{and } B_0(x) = \frac{1}{4}(x^5-x^4-x^3+x^2)$$

$$B_1(x) = x^5-2x^3+3$$

$$B_2(x) = \frac{1}{4}(x^5+x^4-x^3-x^2)$$

$$\text{Hence (1)} \Rightarrow p(x) = 2x^4-x^2+x+1.$$

$$\text{Now } f(-0.5) \cong p(-0.5)$$

$$= 3/8$$

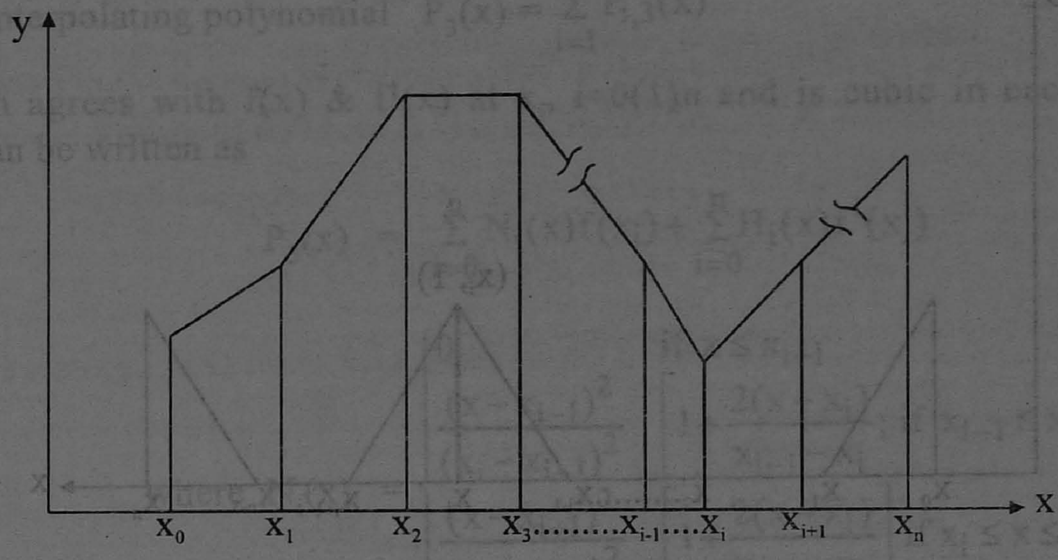
$$\&\ f(0.5) \cong \frac{11}{8}$$

3.14 PIECEWISE AND SPLINE INTERPOLATION :

In order to achieve accurate results, we keep the degree of interpolatng polynomials are small and this can be obtained using.

3.14.1 Piecewise Linear Interpolation :

Given that $a = x_0<x_1<x_2<........<x_n = b$ as n+1 nodal points and we want to find an interpolation which shown in the following fig.



Since the interpolation is linear in each subinterval $[x_{i-1}, x_i]$ and it agrees with the function f(x) at n+1 nodal points. The subintervals or the line segments are called finite elements in one dimension and the nodal points are called knots. Using the linear Lagrange interpolation (3.12), for $x \in [x_{i-1}, x_i]$, we have the piecewise linear interpolation

$$P_{i,1}(x) = \frac{x - x_i}{x_{i-1} - x_i} f(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} f(x_i) \quad i = 1(1)n \qquad \text{------(3.49)}$$

Hence the interpolating polynomial becomes,

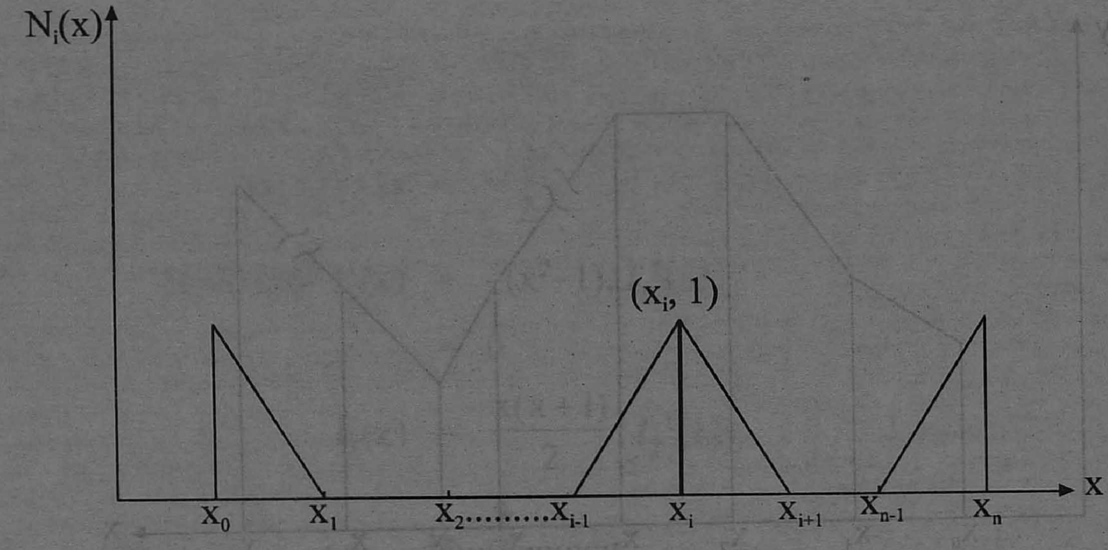
$$P(x) = \sum_{i=0}^{n} P_{i,1}(x) \qquad \text{-----(3.50)}$$

which agrees with f(x) at x_i, i = 0(1)n and also linear in each subinterval $[x_{i-1}, x_i]$ can be written as

$$P(x) = \sum_{i=0}^{n} N_i(x).f(x_i) \qquad \text{------(3.51)}$$

$$\text{where } N_i(x) = \begin{cases} 0 & \text{if } x \le x_{i-1} \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & \text{if } x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & \text{if } x_i \le x \le x_{i+1} \\ 0 & \text{if } x \ge x_{i+1} \end{cases} \qquad \text{------(3.52)}$$

The function $N_i(x)$ is called a shape function and it is shown in the following next page figure.



Note : The error in piecewise linear interpolation is

$$f(x)-P(x) \;=\; \frac{1}{2!}(x-x_{i-1})(x-x_i)f''(\xi_i);\; x_{i-1}<\xi_i<x_i \qquad \text{------(3.53)}$$

3.14.2 Piecewise Cubic Interpolation :

In each interval $[x_{i-1}, x_i]$; $i = 1(1)n$, approximate the function $f(x)$ by a cubic polynomial $P_{i,\,3}(x)$, we get, piecewise cubic interpolation. The cubic polynomial on each interval $[x_{i-1}, x_i]$ can be obtained by using the conditions.

$$\left.\begin{aligned} P_{i,3}(x_{i-1}) = f_{i-1},\ P_{i,\,3}(x_i) = f_i \\ P_{i,3}(x_{i-1}) = f^1_{1-1},\ P^1_{i,3}(x_i) = f^1_i \end{aligned}\right\} \qquad \text{------(3.54)}$$

The obtained polynomial is called piecewise cubic Hermite interpolation.

$$\text{Hence}\quad P_{i,3}(x) \;=\; A_{i-1}(x)f_{i-1}+A_i(x)f_i+B_{i-1}(x)f^1_{i-1}+B_i(x)f^1_i \qquad \text{-----(3.55)}$$

$$\left.\begin{aligned} \text{where}\quad A_{i-1}(x) &= \frac{(x-x_i)^2}{(x_{i-1}-x_i)^2}\left[1+\frac{2(x_{i-1}-x)}{x_{i-1}-x_i}\right] \\ A_i(x) &= \frac{(x-x_{i-1})^2}{(x_{i-1}-x_i)^2}\left[1+\frac{2(x-x_i)}{x_{i-1}-x_i}\right] \\ B_{i-1}(x) &= \frac{(x-x_{i-1})(x-x_i)^2}{(x_{i-1}-x_i)^2} \\ B_i(x) &= \frac{(x-x_i)(x-x_{i-1})^2}{(x_{i-1}-x_i)^2} \end{aligned}\right\} \qquad \text{------(3.56)}$$

The interpolating polynomial $P_3(x) = \sum_{i=1}^{n} P_{i,3}(x)$ ------(3.57)

which agrees with $f(x)$ & $f^1(x)$ at x_i, $i=0(1)n$ and is cubic in each subinterval $[x_{i-1}, x_i]$ can be written as

$$P_3(x) = \sum_{i=0}^{n} N_i(x)f(x_i) + \sum_{i=0}^{n} H_i(x)f^1(x_i) \quad \text{-----(3.58)}$$

$$\text{where } N_i(x) = \begin{cases} 0 & \text{if } x \le x_{i-1} \\ \dfrac{(x-x_{i-1})^2}{(x_i-x_{i-1})^2}\left[1+\dfrac{2(x-x_i)}{x_{i-1}-x_i}\right]; & \text{if } x_{i-1} \le x \le x_i \\ \dfrac{(x-x_{i+1})^2}{(x_{i+1}-x_i)^2}\left[1+\dfrac{2(x-x_i)}{x_{i+1}-x_i}\right] & \text{if } x_i \le x \le x_{i+1} \\ 0 & \text{if } x \ge x_{i+1} \end{cases} \quad \text{---(3.59)}$$

$$\text{and } H_i(x) = \begin{cases} 0 & \text{if } x \le x_{i-1} \\ \dfrac{(x-x_{i-1})^2(x-x_i)}{(x_i-x_{i-1})^2} & \text{if } x_{i-1} \le x \le x_i \\ \dfrac{(x-x_{i+1})^2(x-x_i)}{(x_{i+1}-x_i)^2} & \text{if } x_i \le x \le x_{i+1} \\ 0 & \text{if } x \ge x_{i+1} \end{cases} \quad \text{----(3.60)}$$

Clearly $P_{i-1,3}(x_i) = P^1_{i,3}(x_i) = f_i;\ i = 1(1)n$

and $P^1_{i-1,3}(x_i) = P^1_{i,3}(x_i) = f^1_i;\ i = 1(1)\,n$

Hence $P_3(x)$ is continuously differentiable on [a, b].

The error in the piecewise cubic Hermite polynomial is

$$f(x)-P_{i,3}(x) = \frac{1}{4!}(x-x_{i-1})^2(x-x_i)^2 f^{(4)}(\xi_i) \quad x_{i-1}<\xi_i<x_i \quad \text{------(3.61)}$$

Example E. 3.13 :

Using piecewise cubic Hermite interpolation estimate $f(-0.5)$ & $f(0.5)$ from the following data.

x	f(x)	f'(x)
-1	1	-5
0	1	1
1	3	7

Solution :

$$\text{Here } x_{i-1} = -1,\ x_i=0,\ x_{i+1}=1 \ \&\ n=3.$$

$$\text{Since } x = -0.5 \in [-1, 0] = [x_{i-1}, x_i] \text{ and therefore}$$

$$\begin{aligned} P_3(x) &= [1+2(x+1)]x^2(1)+[1-2(x-0)](x+1)^2(1) \\ &\quad +(x+1)x^2(-5)+x(x+1)^2(1) \\ &= -(4x^3+3x^2-x-1) \end{aligned}$$

$$\text{Hence } f(-0.5) \cong \frac{1}{4}$$

$$\text{Again } x = 0.5 \in [0, 1] = [x_i, x_{i+1}] \text{ then}$$

$$\begin{aligned} P_3(x) &= [1+2(x-0)](x-1)^2(1)+[1-2(x-1)]x^2(3) \\ &\quad +x(x-1)^2(1)+(x-1)x^2(7) \\ &= 4x^3-3x^2+x+1 \end{aligned}$$

$$\text{and therefore } f(0.5) \cong \frac{5}{4}$$

3.15 Spiline Interpolation :

The piece cubic Hermite interpolation method is not applicable when $f^1(x_i)$, $i=0(1)n$ are not known.

The particular choice of the numbers $m_i = f^1(x_i)$, the resulting cubic polynomial $P_3(x)$ will interpolate $f(x)$ at $x_0, x_1, \ldots\ldots, x_n$ and also $P_3(x)$ is continuously differentiable on [a, b].

The second order derivatives of $P_3(x)$ exists but need not be continuous at the knots. It is possible to determine $m_0, m_1, \ldots\ldots\ldots, m_n$ in such a way that the resulting piecewise cubic interpolation is twice continuously differentiable. Such an interpolation is called cubic spline interpolation.

Definition D. 3.3 :

A spline function of degree n with knots $x_0, x_1, \ldots\ldots, x_n$ is a function $F(x)$ with the properties

(i) On each subinterval $[x_{i-1}, x_i]$, $1 \le i \le n$, $F(x)$ is a polynomial of degree n.

(ii) $F(x)$ and its first $(n-1)$ derivatives are continuous on [a, b].

Construction of the Cubic spline function :

Since F(x) is to be a piecewise cubic polynomial, F"(x) is a linear function of x in the interval $x_{i-1} \le x \le x_i$ and hence it can be written as :

$$F''(x) = \frac{x_i - x}{x_i - x_{i-1}} F''(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} F''(x_i) \quad \text{-----(3.62)}$$

Integrating F"(x) two times, with respect to x, we get,

$$F(x) = \frac{(x_i - x)^3}{6h_i} M_{i-1} + \frac{(x - x_{i-1})^3}{6h_i} M_i + C_1 x + C_2 \quad \text{-----(3.63)}$$

where $M_i = F''(x_i)$ and C_1 and C_2 are integrating constants to be determined by using the conditions $F(x_{i-1}) = f(x_{i-1})$ and $F(x_i)=f(x_i)$

$$\therefore \text{ we have } C_1 = \frac{f_i - f_{i-1}}{h_i} - \frac{1}{6}(M_i - M_{i-1})h_i$$

$$C_2 = \frac{x_i f_{i-1} - x_{i-1} f_i}{h_i} - \frac{1}{6}(x_i M_{i-1} - x_{i-1} M_i)h_i \quad \text{-----(3.64)}$$

$$\therefore \text{ (3.63) becomes } F(x) = \left(\frac{(x_i - x)((x_i - x)^2 - h_i^2)}{6h_i}\right) M_{i-1}$$

$$+ \left(\frac{(x - x_{i-1})((x - x_{i-1})^2 - h_i^2)}{6h_i}\right) M_i$$

$$+ \frac{1}{h_i}(x_i - x) f_{i-1} + \frac{1}{h_i}(x - x_{i-1}) f_i \quad \text{------(3.65)}$$

$$\text{and } F^1(x) = \frac{-(x_i - x)^2}{2h_i} M_{i-1} + \frac{(x - x_{i-1})^2}{2h_i} M_i$$

$$- \frac{(M_i - M_{i-1})}{6} h_i + \frac{f_i - f_{i-1}}{h_i} \quad \text{-------(3.66)}$$

Now we require that the derivative $f^1(x)$ be continuous at $x = x_i \pm \in$ as $\in \to 0$.

Putting $F^1(x_i - \in) = F^1(x_i + \in)$ as $\in \to 0$, we get

$$\frac{h_i}{6} M_{i-1} + \frac{h_i}{3} M_i + \frac{1}{h_i}(f_i - f_{i-1})$$

$$= -\frac{h_{i+1}}{3} M_i - \frac{h_{i+1}}{6} M_{i+1} + \frac{1}{h_{i+1}}(f_{i+1} - f_i)$$

which may be written as $\frac{h_i}{6}M_{i-1}+\frac{h_i+h_{i+1}}{3}M_i+\frac{h_{i+1}}{6}M_{i+1}$

$$= \frac{1}{h_{i+1}}(f_{i+1}-f_i)-\frac{1}{h_i}(f_i-f_{i-1}) \qquad \text{------(3.67)}$$

$$i = 1(1)n-1.$$

This gives a system of n–1 linear equations in n+1 unknowns $M_0, M_1, \ldots\ldots, M_n$.

Two more additional conditions may be taken in one of the following forms.

(i) $M_0=M_n=0$. (The spline satisfying these conditions is called natural spline).

(ii) $M_0=M_n$, $M_1=M_{n+1}$, $f_0=f_n$, $f_1=f_{n+1}$, $h_1=h_{n+1}$

(The spline satisfying these conditions is called Periodic spline)

(iii) For a non-periodic spline, we use the following conditions,

$$F^1(a) = f^1(a) = f^1_0 \text{ and}$$

$$F^1(b) = f^1(b) = f^1_n$$

which gives $2M_0+M_1 = \frac{6}{h_1}\left(\frac{h-f_0}{h_1}-f_0{}^1\right)$

and $M_{n-1}+2M_n = \frac{6}{h_n}\left(f_n{}^1-\frac{f_n-f_{n-1}}{h_n}\right)$ -----(3.68)

Note :

For equispaced knots, (i.e.,) hi = h for every i,

$$F(x) = \frac{1}{6h}\left[(x_i-x)^3M_{i-1}+(x-x_{i-1})^3M_i\right]$$

$$+\frac{1}{h}(x_i-x)\left(f_{i-1}-\frac{h^2}{6}M_{i-1}\right)$$

$$+\frac{1}{h}(x-x_{i-1})\left(f_i-\frac{h^2}{6}M_i\right) \qquad \text{------(3.69)}$$

and $M_{i-1}+4M_i+M_{i+1} = \frac{6}{h^2}(f_{i+1}-2f_i+f_{i-1})$ ------(3.70)

Example E. 3.14 :

Obtain the cubic spline approximation for the f(x) given in the following table :

x	:	0	1	2	3	
f(x)	:	1	2	33	244	and M(0) = M(3) = 0

Solution :

Here h = 1 and ∴ the cubic spline interpolation be obtained from

$$M_{i-1}+4M_i+M_{i+1} = \frac{6}{h^2}(f_{i+1}-2f_i+f_{i-1}),\ i = 1, 2.$$

By putting i = 1, 2, we get

$$M_0+4M_1+M_2 = 6(f_2-2f_1+f_0)$$

$$M_1+4M_2+M_3 = 6(f_3-2f_2+f_1)$$

Again substituting $M_0=M_3=0$ & the values of f_0, f_1, f_2, f_3, we get

$$4M_1+M_2 = 180$$

$$M_1+4M_2 = 1080$$

Solving the above two equations, we get

$$M_1 = -24$$

$$M_2 = 276$$

Now we shall find the cubic splines in the corresponding intervals.

In the interval [0, 1], (i.e,) $x_0 = 0$, $x_1 = 1$, then from (3.65), we have,

$$F(x) = \left(\frac{(x-0)\left((x-0)^2-1\right)}{6}\right)(-24)+(1-x)(1)+1(x-0)(2)$$

$$= -4x^3+5x+1$$

Similarly in the interval [1, 2],

$$F(x) = 50x^3-162x^2+167x-53 \text{ and in } [2, 3]$$

$$F(x) = -46x^3+414x^2-985x+715$$

Hence the cubic spline interpolation is givenly

$$F(x) = \begin{cases} -4x^3+5x+1 & \text{if } x \in [0, 1] \\ 50x^3-162x^2+167x-53 & \text{if } x \in [1, 2] \\ -46x^3+414x^2-985x+715 & \text{if } x \in [2, 3] \end{cases}$$

3.16 Bivariate Interpolation :

We shall discuss the interpolating polynomials for the functions of several independent variables. For our convenience we took only two variable and we may extend the same procedure for more than two variables.

3.16.1 Lagrange Bivariate Interpolation :

Let f(x,y) be defined at (m+1)(n+1) distinct points (x_i, y_j); i=0(1)m; j=0(1)n, and let $f_{ij} = f(x_i, y_j)$.

We want to find a polynomial P(x; y) of degree atmost m in x & n in y, such that

$$P(x, y) = f_{ij};\ i = 0(1)m;\ j = 0(1)n \qquad \text{-------(3.71)}$$

Using the Lagrange fundamental polynomial (3.25) of a single variable,

$$\text{we define } X_{mi}(x) = \frac{w(x)}{(x-x_i)w^1(x_i)};\quad i = 0(1)\ m$$

$$\text{and } y_{nj}(y) = \frac{w_1(y)}{(y-y_j)w_1{}^1(y_j)};\quad j = 0(1)n$$

$$\text{where } w(x) = (x-x_1)(x-x_2)..........(x-x_m)$$

$$\text{and } w_1(y) = (y-y_1)(y-y_2)..........(y-y_n)$$

Clearly $X_{mi}{}^{(x)}$, $y_{nj}(y)$ are polynomials of degree m in x and n in y respectively.

Again thee polynomials satisfies

$$X_{mi}(x_k) = \delta_{ik},\ y_{nj}(y_k) = \delta_{jk}$$

Hence the required interpolating bivariate polynomial is

$$P_{mn}(x, y) = \sum_{x=0}^{m}\sum_{y=0}^{n} X_{mi}(x)y_{nj}(y)f_{ij} \qquad \text{-------(3.72)}$$

and the above polynomial (3.72) is called Lagrange bivariate interpolating polynomial.

3.16.2. Newton's Bivariate Interpolation for Equispaced Points :

Let the nodal points of x and y be x_i; i = 0(1)m & y_j; j=0(1)n with

$x_i-x_{i-1}=h$ & $y_j-y_{j-1}=k$.

We define the following :

(i) $\Delta_x f(x, y) = f(x+h, y)-f(x, y)$

$= (E_x-1)f(x, y)$

$$
\begin{aligned}
\text{(ii)} \quad \Delta_y f(x, y) &= f(x, y+k)-f(x, y)\\
&= (E_y-1)f(x, y)\\
\text{(iii)} \quad \Delta_{xx} f(x, y) &= \Delta_x f(x+h, y)-\Delta_x f(x, y)\\
&= [E_x-1]^2 f(x, y)\\
\text{(iv)} \quad \Delta_{yy} f(x, y) &= \Delta_y f(x, y+k) - \Delta_y f(x, y)\\
&= (E_y-1)^2 f(x, y) \qquad \text{and}\\
\text{(v)} \quad \Delta_{xy} f(x, y) &= \Delta_x f(x, y+k) - \Delta_x f(x, y)\\
&= \Delta_y f(x+h, y) - \Delta_y f(x, y)\\
&= (E_x-1)(E_y-1) f(x, y) \text{ and so on.}
\end{aligned}
$$

Again $f(x_0+mh, y_0+nk)$

$$
\begin{aligned}
&= E_x{}^m E_y{}^n f(x_0, y_0)\\
&= (1+\Delta_y)^m(1+\Delta_y)^n f(x_0, y_0)\\
&= [1+mC_1\Delta_x+nC_1\Delta_y+mC_2\Delta_{nx}+mC_1.nC_1\Delta_{xy}+nC_2\Delta_{yy}+.....]f(x_0, y_0)
\end{aligned}
$$

From the above we can able to find an interpolating polynomial

$$
\begin{aligned}
P(x, y) = {} & f(x_0,y_0)+\left(\frac{1}{h}(x-x_0)\Delta_x+\frac{1}{k}(y-y_0)\Delta y\right)f(x_0,y_0)\\
&+\frac{1}{2!}\left[\frac{1}{h^2}(x-x_0)(x-x_1)\Delta_{xx}+\frac{2}{hk}(x-x_0)(y-y_0)\Delta_{xy}\right.\\
&\left.+\frac{1}{k^2}(y-y_0)(y-y_1)\Delta_{yy}\right]f(x_0,y_0)+..... \qquad \text{----(3.73)}
\end{aligned}
$$

The polynomial (3.73) is called Newton's bivariate interpolating polynomial for equispaced points.

Example E. 3.15 :

Using the following data obtain the Lagrange and Newton's bivariate interpolating polynomials.

y \ x	0	1	2
0	1	3	7
1	3	6	11
2	7	11	17

Solution :

$$\text{We have } X_{20} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$= \frac{(x-1)(x-2)}{(0-1)(0-2)}$$

$$= \frac{(x-1)(x-2)}{2}$$

$$\text{Similarly } X_{21} = \frac{x(x-2)}{-1}$$

$$X_{22} = \frac{x(x-1)}{2}$$

$$y_{20} = \frac{(y-1)(y-2)}{2}$$

$$y_{21} = \frac{y(y-2)}{-1}$$

$$y_{22} = \frac{y(y-1)}{2}$$

$\therefore$ By Lagrange interpolating polynomial, we have

$$P_2(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} X_{2i} Y_{2j} f_{ij}$$

$$= \frac{(x-1)(x-2)}{2}(y^2+y+1) - x(x-2)(y^2+2y+3) + \frac{(x^2-x)}{2}(y^2+3y+7)$$

Using Newton's Bivariate Interpolation :

$$\text{Again } \Delta_x f(x_0, y_0) = f(x_{0+h}, y_0) - f(x_0, y_0)$$

$$= (3-1)$$

$$= 2$$

$$\Delta_y f(x_0, y_0) = 2$$

$$\Delta_x f(x_1, y_1) = 7-3 = 4$$

$$\Delta_y f(x_1, y_1) = 7-3 = 4$$

$$\Delta_{xx} f(x_0, y_0) = \Delta_x f(x_{0+h}, y_0) - \Delta_x f(x_0, y_0)$$

$$= 4-2$$

$$= 2$$

$$\Delta_{yy} f(x_0, y_0) = 2$$

$$\Delta_{xy} f(x_0, y_0) = 4-2 = 2$$

Using Newton's interpolating polynomial, we have

$$\begin{aligned}
P(x,y) &= f(x_0,y_0) + \left[\frac{1}{h}(x-x_0)\Delta_x + \frac{1}{k}(y-y_0)\Delta y\right] f(x_0,y_0) \\
&\quad + \frac{1}{2!}\left[\frac{1}{h^2}(x-x_0)(x-x_1)\Delta_{xx} + \frac{2}{hk}(x-x_0)(y-y_0)\Delta_{xy}\right. \\
&\quad \left. + \frac{1}{k^2}(y-y_0)(y-y_1)\Delta_{yy}\right] f(x_0,y_0) + \ldots\ldots \\
&= 1 + \left[\frac{1}{1}(x-0)2 + \frac{1}{1}(y-0)2\right] \\
&\quad + \frac{1}{2}\left[\frac{1}{1}(x-0)(x-1)2 + \frac{2}{1}(x-0)(y-0)2 + \frac{1}{1}(y-0)(y-1)2\right] \\
&= 1 + 2(x+y) + [x^2 - x + xy + y^2 - y] \\
&= x^2 + y^2 + xy + x + y + 1
\end{aligned}$$

The existence of a polynomial function P(x) which approximates any continuous function f(x) on a finite interval [a, b] is guaranteed by the Weierstrass approximation theorem. Here we state that theorem without proof.

"If the function f(x) is continuous on a finite interval [a, b], then given any $\in > 0$, $\exists$ *a n and a polynomial P(x) of degree n such that* $|f(x)-P(x)| < \in$ *for all* $x \in [a, b]$.

3.17 Least Squares Approximations :

Least square approximations are most commonly used approximations for approximating a function f(x) which may be given in tabular form or known explicitly over a given interval.

In this method, the best approximation is defined as that for which the constants $C_1, C_2, \ldots\ldots\ldots, C_n$ are determined so that the aggregate of $W(x)E^2$ over a given domain D is as small as possible where $W(x)>0$ called weight function.

Example E. 3.16 :

Determine the lest squares approximations of the type ax^2+bx+c, to the function 2^x at the points $x_i = 0,1,2,3,4$.

Solution :

Given that

x_i	2^{x_i}	x_i^2	x_i^3	x_i^4
0	1	0	0	0
1	2	1	1	1
2	3	4	8	16
3	8	9	27	81
4	16	16	64	256
Total 10	31	30	100	354

We determine a, b and c such that

$$I = \sum_{i=0}^{4}\left(2^{x_i} - \left(ax_i^2 + bx_i + c\right)\right)^2 \text{ is minimum}$$

Hence we have the following normal equations as

$$\sum_{i=0}^{4}\left(2^{x_i} - ax_i^2 - bx_i - c\right) = 0$$

$$\sum_{i=0}^{4}\left(2^{x_i} - ax_i^2 - bx_i - c\right)x_i = 0$$

$$\sum_{i=0}^{4}\left(2^{x_i} - ax_i^2 - bx_i - c\right)x_i^2 = 0$$

Using the table values, we get, the following equation

$$30a+10b+5c = 31$$

$$100a+30b+10c = 98$$

$$354a+100b+30c = 346$$

Solving the above equations, we have,

$a = 1.143$, $b = -0.971$, $c = 1.286$

$\therefore$ The least squares approximation to 2^x is $y=1.143x^2-0.971x+1.286$

Example E. 3.17 :

A person runs the same race track for five consecutive days and is timed as follows :

Days :	1	2	3	4	5
Times :	15.30	15.10	15.00	14.50	14.00

Make a last square fit to the above data using a function $a+\frac{b}{x}+\frac{c}{x^2}$.

Solution :

Let $I = \sum_{i=0}^{4}\left[y_i - a - \frac{b}{x_i} - \frac{c}{x_i^2}\right]$

Now we want to determine the values of a, b and c so that I is minimum. The normal equations of I are given by

$$\sum_{i=0}^{4} y_i - 5a - b\sum_{i=0}^{4}\frac{1}{x_i} - c\sum_{i=0}^{4}\frac{1}{x_i^2} = 0$$

$$\sum_{i=0}^{4}\frac{y_i}{x_i} - a\sum_{i=0}^{4}\frac{1}{x_i} - b\sum_{i=0}^{4}\frac{1}{x_i^2} - c\sum_{i=0}^{4}\frac{1}{x_i^3} = 0$$

$$\sum_{i=0}^{4}\frac{y_i}{x_i^2} - a\sum_{i-0}^{4}\frac{1}{x_i^2} - b\sum_{i=0}^{4}\frac{1}{x_i^2} - c\sum_{i=0}^{4}\frac{1}{x_i^4} = 0$$

Using the given data, we can easily find that

$$\sum\frac{1}{x_i} = 2.2833333$$

$$\sum\frac{1}{x_i^2} = 1.463611$$

$$\sum_{i=0}^{4}\frac{1}{x_i^3} = 1.18662$$

$$\sum_{i=0}^{4}\frac{1}{x_i^4} = 1.080352$$

$$\sum_{i=0}^{4} y_i = 73.90$$

$$\sum_{i=0}^{4} \frac{y_i}{x_i} = 34.275$$

$$\sum_{i=0}^{4} \frac{y_i}{x_i^2} = 22.207917$$

$\therefore$ The substiting the above values in the normal equations we get,

$$5a + 2.283333b + 1.463611c = 73.90$$

$$2.28333331 + 1.963611b + 1.185662c = 34.275$$

$$1.463611a + 1.185662a + 1.080352c = 22.207917$$

Solving the above equations, we get a=13.0065, b=6.7512, c=–4.4738.

$\therefore$ Required least square approximation is

$$f(x) = 13.0065 + \frac{6.7512}{x} - \frac{4.4738}{x^2}$$

Example E. 3.18 :

Obtain the least square polynomial approximation of degree two for $f(x)=\sqrt{x}$ on [0, 1].

Solution :

Let $I = \int_0^1 \left(\sqrt{x} - a - bx - cx^2\right)^2 dx$

Here we want to find the values of a, b & c in such a way that I is minimum.

$\therefore$ The normal equations are given by

$$\frac{\partial I}{\partial a} = -2\int_0^1 \left(\sqrt{x} - a - bx - cx^2\right)dx$$

$$\frac{\partial I}{\partial b} = -2\int_0^1 \left(\sqrt{x} - a - bx - cx^2\right)x\,dx$$

$$\frac{\partial I}{\partial c} = -2\int_0^1 \left(\sqrt{x} - a - bx - cx^2\right)x^2 dx$$

(i.e.,) $\quad a + \frac{1}{2}b + \frac{1}{3}c = \frac{2}{3}$

$$\frac{1}{2}a + \frac{1}{3}b + \frac{1}{4}c = \frac{2}{5}$$

$$\frac{1}{3}a + \frac{1}{4}b + \frac{1}{5}c = \frac{2}{7}$$

Solving the above equations $a = \frac{6}{35}$, $b = \frac{48}{35}$ and $c = -\frac{20}{35}$.

$\therefore$ The required approximation is

$$P(x) = \frac{1}{35}\left(6 + 48x - 20x^2\right)$$

Definition D. 3.4 :

A set of function $\{\phi_i(x)\}$ is said to the orthogonal over a set of points $\{x_i\}$ with respect to the weight function W(x), if

$$\sum_{i=0}^{N} W(x_i)\phi_j(x_i)\phi_k(x_i) = 0, \quad j \neq k \qquad \text{------(3.74)}$$

Definition D. 3.5 :

A set functions $\{\phi_i(x)\}$ is said to be orthogonal on an interval [a, b] with respect to the weight function if

$$\int_a^b W(x)\phi_i(x)\phi_j(x)dx = 0, \quad i \neq j \qquad \text{-------(3.75)}$$

3.18 Gram-Schmidt Orthogonalizing Process :

Given the polynomial $\phi_i(x)$ of degree i, the polynomials $\phi^*_i(x)$ of degree i which are orthogonal over [a, b] with respect to the weight function W(x) can be generated recursively from the relation

$$\phi^*_i(x) = x^i - \sum_{r=0}^{i-1} a_{ir}\phi^*_r(x), \quad i = 1,2,3,\ldots\ldots,n \qquad \text{-------(3.76)}$$

$$\text{where } a_{ir} = \frac{\int_a^b W(x)x^i\phi^*_r(x)dx}{\int_a^b W(x)\phi^{*2}_r(x)dx} \text{ and } \phi^*_0(x) = 1$$

Just like Gram-Schmidt orthogonalization process the other available orthogonalization polynomial are Legendre Polynomial and Chebyshev polynomials.

3.19 Legendre Polynomials :

The legendre polynomial $P_n(x)$ defined on [–1, 1] are given by

$$P_n(x) = \sum_{m=0}^{M}(-1)^m \frac{(2n-2m)!\,x^{n-2m}}{m!(n-m)!(n-2m)!} \qquad \text{-----(3.77)}$$

where $m = \frac{n}{2}$ or $\frac{n-1}{2}$ whichever is integer.

The Legendre polynomial satisfy the differential equation

$$(1-x^2)y''-2x'+n(n+1)y = 0 \quad \text{------(3.78)}$$

The Legendre polynomial satisfy the recurence relation

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x)-nP_{n-1}(x) \quad \text{------(3.79)}$$

Again the Legendre polynomials has the following properties.

(i) $P_n(x)$ is an even polynomial if n is even & an odd polynomial if n is odd.

(ii) $P_n(x)$ are orthogonal pulynomial and satisfy

$$\int_{-1}^{1} P_m(x).P_n(x)dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n+1} & \text{if } m = n \end{cases} \quad \text{------(3.80)}$$

(iii) $P_n(-x) = (-1)^n P_n(x)$ ------(3.81)

3.20 Chebyshev Polynomials :

The Chebyshevs polynomial of the first kind $T_n(x)$ defined on [–1, 1] are given by

$$T_n(x) = \cos(n\cos^{-1}x) = \cos n\theta \quad \text{-------(3.82)}$$

$$\text{where } \theta = \cos^{-1}x$$

The polynomial in (3.82) satisfy the differential equation

$$(1-x^2)y''-xy'+n^2y = 0 \quad \text{------(3.83)}$$

One dependent solution of (3.83) gives $T_n(x)$ and the second independent solution gives Chebyshev polynomials of the second kind $U_n(x)$.

$$U_n(x) = \sin[(n+1)\cos^{-1}x] \quad \text{------(3.84)}$$

The polynomial (3.82) satisfy the recurrence relation.

$$T_{n+1}(x) = 2xT_n(x)-T_{n-1}(x) \quad \text{------(3.85)}$$

$$\text{where } T_0(x) = 1,\ T_1(x) = x$$

The Chebyshev polynomials $T_n(x)$ has the following properties :

(i) $T_n(x)$ is a polynomial of degree n.

(ii) $T_n(x)$ has n simple zeros given by $x_k=\cos\left(\frac{2k-1}{2n}\pi\right)$; k=1,2,3,......,n on the interval [–1, 1].

(iii) $T_n(x)$ assumes extreme values at n+1 points $x_k = \cos\left(\frac{k\pi}{n}\right)$; k=0,1,2,....,n and the extreme value at x_k is $(-1)^k$.

(iv) $|T_n(x)| \le 1$; $x \in [-1, 1]$

(v) If $P_n(x)$ is any polynomial of degree n with leading coefficient unity and $\tilde{T}_n(x) = T_n(x)/2^{n-1}$ is the monic Chebyshev polynomial then

$\max\limits_{-1\le x\le 1}\left|\tilde{T}_n(x)\right| \le \max\limits_{-1\le x\le 1}\left|p_n(x)\right|$ (this property is called minimax property).

(vi) $T_n(x)$ are orthogonal with respect to the weight function $W(x) = \frac{1}{\sqrt{1-x^2}}$

$$\therefore \int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} 0 & \text{if } m \ne n \\ \frac{\pi}{2} & \text{if } m = n \ne 0 \\ \pi & \text{if } m = n = 0 \end{cases} \qquad \text{------(3.86)}$$

Example E. 3.19 :

Develop the function $f(x) = \frac{1}{2} ln\left(\frac{1+x}{1-x}\right)$ in a series of Chebyshev polynomials.

Solution :

$$\text{Let } f(x) = \sum_{r=0}^{0}{}' a_r T_r(x)$$

where Σ' denotes a summation whose first term is halved. Using orthogonal properties of the Chebyshev polynomials, we get,

$$a_r = \frac{2}{\pi}\int_{-1}^{1}\left(1-x^2\right)^{-1/2} f(x)T_r(x)dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(\cos\theta)\cos r\theta d\theta$$

$$\text{Now } f(x) = \frac{1}{2} ln\left(\frac{1+x}{1-x}\right)$$

$$\therefore f(\cos\theta) = \frac{1}{2} ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right)$$

$$= \frac{1}{2} ln\left(\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}}\right)$$

$$= \ln\left(\cos\frac{\theta}{2}\right)$$

$$\therefore\ a_r = \frac{2}{\pi}\int_0^{\pi}\ln\left(\cot\frac{\theta}{2}\right)\cos r\theta\, d\theta$$

$$= \frac{2}{\pi}\left\{\left[\frac{1}{r}\sin r\theta\, \ln\cot\frac{\theta}{2}\right]_0^{\pi} + \int_0^{\pi}\sin r\theta\tan\frac{\theta}{2}\operatorname{cosec}^2\frac{\theta}{2}\, d\theta\right\}$$

$$= \frac{2}{\pi r}\int_0^{\pi}\frac{\sin r\theta}{\sin\theta}\, d\theta$$

$$= \frac{2}{\pi r} I_r \text{ where } I_r = \int_0^{\pi}\frac{\sin r\theta}{\sin\theta}\, d\theta$$

$$\text{Now } I_r = \int_0^{\pi}\frac{\sin r\theta}{\sin\theta}\, d\theta$$

$$= \int_0^{\pi}\frac{\sin((r-1)\theta+\theta)\, d\theta}{\sin\theta}$$

$$= \int_0^{\pi}\frac{\sin(r-1)\theta\cos\theta + \cos(r-1)\theta\sin\theta}{\sin\theta}\, d\theta$$

$$= \int_0^{\pi}\frac{\sin(r-1)\theta\cos\theta}{\sin\theta}\, d\theta + \int_0^{\pi}\frac{\cos(r-1)\theta\sin\theta}{\sin\theta}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\frac{2\sin(r-1)\theta\cos\theta}{\sin\theta} + 0$$

$$= \frac{1}{2}\int_0^{\pi}\frac{\sin r\theta + \sin(r-2)\theta}{\sin\theta}\, d\theta$$

$$= \frac{1}{2}I_r + \frac{1}{2}I_{r-2}$$

$$\text{Thus } I_r = I_{r-2} = I_{r-4} = \dots = \begin{cases} I_0 & \text{if } r \text{ is even} \\ I_1 & \text{if } r \text{ is odd} \end{cases}$$

$$\text{Thus } a_r = \begin{cases} 0 & \text{if } r \text{ is even} \\ \dfrac{2}{r} & \text{if } r \text{ is odd} \end{cases}$$

$$\therefore\ \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = 2\left[I_1 + \frac{1}{3}I_3 + \frac{1}{5}I_5 + \dots\dots\right].$$

UNIT – 4

DIFFERENTIATION AND INTEGRATION

4.1 Introduction :

If a function f(x) be a given function then we can able to find derivative or integral of that function. When f(x) be some complicated function then it is very difficult to find its derivative or integral value. In such situation we may use numerical differentiation or numerical integration.

4.2 Numerical Differentiation :

4.2.1 Methods Based on Interpolation :

Given the values of f(x) at a set of points x_0, x_1, x_2,........,x_n then first find the interpolating polynomials $P_n(x)$ and their differentiate this polynomial r times (n≥r) to get $P^{(r)}{}_n(x)$. The value of $P^{(r)}{}_n(x_k)$ gives the approximate value of $f^{(r)}(x)$ at the nodal point x_k.

4.2.2 Non-uniform Nodal Points :

If x_0, x_1,.......,x_n be n+1 value assumed by x and the corresponding functional values be f_0, f_1, f_2,.......,f_n. Then we can able to find Lagrange interpolating polynomial differentiate the polynomial r times we get r^{th} derivative of f(x).

4.2.3 Uniform Nodal Points :

When the distinct points x_0, x_1, x_2,......,x_n are equispaced with step length h, we have, $x_i = x_0+ih$, i=1,2,.....,n and $f_i=f(x_i)$

Using linear interpolation, we have

$$f^1(x_0) \cong P^1{}_1(x_0) = \frac{f_1 - f_0}{h}$$

Similarly, using quadratic interpolation we have

$$f^{-1}(x_0) \cong P^1{}_2(x_0) = \frac{-3f_0 + 4f_1 - f_2}{2h}$$

$$\text{and } f''(x_0) \cong P''_2(x_0) = \frac{f_0 - 2f_1 + f_2}{h^2}$$

Definition D. 4.1 :

A numerical differentiation method is said to be of order p if

$$|f^{(r)}(x) - P^{(r)}(x)| \le ch^p \qquad \text{-------}(4.1)$$

where c is a constant independent of h.

Example E. 4.1 :

Using the following data find f'(6) and f''(6.3)

x	:	6.0	6.1	6.2	6.3	6.4
f(x)	:	0.1750	–0.1998	–0.2223	–0.2422	–0.2596

Solution :

$$\text{We know that } f'(6) = \frac{f_1 - f_0}{h}$$

$$= \frac{-0.1998 - 0.1750}{0.1} \quad (\because h = 6.1 - 6.0 - 0.1)$$

$$= -3.748$$

$$\text{and } f''(6.3) = \frac{f_0 - 2h + f_2}{h^2}$$

$$= \frac{f(6.2) - 2f(6.3) + f(6.4)}{h^2}$$

$$= \frac{-0.2223 - 2(-0.2422) + (-0.2596)}{(0.1)^2}$$

$$= 0.25$$

4.3 Methods Based on Finite Differences

$$\text{We know that } Ef(x) = f(x+h)$$

$$= f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \ldots \text{ (by Taylor's expansion)}$$

where $h = x - x_0$

$$= \left(1 + \frac{hD}{1!} + \frac{h^2}{2!} D^2 + \ldots\right) f(x) \quad \text{where } D = \frac{d}{dx}.$$

$$= e^{hD} f(x) \qquad \text{------}(4.2)$$

Here D = d/dx is called differential operator.

Thus (4.2) becomes $E = e^{hD}$

$$\therefore\ hD = \log E$$

$$= \begin{cases} \log(1+\Delta) \\ -\log(1-\nabla) \\ 2\sinh^{-1}\left(\dfrac{\delta}{2}\right) \end{cases}$$

$$= \begin{cases} \Delta - \dfrac{1}{2}\Delta^2 + \dfrac{1}{3}\Delta^3 - \ldots\ldots \\ \nabla + \dfrac{1}{2}\nabla^2 + \dfrac{1}{3}\nabla^3 + \ldots\ldots \\ \delta - \dfrac{1^2}{2^2.3^2}\delta^3 + \ldots\ldots\ldots\ldots \end{cases} \quad \text{------(4.2)}$$

$$\therefore\ hf'(x_k) = hDf(x_k)$$

$$= \begin{cases} \Delta f_k - \dfrac{1}{2}\Delta^2 f_k + \dfrac{1}{3}\Delta^3 f_k - \ldots\ldots \\ \nabla f_k + \dfrac{1}{2}\nabla^2 f_k + \dfrac{1}{3}\nabla^3 f_k + \ldots\ldots \\ \delta f_k - \dfrac{1^2}{2^2.3^2}\delta^3 f_k + \ldots\ldots\ldots\ldots \end{cases} \quad \text{------(4.3)}$$

Similarly to find r^{th} derivative.

$$h^r D^r = \begin{cases} \Delta^r - \dfrac{1}{2}r\Delta^{r+1} + \dfrac{r(3r+5)}{24}\Delta^{r+2} - \ldots\ldots \\ \nabla^r + \dfrac{1}{2}r\nabla^{r+1} + \dfrac{r(3r+5)}{.24}\nabla^{r+2} + \ldots\ldots \\ \mu\delta^2 - \dfrac{r+3}{24}\mu\delta^{r+2} + \dfrac{5r^2+52r+135}{5760}\mu\delta^{r+4} - \ldots\ldots \text{ if } r \text{ is odd---(4.4)} \\ \delta^2 - \dfrac{r}{24}\mu\delta^{r+2} + \dfrac{r(5r+22)}{5760}\delta^{r+4} \ldots\ldots\ldots \text{if } r \text{ is even} \end{cases}$$

where $m = \sqrt{1+\dfrac{\delta^2}{4}}$

Thus we have,

$$hf'(x_k) = \begin{cases} \Delta f_k - \frac{1}{2}\Delta^2 f_k + \frac{1}{3}\Delta^3 f_k - \ldots.. \\ \nabla f_k + \frac{1}{2}\nabla^2 f_k + \frac{1}{3}\nabla^3 f_k + \ldots. \\ \mu\delta f_k - \frac{1}{6}\mu\delta^3 f_k + \frac{1}{30}\mu\delta^5 f_n - \ldots. \end{cases} \quad \text{------(4.5)}$$

and

$$h^2 f''(x_k) = \begin{cases} \Delta^2 f_k - \Delta^3 f_k + \frac{11}{12}\Delta^4 f_k - \ldots\ldots. \\ \nabla^2 f_k + \nabla^3 f_k + \frac{11}{12}\nabla^4 f_k + \ldots\ldots. \\ \delta^2 f_k - \frac{1}{12}\delta^4 f_k + \frac{1}{90}\delta^6 f_k - \ldots\ldots.. \end{cases} \quad \text{------(4.6)}$$

4.4 Partial Differentiation

Let the values of the function f(x, y) be given at a set of points (x_i, y_j) in the (x, y) plane with spacing h and k in x and y directions respectively.

$\therefore\ x_i = x_0 + ih,\ y_j = y_0 + jk,\ i, j = 1, 2, 3, \ldots\ldots.$

We shall define the partial derivatives of f(x, y) at (x_i, y_j) are as follows.

By assuming $\quad f_{i,j} = f(x_i, y_j)$,

$$\left(\frac{\partial f}{\partial x}\right)_{(x_i, y_j)} = \begin{cases} \dfrac{f_{i+1,j} - f_{i,j}}{h} + 0(h) \\ \dfrac{f_{i,j} - f_{i-1,j}}{h} + 0(h) \\ \dfrac{f_{i+1,j} - f_{i-1,j}}{2h} + 0(h^2) \end{cases} \quad \text{------(4.7)}$$

$$\left(\frac{\partial f}{\partial x}\right)_{(x_i, y_j)} = \begin{cases} \dfrac{f_{i,j+1} - f_{i,j}}{k} + 0(k) \\ \dfrac{f_{i,j} - f_{i,j-1}}{k} + 0(h) \\ \dfrac{f_{i,j+1} - f_{i,j-1}}{2k} + 0(k^2) \end{cases} \quad \text{------(4.8)}$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(x_i,y_j)} = \frac{f_{i-1},y-2f_{i,j}+f_{i+1},j}{h^2}+0(h^2) \quad \text{------(4.9)}$$

$$\left(\frac{\partial^2 f}{\partial y^2}\right)_{(x_i,y_j)} = \frac{f_{i,j-1}-2f_{i,j}+f_{i,j+1}}{k^2}+0(k^2) \quad \text{------(4.10)}$$

and $$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(x_i,y_j)} = \frac{f_{i+1,j+1}-f_{i-1,j+1}-f_{i+1,j-1}+f_{i-1,j-1}}{4hk}+0(h^2+k^2) \quad \text{--(4.11)}$$

Example E. 4.2 :

Find the Jacobian matrix for the system of equations $f_1(x, y) = x^2 + y^2 - x = 0$, $f^2(x, y)=x^2-y^2-y = 0$ at the point (x, y).

Solution :

Given that $f_1(x, y) = x^2+y^2-x = 0$

$f_2(x, y) = x^2-y^2-y = 0$ & $h = k = 1$.

The Jacobian matrix is $J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$

Now $$\left(\frac{\partial f_1}{\partial x}\right)_{(1,1)} = \frac{f_1(1+h,1)-f_1(1,1)}{2h}$$

$$= \frac{f_1(2,1)-f(1,1)}{2}$$

$$= 1$$

$$\left(\frac{\partial f_1}{\partial y}\right)_{(1,1)} = \frac{f_1(1,1+k)-f_1(1,1)}{2k}$$

$$= \frac{f_1(1,2)-f_1(1,1)}{2}$$

$$= 2$$

$$\left(\frac{\partial f_2}{\partial x}\right)_{(1,1)} = \frac{f_2(1+h,1)-f_2(1,1)}{2k}$$

$$= \frac{f_2(2,1)-f_2(1,1)}{2} = 2$$

$$\text{and} \quad \left(\frac{\partial f_2}{\partial y}\right)_{(1,1)} = \frac{f_2(1,1+k)-f_2(1,1)}{2k}$$

$$= \frac{f_2(1,2)-f(1,1)}{2} = -3$$

$$\therefore\ J = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$$

4.5 Numerical Integration :

In the numerical integration, we find an approximate value of the integral

$$I = \int_a^b W(x)f(x)dx \qquad \text{------(4.12)}$$

where $W(x)>0$ called weight function in [a, b]. Here we assume that both $W(x)$ & $W(x)f(x)$ are integrable in Riemann sense on [a, b]. The integral I in (4.12) can be approximated by a finite linear combination of values of $f(x)$ in the form

$$I = \int_a^b W(x)f(x)dx \cong \sum_{k=0}^{n}\lambda_k f_k \qquad \text{------(4.13)}$$

where x_k, $k = 0,1,2,\ldots\ldots.n$ are called abscissas or nodes with

$$a = x_0<x_1<x_2<\ldots\ldots<x_n = b$$

and λ_k, $k = 0,1,2,3,\ldots\ldots,n$ are called the weights of the integration rule and the quadrature formula.

The error of approximation is

$$R_n = \int_a^b W(x)f(x)dx - \sum_{k=0}^{n}\lambda_k f_k \qquad \text{------(4.14)}$$

Definition D. 4.2 :

An integration method of the form (4.1) is said to be of order p, if it produces exact results for all polynomials of degree less than or equal to p.

4.6 Methods Based on Interpolation :

Given n+1 abscissae $x_0, x_1,\ldots\ldots\ldots,x_n$ and the corresponding values $f_0, f_1, f_2,\ldots\ldots,f_n$.

Lagrange interpolating polynomial fitting these points (x_i, f_i); $i = 0,1,2,\ldots\ldots,n$ is

$$f(x) = \sum_{k=0}^{n} l_k(x_k)f_k + \frac{\pi(x)}{(n+1)!}f^{n+1}(\xi), \quad x_0 < \xi < x_n$$

$$\text{where } l_k(x_k) = \frac{\pi(x)}{(x-x_k)\pi^1(x_k)} \text{ and}$$

$$\pi(x) = (x-x_0)(x-x_1)......(x-x_n)$$

$$\text{From (4.15), } I = \int_a^b W(x).f(x)dx$$

$$= \sum_{k=0}^{n}\left[\int_a^b W(x).l_k(x)dx\right]f_k + \int_a^b W(x).\frac{\pi(x)}{(n+1)!}f^{(n+1)}(\xi)dx$$

$$= \sum_{k=0}^{n}\lambda_k f_k + R_n \quad \text{------(4.16)}$$

$$\text{where } \lambda_k = \int_a^b W(x).l_k(x)dx \quad \text{------(4.17)}$$

$$R_n = \frac{1}{(n+1)!}\int_a^b W(x).\pi(x).f^{(n+1)}(\xi)dx \quad \text{------(4.18)}$$

Note :

We know that R_n from (4.18) as $\int_a^b W(x)\frac{\pi(x)}{(n+1)!}f^{(n+1)}(\xi)dx$. This can be written as

$$R_n = \frac{f^{(n+1)}(n)}{(n+1)!}\int_a^b W(x).\pi(x)dx \text{ where } \eta\in(a,b) \text{------(4.19)}$$

If $\pi(x)$ chages sign in [a, b], then

$$|R_n| \le \frac{1}{(n+1)!}\int_a^b W(x).|\pi(x)|f^{(n+1)}(\xi)dx$$

$$\le \frac{M_{n+1}}{(n+1)!}\int_a^b W(x)|\pi(x)|dx \quad \text{------(4.20)}$$

where $\left|f^{(n+1)}(x)\right| \le M_{n+1},\ x\in[a, b]$

When the polynomial is of degree $\le$ n then

$R_n = 0$ if $f(x) = x^i$; $i = 0(1)$ and

$R_n \ne 0$ if $f(x) = x^{i+1}$

$\therefore$ The error term is $R_n = \frac{c}{(n+1)!} f^{(n+1)}(\eta)$

$$\text{where } c = \int_a^b W(x)x^{n+1}dx - \sum_{k=0}^{n} \lambda_k x_k^{n+1} \qquad \text{------(4.21)}$$

Here c is called error constant.

If c=0 for $f(x) = x^{n+1}$, then take the next higher degree polynomial.

By omitting error term in (4.16), we get

$$\int_a^b W(x)f(x)dx = \sum_{k=0}^{n} \lambda_k f_k \qquad \text{------(4.22)}$$

4.7 Newton-cotes Method :

If $W(x) = 1$ and $x_i = x_0 + ih$ where $h = \frac{b-a}{h}$ then (4.22) is called Newton-cotes integration method and λ_k's are called cotes numbers.

$$\text{Let } x = x_0 + sh$$

$$\therefore x - x_i = (x_0+sh)-(x_0+ih)$$

$$= (s-i)h.$$

$$\text{Then } \pi(x) = (x-x_0)(x-x_1)\ldots\ldots\ldots(x-x_n)$$

$$= sh.(s-1)h(s-2)h\ldots\ldots.(s-n)h$$

$$= h^{n+1}s(s-1)(s-2)h\ldots\ldots.(s-n)h$$

$$\text{and } l_k(x) = \frac{(-1)^{n-k}}{k!(n-k)!} s(s-1)\ldots.(s-k+1)(s-k-1)\ldots.(s-n)$$

$$\text{and } \lambda_k = \frac{(-1)^{n-k}}{k!(n\text{-}k)!} \int_a^n s(s-1)\ldots(s-k+1)(s-k-1)\ldots(s-n)ds \quad \text{---(4.23)}$$

$$R_n = \frac{h^{n+2}}{(n+1)!} \int_0^n s(s-1)\ldots.(s-n)f^{(n+1)}(\xi)ds \qquad \text{------(4.24)}$$

4.8 Trapezoidal Rule :

For $n = 1$, then $x_0=a$, $x_1=b$ and $h=b-a$

$$\therefore (4.23) \text{ becomes } \lambda_0 = -h\int_0^1 (s-1)ds$$

$$= -h\left[\frac{s^2}{2} - s\right]_0^1$$

$$= \frac{h}{2}$$

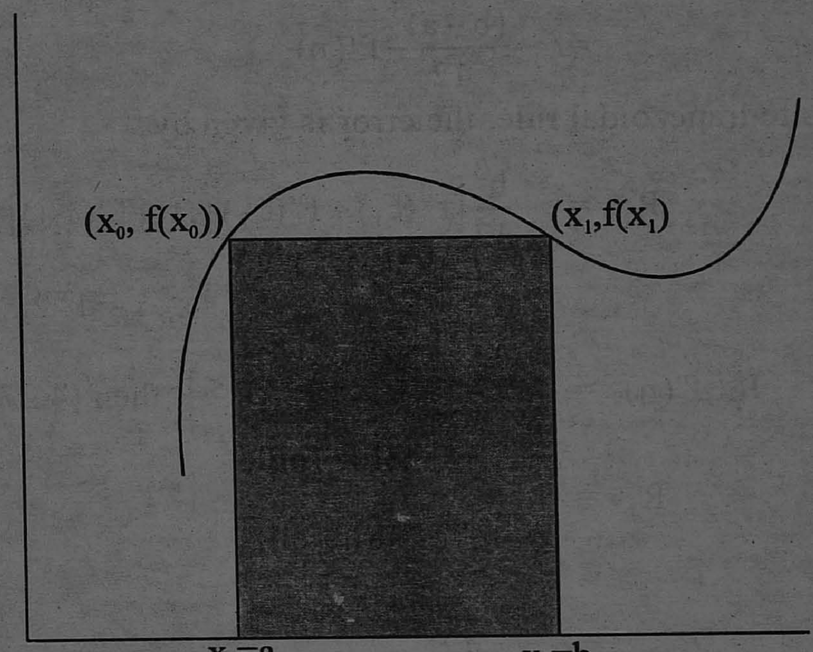
$$\text{and } \lambda_1 = h\int_0^1 s\,ds = \frac{h}{2}$$

$$\therefore \int_a^b f(x)dx = \lambda_0 f_0 + \lambda_1 f_1 = \frac{h}{2}f(x_0) + \frac{h}{2}f(x_1)$$

$$= \frac{h}{2}(f(a) + f(b)) \qquad \text{------(4.25)}$$

Here (4.22) is called **trapezoidal rule.**

Geometrically, (4.22) is the area of the trapezoid with width b–a and ordinates f(a)&f(b).



4.8.1 Composite Trapezoidal Rule :

When we divide [a,b] into N subintervals, each of length $h = \frac{6-a}{N}$ Denote these subintervals as $(x_0, x_1), (x_1, x_2)........., (x_{N-1}, x_N)$.

with x_0=a, x_n=b and $x_i=x_0+ih$; i=1(1)N–1, & fi=f(x_i), i=0(1)N.

$$\therefore \int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \ldots\ldots + \int_{x_{n-1}}^{x_N} f(x)dx$$

$$= \frac{h}{2}(f_0+f_1)+\frac{h}{2}(f_1+f_2)+\ldots.+\frac{h}{2}(f_{N-1}+f_N)$$

$$= \frac{h}{2}\left[(f_0+f_n)+2(f_1+f_2+\ldots..+f_{N-1})\right] \qquad \text{-------(4.25)}$$

This (4.25) formula is called the **composite trapezoidal rule.**

The error in the trapezoidal rule is given by

$$R_1 = \frac{h^3}{2}\int_0^1 s(s-1)f''(\xi)ds$$

Since s(s–1) does not change sign in [0, 1], we get

$$R_1 = \frac{h^3}{2}f''(\eta)\int_0^1 s(s-1)ds,\ \eta \in (0,1)$$

$$= -\frac{h^3}{12}f''(\eta)$$

$$= -\frac{(b-a)^3}{12}f''(\eta) \qquad \text{------(4.26)}$$

For composite trapezoidal rule, the error is given by

$$R_1 = -\frac{h^3}{12}\left[f''(\xi_1)+f''(\xi_2)+\ldots.+f''(\xi_N)\right] \qquad \text{------(4.27)}$$

where $x_{i-1}<\xi_i<x_i$, i = 1, 2,.......,N

If $f''(\eta) = \max_{a\le x\le b}|f''(x)|,\ a<\eta<b$ then (4.27) becomes

$$R_1 = \frac{-h^3N}{12}f''(\eta)$$

$$= \frac{-(b-a)^2N}{12}f''(\eta) \qquad \text{------(4.28)}$$

4.9 Simpson's Rule :

For n=2 then $h = \frac{b-a}{2}$, $x_0=a$, $x_1=\frac{a+b}{2}$, $x_2=b$. Therefore (4.23) becomes

$$\lambda_0 = \frac{h}{2}\int_0^2 (s-1)(s-2)ds$$

$$= \frac{h}{2}\int_0^2 (s^2-3s+2)ds = \frac{h}{3}$$

$$\lambda_1 = -h\int_0^2 s(s-2)ds = 4\frac{h}{3}$$

$$\text{and } \lambda_2 = \frac{h}{2}\int_0^2 s(s-1)ds = \frac{h}{3}$$

$$\therefore \; I = \int_a^b f(x)dx = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

$$= \frac{h}{3}[f(x_0)+4f(x_1)+f(x_2)] \qquad \text{------(4.29)}$$

The formula (4.29) is called **Simpon's 1/3 rule** and the associated error is

$$R_2 = \frac{h^4}{3!}\int_0^2 s(s-1)(s-2).f'''(\xi)d\xi \quad \text{(use (4.24))}$$

Again using (4.21), we have,

$$c = \int_a^b x^3 dx - \frac{h}{3}(f(x_0)+4f(x_1)+f(x_2))$$

$$= \left(\frac{x^4}{4}\right)_a^b - \frac{h}{3}\left(f(a)+af\left(\frac{a+b}{2}\right)+f(b)\right)$$

$$= \frac{b^4-a^4}{4} - \frac{b-a}{6}\left(a^3+4.\left(\frac{a+b}{2}\right)^2+b^3\right) \left\{\because f(x)=x^3\right\}$$

$$= \frac{b^4-a^4}{4} - \frac{b-a}{6}\left[a^3+\frac{(a+b)^3}{2}+b^3\right]$$

$$= \frac{b^4-a^4}{4} - \frac{b-a}{6}\left[a^3+b^3+\frac{a^3+3a^2b+3ab^2+b^3}{2}\right]$$

$$= \frac{b^4-a^4}{4} - \frac{b-a}{6}\left[\frac{2a^3+2b^3+a^3+3a^2b+3ab^2+b^3}{2}\right]$$

$$= \frac{b^4-a^4}{4} - \frac{b-a}{4}\left(a^3+a^2b+ab^2+b^3\right)$$

$$= \frac{b^4-a^4}{4} - \frac{b^4-a^4}{4}$$

$$= 0$$

(i.e.) I is exact for polynomial of degree 3.

$\therefore$ The error term is $R_2 = \frac{c}{4!} f^{(iv)}(\eta), \eta \in (0,2)$

$$\text{where} \qquad c = \int_a^b x^4 dx - \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_1)]$$

$$= \left(\frac{x^5}{5}\right)_a^b - \frac{(b-a)}{6}\left(a^4 + 4\left(\frac{a+b}{2}\right)^4 + b^4\right)$$

$$= \frac{b^5 - a^5}{5} - \frac{(b-a)}{6}\left(a^4 + \frac{(a+b)^4}{4} + b^4\right)$$

$$= -\frac{(b-a)^5}{120}$$

Thus
$$R_2 = \frac{-(b-a)^5}{2880} f^{(iv)}(\eta), \quad \eta \in (0,2) \qquad \text{------(4.30)}$$

4.9.1 Composite Simpson's Rule :

If we divide the interval [a, b] into 2N subintervals each of length $h = \frac{b-a}{2N}$, then we get 2N+1 abscissae $x_0 = a, x_1, x_2, \ldots\ldots, x_{2N} = b$ and $x_i = x_0 + ih$, $i = 1(1)2N-1$.

$$\therefore \quad I = \int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots + \int_{x_{2N-2}}^{x_{2N}} f(x)dx$$

$$= \frac{h}{3}[f_0 + 4f_1 + f_2] + \frac{h}{3}[f_2 + 4f_5 + f_5] + \ldots..$$

$$+ \frac{h}{3}[f_{2N-2} + 4f_{2N-1} + f_{2N}]$$

$$= \frac{h}{3}\{(f_0 + f_{2N}) + 2(f_2 + f_4 + \ldots.. + f_{2N-2})$$

$$+4(f_1 + f_3 + \ldots\ldots + f_{2N-1})\} \qquad \text{------(4.31)}$$

Here (4.31) is called **composite Simpson's rule** and the error is given by

$$R_2 = \frac{-(b-a)}{180} h^4 f^{(iv)}(h), \quad h \in (a,b)$$

Note :

In Simpson's 1/3 rule, we divide the interval [a, b] in 2N subintervals (ie) even number of subintervals.

4.10 Simpson's 3/8 rule :

For n = 3, then $h = \dfrac{b-a}{3}$

$x_0 = a$, $x_1 = \dfrac{2a+b}{3}$, $x_2 = \dfrac{a+2b}{3}$, $x_3 = b$.

$$\text{Then } \lambda_0 = \frac{-h}{6}\int_0^3 (s-1)(s-2)(s-3)ds$$

$$= \frac{3h}{8},$$

$$\lambda_1 = \frac{h}{2}\int_0^3 s(s-2)(s-3)ds$$

$$= \frac{9h}{8},$$

$$\lambda_2 = -\frac{h}{2}\int_0^3 s(s-1)(s-3)ds$$

$$= -\frac{9h}{8}$$

$$\text{and } \lambda_3 = \frac{h}{6}\int_0^3 s(s-1)(s-2)ds$$

$$= \frac{3h}{8}$$

$$\text{Thus } I = \int_a^b f(x)dx = \sum_{k=0}^{3} \lambda_k f_k$$

$$= \lambda_0 f_0 + \lambda\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$$

$$= \frac{3}{8}hf_0 + \frac{9}{8}hf_1 + \frac{9}{8}hf_2 + \frac{3}{8}f_3$$

$$= \frac{3}{8}[f_0 + 3f_1 + 3f_2 + f_3]$$

(i.e.,)
$$I = \frac{3h}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] \quad \text{---(4.32)}$$

This formula (4.32) is called **Simpson's** 3/8 rule and the associated error is

$$R_3 = \frac{h^5}{24}\int_0^3 s(s-1)(s-2)(s-3)f^{(iv)}(\xi)d\xi$$

4.10.1 Composite Simpson's 3/8 rule :

If we divide the interval [a, b] into 3N subintervals eachof length $h=\frac{b-a}{3N}$, then we get 3N+1 abscissae $x_0, x_1, x_2, \ldots\ldots, x_{3N}$ and $xi = x_0+ih$, $i = 1,2,\ldots,3N-1$.

$$\therefore \ I = \int_a^b f(x)dx = \int_{x_0}^{x_3} f(x)dx + \int_{x_3}^{x_6} f(x)x + \ldots\ldots + \int_{x_{3N-3}}^{x_{3N}} f(x)dx$$

$$= \frac{3h}{8}[f_0 + 3f_1 + 3f_2 + f_3] + \frac{3h}{8}[f_3 + 3f_4 + 3f_5 + f_6]$$

$$+ \ldots + \frac{3h}{8}[f_{3N-3} + 3f_{3N-2} + 3f_{3N-1} + f_3N]$$

$$= \frac{3h}{8}[(f_0 + f_{3N}) + 2(f_3 + f_6 + \ldots + f_{3N-3})$$

$$+3(f_1 + f_2 + f_4 + f_5 + \ldots + f_{3N-2} + f_{3N-1})] \quad \text{----(4.33)}$$

This formula (4.33) is called **composite Simpson's 3/8 rule.**

Example E. 4.3 :

Evaluate the integral $I = \int_0^1 \frac{dx}{1+x}$ using (i) composite trapezoidel rule and (ii) composite Simpson's rule with 2, 4 & 8 equal subintervals.

Solution :

When N=2 then $h = \frac{b-a}{N} = \frac{1-0}{2} = \frac{1}{2}$ and the three nodes are 0, ½, 1.

$\therefore$ using trapezoidal rule,

$$I = \int_0^1 f(x)dx \text{ where } f(x) = \frac{1}{1+x}$$

$$= \frac{h}{2}[f_0 + 2f_1 + f_2]$$

$$= \frac{1}{4}[f(0) + 2f(½) + f(1)]$$

$$= \frac{1}{4}\left[\frac{1}{1+0} + 2.\frac{1}{1+½} + \frac{1}{1+1}\right]$$

$$= \frac{17}{24}$$

$$= 0.708333$$

Similarly using Simpson's 1/3 rule, we have

$$I = \int_0^1 f(x)dx$$

$$= \frac{h}{3}[f_0 + 4f_1 + f_2]$$

$$= \frac{1}{6}\left[\frac{1}{1} + 4.\frac{1}{1+½} + \frac{1}{1+1}\right]$$

$$= \frac{25}{36}$$

$$= 0.694444$$

When N = 4, then $h = \frac{1-0}{N} = \frac{1}{4}$

∴ The nodes are 0, ¼, ½, ¾ & 1.

∴ Using trapezoidal rule,

$$I = \int_0^1 f(x)dx$$

$$= \frac{h}{2}[(f_0 + f_1) + 2(f_1 + f_2 + f_3)]$$

$$= \frac{1}{8}\left[(f(0) + f(1)) + 2\left(f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)\right)\right]$$

$$= \frac{1}{8}\left[\left(\frac{1}{1} + \frac{1}{1+1}\right) + 2\left(\frac{1}{1+¼} + \frac{1}{1+½} + \frac{1}{1+¾}\right)\right]$$

$$= 0.697024$$

Using Simpson's 1/3 rule,

$$I = \int_0^1 f(x)dx$$

$$= \frac{h}{3}[(f_0 + f_{2N}) + 2(f_2 + f_4 + \ldots.. + f_{2N-2})$$

$$+4(f_1 + f_3 + f_5 + \ldots\ldots + f_{2N-1})]$$

$$= \frac{1}{12}[(f(x_0) + f(1)) + 27(½) + 4(f(¼) + f(¾)]$$

$$= \frac{1}{12}\left[\left(\frac{1}{1+0}+\frac{1}{1+1}\right)+2\frac{1}{1+½}+4\left(\frac{1}{1+¼}+\frac{1}{1+¾}\right)\right]$$

$$= 0.693254$$

Similarly proceding above when N=8, using trapezoidal & Simpson's 1/3 rule we get the values 0.694122 & 0.603155 respectively.

4.11 Methods Based on Undetermind Coefficients :

In (4.22), the nodes x_k's and weights λ_k's k=0(1)n can be obtained by making the formula exact for polynomials of degree upto n.

If m=n then the nodes are known and the method is called Newton-cotes method. When the nodes are also to be determined and m=2n+1 and the method is called Gaussian integration method.

We know that [−1, 1] and [a, b] are homeomorphic to each other and therefore the integral (4.22) can be written as

$$\int_{-1}^{1} W(x)f(x)dx = \sum_{k=0}^{n} \lambda_k f_k \qquad \text{------(4.34)}$$

4.11.1 Gauss-Legendre Integration Method :

Take W(x) = 1 in (4.34), we get,

$$\int_{-1}^{1} f(x)dx = \sum_{k=0}^{n} \lambda_k f_k \qquad \text{------(4.35)}$$

In (4.35), all the nodes and weights are unknown. This method is called **Gauss-Legendre integration** method.

For n=2, $$\int_{-1}^{1} f(x)dx = \lambda_0 f_0+\lambda_1 f_1+\lambda_2 f_2 \qquad \text{------(4.36)}$$

Clearly in (4.36), three nodes x_0, x_1, x_2 and the three weights λ_0, λ_1, λ_2 are to be determined. Hence (4.36) can be made exact for polynomials of degree upto 5.

For $f(x)=x^i$, i = 0(1)5, we get,

$$\int_{-1}^{1} 1dx = \lambda_0 + \lambda_1 + \lambda_2 \; (\because\; f(x) = 1)$$

$$\text{(i.e.,) } \lambda_0 + \lambda_1 + \lambda_2 = 2 \quad \text{------(1)}$$

$$\text{if } f(x) = x \text{ then } \lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 = 0 \quad \text{------(2)}$$

$$\text{Similarly we have, } \lambda_0 x^2{}_0 + \lambda_1 x^2{}_1 + \lambda_2 x^3{}_2 = 2/3 \quad \text{------(3)}$$

$$\lambda_0 x^3{}_0 + \lambda_1 x^3{}_1 + \lambda_3 x^3{}_2 = 0 \quad \text{------(4)}$$

$$\lambda_0 x^4{}_0 + \lambda_1 x^4{}_1 + \lambda_3 x^4{}_2 = 2/5 \quad \text{------(5)}$$

$$\lambda_0 x^5{}_0 + \lambda_1 x^5{}_1 + \lambda_2 x^5{}_2 = 0 \quad \text{------(6)}$$

Solving the above equations from (1) to (6), we get,

$$x0 = -\sqrt{\frac{3}{5}},\ x_1 = 0,\ x_2 = \sqrt{\frac{3}{5}}$$

$$\text{and } \lambda_0 = \frac{5}{9},\ \lambda_1 = \frac{8}{9} \text{ and } \lambda_2 = \frac{5}{9}$$

Hence (4.36) becomes,

$$\int_{-1}^{1} f(x)dx = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right] \quad \text{------(4.37)}$$

The error term associated with this method is given by

$$R_5 = \frac{c}{6!} f^{(6)}(\xi),\ -1 < \xi < 1 \quad \text{------(4.38)}$$

$$\text{where } c = \int_{-1}^{1} x^6 dx - \left(\lambda_0 x_0{}^6 + \lambda_1 x_1{}^6 + x_1{}^6 + \lambda_2 x_2{}^6\right)$$

$$= \frac{2}{7} - \frac{6}{25}$$

$$= \frac{8}{175}$$

Note : The nodes x_k's are the roots of the Legendre polynomial $P_{n+1}(x)=0$ where

$$P_{n+1}(x) = \frac{1}{2^{n+1}(n+1)!} \frac{d^{n+1}}{dx^{n+1}}\left[\left(x^2-1\right)^{n+1}\right],\ n = 0,1,2,\ldots\ldots \quad \text{----(4.39)}$$

Example E. 4.4 :

Find the value of the integral $I = \int_2^3 \frac{\cos 2x}{1+\sin x} dx$ using Gauss-Legendre two and three point integration.

Solution :

First we transform the interval [2, 3] to the interval [−1, 1] using the transform $x=\frac{b-a}{2}t+\frac{b+a}{2}$.

$$\text{(ie)} \quad x = \frac{t}{2}+\frac{5}{2}$$

$$x = \frac{t+5}{2}$$

$$\therefore \quad dx = \frac{dt}{2}$$

$$\text{Thus } I = \int_{t=-1}^{1} \frac{\cos(t+5)}{1+\sin\left(\frac{t+5}{2}\right)}\frac{dt}{2}$$

Using the Gauss-Legendre two point formula

$$I = \frac{1}{2}\int_{-1}^{1} f(x)dx = \frac{1}{2}\left[f\left(\frac{1}{\sqrt{3}}\right)+f\left(\frac{-1}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{2}(0.56558356-0.15856672)$$

$$= 0.20350842$$

and using the Gauss-Legendre three point formula,

$$I = \frac{1}{2}\int_{-1}^{1} f(x)dx$$

$$= \frac{1}{2}\cdot\frac{1}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right)+8f(0)+5f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{18}[-1.26018515+1.41966661+3.48936886]$$

$$= 0.20271391$$

Example E. 4.5 :

$$\text{Evaluate} \quad I = \int_{0}^{1}\frac{dx}{1+x}$$

Using Gauss-Legendre three point formula.

Solution :

We transform [0, 1] to [−1, 1] using $x = \frac{1}{2}t + \frac{1}{2} = \frac{t+1}{2}$

$$\therefore \ dx = \frac{dt}{2}$$

$$\therefore \ I = \int_{-1}^{1} \frac{1}{1+\frac{t+1}{2}} \frac{dt}{2}$$

$$= \frac{1}{2}.2 \int_{-1}^{1} \frac{dt}{t+3}$$

$$= \int_{-1}^{1} \frac{dt}{t+3}$$

$$= \int_{-1}^{1} g(t)dt \ \text{ where } \ g(t) = \frac{1}{t+3}$$

Using Gauss Legendre three point formula,

$$I = \int_{0}^{1} \frac{dx}{1+x}$$

$$= \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= 0.693122$$

4.11.2 Lobatto Itegration Method :

In (4.34) take W(x) = 1 and two end points −1 and 1 are taken as node points. The remaining n−1 nodes are to be determined.

∴ (4.34) becomes,

$$\int_{-1}^{1} f(x)dx = \lambda_0 f(-1) + \lambda_n f(1) + \sum_{k=1}^{n-1} \lambda_k f_k \qquad \text{------(4.40)}$$

and it is called **Lobatta integration** method.

Since there are n-1 nodes and n+1 weights & hence totally 2n unknowns to be findout and this method can be made exact for polynomial of degree upto 2n−1.

For n=2 then (4.40) becomes

$$\int_{-1}^{1} f(x)dx = \lambda_0 f(-1) + \lambda_1 f(x_1) + \lambda_2 f(x_2) \qquad \text{-------(4.41)}$$

Since (4.41) can be made exact for polynomials of degree upto three,

For $f(x) = x^i$, $i = 0,1,2,3$, then we get,

$$\lambda_0+\lambda_1+\lambda_2 = 2$$
$$-\lambda_0+\lambda_1 x_1+\lambda_2 = 0$$
$$\lambda_0+\lambda_1 x_1^2+\lambda_2 = 2/3$$
$$-\lambda_0+\lambda_1 x_1^3+\lambda_2 = 0$$

Solving the above system of equations, we get,

$x_1=0$, $\lambda_0=\lambda_2=1/3$, $\lambda_1=4/3$.

$\therefore$ (4.41) changes as

$$\int_{-1}^{1} f(x)dx = \frac{1}{3}[f(-1)+4f(0)+f(1)] \qquad \text{-----(4.42)}$$

with the error term $R_3 = \frac{c}{4!} f^{(iv)}(\xi), \ -1<\xi<1$

$$\text{where } c = \int_{-1}^{1} x^4 dx - \left(\lambda_0+\lambda_1 x_1^4+\lambda_2\right)$$
$$= \frac{-4}{15}.$$

4.11.3 Radau Integration Method :

In (4.34), take W(x)=1 & the lower limit -1 is fixed as one node. The remaining n nodes to be determined & the integration

$$\int_{-1}^{1} f(x)dx = \lambda_0 f(-1) + \sum_{k=1}^{n} \lambda_k f_k \qquad \text{------(4.43)}$$

is called **Radau integration** method.

For n=2, we have,

$$\int_{-1}^{1} f(x)dx = \frac{2}{9} f(-1) + \frac{16+\sqrt{6}}{18} f\left(\frac{1-\sqrt{6}}{5}\right) + \frac{16-\sqrt{6}}{18} f\left(\frac{1+\sqrt{6}}{5}\right) \text{--(4.44)}$$

and the error term

$$R_4 = \frac{c}{5!} f^{(v)}(\xi), \quad -1<\xi<1 \text{ where } c = \frac{-37}{225}$$

4.11.4 Gauss-Chebyshev Integration Method :

In (4.34), take $W(x)=\frac{1}{\sqrt{1-x^2}}$ then

$$I = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)dx \;=\; \sum_{k=0}^{n} \lambda_k f_k \qquad \text{------(4.45)}$$

are called **Gauss-chebyshev integration** method.

This method is exact for polynomial of degree upto 2n+1. The nodes x_k's are found to be the root of the chabyshev polynomials.

$$T_{n+1}(x) \;=\; \cos((n+1)\cos^{-1}x) = 0$$

$$\text{(i.e.,)}\; x_k \;=\; \cos\left(\frac{2(k+1)\pi}{2n+1}\right),\; k=0(1)n$$

when n=2 then

$$\int_{-1}^{-1} \frac{f(x)dx}{\sqrt{1-x^2}} \;=\; \frac{\pi}{3}\left[f\left(\frac{\sqrt{3}}{2}\right)+f(0)+f\left(\frac{-\sqrt{3}}{2}\right)\right] \qquad \text{------(4.46)}$$

with the error term $R_5 \;=\; \frac{c}{\angle 6} f^{(vi)}(\xi),\; -1<\xi<1$

where $c \;=\; \frac{\pi}{32}$.

Note : In the above method all weights are equal & $\lambda_k = \frac{\pi}{n+1}$, k=0(1)n.

Example E. 4.6 :

Evaluate the integral $\int_{-1}^{1} \left(1-x^2\right)^{3/2} \cos x dx$ using

(i) Gauss-Legendre three point formula &

(ii) Gauss-Chebyshev three point formula.

Solution :

Using Gauss-Legendre three pt formula is

$$\int_{-1}^{1} f(x)dx \;=\; \frac{1}{9}\left[5f\left(\sqrt{\frac{3}{5}}\right)+8f(0)+5f\left(-\sqrt{\frac{3}{5}}\right)\right]$$

where $f(x) = (1-x^2)^{3/2} \cos x.$

$$\therefore I = \int_{-1}^{1} \left(1-x^2\right)^{3/2} \cos x dx$$

$$= \frac{2}{9}\left[\sqrt{\frac{2}{5}} \cos\left(\sqrt{\frac{3}{5}}\right) + 4 + \sqrt{\frac{2}{5}} \cos\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= 1.08979$$

Using-Chebyshev three point formula,

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{3}\left[f\left(\frac{\sqrt{3}}{2}\right) + f(0) + f\left(\frac{-\sqrt{3}}{2}\right)\right]$$

where $f(x) = (1-x^2)^2 \cos x.$

$$\therefore I = \frac{\pi}{3}\left[\frac{1}{16}\cos\left(\frac{\sqrt{3}}{2}\right) + 1 + \frac{1}{16}\cos\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= 1.132$$

4.11.5 Gauss-Laguerre Integration Methods :

Consider the integral of the form

$$\int_{0}^{\infty} e^{-x} f(x) dx = \sum_{n=0}^{n} \lambda_k f_k \qquad \text{------(4.47)}$$

The nodes x_k's are obtained from the roots of the Laguerre polynomial

$$L_{n+1}(x) = (-1)^{n+1} e^x \frac{d^{n+1}}{dx^{n+1}}\left(e^{-x} x^{n+1}\right)$$

We have $L_0(x) = 1,$

$L_1(x) = x-1,$

$L_2(x) = x^2-4x+2,$

$L_3(x) = x^3-9x^2+18x-6$

The corresponding weight are obtained from the relation

$$\lambda_k = \int_{0}^{\infty} \frac{e^{-x} L_{n+1}(x)}{(x-x_k) L'_{n+1}(x_k)} dx \qquad \text{------(4.48)}$$

This method produces exact results for polynomial of degree upto 2n+1.

4.11.6 Gauss-Hermite Integration Methods :

The method of the form

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)dx = \sum_{k=0}^{n} \lambda_k x_k \quad \text{------(4.49)}$$

are called **Gauss-Hermite integration** methods.

The nodes x_k's are the roots of the hermite polynomial

$$H_{n+1}(x) = (-1)^{n+1} e^{x^2} \frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right)$$

We have, $H_0(x) = 1$,

$$H_1(x) = 2x,$$

$$H_2(x) = 2(2x^2-1),$$

$$H_3(x) = 4(2x^3-3x)$$

The corresponding weights are obtained from

$$\lambda_k = \int_{-\infty}^{\infty} \frac{e^{-x^2}.H_{n+1}(x)dx}{(x-x_k)H^1_{n+1}(x)} \quad \text{-------(4.50)}$$

4.12 Romberg Integration :

Let I_T & I_S represent the values obtained by usings the trapezoidal rule & simpson's rule respectively.

$$\text{Consider } I = \int_a^b f(x)dx$$

The error in the composite trapezoidal rule & composite simpson's rule can be obtained as

$$I = I_T + c_1h^2 + c_2h^4 + c_3h^6 + \ldots\ldots \quad \text{------(4.51)}$$

$$\text{and } I = I_S + d_1h^4 + d_2h^6 + d_3h^8 + \ldots\ldots\ldots \quad \text{------(4.52)}$$

where c's and d's are constant independent of h.

∴ The Romberg Integration formula is given by

$$I_T^{(m)}(h) = \frac{4^m I_T^{(m-1)}(h/2) - I_T^{(m-1)}(h)}{4^m - 1}, \; m=1,2,3,\ldots \quad \text{----(4.53)}$$

$$\text{and } I_S^{(m)}(h) = \frac{4^{m+1} I_S^{(m-1)}.(h/2) - I_S^{(m-1)}(h)}{4^{m+1} - 1}, \; m=1,2,3,\ldots \quad \text{----(4.54)}$$

Example E. 4.7 :

Find the approximate value of the inteval $\int_0^1 \frac{dx}{1+x}$ using (i) composite trapezoidal rule with 2, 3, 5, 9 nodes and Romberg integration and (ii) composite simpson's rule with 3, 5, 9 nodes and Romberg integration.

Solution :

Using composite trapezoidal rule,

$$I = \int_a^b f(x)dx = \int_0^1 f(x)dx \text{ where } f(x) = \frac{1}{1+x}$$

$$= \frac{h}{2}[(f_0 + f_N) + 2(f_1 + f_2 + \ldots\ldots + f_{N-1})]$$

$$\text{when } h = \frac{b-a}{N} = \frac{1-0}{N} = \frac{1}{N}$$

$$\text{If } N = 1 \text{ then } h = 1 \text{ and } I_T = \frac{1}{2}[f_0 + f_1]$$

$$= \frac{1}{2}\left[\frac{1}{1+0} + \frac{1}{1+1}\right]$$

$$= \frac{1}{2}[1.5]$$

$$= 0.75$$

$$\text{If } N=2 \text{ then } h=1/3 \text{ and } I_T = \frac{1}{6}[(f_0 + f_2) + 2f_1]$$

$$= 0.708333$$

$$\text{If } N=4 \text{ then } h = 1/4 \text{ and } I_T = \frac{1}{8}[(f_0 + f_4) + 2(f_1 + f_2 + f_3)]$$

$$= 0.697024$$

$$\text{If } N=8 \text{ then } h=1/8 \text{ and } I_T = \frac{1}{16}[(f_0 + f_8) + 2(f_1 + \ldots\ldots + f_7)]$$

$$= 0.694122$$

We shall tabulate the values of I using Romberg integration.

h	Second Order Method	IV Order Method	VI Order Method	VIII Order Method
1	0.75			
		0.694444		
			0.693175	
1/2	0.708333			0.693148
		0.693254		
			0.693148	
1/4	0.697024			
		0.693155		
1/8	0.694122			

We know that using Romberg integration,

$$I_T^{(m)}(h) = \frac{4^m I_T^{(m-1)}\left(\frac{h}{2}\right) - I_T^{(m-1)}(h)}{4^m - 1}, m = 1,2,3,\ldots\ldots$$

Step 1 :

when m = 1

$$I_T^{(1)}(h) = \frac{4I_T^{(0)}\left(\frac{h}{2}\right) - I_T^{(m-1)}(h)}{4-1}$$

If h = 1 then $I_T^{(1)}(1) = \dfrac{4I_T^{(0)} - I_T^{(0)}(1)}{3}$

$$= \frac{4(0.708333) - 0.75}{3}$$

$$= 0.694444$$

If h = 1/2, then $I_T^{(1)}(½) = \dfrac{4(0.697024) - 0.708333}{3}$

$$= 0.693254$$

If h=1/4 then $I_T^{(1)}(¼) = \dfrac{4(0.694122) - 0.697024}{3}$

$$= 0.693155$$

Step 2 :

$$\text{when } m=2 \text{ then } I_T^{(2)}(h) = \frac{16I^{(1)}\left(\frac{h}{2}\right) - I_T^{(1)}(h)}{16-1}$$

$$\text{If } h=1 \text{ then } I_T^{(2)}(1) = \frac{16I^{(1)}(1/2) - I_T^{(1)}(1)}{15}$$

$$\text{(i.e.) } I_T^{(2)}(1) = \frac{16(0.693254) - 0.694444}{15}$$

$$= 0.693175$$

$$\text{If } h=1/2 \text{ then } I_T^{(2)}(1/2) = \frac{16(0.693155) - 0.693254}{15}$$

$$= 0.693148$$

Step 3 :

$$\text{when } m=3,\ I_T^{(3)}(h) = \frac{64I_T^{(2)}\left(\frac{h}{2}\right) - I_T^{(2)}(h)}{63}$$

$$\text{If } h=1,\ I_T^{(3)}(1) = \frac{64(0.693148) - 0.693175}{63}$$

$$= 0.693148$$

Thus from the table it is clear that $\int_0^1 \frac{dx}{1+x} = 0.693148$ using Romberg integration with the help of trapezoidal rule.

Using Composite Simpson's Rule :

$$\text{Here } I = \int_0^1 f(x)dx \text{ where } f(x) = \frac{1}{1+x}$$

$$= \frac{h}{3}[(f_0 + f_{2N}) + 4(f_1 + f_3 + \ldots + f_{2N-1}) + 2(f_2 + f_4 + \ldots + f_{2N-2})]$$

$$\text{and here } h = \frac{1-0}{2N}$$

If N=1, then h = 1/2

$$\text{and } I_S = \frac{h}{3}[(f_0 + f_2) + 4f_1]$$

$$= \frac{1/2}{3}\left[(f(0)+f(1))+4f(1/2)\right]$$

$$= \frac{1}{6}\left[\frac{1}{1}+\frac{1}{2}+4.\frac{1}{1+½}\right]$$

$$= \frac{25}{36}$$

$$= 0.694444$$

If N=2, then h=¼

$$\therefore\ I_S = \frac{h}{3}\left[(f_0+f_4)+4(f_1+f_3)+2f_2\right]$$

$$= \frac{h}{3}\left[(f(0)+f(1))+4\left(f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)\right)+2.f\left(\frac{1}{2}\right)\right]$$

$$= \frac{1}{12}\left[\left(\frac{1}{1}+\frac{1}{2}\right)+4\left(\frac{1}{1+¼}+\frac{1}{1+¾}\right)+2\frac{1}{1+½}\right]$$

$$= \frac{1}{12}\left[\frac{3}{2}+4\left(\frac{4}{5}+\frac{4}{7}\right)+\frac{4}{3}\right]$$

$$= \frac{1747}{2520}$$

$$= 0.693254$$

If N=3 then h=1/8, and $I_S = \frac{h}{3}\left[(f_0+f_8)+4(f_1+f_3+f_5+f_7)+2(f_2+f_4+f_8)\right]$

$$= 0.693155$$

We shal tabulate the values as follows :

h	IV Order Method	VI Order Method	VIII Order Method
1/2	0.694444		
		0.693175	
1/4	0.693254		0.693148
		0.693148	
1/8	0.693155		

We know that $I_S^{(m)} = \frac{4^{m+1} I_S^{(m-1)}\left(\frac{h}{2}\right) - I_S^{(m-1)}(h)}{4^{m+1}-1}$, m=1,2,3,.....

Step 1 :

when m = 1.

$$\therefore \quad I_S^{(1)}(h) = \frac{4^2 I_S^{(0)}\left(\frac{h}{2}\right) - I_S^{(0)}(h)}{4^2 - 1}$$

$$= \frac{16 I_S^{0}\left(\frac{h}{2}\right) - I_S^{(0)}(h)}{15}$$

$$\text{If } h=1/2 \text{ then } \quad I_S^{(1)}(1/2) = \frac{16(0.693254) - 0.694444}{15}$$

$$= 0.693175$$

$$\text{If } h=1/4 \text{ then } \quad I_S^{(1)}(1/4) = \frac{16(0.693155) - 0.693254}{15}$$

$$= 0.693148$$

Step 2 :

$$\text{when } m=2, \quad I_S^{(2)}(h) = \frac{64 I_S^{(1)}\left(\frac{h}{2}\right) - I_S^{(1)}(h)}{63}$$

$$\text{If } h=1/2 \text{ then } \quad I_S^{(2)}(1/2) = \frac{64(0.693148) - 0.693175}{63}$$

$$= 0.693148$$

$$\text{Thus required } \ I = \int_0^1 \frac{dx}{1+x}$$

$$= 0.693148.$$

UNIT – 5

ORDINARY DIFFERENTIAL EQUATIONS

5.1 Initial Value Problem :

An **ordinary differential equation** is a relation betweeen a function, its derivatives and the variable upon which they depend.

The general form of ordinary differential equation is

$$\varphi(t, y, y', y'', \ldots, y^{(m)}) = 0 \qquad \text{------(5.1)}$$

where m is the highest order derivative, and y and its derivatives are functions of t.

Note :

1) The **order** of the differential equation is the order of its highest derivative and its **degree** is the degree of the derivative of the highest order after the equations has been rationalized.

2) A **linear differential equation** of order m is

$$\sum_{P=0}^{m} \varphi_P(t) y^{(P)}(t) = r(t) \qquad \text{-------(5.2)}$$

where $\varphi_P(t)$ are known functions.

3) The general **non-linear differential equation** of order m is

$$y^{(m)} = F(t, y, y', y'', \ldots\ldots, y^{(m)}) \qquad \text{-------(5.3)}$$

4) (5.3) is called **canonical representation** of (5.1).

5.1.1 Initial Value Problem

A general solution of an ordinary differential equation (6.1) is a relation between y, t and m arbitrary constants which satisfies the equation, but which contains no derivatives. The solutions may be an implicit form

$$w(t, y, C_1, C_2, \ldots\ldots, C_m) = 0 \qquad \text{-------(5.4)}$$

or an explicit form $y = w(t, C_2, C_2, \ldots\ldots, C_m)$ --------(5.5)

The m arbitrary constants in (5.4) or in (5.5) can be determined by prescribing m conditions of the form $y^{(v)}=\eta_v$, $v = 0, 1, 2, \ldots\ldots, m-1$ --------(5.6)

at one point $t=t_0$, which are called **initial conditions**. The differential equation (5.1) together with the initial conditions (5.6) is called an **m^{th} order initial value problem.**

5.2 Boundary Value Problem :

If the m conditions are prescribed at more than one point to determine the m arbitrary constants in the general solution (5.4), these are called **boundary conditions.**

The differential equation (5.1) together with the boundary conditions is known as the **boundary value problem.**

5.3 Reduction of Higher Order Equations to the system of First Order Differential Equations :

The m^{th} order differential equation (5.3) with initial conditions (5.6) may be written as an equivalent system of m first order initial value problem.

$$\begin{aligned}
u_1 &= y \\
u'_1 &= u_2 \\
u'_2 &= u_3 \\
&\bullet \\
&\bullet \\
&\bullet \\
u'_{m-1} &= u_m \\
u'_m &= g(t, u_1, u_2, \ldots\ldots, u_m) \\
u_1(t_0) &= \eta_0,\ u_2(t_0) = \eta_1, \ldots\ldots, u_m(t_0) = \eta_{m-1}
\end{aligned}$$

The above equations be written in a vector notation as

$$\left.\begin{aligned}
\mathbf{u}' &= \mathbf{f}(t, \mathbf{u}), \\
\mathbf{u}(t_0) &= \eta
\end{aligned}\right\} \qquad \text{--------(5.7)}$$

$$\begin{aligned}
\text{where } \mathbf{u} &= [u_1, u_2, \ldots\ldots, u_m]^T \\
\mathbf{f} &= [u_2, u_3, \ldots\ldots, g]^T \\
\text{and } \mathbf{h} &= [\eta_0, \eta_1, \ldots\ldots, \eta_{m-1}]^T
\end{aligned}$$

5.3.1 First Order Initial Value Problem :

The first order initial value problem is

$$\frac{du}{dt} = f(t, u),\ u(t_0) = \eta \qquad \text{--------(5.8)}$$

5.3.2 Existence and Uniqueness :

The existence and uniqueness theorem of the solution of the initial value problem (5.8) is guaranteed by the following result.

Result :

We assume that f(t, u) satisfies the following conditions.

(i) f(t, u) is a real function.

(ii) f(t, u) is defined and continuous in the strip $t \in [t_0, b]$, $u \in (-\infty, \infty)$

(iii) there exists a constant K such that for any $t \in [t_0, b]$ and for any u_1 and u_2, $|f(t, u_1) - f(t, u_2)| \leq K|u_1 - u_2|$ where K is called the **Lipschitz constant**. Then for any u_0 the initial value problem (5.8) has a unique solution u(t) for $t \in [t_0, b]$.

Important Note :

1) In our future discussions, we assume that the existence and uniqueness of the solution and also that f(t, u) has continuous partial derivatives with respect to t and u of as high order as we desire.

2) Partition of an interval $[t_0, b]$ we mean that $t_0 < t_1 < t_2 < \ldots\ldots < t_N = b$.

   Here the points $t_0, t_1, \ldots\ldots, t_N$ are called the **mesh points or the grid points.**

   The spacing between the points is

   $$h_j = t_j - t_{j-1},\ j = 1, 2, \ldots\ldots, N$$

   which is called **mesh spacing or step length.**

   If $h_j = h$ = constant for $j = 1, 2, \ldots\ldots, N$ then $t_j = t_0 + jh$, $j = 0, 1, 2, \ldots\ldots, N$

3) The set of numbers $\{u_j\}$ (ie) $u_0, u_1, \ldots, u_N$ is the **numerical solution** of the initial value problem. The numbers $\{u_j\}$ are determined from a set of algebraic equations called the difference equations.

Using Taylor's series, u(t) can be expanded in the interval $t-h \le t \le t+h$ as

$$\frac{u(t+t)-u(t)}{h} = u'(t)+\frac{h}{2}u''(t)+o(h^2)$$

$$\text{(ie)} \quad \frac{\Delta u(t)}{h} = \frac{du}{dt}+o(h) \qquad \text{-------(5.9)}$$

Similarly, we get,

$$\frac{\nabla u(t)}{h} = \frac{du}{dt}+o(h) \qquad \text{-------(5.10)}$$

$$\text{and} \quad \frac{\mu\delta u(t)}{h} = \frac{du}{dt}+o(h^2) \qquad \text{-------(5.11)}$$

By neglecting error terms, we have the difference approximation to u'(t) at $t = t_j$ as

$$u'(t_j) = \begin{cases} \dfrac{u_{j+1}-u_j}{h} \rightarrow \text{(i)} \\ \dfrac{u_j-u_{j-1}}{h} \rightarrow \text{(ii)} \\ \dfrac{u_{j+1}-u_{j-1}}{h} \rightarrow \text{(iii)} \end{cases} \qquad \text{-------(5.12)}$$

By using these approximation to (5.8) at the mesh point t_j, we have

$$\left.\begin{aligned} &\text{(a)} \quad \frac{u_{j+1}-u_j}{h} = f(t_j,u_j) \\ &\text{(b)} \quad \frac{u_j-u_{j-1}}{h} = f(t_j,u_j) \\ &\text{(c)} \quad \frac{u_{j+1}-u_{j-1}}{h} = f(t_j,u_j) \end{aligned}\right\} \qquad \text{-------(5.13)}$$

Here (5.13) is called **difference equations.**

The methods (5.13(a)) and (5.13(b)) are of first order or **single step method** and (5.13(c)) is of second order difference equations (or) **two-step or multi-step methods.**

5.4 Euler Method :

We write (5.13(a)) as $u_{j+1} = u_j+hf_j$ where $f_j = f(t_j, u_j)$.

This is called the **Euler** or the first order **Adams–Bashforth method.**

Example E.5.1 :

Use the Euler method to solve the initial value problem $u' = -2tu^2$, $u(0) = 1$ with h=0.2; 0.1 and 0.05, on the interval [0, 1].

Solution :

Given that $u' = -2tu^2$, $u(0) = 1$.

(ie) $u' = f(t, u)$, $u(t_0) = 1$ where $f(t, u) = -2tu^2$ and $t_0 = 0$.

Using Euler formula, we have,

$u_{j+1} = u_j + hf_j$ where $f_j = f(t_j, u_j)$, j = 0,1,2,3,4 and h = 0.2.

(Here $t_0=0$, $t_1=0.2$, $t_2=0.4$, $t_3 = 0.6$, $t_4 = 0.8$ & $t_5 = 1$).

$$\text{(ie) } u_{j+1} = u_j - 0.2(2t_j u_j^2)$$

$$\text{(ie) } u_{j+1} = u_j - 0.4t_j.u_j^2, \quad j=0,1,2,3,4 \quad \text{--------(5.14)}$$

when j=0 then (5.14) $\Rightarrow u_1 = u_0 - 0.4t_0.u_0^2$

$$\text{(ie) } u_1 = 1-0.4(0)(1)^2 = 1$$

$$\text{(ie) } u_1 \cong 1$$

When j = 1, then $u_2 = u_1 - 0.4t_1.u_1^2$

$$\text{(ie) } u_2 = 1-0.4(0.2).(1)^2 = 0.92$$

$$\text{(ie) } u(0.4) \cong 0.92$$

when j=2, then $u_3 = u_2 - 0.6t_2.u_2^2$

$$\text{(ie) } u3 = 0.92-0.4(0.4)(0.92)^2 = 0.78458$$

$$\text{(ie) } u(0.6) \cong 0.78458$$

when j=3, then $u_4 = u_3 - 0.4t_3u_3^2$

$$\text{(ie) } u_4 = 0.78458-0.4(0.6)(0.78458)^2 = 0.63684$$

$$\text{(ie) } u(0.8) \cong 0.63684$$

when j=4, then $u_5 = u_4 - 0.4t_4u_4^2$

$$\text{(ie) } u_5 = 0.63684 - 0.4(0.8)(0.63684)^2 = 0.50706$$

similarly when h=0.1, then we have

$u(0) = 1$	$u(0.6) \cong 0.75715$
$u(0.1) \cong 1$	$u(0.7) \cong 0.68835$
$u(0.2) \cong 0.98$	$u(0.8) \cong 0.62202$
$u(0.3) \cong 0.94158$	$u(0.9) \cong 0.56011$
$u(0.4) \cong 0.88839$	$u(1) \cong 0.50364$
$u(0.5) \cong 0.82525$	

Try final u values when h = 0.05.

5.5 Backward Euler Method :

From (5.13(b)) at the mesh point $t=t_{j+1}$, we have, $u_{j+1} = u_j+hf_{j+1}$ ------(5.15)

$$\text{where } f_{j+1} = f(t_{j+1}, u_{j+1})$$

This is called **Backward Euler** or the first order **Adams–Moulton Method.**

Example E.5.2 :

Solve the initial value problem $u' = -2tu^2$, $u(0) = 1$ with h=0.2 on the interval [0,1], using backward Euler method.

Solution :

Given that $u'(t) = -2tu^2$, $u(0) = 1$ and $h = 0.2$.

Here $t_0 = 0$, $t_1 = 0.2$, $t_2 = 0.4$, $t_3 = 0.6$, $t_4 = 0.8$, $t_5 = 1$.

The backward Euler method gives,

$$u_{j+1} = u_j-2ht_{j+1}\, u^2_{j+1}, \quad j = 0,1,2,3,4.$$

$$\text{(ie) } 2ht_{j+1}\, u^2_{j+1}+u_{j+1}-u_j = 0$$

$$\therefore\ u_{j+1} = \frac{-1\pm\sqrt{1+8hu_jt_{j+1}}}{4ht_{j+1}}$$

By considering positive solution, we have,

$$u_{j+1} = \frac{-1+\sqrt{1+8hu_jt_{j+1}}}{4ht_{j+1}}, \quad j=0,1,2,3,4.$$

For $j = 0$, $u_0 = 1$, $t_1 = 0.2$.

$$\therefore\ u(0.2) \cong u_1 = \frac{-1+\sqrt{1+8hu_0t_1}}{4ht_1}$$

$$= \frac{-1+\sqrt{1+8(0.2)(1)(0.2)}}{4(0.2)(0.2)}$$

$$= 0.9307033$$

For $j = 1$, $u_1 = 0.9307033$, $t_2 = 0.4$

$$\therefore\ u(0.4) \cong u_2 = \frac{-1+\sqrt{1+8ht_2u_1}}{4ht_2}$$

$$= \sqrt{\frac{-1+\sqrt{1+8(0.2)(0.4)(0.9307033)}}{4(0.2)(0.4)}}$$

$$= 0.8224701$$

For $j = 2$, then $u_2 = 0.8224701$, $t_3 = 0.6$

$$\therefore\ u(0.6) \cong u_3 = \frac{-1+\sqrt{1+8ht_3u_2}}{8ht_3}$$

$$= \frac{-1+\sqrt{1+8(0.2)(0.6)(0.8224701)}}{8(0.2)(0.6)}$$

$$= 0.7036429$$

For $j = 3$, then $u_3 = 0.7036429$, $t_4 = 0.8$

$$\therefore\ u(0.8) \cong u4 = \frac{-1+\sqrt{1+8ht_4u_3}}{8ht_4}$$

$$= \frac{-1+\sqrt{1+8(0.2)(0.8)(0.7036429)}}{8(0.2)(0.8)}$$

$$= 0.4940135$$

$$\begin{aligned} \text{Thus } u(0) &\cong 1 \\ u(0.2) &\cong 0.9307033 \\ u(0.4) &\cong 0.8224701 \\ u(0.6) &\cong 0.7036429 \\ u(0.8) &\cong 0.5916333 \\ u(1) &\cong 0.4940135 \end{aligned}$$

5.6 Mid-point Method :

Equation (5.13(c)) may be written as

$$u_{j+1} = u_{j-1}+2hf_j \qquad \text{------(5.16)}$$

The equation (5.16) is called **mid-point or the second order Nystrom method.**

The solution values of (5.16) are given by

$$\left.\begin{aligned} u_2 &= u_0+2hf_1 \\ u_3 &= u_1+2hf_2 \\ &\bullet \\ &\bullet \\ &\bullet \\ u_N &= u_{N-2}+2hf_{N-1} \end{aligned}\right\} \qquad \text{------(5.17)}$$

where u_0 be known from the initial conditions. The value u_1 is unknown and must be determined by some other method after which (5.17) is used to determine successively $u_2, u_3, \ldots, u_N$.

Example E.5.3 :

Solve the initial value problem $u' = -2tu^2$, $u(0)=1$ using mid-point method with $h=0.2$ over the interval $[0,1]$.

Solution :

Given that $u' = -2tu^2$, $u(0) = 1$ with $h = 0.2$ we know the midpoint method formula is

$$u_{j+1} = u_{j-1}-4ht_j\, u^2_j, \qquad j = 1,2,3,4 \qquad \text{------(5.18)}$$

where $u_0 = 1$, $t_0 = 0$, $t_1 = 0.2$, $t_2 = 0.4$, $t_3 = 0.6$, $t_4 = 0.8$ and $t_5 = 1$.

We shall calculate u_1 from the exact solution.

$$u(t) = \frac{1}{1+t^2}$$

$$\therefore\ u_1 = u(0.2) = \frac{1}{1+0.4}$$

$$= 0.9615385$$

put $j = 1$ in (5.18), we have,

$$u_2 = u_0 - 4ht_1u^2_1$$

(ie) $u(0.4) = 1-4(0.2)(0.2)(0.9615385)$

$= 0.8520710$

Similarly, put j=2 in (5.18), we have,

$$u_3 = u_1 - 4ht_2u^2{}_2$$

$$\text{(ie) } u(0.6) = 0.9615385-4(0.2)(0.4)(0.8520710)2$$

$$= 0.7292105$$

Again put j = 3 in (5.18), we have

$$u_4 = u_2 - 4ht_3u^2{}_3$$

$$\text{(ie) } u(0.8) = 0.8520710-4(0.2)(0.6).(0.7292105)^2$$

$$= 0.5968320$$

and put j = 4 in (5.18), we get,

$$u_5 = u_3 - 4ht_4u^2{}_4$$

$$\text{(ie) } u(1) = 0.7292105-4(0.2)(0.8)(0.596832)^2$$

$$= 0.5012371$$

5.7 Single Step Method :

A single step method for the solution of the initial value problem $\frac{du}{dt}=f(t, u)$, $u(t_0)=\eta$ is related to the first order difference equation.

$\therefore$ a general single method may be witten as

$$u_{j+1} = uj+h\varphi(t_j, u_j, h) \qquad \text{---------(5.19)}$$

where $\varphi(t, u, h)$ is a function of the t, h, u and in addition depends on f.

5.8 Taylor Series Method :

Assume that the function u(t) can be expended in a Taylor series about any point t_j.

$$\text{(ie) } u(t) = u(t_j)+(t-t_j)u'(t_j)+\frac{1}{2!}(t-t_j)^2u''(t_j)+...$$

$$+\frac{1}{p!}(1-t_j)^p u^{(p)}(t_j)+\frac{1}{(p+1)!}(t-t_j)^{p+1}u^{(p+1)}(t_j+\theta h) \qquad \text{----(5.20)}$$

Now the expansion (5.20) holds good for $t\in(t_0, b)$ and $0<\theta<1$.

Put $t = t_{j+1}$ in (5.20), we get

$$u_{j+1} = u(t_j) + hu'(t_j) + \frac{h^2}{2!}u''(t_j) + ... + \frac{1}{p!}h^p u^{(p)}(t_j)$$

$$+ \frac{1}{(p+1)!}h^{p+1}u^{(p+1)}(t_j + \theta h) \quad (\text{since } h = t_{j+1} - t_j)$$

Consider $h\varphi(t_j, u(t_j), h) = hu'(t_j) + \frac{h^2}{2!}u''(t_j) + ... + \frac{h^p}{p!}u^{(p)}(t_j)$

and $h\varphi(t_j, u_j, h)$ is to be obtained from $\varphi(t_j, u(t_j), h)$ by using approximate value u_j in place of the exact value $u(t_j)$.

$\therefore$ We have, $u_{j+1} = u_j + h\varphi(t_j, u_j, h)$ ------(5.22)

$j = 0,1,2,.....,N-1$

which approximate $u(t_{j+1})$.

The equation (5.22) is called Taylor series method of order p.

Note : Put p=1 in (5.22), we get, $u_{j+1} = u_j + hu'_j$ which is the Euler Method.

Example E.5.4 :

The following initial value problem is given $y' = 2x+3y$, $y(0) = 1$

(i) If the error in y(x) obtained from the first four terms of the Taylor series is to be less than 5×10^{-5} after rounding, find x.

(ii) Use Taylor series second order method to get y(0.4) with step length h=0.1

Solution :

Given that $y' = 2x+3y$, $y(0) = 1$.

(ie) $\frac{dy}{dx} - 3y = 2x$ -------(5.23)

The analytic solution of (5.23) is

$$y(x) = \frac{11}{9}e^{3x} - \frac{2}{9}(3x+1)$$

$$\therefore y'(x) = \frac{11}{3}e^{3x} - \frac{2}{3}$$

$$y''(x) = 11e^{3x},$$

$$y'''(x) = 33e^{3x}$$

$$y^{iv}(x) = 99e^{3x} \text{ and so on.}$$

when $x=0$ then $y(0) = 1,$

$$y'(0) = 3,$$

$$y''(0) = 11$$

$$y'''(0) = 33$$

$$y^{iv}(0) = 99$$

$\therefore$ The Taylor series method with the first four terms becomes

$$y(x) = 1+3x+\frac{11}{2}x^2+\frac{11}{2}x^3$$

The remainder term is $R_4 = \dfrac{x^4}{4!}y^{(4)}(\xi)$

Now $|R4| < 5 \times 10^{-5}$

$$\Rightarrow \left(\frac{x^4}{24}99e^{3x}\right) < 5\times10^{-5}$$

$$\Rightarrow x^4e^{3x} < 0.00001212$$

$$\Rightarrow x<0.056.$$

(ii) We know that the second order Taylor series method is

$$y_{1+1} = y_n + hy'_n + \frac{h^2}{2}y''_n, \qquad n=0,1,2,3$$

Now $y'_n = 2x_n+3y_n$

$$y''_n = 2+3y'_n$$

$$= 2+3(2x_n+3y_n)$$

$$= 2+6x_n+9y_n$$

with $h = 0.1$, the solution procedure be given below :

For $n = 0$, $y_0 = 1$,

$$y''_0 = 2\times0+3y_0$$

$$= 3,$$

$$y''_0 = 2+6\times0+9\times1 = 1$$

$$\therefore\ y'_1 = 1+0.1(3)+\frac{(0.1)^2}{2}\to 11$$

$$= 1.355$$

$$\text{For } n = 2,\ y'_2 = 2\times(0.2)+3(1.855475)$$

$$= 5.966425$$

$$y''_2 = 2+6\times(0.2)+9(1.855475)$$

$$= 19.899275$$

$$y_3 = y_2+hy'_2+\frac{h^2}{2}y''_2$$

$$= 1.855475+0.1(5.966425)+\frac{(0.1)^2}{2}(19.899275)$$

$$= 2.5516138$$

$$\text{For } n = 3,\ y'_3 = 2\times0.3+3(2.5516138)$$

$$= 8.8348414$$

$$y''_3 = 2+6(0.3)+9(2.5516138)$$

$$= 26.764524$$

$$\therefore\ y_4 = 2.5516138+0.1(8.2548414)$$

Thus the solution values are,

$$y(0.1) \cong 1.355,$$

$$y(0.2) \cong 1.85548$$

$$y(0.3) \cong 2.55161$$

$$y(0.4) \cong 3.51092$$

Example E.5.5 :

Compute an approximation to y(1), y'(1) and y''(1) with Taylor's algorithm of order two and steplength h=1 when y(x) is the solution to the initial value problem.

$$y'''+2y''+y'-y = \cos x,\ 0\le x\le 1,\ y(0) = 0,\ y'(0) = 1,\ y''(0) = 2.$$

Solution :

The Taylor series method of order two may be written as

$$\left.\begin{aligned} y(x_0+h) &= y(x_0)+hy'(x_0)+\frac{h^2}{2}y''(x_0) \\ y'(x_0+h) &= y'(x_0)+hy''(x_0)+\frac{h^2}{2}y'''(x_0) \\ y''(x_0+h) &= y''(x_0)+hy'''(x_0)+\frac{h^2}{2}y^{(4)}(x_0) \end{aligned}\right\} \quad \text{-------(5.23)}$$

Given that

$$y'''+2y''+y'-y = \cos x \quad \text{-------(5.24)}$$

Differentiate with respect to x, we get,

$$y^{(4)}+2y'''+y''-y' = -\sin x \quad \text{-------(5.25)}$$

$$\begin{aligned} \text{Also given that } y(0) &= 0, \\ y'(0) &= 1, \\ \text{and } y''(0) &= 2 \end{aligned}$$

Put x = 0 in (5.24), we get,

$$\begin{aligned} y'''(0)+2y''(0)+y'(0)-y(0) &= \cos(0) \\ \text{(ie) } y'''(0)+2(2)+1-0 &= 1 \\ \textbf{(ie) } \mathbf{y''(0)} &= \mathbf{-4} \end{aligned}$$

Put x = 0 in (5.25), we get

$$\begin{aligned} y^{(4)}(0)+2y'''(0)+y''(0)-y'(0) &= -\sin(0) \\ \text{(ie) } y(4)(0)+2(-4)+2-1 &= 0 \\ \textbf{(ie) } \mathbf{y^{(4)}(0)} &= \mathbf{7} \end{aligned}$$

Thus for h = 1 and $x_0 = 0$, we obtain from (5.23) as

$$\begin{aligned} y(1) &\cong y(0)+(1)y'(0)+\frac{(1)^2}{2}y''(0) \\ &= 0+1+\frac{1}{2}(2) = 2 \end{aligned}$$

$$\text{and } y'(1) \cong y'(0)+hy''(0)+\frac{h^2}{2}y'''(0)$$

$$= 1+(1)(2)+\frac{(1)^2}{2}(-4)$$

$$= 1$$

$$\text{and } y''(1) \cong y''(0)+(1)y'''(0)+\frac{h^2}{2}y^{(4)}(0)$$

$$= 2+(-4)+\frac{1}{2}(7)$$

$$= \frac{3}{2}$$

5.9 Runge-Kutta Method :

The Runge-Kutta method with v slopes can be written as

$$K_1 = hf(t_j, u_j)$$

$$K_2 = hf(t_j+C_2h, u_j+a_{21}K_1)$$

$$K_3 = hf(t_j+C_3h, u_j+a_{31}K_1+a_{32}K_2)$$

$$K_4 = hf(t_j+C_4h, u_j+a_{41}K_1+a_{42}K_2+a_{43}K_3)$$

$$\bullet$$

$$\bullet$$

$$\bullet$$

$$K_v = hf\left(t_j+C_vh, u_j+\sum_{i=1}^{C-1} a_{vi}K_i\right)$$

$$\text{and } u_{j+1} = u_j+W_1K_1+W_2K_2+....+W_vK_v \quad \text{-------(5.26)}$$

Here (5.26) is called the explicit **Runge-Kutta Method** with v slopes.

To determine the parameters C's, a's and W's in (5.26), expand u_{j+1} in powers of h such that it agrees with the Taylor series expansion of the solution of the differential equation upto a certain number of terms.

5.9.1 Second Order Method :

Consider the Runge-Kutta method with two slopes.

$$\left.\begin{aligned} K_1 &= hf(t_j, u_j) \\ K_2 &= hf(t_2+C_2h, u_j+a_{21}K_1) \\ u_{j+1} &= u_j+W_1K_1+W_2K_2 \end{aligned}\right\} \quad \text{-------(5.27)}$$

where the parameters C_2, a_{21}, W_1 and W_2 are chosen to make u_{j+1} closer to $u(t_{j+1})$.

Now the Taylor's series is

$$u(t_{j+1}) = u(t_j) + hu'(t_j) + \frac{h^2}{2!}u''(t_j) + \frac{h^3}{3!}u'''(t_j) + ...$$

$$= u(t_j) + hf(t_j, u(t_j)) + \frac{h^2}{2!}(f_t + ff_u)t_j$$

$$+ \frac{h^3}{3!}[f_{tt} + 2ff_{tu} + f^2 f_{uu} + f_u(f_t + ff_u)]_{t_j} + ... \quad \text{----(5.28)}$$

We also have,

$$K_1 = hf_j$$

$$K_2 = hf(t_j+C_2h,\ u_j+a_{21}hf_j)$$

$$= h[f_j + h(C_2 f_t + a_{21} ff_u)_{t_j}$$

$$+ \frac{h^2}{2}(C^2 f_{tt} + 2C_2 a_{21} ff_{uu} + a^2{}_{21} f^2 f_{uu})_{t_j} + ...]$$

substituting the values of K_1 and K_2 in (5.27) we get,

$$u_{j+1} = u_j + (W_1 + W_2)hf_j + h^2(W_2 C_2 f_t + W_2 a_{21} ff_u)_{t_j}$$

$$+ \frac{h^3}{2} W_2 (C_2^2 f_{tt} + 2C_2 a_{21} ff_{uu} + a_{21}^2 f^2 f_{uu})_{t_j} + ...$$

Comparing the co-efficients of like powers of h in (5.28) & (5.29), we get,

$$W_1+W_2 = 1 \quad \text{-------(5.30)}$$

$$C_2W_2 = 1/2 \quad \text{-------(5.31)}$$

$$a_{21}W_2 = 1/2 \quad \text{-------(5.32)}$$

$$\left.\begin{aligned} &\text{From (5.31), } W_2 = \frac{1}{2C_2} \\ &\therefore \text{(5.30) gives us } W_1 = 1 - \frac{1}{2C_2} \\ &\text{\& (5.32) gives as } a_{21} = C_2 \end{aligned}\right\} \quad \text{--------(5.33)}$$

Here $C_2 \neq 0$ and is arbitrary.

From (5.29) and (5.33), we have

$$u_{j+1} = u_j + hf_j + \frac{h^2}{2}(f_j + ff_u)_{t_j}$$

$$+\frac{h^3C_2}{4}(f_{tt} + 2ff_{tu} + f^2f_{uu})_{t_j} + \ldots.. \quad \text{-----(5.34)}$$

Note :

1) If $C_2 = 1/2$ then (5.34) gives us

$$u_{j+1} = u_j + hf\left(t_j + \frac{h}{2}, u_j + \frac{h}{2}f_j\right), \quad j=0,1,2,\ldots.,N-1$$

which is the Euler method with spacing h/2.

2) If C=1, then (5.34) changes as

$$u_{j+1} = u_j + \frac{h}{2}[f(t_j, u_j) + f(t_j + h, u_j + hf_j)],$$
$$j=0,1,2,\ldots..,N-1$$

which is trapezoidal rule when f(t, u) is independent of u.

5.9.2 Fourth Order Method :

$$\begin{aligned} \text{Define } K_1 &= hf(t_j, u_j), \\ K_2 &= hf(t_j+C_2h, u_j+a_{21}K_1) \\ K_3 &= hf(t_j+C_3h, u_j+a_{31}K_1+a_{32}K_2) \\ K_4 &= hf(t_j+C_4h, u_j+a_{41}K_1+a_{42}K_2+a_{43}K_3) \\ \text{and } u_{j+1} &= u_j+W_1K_1+W_2K_2+W_3K_3+W_4K_4 \end{aligned} \quad \text{-------(5.36)}$$

where the parameters C_2, C_3, C_4, $a_{21}, \ldots.., a_{43}$ and W_1, W_2, W_3, W_4 are chosen to make u_{j+1} chosen to $u(t_{j+1})$.

Expaining $u(t_{j+1})$ using Tayler's series and the comparing coefficients of powers of h, we obtain the following system of equation.

$$\begin{aligned} C_2 &= a_{21} \\ C_3 &= a_{31}+a_{32} \\ C_4 &= a_{41}+a_{42}+a_{43} \\ W_1+W_2+W_3+W_4 &= 1 \end{aligned}$$

$$W_2C_2+W_3C_3+W_4C_4 = \frac{1}{2}$$

$$W_2C^2{}_2+W_2C^2{}_3+W_4C^2{}_4 = \frac{1}{3}$$

$$W_2C_2a_{32}+W_4(C_2a_{42}+C_3a_{43}) = \frac{1}{6}$$

$$W_2C^3{}_2+W_3C^3{}_3+W_4C^3{}_4 = \frac{1}{4}$$

$$W_3C^2{}_2a_{32}+W_4(C^2{}_2a_{42}+C^2{}_3a_{43}) = \frac{1}{12}$$

$$W_3C_2C_3a_{32}+W_4(C_2a_{42}+C_3a_{43})C_4 = \frac{1}{8}$$

$$W_4C_2a_{32}a_{43} = \frac{1}{24} \qquad \text{--------(5.37)}$$

In (5.37), we have II equations in 13 unknowns.

Solving the equations in (5.37), we get

$$C_2 = C_3 = \frac{1}{2}$$
$$C_4 = 1$$
$$W_2 = W_3 = \frac{1}{6}$$
$$W_1 = W_4 = \frac{1}{6}$$
$$a_{41} = 0$$
$$a_{42} = 0$$
$$a_{43} = 1$$

$\therefore$ The fourth order method (5.36) becomes,

$$K_1 = hf(t_j, u_j)$$
$$K_2 = hf\left(t_j+\frac{1}{2}h, u_j+\frac{1}{2}K_1\right)$$
$$K_3 = hf\left(t_j+\frac{1}{2}h, u_j+\frac{1}{2}K_2\right)$$
$$K_4 = hf(t_j+h, u_j+K_3)$$

$$\text{and } u_{j+1} = u_j+\frac{1}{6}(K_1+2K_2+2K_3+K_4)$$

Example E.5.6 :

Some the initial value problem u' = $-2tu^2$, u(0)=1 with h=0.2 on the interval [0, 1]. Use the fourth order classical Runge-Kutta method.

Solution :

Given that u' = $-2tu^2$, n(0) = 1.

For j = 0, then $t_0 = 0$, $u_0 = 1$.

$$\therefore K_1 = hf(t_0, u_0)$$

$$= -2(0.2)(0)(1)^2$$

$$= 0$$

$$K_2 = hf\left(t_0 + \frac{h}{2}, u_0 + \frac{1}{2}K_1\right)$$

$$= -2(0.2)\left(\frac{0.2}{2}\right)(1)^2$$

$$= -0.04$$

$$K_3 = hf\left(t_0 + \frac{h}{2}, u_0 + \frac{1}{2}K_2\right)$$

$$= -2(0.2)\left(\frac{0.2}{2}\right)(0.98)^2$$

$$= -0.038416$$

$$K_4 = hf(t_0+h, u_0+K_3)$$

$$= -2(0.2)(0.2)(0.961584)^2$$

$$= -0.0739715$$

$$\therefore u(0.2) \cong u_1 = 1+\frac{1}{6}(0-0.08-0.076832-0.0739715)$$

$$= 0.9615328$$

For j = 1, $t_1 = 0.2$, $u_1 = 0.9615328$

$$K_1 = hf(t_1, u_1)$$

$$= -2(0.2)(0.2)(0.9615328)^2$$

$$= -0.0739636$$

$$K_2 = h\left(t_1+\frac{h}{2},\ u_1+\frac{K_1}{2}\right)$$

$$= -2(0.2)(0.3)(0.924551)^2$$

$$= -0.1025754$$

$$K_3 = hf\left(t_1+\frac{h}{2},\ u_1+\frac{K_2}{2}\right)$$

$$= -2(0.2)(0.3)(0.9102451)^2$$

$$= -0.994255$$

$$K_4 = hf(t_1+h_1 u_1+K_3)$$

$$= -2(0.2)(0.4)(0.8621073)2$$

$$= -0.1189166$$

$$\therefore u(0.4) \cong u_2$$

$$= 0.09615328+1/6(-0.0739636-0.2051508$$
$$-0.1988510-0.1189166)$$

$$= 0.8620525$$

Similarly proceeding like above, we get

$$u(0.6) \cong u_3 = 0.7352784$$

$$u(0.8) \cong u_4 = 0.6097519$$

$$u(1) \cong u_5 = 0.5000073$$

Example E.5.37

Use the classicial Runge-Kutta formula of fourth order to find the numerical solution at x=0.8 for $\frac{dy}{dx} = \sqrt{x+y}$, y(0.4)=0.41. Assume the step length h=0.2.

Solution :

Step 1 :

For n = 0 and h = 0.2, we have,

$x_0 = 0.4$, $y_0 = 0.41$

$$\therefore K_1 = hf(x0, y0) \text{ where } f(x, y) = \sqrt{x+y}$$

$$= h\sqrt{x_0+y_0}$$

$$= 0.2\sqrt{0.4+0.41}$$
$$= 0.18$$
$$K_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{1}{2}K_1\right)$$
$$= h\sqrt{\left(x_0+\frac{h}{2}\right)+\left(y_0+\frac{1}{2}K_1\right)}$$
$$= 0.2\sqrt{0.4+0.1+0.41+\frac{1}{2}(0.18)}$$
$$= 0.2$$
$$K_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{1}{2}K_2\right)$$
$$= h\sqrt{x_0+\frac{h}{2}+y_0+\frac{1}{2}K_2}$$
$$= 0.2\sqrt{0.4+0.1+0.4+0.1}$$
$$= 0.2009975$$
$$\text{and } K_4 = hf(x_0+h, y_0+K_3)$$
$$= h\sqrt{x_0+h+y_0+K_3}$$
$$= 0.2\sqrt{0.4+0.2+0.41+0.2009975}$$
$$= 0.2200906$$
$$\text{Thus } y_1 = y_0+\frac{1}{6}(K_1+2K_2+2K_3+K_4)$$
$$= 0.41+1/6[(0.18+2(0.2)+2(0.2009975)+0.2200906]$$
$$= 0.41+0.6103476$$
$$= 0.6103476$$

Step 2 :

For $n = 1$, $x_1 = x_0+h = 0.4+0.2 = 0.6$ and $y_1 = 0.6103476$

$$\therefore K_1 = hf(x_1, y_1)$$
$$= h\sqrt{x_1+y_1}$$
$$= 0.2\sqrt{0.6+0.6103476}$$
$$= 0.2200316$$

$$K_2 = hf\left(x_1+\frac{h}{2}+y_1+\frac{1}{2}K_1\right)$$

$$= h\sqrt{x_1+\frac{h}{2}+y_1+\frac{1}{2}K_1}$$

$$= 0.2.\sqrt{0.6+0.1+0.6103476+\frac{1}{2}(0.2200316)}$$

$$= 0.2383580$$

$$K_3 = hf\left(x_1+\frac{h}{2}, y_1+\frac{1}{2}K_2\right)$$

$$= h\sqrt{x_1+\frac{h}{2}+y_1+\frac{1}{2}K_2}$$

$$= 0.2.\sqrt{0.2+0.1+0.6103476+\frac{1}{2}(0.2383580)}$$

$$= 0.2391256$$

$$\text{and } K_4 = hf(x_1+h, y_1+K_3)$$

$$= h\sqrt{x_1+h+y_1+K_3}$$

$$= 0.2\sqrt{0.6+0.2+0.6103476+0.2391256}$$

$$= 0.2568636$$

$$\text{Thus } y_2 = y_1+\frac{1}{6}(K_1+2K_2+2K_3+K_4)$$

$$= 0.6103476+1/6(0.2200316+2(0.2383580)+2(0.2391256)+0.2568636)$$

$$= 0.8489912$$

$$\text{Hence } y(0.8) \cong y_2 = 0.84899$$

5.9.3 System of Equations :

The fourth order classical Runge-Kutta mehtod for the system of equations

$$\left.\begin{aligned} \frac{du}{dt} &= f(t_1, u_1, u_2, \ldots, u_n) \\ u(t_0) &= \eta \end{aligned}\right\} \qquad \text{-------(5.38)}$$

be written as

$$\mathbf{u}_{j+1} = \mathbf{u}_j + 1/6(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_4)$$

$$\text{where } \mathbf{K}_1 = \begin{bmatrix} K_{11} \\ K_{21} \\ \vdots \\ K_{n1} \end{bmatrix}, \qquad \mathbf{K}_2 = \begin{bmatrix} K_{12} \\ K_{22} \\ \vdots \\ K_{n2} \end{bmatrix}$$

$$\mathbf{K}_3 = \begin{bmatrix} K_{13} \\ K_{23} \\ \vdots \\ K_{n3} \end{bmatrix}, \qquad \mathbf{K}_4 = \begin{bmatrix} K_{14} \\ K_{24} \\ \vdots \\ K_{n4} \end{bmatrix}$$

$$\text{and } K_{i1} = hf_i(t_j, u_{i,j}, u_{2,j}, \ldots\ldots, u_{n,j})$$

$$K_{i2} = hf_i\left(t_j + \frac{h}{2}, u_{1,j} + \frac{1}{2}K_{11}, u_{2,j} + \frac{1}{2}K_{21}, \ldots\ldots, u_{n,j} + \frac{1}{2}K_{n1}\right)$$

$$K_{i3} = hf_i\left(t_j + \frac{h}{2}, u_{1,j} + \frac{1}{2}K_{12}, u_{2,j} + \frac{1}{2}K_{22}, \ldots\ldots, u_{n,j} + \frac{1}{2}K_{n2}\right)$$

$$K_{i4} = hf_i(t_j + h, u_{i,j} + K_{13}, u_{2,j} + K_{23}, \ldots, u_{n,j} + K_{n3}), \quad i = 1,2,3,\ldots,n$$

$\therefore$ The explicit form of (5.38) is

$$\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{n,j+1} \end{bmatrix} = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{n,j} \end{bmatrix} + \frac{1}{6}\left\{\begin{bmatrix} K_{11} \\ K_{21} \\ \vdots \\ K_{n1} \end{bmatrix} + 2\begin{bmatrix} K_{12} \\ K_{22} \\ \vdots \\ K_{n2} \end{bmatrix} + 2\begin{bmatrix} K_{13} \\ K_{23} \\ \vdots \\ K_{n3} \end{bmatrix} + \begin{bmatrix} K_{14} \\ K_{24} \\ \vdots \\ K_{n4} \end{bmatrix}\right\}$$

Example E.5.7 :

Solve the system of equations

$u' = -3u + 2v$, $u(0) = 0$

$v' = 3u - 4v$, $v(0) = 0.5$ for $h = 0.2$ on the interval $[0, 1]$.

Use the Euler-Cauchy method.

Solution :

Step 1 :

For $j = 0$

$t_0 = 0$, $u_0 = 0$, $v_0 = 0.5$

$$\therefore\ K_{11} = hf1(t_0, u_0, v_0)$$
$$= 0.2(-3\times0+2\times0.5)$$
$$= 0.2$$
$$K_{21} = hf_2(t_0, u_0, v_0)$$
$$= 0.2(3\times0-4\times0.5)$$
$$= -0.4$$
$$K_{12} = hf_1(t_0+h, u_0+K_{11}, v_0+K_{21})$$
$$= 0.2[-3(0+0.2)+2(0.9-0.4)]$$
$$= -0.08$$
$$K_{22} = hf_2(t_0+h, u_0+K_{11}, v_0+K_{21})$$
$$= 0.2[3(0+0.2)-4(0.5-0.4)]$$
$$= 0.04$$
$$\therefore\ u(0.2) \cong u_1 = u_0 + \frac{1}{2}(K_{11} + K_{12})$$
$$= 0.06$$
$$\text{and } v(0.2) \cong v_1 = v_0 + \frac{1}{2}(K_{21} + K_{22})$$
$$= 0.32$$

Step 2 :

For $j = 1$

Here $t_1 = 0.2$, $u_1 = 0.06$, $v_1 = 0.32$

$$\therefore\ K_{11} = hf_1(t_1, u_1, v_1)$$
$$= 0.2[-3\times0.06+2\times0.32]$$
$$= 0.092$$
$$K_{21} = hf_2(t_1, u_1, v_1)$$
$$= 0.2[3\times0.6-4\times0.32]$$
$$= -0.22$$
$$K_{12} = hf_1(t_1+h_1u_1+K_{11}, v_1+K_{21})$$
$$= 0.2[-3(0.06+0.092)+2(0.32-0.22)]$$
$$= -0.0512$$

$$K_{22} = hf_2(t_1+h, u_1+K_{11}, v_1+K_{11})$$
$$= 0.2[3(0.06+0.092)-4(0.32-0.22)]$$
$$= 0.0112$$

$$\therefore u(0.4) \cong u_2 = u_1 + \frac{1}{2}(K_{11}+K_{12})$$
$$= 0.0804$$

$$\text{and } v(0.4) \cong v_2 = v_1 + \frac{1}{2}(K_{21}+K_{22})$$
$$= 0.2156$$

Similarly, we can derive

$u(0.6) \cong u_3 = 0.082152$, $\quad v(0.6) \cong v_3 = 0.152456$

$u(0.8) \cong u4 = 0.079309$, $\quad v(0.8) \cong v_4 = 0.112359$

$u(1) \cong u_5 = 0.069$, $\quad v(1) \cong v_5 = 0.08619$

5.10 Implicit Runge-Kutta Methods :

The implicit Runge-Kutta method using v slopes is defined as

$$K_i = hf\left(t_j + C_i h,\ u_j + \sum_{m=1}^{v} a_{im} K_m\right)$$

$$u_{j+1} = u_j + \sum_{m=1}^{v} W_m K_m \qquad \text{-------(5.39)}$$

$$\text{where } C_i = \sum_{j=1}^{v} a_{ij}, \quad i = 1,2,3,\ldots,v$$

and a_{ij} (i=1,2,3,.....,v, j=1,2,3,.....,v), W_i(i=1,2,3,......,v) are arbitrary constants.

Here the slopes K_m are defined implicitly and the number of unknown parameters are v(v+1).

Now we shall derive implicit Runge-Kutta formula for the cases v=1 & v=2.

$$\left.\begin{aligned} \text{For } v = 1, \text{ then } K_1 &= hf(t_j+C_1h, u_j+a_{11}K_1) \\ u_{j+1} &= u_j+W_1K_1 \end{aligned}\right\} \qquad \text{--------(5.40)}$$

From Taylor series, we have,

$$u(t_{j+1}) = u(t_j) + hu'(t_j) + \frac{h^2}{2}u''(t_j) + \ldots..$$

$$= u(t_j)+hf(t_j,u(t_j))+\frac{h^2}{2}(f_t+ff_u)t_j+....$$

$$\text{and } K_1 = h(f(t_j, u_j)+C_1hf_t+a_{11}K_1f_u+.......)$$

$$= (hf+C_1h^2f_t+ha_{11}f_uK_1)t_j+0(h^3)$$

$$= hf_j+h^2(C_1f_t+a_{11}ft_u)t_j+0(h^3) \qquad \text{-------(5.41)}$$

Thus from (5.40) and (5.41) and comparing the coefficients of h and h^2, we get,

$$C_1 = a_{11},$$

$$W_1 = 1$$

$$W_1C_1 = \frac{1}{2}$$

Hence $W_1 = 1$, $C_1 = a_{11} = \frac{1}{2}$

∴ The second order implicit Runge-Kutta method becomes

$$\left.\begin{aligned} K_1 &= hf\left(t_j+\frac{1}{2}h,\ u_j+\frac{1}{2}K_1\right) \\ u_{j+1} &= u_j+K_1 \end{aligned}\right\} \qquad \text{--------(5.42)}$$

For v=2, the implicit Runge-Kutta method (5.39) becomes,

$$K_1 = hf(t_j+C_1h,\ u_j+a_{11}K_1+a_{12}K_2)$$

$$K_2 = hf(t_j+C_2h,\ u_j+a_{21}K_1+a_{22}K_2)$$

$$u_{j+1} = u_j+W_1K_1+W_2K_2 \qquad \text{-------(5.43)}$$

Where the parameters values are

$$W_1 = \frac{1}{2}, \qquad W_2 = \frac{1}{2},$$

$$C_1 = \frac{3-\sqrt{3}}{6}, \qquad C_2 = \frac{3+\sqrt{3}}{6}$$

$$a_{11} = \frac{1}{4}, \qquad a_{12} = \frac{1}{4}-\frac{\sqrt{3}}{6}$$

$$a_{21} = \frac{1}{4}+\frac{\sqrt{3}}{6} \qquad a_{22} = \frac{1}{4}$$

which had to a fourth order method.

$$\text{Now } K_1 = hf\left(t_j+\frac{3-\sqrt{3}}{6}, u_j+\frac{1}{4}K_1+\frac{3-2\sqrt{3}}{12}K_2\right)$$

$$K_2 = hf\left(t_j+\frac{3+\sqrt{3}}{6}, u_j+\frac{3+2\sqrt{3}}{12}K_1+\frac{1}{4}K_2\right)$$

$$u_{j+1} = u_j+\frac{1}{2}(K_1+K_2) \quad \text{-------(5.44)}$$

Example E.5.8 :

Solve the initial value problem $u' = -2tu^2$, $u(0) = 1$ with $h=0.2$ on the interval $[0, 0.4]$. Use the second order implicit Runge-Kutta method.

Solution :

The Second order implicit Runge-Kutta method is

$$u_{j+1} = u_j+K_1,\ j=0,1,2$$

$$K_1 = hf\left(t_j+\frac{h}{2}, u_j+\frac{1}{2}K_1\right)$$

$$\text{which gives } K_1 = -h(2t_j+h)\left(u_j+\frac{1}{2}K_1\right)^2$$

(ie) $h(2t_j+h)K^2_1+4(1+hu_j(2t_j+h))K_1+4h(2t_j+h)u^2_j = 0$

Solving the above quadratic equaion, we get,

$$K_1 = \frac{-4(1+0.2(0.2))+\sqrt{16+32(0.2)(0.2)}}{2(0.2)(0.2)}$$

$$= -0.0384759$$

$$\therefore u(0.2) \cong u_1 = u_0+K_1$$

$$= 1-0.0384759$$

$$= 0.9615241$$

For $j = 1$, $t_1 = 0.2$, $u_1 = 0.9615241$

$$\therefore K_1 = \frac{-4(1+012(0.9615241)+\sqrt{16+3.84\times 0.9615241}}{0.24}$$

$$= -0.0997344$$

$$\therefore u(0.4) \cong u_2 = u_1+K_1$$

$$= 0.9915241-0.0997344 = 0.8617897$$

Multistep Method :

The general multistep or k-step method is given by

$$u_{j+1} = a_1u_i+a_2u_{j-1}+....+a_ku_{j-k+1}+h(b_0u'_{j+1}+b_1u'_j+.......+b_ku'_{j-k+1}) \quad \text{-------(5.45)}$$

$$\text{(ie)} \quad u_{j+1} = \sum_{j=1}^{k} a_iu_{j-i+1}+h\sum_{i=0}^{k} b_iu'_{j-i+1}$$

Symbolically, we can write (5.45) as

$$\rho(E)u_{j-k+1}-h\sigma(E)u'_{j-k+1} = 0$$

where ρ and σ are polynomials defined by

$$\rho(\xi) = \xi^k-a_1\xi^{k-1}-a_2\xi^{k-2}-......-a_k \text{ and}$$

$$\sigma(\xi) = b_0\xi^k+b_1\xi^{k-1}+.....+b^k$$

Now (5.45) can be used only if the values of the solution u(t) and u'(t) at k successive points are known (ie) $u_j, u_{j-1},.....,u_{j-k+1}$. These k values will be assumed to be given.

Note :

If $b_0 = 0$ then (5.45) is called an **explicit or predictor method.**

If $b_0 \neq 0$ then (5.45) is called an **implicit or corrector method.**

5.11 Adams–Bashforth Methods :

$$\left.\begin{aligned} \text{Let } \rho(\xi) &= \xi^{k-1}(\xi-1) \text{ and} \\ \sigma(\xi) &= \xi^{k-1}\sum_{m=0}^{k-1}\lambda_m(1-\xi^{-1})^m \end{aligned}\right\} \quad \text{-------(5.46)}$$

$$\text{where } \lambda_m+\frac{1}{2}\lambda_{m-1}+....+\frac{1}{m+1}\lambda_0 = 1, \; m=0,1,2,...$$

Here (5.46) is called Adams-Bashforth method.

5.12 Nystrom methods :

$$\left.\begin{aligned} \text{Let } \rho(\xi) &= \xi^{k-2}(\xi^2-1) \text{ and} \\ \sigma(\xi) &= \xi^{k-1}\sum_{m=0}^{k-1}\lambda_m(1-\xi^{-1})^m \end{aligned}\right\} \quad \text{-------(5.47)}$$

where $\lambda_m + \frac{1}{2}\lambda_{m-1} + \dots + \frac{1}{m+1}\lambda_0 = 2$ if $m = 0$

$\lambda_m + \frac{1}{2}\lambda_{m-1} + \dots + \frac{1}{m+1}\lambda_0 = 1$ if $m = 1,2,3,\dots\dots$

Here (5.47) is called Nystrom method.

5.13 Adams-Moulton Method :

$$\left.\begin{aligned} \text{Let } \rho(\xi) &= \xi^{k-1}(\xi-1) \text{ and} \\ \sigma(\xi) &= \xi^k \sum_{m=0}^{k} \lambda_m (1-\xi^{-1})^m \end{aligned}\right\} \quad \text{-------(5.48)}$$

where $\lambda_m + \frac{1}{2}\lambda_{m-1} + \dots + \frac{1}{m+1}\lambda_0 = 1$ if $m = 0$

$\lambda_m + \frac{1}{2}\lambda_{m-1} + \dots + \frac{1}{m+1}\lambda_0 = 0$ if $m = 1,2,3,\dots\dots$

Here (5.48) is called Adams-Moulton method.

5.14 Milne-Simpson Method :

$$\left.\begin{aligned} \text{Let } \rho(\xi) &= \xi^{k-2}(\xi^2-1) \\ \text{and } \sigma(\xi) &= \xi^k \sum_{m=0}^{k} \lambda_m (1-\xi^{-1})^m \end{aligned}\right\} \quad \text{-------(5.49)}$$

$$\text{where } \lambda_m + \frac{1}{2}\lambda_{m-1} + \dots + \frac{1}{m+1}\lambda_0 = \begin{cases} 2 \text{ if } m = 0 \\ -1 \text{ if } m = 1 \\ 0 \text{ if } m = 2,3,4,\dots \end{cases}$$

5.15 Predictor-Corrector Methods :

P(EC)^{m}E Method :

Consider the predictor-corrector set

$$P : u^{(0)}_{j+1} = \sum_{i=1}^{k} a^{(0)}{}_i u_{j-i+1} + h \sum_{i=1}^{k} b^{(0)}{}_i f_{j-i+1} \quad \text{-------(5.50)}$$

$$C : u^{(s+1)}_{(j+1)} = \sum_{i=1}^{k} a_i u_{j-i+1} + hb_0 f(t_{j+1}, u^{(s)}{}_{j+1}) + h \sum_{i=1}^{k} b_i f_{j-i+1} \quad \text{----(5.51)}$$

$$s = 0,1,2,\dots\dots$$

To solve the initial value problem $u' = f(t, u)$, $u(t_0) = \eta$

The predictor - corrector method is given as :

P : Predict some value $u^{(0)}_{j+1}$

E : Evaluate $f(t_{j+1}, u^{(0)}_{j+1})$

$$\text{C : Correct } u^{(1)}_{j+1} = \sum_{j=1}^{k} (a_i u_{j-i+1} + hb_i f_{j-i+1}) + hb_0 f(t_{j+1}, u^{(0)}_{j+1})$$

E : Evaluate $f(t_{j+1}, u^{(1)}_{j+1})$

$$\text{C : Correct } u^{(2)}_{j+1} = \sum_{j=1}^{k} (a_i u_{j-i+1} + hb_i f_{j-i+1}) + hb_0 f(t_{j+1}, u^{(1)}_{j+1})$$

The sequence of operations PECECE.... is denoted by $P(EC)^mE$ and is called predictor-corrector method.

We shall discuss $P(EC)^mE$ method for the equation $u'=\lambda u$ and the P–C set.

$$P : u_{j+1} = u_j + hf_j$$

$$C : u_{j+1} = u_j + \frac{h}{2}(f_{j+1} + f_j)$$

The $P(EC)^mE$ method may be written as

$$\begin{aligned}
u^{(0)}_{j+1} &= u_j + hf_j \\
u^{(s)}_{j+1} &= u_j + \frac{h}{2}(f^{(s-1)}_{j+1} + f_j), \quad s=1,2,3,.....,m \\
u_{j+1} &= u^{(m)}_{j+1} \\
f_{j+1} &= f^{(m)}_{j+1} \qquad \text{-------(5.51)}
\end{aligned}$$

$$\text{where } f^{(s)}_{j+1} = f(t_{j+1}, u^{(s)}_{j+1})$$

$\therefore$ The $P(EC)^mE$ method becomes

$$\begin{aligned}
u^{(0)}_{j+1} &= (1+\lambda h)u_j \\
u^{(1)}_{j+1} &= u_j + \frac{h}{2}[\lambda(1+\lambda h)u_j + \lambda u_j] \\
&= \left(1+\lambda h + \frac{1}{2}(\lambda h)^2\right)u_j
\end{aligned}$$

$$u^{(2)}_{j+1} = u_j + \frac{h}{2}\left[\lambda(1+\lambda h + \frac{1}{2}(\lambda h)^2 u_j + \lambda u_j)\right]$$

$$= \left(1+\lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{4}\right) u_j$$

$$\vdots$$

$$u^{(m)}_{j+1} = \left(1+\lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^2}{4} + \ldots + \frac{(\lambda h)^{m+1}}{2^m}\right) u_j$$

$$= \left(1+\lambda h + \frac{\frac{(\lambda h)^2}{2}\left(1-\left(\frac{\lambda h}{2}\right)^m\right)}{1-\frac{\lambda h}{2}}\right) u_j$$

$$= \left(\frac{1+\left(\frac{\lambda h}{2}\right) - 2\left(\frac{\lambda h}{2}\right)^{m+2}}{1-\frac{\lambda h}{2}}\right) u_j$$

$$\text{Thus } u_{j+1} = \left(\frac{1+\frac{\lambda h}{2} - 2\left(\frac{\lambda h}{2}\right)^{m+2}}{1-\frac{\lambda h}{2}}\right) u_j \qquad \text{------(5.52)}$$

Note :

The truncation error of $P(EC)^mE$ method is

$$T_{j+1} = u(t_{j+1}) - u_{j+1}$$

$$= \left(e^{\lambda h} - \left[\frac{1+\frac{\lambda h}{2} - 2\left(\frac{\lambda h}{2}\right)^{m+2}}{1-\frac{\lambda h}{2}}\right]\right) u(t_j) \qquad \text{-------(5.53)}$$

$$\text{where } f^{(s)}_{j+1} = f(t_{j+1}, u^{(s)}_{j+1})$$

$\therefore$ (5.53) becomes,

$\frac{1}{2}(\lambda h)^2 + 0(|\lambda h|^3)$ for 0 corrector

$\frac{1}{6}(\lambda h)^3 + 0(|\lambda h|^4)$ for 1 corrector

$-\frac{1}{12}(\lambda h)^3 + 0(|\lambda h|^4)$ for 2 corrector

$-\frac{1}{12}(\lambda h)^3 + 0(|\lambda h|^4)$ for 3 corrector

Now we apply P–C set

$$P : u_{j+1} = 2u_j - u_{j-1} + h^2 f_j$$

$$C : u_{j+1} = 2u_j - u_{j-1} + \frac{h^2}{12}(f_{j+1} + 10f_j + f_{j-1})$$

to the second order initial value problem.

$$u'' = -w^2 u$$

$$u(0) = 1, u'(0) = 0$$

The $P(EC)^m E$ method is given by

$$u^{(0)}_{j+1} = 2u_j - u_{j-1} + h^2 f_j$$

$$u^{(s)}_{j+1} = 2u_j - u_{j-1} + \frac{h^2}{12}(f^{(s-1)}_{j+1} + 10f_j + f_{j-1}), \; s=1,2,3,\ldots,m$$

$$u_{j+1} = u^{(m)}_{j+1}$$

$$f_{j+1} = f^{(m)}_{j+1}$$

$$\text{where } f^{(s)}_{j+1} = f(t_{j+1}, u^{(s)}_{j+1})$$

we obtain

$$u^{(0)}_{j+1} = 2u_j - u_j - H^2 u_j$$

$$= (2-H^2)u_j - u_{j-1}$$

$$\text{where } H = wh$$

$$\text{Now } u^{(1)}_{j+1} = 2u_j - u_{j-1}$$

$$+\frac{h^2}{12}[-w^2((2-H^2)u_j-u_{j-1})-10w^2u_j-w^2u_{j-1}]$$

$$= 2u_j-u_{j-1}-\frac{H^2}{12}[12-H^2)u_j]$$

$$= \left(2-H^2+\frac{H^4}{12}\right)u_j-u_{j-1}$$

$$u^{(2)}{}_{j+1} = 2u_j-u_{j-1}-\frac{H^2}{12}\left[u^{(2)}{}_{j+1}+10u_j+u_{j-1}\right]$$

$$= 2u_j-u_{j-1}$$

$$-\frac{H^2}{12}\left[\left(2-H^2+\frac{H^4}{12}\right)u_j-u_{j-1}+10u_j+u_{j-1}\right]$$

$$u^{(3)}{}_{j+1} = \left(2-H^2+\frac{(H^2)^2}{12}-\frac{(H^2)^3}{12^2}\right)u_j-u_{j-1}$$

$$= 2\left(1-\frac{H^2}{2}\left(1-\frac{H^2}{12}+\left(\frac{H^2}{12}\right)^2\right)\right)u_j-u_{j-1}$$

After m correctors, we get,

$$u_{j+1}-2Bu_j+u_{j-1} = 0$$

$$\text{where } B = \frac{1-\frac{5}{12}H^2+(-1)^{m-1}6\left(\frac{H^2}{12}\right)^{m+2}}{1+\frac{H^2}{12}}$$

For convergence, $|B|<1$ as $m\to\infty$.

$$\text{(ie) } \frac{H^2}{12}<1 \text{ and } \left|\frac{1-\frac{5H^2}{12}}{1+\frac{H^2}{2}}\right|<1$$

(ie) $H^2 < 6$.

The truncation error

$$T_{j+1} = 2\cos H - \left[\frac{2\left(1-\frac{5H^2}{12}+(-1)^{m-1}6\left(\frac{H^2}{12}\right)^{m+2}\right)}{1+\frac{H^2}{12}}\right]u(t_j)$$

$$= \begin{cases} \frac{H^2}{12}+0(H^6) \text{ for 0 corrector} \\ \frac{-H^6}{360}+0(H^8) \text{ for 1 corrector} \\ \frac{H^6}{240}+0(H^8) \text{ for 2 corrector} \\ \frac{H^6}{240}+0(H^8) \text{ for 3 corrector} \end{cases}$$

5.17 PM_PCM_C Method :

This method is called modified predictor-corrector method. In this method we use the estimate of the truncation error to modify the predicted and corrected values.

Let the predictor (5.50) and the corrector (5.51) both have the order P. ∴ we have

$$u(t_{j+1})-u^{(P)}_{j+1} = C^*_{j+1}h^{P+1}u^{(P+1)}(t_j)+0(h^{P+2}) \quad \text{--------(5.54)}$$

$$u(t_{j+1})-u^{(C)}_{j+1} = C_{j+1}h^{P+1}u^{(P+1)}(t_j)+0(h^{P+2}) \quad \text{--------(5.55)}$$

where $u^{(P)}_{j+1}$ and $u^{(C)}_{j+1}$ represent the solution values obtained by using the predictor and corrector respectively.

Now (5.55) – (5.54) implies,

$$u^{(P)}_{j+1}-u^{(C)}_{j+1} = (P_{j+1}-C^*_{j+1})h^{P+1}u^{(P+1)}(t_j)+0(h^{P+2}) \quad \text{-------(5.56)}$$

Substituting the values of $h^{P+1}\ u^{P+1}(t_j)$ from (5.56) into (5.54) and (5.55) we obtained predicted and corrected values m_{j+1} and u_{j+1} respectively.

$$\text{as } m_{j+1} = P_{j+1}+C^*_{j+1}(C_{j+1}-C^*_{j+1})^{-1}(P_{j+1}-C_{j+1})$$

$$u_{j+1} = C_{j+1}+C_{j+1}(C_{j+1}-C^*_{j+1})^{-1}(P_{j+1}-C_{j+1})$$

where p_{j+1} and c_{j+1} are the predicted and the corrected values respectively.

Thus the modified P–C method becomes,

$$\left.\begin{aligned}
\text{Predicted value } P_{j+1} &= \sum_{i=1}^{k}\left(a^{(0)}{}_i u_{j-i+1} + hb_i^{(0)} f_{j-i+1}\right) \\
\text{Modified value } m_{j+1} &= P_{j+1} + C^*{}_{j+1}\left(C_{j+1} - C^*{}_{j+1}\right)^{-1}\left(P_j - C_j\right) \\
\text{Corrected value } C_{j+1} &= \sum_{i=1}^{k}\left(a_i u_{j-i+1} + hb_i u'{}_{j-i+1}\right) + hb_0 m'{}_{j+1} \\
\text{Final Value } u_{j+1} &= C_{j+1} + C_{j+1}\left(C_{j-1} - C^*{}_{j+1}\right)^{-1}\left(p_{j+1} - C_{j+1}\right)
\end{aligned}\right\} \quad \text{-----(5.57)}$$

Note: Normally take $p_j - C_j = 0$ ------(5.58) in the first step for the modification of $p_j - C_j$.

Example E.5.9 :

Solve the initial value problem.

$u' = -2tu^2$, $u(0) = 1$ with $h = 0.2$ on the interval [0, 0.4], using the P–C method.

$$P : u_{j+1} = u_j + \frac{h}{2}\left(3u'_j - u'_{j-1}\right)$$

$$C : u_{j+1} = u_j + \frac{h}{2}\left(u'_{j+1} + u'_j\right)$$

Method 1 : (Using $P(EC)^mE$ Method)

To find u(t) and u'(t) at t=0.2. The exact values of u(t), u'(t) at x=0.2 from the exact solution $u(t) = \frac{1}{1+t^2}$ are

$u(0.2) = u_1 = 0.9615385$

$u'(0.2) = u'_1 = -0.3698225$

For $j = 1 : t_0 = 0,\ t_1 = 0.2,\ t_2 = 0.4$

$$\begin{aligned}
P : u^{(0)}{}_2 &= u_1 + \frac{h}{2}(3u'_1 - u'_0) \\
&= 0.9615385 + 0.1(-3\times 0.3698225 - 0) \\
&= 0.8505918
\end{aligned}$$

$E : f(t_1, u^{(0)}{}_2) = u^{(0)}{}_2 = -0.5788051$

$$C : u^{(1)}{}_2 = u_1 + \frac{h}{2}\left(u'^{(0)}{}_2 + u'_1\right)$$

$$= 0.9615385+0.1(-0.5788051-0.3698225)$$

$$= 0.8666757$$

$$E : f(t_2, u^{(1)}{}_2) = -0.6009015$$

$$C : u^{(2)}{}_2 = u_1 + \frac{h}{2}\left(u'^{(1)}{}_2 + u'_1\right)$$

$$= 0.9615385+0.1(-0.6009015-0.3698225)$$

$$= 0.8644661$$

$$\text{Hence } u(0.4) \cong 0.8644661$$

Method 2 : (Using PMPCMC Method)

$$p_{j+1} = u_j + \frac{h}{2}\left(3u'_j - u'_{j-1}\right)$$

$$m_{j+1} = p_{j+1} - \frac{5}{6}\left(P_j - C_j\right)$$

$$C_{j+1} = u_j + \frac{h}{2}\left(m'_{j+1} + u'_j\right)$$

$$u_{j+1} = C_{j+1} + \frac{1}{6}\left(p_{j+1} - C_{j+1}\right),\ j = 1,2,3,\ldots\ldots$$

Now the values $u_1 = 0.9615385$, $u'_1 = -0.3698225$ are obtained from the exact solution of $u(t) = \dfrac{1}{1+t^2}$.

$$\text{Now } t_1 = 0.2$$

$$\text{For } j = 1$$

$$P_2 = u_1 + \frac{h}{2}\left(3u'_1 - u'_0\right)$$

$$= 0.9615385+0.1(-3\times0.3698225-0)$$

$$= 0.8505918$$

$$m_2 = P_2 - \frac{5}{6}(p_1 - C_1)$$

Taking $p_1 - C_1 = 0$, we obtain

$$m_2 = 0.8505918$$

$$m'_2 = -2t2m^2_2$$
$$= -2(0.4)(0.8505918)^2$$
$$= -0.5788051$$

$$C_2 = u_1 + \frac{h}{2}(m'_2 + u'_1)$$
$$= 0.9615385+0.1(-0.5788051-0.3698225)$$
$$= 0.8666757$$

$$P_2 - C_2 = 0.8505918-0.8666757$$
$$= -0.0160839$$

$$\therefore u(0.4) \cong u_2$$
$$= C_2 + \frac{1}{6}(p_2 - C_2)$$
$$= 0.8666757+1/6(-0.0160839)$$
$$= 0.8639951.$$

UNIT – 6

LINEAR EQUATIONS WITH VARIABLE COEFFICIENTS

6.1 Introduction :

Definition D. 6.1 :

A linear differential equation of order n with variable coefficients of the form $a_0(x)y^{(n)}+a_1(x)y^{(n-1)}+....+a_n(x)y = b(x)$, where a_0, a_1,....a_n, b are complex valued functions on some real interval **I**.

Note :

1) If $x \in \mathbf{I}$ such that $a_0(x) = 0$ then x is called singular point.

2) In our future discussions we consider only non-singular points, unless otherwise it is stated. (ie) $a_0(x) \neq 0 \ \forall \ x \in \mathbf{I}$.

3) Hereafter we consider the linear differential equation as

$$y^{(n)}+a_1(x)y^{(n-1)}+...+a_n(x)y = b(x) \qquad \text{------(6.1)}$$

4) When a_0, a_1,......,a_n are constant then (6.1) is denoted by $L(y) = b(x)$ where $L(y) = y^{(n)}+a_1(x)y^{(n-1)}+....+a_n(x)y$.

5) If $b(x) = 0 \ \forall \ x \in \mathbf{I}$ then linear differential equation is called homogenous equation.

6) If we consider L as an operator then $L(\varphi)(x) = \varphi^{(n)}(x)+a_1(x)\varphi^{(n-1)}(x)+....+a_n(x)\varphi(x)$, where φ is a function which has n derivatives.

Definition D. 6.2 :

A function φ on **I** which has n derivatives is called a solution of the linear differential equation if $L(\varphi)=b$.

6.2 Initial value problems for the homogeneous equations.

Theorem 6.1 :

Let a_1, a_2,......,a_n be continuous functions on an interval I containing the point x_0. If α_1, α_2,........,α_n are any n constants, there exists solution φ of $L(y) = y^{(n)}a_1(x)y^{(n-1)}+....+a_n(x)y=0$ on **I** satisfying $\varphi(x_0)=\alpha_1$, $\varphi^1(x_0)=\alpha_2$,...,$\varphi^{(n-1)}(x_0) = \alpha_n$.

We shall discuss the proof of this theorem in later sections.

Definition D. 6.3 :

The norm of $\varphi(x)$ is defined as $\|\varphi(x)\| = [|\varphi(x)|^2+|\varphi^1(x)|^2+......+|\varphi^{(n-1)}(x)|^2]^{1/2}$

Theorem 6.2 :

Let $b_1, b_2,......b_n$ be non-negative constants such that for all x in **I** $|a_j(x)|\leq b_j$ $(j=1,2,3,......,n)$ and define k by $k=1+b_1+.....+b_n$. If $x_0\in$**I** and φ is a solution of $L(y)=0$ on **I**, then $\|\varphi(x_0)\|e^{-k|x-x_0|} \leq \|\varphi(x)\| \leq \|\varphi(x_0)\| e^{k|x-x_0|}$ for all $x\in$**I**.

Proof :

If φ is a solution of $L(y)=0$ then $L(\varphi)=0$

$$\text{(ie) } \varphi^{(n)}(x)+a_1(x)\varphi^{(n-1)}(x)+...+a_n(x)\varphi(x) = 0$$

$$\text{(ie) } \varphi^{(n)}(x) = -a_1(x)\varphi^{(n-1)}(x)-a_2(x)\varphi^{(n-2)}(x)-.....-a_n(x)\varphi(x)$$

$$\therefore |\varphi^{(n)}(x)| \leq b_1|\varphi^{(n-1)}(x)|+b_2|\varphi^{(n-2)}(x)|+...+b_n|\varphi(x)| \quad \text{-----(6.2)}$$

$$\text{Let } u(x) = \|\varphi(x)\|^2$$

$$\therefore u = \|\varphi\|^2$$

$$= \varphi\overline{\varphi}+\varphi\overline{\varphi'}+......+\varphi^{(n-1)}\overline{\varphi^{(n-1)}}$$

$$\text{Hence } u^1 = \varphi'\overline{\varphi}+\varphi\overline{\varphi'}+\varphi\overline{\varphi''}+\varphi'\overline{\varphi'}+...+\varphi^{(n)}\overline{\varphi^{(n-1)}}+\varphi^{(n-1)}\overline{\varphi^{(n)}}$$

$$\therefore |u'| = |\varphi'||\overline{\varphi}|+|\varphi||\overline{\varphi'}|+|\varphi||\overline{\varphi''}|+|\varphi'||\overline{\varphi'}|+...+\left|\varphi^{(n)}\right|\left|\overline{\varphi^{(n-1)}}\right|+\left|\varphi^{(n-1)}\right|\left|\overline{\varphi^{(n)}}\right|$$

$$= 2|\varphi||\varphi'|+2|\varphi'||\varphi''|+....+2|\varphi^{(n-1)}||\varphi^{(n)}| \quad \text{------(6.3)}$$

From (6.2) & (6.3), we have,

$$|u'| \leq 2|\varphi||\varphi'|+2|\varphi'||\varphi''|+....+2|\varphi^{(n-2)}||\varphi^{(n-1)}|+2b_1|\varphi^{(n-1)}|^2$$
$$+2b_2|\varphi^{(n-2)}||\varphi^{(n-2)}|+....+2b_n|\varphi||\varphi^{(n-1)}| \quad \text{------(6.4)}$$

We know that $2|x||y| \leq |x|^2+|y|^2$ and therefore (6.4) changes as

$$|u'| \leq (1+b_1)|\varphi|+(2+b_{n-1})|\varphi'|^2+.....$$
$$+(2+b_2)|\varphi^{(n-2)}|^2+(1+2b_1+b_2+......+b_n)|\varphi^{(n-1)}|^2.$$
$$\leq 2ku$$

$$\text{(ie) } -2ku \leq u' \leq 2ku \quad \text{------(6.5)}$$

Now (6.5) is true for $\forall\ x \in \mathbf{I}$.

$$\therefore\ -2ku(x) \leq \varphi\, u'(x) \leq 2ku(x) \text{ for } x \in \mathrm{I} \quad \text{-------(6.6)}$$

$$\text{Consider } u'(x) \leq 2ku(x).$$

$$\text{(ie) } u'(x) - 2ku(x) \leq 0$$

$$\text{(ie) } e^{-2kx}(u'(x) - 2ku(x)) \leq 0$$

$$\text{(ie) } (e^{-2kx}.u(x))' \leq 0 \quad \text{-------(6.7)}$$

If $x \geq 0$ and integrate (6.7) then we have

$$\int_{t=x_0}^{x} \left(e^{-2kt}u(t)\right)' \leq 0$$

$$\text{(ie) } e^{-2kx}u(x) - e^{-2kx_0}u(x_0) \leq 0$$

$$\text{(ie) } e^{-2kx}u(x) \leq e^{-2kx_0}u(x_0)$$

$$\text{(ie) } u(x) \leq e^{2k(x-x_0)}u(x_0)$$

$$\text{(ie) } \|\varphi(x)\|^2 \leq e^{2k(x-x_0)} \|\varphi(x_0)\| \ (\because u = \|\varphi\|^2)$$

$$\therefore\ \|\varphi(x)\| \leq e^{k(x-x_0)} \|\varphi(x_0)\| \quad \text{-------(6.7)}$$

Similarly if $x \leq x_0$ then we have

$$\|\varphi(x)\| \leq e^{-k(x-x_0)} \|\varphi(x_0)\| \quad \text{-------(6.8)}$$

From (6.7) & (6.8), we have,

$$\|\varphi(x)\| \leq e^{k|x-x_0|}\ \forall\ x \in \mathbf{I} \quad \text{-------(6.9)}$$

Again consider LHS inequality of (6.8) and then proceding above, we get,

$$e^{-k|x-x_0|} \|\varphi(x_0)\| \leq \|\varphi(x)\|\ \forall\ x \in \mathbf{I} \quad \text{-------(6.10)}$$

From (6.9) & (6.10), we get,

$$\|\varphi(x_0)\|\, e^{-|x-x_0|} \leq \|\varphi(x)\| \leq e^{k|x-x_0|} \|\varphi(x_0)\|\ \forall\ x \in \mathbf{I}$$

This proves the theorem.

Theorem 6.3 (Uniqueness Theorem)

Let $x_0 \in \mathbf{I}$ and let $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ be any n constants. There is atmost one solution φ of $L(y) = 0$ on $\mathbf{I}$ satisfying $\varphi(x_0) = \alpha_1$, $\varphi'(x_0) = \alpha_2, \ldots\ldots, \varphi^{(n-1)}(x_0) = \alpha_n$.

Proof :

Let φ, Ψ be two solutions of $L(y) = 0$ on **I** such that

$$\left.\begin{aligned} &\varphi(x_0) = \alpha_1,\ \varphi'(x_0) = \alpha_2, \ldots\ldots, \varphi^{(n-1)}(x_0) = \alpha_n \\ \text{and } &\Psi(x_0) = \alpha_1,\ \Psi'(x_0) = \alpha_2, \ldots\ldots, \Psi^{(n-1)}(x_0) = \alpha_n \end{aligned}\right\} \quad \text{-------(6.11)}$$

$$\text{Let } \Gamma = \varphi - \Psi.$$

Claim :

$\Gamma(x) = 0\ \forall\ x \in I$

Let x be any point in I otherthan x_0.

Let J be any closed bounded interval in I which contains x_0 and x.

Clearly in this interval a_j are bounded.

(ie) $|a_j(x)| \leq b_j$ (j=1,2,3,.....,n) on J for some constants b_j, which may depend on J.

Clearly Γ is defined on J (from theorem 6.2)

Since $L(\Gamma) = L(\varphi) - L(\Psi) = 0$ on J. and therefore $\|\Gamma(x_0)\| = 0$

This implies that $\|\Gamma(x)\| = 0$ because $\|\Gamma(x_0)\|\, e^{-k|x-x_0|} \leq \|\Gamma(x_0)\| \leq \|\Gamma(x)\|\, e^{k|x-x0|}$

$\therefore\ \Gamma(x) = 0\ \forall\ x \in I\ \&\ x \neq x_0$.

$\Rightarrow \varphi(x) = \Psi(x)\ \forall\ x \in I\ \&\ x \neq x_0$.

But by (6.11), $\varphi(x) = \Psi(x)$ at $x = x_0$.

Thus $\varphi(x) = \Psi(x)\ \forall\ x \in I$.

This proves the theorem.

6.3 Solution of the homogenous equations :

Example E. 6.1 :

Prove that any linear combination of solution of L(y)=0 is again a solution of L(y)=0.

Proof :

If $\varphi_1, \varphi_2, \ldots\ldots, \varphi_m$ are any m solutions of the n^{th} order equation L(y)=0 then $L(\varphi_j)=0$, j=1,2,.....,n.

The linear combination of $\varphi_1,\varphi_2,\ldots,\varphi_m$ is $C_1\varphi_1+C_2\varphi_2+\ldots+C_m\varphi_m$ where $C_1,C_2,\ldots,C_m$ are constants.

$$\text{Now } L(C_1\varphi_1+C_2\varphi_2+\ldots+C_n\varphi_n) = C_1L(\varphi_1)+C_2L(\varphi_2)+\ldots+C_nL(\varphi_n)$$
$$= 0$$

$\therefore C_1\varphi_1+C_2\varphi_2+\ldots+C_n\varphi_n$ is a solution of $L(y) = 0$

This proves the problem.

Definition D. 6.3 :

n functions $\varphi_1, \varphi_2,\ldots,\varphi_n$ defined an interval I are said to be linearly independent if $C_1\varphi_1+C_2\varphi_2+\ldots+C_n\varphi_n=0$ on I then $C_1=C_2=\ldots=C_n=0$

Theorem 6.4 :

There exists n linearly independent solutions of $L(y)=0$ on I.

Proof :

Let $x_0\in 1$.

We know that by existence theorem there is a solution φ_1 of $L(y)=0$ satisfying $\varphi_1(x_0)=1, \varphi_1'(x_0)=0,\ldots,\varphi_1^{(n-1)}(x_0)=0$.

In general for each $i=1,2,3,\ldots,n$ there is a solution φ_i satisfying $\varphi_i^{(i-1)}(x_0)=1$, $\varphi_i^{(j-1)}(x_0) = 0, j\neq i$ ------(6.12)

Claim : $\varphi_1,\varphi_2,\ldots,\varphi_n$ are linearly independent.

Let $C_1\varphi_1(x)+C_2\varphi_2(x)+\ldots+C_n\varphi_n(x) = 0$ --------(6.13)

for all $x\in I$ and $C_1,C_2,\ldots,C_n$ are constants.

Differentiating, we get,

$$\left.\begin{array}{l} C_1\varphi_1'(x)+C_2\varphi_2'(x)+\ldots+C_n\varphi_n'(x) = 0 \\ C_2\varphi_1''(x)+C_2\varphi_2''(x)+\ldots+C_n\varphi_n''(x) = 0 \\ \bullet \\ \bullet \\ \bullet \\ C_1\varphi_1^{(n-1)}(x)+C_2\varphi_2^{(n-1)}(x)+\ldots+C_n\varphi_n^{(n-1)}(x) = 0 \end{array}\right\} \text{------(6.14)}$$

From (6.13) & (6.14), and for $x=x_0$, we have, $C_1=C_2=.....=C_n=0$.

Therefore $\varphi_1, \varphi_2,.....,\varphi_n$ are linearly independent.

Theorem 6.5 :

Let $\varphi_1, \varphi_2,.....,\varphi_n$ be the n solutions of $L(y)=0$ on I satisfying $\varphi_i^{(i-1)}(x_0)=1$, $\varphi_i^{(j-1)}(x_0)=0$, $j\neq i$. If φ is any solution of $L(y)=0$ on I, there are n constants $C_1, C_2,....,C_n$ such that $\varphi = C_1\varphi_1+C_2\varphi_2+......+C_n\varphi_n$.

Proof :

Let $\varphi(x_0) = \alpha_1$, $\varphi'(x_0) = \alpha_2,......,\varphi^{(n-1)}(x_0) = \alpha_n$ and let $\Psi = \alpha_1\varphi_1+\alpha_2\varphi_2+......+\alpha_n\varphi_n$.

$$\text{Now } L(\Psi) = L\left(\sum_{i=1}^{n}\alpha_i\varphi_i\right)$$

$$= \sum_{i=1}^{n}\alpha_i L(\varphi_i)$$

$$= 0$$

$\therefore$ Ψ is a solution of $L(y) = 0$.

$$\text{Again } \Psi(x_0) = \alpha_1\varphi_1(x_0)+\alpha_2\varphi_2(x_0)+.....+\alpha_n\varphi_n(x_0)$$

$$= \alpha_1 \text{ because } \varphi_1(x_0) = 1, \varphi_2(x_0) = 0,....., \varphi_n(x_0)=0.$$

Similarly we can prove $\Psi'(x_0)=\alpha_2$, $\Psi''(x_0)=\alpha_3,.......,\Psi^{(n-1)}(x_0)=\alpha_n$.

Hence Ψ is a solution of $L(y) = 0$ having the same initial condition at x_0 as φ.

By uniqueness theorem, $\varphi = \Psi$.

(ie) $C_1=\alpha_1$, $C_2=\alpha_2,.......,C_n=\alpha_n$

$\therefore$ $\varphi = C_1\varphi_1+C_2\varphi_2+......+C_n\varphi_n$ is a solution of $L(y)=0$.

This proves the theorem.

Note : The set of all solutions of $L(y) = 0$ on an interval I is a linear space of functions.

6.4 The Wronskian and Linear Independence

Definition D. 6.4 :

The Wronskian $W(\varphi_1, \varphi_2,.....\varphi_n)$ of n functions $\varphi_1, \varphi_2,....,\varphi_n$ having n−1 derivatives on an interval I is defined as

$$W(\varphi_1, \varphi_2, \ldots\ldots, \varphi_n) = \begin{vmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \ldots\ldots & \varphi_n \\ \varphi_1' & \varphi_2' & \varphi_3' & \ldots\ldots & \varphi_n' \\ \varphi_1'' & \varphi_2'' & \varphi_2'' & \ldots\ldots & \varphi_n'' \\ \ldots\ldots & & & & \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \varphi_3^{(n-1)} & & \varphi_n^{(n-1)} \end{vmatrix}$$

and its value at $x \in I$ is $W(\varphi_1, \varphi_2, \ldots\ldots, \varphi_n)(x)$

Theorem 6.6 :

If $\varphi_1, \varphi_2, \ldots\ldots, \varphi_n$ are n solutions of $L(y) = 0$ on an interval I, they are linearly independent there if and only if $W(\varphi_1, \varphi_2, \ldots\ldots, \varphi_n)(x) \neq 0$ for all $x \in I$.

Proof :

Let $W(\varphi_1, \varphi_2, \ldots\ldots, \varphi_n)(x) \neq 0 \ \forall \ x \in I$ -------(6.15)

Let $C_1, C_2, \ldots\ldots, C_n$ be n constants such that $C_1\varphi_1(x)+C_2\varphi_2(x)+\ldots.+C_n\varphi_n(x)=0 \ \forall \ x \in I$.

$\therefore$ we have,

$$\left.\begin{aligned} &C_1\varphi_1'(x)+C_2\varphi_2'(x)+\ldots\ldots.+C_n\varphi_n'(x) = 0 \\ &C_1\varphi_1''(x)+C_2\varphi_2''(x)+\ldots\ldots.+C_n\varphi_n''(x) = 0 \\ &\bullet \\ &\bullet \\ &C_1\varphi_1^{(n-1)}(x)+C_2\varphi_2^{(n-1)}(x)+\ldots\ldots.+C_n\varphi_n^{(n-1)}(x) = 0 \end{aligned}\right\} \text{-------(6.16)}$$

Now fix $x \in I$.

Then from (6.15) & (6.16), we have, n linear homogeneous equations satisfied by $C_1, C_2, \ldots., C_n$.

Hence there is only one solution to this system, namely, $C_1=C_2=\ldots..=C_n=0$.

(ie) $\varphi_1, \varphi_2, \ldots\ldots, \varphi_n$ are linearly independent.

Conversly, suppose $\varphi_1, \varphi_2, \ldots\ldots, \varphi_n$ are linearly independent on I.

Claim : $W(\varphi_1, \varphi_2, \ldots\ldots, \varphi_n)(x) \neq 0 \ \forall \ x \in I$.

Suppose there exist $x_0 \in I$ such that $W(\varphi_1, \ldots., \varphi_n)(x_0) = 0$

This implies that the system of n linear equations

$$\left.\begin{array}{l} C_1\varphi_1(x_0)+C_2\varphi_2(x_0)+.......+C_n\varphi_n(x_0) = 0 \\ C_1\varphi_1'(x_0)+C_2\varphi_2'(x_0)+.......+C_n\varphi_n'(x_0) = 0 \\ \bullet \\ \bullet \\ C_1\varphi_1^{(n-1)}(x_0)+C_2\varphi_2^{(n-1)}(x_0)+.......+C_n\varphi_n^{(n-1)}(x_0) = 0 \end{array}\right\} \quad \text{------}(6.17)$$

has a solution $C_1, C_2, ..., C_n$ where not all the constants, $C_1, C_2, ..., C_n$ are zero.

Let $C_1, C_2, ..., C_n$ be such a solution.

Let $\Psi = C_1\varphi_1+C_2\varphi_2+......+C_n\varphi_n$

Clearly $\Psi(x_0) = 0$,

(ie) $C_1\varphi_1(x_0)+C_2\varphi_2(x_0)+.....+C_n\varphi_n(x_0) = 0$

$$\begin{aligned} \text{Similarly } \Psi'(x_0) &= C_1\varphi_1'(x_0)+C_2\varphi_2'(x_0)+......+C_n\varphi_n'(x_0) = 0 \\ \Psi''(x_0) &= C_1\varphi_1''(x_0)+C_2\varphi_2''(x_0)+.........+C_n\varphi_n''(x0) = 0 \\ &.... \\ &.... \\ \Psi^{(n-1)}(x_0) &= C_1\varphi_1^{(n-1)}(x_0)+C_2\varphi_2^{(n-1)}(x_0)+.....+C_n\varphi_n^{(n-1)}(x_0) = 0 \end{aligned}$$

Then by uniqueness theorem, $\Psi(x) = 0 \ \forall \ x \in I$

(ie) $C_1\varphi_1(x)+C_2\varphi_2(x)+.....+C_n\varphi_n(x) = \forall \ x \in I$

$\Rightarrow \varphi_1, \varphi_2, ..., \varphi_n$ are linearly independent

$\Rightarrow\Leftarrow$ to $\varphi_1, \varphi_2, ..., \varphi_n(x) \neq 0 \ \forall \ x \in I$

This proves the converse part. Hence the theorem be proved.

Theorem 6.7 :

Let $\varphi_1, \varphi_2, ...\varphi_n$ be n linearly independent solutions of $L(y)=0$ on an interval I. If φ is any solution of $L(y) = 0$ on I, it can be represented in the form $\varphi=C_1\varphi_1+C_2\varphi_2+....+C_n\varphi_n$ where $C_1, C_2, ..., C_n$ are constants. Thus any set of n linearly independent solutions of $L(y)=0$ on I is a basis for the solutions of $L(y)=0$ on I.

Proof :

Let $x_0 \in I$ and let $\varphi(x_0)=\alpha_1, \varphi'(x_0)=\alpha_2, ..., \varphi^{(n-1)}(x_0)=\alpha_n$.

Claim : There exists unique constants $C_1, C_2, \ldots, C_n$ such that $\Psi = C_1\varphi_1 + C_2\varphi_2 + \ldots + C_n\varphi_n$ is a solution of $L(y)=0$ satisfying $\Psi(x_0)=\alpha_1$, $\Psi'(x_0)=\alpha_2, \ldots, \Psi^{(n-1)}(x_0)=\alpha_n$.

Proof of the claim :

Let $\Psi = C_1\varphi_1 + C_2\varphi_2 + \ldots + C_n\varphi_n$.

By the uniqueness theorem $\Psi = \varphi$. This proves the claim.

$\therefore \varphi = C_1\varphi_1 + C_2\varphi_2 + \ldots + C_n\varphi_n$.

Again

$$\left.\begin{array}{l} \varphi(x_0) = C_1\varphi_1(x_0) + C_2\varphi_2(x_0) + \ldots + C_n\varphi_n(x_0) = \alpha_1 \\ \varphi'(x_0) = C_1\varphi_1'(x_0) + C_2\varphi_2'(x_0) + \ldots + C_n\varphi_n'(x_0) = \alpha_2 \\ \bullet \\ \bullet \\ \varphi^{(n-1)}(x_0) = C_1\varphi_1^{(n-1)}(x_0) + C_2\varphi_2^{(n-1)}(x_0) + \ldots + C_n\varphi_n^{(n-1)}(x_0) = \alpha_n \end{array}\right\} \text{------}(6.18)$$

Now (6.18) is a set of n linear equations in $C_1, C_2, \ldots, C_n$.

Since $\varphi_1, \varphi_2, \ldots, \varphi_n$ are linearly independent, we have

$$\begin{vmatrix} \varphi_1(x_0) & \varphi_2(x_0) & \ldots & \varphi_n(x_0) \\ \varphi_1'(x_0) & \varphi_2'(x_0) & \ldots & \varphi_n(x_0) \\ \ldots & & & \\ \varphi_1^{(n-1)}(x_0) & \varphi_2^{(n-1)}(x_0) & \ldots & \varphi_n^{(n-1)}(x_0) \end{vmatrix} \neq 0$$

(ie) $W(\varphi_1, \varphi_2, \ldots, \varphi_n)(x_0) \neq 0$.

Hence (6.18) has a unique solution for $C_1, C_2, \ldots, C_n$.

$\therefore \varphi$ is written uniquely as a linear combination of $\varphi_1, \varphi_2, \ldots, \varphi_n$.

This proves the theorem.

Theorem 6.8 :

Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be n solutions of $L(y) = 0$ on an interval I, and let x_0 be any point in I. Then $W(\varphi_1, \varphi_2, \ldots, \varphi_n)(x) = \exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right] W(\varphi_1, \varphi_2, \ldots \varphi_n)(x_0)$.

Proof :

Step 1 :

First we shall prove the theorem for the case n=2.

(ie) φ_1, φ_2 are two solutions of L(y)=0 on an interval I, $x_0 \in I$ then we shall prove that

$$W(\varphi_1, \varphi_2)(x) = \exp\left[-\int_{x_0}^{x} a_1(t)dt\right] W(\varphi_1, \varphi_2)(x_0)$$

We know that $W(\varphi_1, \varphi_2) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$

$$= \varphi_1\varphi_2' - \varphi_1'\varphi_2 \quad \text{-------(6.19)}$$

$$\therefore W'(\varphi_1, \varphi_2) = \varphi_1\varphi_2'' + \varphi_1'\varphi_2' - \varphi_2\varphi_1'' - \varphi_2'\varphi_1'$$

$$= \varphi_1\varphi_2'' - \varphi_1''\varphi_2 \quad \text{-------(6.20)}$$

Since φ_1, φ_2 are the solutions of the second order linear differential equation

$$L(y) = y''(x) + a_1(x)y'(x) + a_2(x)y(x) = 0 \ \forall\ x \in I;$$

We have, $\varphi_1'' + a_1\varphi_1' + a_2\varphi_1 = 0$

$$\Rightarrow \varphi_1'' = -a_1\varphi_1' - a_2\varphi_1 \quad \text{-------(6.21)}$$

Similarly $\varphi_2'' = -a_1\varphi_2' - a_2\varphi_2$ -------(6.22)

From (6.20), ((6.21) & (6.22), we have,

$$W'(\varphi_1, \varphi_2) = \varphi_1(-a_1\varphi_2' - a_2\varphi_2) - \varphi_2(-a_1\varphi_1' - a_2\varphi_1')$$

$$= -a_1\varphi_1\varphi_2' - a_2\varphi_1\varphi_2 + a_1\varphi_1'\varphi_2 + a_2\varphi_1\varphi_2$$

$$= -a_1(\varphi_1\varphi_2' - \varphi_1'\varphi_2)$$

$$= -a_1 W(\varphi_1, \varphi_2)$$

(ie) $W'(\varphi_1, \varphi_2) + a_1 W(\varphi_1, \varphi_2) = 0$

$\therefore W(\varphi_1, \varphi_2)$ is a solution of the linear differential equation $j^1 + a_1 y = 0$

Now we shall solve $y' + a_1 y = 0$.

Now $y' + a_1 y = 0$

$$\Rightarrow \quad y' = -a_1 y$$

$$\Rightarrow \quad \frac{dy}{dx} = -a_1(x)y$$

$$\Rightarrow \quad \frac{dy}{y} = -a_1(x)dx$$

Integrating on both sides from x_0 to x, we get, $\log y = -\int_{x_0}^{x} a_1(t)dt + \log C$ where log c is the integrating constant.

$$\text{(ie)} \quad \log\left(\frac{y}{C}\right) = -\int_{x_0}^{x} a_1(t)dt$$

$$\text{(ie)} \quad y = C.\exp\left(-\int_{x_0}^{x} a_1(t)dt\right) \qquad \text{------(6.23)}$$

When $x = x_0$, then $y = W(\varphi_1, \varphi_2)(x_0)$

$$\therefore (6.23) \Rightarrow W(\varphi_1, \varphi_2)(x_0) = C(1)$$

$$\text{(ie)} \quad C = W(\varphi_1, \varphi_2)(x_0)$$

$\therefore$ (6.23) becomes $W(\varphi_1, \varphi_2)(x) = \exp\left(-\int_{x_0}^{x} a_1(t)dt\right) w(\varphi_1, \varphi_2)(x_0)$

Thus we proved the theorem for n=z. This proves step 1.

Step 2 :

We shall prove the general case.

Denote $W(\varphi_1, \varphi_2, \ldots\ldots, \varphi_n)$ as W.

(ie) $W = W(\varphi_1, \varphi_2, \ldots\ldots, \varphi_n)$.

$\therefore$ $W' = V_1 + V_2 + \ldots\ldots + V_n$ where V_k is the determinant obtained from W by differentiating k^{th} row and retaining the remaining rows.

$\therefore$ $W' = V_1 + V_2 + \ldots\ldots + V_n$

$$= \begin{vmatrix} \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \cdots \\ \cdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1'' & \varphi_2'' & \cdots & \varphi_n'' \\ \cdots \\ \vdots \\ \cdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix}$$

$$+ \cdots + \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \cdots \\ \cdots \\ \varphi_1^{(n)} & \varphi_2^{(n)} & \cdots & \varphi_n^{(n)} \end{vmatrix}$$

In (6.24) the values of first (n–1) determinants is zero because any two rows of a determinant are identitical then its value is zero.

$$\therefore W' = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \varphi_1'' & \varphi_2'' & \cdots & \varphi_n'' \\ \cdots \\ \varphi_1^{(n-2)} & \varphi_2^{(n-2)} & \cdots & \varphi_n^{(n-2)} \\ \varphi_1^{(n)} & \varphi_2^{(n)} & \cdots & \varphi_n^{(n)} \end{vmatrix} \qquad \text{-------(6.25)}$$

Since φ_i is a solution of $L(y) = 0$, we have, $L(\varphi_i) = 0$

(ie) $\varphi_i^{(n)} + a_1\varphi_1^{(n-1)} + \cdots + a_n\varphi_i = 0$

(ie) $\varphi_i^{(n)} = -a_1\varphi_i^{(n-1)} - a_2\varphi_i^{(n-2)} - \cdots - a_n\varphi_i$

$= -\sum_{j=0}^{n-1} a_{n-j}\varphi_i^{(j)}$ where $\varphi_i^{(0)} = \varphi_i$

$$\therefore (6.25) \Rightarrow W' = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \cdots \\ \varphi_1^{(n-2)} & \varphi_2^{(n-2)} & \cdots & \varphi_n^{(n-2)} \\ -\sum_{j=0}^{n-1} a_{n-j}\varphi_1^{(j)} & -\sum_{j=0}^{n-1} a_{n-j}\varphi_2^{(j)} & \cdots & -\sum_{j=0}^{n-1} a_{n-j}\varphi_n^{(j)} \end{vmatrix}$$

$$R_n \to a_nR_1 + a_{n-1}R_2 + \cdots + R_n$$

$$= \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \cdots & & & \\ \varphi_1^{(n-2)} & \varphi_2^{(n-2)} & \cdots & \varphi_n^{(n-2)} \\ -a_1\varphi_1^{(n-1)} & -a_1\varphi_2^{(n-1)} & \cdots & -a_1\varphi_n^{(n-1)} \end{vmatrix}$$

$= -a_1 W.$

(ie) $W' + a_1 W = 0$

$\therefore$ W is a solution of $y' + a_1(x)y = 0$

We know that the solution of the above linear differential equation is

$$W(x) = \exp\left(-\int_{x_0}^{x} a_1(t)dt\right) W(x_0).$$

$$\text{(ie) } W(\varphi_1,\varphi_2,\ldots,\varphi_n)(x) = \exp\left(-\int_{x_0}^{x} a_1(t)dt\right) W(\varphi_1,\varphi_2,\ldots,\varphi_n)(x_0)$$

This prove the general case.

Hence the theorem.

6.5 Reduction of the order of a homogenous equation.

Theorem 6.9 :

Let φ_1 be a solution of $L(y) = 0$ on an interval I, and suppose $\varphi_1(x) \neq 0$ on I. If $v_2, v_3, \ldots, v_n$ is any basis on I for the solution of the linear equations of order $(n-1)$ and if $v_k = u_k'$ then $\varphi_1, u_2\varphi_1, u_3\varphi_1, \ldots, u_n\varphi_1$ is a basis for the solutions of $L(y)=0$ on I.

Proof :

Consider the linear differential equation

$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = 0$ -------(6.26)

whose one of the solution be φ_1.

Claim : To find solution φ of $L(\varphi) = 0$ of the form $\varphi = u\varphi_1$ where u is some function.

If $\varphi = u\varphi_1$ is to be a solution of (6.26) then we have, $\varphi^{(n)} + a_1\varphi^{(n-1)} + \ldots + a_n\varphi = 0$

(ie) $(u\varphi_1)^{(n)}+a_1(u\varphi_1)^{(n-1)}+.....+a_n(u\varphi_1) = 0$

(ie) $(u^{(n)}\varphi_1+......+nu'\varphi^{(n-1)}+u\varphi_1^{(n)}) + a_1(u^{(n-1)}\varphi_1+(n-1)u^{(n-2)}\varphi_1'+....+u\varphi_1^{(n-1)})$

$$+.....+a_{n-1}(u'\varphi_1+u\varphi_1')+a_n(u\varphi_1) = 0$$

(ie) $u^{(n)}\varphi_1+(n\varphi_1'+a_1\varphi_1)u^{(n-1)}+....+(n\varphi_1^{(n-1)}+(n-1)a_1\varphi_1^{(n-2)}+.....+a_{n-1}\varphi_1)u'$

$$+(\varphi_1^{(n)}+a_1\varphi_1^{(n-1)}+......+a_{n-1}\varphi_1'+a_n\varphi_1)u = 0 \quad \text{-----(6.27)}$$

Since φ_1 is a solution of $L(y) = 0$, we have,

$$\varphi_1^{(n)}+a_1\varphi_1^{(n-1)}+....+a_n\varphi_1 = 0 \quad \text{-------(6.28)}$$

$\therefore$ From (6.27) & (6.28), we have,

$$u^{(n)}\varphi_1+(n\varphi_1'+a_1\varphi_1)u^{(n-1)}+....+(n\varphi_1^{(n-1)}+(n-1)a_1\varphi_1^{(n-2)}+....+a_{n-1}\varphi_1)u'=0 \quad \text{------(6.29)}$$

If $v=u'$ then $v'=u''$, $v''=u'''$,.......$v^{(n-1)} = u^{(n)}$.

$\therefore$ (6.29) becomes $v^{(n-1)}\varphi_1+(n\varphi_1'+a_1\varphi_1)v^{(n-2)}+.....+(n\varphi_1^{(n-1)}+(n-1)a_1\varphi_1^{(n-2)}$

$$+...+a_{n-1}\varphi_1)v=0 \quad \text{-------(6.30)}$$

In (6.30) the coefficient of $v^{(n-1)}$ is φ_1.

If $\varphi_1(x)\neq 0$ on the interval I, then (6.30) has $(n-1)$ linearly independent solutions say, $v_2, v_3,....,v_n$ on I.

If $x_0 \in I$ and if $u_k(x) = \int_{x_0}^{x} v_k(t)dt$, $\quad k=2,3,......,n$ then $u_k' = v_k$.

Clearly, $\varphi_1, u_2\varphi_1, u_3\varphi_1,........,u_n\varphi_1$ ------(6.31)

are solutions of $L(y) = 0$ on I.

Claim : $\varphi_1, u_1\varphi_1, u_2\varphi_1,........,u_n\varphi_n$ form a basis for the solutions of $L(y) = 0$ on I.

It is enough to prove that $\varphi_1, u_2\varphi_1, u_3\varphi_1,.......,u_n\varphi_1$ are linearly independent on I.

Suppose $C_1\varphi_1+C_2u_2\varphi_1+.....+C_nu_n\varphi_n = 0$ ------(6.32)

where $C_1, C_2,.....,C_n$ are constants.

Since $\varphi_1(x)\neq 0$ on I, then $C_1+C_2u_2+.....+C_nu_n = 0$ ------(6.33)

Differentiate (6.33) w.r.t. x, we have,

$C_2u_2'+C_3u_3'+......+C_nu_n' = 0$

(ie) $C_2v_2+C_3v_3+.....+C_nv_n = 0$

$\Rightarrow C_2 = C_3 =........= C_n = 0$ because $v_2,......v_n$ are linearly independent.

From (6.33), $C_1 = 0$,

Thus $C_1=C_2=......=C_n=0$ and therefore φ_1, $u_2\varphi_1$, $u_3\varphi_1,.....u_n\varphi_1$ are linearly independent.

Hence φ_1, $u_2\varphi_1,.......,u_n\varphi_1$ form a basis for the solutions of $L(y) = 0$ on I.

This proves the theorem.

Theorem 6.10 :

If φ_1 is a solution of $y''+a_1(x)y'+a_2(x)y = 0$ ------(6.34)

on an interval I and $\varphi_1(x)\neq 0$ on I, a second solution φ_2 of (6.34) on I is given by

$$\varphi_2(x) = \varphi_1(x). \int_{s=x_0}^{x} \frac{1}{(\varphi_1(x))^2} \exp\left(- \int_{t=x_0}^{s} a_1(t)dt \right)ds$$

Further the functions φ_1 & φ_2 form the basis for the solutions of (6.34) on I.

Proof :

Let $L(y) = y''+a_1(x)y'+a_2(x)y=0$ be the given differential equation on I and let φ_1 be a solution of $L(y)=0$.

Let φ_2 be the second solution of (6.34). Then $\varphi_2=u\varphi_1$ where u is some function.

If φ_2 is a solution of $L(y) = 0$ then $L(\varphi_2)=0$.

(ie) $\varphi_2''+a_1(x)\varphi_2'+a_2(x)\varphi_2 = 0 \quad \forall\, x\in I$

(ie) $(u\varphi_1)''+a_1(u\varphi_1)'+a_2(x)(u\varphi_1) = 0 \quad \forall\, x\in I$

(ie) $(\varphi_1u''+2\varphi_1'u'+\varphi_1''u)+a_1(u'\varphi_1+\varphi_1'u)+a_2u\varphi_1 = 0$

(ie) $\varphi_1u''+(2\varphi_1'+a_1\varphi_1)u'+0 = 0$ ($\because \varphi_1$ is a solution of $L(y)=0$)

(ie) $\varphi_1v'+(2\varphi_1'+a_1\varphi_1)v = 0$ where $v=u'$. ------(6.35)

Multiply (6.35) by φ_1, we get,

$$\varphi_1^2v'+(2\varphi_1\varphi_1'+a_1\varphi_1^2)v = 0$$

(ie) $(\varphi_1^2v'+2_1\varphi_1\varphi_1'v)+a_1\varphi_1^2v = 0$

(ie) $(\varphi_1^2v)'+a_1(\varphi_1^2v) = 0$

(ie) $w'+a_1w = 0$ where $w = \varphi_1^2v$

(ie) $\dfrac{w'}{w}+a_1 = 0$

Integrating on both sides, we get,

$$\log w + \int_{t=x_0}^{x} a_1(t)dt = \log C$$

$$\text{(ie) } \log\left(\frac{w}{C}\right) = -\int_{t=x_0}^{x} a_1(t)dt$$

$$\text{(ie) } \quad w = C\exp\left(-\int_{t=x_0}^{x} a_1(t)dt\right)$$

$$\text{(ie) } \varphi_1^2v = C\exp\left(-\int_{t=x_0}^{x} a_1(t)dt\right) \qquad \text{------(6.36)}$$

Since any constant multiple of a solution of (6.35) is again a solution of (6.35) and therefore (6.36) can be taken as

$$v(x) = \frac{1}{\varphi_1^2(x)}.\exp\left(-\int_{t=x_0}^{x} a_1(t)dt\right) \qquad \text{------(6.37)}$$

Clearly (6.37) is a solution of (6.35) and therefore the two independent solution of (6.34) on I are φ_1, φ_2 where $\varphi_2=u\varphi_1$ where

$$u(x) = \int_{s=x_0}^{x} v(s)ds$$

$$= \int_{s=x_0}^{x}\left[\frac{1}{\varphi_1^2(s)}.\exp\left(-\int_{t=x_0}^{s} a_1(t)dt\right)\right]ds$$

$$\text{Thus } \varphi_2(x) = \varphi_1(x).\int_{s=x_0}^{x}\left[\frac{1}{\varphi_1^2(s)}\exp\left(-\int_{t=x_0}^{s} a_1(t)dt\right)\right]ds$$

This proves the theorem.

Example E.6.2 :

Verify that the function $\varphi_1(x) = x^3$, $x>0$ satisfies the equation $x^2y''-7xy'+15y = 0$ and also find a second independent solution.

Solution :

Let $L(y) = x^2y''-7xy'+15y = 0$ is a differential equation and let $\varphi_1(x)=x^3$, $x>0$.

Step 1 : To check φ_1 is a solution of $L(y) = 0$

$$\begin{aligned} \text{Now } L(\varphi_1) &= x^2\varphi_1''-7x\varphi_1 1+15\varphi_1 \\ &= x^2.6x-7x.3x^2+15x^3 \\ &= 0 \end{aligned}$$

$\therefore \varphi_1(x) = x^3$, $x>0$ is a solution of $L(y) = 0$.

Step 2 : To find the second solution φ_2 of $L(y) = 0$

Let $\varphi_2 = u\varphi_1$ where u is some function.

If φ_2 is a solution of $L(y) = 0$ then $L(\varphi_2) = 0$

(ie) $x^2\varphi_2''-7x\varphi_2'+15\varphi_2 = 0$

(ie) $x^2(u.6x+2u'.3x^2+u''x^3)-7x(u.3x^2+u'x^3)+15ux^3 = 0$

$(\because \varphi_2'' = u6x+2u'.3x^2+u''x^3 \ \& \ \varphi_2' = u3x^3+u'x^3)$

(ie) $u''x^5-u'x^4 = 0$

(ie) $u''x-u' = 0$ $(\because x>0)$

(ie) $v'x-v = 0$ where $v=u'$.

(ie) $\dfrac{v'}{v} = \dfrac{1}{x}$

Integrating on both sides, we get $\log v = \log x + \log c$

(ie) $v = x.c$

(ie) $u' = x.c$

(ie) $\dfrac{du}{dx} = c.x$

(ie) $du = cxdx$

Integrating on both sides, we get

$$u = C.\frac{x^2}{2}$$

$$(ie)\ u = C_1x^2 \text{ where } C_1=C$$

$$\therefore \varphi_2 = u\varphi_1$$

$$= C_1x^2x^3$$

$$= C_1x^5 \text{ which is the second solution of } L(y) = 0$$

Conveniently select $\varphi_2 = x^5$.

Step 3 : To check φ_1, φ_2 are independent. Now

$$W(\varphi_1, \varphi_2)(x) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$= \begin{vmatrix} x^3 & x^5 \\ 3x^2 & 5x^4 \end{vmatrix}$$

$$= 5x^7-3x^7$$

$$= 2x^7$$

$$\neq 0\ (\because x>0)$$

Since $W(\varphi_1, \varphi_2)(x)\neq 0$ then φ_1, φ_2 are independent.

Thus $\varphi_2=x^5$ is the required solution of $L(y)=0$.

Example E. 6.3 :

Verify that $\varphi_1(x)=e^x$, $(x>0)$ is a solution of $xy''-(x+1)y'+y=0$. Also find a second independent solution.

Solution :

Let $L(y) = xy''-(x+1)y'+y = 0$ be the given differential equation and

let $\varphi_1(x)=e^x$, $(x>0)$.

Step 1 : To check φ_1 is a solution of $L(y) = 0$.

$$\text{Now}\quad L(\varphi_1) = x\varphi_1''-(x+1)\varphi'+\varphi$$
$$= xe^x-(x+1)e^x+e^x$$
$$= 0$$

$\therefore$ φ_1 is a solution of $L(y) = 0$.

Step 2 : To find the second solution.

Let $\varphi_2 = u\varphi_1$ where u is some function.

If φ_2 is a solution of $L(y) = 0$ then $L(\varphi_2) = 0$.

(ie) $x\varphi_2''-(x+1)\varphi_2'+\varphi_2 = 0$

(ie) $x(u\varphi_1)''-(x+1)(u\varphi_1)'+(u\varphi_1) = 0$

(ie) $x(u''e^x+2u'e^x+ue^x)-(x+1)(u'e^x+ue^x)+ue^x = 0$

(ie) $u''xe^x+(xe^x-e^x)u' = 0$ ($\because$ $xe^x-(x+1)e^x+e^x=0$)

(ie) $v'xe^x+(xe^x-e^x)v = 0$ where $v=u'$.

Divide e^x on both sides, we get,

$v'x+(x-1)v = 0$

(ie) $v'x = (1-x)v$

(ie) $\dfrac{v'}{v} = \dfrac{1}{x} - 1$

Integrating on both sides, we get

$\log v = \log x - x + \log C$

(ie) $\log\left(\dfrac{v}{Cx}\right) = -x$

(ie) $y = Cxe^{-x}$

(ie) $u' = Cxe^{-x}$

(ie) $du = Cxe^{-x}dx$

Integrating on both sides, we get,

$$\int du = C\int xe^{-x}dx$$

$$\text{(ie)} \quad u = C[x(-e^{-x})+1.(-e^{x})]$$

$$= -Ce^{-x}(1+x)$$

$$\therefore \quad u\varphi_1 = -Ce^{-x}(1+x).e^{x}$$

$$= -C(1+x)$$

For conveniently choose $\varphi_2 = -(1+x)$.

which the required second solution of $L(y) = 0$.

Step 3 : To check φ_1, φ_2 are linearly independent.

$$\text{Now } W(\varphi_1, \varphi_2)(x) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^x & -(1+x) \\ e^x & -1 \end{vmatrix}$$

$$= xe^x$$

$$\neq 0 \ (\because x>0)$$

$\therefore \varphi_1, \varphi_2$ are linearly independent.

Example E. 6.4 :

Consider the differential equation $y''+\frac{1}{x}y'-\frac{1}{x^2}y=0$ for $x>0$.

(a) Show that there is a solution of the form x^r where r is a constant.

(b) Find two solutions of the differential equation and prove that they are linearly independent.

Solution :

Given that $L(y) = y''+\frac{1}{x}y'-\frac{1}{x^2}y=0$ is a differential equation.

Step 1 : To prove that $L(y)=0$ has a solution of the form x^r where r is a constant.

Let $\varphi(x) = x^r$.

If φ is a solution of $L(y) = 0$ then $L(\varphi) = 0$

(ie) $\varphi''+\frac{1}{x}\varphi'-\frac{1}{x^2}\varphi=0$

(ie) $r(r-1)x^{r-2}+\frac{1}{x}.rx^{r-1}-\frac{1}{x^2}x^r=0$

(ie) $(r(r-1)+r-1)x^{r-2}=0$

Since $x>0$ then $x^{r-2}>0$ and hence $r(r-1)+r-1=0$

(ie) $r^2=1$

(ie) $r=\pm 1$

$\therefore$ x, 1/x are the solution of L(y) = 0.

Let $\varphi_1(x)=x$, $\varphi_2(x)=1/x$.

Step 2 : To check φ_1, φ_2 are linearly independent.

$$\text{Now } W(\varphi_1,\varphi_2)(x) = \begin{vmatrix}\varphi_1 & \varphi_2\\ \varphi_1' & \varphi_2'\end{vmatrix}$$

$$= \begin{vmatrix}x & 1/x\\ 1 & -1/x^2\end{vmatrix}$$

$$= \frac{-2}{x} \neq 0 \quad (x>0)$$

$\therefore$ φ_1, φ_2 are linearly independent.

Hence x, 1/x are the required linearly independent solutions of L(y)=0.

Example E. 6.4 :

Find two linearly independent solutions of the equations

$(3x-1)^2y''+(9x-3)y'-9y=0$ for $x>1/3$.

Solution :

Given that $(3x-1)^2y''+(9x-3)y'-9y=0$ is a given differential equation for $x>1/3$.

Now $x>1/3 \Rightarrow 3x-1>0$.

Step 1 : To find the solution of the given differential equation.

$\therefore$ Given differential equation can be rewritten as

$$y''+\frac{3}{3x-1}y'-\frac{9}{(3x-1)^2}=0$$

(ie) $L(y) = 0$ where $L(y) = y''+\frac{3}{3x-1}y'-\frac{9}{(3x-1)^2}$

Let $X = 3x-1$, $Y = y$.

$$\therefore \frac{dX}{dx} = 3$$

$$\text{Now } y' = \frac{dy}{dx}$$

$$= \frac{dY}{dx}$$

$$= \frac{dY}{dx}.\frac{dX}{dx}$$

$$= 3\frac{dy}{dx}$$

$$\text{and } y'' = \frac{d^2y}{dx^2}$$

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dX}\left(3\frac{dY}{dX}\right).\frac{dX}{dx}$$

$$= 9\frac{d^2y}{dx^2}$$

Hence $L(y) = 0$ becomes $9\frac{d^2Y}{dX^2}+\frac{3}{x}.3\frac{dY}{dX}-\frac{9}{X^2}Y=0$

(ie) $\frac{d^2Y}{dX^2}+\frac{1}{X}\frac{dY}{dX}-\frac{Y}{X^2}=0$ for $x>0$ ------(6.38)

Let $\varphi = x^r$ be a solution of (6.38)

Then $\varphi'' + \frac{1}{X}\varphi' - \frac{1}{X^2}\varphi = 0$

(ie) $r(r-1)X^{r-2} + \frac{1}{X}rX^{r-1} - \frac{1}{X^2}X^r = 0$

(ie) $r^2 = 1$ ($\because$ x>0)

(ie) $r = \pm 1$

$\therefore$ X, 1/X are the solutions of (6.38).

Let $\varphi_1(x) = X = 3x-1$ and $\varphi_2(x) = \frac{1}{X} = \frac{1}{3x-1}$.

Step 2 : To verify that φ_1, φ_2 are linearly independent.

$$\text{Now } W(\varphi_1, \varphi_2)(x) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$= \begin{vmatrix} 3x-1 & \frac{1}{3x-1} \\ 3 & \frac{-3}{(3x-1)^2} \end{vmatrix}$$

$$= \frac{-6}{3x-1}$$

$$\neq 0 \ (3x-1 > 0)$$

$\therefore$ φ_1, φ_2 are linearly independent.

Hence $\varphi_1(x) = 3x-1$, $\varphi_2(x) = \frac{1}{3x-1}$ are the required independent solutions of L(y)=0.

Practice Problem :

If x^2, (x>0) is one solution of $y'' - \frac{2}{x^2}y = 0$, find the other solution such that the solutions are linearly independent.

6.6 The non-homogeneous equation :

Theorem 6.11 :

Let $L(y) = y^{(n)}+a_1(x)y^{(n-1)}+....+a_n(x)y = b(x)$

where $a_1, a_2,......,a_n$ b be continuous functions on an interval I. Let $\varphi_1, \varphi_2,.....,\varphi_n$ be a basis for the solutions of L(y)=0 on I. Then every solution Ψ of L(y) = b(x) can be written as

$\Psi = \Psi_p+C_1\varphi_1+C_2\varphi_2+.......+C_n\varphi_n$ where Ψ_p is a particular solution of L(y) = b(x) and $C_1, C_2,.....C_n$ are constants. Every such Ψ is a solution of L(y)=b(x). A particular solution Ψ_p is given by

$$\Psi_p(x) = \sum_{k=1}^{n}\varphi_k(x)\int_{x_0}^{x}\frac{W_k(t)b(t)}{W(\varphi_1,\varphi_2....,\varphi_n)(t)}dt \quad \text{where}$$

W_k is the determinant obtained from $W(\varphi_1, \varphi_2,......,\varphi_n)$ by replacing k^{th} column $(\varphi_k, \varphi_k',.....,\varphi_k^{(n-1)})$ by (0, 0, 0,.....,0, 1)

Proof :

Given that $L(y) = y^{(n)}+a_1(x)y^{(n-1)}+....+a_n(x)y = b(x)$

where $a_1, a_2,...,a_n$ be are continuous functions on an interval **I** and let Ψ_p is a particular solution of L(y) = b(x).

$\therefore L(\Psi_p) = b(x)$

If Ψ is any solution of L(y) = b(x) then $L(\Psi) = b(x)$.

$$\text{Now } L(\Psi-\Psi_p) = L(y)-L(\Psi_p)$$
$$= b(x)-b(x)$$
$$= 0$$

(ie) $\Psi-\Psi_p$ is a solution of L(y) = 0.

If $\varphi_1, \varphi_2,......,\varphi_n$ are a set of n linear independent solutions of L(y) = 0. Then

$\Psi-\Psi_p = C_1\varphi_1+C_2\varphi_2+.....+C_n\varphi_n$ where $C_1, C_2,....,C_n$ are constants.

(ie) $\Psi = \Psi_p+C_1\varphi_1+C_2\varphi_2+....+C_n\varphi_n$

Now we shall find the particular solution Ψ_p.

Let $\Psi_p = u_1\varphi_1+u_2\varphi_2+....+u_n\varphi_n$ ------(6.39)

be a particular solution of $L(y) = b(x)$ where $u_1, u_2,...,u_n$ are function of x.

Differentiate (6.39) with respect to x, we get,

$$\Psi_p' = (u_1\varphi_1'+u_1'\varphi_1)+(u_2\varphi_2'+u_2'\varphi_2)+....+(u_n\varphi_n'+u_n'\varphi_n)$$

$$= (u_1\varphi_1'+u_2\varphi_2'+.....+u_n\varphi_n')+(u_1'\varphi_1+u_2'\varphi_2+....+u_n'\varphi_n)$$

Thus if $u'\varphi_1+u_2'\varphi_2+......+u_n'\varphi_n = 0$ then $\Psi_p' = u_1\varphi_1'+u_2\varphi_2'+.....+u_n\varphi_n'$.

Similarly if $u_1'\varphi_1'+u_2'\varphi_2'+......+u_n'\varphi_n' = 0$ then $\Psi_p'' = u_1\varphi_1''+u_2\varphi_2''+....+u_n\varphi_n''$

Similarly proceeding above, we get,

$u_1\varphi_1^{(n-2)}+.....u_n\varphi_n^{(n-2)} = 0$ and

$u_n'\varphi_1^{(n-1)}+u_2'\varphi_2^{(n-1)}+......+u_n'\varphi_n^{(n-1)} = b$

Hence $u_1', u_2',......,u_n'$ satisfy

$$\left.\begin{array}{l} u_1'\varphi_1+u_2'\varphi_2+......+u_n'\varphi_n = 0 \\ u_1'\varphi_1'+u_2'\varphi_2'+......+u_n'\varphi_n' = 0 \\ u_1'\varphi_1''+u_2'\varphi_2''+......+u_n'\varphi_n'' = 0 \\ \bullet \\ \bullet \\ u_1'\varphi_1^{(n-2)}+u_2'\varphi_2^{(n-2)}+......+u_n'\varphi_n^{(n-2)} = 0 \\ u_1'\varphi_1^{(n-1)}+u_2'\varphi_2^{(n-1)}+......+u_n'\varphi_n^{(n-1)} = b \end{array}\right\} \qquad \text{------(6.40)}$$

and hence $\Psi_p = u_1\varphi_1+u_2\varphi_2+....+u_n\varphi_n$,

$\Psi_p' = u_1\varphi_1'+u_2\varphi_2'+....+u_n\varphi_n'$

$\Psi_p'' = u_1\varphi_1''+u_2\varphi_2''+....+u_n\varphi_n''$

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$\Psi_p^{(n-1)} = u_1\varphi_1^{(n-1)}+u_2\varphi_2^{(n-1)}+....+u_n\varphi_n^{(n-1)}$

& $\Psi_p^{(n)} = u_1\varphi_1^{(n)}+u_2\varphi_2^{(n)}+....+u_n\varphi_n^{(n)}+b$

$$\text{Again } L(\Psi_p) = u_1L(\varphi_1)+u_2L(\varphi_2)+\ldots.+u_nL(\varphi_n)+b$$

$$= b$$

$$\therefore L(\Psi_p) = b(x), x\in I.$$

Now we shall find the functions $u_1, u_2, \ldots\ldots, u_n$ satisfying (6.40).

Solving the equation in (6.40), we get,

$$u_1' = \begin{vmatrix} 0 & \varphi_2 & \varphi_3 & \ldots\ldots & \varphi_m \\ 0 & \varphi_2' & \varphi_3' & \ldots\ldots & \varphi_n' \\ \ldots\ldots & & & & \\ 0 & \varphi_2^{(n-2)} & \varphi_3^{(n-2)} & \ldots\ldots & \varphi_n^{(n-2)} \\ 1 & \varphi_2^{(n-1)} & \varphi_3^{(n-1)} & \ldots\ldots & \varphi_n^{(n-1)} \end{vmatrix} \div W$$

where $W = W(\varphi_1, \varphi_2, \ldots\ldots, \varphi_n)(x)$

$$\text{(ie) } u_1' = \frac{b(x)W_1}{W} \text{ where } W_1 = \begin{vmatrix} 0 & \varphi_2 & \varphi_3 & \ldots\ldots & \varphi_m \\ 0 & \varphi_2' & \varphi_3' & \ldots\ldots & \varphi_n' \\ \ldots\ldots & & & & \\ 0 & \varphi_2^{(n-2)} & \varphi_3^{(n-2)} & \ldots\ldots & \varphi_n^{(n-2)} \\ 1 & \varphi_2^{(n-1)} & \varphi_3^{(n-1)} & \ldots\ldots & \varphi_n^{(n-1)} \end{vmatrix}$$

Note that W_1 be obtained from W by replacing the first column of W by (0, 0, 0,.....,1)

If x_0 is a point in I then

$$u_1 = \int_{x_0}^{x} \frac{b(t)W_1(t)}{W(t)}\,dt$$

$$\text{In general, } u_k = \int_{x_0}^{x} \frac{b(t)W_k(t)}{W(t)}\,dt$$

where $W_k(t)$ is the determinent obtained from W by replacing k^{th} column of W by (0, 0, 0,.....,0, 1)

$$\therefore \Psi_p = u_1\varphi_1+u_2\varphi_2+\ldots.+u_n\varphi_n$$

$$= \sum_{k=1}^{n} u_k\varphi_k$$

$$= \sum_{k=1}^{n} \varphi_k \int_{x_0}^{x} \frac{b(t)W_k(t)}{W(t)} dt$$

This proves the theorem.

Example E. 6.5 :

One solution of $x^2y''-2y = 0$ on an interval $0<x<\infty$ is $\varphi_1(x)=x^2$. Find all solution of $x^2y''-2y = 2x-1$ on the interval $0<x<\infty$.

Solution :

Given that $\varphi_1(x) = x^2$, $(0<x<\infty)$ is a solution of $x^2y''-2y=0$

Step 1 : To find the second solution of $L(y) = x^2y''-2y = 0$.

Let $\varphi_2 = u\varphi_1$ where u is some function of x.

If φ_2 is a solution of $L(y) = 0$ then $L(\varphi_2) = 0$.

(ie) $x^2\varphi_2''-2\varphi_2 = 0$

(ie) $x^2(u\varphi_1)''-2(u\varphi_1) = 0$

(ie) $x^2(u''x^2+2u'2x+2u)-2ux^2 = 0$

(ie) $u''x^2+4u'x = 0$

(ie) $v'x^2+4xv = 0$ where $v = u'$

(ie) $\frac{v'}{v} = \frac{-4}{x}$

Integrating an both sides, we get,

$\log v = \log(Cx^{-4})$

(ie) $v = Cx^{-4}$

(ie) $u' = Cx^{-4}$

(ie) $du = Cx^{-4}dx$

Again integrating on both sides, we get,

$$u = C\frac{1}{-3x^3}$$

$$= C_1 \cdot \frac{1}{x^3} \text{ where } C_1 = \frac{-C}{3}$$

For our convenience consider $u = \frac{1}{x^3}$

$$\therefore \varphi_2(x) = u(x).\varphi_1(x)$$
$$= \frac{1}{x^3}.x^2$$
$$= \frac{1}{x}$$

(ie) $x^2, \frac{1}{x}$ are the solution of $x^2y''-2y = 0$

Step 2 :

To find the solutions of the non-homogeneous differential equation $x^2y''-2y=2x-1$.

$$\text{(ie) } y''-\frac{2}{x^2}y = \frac{2}{x}-\frac{1}{x^2} \qquad \text{------(6.41)}$$

From step 1, $x^2, \frac{1}{x}$ are the solution of $y''-\frac{2}{x^2}y=0$.

$\therefore$ a solution Ψ_p of (6.41) is of the form

$\Psi_p = u_1\varphi_1+u_2\varphi_2$ where u_1', u_2' satisfy the equations

$$\left.\begin{aligned} u_1'\varphi_1+u_2'\varphi_2 &= 0 \\ \text{and } u_1'\varphi_1'+u_2'\varphi_2' &= b(x) \end{aligned}\right\} \qquad \text{------(6.42)}$$

where $b(x) = \frac{2}{x}-\frac{1}{x^2}$

Here $\varphi_1(x) = x^2, \varphi_2(x) = \frac{1}{x}$

$\therefore \varphi_1'(x) = 2x, \varphi_2'(x) = \frac{-1}{x^2}$

$\therefore$ (6.42) becomes,

$$\left.\begin{aligned} x^2u'+\frac{1}{x}u_2' &= 0 \\ 2xu'-\frac{1}{x^2}u_2' &= 0 \end{aligned}\right\} \qquad \text{------(6.43)}$$

Now we shall solve the equations in (6.43), we get,

$$W = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & \frac{1}{x} \\ 2x & \frac{-1}{x^2} \end{vmatrix}$$

$$= -3 \ (\neq 0)$$

$$\text{Again } W_1(x) = \begin{vmatrix} 0 & \frac{1}{x} \\ 1 & \frac{-1}{x^2} \end{vmatrix}$$

$$= \frac{-1}{x}$$

$$\text{and } W_2(x) = \begin{vmatrix} x^2 & 0 \\ 2x & 1 \end{vmatrix}$$

$$= x^2$$

$$\therefore \ u_1' = \frac{b(x)W_1(x)}{W}$$

$$= \frac{\left(\frac{2}{x} - \frac{1}{x^2}\right)\left(\frac{-1}{x}\right)}{-3}$$

$$= \frac{1}{3}\left[\frac{2}{x^2} - \frac{1}{x^3}\right]$$

$$\therefore \ u_1 = \frac{1}{3}\int\left(\frac{1}{x^2} - \frac{1}{x^3}\right)dx$$

$$= \frac{1}{3}\left(-\frac{2}{x} + \frac{1}{2x^2}\right)$$

$$= -\frac{2}{3x} + \frac{1}{6x^2}$$

$$\text{Similarly } u_2' = \frac{b(x).W_2(x)}{W(x)}$$

$$= \frac{\left(\frac{2}{x} - \frac{1}{x^2}\right)(x^2)}{-3}$$

$$= \frac{-1}{3}(2x-1)$$

$$\therefore \quad u_2 = \frac{-1}{3}\int(2x-1)dx$$

$$= \frac{-1}{3}[x^2-x]$$

$$\text{Hence } \Psi_p = u_1\varphi_1+u_2\varphi_2$$

$$= \left(\frac{-2}{3x}+\frac{1}{6x^2}\right)x^2+\left(\frac{-1}{3}\right)\left(x^2-x\right)\left(\frac{1}{x}\right)$$

$$= \frac{1}{2}-x$$

$\therefore$ The required solution of the non-homogenous differential equation is

$$\Psi = \Psi_p+C_1\varphi_1+C_2\varphi_2$$

$$\text{(ie) } \Psi = C_1x^2+C_2\frac{1}{x}-x+\frac{1}{2}$$

Example E.6.16 :

One solution of $x^2y''-2y=0$ on an interval $0<x<\infty$ is $\varphi_1(x)=x^2$, find all the solutions of $x^2y''-2y=x^3$ on the interval $0<x<\infty$.

Solution :

Given that $\varphi_1(x)=x_2$, $(x>0)$ is one of the solutions of $x^2y-2y=0$.

Step 1 :

First we shall find the second solution of the homogeneous differential equation of $x^2y''-2y=0$.

Let $\varphi_2=u\varphi_1$ where u_1 is some function of x. If φ_2 is a solution of $L(y) = x^2y''-2y=0$ then $L(\varphi_2)=0$.

(ie) $x^2\varphi_2''-2\varphi_2 = 0$

(ie) $x^2(u\varphi_1)''-2(u\varphi_1) = 0$

(ie) $x^2(ux^2)''-2(ux^2) = 0$

(ie) $u''x^2+4u'x = 0$

(ie) $v'x^2+4vx = 0$ where $v=u'$.

On solving the above differential equation, we get,

$$\therefore \quad v = Cx^{-4}$$

$$\text{(ie)} \quad u' = Cx^{-4}$$

$$\therefore \quad u = C.\frac{x^{-3}}{-3}$$

$$= C_1x^{-3} \text{ where } C_1 = \frac{-C}{3}$$

For our convenience choose $u = 1/x^3$,

$$\therefore \varphi_2 = u\varphi_1 = 1/x$$

Hence x^2, $1/x$ are the solutions of the homogeneous differential equation $L(y)=0$

Step 2 :

To find the solutions of non-homogeneous differential equation $x^2y''-2y = x^3$

$$\text{(ie)} \qquad y''-\frac{2}{x^2}y = x \qquad \text{------(6.44)}$$

Let Ψ_p be a solution of (6.44)

Here $\Psi_p = u_1\varphi_1+u_2\varphi_2$ where u_1', u_2' satisfy the equations,

$$\left.\begin{aligned} u_1'\varphi_1+u_2'\varphi_2 &= 0 \\ u_1'\varphi_1'+u_2'\varphi_2' &= b(x) \end{aligned}\right\} \qquad \text{-------(6.45)}$$

Here $\varphi_1(x) = x2$, $\varphi_2(x) = \frac{1}{x}$, $b(x) = x$

$$\therefore \varphi_1'(x) = 2x, \ \varphi_2'(x) = \frac{-1}{x^2}$$

$\therefore$ (6.45) becomes

$$\left.\begin{aligned} x^2u_1'+\frac{1}{x}u_2' &= 0 \\ 2xu_1'-\frac{1}{x^2}u_2' &= 0 \end{aligned}\right\} \qquad \text{-------(6.46)}$$

Solving the equation in (6.46), we get,

$$W(x) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & \frac{1}{x} \\ 2x & \frac{-1}{x^2} \end{vmatrix}$$

$$= -3$$

$$W_1(x) = \begin{vmatrix} 0 & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

$$= -\frac{1}{x}$$

$$W_2(x) = \begin{vmatrix} x^2 & 0 \\ 2x & 1 \end{vmatrix} = x^2$$

$$\text{Now } u_1' = \frac{b(x)W_1(x)}{W}$$

$$= \frac{x\left(-\frac{1}{x}\right)}{-3}$$

$$= \frac{1}{3}$$

$$\therefore \quad u_1 = \frac{1}{3}x$$

$$\text{and } \quad u_2' = \frac{b(x).W_2(x)}{W}$$

$$= \frac{x(x^2)}{-3}$$

$$= -\frac{1}{3}x^3$$

$$\therefore \quad u_2 = -\frac{x^4}{12}$$

$$\text{Thus } \Psi_p = u_1\varphi_1 + u_2\varphi_2 = \frac{x^3}{4}$$

Hence the solutions of the non-homogeneous equation $x^2y''-2y=x^3$ in the interval $0<x<\infty$ are given by $\Psi = \Psi_p+C_1\varphi_1+C_2\varphi_2$ (ie) $\Psi = C_1x^2+C_2.\frac{1}{x}+\frac{x^3}{4}$ where C_1, C_2 are constants.

Example E. 6.7 :

One solution of $x^2y''-xy'+y = 0$ is $\varphi_1(x)=x$ where $x>0$. Find the solution Ψ of $x^2y''-xy'+y=x^2$ satisfying $\Psi(1)=1$, $\Psi'(1)=0$.

Solution :

Step 1 : To find the solution of the homogeneous differential equation $x^2y''-xy'+y=0$.

Given that $\varphi_1(x)=x$ is one of the solutions of $x^2y''-xy'+y=0$, $(x>0)$

Let $\varphi_2=u\varphi_1$, where u is some function of x.

If φ_2 is a solution of $x^2y''-xy'+y = 0$ then we have, $x^2\varphi_2''-x\varphi_2'+\varphi_2 = 0$.

(ie) $x^2(ux)''-x(ux)'+(ux) = 0$

(ie) $x^2(u''x+2u')-x(u'x+u)+ux = 0$

(ie) $u''x^3+u'x^2 = 0$

(ie) $v'x3+vx^2 = 0$ where $v=u'$.

(ie) $\frac{v'}{v} = -\frac{1}{x}$

Integrating on both sides, we get, $\log v = -\log x + \log C$.

(ie) $v = \frac{C}{x}$

(ie) $u' = \frac{C}{x}$

(ie) $du = \frac{C}{x}dx$

Integrating on both sides, we get,

$\int du = C\int\frac{dx}{x}$

(ie) $u = C \log x$.

For our convenience, take $u = \log x$ and therefore $\varphi_2 = u\varphi_1 = x \log x$

Hence x, $x \log x$ are the solutions of $x^2y''-xy'+y = 0$

Step 2 : To find the solution of the non-homogenous solution of $x^2y''-xy'+y = x^2$.

(ie) $y'' - \frac{1}{x}y' + \frac{y}{x^2} = 1$ -------(6.47)

Let Ψ_p be the particular solution of (6.47).

Then $\Psi_p = u_1\varphi_1+u_2\varphi_2$ which u_1', u_2' satisfy the equations

$$\left.\begin{aligned} u_1'\varphi_1+u_2'\varphi_2 &= 0 \\ \text{and} \quad u_1'\varphi_1'+u_2'\varphi_2' &= 0 \end{aligned}\right\} \qquad \text{-------(6.49)}$$

Solve the equations in (6.49), we have,

$$W = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$= \begin{vmatrix} x & x \log x \\ 1 & 1+\log x \end{vmatrix} = x$$

$$\therefore \quad W_1 = \begin{vmatrix} 0 & \varphi_2 \\ 1 & \varphi_2' \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x \log x \\ 1 & 1+\log x \end{vmatrix} = -x \log x$$

$$\text{and} \quad W_2 = \begin{vmatrix} \varphi_1 & 0 \\ \varphi_1' & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x & 0 \\ 1 & 1 \end{vmatrix} = x$$

$$\text{Hence } u_1' = \frac{b(x)W_1(x)}{W(x)}$$

$$= \frac{1(-x \log x)}{x}$$

$$= -\log x$$

$$\text{(ie)}\ du_1 = -\log x\, dx$$

Integrating on both sides, we get,

$$\int du_1 = -\int \log x\, dx$$

$$\text{(ie)}\ u_1 = -[x \log x - x]$$

$$\text{Again}\ u_2' = \frac{b(x) W_2(x)}{W(x)}$$

$$= \frac{1(x)}{x}$$

$$= 1$$

$$\therefore\ du_2 = dx$$

$$\Rightarrow \int du_2 = \int dx$$

$$\Rightarrow u_2 = x$$

$$\therefore\ \Psi_p = u_1\varphi_1 + u_2\varphi_2 = x^2.$$

Hence the general solution of (6.47) is

$$\Psi = \Psi_p + C_1\varphi_1 + C_2\varphi_2 \text{ where } C_1, C_2 \text{ are constants.}$$

$$\text{(ie)}\ \Psi = C_1 x + C_2 x \log x + x^2 \qquad \text{------(6.50)}$$

Step 3 : To find the values of C_1 & C_2.

$$\text{Now}\ \Psi = C_1 x + C_2 x \log x + x^2$$

$$\therefore\ \Psi' = C1 + C_2(1+\log x) + 2x$$

$$\text{Given that}\ \Psi(1) = 1$$

$$\text{(ie)}\ C_1 + C_2(1) \log x + 1 = 1$$

$$\text{(ie)}\ \mathbf{C_1 = 0}$$

$$\text{Again given that}\ \Psi'(1) = 0$$

$$\text{(ie)}\ C_1 + C_2(1+\log 1) + 2(1) = 0$$

$$\text{(ie)}\ 0 + C_2 = -2$$

$$\text{(ie)}\ \mathbf{C_2 = -2}$$

Hence $\Psi = x^2 - 2x \log x$ which is the required solution of (6.47).

Example E. 6.8 :

Consider the equation $y''+y=b(x)$ where $b(x)$ is a continuous function on $1\le x<\infty$ satisfying $\int_1^\infty |b(t)|dt < \infty$

(a) Show that a particular solution Ψ_p is given by $\Psi_p(x) = \int_1^x \sin(x-t)b(t)dt$.

(b) Show that any solution is bounded on this interval $1\le x<\infty$.

Proof :

Consider the homogeneous differential equation $y''+y=0$ ------(6.51)

The auxiliary equation is $m^2+1 = 0$

(ie) $m = \pm i$

$\therefore$ The solution of (6.51) is $y = A\cos x+B\sin x$.

(ie) $A\varphi_1(x)+B\varphi_2(x)$ where $\varphi_1(x)=\cos x$, $\varphi_2(x)=\sin x$.

let Ψ_p be a particular solution of the non-homogeneous equation

$$y''+y = b(x) \quad \text{------(6.52)}$$

Let $\Psi_p = u_1\varphi_1+u_2\varphi_2$ where u_1', u_2' satisfy the equations

$$\left.\begin{aligned} u_1'\varphi_1+u_2'\varphi_2 &= 0 \\ u_1'\varphi_1'+u_2'\varphi_2' &= b(x) \end{aligned}\right\} \quad \text{------(6.53)}$$

Here $\varphi_1(x) = \cos x$, $\varphi_2(x) = \sin x$

$\therefore \varphi_1'(x) = -\sin x$, $\varphi_2'(x) = \cos x$.

Thus (6.53) becomes

$$\left.\begin{aligned} \cos x.u_1'+\sin x\, u_2' &= 0 \\ -\sin x\, u_1'+\cos x\, u_2' &= b(x) \end{aligned}\right\} \quad \text{------(6.54)}$$

Solving the equations in (6.54), we have,

$$W = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos x & \sin x \\ \sin x & \cos x \end{vmatrix} = \cos^2x+\sin^2x = 1$$

$$\text{Now } W_1(x) = \begin{vmatrix} 0 & \varphi_2 \\ 1 & \varphi_2' \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \sin x \\ 1 & \cos x \end{vmatrix} = -\sin x$$

$$\text{and } W_2(x) = \begin{vmatrix} \varphi_1 & 0 \\ \varphi_1' & 1 \end{vmatrix} = \varphi_1 = \cos x$$

$$\therefore u_1' = \frac{b(x).W_1(x)}{W(x)}$$

$$= -\sin x.b(x)$$

$$\text{Hence } u_1 = -\int_1^x \sin t.b(t)dt$$

$$\text{Similarly, } u_2' = \frac{b(x)W_2(x)}{W(x)}$$

$$= b(x)\cos x$$

$$\therefore u_2 = \int_1^x b(t).\cos t dt$$

$$\text{Thus } \Psi_p = u_1\varphi_1+u_2\varphi_2$$

$$= -\cos x\int_1^x b(t)\sin t dt + \sin x\int_1^x b(t)\cos t dt$$

$$= \int_1^x (\sin x\cos t - \cos t\sin t)b(t)dt$$

$$= \int_1^x \sin(x-t).b(t)dt$$

This proves (a).

Proof of (b) :

Any solution Ψ of $L(y) = b(x)$ is of the form $\Psi = \Psi_p+C_1\varphi_1+C_2\varphi_2$ where C_1, C_2 are constants.

$$\text{Now } |\Psi| \leq |\Psi_p|+|C_1||\varphi_1|+|C_2||\varphi_2|$$

$$\leq \int_1^x |b(t)|dt+|C_1|+|C_2| \quad (\because |\sin x|\leq 1, |\cos x|\leq 1)$$

$$< \infty \quad \left(\because \int_1^x |b(t)|dt < \infty\right)$$

(ie) Ψ is bounded in the interval $1\leq x<\infty$.

This proves the problem.

6.7 Justification of the power series method :

Theorem 6.12 : (Existence Theorem for Analytic Coefficients)

Let x_0 be a real number and suppose that the co-efficients $a_1, a_2, \ldots, a_n$ in $L(y)=y^{(n)}+a_1(x)y^{(n-1)}+\ldots+a_n(x)y$ have convergent power series expansions in powers of $x-x_0$ on an interval $|x-x_0|<r_0$, $r_0>0$.

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are any n constants, there exits a solution φ of the problem $L(y)=0$, $y(x_0)=\alpha_1$, $y'(x_0)=\alpha_2, \ldots, y^{(n-1)}(x_0)=\alpha_n$, with a power series expansion

$$\varphi(x) = \sum_{k=0}^{\infty} C_k(x-x_0)^k \qquad \text{------(6.55)}$$

convergent for $|x-x_0|<r_0$. Further we have,

$$k!.C_k = \alpha_{k+1}, \ (k=0,1,2,3,\ldots\ldots,n-1) \text{ and}$$

C_k for $k\geq n$ may be computed in terms of $C_0, C_1, \ldots, C_{n-1}$ by substituting the series (6.55) into $L(y)=0$.

Proof :

We shall prove the theorem for n=2 and x=0 and the general case is similar to this particular case.

$$\text{Let } L(y) = y''+a(x)y'+b(x)y = 0 \qquad \text{------(6.56)}$$

$$\left.\begin{aligned} \text{where } a(x) &= \sum_{k=0}^{\infty} \alpha_k x^k, \\ b(x) &= \sum_{k=0}^{\infty} \beta_k x^k \end{aligned}\right\} \qquad \text{------(6.57)}$$

which converge for $|x|<r_0$ for some $r_0>0$.

Claim : For any given two constants a_1, a_2 we want to produce a solution of (6.56) satisfying $\varphi(0)=a_1$, $\varphi'(0)=-a_2$ and $\varphi(x)=\sum_{k=0}^{\infty} c_k x^k$ --------(6.58)

where the series converges for $|x|<r_0$.

If (6.58) is convergent then we have, $c_0=a_1$, $c_1=a_2$ and the constants $c_k(k\geq 2)$ must satisfy a recursion relation.

A low differentiate (6.58) twice with respect to x, we have,

$$\varphi'(x) = \sum_{k=0}^{\infty}(k+1)c_{k+1}x^k$$

$$\text{and } \varphi''(x) = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k \qquad \text{------(6.59)}$$

$$\therefore \quad a(x).\varphi'(x) = \left(\sum_{k=0}^{\infty}\alpha_k x^k\right)\left(\sum_{k=0}^{\infty}(k+1)\,c_{k+1}x^k\right)$$

$$= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\alpha_{k-j}(j+1).\,c_{j+1}\right)x^k \qquad \text{------(6.60)}$$

$$\text{and} \quad b(x).\varphi(x) = \left(\sum_{k=0}^{\infty}\beta_k x^k\right)\left(\sum_{k=0}^{\infty}c_k x^k\right)$$

$$= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\beta_{k-j}\,c_j\right)x^k \qquad \text{------(6.61)}$$

Adding (6.59), (6.60) & (6.61), we have,

$$L(\varphi)(x) = \sum_{k=0}^{\infty}\left[(k+2)(k+1)\,c_{k+2} + \sum_{j=0}^{k}\left(\alpha_{k-j}(j+1)\,c_{j+1} + \beta_{k-j}\,c_j\right)\right]x^k = 0$$

Equating the coefficients of x^k is zero, we get,

$$(k+2)(k+1)c_{k+2} = -\sum_{j=0}^{k}\left[\alpha_{k-j}(j+1)c_{j+1} + \beta_{k-j}c_j\right] \qquad \text{------(6.62)}$$

$$(k=0,1,2,.....)$$

Claim : $\sum_{k=0}^{\infty}C_k x^k$ is convergent for $|x| < r_0$ ------(6.63)

Let r be any number satisfying $0<r<r_0$.

Since the series $\sum_{k=0}^{\infty}\alpha_k x^k$, $\sum_{k=0}^{\infty}\beta_k x^k$ are convergent for $|x|=r$, we have a real constant M>0 such that $|\alpha_j|r^j \le M$ and $|\beta_j|r^j \le M$; (j=0,1,2,.......) ------(6.64)

From (6.62) & (6.64), we have,

$$(k+2)(k+1)|c_{k+2}| \le \frac{M}{r^k}\sum_{j=0}^{k}\left[(j+1)|c_{j+1}| + |c_j|\right]r^j$$

$$\leq \frac{M}{r^k}\sum_{j=0}^{k}\left[(j+1)|c_{j+1}|+|c_j|\right]r^j + M|c_{k+1}|r \qquad \text{------(6.65)}$$

Let us define $c_0 = |c_0|$, $C_1=|c_1|$ and C_k for $k\geq 2$ by

$$(k+2)(k+1)c_{k+2} = \frac{M}{r^k}\sum_{j=0}^{k}\left[(j+1)C_{j+1}+C_j\right]r^j + MC_{k+1}r \qquad \text{------(6.66)}$$

$$(k=0,1,2,\ldots\ldots)$$

Comparing (6.65) & (6.66), we have,

$$|C_k| \leq C_k,\ C_k \geq 0\ (k=0,1,2,\ldots\ldots) \qquad \text{------(6.67)}$$

We shall find that values for x the series

$$\sum_{k=0}^{\infty} c_k x^k \qquad \text{------(6.68)}$$

is convergent.

From (6.66), we find that,

$$(k+1)kc_{k+1} = \frac{M}{r^{k-1}}\sum_{j=0}^{k-1}\left[(j+1)C_{j+1}+C_j\right]r^j + MC_k r$$

$$\text{and } k(k-1)C_k = \frac{M}{r^{k-2}}\sum_{j=0}^{k-2}\left[(j+1)C_{j+1}+C_j\right]r^j + MC_{k-1}r, \text{ for large k.}$$

From these expression we obtain

$$r(k+1)kC_{k+1} = \frac{M}{r^{k-2}}\sum_{j=0}^{k-2}\left[(j+1)C_{j+1}+C_j\right]r^j + M\left[kC_k + C_{k-1}\right]r + MC_k r^2$$

$$= k(k-1)C_k - MC_{k-1}r + M_kC_kr + MC_{k-1}r + MC_kr^2$$

$$= [k(k-1)+Mkr+Mr^2]C_k$$

$$\therefore \left|\frac{C_{k+1}x^{k+1}}{C_kx^k}\right| = \left[\frac{k(k-1)+Mkr+Mr^2}{r(k+1)k}\right]|x|.$$

which tend to $\frac{|x|}{r}$ as $k\to\infty$.

Hence by ratio test, the series (6.68) is converges for $|x|<r$.

Which implies that the series (6.63) converges for $|x|<r$.

Since $0<r<r_0$ is arbitrary, the series (6.63) converges for $|x|<r_0$.

This proves the theorem.

Example E. 6.9 :

Find two linearly independent power series solution of $y''-xy=0$.

Solution :

Given that $y''-xy=0$.

Here $a_1(x)=0$, $a_2(x)=-x$, which are analytic for all real x_0.

$$\text{Let } \varphi(x) = \sum_{k=0}^{\infty} C_k x^k$$

$$= C_0+C_1x+C_2x^2+\ldots\ldots \text{ be a solution of } L(y)=y''-xy=0$$

$$\therefore L(\varphi) = 0$$

$$\text{(ie) } \varphi''-x\varphi = 0 \qquad \text{-------(6.69)}$$

$$\text{Now } \varphi(x) = \sum_{k=0}^{\infty} C_k x^k$$

$$= \varphi'(x) = \sum_{k=1}^{\infty} kC_k x^{k-1}$$

$$= \sum_{k=0}^{\infty} (k+1)C_{k+1}x^k$$

$$\text{and } \varphi''(x) = \sum_{k=0}^{\infty} k(k-1)C_k x^{k-2}$$

$$= \sum_{k=0}^{\infty} (k+2)(k+1)C_{k+2}x^k$$

$$\text{Now } x\varphi(x) = x\sum_{k=0}^{\infty} C_k x^k$$

$$= \sum_{k=0}^{\infty} C_k x^{k+1}$$

$$= \sum_{k=0}^{\infty} C_{k-1} x^k$$

$\therefore$ (6.69) becomes,

$$\sum_{k=0}^{\infty}(k+2)(k+1)C_{k+2}x^k - \sum_{k=1}^{\infty} C_{k-1}x^k = 0.$$

$$\text{(ie) } (2)(1)\,C_2 + \sum_{k=1}^{\infty}(k+2)(k+1)C_{k+2}x^k - \sum_{k=1}^{\infty} C_{k-1}x^k = 0$$

$$\text{(ie) } 2C_2 + \sum_{k=1}^{\infty}\left[(k+2)(k+1)C_{k+2} - C_{k-1}\right]x^k = 0$$

Comparing the coefficients of x^k on both sides, we get,

$2C_2 = 0$, and $(k+2)(k+1)C_{k+2} - C_{k-1} = 0$

(ie) $C_2 = 0$ and $C_{k+2} = \dfrac{C_{k-1}}{(k+2)(k+1)}$, k=1,2,3,.....

when k=1, then $C_3 = \dfrac{C_0}{3.2}$

when k=2, then $C_4 = \dfrac{C_1}{4.3}$

when k=3, then $C_5 = \dfrac{C_2}{5.4} = 0.\ (\because\ C_2 = 0)$

when k=4, then $C_6 = \dfrac{C_3}{6.5} = \dfrac{C_0}{6.5.3.2}$

when k=5, then $C_7 = \dfrac{C_4}{7.6} = \dfrac{C_1}{7.6.4.3}$

when k=5, then $C_8 = \dfrac{C_5}{8.7} = 0$

$\therefore$ By induction,

$$C_{3m} = \frac{C_0}{2.3.5.6.........(3m-1).3m}, \quad m = 1,2,3,....$$

$$C_{3m+1} = \frac{C_1}{3.4.6.7.......3m(3m+1)}, \quad m = 1,2,3,.......$$

$$\text{and } C_{m+2} = 0; \quad m = 0,1,2,3,......$$

$$\text{Thus } \varphi(x) = C_0\left[1+\frac{x^3}{3.2}+\frac{x^6}{6.5.3.2}+.....\right]+C_1\left[x+\frac{x^4}{4.3}+\frac{x^7}{7.6.4.3}+....\right]$$

$$= C_0\varphi_1(x)+C_1\varphi_2(x)$$

$$\text{where } \varphi_1(x) = 1+\sum_{m=1}^{\infty}\frac{x^{3m}}{2.3.5.6.....(3m-1).3m}$$

$$\text{and } \varphi_2(x) = x+\sum_{m=1}^{\infty}\frac{x^{3m+1}}{3.4.6.7......3m(3m+1)}$$

which are the required linearly independent power series solutions of L(y)=0, because

$$W(\varphi_1, \varphi_2)(0) = \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix} = 1 \neq \infty.$$

Example E. 6.9 :

Find two linearly independent power series solutions (in powers of x) of the equation $y''-xy'+y=0$.

Solution :

Given that $y''-xy'+y=0$

Here $a_1(x)=-x$, $a_2(x)=1$ which are analytic for all real x_0.

Let $\varphi(x) = \sum_{k=0}^{\infty} C_k x^k$ be a solution of $L(y) = y''-xy'+y=0$.

$$\therefore L(\varphi) = 0$$

$$\text{(ie) } \varphi''-x\varphi'+\varphi = 0 \qquad \text{-------(6.70)}$$

$$\text{Now } \varphi(x) = \sum_{k=0}^{\infty} C_k x^k$$

$$\therefore \varphi'(x) = \sum_{k=1}^{\infty} kC_k x^{k-1}$$

$$\text{and } \varphi''(x) = \sum_{k=2}^{\infty} k(k-1)C_k x^{k-2}$$

$$\therefore x\varphi'(x) = \sum_{k=1}^{\infty} kC_k x^k$$

and hence (6.70) changes as

$$\sum_{k=0}^{\infty}(k+2)(k+1)C_{k+2}x^k - \sum_{k=1}^{\infty} kC_k x^k + \sum_{k=0}^{\infty} C_k x^k = 0$$

$$\text{(ie) } 2C_2 + C_0 + \sum_{k=1}^{\infty}\{(k+2)(k+1)C_{k+2} - kC_k + C_k\}x^k = 0$$

Comparing constant terms and the coefficients of x^k on both sides, we get,

$$2C_2+C_0 = 0$$

$$\text{(ie) } C_2 = \frac{-C_0}{2} \qquad \text{------(6.71)}$$

and $(k+2)(k+1)C_{k+2}-(k-1)C_k = 0, \quad k = 1,2,3,\ldots\ldots$

$$\text{(ie) } C_{k+2} = \frac{(k-1)C_k}{(k+2)(k+1)}, \quad k=1,2,3,\ldots\ldots$$

$$\text{when } k=1, \text{ then } C_3 = 0$$

$$\text{when } k=2, \text{ then } C_4 = \frac{C_2}{4.3}$$

$$= \frac{-C_0}{4.3.2}$$

$$= \frac{-C_0}{2^2(1.2)3}$$

$$= \frac{-C_0}{2^2.2!3!}$$

$$\text{when } k = 3, \text{ then } C_5 = \frac{2C_3}{5.4} = 0 \quad (\because C_3 = 0)$$

$$\text{when } k = 4, \text{ then } C_6 = \frac{3C_4}{6.5}$$

$$= \frac{-3.C_0}{2^2.2!.3.6.5}$$

$$= \frac{-C_0}{2^3.3!.5}$$

$$\text{when } k = 5, \text{ then } C_7 = \frac{5C_6}{7.6} = 0$$

$$\text{when } k = 6, \text{ then } C_8 = \frac{5C_6}{8.7}$$

$$= \frac{-5.C_0}{8.7.2^3.3!.5}$$

$$= \frac{-C_0}{2^4.4!.7} \text{ and so on.}$$

In general $C_{2m+1} = 0$, $m = 0, 1, 2, \ldots\ldots$

$$\text{and } C_{2m} = \frac{-C_0}{2^m.m!(2m-1)}, \quad m=1,2,3,\ldots\ldots$$

$$\therefore \quad \varphi = C_0+C_1x+C_2x^2+\ldots\ldots\ldots\ldots$$

$$= C_0+C_1x+C_2x^2+C_4x^4+\ldots\ldots$$

$$= C_0 + C_1x + \sum_{m=1}^{\infty} C_{2m}x^{2m}$$

$$= C_0 + C_1x + (-C_0) \sum_{m=1}^{\infty} \frac{1}{2^m.m!(2m-1)} x^{2m}$$

$$= C_1x + C_0\left[1 - \sum_{m=1}^{\infty} \frac{x^{2m}}{2^m(2m-1)m!}\right]$$

$$= C_1\varphi_1(x) + C_0\varphi_2(x)$$

where $\varphi_1(x) = x$ and

$$\varphi_2(x) = 1 - \sum_{m=1}^{\infty} \frac{1}{2^m(2m-1)m!}$$

$$\text{Since } W(\varphi_1, \varphi_2)(0) = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$

$$= 1$$

$$\neq 0.$$

& $\therefore$ φ_1, φ_2 are the required two independent solutions of $L(y) = 0$.

6.8 LEGENDRE EQUATION

Definition D. 6.5 :

The equation $(1-x^2)y''-2xy'+\alpha(\alpha+1)y = 0$ where α is a constant is called **Legendre equation.**

Example E. 6.10 :

Find two linearly independent solutions of the Legendre equation where $|x|<1$.

Solution :

We know that the Legendre equation as $(1-x^2)y''-2xy'+\alpha(\alpha+1)y = 0$ ------(6.72)

Let $\varphi(x) = \sum_{k=0}^{\infty} C_k x^k$ be the power series solution of (6.72)

$$\text{Now} \quad \varphi'(x) = \sum_{k=1}^{\infty} kC_k x^{k-1} \text{ and}$$

$$\varphi''(x) = \sum_{k=2}^{\infty} k(k-1)C_k x^{k-2}$$

If φ is a solution of (6.72) then $(1-x^2)\varphi''-2x\varphi'+\alpha(\alpha+1)\varphi = 0$

$$\text{(ie)} \quad (1-x^2)\sum_{k=2}^{\infty} k(k-1)C_k x^{k-2} - 2x\sum_{k=1}^{\infty} kc_k x^{k-1} + \alpha(\alpha+1)\sum_{k=1}^{\infty} C_k x^k = 0$$

(ie) $\sum\limits_{k=2}^{\infty} k(k-1)C_k x^{k-2} - \sum\limits_{k=2}^{\infty} k(k-1)C_k x^k - 2\sum\limits_{k=1}^{\infty} kC_k x^k + \alpha(\alpha+1)\sum\limits_{k=0}^{\infty} C_k x^k = 0$

(ie) $\sum\limits_{k=0}^{\infty}(k+2)(k+1)C_{k+2}x^k - 2\sum\limits_{k=1}^{\infty} kC_k x^k + \alpha(\alpha+1)\sum\limits_{k=2}^{\infty} C_k x^k = 2(1)C_1 x = 0$

(ie) $2C_2 + 6C_3 x - 2C_1 x + \sum\limits_{k=2}^{\infty}\{(k+2)(k+1)C_{k+2} - 2kC_k + \alpha(\alpha+1)C_k\}x^k = 0$

Equating constants and coefficients of like powers on both sides, we get,

$$\left.\begin{aligned} 2C_2+\alpha(\alpha+1)C_0 &= 0 \\ 6C_3-2C_1+\alpha(\alpha+1)C_1 &= 0 \\ (k+2)(k+1)C_{k+2}-k(k-1)C_k-2kC_k+\alpha(\alpha+1)C_k &= 0,\ k=2,3,4,\ldots\ldots. \end{aligned}\right\} \quad \text{-------}(6.73)$$

Now $2C_2+\alpha(\alpha+1)C_0 = 0$

$$\text{(ie)} \quad C_2 = \frac{-\alpha(\alpha+1)C_0}{2}$$

and $6C_3-2C_1+\alpha(\alpha+1)C_1 = 0$

$$\text{(ie)} \quad 6C_3 = 2C_1-\alpha(\alpha+1)C_1$$

$$\text{(ie)} \quad C_3 = \frac{1}{6}[2C_1 - \alpha(\alpha+1)C_1]$$

$$= \frac{1}{6}[2-\alpha^2+\alpha]C_1$$

$$= -\frac{1}{6}[\alpha^2-\alpha-2]C_1$$

$$= -\frac{1}{3!}(\alpha^2-\alpha-2)C_1$$

Again $(k+2)(k+1)C_{k+2}-[k(k-1)+2k-\alpha(\alpha+1)]C_k = 0$

$$\text{(ie)} \quad C_{k+2} = \frac{[k(k-1)+2k-\alpha(\alpha+1)]C_k}{(k+2)(k+1)}$$

$$= \frac{k^2+k-\alpha^2-\alpha}{(k+2)(k+1)}C_k$$

$$= \frac{-(\alpha^2+\alpha-k^2-k)}{(k+2)(k+1)}C_k$$

$$= \frac{-[(\alpha+k)(\alpha-k)+(\alpha-k)]}{(k+2)(k+1)}C_k$$

$$= \frac{-(\alpha-k)(\alpha+k+1)}{(k+2)(k+1)} C_k$$

When k = 2, then $C_4 = \frac{-(\alpha+3)(\alpha-2)}{4.3} C_2$

$$= \frac{(-1)^2(\alpha+3)(\alpha+1)(\alpha)(\alpha-2)}{4!} C_0$$

When k = 3, then $C_5 = \frac{-(\alpha+4)(\alpha-3)C_3}{5.4}$

$$= \frac{(-1)^2(\alpha+4)(\alpha+2)(\alpha-1)(\alpha-3)}{5!}$$

In general, $C_{2m} = \frac{[(-1)^m(\alpha+2m-1)....(\alpha+3)(\alpha+1)\alpha(\alpha-2)....(\alpha-2m+2)]}{2m!} C_0$

and $C_{2m+1} = \frac{[(-1)^m(\alpha+2m)....(\alpha+2)(\alpha-1)(\alpha-3)....(\alpha-2m+1)]}{(2m+1)!} C_0$

for m = 1,2,3,.......

Now $\varphi(x) = \sum_{k=0}^{\infty} C_k x^k$

$$= C_0+C_1x+C_2x^2+.......$$

$$= C_0+C_1x+\sum_{m=1}^{\infty} C_{2m}x^{2m} + \sum_{m=1}^{\infty} C_{2m+1}x^{2m+1}$$

$$= C_0\varphi_1(x)+C_1\varphi_2(x) \text{ where}$$

$$\varphi_1(x) = 1+\sum_{m=1}^{\infty}\frac{1}{(2m)!}\Big[(-1)^m.(\alpha+2m-1)(\alpha+2m-3)....(\alpha+3)(\alpha+1)(\alpha-2)....(\alpha-2m+2)\Big]x^{2m}$$

and $\varphi_2(x) = x+\sum_{m=1}^{\infty}\frac{1}{(2m+1)!}(-1)^m(\alpha+2m)....(\alpha+2)(\alpha-1)......(\alpha-2m+1)x^{2m+1}$

Clearly $W(\varphi_1, \varphi_2)(0) = \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix} = 1\ (\neq 0).$

$\therefore$ φ_1, φ_2 are the required independent powre series solution of the Legendre equation.

Example E. 6.11 :

Find the second solution of the given $(1-x^2)y''-2xy'+2y=0$ and $\varphi_1(x)=x$, $0<x<1$.

Solution :

Givent that $(1-x^2)y''-2xy'+2y = 0$ and $\varphi_1(x) = x$. Let $\varphi_2 = u\varphi_1(x)$

$$\text{(ie) } \varphi_2 = ux$$

$$\text{(ie) } (1-x^2)\varphi_2''-2x\varphi_2'+2\varphi_2 = 0$$

$$(1-x^2)(u\varphi_1)''-2x(u\varphi_1)'+2u\varphi_1 = 0$$

$$(1-x^2)(u''x+2u')-2x(u'x)-2xu+2xu = 0$$

$$\text{(ie) } u''(x-x^3)+u'(2-4x^2) = 0$$

$$\text{(ie) } v'(x-x^3)+v(2-4x^2) = 0 \qquad \text{(where } v=u')$$

$$\text{(ie) } \frac{dv}{v} = -\frac{2(1-2x^2)}{x(1-x^2)}dx$$

Integrating on both sides, we get,

$$\int\frac{dv}{v} = -\int\frac{2(1-2x^2)}{x(1-x^2)}dx \qquad \text{-------(6.74)}$$

$$\text{Let } -\frac{2(1-2x^2)}{x(1-x^2)} = \frac{A}{x}+\frac{B}{1+x}+\frac{C}{1-x}$$

$$-2(1-2x^2) = A(1+x)(1-x)+Bx(1-x)+Cx(1+x) \qquad \text{-------(6.75)}$$

Put $x = 1$ in (6.75) we have $2 = 2C \Rightarrow C = 1$.

Put $x = -1$ in (6.75) we have $2 = -2B \Rightarrow B = -1$.

Put $x = 0$ in (6.75) we have $-2 = A \Rightarrow A = -2$.

$$\text{(ie) } \log v = -2\int\frac{dx}{x}-\int\frac{dx}{1+x}+\int\frac{dx}{1-x}$$

$$\log v = -2\log x - \log(1+x)+\log(1-x)+\log C$$

$$\text{(ie) } \log[vx^2(1+x)(1-x)] = \log C$$

$$\text{(ie) } v = \frac{C}{x^2(1+x)(1-x)}$$

$$\text{(ie) } v = \frac{C}{x^2(1-x^2)}$$

$$\text{(ie)}\quad u' = \frac{C}{x^2(1-x^2)}$$

$$\frac{du}{dx} = \frac{C}{x^2(1-x^2)}$$

$$du = \frac{Cdx}{x^2(1-x^2)}$$

Integrating on both sides, we get

$$\int du = \int \frac{Cdx}{x^2(1-x^2)} \qquad \text{-------(6.76)}$$

$$\text{Let}\quad \frac{1}{x(1-x^2)} = \frac{1}{x^2(1+x)(1-x)}$$

$$= \frac{Ax+B}{x^2} + \frac{C}{1+x} + \frac{D}{1-x}$$

$$\therefore 1 = (Ax+B)(1+x)(1-x)+Cx^2(1-x)+D(1+x)x^2 \qquad \text{-------(6.77)}$$

Put $x = 1$ in (4) we get

$$1 = D(2) \Rightarrow \mathbf{D = 1/2}$$

Put $x = -1$ in (4) we get

$$1 = 2C \Rightarrow \mathbf{C = 1/2}$$

Put $x = 0$ in (6.77) we get

$$1 = B \Rightarrow \mathbf{B = 1}$$

Put $x = 2$ in (4) we get

$$1 = (2A+1).3(-1)+C.4(-1)+D.4.3$$

$$1 = (2A+1)-3-½.4+12×½$$

$$1 = -6A-3-2+6$$

$$1 = -6A+1 \Rightarrow \mathbf{A = 0}$$

$\therefore$ (6.77) becomes

$$\frac{1}{x^2(1-x^2)} = \frac{1}{x^2} + \frac{1/2}{(1+x)} + \frac{1/2}{(1-x)}$$

$$\therefore \int du = C\int\left\{\frac{1}{x^2} + \frac{1/2}{(1+x)} + \frac{1/2}{(1-x)}\right\}dx$$

$$u = C\left\{\frac{-1}{x}+\frac{1}{2}\log(1+x)-\frac{1}{2}\log(1-x)\right\}$$

$$= C\left\{-\frac{1}{x}+\frac{1}{2}\log\left(\frac{1+x}{1+x}\right)\right\}$$

For our convenience choose

$$u = \frac{-1}{x}+\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\therefore\ u\varphi_1 = -1+\frac{x}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\therefore\ \varphi_2 = \frac{x}{2}\log\left(\frac{1+x}{1-x}\right)-1$$

Example E.6.12 :

Find the two linear independent power series solution of $y''+x^3y'+x^2y = 0$

Solution :

Given that $y''+x^3y'+x^2y = 0$

Here $a_1(x)=x^3$ $a_2(x)=x^2$ which are analytic for all real x_0.

$$\text{Let } \varphi(x) = \sum_{k=0}^{\infty} C_k x^k \text{ be a solution of } L(y) = y''+x^3y'+x^2y = 0$$

$$\therefore\ L(\varphi) = 0$$

$$\text{(ie) } \varphi''+x^3\varphi'+x^2\varphi = 0 \qquad \text{------(6.78)}$$

$$\text{Now } \varphi(x) = \sum_{k=0}^{\infty} C_k x^k$$

$$\therefore\ \varphi_1'(x) = \sum_{k=1}^{\infty} kC_k x^{k-1}$$

$$\varphi_1'' = \sum_{k=2}^{\infty} k(k-1)C_k x^{k-2}$$

$$\therefore\ (6.78) \Rightarrow \sum_{k=2}^{\infty} k(k-1)C_k x^{k-2}+x^3\sum_{k=1}^{\infty} kC_k x^{k-1}+x^2\sum_{k=0}^{\infty} C_k x^k = 0$$

(ie) $\sum_{k=0}^{\infty}(k+2)(k+1)C_{k+2}x^k + \sum_{k=3}^{\infty}(k-2)C_{k-2}x^k + \sum_{k=2}^{\infty}C_{k-2}x^k = 0$

(ie) $2C_2 + 3.2C_3x + 4.3C_4x^2 + C_0x^2 + \sum_{k=3}^{\infty}\{(k+2)(k+1)C_{k+2} + (k-2)C_{k-2} + C_{k-2}\}x^k = 0$

Comparing like powers of x on both sides, we have

$$\mathbf{C_2 = 0, \quad C_3 = 0}$$

$$4.3C_4 + C_0 = 0$$

$$4.3C4 = -C0$$

$$\therefore \mathbf{C4 = \frac{-C_0}{4.3}}$$

and $(k+2)(k+1)C_{k+2}+(k-1)C_{k-2} = 0$, $k = 3, 4, 5,\ldots\ldots\ldots$ --------(6.79)

$$\therefore (6.79) \Rightarrow C_{k+2} = \frac{-(k-2)C_{k-2}}{(k+2)(k+1)} \text{ where } k = 3,4,5 \qquad \text{--------(6.80)}$$

Put k = 3 in (6.80), we get

$$\mathbf{C_5 = \frac{-2C_1}{5.4}}$$

k = 4 in (6.80)

$$\mathbf{C_6 = \frac{-3.C_2}{6.5} \Rightarrow C_6 = 0 \quad (\because C_2 = 0)}$$

k = 5 in (6.80)

$$C7 = \frac{-4C_3}{7.6} = 0 \qquad \mathbf{C_7 = 0}$$

k = 6 in (6.80)

$$C_8 = \frac{-5C_4}{8.7} = \frac{5C_0}{8.7.4.3}$$

put k = 7 in (6.80) we get

$$C_9 = \frac{-6C_5}{9.8} = \frac{6.2C_1}{9.8.5.4} \qquad \left\{\because C_5 = \frac{-2C_1}{5.4}\right\}$$

$$\therefore \mathbf{C_9 = \frac{6.2C_1}{9.8.5.4}}$$

put k = 8 in (6.80) we get

$$C_{10} = \frac{-7.C_6}{10.9} = 0 \quad (\because C_6 = 0)$$

$$\mathbf{C_{10} = 0}$$

put k = 9 in (6.80) we get

$$C_{11} = \frac{-8.C_7}{11.10} = 0 \quad (\because\ C_7 = 0)$$

$$\therefore\ C_{11} = 0$$

put k = 10 in (6.80) we get

$$C_{12} = \frac{-9C_8}{12.11} = \frac{-9.5C_0}{12.11.8.7.4.3} \quad \left(\because C_8 = \frac{5C_0}{8.7.4.3}\right)$$

$$\therefore\ \mathbf{C_{12}} = \frac{\mathbf{-9.5C_0}}{\mathbf{12.11.8.7.4.3}}$$

put k = 11 in (6.80) we get

$$C_{13} = \frac{-10C_9}{13.12} = \frac{-10.6.2.C_1}{13.12.9.8.5.4} \left(\because C_9 = \frac{6.2.C_1}{9.8.5.4}\right)$$

In general

$\therefore\ C_{4m-2} = 0$, and $C_{4m-1} = 0$ where m=1,2,3,.......

$$C_{4m} = (-1)^m\left[\frac{1.5.9.......(4m-3)C_0}{3.4.7.8.11.12(4m-1)(4m)}\right] \text{ where m=1,2,3,....}$$

$$C_{4m+1} = (-1)^m\left[\frac{2.6.10.....(4m-1)C_1}{4.5.8.9....(4m)(4m+1)}\right] \text{ where m=1,2,3,....}$$

$$\text{Now } \varphi(x) = \sum_{k=0}^{\infty} C_k x^k$$

$$= C_0 + C_1 x + \sum_{m=1}^{\infty} C_{4m-2}x^{4m-2} + \sum_{m=1}^{\infty} C_{4m-1}x^{4m-1} + \sum_{m=1}^{\infty} C_{4m}x^{4m} + \sum_{m=1}^{\infty} C_{4m+1}x^{4m+1}$$

$$= C_0 + C_1 x + \sum_{m=1}^{\infty}(-1)^m\left[\frac{1.5.9.....(4m-3)}{3.4.7...(4m-1)(4m)}\right]x^{4m} + \sum_{m=1}^{\infty}(-1)^m\left[\frac{2.6.10....(4m-3)}{4.5.8.9.....(4m)(4m+1)}\right]x^{4m+1}$$

$$\{\because\ C_{4m-2} = C_{4m-1} = 0\}$$

$$= C_0\left[1+\sum_{m=1}^{\infty}\left(\frac{1.5.9....(4m-3)C_0}{3.4.7.(4m-1)(4m)}\right)x^{4m}\right]$$

$$+C_1\left[x+\sum_{m=1}^{\infty}(-1)^m\left(\frac{2.6.10....(4m-2)}{4.5.8.9....(4m)(4m+1)}\right)x^{4m+1}\right]$$

$$= C_0\varphi_1(x)+C_1\varphi_2(x)$$

$$\text{where } \varphi_1(x) = \left[1+\sum_{m=1}^{\infty}(-1)^m\left(\frac{1.5.9....(4m-3)}{3.4.7....(4m-1)(4m)}\right)\right]$$

$$\varphi_2(x) = \left[x+\sum_{m=1}^{\infty}(-1)^m\left(\frac{2.6.10.....(4m-2)}{4.5.8.9....(4m)(4m+1)}\right)x^{4m+1}\right]$$

$$\text{Now } W(\varphi_1,\varphi_2)(0) = \begin{vmatrix}\varphi_1 & \varphi_2\\ \varphi_1' & \varphi_2'\end{vmatrix}$$

$$= \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}$$

$$= 1 \neq 0.$$

$\therefore$ φ_1 and φ_2 are independent solution of $L(y) = 0$ which are the required solution.

Example E.6.13 :

Find two linearly independent power series solutions (in powers of x) of the following equations :

$$y''+x^3y'+x^2y = 0$$

Solution :

Given that $y''x^3y'+x^2y = 0$

Here $a_1(x) = x^3$, $a_2(x) = x^2$ which are analytic for all real x_0.

$$\text{Let } \phi(x) = \sum_{k=0}^{\infty}C_kx^k \text{ be a solution of } L(y) = y''x^3y'+x^2y = 0$$

$$\therefore\ L(\phi) = 0$$

$$\text{(ie) } \phi''+x^3\phi'+x^2\phi = 0 \qquad\text{--------(6.81)}$$

$$\text{Now } \phi(x) = \sum_{k=0}^{\infty}C_kx^k = C_0+C_1x+C_2x^2+.......$$

$$\therefore\ \phi'(x) = C_1+2C_2x+3C_3x^2+....... = \sum_{k=1}^{\infty}kC_kx^{k-1}$$

$$\& \ \phi''(x) = 2C_2+6.C_3x+\ldots\ldots = \sum_{k=2}^{\infty} k(k-1)C_k x^{k-2}$$

$$(6.81) \Rightarrow \sum_{k=2}^{\infty} k(k-1)C_k x^{k-2} + x^3 \sum_{k=1}^{\infty} kC_k x^{k-1} + x^2 \sum_{k=0}^{\infty} C_k x^k = 0$$

$$\text{(ie)} \quad \sum_{k=2}^{\infty} k(k-1)C_k x^{k-2} + \sum_{k=1}^{\infty} kC_k x^{k+2} + \sum_{k=0}^{\infty} C_k x^{k+2} = 0$$

$$\text{(ie)} \quad \sum_{k=0}^{\infty}(k+2)(k+1)C_{k+2}x^k + \sum_{k=3}^{\infty}(k-2)C_{k-2}x^k + \sum_{k=2}^{\infty} C_{k-2}x^k = 0$$

$$\text{(ie)} \quad 2C_2 + 3.2C_3x + 4.3C_4x^2 + 4.3C_4x^2 + \sum_{k=3}^{\infty}(k+2)(k+1)C_{k+2}x^k$$

$$+ \sum_{k=3}^{\infty}(k-2)C_{k-2}x^k + C_0x^2 + \sum_{k=3}^{\infty} C_{k-2}x^k = 0$$

$$\text{(ie)} \quad 2C_2 + 3.2.C_3x + 4.3C_4x^2 + C_0x^2 + \sum_{k=3}^{\infty}\{(k+2)(k+1)C_{k+2}[(k-2)+1]C_{k-2}\}x^k = 0$$

Comparing coefficients of like powers of x on both sides, we have

$$\left.\begin{aligned} &2C_2 = 0, \quad 3.2.C_3 = 0, \quad 4.3C_4+C_0 = 0 \\ &\& \ (k+2)(k+1)C_{k+2}[(k-2)+1]C_{k-2}=0, \ k = 3,4,5,\ldots.. \end{aligned}\right\} \quad \text{--------(6.82)}$$

From (6.82),

$$2C_2 = 0$$

$$\mathbf{C_2 = 0}$$

$$3.2.C_3 = 0$$

$$\mathbf{C_3 = 0}$$

$$4.3.C_4+C_0 = 0$$

$$\mathbf{C_4 = \frac{-C_0}{4.3}}$$

$$\& \ (k+2)(k+1)C_{k+2}+[k-1]C_{k-2} = 0$$

$$C_{k+2} = \frac{-(k-1)C_{k-2}}{(k+2)(k+1)}, \quad k=3,4,5,\ldots. \qquad \text{------(6.83)}$$

Put k = 3 in (6.83), we get

$$\mathbf{C_5 = \frac{-2C_1}{5.4}}$$

Put k = 4 in (6.83), we get

$$C_6 = \frac{-3.C_2}{6.5} = 0 \quad [\because\ C_2 = 0]$$

Put k = 5 in (6.83), we get

$$C_7 = \frac{-4.C_3}{7.6} = 0 \quad [\because\ C_3 = 0]$$

Put k = 6 in (6.83), we get

$$C_8 = \frac{-5.C_4}{8.7} = \frac{5.C_0}{8.7.4.3}$$

Put k = 7 in (6.83), we get

$$C_9 = \frac{-6.C_5}{9.8} = \frac{6.2.C_1}{9.8.5.4}$$

Put k = 8 in (6.83), we get

$$C_{10} = \frac{-7C_6}{10.9} = 0$$

Put k = 9 in (6.83), we get

$$C_{11} = \frac{-8.C_7}{11.10} = 0$$

Put k = 10 in (6.83), we get

$$C_{12} = \frac{-9.C_8}{12.11} = \frac{-9.5.C_0}{12.11.8.7.4.3}$$

Put k = 11 in (6.83), we get

$$C_{13} = \frac{-10.C_9}{13.12} = \frac{-10.6.2.C_1}{13.12.9.8.5.4}$$

In general

$\therefore C_{4m-2} = 0, C_{4m-1} = 0$ for m = 1, 2, 3,.....

$$C_{4m} = (-1)^m\left[\frac{1.5.9....(4m-3)C_0}{3.4.7.8.11.......(4m-1)(4m)}\right] \text{ for } m = 1, 2, 3,.....$$

$$C_{4m+1} = (-1)^m\left[\frac{2.6.10....(4m-2)C_1}{4.5.8.9....(4m)(4m+1)}\right] \text{ for } m = 1, 2, 3,.....$$

$$\text{Now } \phi(x) = \sum_{k=0}^{\infty} C_k x^k$$

$$= C_0 + C_1x + \sum_{m=1}^{\infty} C_{4m-2}x^{4m-2} + \sum_{m=1}^{\infty} C_{4m-1}x^{4m-1}$$

$$+ \sum_{m=1}^{\infty} C_{4m}x^{4m} + \sum_{m=1}^{\infty} C_{4m+1}x^{4m+1}$$

$$= C_0 + C_1x + \sum_{m=1}^{\infty}(-1)^m\left[\frac{1.5.9....(4m-3)C_0}{3.4.7....(4m-1)(4m)}\right]x^{4m}$$

$$+ \sum_{m=1}^{\infty}(-1)^m\left[\frac{2.6.10....(4m-2)C_1}{4.5.8.9....(4m)(4m+1)}\right]x^{4m+1}$$

$$(\because C_{4m-2} \,\&\, C_{4m-1} = 0)$$

$$= C_0\left[1 + \sum_{m=1}^{\infty}(-1)^m\left[\frac{1.5.9....(4m-3)C_0}{3.4.7.8....(4m-1)(4m)}\right]x^{4m}\right]$$

$$+C_1\left[x + \sum_{m=1}^{\infty}(-1)^m\left(\frac{2.6.10....(4m-2)}{4.5.8.9....(4m)(4m+1)}\right)x^{4m+1}\right]$$

$$= C_0\phi_1(x) + C_1\phi_2(x)$$

$$\text{where } \phi_1(x) = 1 + \sum_{m=1}^{\infty}(-1)^m\left(\frac{1.5.9......(4m-3)C_0}{3.4.7.8....(4m-1)(4m)}\right)$$

$$\phi_2(x) = x + \sum_{m=1}^{\infty}(-1)^m\left(\frac{2.6.10.....(4m-2)}{4.5.8.9.....(4m)(4m+1)}\right)x^{4m+1}$$

$$\text{Now } W(\phi_1, \phi_2)(0) = \begin{vmatrix}\phi_1 & \phi_2 \\ \phi_1' & \phi_2'\end{vmatrix}$$

$$= \begin{vmatrix}1 & 0 \\ 0 & 1\end{vmatrix}$$

$$= 1 \neq 0.$$

$\therefore$ ϕ_1 & ϕ_2 are independent solution of $L(y) = 0$.

Which are the required solution.

UNIT – 7

LINEAR EQUATION WITH REGULAR SINGULAR POINTS

7.1 Introduction :

Definition D. 7.1 :

A point x_0 such that $a_0(x_0) = 0$ is called a singular point of the equation

$$a_0(x)y^{(n)}+a_1(x)y^{(n-1)}+.....+a_n(x)y = 0 \qquad \text{-------(7.1)}$$

Note :

x_0 is a regular singular point for (7.1) if the equation can be written as $(x-x_0)^n y^{(n)}+b_1(x)(x-x_0)^{n-1} y^{(n-1)}+....+b_n(x)y = 0$ where $b_1, b_2,......,b_n$ are analytic at x_0.

Definition D.7.2 :

An equation of the form $C_0(x)(x-x_0)^n y^{(n)}+C_1(x)(x-x_0)^{n-1}y^{(n-1)}+....+C_n(x)y = 0$ has a regular singular point at x_0 if $C_0, C_1,......C_n$ are analytic at x_0 and $C_0(x_0)\neq 0$.

Theorem 7.1 :

Consider the second order Euler equation $x^2y''+axy'+by=0$ (a, b are constants) and the polynomial q given by $q(r) = r(r-1)+ar+b$. A basis for the solutions of the Euler equation on any interval not containing x=0 is given by $\varphi_1(x)=|x|^r$, $\varphi_2(x)=|x|^{r_2}$ if r_1, r_2 are distinct roots of q and by $\varphi_1(x)=|x|^{r_1}$, $\varphi_2(x)=|x|^{r_1}\log(x)$ if r_1 is a root of q of multiplicity two.

Proof :

Given that $x^2y''+axy'+by = 0$ is a second degree differential equation where a, b are constants.

Case (i) Let $x > 0$.

$$\text{Let } L(y) = x^2y''+axy'+by$$

$$\text{Now } L(x^r) = x^2(x^r)''+ax(x^r)'+bx^r$$

$$= x^2.r(r-1)x^{r-2}+ax.r.x^{r-1}+bx^r$$

$$= [r(r-1)+ra+b]x^r$$

$$= q(r).x^r \text{ where } q(r) = r(r-1)+ar+b$$

Let r_1 be a root of $q(r) = 0$.

$$\text{Then } q(r_1) = 0$$

$$\therefore L(r_1) = q(r_1).x^{r_1} = 0$$

(ie) x^{r_1} is a solution of $L(y) = 0$

$$\text{Let } \varphi_1(x) = x^{r_1}$$

Similarly if r_2 is a root of $q(r) = 0$ then $\varphi_2(x) = x^{r_2}$.

If $r_1 \neq r_2$, then the solutions of $L(y) = 0$ are $\varphi_1(x)=x^{r_1}$, and $\varphi_2(x)=x^{r_2}$.

If $r_1 = r_2$ then $q(r)=0$ has two equal roots. (ie) r_1 is a root of $q(r)=0$ of multiplicity two.

$$\therefore q(r_1) = 0 \;\&\; q'(r_1) = 0$$

$$\text{Now } \frac{\partial}{\partial r}(x^r) = \frac{\partial}{\partial r}(e^{\log x^r})$$

$$= \frac{\partial}{\partial x}(e^r.\log x)$$

$$= \log x \,.\, e^{r \log x}$$

$$= x^r.\log x$$

$$\text{(ie) } \frac{\partial}{\partial r}(x^r) = x^r \log r.$$

Again we know that,

$$\frac{\partial}{\partial r}(L(x^r)) = L\left(\frac{\partial}{\partial r}(x^r)\right)$$

$$= L(x^r \log x)$$

$$= x^2[x^r \log x]''+ax(x^r \log x)'+bx^r \log x$$

$$= x^2\left[x^r\frac{1}{x}+rx^{r-1}\log x\right]^1+ax\left[x^2.\frac{1}{x}+rx^{r-1}.\log x\right]+bx^r\log x$$

$$= x^2[x^{r-1}+rx^{r-1}\log x]'+ax[x^{r-1}+rx^{r-1}\log x]+bx^r \log x$$

$$= x^2\left[(r-1)x^{r-2} + rx^{r-1}.\frac{1}{x} + r(r-1)x^{r-2}.\log x\right]$$

$$+ ax.x^{r-1}[a + r\log x] + bx^r \log x$$

$$= [2r-1+a+\{r(r-1)+ar+b\}\log x]x^r.$$

$$\therefore \frac{\partial}{\partial r}L(x^r) = [q'(r)+q(r)\log x]x^r.$$

$$\text{(ie)}\ L(x^r \log x) = [q'(r)+q(r)\log x]x^r.$$

When $r = r_1$, then $L(x^{r_1} \log x) = 0$

$\therefore$ The second solution is $\varphi_2(x) = x^{r_1}. \log x$.

Case(2) : Let $x < 0$.

$$\text{Now } [(-x)^r]' = r(-x)^{r-1}(-1)$$

$$= -r(-x)^{r-1}$$

$$\text{and } [(-x)^r]'' = r(r-1)(-x)^{r-2}$$

$$\text{Now } L[(-x)^r] = x^2[(-x)^r]''+ax[(-x)^r]'+b(-x)^r$$

$$= r.(r-1).(-x)^r+ar(-x)^r+b(-x)^r$$

$$= [r(r-1)+ar+b](-x)^r$$

$$= q(r)(-x)^r \text{ where } q(r) = r(r-1)+ar+b.$$

$$\text{Again } \frac{\partial}{\partial r}((-x)^r) = (-x)^r \log(-x).$$

As in case (1), when $r_1 \neq r_2$ then the solution are given by $\varphi_1(x)=(-x)^{r_1}$ and $\varphi_2(x)=(-x)^{r_2}$.

When $r_1 = r_2$, then the solution of $L(y)=0$ are given by $\varphi_1(x) = (-x)^{r_1}$, $\varphi_2(x)=(-x)^{r_1} \log(-x)$.

Hence by case (1) & (2), the solutions of $L(y)=0$ are given by

$\varphi_1(x) = |x|^{r_1}$, $\varphi_2(x) = |x|^{r_2}$ if $r_1 \neq r_2$

and $\varphi_1(x) = |x|^{r_1}$, $\varphi_2(x) = |x|^{r_1} \log |x|$ if $r_1 = r_2$

This proves the theorem.

Note : The equation $q(r)$ is called **indicial polynomial** of $L(y)=0$

Example E. 7.1 :

Find all solutions of the equations $x^2y''+2xy'-6y = 0$ for $x>0$.

Solution :

Given that $x^2y''+2xy'-6y = 0$ for $x>0$ ------(7.2)

Comparing the above equation with the Euler's equation $x^2y''+2xy'-6y = 0$, we get, $a=2$, $b=-6$.

$\therefore$ The indicial polynomial of (7.2) is

$$\begin{aligned} q(r) &= r(r-1)+ar+b \\ &= r^2-r+2r-6 \\ &= r^2+r-6 \end{aligned}$$

Thus the roots of $q(r) = 0$ is $r^2+r-6 = 0$

(ie) $(r+3)(r-2) = 0$

(ie) $r+3 = 0$ or $r-2 = 0$

(ie) $r = -3$ or $r = 2$

$\therefore$ The solutions are given by $\varphi_1(x)=x^{-3}$, $\varphi_2(x)=x^2$ & hence the solution of (7.1) is $\varphi(x) = C_1\varphi_1+C_2\varphi_2$

(ie) $\varphi(x) = C_1x^{-3}+C_2x^2$

Example E. 7.2 :

Find all solutions of given equation x2y''–5xy'+9y = x3.

Solution :

Step 1 : To find the solution of $x^2y''-5xy'+9y = 0$

Given that $x^2y''-5xy'+9y = 0$ ------(7.3)

Comparing with Euler's equation $x^2y''+ax'+by = 0$

We get, $a = -5$, $b = 9$.

Now the indicial equation is

$$q(r) = r(r-1)+ar+b$$
$$= r^2-r-5r+9$$
$$= r^2-6r+9$$

If $q(r) = 0$ then $r^2-6r+9 = 0$

$$\text{(ie)}\ r = 3 \text{ or } 3.$$

(ie) the roots of $q(r) = 0$ are 3, 3.

Hence the solutions of (7.3) is $\varphi_1(x)=x^3$, $\varphi_2(x)=x^3\log x$ & $\therefore$ the general solution of (7.3) is

$$\varphi(x) = C_1x^3+C_2x^3\log x.$$

Step 2 : To find the solution of $x^2y''-5xy'+9y = x^3$.

Now $x^2y''-5xy'+9y = x^3$ can be rewritten as

$$y''-\frac{5}{x}y'+\frac{9}{x^2}y = x$$

To find Ψ_p

Let $\Psi_p = u_1\varphi_3+u_2\varphi_4$ where u_1', u_2' satisfy the equations $u_1'\varphi_3+u_2'\varphi_4 = 0$

and $u_1'\varphi_3'+u_2'\varphi_4' = b(x)$

$$\left.\begin{aligned} \text{(ie)}\ x^3u_1'+x^3\log x\, u_2' &= 0 \\ 3x^2u_1'+(x^2+3x^2\log x)u_2' &= x \end{aligned}\right\} \qquad \text{-------(7.4)}$$

Solving the equations (7.4), we get,

$$W(x) = \begin{vmatrix} \varphi_3 & \varphi_4 \\ \varphi_3' & \varphi_4' \end{vmatrix}$$

$$= \begin{vmatrix} x^3 & x^3\log x \\ 3x^2 & x^2+3x^2\log x \end{vmatrix}$$

$$= x^5+3x^5\log x-3x^5\log x = x^5.$$

$$\text{Again } W_1(x) = \begin{vmatrix} 0 & x^3\log x \\ 1 & x^2+3x^2\log x \end{vmatrix}$$

$$= -x^3 \log x.$$

$$\text{and } W_2(x) = \begin{vmatrix} x^3 & 0 \\ 3x^2 & 1 \end{vmatrix}$$

$$= x^3.$$

$$\therefore\ u_1' = \frac{b(x).W_1(x)}{W(x)}$$

$$= \frac{x(-x^3\log x)}{x^5}$$

$$= -\frac{\log x}{x}$$

$$\text{(ie) } du_1 = -\frac{\log x}{x}dx$$

Integrating on both sides, we get,

$$u_1 = -\int \frac{1}{x}\log x dx$$

$$= -\int \log d(\log x)$$

$$= -\frac{1}{2}(\log x)^2$$

$$\text{and } u_2' = \frac{b(x).W_2(x)}{W(x)}$$

$$= \frac{x.x^3}{x^5}$$

$$= \frac{1}{x}$$

$$\text{(ie) } du_2 = \frac{1}{x}dx$$

Integrating on both sides, we get,

$$u_2 = \log x$$

$$\therefore\ \Psi_p = u_1\varphi_1 + u_2\varphi_2$$

$$= -\frac{x^3}{x}(\log x)^2 + x^3(\log x)^2$$

$$= \frac{x^3}{2}(\log x)^2$$

Thus the required solution is $\varphi(x) = C_1\varphi_1+C_2\varphi_2+\Psi_p$

(ie) $\varphi(x) = C_1x^3+C_2x^3 \log x + \frac{x^2}{2}(\log x)^2$

Example E. 7.3 :

Find all solutions of the equation $x^3y'''+2x^2y''-xy'+y = 0$ for $x>0$.

Solution :

Given that $x^3y'''+2x^2y''-xy'+y = 0$ --------(7.4) is a differential equation.

Comparing (7.4) with the Euler equation of degree 3 which is given by $x^3y'''+ax^2y''-bxy'+cy = 0$, we have, $a=2$, $b=-1$, $c=1$.

$\therefore$ The indicial polynomial is

$$\begin{aligned} q(r) &= r(r-1)(r-2)+ar(r-1)+br+c \\ &= r(r-1)(r-2)+2r(r-1)-r+1 \\ &= (r-1)(r^2-1) \end{aligned}$$

Thus the roots of $q(r) = 0$ are $1, \pm 1$.

$\therefore$ The solutions of (7.4) are given by

$\varphi_1(x) = x^{-1}$, $\varphi_2(x) = x$, $\varphi_3(x) = x \log x$

and the general solution is

$$\varphi(x) = C_1\frac{1}{x}+C_2x+C_3x\log x$$

Example E. 7.4 :

Find all solutions of the equation $x^2y''-(2+i)xy'+3iy = 0$ for $|x|>0$.

Solution :

Given that $x^2y''-(2+i)xy'+3iy = 0$ -------(7.5)

is a differential equation in **C**.

Comparing (7.5) with the Euler equation, $x^2y''-axy'+by = 0$, we get,

$a = -(2+i)$, $b = 3i$

Here the indicial polynomial is $q(x) = r(r-1)+ar+b$

$$\text{(ie)} \quad q(r) = r^2-r-(2+i)r+3i$$

$$= r^2-3r-ir+3i$$

If $q(r) = 0$ then $r^2-3r-ir+3i = 0$

$$\Rightarrow r^2-(3+i)r+3i = 0$$

$$\Rightarrow r = \frac{(3+i)\pm\sqrt{(3+i)^2-4(3i)}}{2}$$

$$= 3, i$$

$\therefore$ The solutions of (7.5) are $\varphi_1(x)=|x|^3$, $\varphi_2(x)=|x|^i$.

7.2 Second order equation with regular singular points – an example.

Definition D. 7.3 :

A second order equation with a regular singular point at x_0 has the form

$(x-x_0)^2y''+a(x)(x-x_0)y'+b(x)y = 0$ where $a(x)$, $b(x)$ are analytic at x_0.

We have the following result without proof.

Result :

Consider the equation $x^2y''+a(x)y'+b(x)y = 0$ where a, b are convergent power series expansion for $|x|<r_0$ where $r_0>0$.

Let r_1, r_2 be the roots of the indicial polynomial $q(r) = r(r-1)+a(0)r+b(0)$.

For $0<|x|<r_0$, there is a solution φ_1 of the form $\varphi_1(x) = |x|^{r_1} \sum\limits_{k=0}^{\infty} C_k x^k$, $(C_0=1)$, where the series converges for $|x|<r_0$.

If $r_1-r_2 \neq 0$ or positive integer, there is a solution φ_2 for $0<|x|<r_0$ of the form $\varphi_2(x) = |x|^{r_2} \sum\limits_{k=0}^{\infty} \overline{C}_k x^k$ where $\overline{C}_k = 1$ where the series converges for $|x|<r_0$.

Example E. 7.5 :

Find the singular points of the equation $x^2y''+(x+x^2)y'-y = 0$ and determine whether regular singular point (or) not.

Solution :

Given that $x^2y''+(x+x^2)y'-y = 0$ --------(7.6)

is a differential equation.

Clearly $x=0$ is a singular point of (7.6)

$\therefore$ The given equation has a regular singular point because (7.6) can be written as

$(x-0)^2y''+(1+x)(x-0)y'-y = 0$

Example E. 7.6 :

Find the singular points of the equation $(1-x^2)y''-2xy'+2y = 0$ and determine whether regular singular point or not.

Solution :

Given that $(1-x^2)y''-2xy'+2y = 0$ --------(7.7)

If $1-x^2 = 0$ then $x = \pm 1$.

$\therefore$ ± 1 are singular points of (7.7)

Now we shall find the regular singular point at $x=1$.

Now (7.7) can be written as

$(1+x)(1-x)y''-2xy'+2y = 0$

(ie) $(1-x)y''-\dfrac{2x}{1+x}y'+\dfrac{2}{1+x}y=0$

(ie) $-(x-1)y''-\dfrac{2x}{1+x}y'+\dfrac{2y}{1+x}=0$

(ie) $(x-1)y''+\dfrac{2x}{1+x}y'-\dfrac{2y}{1+x}=0$

(ie) $(x-1)^2y''+\dfrac{2x(x-1)}{1+x}y'-\dfrac{2(1-x)}{(1+x)}y=0$

Similarly the regular singular point of x=–1 is obtained as follows :

$$(1+x)(1-x)y''-2xy'+2y=0$$

(ie) $(1+x)y''-\dfrac{2x}{1-x}y'+\dfrac{2y}{1+x}=0$

(ie) $(x+1)^2y''-\dfrac{2x(x+1)}{1-x}y'-\dfrac{2(x+1)}{1-x}y=0$

Thus the given equation (7.7) has regular singular point.

Example E.7.7 :

Find the singular points of the equation $(x^2+x-2)^2y''+3(x+2)y'+(x-1)y=0$ and determine whether they are regular singular points or not.

Solution :

Given that $(x^2+x-2)^2y''+3(x+2)y'+(x-1)y=0$ --------(7.8)

if $x^2+x-2=0$ then $x=1,-2$.

Again (7.8) can be written as

$(x+2)^2(x-1)^2y''+3(x+2)y'+(x-1)y=0$ -------(7.9)

Now divide $(x-1)^2$ on both sides of (7.9), we get,

$$(x+2)^2+\frac{3(x+2)}{(x-1)^2}y'+\frac{(x-1)}{(x+2)^2}y=0$$

(ie) $[x-(-2)]^2y''+\dfrac{3}{(x-1)^2}[x-(-2)]y'+\dfrac{1}{x-1}y=0$

$\therefore$ $x=-2$ is a regular singular point.

Again divide $(x+2)^2$ on both sides of (7.9), we get,

$$(x-1)^2y''+\frac{3}{(x+2)}y'+\frac{(x-1)}{(x+2)^2}y=0$$

(ie) x=1 is not a regular singular point.

7.3 Second order equations with regular singular points–the general case.

Definition D.7.5 :

The equation $xy''+(1-x)y'+\alpha y = 0$ is called Laguerre equation where α is a constant.

Example E.7.8 :

Find the singular points of the Lagurre equation and determine it is regular point or not.

Solution :

We know that the Lagurre equation is $xy''+(1-x)y'+\alpha y = 0$ -------(7.10)

(ie) $x^2y''+x(1-x)y'+\alpha xy = 0$

If $x^2=0$ then $x=0$ which is the singular point of (7.10).

Moreover (7.10) can be written as

$(x-0)^2y''+(1-x)(x-0)y'+\alpha(x-0)y = 0$

$\therefore$ $x = 0$ is a regular singular point of (7.10).

Example E. 7.9 :

Let L_n denote the polynomial $L_n(x) = e^x\dfrac{d^n}{dx^n}(x^ne^{-x})$. Show that L_n satisfies the Lagurre equation if $\alpha = n$.

Solution :

$$\text{Let } u(x) = x^ne^{-x}$$

$$\therefore e^x.u(x) = x^n$$

Differentiate with respect to x on both sides,

$e^xu'(x)+e^xu(x) = nx^{n-1}$

(ie) $e^x(u'(x)+u(x)) = nx^{n-1}$

(ie) $u'(x)+u(x) = nx^{n-1}e^{-x}$.

(ie) $x(u'(x)+u(x)) = nu(x)$

(ie) $xu'(x)+xu(x) = nu(x)$

Differentiate with respect to x (n+1) times, we get,

$[xu'(x)]^{(n+1)}+[xu(x)]^{(n+1)} = nu^{(n+1)}(x)$

(ie) $xu^{(n+2)}(x)+(n+1)C_1u^{(n+1)}(x)+xu^{(n+1)}(x)+(n+1)C_1u^{(n)}(x) = nu^{n+1}(x)$

$$\text{(ie) } x\left[\frac{d^n}{dx^n}u(x)\right]''+(n+1)\left[\frac{d^n}{dx^n}u(x)\right]'+x\left[\frac{d^n}{dx^n}u(x)\right]'+(n+1)\left[\frac{d^n}{dx^n}u(x)\right]=n\left[\frac{d^n}{dx^n}u(x)\right]'$$

$$\text{(ie) } x\left[\frac{d^n}{dx^n}u(x)\right]''+(1+x)e^x\left[\frac{d^n}{dx^n}u(x)\right]+(n+1)e^x\left[\frac{d^n}{dx^n}u(x)\right]=0 \quad \text{------(7.11)}$$

$$\text{Now } L_n(x) = e^x\frac{d^n}{dx^n}\left[x^ne^{-x}\right]$$

$$= e^x\frac{d^n}{dx^n}(u(x)) \text{ because } u(x) = x^n e^{-x}.$$

$$\therefore L_n'(x) = e^x\left[\frac{d^n}{dx^n}(u(x))\right]+e^x\left[\frac{d^n}{dx^n}(u(x))\right]'$$

$$\text{(ie) } L_n'(x) = L_n(x)+e^x\left[\frac{d^n}{dx^n}(u(x))\right]' \quad \text{-------(7.12)}$$

Again differentate with written to x, we get,

$$L_n''(x) = L_n'(x)+e^x\left[\frac{d^n}{dx^n}(u(x))\right]'+e^x\left[\frac{d^n}{dx^n}(u(x))\right]''$$

$$\therefore xL_n''(x)-xL_n'(x) = xe^x\left[\frac{d^n}{dx^n}(u(x))\right]'+xe^x\left[\frac{d^n}{dx^n}(u(x))\right]'' \quad \text{-------(7.13)}$$

$$\text{Now (1)} \Rightarrow xe^x\left[\frac{d^n}{dx^n}(u(x))\right]''+xe^x\left[\frac{d^n}{dx^n}(u(x))\right]'+e^x\left[\frac{d^n}{dx^n}(u(x))\right]'$$

$$+ne^x\left[\frac{d^n}{dx^n}(u(x))\right]+e^x\left[\frac{d^n}{dx^n}(u(x))\right]=0$$

(ie) $xL_n''(x)-xL_n'(x)+L_n'(x)-L_n(x)+nL_n(x)+L_n(x) = 0$

(ie) $xL_n''(x)+(1-x)L_n'(x)+nL_n(x) = 0$

(ie) L_n satisfies the Laguerre equation if $\alpha = n$.

Example E. 7.10 :

Find the solutions of the Laguerre equation in the form of $\varphi(x) = x^r \sum_{k=0}^{\infty} C_k x^k$

Solution :

Let the Laguerre equation be $xy''+(1-x)y'+\alpha y = 0$.

$$\begin{aligned}
\text{Let } \varphi(x) &= x^r \sum_{k=0}^{\infty} C_k x^k \\
&= x^2(C_0+C_1x+C_2x^2+\ldots\ldots) \\
&= C_0x^r+C_1x^{r+1}+C_2x^{r+2}+\ldots\ldots \\
\therefore\ \varphi'(x) &= C_0rx^{r-1}+C_1(r+1)x^r+C_2(r+2)x^{r+1}+\ldots\ldots \\
\text{and } \varphi''(x) &= C_0r.(r-1)x^{r-2}+C_1(r+1)rx^{r-1}+C_2(r+2)(r+1)x^r+\ldots\ldots \\
\text{Now } x\varphi''(x) &= C_0r(r-1)x^{r-1}+C_1(r+1)rx^r+C_2(r+2)(r+1)x^{r+1}+\ldots\ldots \\
\text{and } (1-x)\varphi'(x) &= \varphi'(x)-x\varphi'(x) \\
&= C_0rx^{r-1}-C_0rx^r+C_1(r+1)x^r-C_1(r+1)x^{r+1}+C_2(r+2)x^{r+1} \\
&\qquad -C_2(r+2)x^{r+2}+\ldots\ldots \\
\text{Thus } \alpha\varphi(x) &= C_0\alpha x^r+C_1\alpha x^{r+1}+C_2\alpha x^{r+2}+\ldots\ldots
\end{aligned}$$

Since it is given that φ satisfies the Laguerr equation, we have,

$x\varphi''(x)+(1-x)\varphi'(x)+\alpha\varphi(x) = 0$

(ie) $C_0x^{r-1}[r^2-r+r]+x^r[C_1r(r+1)-C_0r+C_1(r+1)+C_0\alpha] + x^{r+1}[C_2(r+2)(r+1)-C_1(r+1)$

$+C_2(r+2)+C_1\alpha]+\ldots\ldots\ldots\ldots = 0$

(ie) $r^2C_0x^{r-1}+x^r[C_1r^2+C_1r-C_0r+C_1r+C_1+C_0r]$

$+ x^{r+1}[C_2(r+2)(r+2)-C_1(r+1)+C_1\alpha]+\ldots\ldots\ldots\ldots = 0$

(ie) $r^2C_0x^{r-1}+x^r[C_1(r+1)^2-C_0r+C_0\alpha] + x^{r+1}[C_2(r+2)^2-C_1(r+1)+C_1\alpha]+\ldots\ldots = 0$

Let $q(r) = r^2$

$\therefore\ q(r+1) = (r+1)^2$ and $q(r+2) = (r+2)^2$.

Now $C_0q(r)x^{r-1}+x^r[q(r+1)C_1+(\alpha-r)C_0]+x^{r+1}[q(r+2)C_2+\{\alpha-(r+1)C_1\}+\ldots.. = 0$

(ie) $C_0q(r)x^{r-1}+x^r\sum_{r=1}^{\infty}[q(r+k)C_k+\{\alpha-(r+k-1)\}C_{k-1}]x^{k-1}=0$

Equating the coefficients of like powers of x on both sides, we get,

$$C_0q(r) = 0$$

$$\text{(ie)}\quad q(r) = 0$$

$$\text{(ie)}\ r^2 = 0 \Rightarrow r=0$$

Also $q(r+k)C_k+\{\alpha-(r+k-1)\}C_{k-1} = 0$

$$\text{(ie)}\quad C_k = \frac{\{r+k-1-\alpha\}C_{k-1}}{q(r+k)},\quad k=1,2,3,\ldots\ldots$$

With loss of generality assume that $C_0 = 1$

When $k = 1$, then $C_1 = \dfrac{-\alpha}{q(0+1)} = \dfrac{-\alpha}{1^2}$

When $k=2$, then $C_2 = \dfrac{(-\alpha)(1-\alpha)}{1^2-2^2}$

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In general, $C_k = \dfrac{(-\alpha)(1-\alpha)\ldots\ldots(k-1-\alpha)}{1^22^2\ldots\ldots..k^2}$

$$= \frac{(-\alpha)(1-\alpha)\ldots..(k-1-\alpha)}{(k)!^2}$$

$$\text{Thus } \varphi(x) = x^r\sum_{k=0}^{\infty}C_kx^k$$

$$= \sum_{k=0}^{\infty}C_kx^k\quad(\because r=0)$$

$$= 1+\sum_{k=1}^{\infty}\frac{(-\alpha)(1-\alpha)\ldots..(k-1-\alpha)}{(k)!^2}x^k$$

7.4 Bessel Equation :

Definition D.7.6 :

The Bessel equation of order α is given by $x^2y''+xy'+(x^2-\alpha^2)y=0$ where α is a constant with $\text{Re}\alpha \geq 0$.

Result :

Consider the equation $x^2y''+a(x)y'+b(x)y = 0$ where a, b have power series expansions which are convergent for $|x|<r_0$. Let r_1, r_2 be the roots of the indicial polynomial $q(r)=r(r-1)+a(0)+b(0)$. If $r_1=r_2$, then the two linearly independent power series solution φ_1, φ_2 of the form

$$\varphi_1(x) = |x|^{r_1}\sigma_1(x),\ \varphi_2(x)=|x|^{r_1+1}\sigma x(x)+(\log|x|).\varphi_1(x).$$

where σ_1, σ_2 have power series expansion which are convergent for $|x|<r_0$ and $\sigma_1(0)\neq 0$.

If r_1-r_2 is a positive integer, then there are two linearly independent solution φ_1, φ_2 of the form $\varphi_1(x)=|x|^{r_1}\sigma_1(x)$, $\varphi_2(x)=|x|^{r_2}\sigma_2(x)+C.\varphi_1(x).\log|x|$ where C is a constant.

Theorem 7.2 :

The Bessel function of zero order of the first kind is J_0 which is given by

$$J_0 = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m)!^2}\left(\frac{x}{2}\right)^{2m}$$

Proof :

We know that the Bessel equation of order α is given by

$$x^2y''+xy'+(x^2-\alpha^2)y = 0 \qquad \text{-------(7.14)}$$

where α is a constant with $\text{Re}\alpha \geq 0$.

Comparing the equation (7.14) with $x^2y''+a(x)xy'+b(x)y=0$, we have, $a(x)=1$, $b(x)=x^2-\alpha^2$.

$\therefore\ a(0) = 1,\ b(0) = -\alpha^2$.

Now the indicial polynomial q(r) is given by

$$q(r) = r(r-1)+a(0)r+b(0)$$
$$= r^2-\alpha^2$$

If $q(r) = 0$ then $r = \pm\alpha$.

(ie) $\pm\alpha$ arethe roots of $q(r)=0$

Let $x>0$.

Consider $\alpha = 0$, then the roots of $q(r)=0$ are equal

$\therefore \varphi_1(x)=\sigma_1(x)$, $\varphi_2(x)=\sigma_2(x)+(\log x)\,\varphi_1(x)$

where σ_1, σ_2 are power series expansion of x_1

The Bessel equation of order zero is $x^2y''+xy'+x^2y = 0$

(ie) $L(y)=0$ where $L(y)=x^2y''+xy'+x^2y$

$$\text{Suppose } \sigma_1(x) = \sum_{k=0}^{\infty} C_k x^k \text{ with } C_0 \neq 0$$

$$\therefore \sigma_1'(x) = \sum_{k=1}^{\infty} kC_k x^{k-1}$$

$$\text{and } \sigma_1''(x) = \sum_{k=2}^{\infty} k(k-1)C_k x^{k-2}$$

If $\sigma_1(x)$ is a solution of $L(y) = 0$ then $L(\sigma_1) = 0$

(ie) $x^2\sigma_1''(x)+x\sigma_1'(x)+x^2\sigma_1(x) = 0$

$$\text{(ie) } \sum_{k=2}^{\infty} k(k-1)C_k x^k + \sum_{k=1}^{\infty} kC_k x^k + \sum_{k=0}^{\infty} C_k x^{k+2} = 0$$

$$\text{(ie) } \sum_{k=2}^{\infty} k(k-1)C_k x^k + \sum_{k=1}^{\infty} kC_k x^k + \sum_{k=2}^{\infty} C_{k-2} x^k = 0$$

$$\text{(ie) } C_1x + \sum_{k=2}^{\infty} [k(k-1)C_k + kC_k + C_{k-2}]x^k = 0$$

Comparing the coefficient of like powers of x^k on both sides, we get,

$$C_1 = 0$$

$$\text{and } k(k-1)C_k + kC_k + C_{k-2} = 0$$

$$k^2C_k + C_{k-2} = 0$$

$$\text{(ie) } C_k = \frac{-C_{k-2}}{k^2} \text{ where } k=2,3,4,\ldots\ldots$$

when k=2 then $C_2 = \dfrac{-C_0}{2^2}$

when k=3, then $C_3 = 0$

when k=4, then $C_4 = \dfrac{-C_2}{4^2} = \dfrac{(-1)^2 C_0}{2^2.4^2}$

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In general, $C_{2m} = \dfrac{(-1)^m C_0}{2^2.4^2\ldots.(2m)^2}$

Without loss of generality choose $C_0 = 1$.

$\therefore C_{2m} = \dfrac{(-1)^m}{(m!)^2 2^{2m}}$, m=1,2,3,........

and $C_{2m+1} = 0$, m=0,1,2,......

$$\therefore \sigma_1(x) = \sum_{k=0}^{\infty} C_k x^k$$

$$= C_0 + \sum_{m=1}^{\infty} C_{2m}x^{2m} + \sum_{m=0}^{\infty} C_{2m+1}x^{2m+1}$$

$$= 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2 2^{2m}} x^{2m}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(m!)^2 2^{2m}}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m}$$

Hence $\sigma_1(x)$ is the Bessel function of zero order of the first kind which denoted by J_0.

Hence $J_0 = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}$

This proves the theorem.

Theorem 7.3 :

Bessel function of order α of the first kind is

$$J_\alpha(x) = \left(\frac{x}{2}\right)^{\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m}$$

Proof :

The Bessel equation of order α is given by $x^2y''+xy'+(x^2-\alpha^2)y=0$ where α is a constant with $Re\alpha\geq 0$.

Comparing with the equation $x^2y''+a(x)xy'+b(x)y=0$, we have, $a(x)=1$, $b(x)=x^2-\alpha^2$.

$\therefore a(0) = 1, b(0) = -\alpha^2$.

The roots of indicial equation are given by $q(r)=0$

(ie) $r(r-1)+a(0)r+b(0) = 0$

(ie) $r^2-\alpha^2 = 0$

(ie) $r = \pm\alpha$.

Assume that $x>0$.

$$\text{Let } \varphi_1(x) = x^{\alpha} \sum_{k=0}^{\infty} C_k x^k \text{ where } C_0 \neq 0.$$

$$= C_0x^{\alpha}+C_1x^{\alpha+1}+C_2x^{\alpha+2}+\ldots\ldots$$

$$\left.\begin{aligned} \therefore \varphi_1'(x) &= \alpha C_0x^{\alpha-1}+(\alpha+1)C_1x^{\alpha}+(\alpha+2)C_2x^{\alpha+1}+\ldots\ldots \\ \text{and } \varphi_1''(x) &= C_0\alpha(\alpha-1)x^{\alpha-2}+(\alpha+1)\alpha C_1x^{x-1}+(\alpha+2)(\alpha+1)C_2x^{\alpha}+\ldots\ldots \end{aligned}\right\} \text{------}(7.15)$$

We know that the Bessel equation of order α is $x^2y''+xy'+(x^2-\alpha^2)y = 0$

If φ_1 is a solution of $L(y) = 0$ where

$L(y) = x^2y''+xy'+(x^2-\alpha^2)y$, then we have, $L(\varphi_1) = 0$

(ie) $x^2\varphi_1''+x\varphi_1'+(x^2-\alpha^2)\varphi_1 = 0$

Substituting φ_1'' & φ_1' values from (7.15), we get after simplification as

$$C_1 x^{\alpha+1}\left[(\alpha+1)^2-\alpha^2\right]+x^{\alpha}\sum_{k=2}^{\infty}\left[\left\{(x+k)^2-\alpha^2\right\}C_k+C_{k-2}\right]x^k=0$$

Comparing the coefficients of like powers of x on both sides, we get,

$$C_1 = 0$$

$$\text{and } \{(\alpha+k)^2-\alpha^2\}C_k+C_{k-2} = 0$$

$$\text{(ie) } C_k = \frac{-C_{k-2}}{k(k+2\alpha)},\quad k=2,3,\ldots\ldots$$

$$\text{When } k=2, \text{ then } C_2 = \frac{-C_0}{2(2+2\alpha)}$$

$$= \frac{-C_0}{2^2(1+\alpha)}$$

$$\text{When } k=3, \text{ then } C_3 = 0$$

$$\text{When } k=4, \text{ then } C_4 = \frac{-C_2}{4(4+2\alpha)}$$

$$= \frac{(-1)^2 C_0}{2^4(2!)(\alpha+1)(\alpha+2)}$$

$$\text{When } k=5, \text{ then } C_5 = 0$$

$$\text{When } k=6 \text{ then } C_6 = \frac{-C_4}{6(6+2\alpha)}$$

$$= \frac{(-1)^3 C_0}{2^6\ 3!(\alpha+1)(\alpha+2)(\alpha+3)}$$

$$\text{In general } C_{2m} = \frac{(-1)^m C_0}{2^{2m}\ m!(\alpha+1)(\alpha+2)\ldots(\alpha+n)},\quad m=1,2,3,\ldots\ldots$$

$$\text{and } C_{2m+1} = 0,\ m=0,1,2,3,\ldots\ldots$$

$$\text{Now } \varphi_1(x) = x^{\alpha}\sum_{k=0}^{\infty}C_k x^k$$

$$= x^{\alpha}\left[C_0+\sum_{m=1}^{\infty}C_{2m}x^{2m}+\sum_{m=1}^{\infty}C_{2m+1}x^{2m+1}\right]$$

$$= x^{\alpha}\left[C_0 + \sum_{m=1}^{\infty} \frac{(-1)^m C_0 x^{2m}}{2^{2m}\; m!(\alpha+1)(\alpha+2)....(\alpha+m)}\right]$$

$$= C_0 x^{\alpha} + C_0 x^{\alpha} \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(\alpha+1)(\alpha+2)...(\alpha+m)}\left(\frac{x}{2}\right)^{2m}$$

Choose $C_0 = \dfrac{1}{2^{\alpha}\Gamma(\alpha+1)}$

$$\varphi_1(x) = \frac{x^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)} + \frac{x^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)} \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(\alpha+1)(\alpha+2)...(\alpha+m)}\left(\frac{x}{2}\right)^{2m}$$

$$= \left(\frac{x}{2}\right)^{\alpha} \cdot \frac{1}{\Gamma(\alpha+1)} + \left(\frac{x}{2}\right)^{\alpha} \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(\alpha+1)....(\alpha+m)\Gamma(\alpha+1)}\left(\frac{x}{2}\right)^{2m}$$

$$= \left(\frac{x}{2}\right)^{\alpha} \frac{1}{\Gamma(\alpha+1)} + \left(\frac{x}{2}\right)^{\alpha} \sum_{m=1}^{\infty} \frac{(-1)^m\left(\frac{x}{2}\right)^{2m}}{m!\; 1.2.3.....\alpha(\alpha+1)....(\alpha+m)}$$

$$= \left(\frac{x}{2}\right)^{\alpha} \frac{1}{\Gamma(\alpha+1)} + \left(\frac{x}{2}\right)^{\alpha} \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\Gamma(\alpha+m+1)}\left(\frac{x}{2}\right)^{2m}$$

$$= \left(\frac{x}{2}\right)^{\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m}$$

Here the function $\varphi_1(x)$ is called Bessel function of order α of the first kind and it is denoted by $J_{\alpha}(x)$.

Hence $\mathbf{J_{\alpha}(x)} = \left(\frac{x}{2}\right)^{\alpha} \sum_{m=0}^{\alpha} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m}$

Note that $\mathbf{J_{-\alpha}(x)} = \left(\frac{x}{2}\right)^{-\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-\alpha+1)}\left(\frac{x}{2}\right)^{2m}$

Example E. 7.11 :

Show that $x^{-1/2} J_{-1/2}(x) = \dfrac{\sqrt{2}}{\Gamma(1/2)} \cos x$

Proof :

We know that

$$J_{-\alpha}(x) = \left(\frac{x}{2}\right)^{-\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-\alpha+1)}\left(\frac{x}{2}\right)^{2m}$$

$$\therefore \; J_{-\alpha}(x) = \left(\frac{x}{2}\right)^{-½} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\sqrt{m+(1/2)}}\left(\frac{x}{2}\right)^{2m}$$

$$= \frac{x^{-½}}{2^{-½}}\left[\frac{1}{\sqrt{(1/2)}} - \frac{1}{1!\sqrt{(3/2)}} + \frac{1}{2!\sqrt{(5/2)}}\left(\frac{x}{2}\right)^4 - \ldots\right]$$

$$\text{(ie) } x^{½}J_{-½}(x) = \sqrt{2}\left[\frac{1}{\sqrt{(1/2)}} - \frac{x^2}{2^2.1/2.\sqrt{(1/2)}} + \frac{x^4}{2^5.(3/2).(1/2).\sqrt{(1/2)}} - \ldots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{(1/2)}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{(1/2)}}\cos x$$

This proves the problem.

Example E. 7.12 :

Prove that $\sqrt{x}.J_{1/2}(x) = \frac{\sqrt{2}}{\sqrt{\pi}}\sin x$

Proof :

We know that

$$J_{\alpha}(x) = \left(\frac{x}{2}\right)^{\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m}$$

$$\therefore \; J_{½}(x) = \left(\frac{x}{2}\right)^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+(3/2))}\left(\frac{x}{2}\right)^{2m}$$

$$= \frac{x^{1/2}}{2^{1/2}}\left[\frac{1}{\sqrt{(3/2)}} - \frac{1}{1!\Gamma(5/2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\Gamma(7/2)}\left(\frac{x}{2}\right)^4 \ldots\right]$$

$$= \frac{x^{1/2}}{2^{1/2}}\left[\frac{1}{\frac{1}{2}\sqrt{\frac{1}{2}}} - \frac{x^2}{2^2.\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)} + \frac{x^4}{2^5.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)} - \ldots\right]$$

$$= \frac{x^{1/2}}{x.2^{1/2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left[x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots\right]$$

$$= \frac{\sqrt{2}}{x^{1/2}\Gamma\left(\frac{1}{2}\right)}\sin x$$

$$\text{(ie)}\quad x^{1/2}J_\alpha(x) = \frac{\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)}\sin x$$

This proves the problem.

Example E.7.13 :

Prove that $J_0'(x) = -J_1(x)$.

Proof :

$$\text{We know that } J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m}$$

$$\therefore\ J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+1)}\left(\frac{x}{2}\right)^{2m}$$

$$\text{and } J_1(x) = \frac{x}{2}\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+2)}\left(\frac{x}{2}\right)^{2m}$$

$$\text{Now } J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+1)}2m\left(\frac{x}{2}\right)^{2m-1}\left(\frac{1}{2}\right)$$

$$\therefore\ xJ_0'(x) = 2\sum_{m=1}^{\infty} \frac{(-1)^m}{(m-1)!\,\Gamma(m+1)}\left(\frac{x}{2}\right)^{2m-1}\frac{x}{2}$$

$$= 2\sum_{m=1}^{\infty} \frac{(-1)^m}{(m-1)!\,\Gamma(m+1)}\left(\frac{x}{2}\right)^{2m}$$

$$= 2\sum_{t=0}^{\infty} \frac{(-1)^{t+1}}{t!\,\Gamma(t+2)}\left(\frac{x}{2}\right)^{2(t+1)}$$

$$= -2\sum_{t=0}^{\infty} \frac{(-1)^t}{t!\,\Gamma(t+2)}\left(\frac{x}{2}\right)^{2t+2} \qquad \text{------(7.16)}$$

$$\text{Now } J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+2)}\left(\frac{x}{2}\right)^{2m+1}$$

$$\therefore \quad (x/2)\, J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+2)} \left(\frac{x}{2}\right)^{2m+2} \qquad \text{------(7.17)}$$

Thus from (7.16) & (7.17), we have, $J_0'(x) = -J_1(x)$.

This proves the problem.

Example E.7.14 :

With the usual notations, prove that

$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$.

Solution :

To prove the result, first we shall prove the following two results.

(i) $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$ and

(ii) $xJ_n'(x) = nJ_n(x) + x.J_{n-1}(x)$

Proof of (i) :

We know that $J_n(x) = \sum\limits_{m=0}^{\infty} \dfrac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\dfrac{x}{2}\right)^{2m+n}$

$$\therefore \quad J_n'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m/(2m+n)}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n-1} \left(\frac{1}{2}\right)$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n-1} + \sum_{m=0}^{\infty} \frac{(-1)^m n}{m!\,\Gamma(m+n-1)} \left(\frac{x}{2}\right)^{2m+n-1} \frac{1}{2}$$

$$\therefore \; xJ_n'(x) = x \sum_{m=1}^{\infty} \frac{(-1)^m}{(m-1)!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n-1} + n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n}$$

$$= x \sum_{t=0}^{\infty} \frac{(-1)^{t+1}}{t!\,\Gamma(t+n+2)} \left(\frac{x}{2}\right)^{2(t+1)+n-1} + nJ_n(x) \text{ (where } t=m-1)$$

$$= -x \sum_{t=0}^{\infty} \frac{(-1)^t}{t!\,\Gamma(t+(n+1)+1)} \left(\frac{x}{2}\right)^{2t+n+1} + nj_n(x)$$

$$= -xJ_{n+1}(x) + nJ_n(x)$$

$$\text{(ie)} \quad xJ_n'(x) = -xJ_{n+1}(x) + nJ_n(x) \qquad \text{-------(7.18)}$$

This proves (i).

Proof (ii) :

We know that $J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n}$

$$\therefore\ J_n'(x) = \sum_{m=0}^{\infty} \frac{(-1)^m(2m+n)}{m!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n-1} \cdot \frac{1}{2}$$

$$\text{(ie) } xJ_n'(x) = \sum_{m=0}^{\infty} \frac{(-1)^m(2m+2n-n)}{m!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m(2m+2n)}{m!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n} - n\sum_{m=0}^{\infty} \frac{(-1)^m(x/2)^{2m+n}}{m!\Gamma(m+n+1)}$$

$$= 2\sum_{m=0}^{\infty} \frac{(-1)^m(m+n)}{m!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n} - nJ_n(x)$$

$$= 2\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+n)}\left(\frac{x}{2}\right)^{2m+n} - nJ_n(x)$$

$$\{\because\ \Gamma(n) = n(n-1)!$$

$$= 2\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+(n-1)+1)}\left(\frac{x}{2}\right)^{2m+n-1} \cdot \left(\frac{x}{2}\right) - nJ_n(x)$$

$$= xJ_{n-1}(x) - nJ_n(x)$$

$$\text{(ie) } xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x) \qquad \text{------(7.19)}$$

This proves (7.19).

Proof of the given equation.

Now (7.18) + (7.19) gives us

$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$

This proves the problem.

UNIT – 8

EXISTENCE AND UNIQUENESS OF SOLUTIONS TO FIRST ORDER EQUATIONS

8.1 Introduction :

In this unit we shall discuss the general first order equation $y' = f(x, y)$ ------(8.1)

where f is some continuous function.

We know that any solution φ of $y' = g(x)y+h(x)$ (where g, h are continuous on some interal **I**) can be written in the form

$$\varphi(x) = e^{Q(x)} \int_{x_0}^{x} e^{-Q(t)} h(t)dt + Ce^{Q(x)}$$

$$\text{where } Q(x) = \int_{x_0}^{x} g(t)dt, \quad x_0 \in I \text{ are C is a constant.}$$

8.2 Equations with variables seperated :

Theorem 8.1 :

Let g, h be continuous real-valued functions for $a \le x \le b$, $c \le y \le d$ respectively, and consider the equation $h(y)y' = g(x)$ -------(8.2)

If G, H are any functions such that $G' = g$, $H' = h$ and C is any constant such that the relation $H(y) = G(x)+C$ defines a real-valued differential function φ for x in some interval I contained in $a \le x \le b$, then φ will be a solution of (8.2) on **I**. Conversely, if φ is a solution of (8.2) on **I**, it satisfies the relation $H(y)=G(x)+C$ on **I**, for some constant C.

Proof :

Given that $h(y).y' = g(x)$ ------(8.2)

(ie) $h(y)\, dy = g(x)dx$

Integrating on both sides, we get

$$\int h(y)dy = \int g(x)dx + C \quad \text{-------(8.3)}$$

where C is a constant.

Let H and G are two functions such that $H' = h$ & $G' = g$.

$$\therefore (8.3) \Rightarrow H(y) = G(x) + C$$

Hence any differentiable function φ which is defined implicitly by the relation

$$H(y) = G(x)+C' \qquad \text{--------(8.4)}$$

will be a solution of (8.2).

Conversly, let φ be a solution of (8.2) on I then clearly φ satisfies (8.4) on I.

This proves the theorem.

Definition D.8.1 :

A first order equation $y' = f(x, y)$ is said to have variables seperated if f can be written in the form $f(x,y) = \dfrac{g(x)}{h(y)}$ where g, h are function of x & y alone.

Example E. 8.1 :

Find all real-valued solutions of the followings differential equation.

$$y' = \frac{x+x^2}{y-y^2}$$

Solution :

$$\text{Given that } y' = \frac{x+x^2}{y-y^2} \qquad \text{-------(8.5)}$$

$$\text{(ie)} \quad \frac{dy}{dx} = \frac{x+x^2}{y-y^2}$$

$$\text{(ie)} \quad (y-y^2)dy = (x+x^2)dx$$

Integrating on both sides, we get,

$$\int(y-y^2)dy = \int(x+x^2)dx$$

$$\text{(ie)} \quad \frac{y^2}{2}-\frac{y^3}{3} = \frac{x^2}{2}+\frac{x^3}{3}+C$$

$$\text{(ie)} \quad 3y^2-2y^3 = 3x^2+2x^2+k, \text{ where } k = 2C$$

which is the required solution of (8.5).

Example E. 8.2 :

Show that the solution φ of $y' = y^2$ which passes through (x_0, y_0) is given by

$$\varphi(x) = \frac{y_0}{1-y_0(x-x_0)}$$

Proof :

$$\text{Given that } y' = y^2 \qquad \text{--------(8.6)}$$

$$\text{(ie) } \frac{dy}{dx} = y2$$

$$\therefore \frac{dy}{y^2} = dx$$

Integrating on both sides, we get,

$$\int \frac{dy}{y^2} = \int dx$$

$$\text{(ie) } \frac{-1}{y} = x+C \qquad \text{------(8.7)}$$

If (8.7) passes through (x_0, y_0), then we have, $C = -x_0 - \frac{1}{y_0}$.

$$\therefore \text{ (8.7) becomes, } \frac{-1}{y} = x - x_0 - \frac{1}{y_0}$$

$$\text{(ie) } \frac{1}{y} = -x + x_0 + \frac{1}{y_0}$$

$$= \frac{-y_0(x-x_0)+1}{y_0}$$

$$\text{(ie) } y = \frac{y_0}{1-y_0(x-x_0)} \qquad \text{-------(8.8)}$$

$$\text{Let } \varphi(x) = \frac{y_0}{1-y_0(x-x_0)}$$

From (8.8) $\varphi(x)$ is a solution of (8.6).

Definition D. 8.2 :

A function of defined for real x, y is said to be homogeneous of degree k if $f(tx, ty) = t^k f(x, y)$ for all t, x, y.

Note :

If $y' = f(x, y)$ be a differential equation where $f(x, y)$ is a homogeneous function of degree k, then it can be reduced to variable seperated differential equation.

$$\text{Let } y' = f(x, y) \qquad \text{-------(8.9)}$$

where $f(x, y)$ is a homogeneous function.

$$\text{Put } y = ux$$

$$\therefore\ y' = u+xu'$$

$$\text{Thus (8.9) becomes } u+xu' = f(x, xu')$$

$$= f(1, u)$$

$$\therefore\ u' = \frac{f(1,u)-u}{x}$$

which is variable seperated differential equation.

Example E.8.3 :

Find all real-valued solutions of $y' = \dfrac{x+y}{x-y}$.

Solution :

$$\text{Given that } y' = \frac{x+y}{x-y} \qquad \text{------(8.10)}$$

$$\text{(ie) } y' = f(x, y) \text{ where } f(x, y) = \frac{x+y}{x-y}$$

$$\text{Now } f(tx, ty) = \frac{tx+ty}{tx-ty}$$

$$= \frac{t(x+y)}{t(x-y)}$$

$$= t^0 f(x, y)$$

$\therefore$ $f(x, y)$ is a homogeneous function of degree zero.

$$\text{Thus put } y = ux$$

$$\therefore\ y' = u+xu'$$

Hence (8.10) becomes,

$$u+xu' = \frac{x+ux}{x-ux}$$

$$\text{(ie) } u+xu' = \frac{1+u}{1-u}$$

$$\therefore\ xu' = \frac{1+u}{1-u}-u$$

$$\text{(ie) } x\frac{du}{dx} = \frac{1+u-u+u^2}{1-u}$$

$$= \frac{1+u^2}{1-u}$$

$$\therefore\ \frac{1-u}{1+u^2}du = \frac{dx}{x}$$

Integrating on both sides, we get,

$$\int\frac{1-u}{1+u^2}du = \int\frac{dx}{x}$$

$$\text{(ie) } \int\frac{dx}{x} = \int\frac{1-u}{1+u^2}du$$

$$\text{(ie) } \log x = \int\frac{du}{1+u^2}-\int\frac{udu}{1+u^2}$$

$$= \tan^{-1}(u)-\frac{1}{2}\log(1+u^2)+\log C$$

$$\text{(ie) } \tan^{-1}(u) = \log x+\frac{1}{2}\log(1+u^2)-\log C$$

$$\text{(ie) } \tan^{-1}(u) = \log\left(\frac{x\sqrt{1+u^2}}{C}\right)$$

$$\text{(ie) } \tan^{-1}\left(\frac{y}{x}\right) = \log\left(C_1.x\sqrt{1+\left(\frac{y}{x}\right)^2}\right), \text{ where } C_1 = \frac{1}{C}$$

$$= \log\left(C_1.\sqrt{x^2+y^2}\right)$$

which is the required solution of (8.10).

Example E.8.4 :

Find all real valued solutions of $y' = \dfrac{x^2+xy+y^2}{x^2}$

Solution :

$$\text{Given that } y' = \frac{x^2+xy+y^2}{x^2} \qquad \text{-------(8.11)}$$

$$\text{(ie) } y' = (x, y) \text{ where } f(x, y) = \frac{x^2+xy+y^2}{x^2}$$

$$\text{Now } f(tx, ty) = \frac{t^2x^2+txty+t^2y^2}{t^2x^2}$$

$$= \frac{t^2(x^2+xy+y^2)}{t^2x^2}$$

$$= t^o f(x, y)$$

(ie) $f(x, y)$ is a homogeneous function of degree 0.

Hence put $y = ux$

$$\therefore y' = u+xu'$$

Thus (8.11) changes as,

$$u+xu' = \frac{x^2+xux+u^2x^2}{x^2}$$

$$\text{(ie) } u+xu' = 1+u+u^2$$

$$\text{(ie) } x\frac{du}{dx} = 1+u^2$$

$$\text{(ie) } \frac{du}{1+u^2} = \frac{dx}{x}$$

Integrating on both sides, we get,

$$\int\frac{du}{1+u^2} = \int\frac{dx}{x}$$

$$\text{(ie) } \tan^{-1}(u) = \log x + \log c$$

$$\text{(ie) } \tan^{-1}(u) = \log(cx)$$

$$\text{(ie) } \tan^{-1}\left(\frac{y}{x}\right) = \log(cx) \quad (\text{since } y = ux)$$

which is the required solution of (8.11).

Note :

$$\text{The equation } y' = \frac{a_1x+b_1y+c_1}{a_2xb_2y+c_2} \qquad \text{------(8.12)}$$

where a_1, b_1, c_1, a_2, b_2, c_2 are constants and c_1, c_2 not both zero can be reduced to a homogeneous equation by assuming

$x = X+h$ and $y = Y+k$, where h, k are constants.

$\therefore$ (8.12) becomes,

$$\frac{dY}{dX} = \frac{a_1X+b_1Y+(a_1h+b_1k+c_1)}{a_2X+b_2Y+(a_2h+b_2k+c_2)} \qquad \text{-------(8.13)}$$

Choose the constants h, k so that

$$\left.\begin{aligned} a_1h+b_1k+c_1 &= 0 \quad \text{and} \\ a_2h+b_2k+c_2 &= 0 \end{aligned}\right\} \qquad \text{-------(8.14)}$$

then (8.12) becomes a homogeneous function.

If such constants h, k are not exists, (ie) (8.14) has no solution then $a_1b_2-a_2b_1=0$.

In this case consider either $u = a_1x+b_1y+c_1$ or $u = a_2x+b_2y+c_2$ then (8.12) reduced to a seperation of variables.

Example E.8.5 :

Solve the following differential equation $y = \dfrac{x-y+2}{x+y-1}$

Solution :

$$\text{Given that } y' = \frac{x-y+2}{x+y-1} \qquad \text{-------(8.15)}$$

Take $x = Y+h$ and $y = Y+k$

$$\therefore (8.15) \Rightarrow \frac{dY}{dX} = \frac{X+h-Y-k+2}{X+h+Y+k-1}$$

$$\text{(ie)} \ \frac{dY}{dX} = \frac{X-Y+(h-k+2)}{X+Y+(h+k-1)} \qquad \text{------(8.16)}$$

Consider $h-k+2 = 0$ and $h+k-1 = 0$.

By solving the above equations, using cross multiplication method, we have,

$$\frac{h}{+1-2} = \frac{k}{2+1} = \frac{1}{+1+1}$$

$$\text{(ie)}\ \frac{h}{-1} = \frac{k}{3} = \frac{1}{2}$$

$$\therefore\ h = \frac{-1}{2},\ k = \frac{3}{2}$$

Hence (8.16) becomes, $$\frac{dY}{dX} = \frac{X-Y}{X+Y} \quad \text{------(8.17)}$$

Clearly (8.17) is a homogeneous differential equation of order zero.

$$\text{Put } Y = uX$$

$$\therefore\ \frac{dY}{dX} = u+X\frac{dY}{dX}$$

Hence (8.17) becomes,

$$u+X\frac{dY}{dX} = \frac{X-uX}{X+uX}$$

$$\text{(ie)}\ u+X\frac{dY}{dX} = \frac{1-u}{1+u}$$

$$\text{(ie)}\ X\frac{dY}{dX} = \frac{1-u}{1+u}-u$$

$$= \frac{1-u-u-u^2}{1+u}$$

$$= \frac{1-2u-u^2}{1+u}$$

$$\text{(ie)}\ \frac{1+u}{1-2u-u^2}du = \frac{dX}{X}$$

Integrating on both sides, we get,

$$\int\frac{dX}{X} = \int\frac{1+u}{1-2u-u^2}du$$

$$\text{(ie)}\ \log X = \frac{-1}{2}\int\frac{-2(1+u)}{1-2u-u^2}du$$

$$= \frac{-1}{2}\log(1-2u-u^2)+\log c$$

$$\text{(ie)}\ \log X = \log\left(\frac{c}{\sqrt{1-2u-u^2}}\right)$$

$$\therefore \quad X = \frac{c}{\sqrt{1-2u-u^2}}$$

$$\text{(ie)} \quad X.\sqrt{1-2u-u^2} = c$$

$$\text{(ie)} \quad X.\sqrt{1-2\frac{y}{x}-\frac{y^2}{x^2}} = c$$

$$\text{(ie)} \quad \sqrt{X^2-2XY-Y^2} = c$$

$$\text{(ie)} \quad X^2-2XY-Y^2 = c_1 \quad , \text{where } c_1 = c^2 \qquad \text{--------(8.18)}$$

$$\text{Since } x = X+h \Rightarrow X = x-\frac{1}{2}$$

$$\Rightarrow X = x+\frac{1}{2}$$

$$\text{and } y = Y+k \Rightarrow Y = y-k$$

$$\Rightarrow Y = y-\frac{3}{2}$$

$$\therefore (8.18) \Rightarrow \left(x+\frac{1}{2}\right)^2 - 2\left(x+\frac{1}{2}\right)\left(y-\frac{3}{2}\right) - \left(y-\frac{3}{2}\right)^2 = c_1$$

which is the required solution.

Example E.8.6 :

$$\text{Solve } y' = \frac{2x+3y+1}{x-2y-1}$$

Solution :

$$\text{Given that } y' = \frac{2x+3y+1}{x-2y-1} \qquad \text{------(8.19)}$$

Clearly both Nr, Dr on the right hand side of (8.19) are linear expressions in x & y and therefore take $x = X+h$ and $y = Y+k$

$$\text{Thus (8.19) changes as } \frac{dY}{dX} = \frac{2X+3Y+(2h+3k+1)}{X-2Y+(h-2k-1)} \qquad \text{------(8.20)}$$

Consider the equation $2h+3k+1 = 0$ & $h-2k-1 = 0$

Solving the above equations, we get,

$$\frac{h}{-3+2} = \frac{k}{1+2} = \frac{1}{-4-3}$$

$$\frac{h}{-1} = \frac{k}{3} = \frac{1}{-7}$$

(ie) $h = \frac{1}{7}, \quad k = \frac{-3}{7}$

$\therefore x = X + \frac{1}{7}$ & $y = Y - \frac{3}{7}$

Hence (8.20) becomes, $\frac{dY}{dX} = \frac{2X+3Y}{X-2Y}$ -------(8.21)

Clearly (8.21) is a homogeneous linear differential equation

$$\text{put } Y = uX$$

$$\therefore \frac{dY}{dX} = u + X\frac{dY}{dX}$$

Hence (8.21) $\Rightarrow u + X\frac{dY}{dX} = \frac{2X+3uX}{x-2uX}$

$$\text{(ie)} \quad X\frac{du}{dX} = \frac{2+3u}{1-2u} - u$$

$$= \frac{2+3u-u+2u^2}{1-2u}$$

$$= \frac{2+2u+2u^2}{1-2u}$$

$$\text{(ie)} \quad \frac{1-2u}{2(1+u+u^2)}du = \frac{dX}{X}$$

Integrating on both sides, we get,

$$\int \frac{1-2u}{1+u+u^2}du = 2\int \frac{dX}{.X}$$

$$\text{(ie) } 2\log X = I \quad \text{-------(8.22)}$$

$$\text{where } I = \int \frac{1-2u}{1+u+u^2}du$$

$$\text{Let } 1-2u = A\frac{d}{du}(Dr)+B$$

$$\text{(ie) } 1-2u = A(1+2u)+B \qquad \text{-------(8.22)}$$

put $u=-\frac{1}{2}$ in (8.23), we have,

$$1+1 = B \Rightarrow \mathbf{B} = \mathbf{2}$$

put u = 0 in (8.23), we have

$$\begin{aligned} 1 = A+B \Rightarrow A &= 1-B \\ &= 1-2 \\ &= -1 \\ \text{(ie) } \mathbf{A} &= \mathbf{-1} \end{aligned}$$

$\therefore$ (8.23) becomes $1-2u = -(1+2u)+2$

$$\begin{aligned} \text{Hence } I &= \int \frac{-(1+2u)+2}{1+u+u^2}du \\ &= -\int \frac{1+2u}{1+u+u^2}du + 2\int \frac{du}{1+u+u^2} \\ &= -\log(1+u+u^2)+2I_1 \end{aligned}$$

$$\text{where } I_1 = \int \frac{du}{1+u+u^2}$$

$$\begin{aligned} \text{Now } 1+u+u^2 &= \left(u+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 \\ &= \left(u+\frac{1}{2}\right)^2 - \frac{1}{4} + 1 \\ &= \left(u+\frac{1}{2}\right)^2 + \frac{3}{4} \\ &= \left(u+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 \end{aligned}$$

$$\therefore I_1 = \int \frac{du}{\left(u+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{u+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right)$$

$$\text{Hence I} = -\log(1+u+u^2)+\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right)$$

$$\text{Thus (8.22)} \Rightarrow 2\log X = -\log(1+u+u^2)+\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right)+c$$

$$\text{(ie) } \log\left(\frac{X^2}{1+u+u^2}\right) = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right)+c$$

$$\text{(ie) } \log\left(\frac{X^2}{1+\frac{Y}{X}+\frac{Y^2}{X^2}}\right) = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2\frac{Y}{X}+1}{\sqrt{3}}\right)+c$$

$$\text{(ie) } \log\left(\frac{X^4}{X^2+XY+Y^2}\right) = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2Y+X}{\sqrt{3}X}\right)+c$$

$$\text{(ie) } \log\left(\frac{\left(x-\frac{1}{7}\right)^4}{\left(x-\frac{1}{7}\right)^2+\left(x-\frac{1}{7}\right)\left(y+\frac{3}{7}\right)+\left(y+\frac{3}{7}\right)^2}\right) = \frac{2}{\sqrt{3}}\left[\frac{2\left(y+\frac{3}{7}\right)+\left(x-\frac{1}{7}\right)}{\sqrt{3}\left(x-\frac{1}{7}\right)}\right]+c$$

$$\left(\because x = X+\frac{1}{7},\ y = Y-\frac{3}{7}\right)$$

which is the required solution of (8.19).

8.3 Exact Equations :

The first order differential equation $M(x,y)+N(x,y).y' = 0$ where M, N are real-valued functions defined for real x, y on some rectangle R and is said to be **exact** in R if there exists a function F having continuous first partial derivatives in R such that $\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$ in R.

Theorem 8.2 :

Let M, N be two real-valued functions which have continuous first partial derivatives on some rectangle R : $|x-x_0|\le a$, $|y-y_0|\le b$. Then the equation $M(x,y)+N(x,y)y'=0$ is exact if and only if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

Proof :

Let $M(x, y)+N(x, y)y' = 0$ be exact in R.

$\therefore$ there exists a functions f which has continuous second derivatives such that

$$\frac{\partial F}{\partial x} = M \text{ and } \frac{\partial F}{\partial y} = N$$

$$\therefore \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial M}{\partial y}, \quad \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

Since F is continuous, we have

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$$

$$\text{(ie)} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Conversly assume that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ be satisfied in R.

Now we shall find a function F satisfying

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N.$$

$$\begin{aligned} \text{Now } F(x, y) - F(x_0, y_0) &= F(x, y)-F(x_0, y)+F(x_0, y)-F(x_0, y_0) \\ &= \int_{x_0}^{x} \frac{\partial F(s,y)}{\partial x}.ds+\int_{y_0}^{y} \frac{\partial F(x_0,t)}{\partial y}dt \\ &= \int_{x_0}^{x} M(s,y)ds+\int_{y_0}^{y} N(x_0,t)dt \end{aligned}$$

Similarly,

$$\begin{aligned} F(x, y)-F(x_0, y_0) &= F(x, y)-F(x, y_0)+F(x, y_0)-F(x_0, y_0) \\ &= \int_{y_0}^{y} \frac{\partial F}{\partial y}(x,t).dt + \int_{x_0}^{x} \frac{\partial F}{\partial x}(s,y_0)ds \\ &= \int_{y_0}^{y} N(x,t)dt + \int_{x_0}^{x} M(s,y_0)ds. \qquad \text{------(8.23)} \end{aligned}$$

Define a function F as

$$F(x, y) = \int_{x_0}^{x} M(s,y)ds + \int_{y_0}^{y} N(x_0,t)dt \qquad \text{------(8.24)}$$

Clearly $F(x_0, y_0) = 0$ and $\frac{\partial F}{\partial x}(x,y) = M(x, y)$ for all (x, y) in R.

Claim :

$$F(x, y) = \int_{y_0}^{y} N(x,t)dt + \int_{x_0}^{x} M(s,y_0)ds \qquad \text{------(8.25)}$$

Now $F(x,y) - \left[\int_{x_0}^{x} M(s,y_0)ds + \int_{y_0}^{y} N(x,t)dt\right]$

$$\begin{aligned} &= \left[\int_{x_0}^{x} M(s,y)ds + \int_{y_0}^{y} N(x_0,t)dt\right] - \left[\int_{x_0}^{x} M(s,y_0)ds + \int_{y_0}^{y} N(x,t)dt\right] \\ &= \int_{x_0}^{x} [M(s,y_0) - M(s,y_0)]ds - \int_{y_0}^{y} [N(x,t) - N(x_0,t)]dt \\ &= \int_{x_0}^{x}\left[\int_{y_0}^{y} \frac{\partial M}{\partial y}(s,t).dt\right]ds - \int_{y_0}^{y}\left[\int_{x_0}^{x} \frac{\partial N}{\partial x}(s,t).ds\right]dt \\ &= \int_{x_0}^{x}\int_{y_0}^{y}\left[\frac{\partial M}{\partial y}(s,t) - \frac{\partial N}{\partial x}(s,t)\right]ds.dt \\ &= 0 \quad \left(\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right) \end{aligned}$$

$$\therefore F(x, y) = \int_{y_0}^{y} N(x,t)dt + \int_{x_0}^{x} M(s,y_0)ds$$

Hence the function $M(x, y)+N(x,y)y' = 0$ is exact.

This proves the theorem.

Example E.8.7 :

Verify that the equation $y' = \dfrac{3x^2 - 2xy}{x^2 - 2y}$ is exact or not. If it is exact solve it.

Solution :

Given that $y' = \dfrac{3x^2 - 2xy}{x^2 - 2y}$

(ie) $\dfrac{dy}{dx} = \dfrac{3x^2 - 2xy}{x^2 - 2y}$

(ie) $(x^2-2y)dy = (3x^2-2xy)dx$

(ie) $(3x^2-2xy)dx-(x^2-2y)dy = 0$ ------(8.25)

Comparing (8.25) with $M(x, y)dx+N(x,y)dy = 0$, we have,

$M(x, y) = 3x^2-2xy$, $N(x, y) = -(x^2-2y)$

$\therefore \dfrac{\partial M}{\partial y} = -2x$ & $\dfrac{\partial N}{\partial x} = -2x$.

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and therefore the given equation is exact.

To find the solution

(ie) we want to find a function F such that $\dfrac{\partial F}{\partial x} = M$ and $\dfrac{\partial F}{\partial y} = N$.

If $\dfrac{\partial F}{\partial x} = M$, then $\dfrac{\partial F}{\partial x} = 3x^2-2xy$

$$\text{(ie) } \partial F = (3x^2-2xy)\partial x$$

$$\therefore \int \partial F = \int (3x^2 - 2xy)\partial x$$

$$\text{(ie) } F = x^3-x^2y+f(y) \quad \text{------(8.26)}$$

Now differentiate F partially with respect to y, we get,

$$\frac{\partial F}{\partial y} = -x2+f'(y)$$

$$\text{If } \frac{\partial F}{\partial y} = N \text{ then } 2y-x^2 = -x^2+f'(y).$$

$$\text{(ie) } f'(y) = 2y$$

$$\text{Hence } f(y) = y^2+c.$$

Thus (8.26) becomes,

$$F = x^3-x^2y+y2+c$$

$\therefore$ The required solution is

$$x^3-x^2y+y^2+c = 0$$

Example E.8.8 :

Verify that the equation whether exact or not, if it is exact, find the solution

$(2ye^{2x}+2x\cos y)dx+(e^{2x}-x^2\sin y)dy = 0$

Solution :

Given that $(2ye^{2x}+2x\cos y)dx+(e^{2x}-x^2\sin y)dy = 0$ --------(8.27)

Comparing (8.27) with $Mdx+Ndy = 0$, we have,

$M = 2ye^{2x}+2x\cos y$ and $N = e^{2x}-x^2\sin y$

$$\therefore \frac{\partial M}{\partial y} = 2e^{2x}-2x\sin y \text{ and } \frac{\partial N}{\partial x} = 2e^{2x}-2x\sin y$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and therefore (8.27) is exact.

Now we shall find the solution of (8.27).

Let F be a function such that $\frac{\partial F}{\partial x} = M$ & $\frac{\partial F}{\partial y} = N$.

If $\frac{\partial F}{\partial x} = M$

(ie) $\frac{\partial F}{\partial x} = 2ye^{2x}+2x\cos y$

(ie) $\partial F = (2ye^{2x}+2x\cos y)\partial x$

Integrating with respect to x by treating y as constant, we get,

$$\int \partial F = \int\left(2ye^{2x}+2x\cos y\right)dx$$

$$\text{(ie) } F = ye^{2x}+x\cos y+f(y) \qquad \text{------(8.28)}$$

Now differentiate (8.28) partially with respect to y, we have,

$$\frac{\partial F}{\partial y} = e^{2x} - x\sin y + f'(y) \qquad \text{-------(8.29)}$$

If $\frac{\partial F}{\partial y}$=N and by (8.29), we have,

$$e^{2x} - x\sin y + f'(y) = e^{2x} - x^2\sin y$$

$$\text{(ie)} \quad f'(y) = x\sin y - x^2\sin y$$

$$= (x - x^2)\sin y$$

Integrating partially with respect to y, we get,

$$F = (x - x^2)(-\cos y) + c$$

$$= (x^2 - x)\cos y + c$$

$\therefore$ (8.28) becomes $F = ye^{2x} + x\cos y + x^2\cos y - x\cos y$

$$\text{(ie)} \quad F = ye^{2x} + x^2\cos y + c$$

Hence the required solution is $ye^{2x} + x^2\cos y + c = 0$

Definition D. 8.3 :

Suppose the differentiate equation M(x,y)dx+N(x,y)dy = 0 is not exact, then there is a function u(x,y) called *integrating factor* if

u(x,y)[M(x,y)dx+N(x,y)dy] = 0 is exact.

Example E.8.9 :

Consider the equation Mdx+Ndy = 0 where M, N have continuous first partial derivatives on some rectangle R. Prove that a function u on R, having continuous first partial derivatives, is an integrating factor iff

$$u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y} \text{ on R.}$$

Proof :

Suppose Mdx+Ndy = 0 is not exact, then we find an integrating factor u(x, y) such that u[Mdx+Ndy] = 0 is exact.

Let $M_1(x, y) = u(x,y).M(x,y)$ and

$N_1(x, y) = u(x,y).N(x,y)$

$$\text{Now } M_1dx+N_1dy = 0 \text{ is exact}$$

$$\text{iff } \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

$$\text{iff } \frac{\partial}{\partial y}(uM) = \frac{\partial}{\partial x}(uN)$$

$$\text{iff } \frac{\partial u}{\partial y}M + u\frac{\partial M}{\partial y} = \frac{\partial u}{\partial x}N + u\frac{\partial N}{\partial x}$$

$$\text{iff } u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N.\frac{\partial u}{\partial x} - M.\frac{\partial u}{\partial y}$$

Example E.8.10 :

Show that if the equation Mdx+Ndy = 0 has an integrating factor u which is a function of x alone, then $p = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$. Again if p is continuous and independent of y, show that an integrating factor is given by $u(x) = e^{P(x)}$, where P is any function satisfying $P' = p$.

Proof :

Let the equation Mdx+Ndy = 0 has an integrating factor u, which is a function of x alone.

$$\text{(ie) } uMdx + uNdy = 0$$

$$\text{(ie) } M_1dx+N_1dy = 0 \qquad \text{-------(8.30)}$$

where $M_1 = uM$ and $N_1 = uN$.

$$\text{Now (8.30) is exact if } \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

$$\text{(ie) } \frac{\partial}{\partial y}(uM) = \frac{\partial}{\partial x}(uN)$$

$$\text{(ie) } u\frac{\partial M}{\partial y} + \frac{\partial u}{\partial y}M = \frac{\partial u}{\partial x}N + u\frac{\partial N}{\partial x}$$

$$\text{(ie) } u\frac{\partial M}{\partial y} = N\frac{\partial u}{\partial x} + u\frac{\partial N}{\partial x} \quad \left(\because \varphi \text{ is a function of x alone} \Rightarrow \frac{\partial u}{\partial y} = 0\right)$$

$$\text{(ie)}\quad u\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = N\frac{\partial u}{\partial x}$$

$$\text{(ie)}\quad \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = \frac{1}{u}\frac{\partial u}{\partial x}$$

$$\text{(ie)}\quad p = \frac{1}{u}\frac{\partial u}{\partial x} \quad \text{where } p = \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$$

which is a function of x alone.

$$\text{Again } p = \frac{1}{u}\frac{\partial u}{\partial x}$$

$$\text{(ie)}\quad \frac{1}{u}\partial u = p\partial x$$

Integrating with respect to x, on both sides, we get,

$$\int\frac{1}{u}\partial u = \int p\partial x$$

$$\text{(ie)}\quad \log u(x) = P(x) \text{ where } P(x) = \int p\partial x$$

$$\text{(ie)}\quad u(x) = e^{P(x)}$$

$$\therefore u = e^{P}$$

This proves the problem.

Note :

If the equation $Mdx+Ndy=0$ has an integrating factor u, which is a function of y alone, then $q = \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$ is a continuous function. Again if q is continuous and independent of x, then the integrating factor is given by $u(y)=e^{Q(y)}$ where Q is any function satisfying $Q' = q$.

Example E.8.11 :

Find an integrating factor of $(e^y+xe^y)dx+xe^ydy = 0$ and hence solve it.

Solution :

Given that $(e^y+xe^y)dx+xe^ydy = 0$ ------(8.31)

Comparing (8.31) with $Mdx+Ndy = 0$, we have, $M = e^y+xe^y$ and $N = xe^y$.

$$\therefore \frac{\partial M}{\partial y} = e^y + xe^y \text{ and } \frac{\partial N}{\partial x} = e^y.$$

$$\text{Now } p = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$= \frac{1}{xe^y}(e^y + xe^y - e^y)$$

$$= 1$$

$$\therefore \text{ I.F } = u(x)$$

$$= e^{\int p dx}$$

$$= e^{\int 1.dx} = e^x$$

$$\text{Thus } u(x) = e^x$$

Multiply the I.F. to (8.31), we have,

$$e^x[e^y + xe^y]dx + x.e^x.e^y dy = 0$$

$$\text{(ie) } e^x e^y(1+x)dx + xe^x.e^y dy = 0$$

Divide xe^xe^y on both sides, we get,

$$\frac{1+x}{x}dx + dy = 0$$

Integrating an both sides, we get,

$$\int\left(\frac{1}{x}+1\right)dx + \int dy = 0$$

$$\text{(ie) } \log x + x + y = c$$

which is the required solution.

8.4 The method of successive approximations

Consider the linear differentiate equation $y' = f(x,y)$ -------(8.32)

where f is any continuous real-valued function defined on some rectangle R:$|x-x_0| \le a$, $|y-y_0| \le b$ (a, b>0) in the real (x, y) plane.

A real valued differentiable function φ satisfying $\varphi(x_0) = y_0$ such that the points $(x, \varphi(x))$ are in R for x in **I** and $\varphi'(x) = f(x, \varphi(x))$ for all x in **I** then φ is called solution of (8.32).

Also φ is called a solution to the initial value problem $y'=f(x, y)$, $y(x_0)=y_0$----(8.33) on **I**.

Theorem 8 :

A function φ is a solution of the initial value problem $y' = f(x, y)$, $y(x_0)=y_0$ on an interval **I** if and only if it is a solution of the integral equation $y = y_0 + \int_{x_0}^{x} f(t,y)dt$ on **I**.

Proof :

Suppose φ is a solution of the initial value problem on **I**.

Then $\varphi' = f(t, \varphi(t))$ -------(8.34) is defined on **I**.

Since φ is continuous on **I**, and f is continuous on R, the function F defined by $F(t) = f(t, \varphi(t))$ is continuous on **I**.

Integrating (8.34) from x_0 to x, we get,

$$\varphi(x) = \varphi(x_0) + \int_{x_0}^{x} f(t,\varphi(t))dt$$

Now $\varphi(x_0) = y_0$

Hence φ is a solution of $\varphi(x) = y_0 + \int_{x_0}^{x} f(t,\varphi(t))dt$

Conversly, suppose φ satisfies

$$\varphi(x) = y_0 + \int_{x_0}^{x} f(t,\varphi(t))dt \quad \text{on } \mathbf{I} \qquad \text{-------(8.35)}$$

Differentiate (8.35) and using fundamental theorem of integral calculus, we have, $\varphi'(x) = f(x, \varphi(x))$ for all x on **I**.

Moreover from (8.35), it is clear that $\varphi(x_0)=y_0$ and hence φ is a solution of the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$.

This proves the theorem.

Note :

We shall find a successive approximations to a solution of

$$y = y_0 + \int_{x_0}^{x} f(t,y)dt \qquad \text{-------(8.36)}$$

Let φ_0 be a function defined by $\varphi_0(x) = y_0$ Clearly $\varphi_0(x_0) = y_0$ but φ_0 need not satisfy (8.36).

$$\text{Let } \varphi_1(x) = y_0 + \int_{x_0}^{x} f(t,\varphi_0(t))dt$$

$$= y_0 + \int_{x_0}^{x} f(t,y_0)dt$$

Similarly proceding, we get

$$\varphi_2(x) = y_0 + \int_{x_0}^{x} f(t,\varphi_1(t))dt$$

$$\text{In general, } \varphi_{k+1} = y_0 + \int_{x_0}^{x} f(t,\varphi_k(t))dt, \quad (k = 0,1,2,\ldots\ldots) \text{ -------}(8.37)$$

$$\text{with } \varphi_0(x) = y_0$$

Definition D. 8.4 :

Let $y' = f(x, y)$, $y(x_0) = y_0$ be a initial value problem and $y = y_0 + \int_{x_0}^{x} f(t,y)dt$ be the integral equation $y = y_0 + \int_{x_0}^{x} f(t,y)dt$.

The functions $\varphi_0, \varphi_1, \varphi_2, \ldots..$ defined by

$$\varphi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t,\varphi_k(t))dt \quad (k=0,1,2,\ldots..)$$

with $\varphi_0(x)=y_0$ is called successive approximations to a solution of the integral equation or the initial value problem.

Example 8.12 :

Compute the first four successive approximation $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ for $y' = x^2+y^2$, $y(0)=0$.

Solution :

Given that $y(0) = 0$

(ie) $y_0 = 0$

$\therefore \varphi_0(x) = y_0 = 0$

We know that $\varphi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t,\varphi_k(t))dt \quad k=0,1,2,\ldots\ldots$ with $y_0 = \varphi_0(x)$

$$\text{(ie)}\ \varphi_{k+1}(x) = \int_{0}^{x} f(t,\varphi_k(t))dt, \quad k=0,1,2,\ldots\ldots$$

$$\text{when } k = 0, \text{ then } \varphi_1(x) = \int_{t=0}^{x} f(t,\varphi_0(t))dt$$

$$= \int_{t=0}^{x} f(t,0)dt$$

$$= \int_{t=0}^{x} t^2 dt$$

$$= \left[\frac{t^3}{3}\right]_{t=0}^{x}$$

$$= \frac{x^3}{3}$$

$$\text{(ie)}\ \varphi_1(x) = \frac{x^3}{3}.$$

$$\text{when } k = 1, \text{ then } \varphi_2(x) = \int_{t=0}^{x} f(t,\varphi_1(t))dt$$

$$= \int_{t=0}^{x} f\left(t,\frac{t^3}{3}\right)dt$$

$$= \int_{t=0}^{x} \left(t^2+\frac{t^6}{9}\right)dt$$

$$= \left[\frac{t^3}{3}+\frac{t^7}{9\times 7}\right]_{t=0}^{x}$$

$$= \frac{x^3}{3}+\frac{x^7}{63}$$

$$\text{(ie)}\ \varphi_2(x) = \frac{x^3}{3}+\frac{x^7}{63}$$

$$\text{when } k = 2, \text{ then } \varphi_3(x) = \int_{t=0}^{x} f(t,\varphi_2(t))dt$$

$$= \int_{t=0}^{x} f\left(f,\frac{t^3}{3}+\frac{t^7}{63}\right)dt$$

$$= \int_{t=0}^{x} \left(t^2 + \left(\frac{t^3}{3} + \frac{t^7}{63} \right)^2 \right) dt$$

$$= \int_{t=0}^{x} \left(t^2 + \frac{t^6}{9} + \frac{t^{14}}{3969} + \frac{2t^{10}}{189} \right) dt$$

$$= \left(\frac{t^3}{3} + \frac{t^7}{63} + \frac{t^{15}}{59535} + \frac{2t^{11}}{2046} \right)_0^x$$

$$= \frac{x^3}{3} + \frac{x^7}{63} + \frac{x^{15}}{59535} + \frac{2x^{11}}{2046}$$

$$\text{(ie)} \quad \varphi_3(x) = \frac{x^3}{3} + \frac{x^7}{63} + \frac{2x^{11}}{2046} + \frac{x^{15}}{59535}$$

Example 8.13 :

Compute the first four successive aproximations φ_0, φ_1, φ_2, φ_3, of $y'=1+xy$, $y(0)=1$.

Solution :

$$\text{Given that } y(0) = 1$$

$$\text{(ie) } y(0) = 1$$

$$\therefore \varphi_0(x) = y_0 = 1$$

$$\text{We know that } \varphi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \varphi_k(t)) dt$$

$$k = 0,1,2,3,\ldots\ldots \text{ with } y_0 = \varphi_0(x)$$

$$\text{(ie)} \quad \varphi_{k+1}(x) = 1 + \int_{t=0}^{x} f(t, \varphi_k(t)) dt$$

$$\text{when } k = 0 \text{ then } \varphi_1(x) = 1 + \int_{t=0}^{x} f(t, \varphi_0(t)) dt$$

$$= 1 + \int_{t=0}^{x} f(t, 1) dt$$

$$= 1 + \int_{t=0}^{x} (1+t) dt$$

$$= 1 + \left[t + \frac{t^2}{2} \right]_0^x$$

$$= 1 + x + \frac{x^2}{2}$$

$$\text{when } k = 1, \text{ then } \varphi_2(x) = 1 + \int_{t=0}^{x} f(t, \varphi_1(t))dt$$

$$= 1 + \int_{t=0}^{x} f\left(t, 1+t+\frac{t^2}{2}\right)dt$$

$$= 1 + \int_{t=0}^{x} \left(1+t\left(1+t+\frac{t^2}{2}\right)\right)dt$$

$$= 1 + \int_{t=0}^{x} \left(1+t+t^2+\frac{t^3}{2}\right)dt$$

$$= 1 + \left(t+\frac{t^2}{2}+\frac{t^3}{3}+\frac{t^4}{8}\right)_0^x$$

$$= 1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}$$

$$\text{when } k = 2, \text{ then } \varphi_3(x) = 1 + \int_{t=0}^{x} f(t, \varphi_2(t))dt$$

$$= 1 + \int_{t=0}^{x} f\left(t, 1+\frac{t^2}{2}+\frac{t^3}{3}+\frac{t^4}{8}\right)dt$$

$$= 1 + \int_{t=0}^{x} \left(1+t+t^2+\frac{t^3}{2}+\frac{t^4}{3}+\frac{t^5}{8}\right)dt$$

$$= 1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}+\frac{x^5}{15}+\frac{x^6}{48}$$

Thus we found φ_0, φ_1, φ_2, & φ_3.

Example E. 8.14 :

Compute the first three successive approximations φ_0, φ_1, φ_2, φ_3 of $y'=xy$, $y(0)=1$. Also show that the series $\varphi_k(x) \to \varphi(x)$.

Solution :

$$\text{Given that } y(0) = 1$$

$$\text{(ie) } y_0 = 1$$

We know that successive approximations of solutions of $y' = xy$ is

$$\varphi_{k+1}(x) = y_0 + \int_{t=0}^{x} (t, \varphi_k(t))dt, \quad k = 0,1,2,3,......$$

$$\text{with } \varphi_0(x) = y_0$$

$$\therefore \varphi_0(x) = y_0 = 1$$

$$\text{Hence } \varphi_{k+1}(x) = 1+\int_{t=0}^{x} f(t,\varphi_0(t))dt, \ k=0,1,2,3,.........$$

$$\text{when } k = 0 \text{ then } \varphi_1 = 1+\int_{t=0}^{x} f(t,\varphi_0(t))dt$$

$$= 1+\int_{t=0}^{x} f(t,1)dt$$

$$= 1+\int_{t=0}^{x} t dt$$

$$= 1+\left(\frac{t^2}{2}\right)_0^x$$

$$= 1+\frac{x^2}{2}$$

$$\text{when } k = 1 \text{ then } \varphi_2(x) = 1+\int_{t=0}^{x} f(t,\varphi_,(t))dt$$

$$= 1+\int_{t=0}^{x} f\left(t,1+\frac{t^2}{2}\right)dt$$

$$= 1+\int_{t=0}^{x}\left(t+\frac{t^3}{2}\right)dt$$

$$= 1+\left(\frac{t^2}{2}+\frac{t^4}{8}\right)_0^x$$

$$= 1+\frac{x^2}{2}+\frac{x^4}{2^3}$$

$$= 1+\frac{x^2}{2}+\frac{1}{2}\left(\frac{x^2}{2}\right)^2$$

$$\text{when } k = 2 \text{ then } \varphi_3(x) = 1+\int_{t=0}^{x} f(t,\varphi_2(t))dt$$

$$= 1+\int_{t=0}^{x} f\left(t,1+\frac{t^2}{2}+\frac{t^4}{8}\right)dt$$

$$= 1+\int_{t=0}^{x}\left(t+\frac{t^3}{2}+\frac{t^5}{8}\right)dt$$

$$= 1+\left[\frac{t^2}{2}+\frac{t^4}{8}+\frac{t^6}{48}\right]_0^x$$

$$= 1+\frac{x^2}{2}+\frac{x^4}{8}+\frac{x^6}{48}$$

$$= 1+\frac{x^2}{2}+\frac{1}{2}\left(\frac{x^2}{2}\right)^2+\frac{1}{3!}\cdot\left(\frac{x^2}{2}\right)^3$$

In general, $$\varphi_k(x) = 1+\left(\frac{x^2}{2}\right)+\frac{1}{2!}\left(\frac{x^2}{2}\right)^2+\frac{1}{3!}\left(\frac{x^2}{2}\right)^3+\dots+\frac{1}{k!}\left(\frac{x^2}{2}\right)^k$$

$$\therefore \lim_{k\to\infty}\varphi_k(x) = 1+\left(\frac{x^2}{2}\right)+\frac{1}{2!}\left(\frac{x^2}{2}\right)^2+\frac{1}{3!}\left(\frac{x^2}{2}\right)$$

$$= e^{x^2/2}$$

Choose $\varphi(x) = e^{x^2/2}$

Thus $\varphi_k(x) \to \varphi(x)$ as $k\to\infty$.

8.5 THE LIPSCHITZ CONDITION :

Definition D. 8.5 :

Let f be a function defined for (x, y) in a set S. This function f is said to satisfy Lipschitz condition on S if there exists a constant K>0 such that

$|f(x, y_1)-f(x, y_2) \le K|y_1-y_2|$ for all (x, y_1), (x, y_2) in S.

Note : The constant K is called a Lipschitz constant.

Theorem 8.3 :

Suppose S is either a rectangle $|x-x_0|\le a$, $|y-y_0|\le b$, (a, b>0) or a strip $|x-x_0|\le a$, $|y|<\infty$, (a>0), and that f is a real valued function defined on S such that $\partial f/\partial y$ exists, is continuous on S and $\left|\frac{\partial f}{\partial y}(x,y)\right| \le K$ ((x, y) in S) for some K>0. Then f satisfies a Lipschitz condition on S with Lipschitz constant K.

Proof :

Let $(x, y_1), (x, y_2) \in S$

$$\text{Then } f(x, y_1)-f(x, y_2) = \int_{t=y_2}^{y_1} \frac{\partial f}{\partial y}(x,t)dt$$

$$\therefore |f(x, y_1)-f(x, y_2)| = \left|\int_{t=y_2}^{y_1} \frac{\partial f}{\partial y}(x,t)dt\right|$$

$$\leq \int_{t=y_2}^{y_1} \left|\frac{\partial f}{\partial y}(x,t)dt\right|$$

$$\leq K \int_{t=y_2}^{y_1} |dt|$$

$$\leq K|y_1-y_2|$$

$$\therefore |f(x, y_1)-f(x, y_2)| \leq K|y_1-y_2| \; \forall \; (x, y_1), (x, y_2)\in S.$$

(ie) f satisfies the Lipschitz condition.

This proves the theorem.

Example E.8.15 :

Verify that the function $f(x, y)=4x^2+y^2$ satisfies Lipechitz condition on $S:|x|\leq 1, |y|\leq 1$.

Verification :

Given that $f(x, y) = 4x^2+y^2$ where $(x, y)\in S : |x|\leq 1, |y|\leq 1$

Let $(x, y_1), (x, y_2) \in S$.

Now $|f(x, y_1)-f(x, y_2)|$

$$= |(4x^2+y_1^2)-(4x^2+y_2^2)|$$

$$= |y_1^2-y_2^2|$$

$$= |(y_1+y_2)(y_1-y_2)|$$

$$= |y_1+y_2| \, |y_1-y_2|$$

$$\leq (|y_1|+|y_2|) \, |y_1-y_2|$$

$$\leq (1+1)|y_1-y_2| \; (\because \; |y|\leq 1)$$

$$= 2|y_1-y_2|$$

(ie) $f(x, y)$ satisfies Lipschitz condition with Lipschitz constant 2.

Theorem 8.4 : (Existence Theorem)

Let f be a continuous real-valued function on the rectangule $R:|x-x_0|\leq a,\ |y-y_0|\leq b$ $(a, b>0)$ and let $|f(x, y)|\leq M$ for all (x, y) in R. Further suppose that f satisfies a Lipschitz condition with constant k in **R**. Then the successive approximations $\varphi_0(x) = y_0$.

$$\varphi_{K+1}(x) = y_0 + \int_{x_0}^{x} f(t,\varphi_k(t))dt,\quad (k=0,1,2,3......)$$ converge on the interval $I : |x-x_0| \leq \alpha = \min\left\{a, \frac{b}{M}\right\}$ to a solution φ of the initial value problem $y' = f(x,y),\ y(x_0) = y_0$ on I.

Proof :

Step 1 : First we shall prove that $\{\varphi_k(x)\}$ is a convergence sequence.

Proof of step 1

$$\text{Now } \varphi_k = \varphi_0+(\varphi_1-\varphi_0)+(\varphi_2-\varphi_1)+....+(\varphi_k-\varphi_{k-1})$$

$$\text{(ie) } \varphi_k(x) = \varphi_0(x)+ \sum_{p=1}^{k}\left[\varphi_p(x)-\varphi_{p-1}(x)\right]$$

$\therefore$ φ_k is a partial sum for the series

$$\varphi_0(x)+ \sum_{p=1}^{\infty}\left[\varphi_p(x)-\varphi_{p-1}(x)\right] \qquad \text{------(8.38)}$$

Claim 1 : (8.38) is a convergent series.

Clearly all φ_p exists and continuous on I and $(x, \varphi_p(x))$ is in R for x in 1.

Again $|\varphi_1(x)-\varphi_0(x)|$

$$= \left|y_0+ \int_{t=x_0}^{x} f(t,y_0)dt - y_0\right| \quad \{\text{since } y_0 = \varphi_0(x)\}$$

$$= \left|\int_{x_0}^{x} f(t,y_0)dt\right|$$

$$\leq \int_{x_0}^{x} |f(t,y_0)|\,|dt|$$

$$\leq M|x-x_0| \quad (\because\ |f(x, y)|\leq M)$$

(ie) $|\varphi_1(x)-\varphi_0(x)| \leq M|x-x_0|$ -------(8.39)

$$\text{Again } \varphi_2(x)-\varphi_1(x) = \left(y_0+\int_{x_0}^{x} f(t,\varphi_1(x))dt\right)-\left(y_0+\int_{x_0}^{x} f(t,\varphi_0(t))dt\right)$$

$$= \int_{x_0}^{x}\left[f(t,\varphi_1(t))-f(t,\varphi_0(t))\right]dt$$

$$\therefore |\varphi_2(x)-\varphi_1(x)| \leq \left|\int_{x_0}^{x}\left|f(t,\varphi_1(t))-f(t,\varphi_0(t))\right|dt\right|$$

$$\leq \left|\int_{x_0}^{x} K|\varphi_1(t)-\varphi_0(t)|dt\right| \text{ (since f satisfies Lipschitz condition)}$$

$$\leq \left|\int_{x_0}^{x} KM|t-x_0|dt\right| \text{ \{from 8.39\}}$$

If $x \geq x_0$ then

$$\therefore |\varphi_2(x)-\varphi_1(x)| = KM\left|\int_{x_0}^{x}(t-x_0)dt\right|$$

$$= KM\frac{(x-x_0)^2}{2}$$

Similarly if $x \leq x_0$ then also we have

$$|\varphi_2(x)-\varphi_1(x)| \leq KM\frac{(x-x_0)^2}{2}$$

Claim 2 : Using induction we shall prove

$$|\varphi_p(x)-\varphi_{p-1}(x)| \leq MK^{p-1}\frac{|x-x_0|^p}{p!} \text{ for x in I} \qquad \text{------(8.40)}$$

Assume the result is true for $p = m$

$$\text{(ie) } |\varphi_m(x)-\varphi_{(m-1)}(x)| \leq \frac{MK^{m-1}}{m!}|x-x_0|^m$$

Let $x \geq x_0$.

$$\text{Now } \varphi_{m+1}(x)-\varphi_m(x) = \int_{x_0}^{x}\left|f(t,\varphi_m(t))-f(t,\varphi_{m-1}(t))\right|dt$$

$$= \int_{x_0}^{x} \left[f(t, \varphi_m(t)) - f(t, \varphi_{m-1}(t)) \right] dt$$

$$\therefore |\varphi_m(x) - \varphi_{m-1}(x)| \leq \int_{x_0}^{x} \left| f(t, \varphi_m(t)) - f(t, \varphi_{m-1}(t)) \right| dt$$

$$\leq K \int_{x_0}^{x} |\varphi_m(t) + \varphi_{m-1}(t)| dt \quad \{\text{by Lipschitz condition}\}$$

$$\leq K . \frac{MK^{m-1}}{m!} \int_{x_0}^{x} (x - x_0)^m dt \quad \{\text{by induction hypothesis}\}$$

$$= \frac{M.K^m}{m!} \frac{(x - x_0)^{m+1}}{m+1}$$

$$= \frac{MK^m}{(m+1)!} (x - x_0)^{m+1}$$

Similarly if $x \leq x_0$ then

$$|\varphi_{m+1}(x) - \varphi_m(x)| \leq \frac{MK^m}{(m+1)!} (x - x_0)^{m+1}$$

$$\therefore |\varphi_{m+1}(x) - \varphi_m(x)| \leq \frac{MK^m}{(m+1)!} (x - x_0)^{m+1} \text{ for x in I.}$$

Hence the result is true for $p = m+1$.

Thus by induction, $|\varphi_p(x) - \varphi_{p-1}(x)| \leq \frac{MK^{p-1}}{p!} |x - x_0|^p$

This proves cliam–2.

Again from claim-2, we have the series

$$\varphi_0(x) + \sum_{p=1}^{\infty} \left[\varphi_p(x) - \varphi_{p-1}(x) \right] \quad \text{------- (8.41)}$$

is absolutely convergent on **I**.

From (8.40), we have,

$$|\varphi_p(x) - \varphi_{p-1}(x)| \leq \frac{MK^p}{K} \frac{|x - x_0|^p}{p!}$$

which shows that the p^{th} term of the series (8.14) is less than or equal to M/K times the p^{th} term of the power series for $e^{K|x-x0|}$.

Since the power series for $e^{K|x-x_0|}$ is convergent & hence the series (8.41) is convergent for x in I.

This implies that the series (8.38) is convergent.

This proves claim-1.

Since the k^{th} partial sum of (8.38) is $\varphi_k(x)$ and therefore $\varphi_k(x) \to \varphi(x)$ as $k\to\infty$ for each x in I.

This proves step 1.

Step 2 : Now we shall prove that φ is continuous on I.

Let x_1, x_2 be in I.

$$\therefore |\varphi_{k+1}(x_1)-\varphi_{k+1}(x_2)| = \left|\int_{x_2}^{x_1} f(t,\varphi_t(t))dt\right|$$

$$\leq M|x_1-x_2|$$

Taking $k\to\infty$ on both sides, we have

$$|\varphi(x_1)-\varphi(x_2)| \leq M|x_1-x_2|$$

(ie) φ is continuous on I.

This proves step 2.

From step 2, we have,

$$|\varphi(x)-\varphi(x_0)| \leq M|x-x_0| \quad \text{-------(8.42)} \quad (x \text{ in I})$$

Step 3 : Now we shall estimate for $|\varphi(x)-\varphi_k(x)|$

$$\text{Now } \varphi(x) = \varphi_0(x)+\sum_{p=1}^{\infty}\left[\varphi_p(x)-\varphi_{p-1}(x)\right]$$

$$\text{and } \varphi_k(x) = \varphi_0(x)+\sum_{p=1}^{k}\left[\varphi_p(x)-\varphi_{p-1}(x)\right]$$

$$\text{Hence } |\varphi(x)-\varphi_k(x)| = \left|\sum_{p=k+1}^{\infty}\left[\varphi_p(x)-\varphi_{p-1}(x)\right]\right|$$

$$\leq \sum_{p=k+1}^{\infty}\left[\varphi_p(x)-\varphi_{p-1}(x)\right]$$

$$\leq \frac{M}{k}\sum_{p=k+1}^{\infty}\left(\frac{(K\alpha)^p}{p!}\right) \quad \{\text{by } (8.40)\}$$

$$= \frac{M}{k}\frac{(K\alpha)^{k+1}}{(k+1)!}e^{k\alpha} \qquad \text{-------}(8.43)$$

Letting $\in_k = \dfrac{(k\alpha)^{k+1}}{(k+1)!}$, then $\in_k \to 0$ as $k\to\infty$.

$\therefore$ (8.43) becomes $|(\varphi(x)-\varphi_k(x)| \leq \dfrac{M}{k}e^{k\alpha}.\in_k$ ------(8.44)

Step 4 : Now we shall prove that φ is a solution of $y' = f(x, y)$.

Claim – 3 : We shall prove that $\varphi(x) = y_0 + \int_{x_0}^{x} f(t,\varphi(t))dt$ ---------(8.45)

for all x in I.

Since φ is continuous on I & f is continuous on R and therefore the function $F(t) = f(t, \varphi(t))$ is continuous on I.

Now $\varphi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t,\varphi_k(t))dt$ and $\varphi_{k+1}(x) \to \varphi(x)$ as $k\to\infty$.

To prove (8.45), it is enough to prove that,

$$\int_{x_0}^{x} f(t,\varphi_k(t))dt \to \int_{x_0}^{x} f(t,\varphi(t))dt \quad \text{as } k\to\infty \qquad \text{-------}(8.46)$$

for each x in I.

Now $\left|\int_{x_0}^{x} f(t,\varphi(t))dt - \int_{x_0}^{x} f(t,\varphi_k(t))dt\right|$

$$\leq \left|\int_{x_0}^{x} \left|f(t,\varphi(t)) - f(t,\varphi_k(t))\right|dt\right|$$

$$\leq \quad K\left|\int_{x_0}^{x}\left|\varphi(t)-\varphi_k(t)\right|dt\right|$$

$$\leq \quad K.\frac{M}{K}e^{K\alpha}.\in_k|x-x_0| \quad \{\text{by } 8.44\}$$

$$= \quad Me^{K\alpha}.\in_k$$

which tends to zero as $k\to\infty$.

This proves (8.46) and hence φ satisfies (8.45).

$\therefore$ φ is a solution of $y' = f(x, y)$, $y_0 = \varphi_0(x)$.

This proves the theorem.

UNIT – 9

PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

9.1 Introduction :

In geometry or Physics when the number of independent variables in the problem under discussion is two or more. In such case, any dependent variable is likely to be a function of more than one variable, so that it possesses not ordinary derivatives with respect to single variable but partial derivatives with respect to several variables. For example, in the study of thermal effects in a solid body the temperature θ may vary from point to point in the solid as well as from time to time, and, as a consequence, the derivatives, $\frac{\partial\theta}{\partial x},\frac{\partial\theta}{\partial y},\frac{\partial\theta}{\partial z},\frac{\partial\theta}{\partial t}$ will, in general, be non-zero.

Further in some problem it may happen that higher derivatives of type $\frac{\partial^2\theta}{\partial x^2},\frac{\partial^2\theta}{\partial x\partial t},\frac{\partial^2\theta}{\partial x^2\partial t}$,.... may be involved and have some physical meaning.

Definition D. 9.1 :

The relation betweent he partial derivatives of kind

$F\left(\frac{\partial\theta}{\partial x},\ldots\frac{\partial^2\theta}{\partial x^2},\ldots\frac{\partial^2\theta}{\partial x\partial t},\ldots\right)=0$ ------(9.1) is called partial differential equation.

Definition D. 9.2 :

The highest higher derivative of the dependent variable is called order of the partial differential equation.

For example, $\frac{\partial^2\theta}{\partial x^2}=\frac{\partial\theta}{\partial t}$ is of order two and therefore it is called second order partial differential equation in two variables.

Again $\left(\frac{\partial\theta}{\partial x}\right)^5+\frac{\partial\theta}{\partial t}=0$ is first order partial differential equation in two variables.

Note : In this unit, we use the notation $\mathbf{p}=\frac{\partial z}{\partial x}$, $\mathbf{q}=\frac{\partial z}{\partial y}$.

9.2 Origins of first-order partial differential equation :

Suppose that we consider the equation, $x^2+y^2+(z-c)^2 = a^2$ -------(9.2)

in which the constants `a' and `c' are arbitrary. Then equation (9.2) represents the set of all spheres whose centers lie along the z-axis. If we differentiate this equation with respect to x, we obtain the relation.

$$x+p(z-c) = 0$$

while if we differentiate it with respect to y, we find that

$$y+q(z-c) = 0$$

Eliminating the arbitrary constant c from these two equations, we obtain the partial differential equation

$$yp-xq = 0 \qquad \text{------(9.3)}$$

which is of first order. In some sense, then the set of all spheres with centers on the z-axis is characterized by the partial differential equation (9.3).

However, other geometrical entities can be described by the same equation. For example, the equation,

$$x^2+y^2 = (z-c)^2 \tan^2\alpha. \qquad \text{------(9.4)}$$

in which both of the constants c and α are arbitrary, represents the set of all right circular cones whoes axes coincide with the line oz. If we differentiate equation (9.4) first with respect to x and then with respect to y, we find that

$$\left.\begin{aligned} p(z-c)\tan^2\alpha &= x, \\ q(z-c)\tan^2\alpha &= y \end{aligned}\right\} \qquad \text{------(9.4)}$$

and, upon eliminating c and α from these relations, we see that for these cones also the equation (9.3) is satisfied.

Now what the spheres and cones have in common is that they are scufaces of revolution which have the line oz as axes of symmetry. All surfaces of revolution with this property are characterized by an equation of the form.

$$z = f(x^2+y^2) \qquad \text{------(9.5)}$$

where the function f is arbitrary. Now if we write. $x^2+y^2 = u$ and differentiate equation (9.5) with respect to x and y, respectively, we obtain the relations,

$$p = 2xf'(u)$$

$$q = 2yf'(u)$$

where $f'(u) = \frac{df}{du}$, from which we obtain equation -------(9.3)

by eliminating the arbitrary function f(u).

Thus we see that the function z defined by each of the equations (9.2), (9.4) and (9.5) is, in some sense a "solution" of the equation (9.3).

We shall now generalize this argument slightly. The relations (9.2) and (9.4) are both of the type

$$F(x, y, z, a, b) = 0 \qquad \text{------(9.6)}$$

where a and b denote arbitrary constants. If we differentiate this equation with respect to x, we obtain the relation.

$$\left.\begin{aligned} \frac{dF}{dx}+p\frac{\partial F}{\partial z} &= 0 \\ \frac{dF}{dy}+q\frac{\partial F}{\partial z} &= 0 \end{aligned}\right\} \qquad \text{------(9.7)}$$

The set of equation (9.7) and (9.8) constitute three equations involving two arbitrary constants a & b, and in the general case, it will be possible to eliminate a & b from these equations, to obtain a relation of the kind

$$f(x, y, z, p, q) = 0 \qquad \text{------(9.9)}$$

Showing that the system of surfaces (9.2) gives rise to a partial differential equation (9.9) of the first order.

The obvious generalization of the relation (9.5) is a relation between x, y, & z of the type.

$$F(u, v) = 0 \qquad \text{------(9.10)}$$

where u and v are known functions of x, y, z and F is an arbitrary function of u and v. If we differentiate equation (9.10) with respect to x and y, respectively, we obtain equation

$$\frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right\}+\frac{\partial F}{\partial v}\left\{\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right\}=0$$

$$\frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right\}+\frac{\partial F}{\partial v}\left\{\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right\}=0$$

and if we now eliminate $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from these equations, we obtain the equation.

$$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)} \qquad \text{------(9.11)}$$

which is a partial differential equation of the type (9.9).

It should be observed, however, that the partial differential equation (9.11) is a linear equation. (ie), the powers of p and q are both unity, where as equation (9.9) need not be linear.

For example, the equation,

$$(x-a)^2+(y-b)^2+z^2 = 1$$

which represents the set of all spheres of unit radius with center in the plane xoy, leads to the first-order nonlinear differential equation

$$z^2(1+p^2+q^2) = 1$$

9.3 Cauchy's problem for first-order equations :

An existence theorem is to establish conditions under which we can assert whether or not a given partial differential equation has a solution at all; the further steps of proving that the solution, when it exists, is unique requires a uniqueness theorem. The conditions to be satisfied in the case of a first order partial differential equation are conveniently crystallized in the classic problem of Cauchy, which in the case of two independent variables may be states as follows.

Cauchy Problem : If

(a) If $x_0(u)$, $y_0(u)$ and $z_0(u)$ are functions which, together with their first derivatives, are continuous in the interval M defined by $U_1<u<U_2$;

(b) And if F(x, y, z, p, q) is a continuous function of x, y, z, p, and q in a certain region U of the xyzpq space, then it is required to establish the existence of a function $\phi(x, y)$ with the following properties :

(1) $\phi(x, y)$ and its partial derivatives with respect to x and y are continuous functions of x and y in a region R of the xy space.

(2) For all values of x and y lying in R, the point $\{x, y, \phi(x, y), \phi_x(x, y), \phi_y(x, y)\}$ lies in U and $F[x, y, \phi(x, y), \phi_x(x, y), \phi_y(x, y)]=0$

(3) For all U belonging to the interval M, the point $\{x_0(U), y_0(U)\}$ belongs to the region R & $\phi\{x_0(U), y_0(U)\}=z_0$.

Stated geometrically, what we wish to prove is that there exists a surface $z=\phi(x, y)$ which passes through the curve Γ whose parametric equations are

$$x=x_0(u),\ y=y_0(u),\ z=z_0(u) \qquad \text{-------(9.12)}$$

and at every point of which the direction $(p, q, -1)$ of the normal is such that

$$F(x, y, z, p, q) = 0 \qquad \text{-------(9.13)}$$

9.4 Linear equations of the first order

Theorem 9.1 :

The general solution of the linear partial differential equation,

$$P_p+Q_q = R \qquad \text{-------(9.14)}$$

$$\text{if } F(u, v) = 0 \qquad \text{-------(9.15)}$$

where F is an arbitary function and $u(x, y, z)=C_1$ and $v(x, y, z)=C_2$ from a solution of the equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \qquad \text{-------(9.16)}$$

Step 1

We shall show that all integral surfaces of the equation (9.14) are generated by the integral curves of the equation (9.16).

Proof of step 1 :

If we are given that $z = f(x,y)$ is an integral surface of the partial differential equation (9.14), then the normal to this surface has direction cosines proportional to $(p, q-1)$, and the differential equation (9.14) is no more than an analytical statement of the fact that this normal is perpendicular to the direction defined by the direction ratios (P, Q, R).

In otherwords, the direction (P, Q, R) is tangential to the integral surface z=f(x, y).

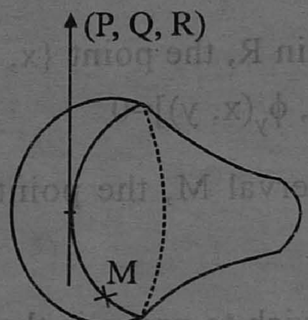


Fig.1

If, therefore, we start from an arbitrary point M on the surface (Fig. 1) and move in such a way that the direction of motion is always (P, Q, R) we trace out an integral curve of the equations (9.16) and since P, Q, and R are assumed to be unique there will be only one such curve through M. Further, since (P, Q, R) is always tangential to the surface, we never leave the surface. In other words, this integral curve of the equations (9.16) lies completely on the surface.

We have therefore shown that through each point M of the surface there is one and only one integral curve of the equation (9.16) and that this curve lies entirely on the surface. That is, the integral surface of the equation (9.14) is generated by the integral curves of the equation (9.16).

Step 2

We shall prove that all surfaces generated by integral curves of the equation (9.16) are integral surfaces of the equation (9.14).

If we are given that the surface z=f(x, y) is generated by integral curves of the equation (9.16) then we notice that its normal at a general point (x, y, z) which is in the direction $(\partial x/\partial x, \partial x/\partial y, -1)$ will be perpendicular to the direction (P, Q, R) of the curves generating the surface.

Therefore, $P\dfrac{\partial z}{\partial x}+Q\dfrac{\partial z}{\partial y}-R=0$

which is just another way of saying that z=f(x, y) is an integral surface of equation (9.13).

To complete the proof of the theorem we have still to prove that any surface generated by the integral curves of the equation (9.16) has an equation of the form (9.15). Let any curve on the surface which is not a particular member of the system.

$$u(x, y, z) = C_1, v(x, y, z) = C_2 \quad \text{-------(9.17)}$$

have equations,

$$\phi(x, y, z) = 0, \quad \Psi(x, y, z) = 0 \quad \text{-------(9.18)}$$

If the curve (9.17) is a generating curve of the surface, if will intersect the curve (9.18). The condition that it should do so will be obtained by climinating x, y, and z from the four equations (9.17) and (9.18). This will be a relation of the form

$$F(C_1, C_2) = 0 \quad \text{-------(9.19)}$$

between the constants C_1 and C_2. The surface is therefore generated by curves (9.17) which obey the condition (9.19) and will therefore have an equationof the form F(u, v)=0 (9.15). Conversely, any surface of the form (9.15) is generated by integral curves (9.17) of the equation for it is that surface generated by those curves of the system (9.17) which satisfy the relation (9.19).

This completes the proof of the theorem.

Theorem 9.2 :

If $u_i(x_1, x_2, \ldots\ldots x_n, z) = C_i$ (i=1,2,3,....,n) are independent solutions of the equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \ldots\ldots = \frac{dx_n}{P_n} = \frac{dz}{R}$$

Then the relation $\phi(u_1, u_2, \ldots\ldots, u_n) = 0$ in which the functions ϕ is arbitrary, is a general solution of the linear partial differential equation

$$P_1\frac{\partial z}{\partial x_1} + P_2\frac{\partial z}{\partial x_2} + \ldots + P_n\frac{\partial z}{\partial x_n} = R$$

To prove this theorem we first of all note that if the solutions of the equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \ldots\ldots = \frac{dx_n}{P_n} = \frac{dz}{R} \quad \text{------(9.20)}$$

are $u_i(x_1, x_2, \ldots\ldots, x_n, z) = C_i, \quad i=1,2,3,\ldots\ldots,n$ -------(9.21)

then the n equation.

$$\sum_{j=1}^{n} \frac{\partial u_i}{\partial x_j} dx_j + \frac{\partial u_i}{\partial z} \partial z = 0, \quad i=1,2,3,\ldots\ldots,n \qquad \text{--------(9.22)}$$

must be compatible with the equation (9.20). In other words, we must have

$$\sum_{j=1}^{n} P_j \frac{\partial u_i}{\partial x_j} + R \frac{\partial u_i}{\partial z} = 0 \qquad \text{-------(9.23)}$$

Solving the set of n equations (9.23) for P_i we find that

$$\frac{P_i}{\dfrac{\partial(u_1, u_2, \ldots\ldots, u_n)}{\partial(x_1, \ldots\ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n)}} = \frac{R}{\dfrac{\partial(u_1, u_2, \ldots\ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)}} \qquad \text{--------(9.24)}$$

$$i = 1,2,3,\ldots\ldots,n$$

where $\dfrac{\partial(u_1, u_2, \ldots\ldots, u_n)}{\partial(x_1, x_2, \ldots\ldots, x_n)}$ denotes the Jacobian

$$\begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \cdots & \dfrac{\partial u_1}{\partial x_n} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \cdots & \dfrac{\partial u_2}{\partial x_n} \\ \vdots & & & \\ \dfrac{\partial u_n}{\partial x_1} & \dfrac{\partial u_n}{\partial x_2} & \cdots & \dfrac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Consider the relation.

$$\phi(u_1, u_2, u_3, \ldots\ldots u_n) = 0 \qquad \text{-------(9.25)}$$

It with respect to x_i, we obtain the equation

$$\sum_{j=1}^{n} \left(\frac{\partial \phi}{\partial u_j} \cdot \frac{\partial u_j}{\partial x_i} + \frac{\partial u_j}{\partial z} \cdot \frac{\partial z}{\partial x_i} \right) = 0 \text{ and}$$

these are n such equations, one for each value of i. Eliminating the n quantities $\dfrac{\partial \phi}{\partial u}, \ldots\ldots, \dfrac{\partial \phi}{\partial u_n}$ from these equations, we obtain the relation

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} + \sum_{j=1}^{n} \frac{\partial z}{\partial x_j} \frac{\partial(u_1, u_2, \ldots, u_{j-1}, u_j, u_{j+1}, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_{j-1}, z, x_{j+1}, \ldots, x_n)} = 0 \qquad \text{--------(9.26)}$$

Substituting from equations (9.24) into the equation (9.26) we see that the function z defined by the relation (9.25) is a solution of the equation

$$P_1\frac{\partial z}{\partial x_1}+P_2\frac{\partial z}{\partial x_2}+\ldots+P_n\frac{\partial z}{\partial x_n}=R \qquad \text{-------(9.27)}$$

which is the required result.

9.5 Surface orthogonal to a given system of surface :

Suppose we are given a one-parameter family of surfaces characterized by the equation $f(x, y, z) = C$ -------(9.56)

and that we wish to find a system of surfaces which cut each of these given surfaces at right angles (cf. fig. 2).

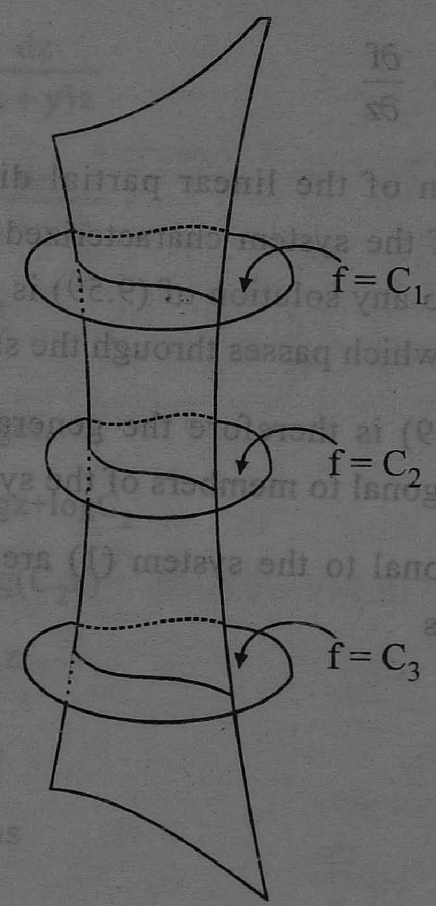


Fig. 2

The normal at the point (x, y, z) to the surface of the system (9.56) which passes through that point is the direction given by the directions ratios

$$(P, Q, R) = \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right) \qquad \text{-------(9.57)}$$

If the surface with equation $z = \phi(x, y)$ -------(9.58)

cuts each surface of the given system orthogonally then its normal at the point (x, y, z) which is in the direction $\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$ is perpendicular to the direction (P, Q, R) of the normal to the surface of the set (9.56) at that point. We therefore have the linear partial differential equation

$$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R \qquad \text{-------(9.59)}$$

for the determination of the surfaces (9.58) substituting from equations (9.57), we see that this equation is equivalent to

$$\frac{\partial f}{\partial x}.\frac{\partial z}{\partial x} + \frac{\partial f}{\partial y}.\frac{\partial z}{\partial y} = \frac{\partial f}{\partial z}$$

Conversely, any solution of the linear partial differential equation (9.59) is orthogonal to every surface of the system characterized by equation (9.56) for (9.59) simply states that the normal to any solution of (9.59) is perpendicular to the normal to that member of the system (1) which passes through the same point.

The linear equation (9.59) is therefore the general partial differential equation determining the surfaces orthogonal to members of the system (9.56).

(ie) The surfaces orthogonal to the system (1) are the surfaces generated by the integral curves of the equations

$$\frac{dx}{\frac{\partial f}{\partial x}} = \frac{dy}{\frac{\partial f}{\partial y}} = \frac{dz}{\frac{\partial f}{\partial z}}.$$

Example E. 9.1 :

Find the general solution of the differential equation $x^2p+y^2q = (x+y)z$

Solution :

Given that $x^2p+y^2q = (x+y)z$ -------(9.28)

$\therefore$ The integral surfaces of this equation are generated by the integral curves of the equations

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} \quad \text{-------(9.29)}$$

From the first two ratios of (9.29), we have

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Integrating on both sides, we get,

$$\frac{1}{x} = \frac{1}{y} + C_1$$

$$\text{(ie)} \quad \frac{1}{x} - \frac{1}{y} = C_1 \quad \text{-------(9.30)}$$

Again from (9.2),

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z}$$

$$\text{(ie)} \quad \frac{dx - dy}{(x+y)(x-y)} = \frac{dz}{(x+y)z}$$

$$\text{(ie)} \quad \frac{d(x-y)}{x-y} = \frac{dz}{z}$$

Integrating on both sides, we get,

$$\log(x-y) = \log z + \log C_2$$

$$\text{(ie)} \log(x-y) = \log(C_2 z)$$

$$\text{(ie)} \ x-y = C_2 z$$

$$\text{(ie)} \quad \frac{x-y}{z} = C_2 \quad \text{-------(9.31)}$$

Further (9.30) can be written as

$$\frac{y-x}{xy} = C1$$

$$\text{(ie)} \quad \frac{-C_2 z}{xy} = C_1 \text{ (using (9.31))}$$

$$\text{(ie)} \quad \frac{xy}{-C_2 z} = C1$$

$$\text{(ie)} \quad \frac{xy}{z} = -C_1 C_2$$

$$\text{(ie)}\quad \frac{xy}{z} = C_3 \text{ where } C_3 = -C_1C_2$$

$\therefore$ The curves (9.31) and (9.32) generate the surface $F\left(\frac{xy}{z}, \frac{x-y}{z}\right) = 0$ where the function F is arbitrary.

Example E. 9.2 :

Find the general integrals of the linear partial differential equations

(i) $z(xp-yq) = y^2-x^2$

(ii) $(y+xz)p-(x+yz)q = x^2-y^2$

(iii) $px(x+y) = qy(x+y)-(x-y)(2x+2y+z)$

Solution to (E. 9.2(i)) :

Given that $z(xp-yq) = y^2-x^2$. (ie) $xzp-yzq = y^2-x^2$ which is of the form $P_p+Q_q=R$

Here $P = xz$, $Q = -yz$ & $R = y^2-x^2$.

$\therefore$ The auxiliary equation is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

$$\text{(ie)}\quad \frac{dx}{xz} = \frac{dy}{-yz} = \frac{dz}{y^2-x^2} \qquad \text{------(9.33)}$$

From the first two ratios of (9.34), we have, $\frac{dx}{xz} = \frac{dy}{-yz}$

$$\text{(ie)}\quad \frac{dx}{x} = \frac{-dy}{y}$$

Integrating on both sides, we get,

$$\int \frac{dx}{x} = -\int \frac{dy}{y}$$

$$\text{(ie)}\quad \log x+\log y = \text{loc } C_1$$

$$\text{(ie)}\quad \log (xy) = \log C_1$$

$$\text{(ie)}\quad xy = C_1 \qquad \text{--------(9.35)}$$

Again from (9.34), $\frac{dx+dy}{zx-yz} = \frac{dz}{y^2-x^2}$

$$\text{(ie)} \quad \frac{dx+dy}{z(x-y)} = \frac{-dz}{(x-y)(x+y)}$$

$$\text{(ie)} \quad (x+y)d(x+y) = -z \log z$$

Integrating on both sides, we get,

$$\int (x+y)d(x+y) = -\int zdz$$

$$\text{(ie)} \quad \frac{(x+y)^2}{2} = \frac{-z^2}{2} + \frac{C_2}{2}$$

$$\text{(ie)} \quad (x+y)^2+z^2 = C_2$$

$$\text{(ie)} \quad x^2+y^2+2xy+z^2 = C_2$$

$$\text{(ie)} \quad x^2+y^2+2C_1+z^2 = C_2 \text{ (from (9.7))}$$

$$\text{(ie)} \quad x^2+y^2+z^2 = C_3 \qquad \text{--------(9.36)}$$

where $C_3 = C_2 - 2C_1$

From (9.35) & (9.36), the general solution is $F(xy, x+y^2+z^2) = 0$.

Solution to 9.2 (ii) :

Given that $(y+xz)p-(x+yz)q = x^2-y^2$ --------(9.38)

Comparing (9.39) with the Lagrange's equation $P_p+Q_q = R$, we have, $P=y+xz$, $Q=-(x+yz)$, $R=x^2-y^2$.

$\therefore$ The auxiliary equation is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\text{(ie)} \quad \frac{dx}{y+xz} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2} \qquad \text{--------(9.40)}$$

Now from (9.10),

$$\frac{ydx+xdy}{y^2+xyz-x^2-xyz} = \frac{dz}{x^2-y^2}$$

$$\text{(ie)} \quad \frac{ydx+xdy}{y^2-x^2} = \frac{dz}{x^2-y^2}$$

$$\text{(ie)} \quad d(xy) = -dz$$

Integrating on both sides, we get,

$$\int d(x,y) = -\int dz \qquad \text{-------(9.41)}$$

Again from (9.40), we have,

$$\frac{dx+dy}{y+xz-x-yz} = \frac{dz}{x^2-y^2}$$

(ie) $$\frac{dx+dy}{z(x-y)-(x-y)} = \frac{dz}{(x^2-y^2)}$$

(ie) $$\frac{dx+dy}{(x-y)(z-1)} = \frac{dz}{(x-y)(x+y)}$$

(ie) $(x+y)d(x+y) = (z-1)dz$

Integrating on both sides, we get,

$$\int(x+y)d(x+y) = \int(z-1)dz$$

(ie) $$\frac{(x+y)^2}{2} = \frac{z^2}{2}-z+C_2$$

(ie) $x2+y2+2xy = z^2-2z+2C_2$

(ie) $x^2+y^2+2(C_1-z)-z^2+2z = 2C_2$ {from 9.11}

(ie) $x^2+y^2-z^2 = 2C_2-2C_1$

(ie) $x^2+y^2-z^2 = C_3$ -------(9.42)

where $C_3 = 2C_2-C_1$

$\therefore$ From (9.41) & (9.42), the general solution is $F(xy+z, x^2+y^2-z) = 0$.

Solution to 9.2 (iii) :

Given that $Px(x+y) = qy(x+y)-(x-y)(2x+2y+z)$

(ie) $px(x+y)-qy(x+y) = -(x-y)(2x+2y+z)$ -------(9.43)

Comparing (9.13) with the Lagrange's equation, we get,

$P = x(x+y)$, $Q = -y(x+y)$ and $R = -(x-y)(2x+2y+z)$

The auxiliary equation is $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$.

(ie) $$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{-(x-y)(2x+2y+z)}$$ -------(9.42)

From first two ratios of (9.42), we have,

$$\frac{dx}{x} = \frac{dy}{-y}$$

Integrating on both sides, we get,

$$\int \frac{dx}{x} = -\int \frac{dy}{y}$$

$$\text{(ie) } \log x = -\log y + \log C_1$$

$$\text{(ie) } \log(xy) = \log C_1$$

$$\text{(ie) } xy = C_1 \qquad \text{-------(9.43)}$$

Again from (9.42), we have

$$\frac{dx+dy}{(x-y)(x+y)} = \frac{dx+dy+dz}{(x-y)(x+y)-(x-y)(2x+2y+z)}$$

$$\text{(ie) } \frac{dx+dy}{(x-y)(x+y)} = \frac{dx+dy+dz}{(x-y)(x+y-2x-2y-z)}$$

$$\text{(ie) } \frac{d(x+y)}{x+y} = \frac{d(x+y+z)}{-(x+y+z)}$$

Integrating on both sides, we get

$$\int \frac{d(x+y)}{x+y} = \int \frac{d(x+y+z)}{-(x+y+z)}$$

$$\text{(ie) } \log(x+y) = -\log(x+y+z) + \log C_2$$

$$\text{(ie) } \log[(x+y)(x+y+z)] = \log C_2$$

$$\text{(ie) } (x+y)(x+y+z) = C_2 \qquad \text{------(9.43)}$$

From (9.42) & (9.43), we have,

$$F(xy, (x+y)(x+y+z)) = 0$$

9.6 Integral surfaces passing through a given curve :

Example E. 9.3 :

Find the integral surface of the partial differential equation.

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$$

which contains the striaght line $x+y=0$, $z=1$.

Solution :

Given that $x(y^2+z)p-y(x^2+z)q = (x^2-y^2)z$ -------(9.44)

Comparing (9.17) with the Lagranges equation, we have $P=x(y^2+x)$, $Q=-y(x^2+z)$ & $R=(x^2-y^2)z$.

$\therefore$ The auxiliary equation is $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$.

$$\text{(ie)}\ \frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \qquad \text{-------(9.45)}$$

Now (9.45) can be written as,

$$\frac{\frac{dx}{x}}{y^2+z} = \frac{\frac{dy}{y}}{-(x^2+z)} = \frac{\frac{dz}{z}}{x^2-y^2} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y^2+z-x^2+z+x^2-y^2}$$

$$\text{(ie)}\ \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z} = 0$$

Integrating on both sides, we get,

$$\int\frac{dx}{x}+\int\frac{dy}{y}+\int\frac{dz}{z} = 0$$

$$\text{(ie)}\ \log x+\log y+\log z = \log C_2$$

$$\text{(ie)}\ \log(xyz) = \log C_2$$

$$\text{(ie)}\ xyz = C_2 \qquad \text{-------(9.46)}$$

Again from (9.44), we have

$$\frac{xdx+ydy}{x^2(y^2+z)-y^2(x^2+z)} = \frac{dz}{(x^2-y^2)z}$$

$$\text{(ie)}\ \frac{xdx+ydy}{z(x^2-y^2)} = \frac{dz}{z(x^2-y^2)}$$

$$\text{(ie)}\ xdx+ydx = zdz$$

Integrating on bothsides, we get,

$$\int xdx+\int ydy = \int zdz$$

$$\text{(ie)}\ \frac{x^2}{2}+\frac{y^2}{z}-\frac{z^2}{z} = C_2$$

(ie) $x^2+y^2-z^2 = C_2$ -------(9.46)

where $C_3 = 2C_2$

Given that the equation of striaght line as $x+y=0$, $z=1$.

Put $x=t$, $y=-t$, $z=1$ in (9.18) & (9.20), we get,

$-t^2 = C_1$ and $t^2+t^2-2=C_2$

(ie) $-t^2 = C_1$ and $2t^2-2 = C_2$

(ie) $-2C_1-2 = C_2$

(ie) $2C_1+C_2+2 = 0$

$\therefore$ The required solution is $2xyz+x^2+y^2-2z+2 = 0$

(ie) $x^2+y^2+2xyz-2z+2 = 0$

Example E. 9.4 :

Find the equation of the integral surface of the partial differential equation $2y(z-3)p+(2x-z)q = y(2x-3)$ which pases through the circle $z=0$, $x^2+y^2 = 2x$.

Solution :

Given that $2y(z-3)p+(2x-z)q = y(2x-3)$ -------(9.47)

Comparing (9.22) with the Lagranges equation, are have, $P=2y(z-3)$, $Q=2x-z$ & $R=y(2x-3)$.

$\therefore$ The auxiliary equation is $\dfrac{dx}{2y(z-3)} = \dfrac{dy}{2x-z} = \dfrac{dz}{y(2x-3)}$ -------(9.48)

From (9.48), we have,

$$\frac{dx}{2y(z-3)} = \frac{dz}{y(2x-3)}$$

$$\text{(ie)} \quad \frac{dx}{2(z-3)} = \frac{dz}{(2x-3)}$$

$$\text{(ie)} \quad (2x-3)dx = 2(z-3)dz$$

Integrating on both sides, we get,

$$\int(2x-3)dx = 2\int(z-3)dz$$

$$\text{(ie)} \quad \frac{2x^2}{2}-3x = 2\left[\frac{z^2}{2}-3z\right]+C_1$$

$$\text{(ie) } x^2-6x = z^2-6z+C_1$$

$$\text{(ie) } x^2-z^2-6x+6z = C_1 \qquad \text{--------(9.49)}$$

Again (9.48), we have,

$$\frac{dy}{2y(z-3)} = \frac{ydy-dz}{2xy-yz-2xy+3y}$$

$$\text{(ie) } \frac{dx}{2y(z-3)} = \frac{ydy-dz}{-y(z-3)}$$

$$\text{(ie) } \frac{dx}{2} = \frac{ydy-dz}{-1}$$

$$\text{(ie) } dx = -2(ydy-dz)$$

Integrating on both sides, we get,

$$\int dx = -2\int(ydy-dz)$$

$$\text{(ie) } x = -2\left[\frac{y^2}{2}-z\right]+C_2$$

$$\text{(ie) } x = -(y^2-2z)+C_2$$

$$\text{(ie) } x+y^2-2z = C_2 \qquad \text{--------(9.50)}$$

Given that the equation of the circle is $z=0$, $x^2+y^2=2x$. Put $x=t$ in the above equation, we have $y^2=2t-t^2$.

Thus (9.49) & (9.50) changes as

$$t^2-3t = C_1 \text{ and } t+2t-t^2 = C_2$$

$$\text{(ie) } 3t-t^2 = C_2$$

$$\text{(ie) } -C1 = C_2$$

$$\text{(ie) } C_1+C_2 = 0$$

$\therefore$ The required solution is $x^2-z^2-3x+6z+x+y^2-2z = 0$

$$\text{(ie) } x^2+y^2-z^2-2x+4z = 0$$

Example E. 9.5 :

Find the integral surface of $(x-y)y^2p+(y-x)x^2q = (x^2+y^2)z$ through the curve $xz=a^3$, $y=0$.

Solution :

Given that $(x-y)y^2p+(y-x)x^2q = (x^2+y^2)z$ --------(9.51)

Comparing (9.26) with the Lagranges equation, we have,

$P=(x-y)y^2$, $Q=(y-x)x^2$, $R=(x^2+y^2)z$

$$\text{(ie)}\ \frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{(x^2+y^2)z} \quad \text{--------(9.52)}$$

From first two ratios of (9.52), we have

$$\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2}$$

$$\text{(ie)}\ x^2dx = -y^2dy$$

Integrating on both sides, we get,

$$\int x^2dx = -\int y^2dy$$

$$\text{(ie)}\ \frac{x^3}{3} = \frac{-y^3}{3} + \frac{C_1}{3}$$

$$\text{(ie)}\ x^3+y^3 = C_1 \quad \text{-------(9.53)}$$

Again from (9.52)

$$\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{(x^2+y^2)z}$$

$$\therefore \frac{dx-dy}{(x-y)y^2-(y-x)x^2} = \frac{dz}{(x^2+y^2)z}$$

$$\text{(ie)}\ \frac{d(x-y)}{(x-y)(x^2+y^2)} = \frac{dz}{(x^2+y^2)z}$$

$$\text{(ie)}\ \frac{d(x-y)}{x-y} = \frac{dz}{z}$$

Integrating on both sides, we get,

$$\int \frac{d(x-y)}{x-y} = \int \frac{dz}{z}$$

$$\text{(ie)}\ \log(x-y) = \log z+\log C_2$$

$$\text{(ie)}\ \log(x-y) = \log(C_2.z)$$

$$\text{(ie) } x-y = C_2 z$$

$$\text{(ie) } \frac{x-y}{z} = C_2 \quad \text{--------(9.54)}$$

Given curve is $xy = a^3$, $y = 0$,

$$\text{put } x = t$$

$$\therefore \; z = \frac{a^3}{t}$$

From (9.53) & (9.54), we have,

$$t^3 = C_1 \text{ and } \frac{t}{(a^3/t)} = C_2$$

(ie) $t^3 = C_1$ and $t^2 = a^3 C_2$

and $t^6 = C_1^{\,2}$, $t^6 = a^9 C_2^{\,3}$

$\therefore \; C_1^{\,2} = a^9 C_2^{\,3}$

Now $\dfrac{(9.28)^2}{(9.29)^3}$ gives us $\dfrac{(x^3+y^3)^2}{\left(\dfrac{x-y}{z}\right)^3} = \dfrac{C_1^2}{C_2^3}$

$$\text{(ie) } \frac{(x^3+y^3)^2 . z^3}{(x-y)^3} = a^9$$

$$\text{(ie) } (x^3+y^3)^2 . z^3 = a^9 (x-y)^3$$

which is the required integral surface.

Example E.9.6 :

Find the surface which intersect the surface of the system $z(x+y) = C(z+1)$ orthogonally and which passes through the circle $x^2+y^2=1$, $z=1$.

Solution :

$$\text{Given that } z(x+y) = C(3z+1)$$

$$\text{(ie) } \frac{z(x+y)}{3z+1} = C$$

$$\text{(ie) } f(x, y, z) = C \text{ where } f(x, y, z) = \frac{z(x+y)}{3z+1}$$

$$\text{Now } \frac{\partial f}{\partial x} = \frac{z}{3z+1}$$

$$\frac{\partial f}{\partial y} = \frac{z}{3z+1}$$

$$\text{and } \frac{\partial f}{\partial z} = (x+y)\left[\frac{(3z+1)-3z}{(3z+1)^2}\right]$$

$$= \frac{x+y}{(3z+1)^2}$$

The integral surface of the equation are given by $\dfrac{dx}{\frac{\partial f}{\partial x}} = \dfrac{dy}{\frac{\partial f}{\partial y}} = \dfrac{dz}{\frac{\partial f}{\partial z}}$.

$$\text{(ie) } \frac{dx}{\frac{z}{3z+1}} = \frac{dy}{\frac{z}{3z+1}} = \frac{dz}{\frac{x+y}{(3z+1)^2}}. \quad \text{-------(9.60)}$$

From first two ratios of (9.60), we have,

$$dx = dy$$

Integrating on both sides, we get,

$$\int dx = \int dy$$

$$\text{(ie) } x-y = C_1 \quad \text{-------(9.61)}$$

Again (9.60) can be rewritten as

$$\frac{dx}{\frac{z(3z+1)}{(3z+1)^2}} = \frac{dy}{\frac{z(3z+1)}{(3z+1)^2}} = \frac{dz}{\frac{x+y}{(3z+1)^2}}$$

$$\text{(ie) } \frac{dx}{z(3z+1)} = \frac{dy}{z(3z+1)} = \frac{dz}{x+y}$$

$$\therefore \frac{dx+dy}{2z(3z+1)} = \frac{dz}{x+y}$$

$$\text{(ie) } (x+y)d(x+y) = 2z(3z+1)dz$$

Integrating on both sides, we have,

$$\int (x+y)d(x+y) = \int (6z^2+2z)dz$$

$$\text{(ie)} \quad \frac{(x+y)^2}{2} = 2z^3+z^2+C_2$$

$$\text{(ie)} \quad \frac{(x+y)^2}{2} - 2z^3 - z^2 = C_2 \qquad \text{-------(9.62)}$$

To find the surface passes through the circle $x^2+y^2 = 1$, $z = 1$.

Let $x = \cos t$, $y = \sin t$.

$\therefore$ (9.61), (9.62) becomes, $\cos t - \sin t = C_1$

$$\text{and} \left(\frac{(\cos t+\sin t)^2}{2}\right) - 2 - 1 = C_2$$

$$\text{(ie)} \quad \frac{\cos^2 t+\sin^2 t+\sin t\cos t}{2} - 3 = C_2$$

$$\text{(ie) } 2\sin t\cos t = 2C_2+5 \qquad \text{--------(9.63)}$$

$$\text{Now } \cos t - \sin t = C_1$$

Squaring on both sides, we have,

$$1-2\sin t\cos t = C_1^2$$

$$\text{(ie) } 2\sin t\cos t = 1-C_1^2 \qquad \text{--------(9.64)}$$

From (9.63) & (9.64), we have,

$$2C_2+5 = 1-C_1^2$$

$$\text{(ie) } C_1^2+2C_2+4 = 0$$

$$\text{(ie) } (x-y)^2+(x+y)^2-4z^3-2z^2+4 = 0$$

$$\text{(ie) } x^2+y^2-2z^3-z^2+2 = 0$$

which is the required orthogonal system of equations.

Example E.9.7 :

Find the surface which is orthogonal to the one parameter system $z=cxy(x^2+y^2)$ and which passes through the hyperbola $x^2-y^2=a^2$, $z=0$.

Solution :

Given that $z = cxy(x^2+y^2)$

$$\text{(ie)} \quad \frac{z}{xy(x^2+y^2)} = c$$

$$\text{(ie)}\quad f(x, y, z) = c \text{ where } f(x, y, z) = \frac{z}{xy(x^2+y^2)}$$

$$\therefore \frac{\partial z}{\partial x} = \frac{z}{y}\left\{\frac{0-\left(2x^2+x^2+y^2\right)}{x^2\left(x^2+y^2\right)^2}\right\}$$

$$= \frac{-z(3x^2+y^2)}{yx^2(x^2+y^2)^2}$$

$$\text{Similarly } \frac{\partial f}{\partial y} = \frac{-z(x^2+3y^2)}{xy^2(x^2+y^2)^2}$$

$$\text{and } \frac{\partial f}{\partial z} = \frac{1}{xy(x^2+y^2)}$$

$\therefore$ The integral surface of the equations are given by

$$\frac{dx}{\frac{\partial f}{\partial x}} = \frac{dy}{\frac{\partial f}{\partial y}} = \frac{dz}{\frac{\partial f}{\partial z}}$$

$$\text{(ie)}\quad \frac{dx}{\frac{-z(3x^2+y^2)}{yx^2(x^2+y^2)}} = \frac{dz}{\frac{-z(x^2+3y^2)}{xy^2(x^2+y^2)^2}} = \frac{dz}{\frac{1}{xy(x^2+y^2)}}$$

$$\text{(ie)}\quad \frac{dx}{-yz(3x^2+y^2)} = \frac{dy}{-xy(x^2+3y^2)} = \frac{dz}{xy(x^2+y^2)} \qquad \text{-------(9.65)}$$

From (9.65), we have,

$$\frac{dz}{xy(x^2+y^2)} = \frac{xdx+ydy+zdz}{xyz(-3x^2-y^2-x^2-3y^2+x^2+y^2)}$$

$$\text{(ie) } dz = \frac{xdx+ydy+zdz}{-3z}$$

$$\text{(ie) } -3zdz = xdx+ydy+zdz$$

$$\text{(ie) } xdx+ydy+4zdz = 0$$

Integrating on both sides, we get,

$$\int(xdx+ydy+4zdz) = \frac{C_1}{2}$$

(ie) $\frac{x^2}{2}+\frac{y^2}{2}+\frac{4z^2}{2} = \frac{C_1}{2}$

(ie) $x^2+y^2+4z^2 = C_1$ --------(9.66)

Again from (9.65), we have,

$$\frac{xdx+ydy}{-4xyz(x^2+y^2)} = \frac{xdx-ydy}{xyz(-3x^2-y^2+x^2+3y^2)}$$

(ie) $\frac{xdx+ydy}{2(x^2+y^2)} = \frac{xdx-dy}{x^2-y^2}$

Integrating on both sides, we get,

$$\frac{1}{2}\int\frac{xdx+ydy}{x^2+y^2} = \int\frac{xdx-ydy}{x^2-y^2}$$

(ie) $\frac{1}{2}\int\frac{2xdx+2ydy}{x^2+y^2} = \int\frac{2xdx-2ydy}{x^2-y^2}$

(ie) $\frac{1}{2}\log(x^2+y^2) = \log(x^2-y^2)+\log C_2$

(ie) $\sqrt{x^2+y^2} = C_2(x^2-y^2)$

(ie) $\frac{\sqrt{x^2+y^2}}{x^2-y^2} = C_2$ --------(9.67)

To find the orthogonal surface which passes through $x^2-y^2=a^2$, $z=0$

Let $x=a\sec t$, $y=a\tan t$.

$\therefore$ (9.66) & (9.67) changes as

$$a^2[\sec^2 t+\tan^2 t] = C_1 \text{ and}$$

$$\frac{\sqrt{a^2(\sec^2 t+\tan^2 t)}}{a^2} = C_2$$

(ie) $\sqrt{C_1} = a^2C_2$.

(ie) $C_1 = C_2^2a^4$

(ie) $x^2+y^2+4z^2 = \frac{a^4(x^2+y^2)}{(x^2-y^2)^2}$

(ie) $(x^2-y^2)^2(x^2+y^2+4z) = a^4(x^2+y^2)$.

Example E. 9.14 :

Find the general integrals of the Linear partial differential equation.

$(y+zx)p-(x+yz)q = (x^2-y^2)$

Solution :

We know that the Lagrangian equation of the form. $Pp+Qq = R$ where P, Q, R are the function of x, y, z.

Given partial differential equation $(y+zx)p-(x+yz)q = x^2-y^2$

Here $P = y+zx$; $Q = -(x+yz)$; $R = x^2-y^2$

The auxiliary equation is $\dfrac{dx}{P}=\dfrac{dy}{Q}=\dfrac{dz}{R}$

$$\Rightarrow \frac{dx}{y+xz}=\frac{dy}{-(x+yz)}=\frac{dz}{x^2-y^2} \quad \text{-------(6.84)}$$

Comparing the first two ratios,

$$\frac{xdx+ydy}{x(y+zx)-y(x+yz)} = \frac{dz}{x^2-y^2}$$

$$\frac{xdx+ydy}{xy+x^2z-xy-y^2z} = \frac{dz}{x^2-y^2}$$

$$\frac{xdx+ydy}{z(x^2-y^2)} = \frac{dz}{x^2-y^2}$$

$$xdx+ydy = zdz$$

Integrating on both sides, we get,

$$\frac{x^2}{2}+\frac{y^2}{z} = \frac{z^2}{2}+C$$

$x^2+y^2-z^2 = C_1$ where $C_1 = 2C$

$\therefore u = C_1$ where $u = x^2+y^2-z^2$

Next, $\dfrac{ydx+xdy}{y(y+zx)-x(x+yz)} = \dfrac{dz}{x^2-y^2}$

$$\frac{ydx+xdy}{y^2+zyx-x^2-xyz} = \frac{dz}{x^2-y^2}$$

$$\frac{ydx + xdy}{y^2 - x^2} = \frac{dz}{x^2 - y^2}$$

$$\frac{ydx + xdy}{-(x^2 - y^2)} = \frac{dz}{x^2 - y^2}$$

$$ydx+xdy = -dz$$

$$d(xy)+dz = 0$$

Integrating on both sides, we get,

$$xy+z = C_2$$

$$\therefore \quad V = C_2 \text{ wehre } v = xy+z$$

$\therefore$ The solution is $F(u, v) = 0$

$$F(x^2+y^2-z^2, xy+z) = 0$$

Example E.9.15 :

Find the equation of the integral surface of the differential equation.

$2y(z-3)p+(2x-z)q = y(2x-3)$, which passes through the circle $z=0$, $x^2+y^2 = 2x$

Solution :

Given that $2y(z-3)p+(2x-z)q = y(2x-3)$ -------(A)

We know that the Lagrange's equation of the form $Pp+Qq = R$.

Where P, Q, R are the function of x, y, z. Here $P = 2y(z-3)$, $Q=2x-z$, $R=y(2x-3)$.

The A.E. of (A) is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(ie) $\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)}$

Take First & Last ratio.

$$\frac{dx}{2y(z-3)} = \frac{dz}{y(zx-3)}$$

$$(2x-3)dx = 2(z-3)dz$$

Integrating on both sides, we get,

$$\int(2x-3)dx = 2\int(z-3)dz$$

$$x^2-3x = 2\left[\frac{z^2}{2} - 3z\right] + C_1$$

$$x^2-3x-z^2+6z = C_1 \quad \text{------(1)}$$

$$\text{Each ratio} = \frac{xdx+3ydy-zdz}{2xy(z-3)+3y(2x-z)-yz(2x-3)}$$

$$= \frac{xdx+3ydy-zdz}{0}$$

$$\Rightarrow xdx+3ydy-zdz = 0$$

Integrating on both sides, we get,

$$\frac{x^2}{2}+\frac{3y^2}{2}-\frac{z^2}{2} = C_2$$

$$x^2+3y^2-z^2 = C_3 \quad \text{where } C_3 = 2C_2 \quad \text{--------(2)}$$

The given equation of circle $z = 0$, $x^2+y^2=2x$.

For this circle the parametric equation is $x = 1+\cos t$, $y = \sin t$, $z = 0$

Substitute in (1) & (2), we get

$$(1+\cos t)^2-3(1+\cos t) = C_1$$

$$1+\cos^2 t+2\cos t-3-3\cos t = C_1$$

$$\mathbf{\cos^2 t-\cos t-2 = C_1} \quad \text{-------(3)}$$

and $(1+\cos t)^2+3\sin^2 t-0= C_3$

$$(1+\cos^2 t)+2\cos t+3\sin^2 t = C_3$$

$$\cos^2 t+\sin^2 t+2\sin^2 t+2\cos t+1 = C_3$$

$$1+2\sin^2 t+2\cos t+1 = C_3$$

$$\mathbf{2\sin^2 t+2\cos t+2 = C_3}$$

Eliminating `t' from (3) & (4)

$$(3)\times 2 \Rightarrow \quad 2\cos^2 t-2\cos t-4 = 2C1$$

$$(4) \Rightarrow \quad 2\sin^2 t+2\cos t+2 = C_3$$

$$\text{Adding,} \quad 2(\cos^2 t+\sin^2 t)-2 = 2C_1+C_3$$

$$\mathbf{2C_1+C_3 = 0} \quad \text{-------(5)}$$

Substitute (1) & (2) in (5)

$$2(x^2-3x-z^2+6z)+x^2+3y^2-z = 0$$

$$3x^2+3y^2-3z^2-6x+12z = 0$$

$$\mathbf{x^2+y^2-z^2-2x+4z = 0}$$

which is a required equation of integral surface.

UNIT – 10

NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

10.1 Non-linear Partial Differential Equations of the first order :

Consider a partial differential equations $F(x, y, z, p, q) = 0$ -------(10.1)

Here F need not be linear in p & q.

We know that the partial differential equations of the two parameter system is

$$f(x, y, z, a, b) = 0 \qquad \text{-------(10.2)}$$

(i) Two parameter system of surfaces $f(x, y, z, a, b)=0$ such an integral is called a **complete integral**.

(ii) If we consider one-parameter subsystem $f(x, y, z, a, \varphi(a))=0$ of the system (10.2) and form its envelop, we obtain a solution of (10.1). When the function $\varphi(a)$ which defines this subsystem is arbitrary, the solution obtained is called the **general integral** of (10.1) corresponding to the complete integral (10.2).

(iii) If the envelop of the two-parameter system (10.2) exists, it is also a solution of the equation (10.1), it is called the **singular integral** of the equation.

We shall discuss various integrals with the help of an example.

Consider the partial differential equation $z^2(1+p^2+q^2)=1$ -------(10.3)

We already discussed in section (9.2) that

$$(x-a)^2+(y-b)^2+z^2 = 1 \qquad \text{-------(10.4)}$$

was a solution of this equation with arbitrary a and b.

Since it contains two arbitrary constants, the solution (10.4) is a complete integral of the equation (10.3).

Putting b=a in (10.3) we obtain the one-parameter subsystem $(x-a)^2+(y-a)^2+z^2=1$ whose envelop is obtained by eliminating `a' between this equation and $x+y-2a=0$. So we have the equation $(x-y)^2+2z^2 = 2$ -------(10.5)

Differentiate (10.5) partially with respect to x & y, we get, $2zp=y-x$ and $2zq=x-y$.

These equations follows that (10.5) is an integral surface of the equation (10.3) that is, it is a general integral of (10.3).

The envelope of the two-parameter system (10.3) is obtained by eliminating a & b from (10.4) and the two equations $x-a=0$, $y-b=0$ (ie) the envelop consists of the pair of planes $z = \pm 1$ which is singular integral of the equation.

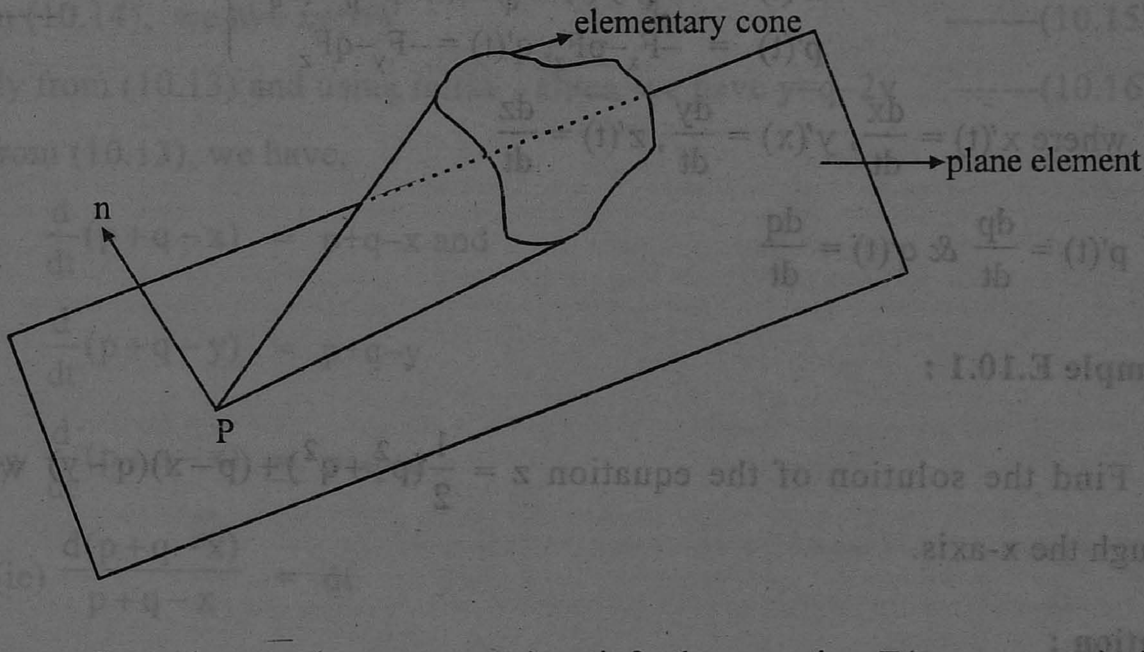
10.2 Cauchy's Method of Characteristics :

Theorem 10.1 :

A necessary and sufficient condition that a surface be an integral surface of the partial differential equation is that at each point its tangent element should touch the elementary cone of the equation.

Proof : The plane passing through the point. $P(x_0, y_0, z_0)$ with its normal parallel to the direction n defined by the direction ratios p_0, q_0, -1 is uniquely specified by the set of numbers $D(x_0, y_0, z_0, p_0, q_0)$.

Conversly any such set of five real numbers defines a plane in three-dimensional space and call a set of five numbers $D(x, y, z, p, q)$ by **plane element** of the space.



Now a plane element (x_0,y_0,z_0,p_0,q_0) satisfy the equation $F(x, y, z, p, q)=0$ --(10.6) is called an integral element of the equation (10.6) at he point (x_0, y_0, z_0)

It is theoretically possible to solve the equation of the type (10.6) to obtain an expression $q=G(x, y, z, p)$---------(10.7) from which to calculate q when x, y, z & p are known.

Keeping x_0, y_0 and z_0 fixed and varying p, we obtain a set of plane elements $\{x_0, y_0, z_0, p, G(x_0, y_0, z_0, p\}$ which depend on the single parameter p.

As p varies, we obtain a set of plane elements all of which pass through the point p and which therefore envelop a cone with vertex P, and the cone so generated is called the elementary cone of the equation (10.6) at the point P.

Consider a surface S whose equation is $z = g(x, y)$ --------(10.8)

If the function $g(x, y)$ and its first partial derivatives $g_x(x, y)$, $g_y(x, y)$ are continuous in a certain region R of the xy plane, then the targent plane at each point of S determines a plane element of the type $\{x_0, y_0, g(x_0, y_0), g_x(x_0, y_0)\}$ -------(10.9)

which we call the tangent element of the surface at the point $\{x_0, y_0, g(x_0, y_0)\}$.

This proves the theorem.

A curve C with parametric equations $x=x(t)$, $y=y(t)$, $z=z(t)$ -------(10.10)

lies on the surface $z = g(x, y)$ if $z(t) = g\{x(t), t(t)\}$

The **characteristic equations** of the differential equation $F(x, y, z, p, q)=0$ are

$$\left.\begin{aligned} x'(t) &= F_p,\ y'(t)=F_q,\ z'(t)=pF_p+qF_q \\ p'(t) &= -F_x-pF_z,\ q'(t) = -F_y-qF_z \end{aligned}\right\} \quad \text{--------(10.11)}$$

where $x'(t) = \dfrac{dx}{dt}$, $y'(x) = \dfrac{dy}{dt}$, $z'(t) = \dfrac{dz}{dt}$

$p'(t) = \dfrac{dp}{dt}$ & $q'(t) = \dfrac{dq}{dt}$

Example E.10.1 :

Find the solution of the equation $z = \frac{1}{2}(p^2+q^2)+(p-x)(q-y)$ which passes through the x-axis.

Solution :

Given that $z = \frac{1}{2}(p^2+q^2)+(p-x)(q-y)$ --------(10.12).

Clearly the initial values are $x_0=v$, $y_0=0$, $z_0=0$, $p_0=0$, $q_0=2v$, $t_0=0$

(since any point on x-axis is of the form (v, 0, 0)

The characteristic equation of (10.12) are

$$\left.\begin{aligned}
\frac{dx}{dt} &= p+q-y \\
\frac{dy}{dt} &= p+q-x \\
\frac{dz}{dt} &= p(p+q-y)+q(p+q-x) \\
\frac{dp}{dt} &= p+q-y \\
\frac{dq}{dt} &= p+q-x
\end{aligned}\right\} \qquad \text{-------(10.13)}$$

From (10.13), $\frac{dx}{dt} = \frac{dp}{dt}$

Integrating on both sides, we get,

$$\int dx = \int dp$$

$$\text{(ie) } x = p+C_1 \qquad \text{-------(10.14)}$$

At $(x_0, y_0, z_0, p_0, q_0)$, (10.14) gives $C_1 = v$.

$\therefore$ From (10.14), we hve $x=p+v$ -------(10.15)

Similarly from (10.13) and using initial values, we have $y=q-2v$ -------(10.16)

Again from (10.13), we have,

$$\frac{d}{dt}(p+q-x) = p+q-x \text{ and}$$

$$\frac{d}{dt}(p+q-y) = p+q-y$$

Now $\frac{d}{dt}(p+q-x) = p+q-x$

$$\text{(ie) } \frac{d(p+q-x)}{p+q-x} = dt$$

Integrating on both sides, we get,

$$\int \frac{d(p+q-x)}{p+q-x} = \int dt$$

$$\log(p+q-x) = t+\log C_2$$

$$\text{(ie) } \log\left(\frac{p+q-x}{C_2}\right) = t \qquad \text{-------(10.17)}$$

Using initial conditions, (10.17) gives us

$$\log\left(\frac{0+2v-v}{C_2}\right) = 0$$

$$\text{(ie) } \frac{v}{C_2} = 1$$

$$\text{(ie) } C_2 = v$$

$$\therefore (10.17) \Rightarrow \frac{p+q-x}{v} = e^t$$

$$\text{(ie) } p+q-x = ve^t \quad \text{--------(10.18)}$$

Similarly $\frac{d}{dt}(p+q-y) = p+q-y$ gives us $p+q-y = 2ve^t$ -------(10.19)

From (10.15) to (10.19), we have,

$$\left.\begin{aligned} x &= v(2e^t-1),\ y = v(e^t-1), \\ p &= 2v(e^t-1),\ q = v(e^t+1) \end{aligned}\right\} \quad \text{-------(10.20)}$$

Thus from (10.13), we have,

$$\frac{dz}{dt} = 5v^2e^{2t}-3v^2e^t$$

$$\text{(ie) } z = \frac{5}{2}v^2(e^{2t}-1)-3v^2(e^t-1) \quad \text{-------(10.21)}$$

Again from (10.20), $e^t = \frac{y-x}{2y-x}$, $v = x-2y$

$\therefore$ (10.21) becomes $z = \frac{1}{2}y(4x-3y)$ which is the required solution.

10.3 Compatible systems of First-order equations :

Definition 10.1 :

If every solution of partial differentiate equation $f(x, y, z, p, q)=0$ ---------(10.22) also satisfies the partial differentiate equation $g(x, y, z, p, q)=0$ ---------(10.23) then the partial differentiate equations (10.22), (10.23) form a compatiable system of equations

Find the condition for the pair of equations (10.22) & (10.23) to be compatible.

If $j = \dfrac{\partial(f,g)}{\partial(p,q)} \neq 0$, then the equations (10.22) & (10.23) can be solved explicitly to given $p = \varphi(x, y, z)$ and $q = \Psi(x, y, z)$ -------(10.24)

The pair of (10.22) & (10.23) equations will be compatible if the pair of equations (10.24) is completely integrable.

For this, the equation $dz = pdx+qdy$

$$\text{(ie) } dz = \varphi dx + \Psi dy$$

$$\text{(ie) } \varphi dx+\Psi dy-dz = 0 \text{ should be integrable.}$$

The condition of integrability is $\varphi(-\Psi_z)+\Psi\varphi_z-(\Psi_x-\varphi_y) = 0$

$$\text{(ie) } \Psi_x+\varphi\Psi_z = \varphi_y+\Psi\varphi_z \quad \text{-------(10.25)}$$

Differentiate (10.22) with respect to x and z respectively and using (10.24), we get

$$f_x+f_p\varphi_x+f_q\Psi_x = 0, \quad \text{-------(10.26)}$$

$$f_z+f_p\varphi_z+f_q\Psi_z = 0 \quad \text{-------(10.27)}$$

Now φ.(10.27)+(10.26) we get,

$$f_x+\varphi f_z+f_p(\varphi_x+\varphi\varphi_z)+f_q(\varphi_x+\varphi\varphi_z) = 0 \quad \text{-------(10.28)}$$

Similarly from (10.23), we have

$$g_x+\varphi g_z+g_p(\varphi_x+\varphi\varphi_z)+g_q(\varphi_x+\varphi\varphi_z) = 0 \quad \text{--------(10.29)}$$

Now solving (10.28) & (10.29), we get

$$\Psi_x+\varphi\Psi_z = \frac{1}{J}\left[\frac{\partial(f,g)}{\partial(x,p)}+\varphi\frac{\partial(f,g)}{\partial(z,p)}\right] \quad \text{-------(10.30)}$$

Now again differentiating (10.22) with respect to y and z using (10.24), we obtain,

$$\varphi_y+\Psi\varphi_z = \frac{-1}{J}\left[\frac{\partial(f,g)}{\partial(y,q)}+\psi\frac{\partial(f,g)}{\partial(z,q)}\right] \quad \text{-------(10.31)}$$

Putting (10.30), (10.31) in (10.25), we get the required condition as [f, g]=0 ----(10.32)

$$\text{where } [f, g] = \frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)} \quad \text{------(10.32)}$$

Which is the required condition for a pair of partial differential equation are compatible.

Example E. 10.2 :

Show that the system of equations $xp = yq$, $z(xp+yq) = 2xy$ are compatible and solve them.

Solution :

Given equation are

$$f(x, y, z, p, q) = xp-yq = 0 \text{ and}$$

$$g(x, y, z, p, q) = z(px+yq)-2xy.$$

$$\therefore \frac{\partial f}{\partial x} = p, \quad \frac{\partial f}{\partial p} = x,$$

$$\frac{\partial g}{\partial x} = zp-2y, \quad \frac{\partial g}{\partial p} = zx$$

$$\text{Now } \frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix}$$

$$= \begin{vmatrix} p & x \\ 3p-2y & zx \end{vmatrix}$$

$$= 2xy$$

$$\text{Similarly, } \frac{\partial(f,g)}{\partial(z,p)} = -x^2p,$$

$$\frac{\partial(f,g)}{\partial(y,q)} = -2xy \text{ and}$$

$$\frac{\partial(f,g)}{\partial(z,q)} = xyp$$

$$\text{Now } [f, g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,q)}{\partial(z,q)}$$

$$= 2xy-p^2x^2-2xy+xypq$$

$$= 0 \text{ because } xp = yq.$$

$\therefore$ Given equations are compatible.

$$\text{Now } xp = yq$$

$$\Rightarrow z(xp+xp) = 2xy$$

$$\Rightarrow p = y/z$$

$$\text{\& hence } q = x/z$$

We know that $dz = pdx + qdy$

$$\text{(ie) } dz = \frac{y}{z}dx + \frac{x}{z}dy$$

$$\text{(ie) } zdz = ydx+xdy$$

$$\text{(ie) } zdz = d(xy)$$

Integrating on both sides, we get,

$$\int zdz = \int d(xy)$$

$$\text{(ie) } \frac{z^2}{2} = xy+C_1$$

$$\text{(ie) } z^2 = 2xy+2C_1$$

$$\text{(ie) } z^2 = 2xy+C_2 \text{ where } C_2 = 2C_1$$

which is the required solution.

Example E.10.3 :

Show that $xp-yq = x$, $x^2p+q = xz$ are compatible and find their solution.

Solution :

$$\text{Let } f = xp-yq-x$$

$$\text{and } g = x^2p+q-xz$$

$$\therefore f_x = p-1 \qquad g_x = 2xp-z$$
$$f_y = -q \qquad g_y = 0$$
$$f_z = 0 \qquad g_z = -x$$
$$f_p = x \qquad g_p = x^2$$
$$f_q = -y \qquad g_q = 1$$

$$\text{and } \frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix} = -x^2p-x^2+xz$$

$$= \begin{vmatrix} p-1 & x \\ 2xp-z & x^2 \end{vmatrix}$$

$$\text{and } \frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix}$$

$$= \begin{vmatrix} -q & -y \\ 0 & 1 \end{vmatrix}$$

$$= -q$$

$$\text{Similarly } \frac{\partial(f,g)}{\partial(z,p)} = x^2$$

$$\text{and } \frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} 0 & -y \\ -x & 1 \end{vmatrix}$$

$$= -xy.$$

$$\text{Now } [f, g] = \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + \varphi \frac{\partial(f,g)}{\partial(z,p)} + \psi \frac{\partial(f,g)}{\partial(z,q)}$$

$$= -x^2p - x^2 + xz - q + px^2 - qxy$$

$$= -x^2 + x^2p + q - q - qxy$$

$$= -x^2 + x(xp - yq)$$

$$= -x^2 + x^2$$

$$= 0$$

$\therefore$ f and g are compatible.

Now we shall solve the equations

$$\text{Since } xp - yq = x,$$

$$\text{(ie) } yq = xp - x$$

$$\therefore q = \frac{x}{y}(p-1)$$

$$\text{Thus } x^2p + q = xz$$

$$\Rightarrow x^2p + \frac{x}{y}(p-1) = z$$

$$\Rightarrow xyp + p - 1 = yz$$

$$\Rightarrow p(xy+1) = 1 + yz$$

$$\Rightarrow p = \frac{1+yz}{1+xy}$$

$$\therefore q = \frac{x}{y}(p-1)$$

$$\Rightarrow q = \left(\frac{1+yz}{1+xy} - 1\right)$$

$$\Rightarrow q = \frac{x(z-x)}{1+xy}$$

$$\text{Now } dz = pdx+qdy$$

$$= \frac{(1+yz)}{1+xy}dx + \frac{x(z-x)}{1+xy}dy$$

$$\text{(ie) } (1+xy)dz = (1+yz)dx+x(z-x)dy$$

$$\text{(ie) } (1+xy)dz-xydx = dx+yzdx-xydx+x(z-x)dy$$

$$\text{(ie) } dz+xy(dz-dx) = dx+yzdx-xydx+xzdy-x^2dy$$

$$\text{(ie) } dz+xyd(z-x) = dx+z[ydx+xdy]-x[ydx+xdy]$$

$$\text{(ie) } dz+xyd(z-x) = dx+(z-x)(ydx+xdy)$$

$$\text{(ie) } dz+xyd(z-x) = dx+(z-x)d(xy)$$

$$\text{(ie) } dz-dx+xyd(z-x) = (z-x)d(xy)$$

$$\text{(ie) } d(z-x)+xyd(z-x) = (z-x)d(xy)$$

$$\text{(ie) } \frac{d(z-x)}{z-x} = \frac{d(1+xy)}{1+x}$$

Integrating on both sides, we get,

$$\int\frac{d(z-x)}{z-x} = \int\frac{d(1+xy)}{1+xy}$$

$$\text{(ie) } \log(z-x) = \log(1+xy)+\log C$$

$$\text{(ie) } \log(z-x) = \log(C(C_1+xy))$$

$$\text{(ie) } z-x = C(1+xy)$$

which is the required solution.

10.4 Charpit's Method :

Now we shall discuss the charpit's method for solving a partial differentiate equation f(x, y, z, p, q) = 0.

Consider a partial differentiate equation f(x, y, z, p, q) = 0 --------(10.34)

Also we have, dz = pdx+qdy --------(10.35)

Now we shall find one more relation in x, y, z, p, q say F(x, y, z, p, q)=0----(10.36)

such that when the values of p and q derived from it and the given equation (10.34) are substituted in (10.35), it becomes integrable.

Clearly the integral of (10.35) will satisfy (10.34) because the values of p & q derived from it are the same as the values of p and q in (10.34).

Now let us assume that (10.36) is the relation which when taken along with (10.34) gives those values of p and q which make (10.35) integrable.

Hence we may consider z, p, q, expressed as functions of x and y such that when these values are substituted in F=0 and f=0, they are satisfied identically. From this it follows that their derivatives with respect to x and y will vanish.

Now differentiating (10.34) and (10.36) with respect to x and y, we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}.\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}.\frac{\partial q}{\partial x}=0$$

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}p+\frac{\partial F}{\partial p}.\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}.\frac{\partial q}{\partial x}=0$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q+\frac{\partial f}{\partial p}.\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}.\frac{\partial q}{\partial y}=0$$

$$\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}q+\frac{\partial F}{\partial p}.\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}.\frac{\partial q}{\partial y}=0$$

Eliminating $\frac{\partial p}{\partial x}$ from the first pair and $\frac{\partial q}{\partial y}$ from the second pair, we have,

$$\left(\frac{\partial f}{\partial x}.\frac{\partial F}{\partial p}-\frac{\partial f}{\partial p}.\frac{\partial F}{\partial x}\right)+p\left(\frac{\partial f}{\partial z}.\frac{\partial F}{\partial p}-\frac{\partial f}{\partial p}.\frac{\partial F}{\partial z}\right)+\frac{\partial q}{\partial x}\left(\frac{\partial f}{\partial q}.\frac{\partial F}{\partial p}-\frac{\partial f}{\partial p}.\frac{\partial F}{\partial q}\right)=0 \quad \text{------(10.37)}$$

$$\left(\frac{\partial f}{\partial y}.\frac{\partial F}{\partial q}-\frac{\partial f}{\partial q}.\frac{\partial F}{\partial y}\right)+p\left(\frac{\partial f}{\partial z}.\frac{\partial F}{\partial q}-\frac{\partial f}{\partial q}.\frac{\partial F}{\partial z}\right)+\frac{\partial p}{\partial y}\left(\frac{\partial f}{\partial p}.\frac{\partial F}{\partial q}-\frac{\partial f}{\partial q}.\frac{\partial F}{\partial p}\right)=0 \quad \text{------(10.38)}$$

$$\text{Now } \frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y} \quad \text{------(10.39)}$$

∴ adding (10.37), (10.39), we get, after rearranign terms,

$$\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z}$$

$$+\left(-\frac{\partial f}{\partial p}\right)\left(\frac{\partial F}{\partial x}\right)+\left(-\frac{\partial F}{\partial q}\right)\left(\frac{\partial F}{\partial q}\right)=0 \quad \text{-------(10.40)}$$

Clearly (10.40) is a Lagrange's linear equation of the first order with x, y, z, p, q as independent variable.

$\therefore$ The axuilary equation is

$$\frac{dx}{\left(-\frac{\partial F}{\partial p}\right)}=\frac{dy}{\left(-\frac{\partial y}{\partial q}\right)}=\frac{dz}{-\left(p\frac{\partial f}{\partial p}+q\frac{\partial f}{\partial q}\right)}=\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}$$

(ie) $$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-(f_x+pf_z)}=\frac{dq}{-(f_y+qf_z)}$$

which is known as Charpit's equation.

Example E.10.5 :

Find a complete integral of the equation $p^2x+q^2y = z$.

Solution :

Given that $p^2x+q^2y = z$ --------(10.41)

(ie) $p^2x+q^2y-z = 0$

(ie) $f(x, y, z, p, q) = 0$ where $f(x, y, z, p, q) = p^2x+q^2y-z$

$\therefore f_x = p^2, f_y = q^2, f_p = 2px, f_q = 2qy$ and $f_z = -1$.

The Charpit's equation is

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-(f_x+pf_z)}=\frac{dq}{-(f_y+qf_z)}$$

(ie) $$\frac{dx}{2px}=\frac{dy}{2qy}=\frac{dz}{2p^2x+2q^2y}=\frac{dp}{-(p^2-p)}=\frac{dq}{-(q^2-q)}$$

$$\frac{dx}{2px}=\frac{dy}{2qy}=\frac{dz}{2z}=\frac{dp}{p-p^2}=\frac{dq}{q-q^2}$$ -------(10.42)

Now from (10.42)

$$\frac{p^2dx+2pxdp}{p^2x}=\frac{q^2dy+2qydq}{q^2y}$$

(ie) $$\frac{d(p^2x)}{p^2x}=\frac{d(q^2y)}{q^2y}$$

Integrating on both sides, we get,

$$\int \frac{d(p^2x)}{p^2x} = \int \frac{d(q^2y)}{q^2y}$$

$$\text{(ie)} \quad \log(p^2x) = \log(q^2y)+\log a$$

$$\text{(ie)} \quad p^2x = aq^2y \qquad \text{--------(10.43)}$$

From (10.41) & (10.43), we have,

$$aq^2y+q^2y = z$$

$$\Rightarrow q^2 = \frac{z}{y(1+a)}$$

$$\Rightarrow q = \sqrt{\frac{z}{y(1+a)}} \qquad \text{-------(10.44)}$$

From (10.41) and (10.44), we have,

$$p^2x+\frac{z}{1+a} = z$$

$$\text{(ie)} \quad p^2x = z-\frac{z}{1+a}$$

$$\Rightarrow p^2x = \frac{az}{1+a}$$

$$\Rightarrow p = \sqrt{\frac{az}{(1+a)x}} \qquad \text{-------(10.45)}$$

We know that $dz = pdx+qdy$

$\therefore$ From (10.44) & (10.45), we have

$$dz = \sqrt{\frac{az}{(1+a)x}}.dx+\sqrt{\frac{z}{(1+a)y}}dy$$

$$\text{(ie)} \quad \frac{dz}{\sqrt{z}} = \sqrt{\frac{a}{1+a}}.x^{-1/2}dx+\sqrt{\frac{1}{1+a}}.y^{-1/2}dy$$

Integrating on both sides, we get,

$$\int z^{-1/2}dz = \sqrt{\frac{a}{1+a}}\int x^{-1/2}dx+\sqrt{\frac{1}{1+a}}\int y^{-1/2}dy$$

$$\text{(ie)} \quad \sqrt{1+a}.\sqrt{z} = \sqrt{a}.\sqrt{x}+\sqrt{y}+b$$

which is the required solution.

Example E. 10. 6 :

Using Charpit's method, solve $(p^2+q^2)y = qz$.

Solution :

$$\text{Given } (p^2+q^2)y = qz \qquad \text{-------(10.46)}$$

$$\text{(ie) } (p^2+q^2)y-qz = 0$$

$$\text{(ie) } f(x, y, z, p, q) = 0 \text{ where } f(x, y, z, p, q) = (p^2+q^2)y-qz.$$

$\therefore f_x = 0, f_y = p^2+q^2, f_z = -q, f_p = 2py, f_q = 2qy-z$

We know that the Charpit's equation

$$\frac{dx}{f_p} = \frac{dy}{fq} = \frac{dz}{pf_p+qf_q} = \frac{dp}{-(f_x+pf_z)} = \frac{dq}{-(f_y+qf_z)}$$

$$\text{(ie) } \frac{dx}{2py} = \frac{dy}{2qy-z} = \frac{dz}{2p^2y+2q^2y-qz} = \frac{dp}{pq} = \frac{dq}{-(p^2+q^2-q^2)}$$

$$\text{(ie) } \frac{dx}{2py} = \frac{dy}{2qy-z} = \frac{dz}{qz} = \frac{dp}{pq} = \frac{dq}{-p^2} \qquad \text{-------(10.47)}$$

$$\text{Take } \frac{dp}{pq} = \frac{-dq}{p^2}$$

$$\text{(ie) } pdp = -qdq$$

Integrating on both sides, we get,

$$\int pdp = -\int qdq$$

$$\Rightarrow \frac{p^2}{2} = -\frac{q^2}{2}+\frac{a^2}{2}$$

$$\Rightarrow p^2+q^2 = a^2 \qquad \text{-------(10.48)}$$

$\therefore$ From (10.46) & (10.48), we have, $a^2y = qz$

$$\text{(ie) } q = \frac{a^2y}{z} \qquad \text{-------(10.49)}$$

From (10.46) and (10.49),we have

$$p^2+\frac{a^4y^2}{z^2} = a^2$$

$$\text{(ie) } p^2 = a^2-\frac{a^4y^2}{z^2}$$

$$\text{(ie) } p^2 = \frac{a^2}{z^2}\left(z^2 - a^2y^2\right)$$

$$\text{(ie) } p = \frac{a}{z}\sqrt{z^2 - a^2y^2} \qquad \text{-------(10.50)}$$

We know that $dz = pdx+qdy$ -------(10.51)

From (10.49), (10.50) & (10.51), we have,

$$dz = \frac{a}{z}\sqrt{z^2 - a^2y^2}dx + \frac{a^2y}{z}dy$$

$$\text{(ie) } zdz = a\sqrt{z^2 - a^2y^2}dx + a^2ydy$$

$$\text{(ie) } \frac{zdz - a^2ydy}{\sqrt{z^2 - a^2y}} = adx$$

Integrating on both sides, we have,

$$\int \frac{zdz - a^2ydy}{\sqrt{z^2 - a^2y}} = a\int dx \qquad \text{-------(10.52)}$$

To find $\int \frac{zdz - a^2ydy}{\sqrt{z^2 - a^2y}}$

$$\text{Let } t^2 = z^2 - a^2y^2$$

$$2tdt = 2zdz - 2a^2ydy$$

$$\text{(ie) } tdt = zdz - a^2ydy$$

$$\therefore \int \frac{zdz - a^2ydy}{\sqrt{z^2 - a^2y^2}} = \int \frac{tdt}{t}$$

$$= t$$

$$= \sqrt{z^2 - a^2y^2}$$

$\therefore$ (10.52) changes as $\sqrt{z^2 - a^2y^2} = ax+b$. Which is the required solution.

10.5 Special types of first order equation.

Type 1 :

Equation involving on p, q.

$$\text{(ie) } f(p, q) = 0$$

In this type put p=a, we get, $q = \varphi(a)$ and substitute in dz=pdx+qdy

$$\Rightarrow \int dz = a\int dx + \varphi(a)\int dy$$

$$\Rightarrow z = ax+\varphi(a)y+b$$

Example E. 10.7 :

Find the complete integral of p+q = pq

Solution :

$$\text{Given that } p+q = pq \qquad \text{-------(10.53)}$$

$$\text{put } p = a \text{ in (10.53), we get,}$$

$$a+q = aq$$

$$\text{(ie) } q(a-1) = a$$

$$\text{(ie) } q = \frac{a}{a-1}$$

$$\therefore dz = pdx+qdy \text{ changes as}$$

$$dz = adx + \frac{a}{a-1}dy$$

Integrating on both sides, we get,

$$\int dz = a\int dx + \frac{a}{a-1}\int dy$$

$$\text{(ie) } z = ax + \frac{a}{a-1}y + b$$

which is the required solution.

Type 2 :

Equation not involving independent variables (ie) f(z, p, q) = 0

To solve the differential equation put p = aq

From dz = pdx+qdy we get the solution.

Example E. 10.8 :

Find the complete integral of $p^2z^2+q^2 = 1$.

Solution :

$$\text{Given that } p^2z^2+q^2 = 1 \qquad \text{-------(10.54)}$$

put $p = aq$ in (10.54), we get,

$$a^2q^2z^2+q^2 = 1$$

$$\text{(ie) } q^2(a^2z^2+1) = 1$$

$$\text{(ie) } q = \frac{1}{\sqrt{a^2z^2+1}}$$

$$\therefore\ p = \frac{a}{\sqrt{a^2z^2+1}}$$

Now $dz = pdx+qdy$ changes as

$$dz = \frac{a}{\sqrt{a^2z^2+1}}dx + \frac{1}{\sqrt{a^2z^2+1}}dy$$

$$\text{(ie) } \sqrt{a^2z^2+1}\,dz = adx+dy$$

Integrating on both sides, we get,

$$\int\sqrt{a^2z^2+1}\,dz = a\int dx + \int dy$$

$$\text{(ie) } \frac{1}{2a}\left[az\sqrt{1+a^2z^2} + \log\left(az+\sqrt{1+a^2z^2}\right)\right] = ax+y+b$$

Which is the required solution.

Type 3 :

Seperable equation in the form $f(x,.p) = q(y, q)$. In this type put $f(x, p) = a$, find the values of p and q and using $dz=pdx+qdy$, find the solution.

Example E.10.9 :

Find the complete integral of $p^2q(x^2+y^2) = p^2+q$.

Solution :

$$\text{Given that } p^2q(x^2+y^2) = p^2+q \qquad \text{-------(10.55)}$$

$$\text{(ie) } x^2+y^2 = \frac{p^2+q}{p^2q}$$

$$\text{(ie) } x^2+y^2 = \frac{1}{q}+\frac{1}{p^2}$$

$$\text{(ie)} \quad \frac{1}{p^2} - x^2 = y^2 - \frac{1}{q} = a^2 \text{ (say)}$$

$$\text{Now } \frac{1}{p^2} - x^2 = a^2$$

$$\Rightarrow \frac{1}{p^2} = a^2+x^2$$

$$\Rightarrow p^2 = \frac{1}{a^2+x^2}$$

$$\Rightarrow p = \frac{1}{\sqrt{a^2+x^2}}$$

$$\text{Again } y^2 - \frac{1}{q} = a^2$$

$$\Rightarrow \frac{1}{q} = y2-a2$$

$$\Rightarrow q = \frac{1}{y^2-a^2}$$

$$\text{Now } dz = pdx+qdy$$

$$\Rightarrow dz = \frac{dx}{\sqrt{a^2+x^2}} + \frac{dy}{y^2-a^2}$$

Integrating on both sides, we get

$$\int dz = \int \frac{dx}{\sqrt{a^2+x^2}} + \int \frac{dy}{y^2-a^2}$$

$$\text{(ie)} \quad z = \sinh^{-1}\left(\frac{x}{a}\right) + \frac{1}{2a}\log\left(\frac{y-a}{y+a}\right) + b$$

Which is the required solution.

Type 4 :

A first order partial differentiate equation in the form of $z = px+qy+f(p, q)$

In this type put $p=a$, $q=b$.

Example E.10.9 :

Find the complete integral of $(p+q)(z-xp-yq) = 1$.

Solution :

Given that (p+q)(z–xp–yq) = 1

$$\text{(ie) } z = xp + yq + \frac{1}{p+q} \qquad \text{-------(10.47)}$$

put p = a, q = b in (10.47), we get

$$z = ax + by + \frac{1}{a+b}$$

which is the required solution.

Example E.10.10 :

Find the integral of $pqz = p^2(xq+p^2)+q^2(yp+q^2)$

Solution :

Given that

$$pqz = p^2(xq+p^2)+q^2(yp+q^2)$$

$$\text{(ie) } z = \frac{p^2(xq+p^2)}{pq} + \frac{q^2(yp+q^2)}{pq}$$

$$z = xp + yq + \frac{p^3}{q} + \frac{q^3}{p} \qquad \text{-------(10.47)}$$

put p = a, q = b in (10.47), we have,

$$z = ax + by + \frac{a^3}{b} + \frac{b^3}{a}$$

which is the required solution.

MADURAI KAMARAJ UNIVERSITY

MODEL QUESTION PAPER

NUMERICAL METHODS AND DIFFERENTIAL EQUATIONS

Time : 3 Hrs. **Max : 100 Marks**

PART A

Answer any FOUR questions (4 × 10 = 40 marks)

1) Use the Secant and Regula-Falsi methods to determine the root of the equation $\cos x - xe^x = 0$.

2) Find the inverse of the coefficient matrix of the system.

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

by the Gauss-Jordan method with partial pivoting and hence solve the system.

3) Find the unique polynomial of degree 2 or less, such that f(0)=1, f(1)=3, f(3)=55 using (i) the Lagrange interpolation and (ii) the Newton-divided difference interpolation.

4) Write short notes on Gauss-Legendre Interpolation method and hence evaluate the following integral using three-point formula

$$\int_0^1 \frac{dx}{1+x}$$

5) Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be n solutions of L(y)=0 on an interval I, and let x_0 be any point in I. Prove that

$$W(\varphi_1, \varphi_2, \ldots, \varphi_n)(x) = \exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right] W(\varphi_1, \varphi_2, \ldots, \varphi_n)(x_0)$$

6) Find the singular points of the following differential equation $x^2y'' + (x+x^2)y' - y = 0$ and also find which are regular singular points.

7) Compute the first four successive approximations of $y' = 1+xy$, $y(0)=1$.

8) Show that the equations $xp - yq = x$, $x^2p + q = xz$ are compatible and find their solution.

PART B

Answer any THREE questions (3 × 20 = 60 marks)

9) a) Perform two iteration by Birge Vieta method to find the smallest positive root of the equation $x^4-3x^3+3x^2-3x+2 = 0$

b) Find all the roots of the polynomial $x^4-x^3+3x^2+x-4=0$

10) a) Determine the inverse of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$ using the partition method.

Hence find the solution of the system of equations

$$x_1+ x_2+ x_3 = 1$$
$$4x_1+3x_2- x_3 = 6$$
$$3x_1+5x_2+3x_3 = 4$$

b) Using Gauss-Seidel iterative method, preform three iterations to the following equations.

$$2x_1- x_2 = 7$$
$$- x_1+2x_2- x_3 = 1$$
$$- x_2+2x_3 = 1$$

11) a) Solve the initial value problem $u' = -2tu^2$, $u(0) = 1$ with h=0.2 on the interval [0, 1]. Use the forth order Runge-Kutta method.

b) Solve the initial value problem $u'=-2tu^2$, $u(0)=1$ using backward Euler method.

12) a) Find the solutions of Legendre equation

b) Show that $\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2^{n+1}}$

13) a) Consider the second order Euler equation $x^2y''+axy'+b=0$, (a, b constants) and the polynomial q given by $q(r)=r(r-1)+ar+b$. Prove that a basis for the solutions of the Euler equations on any interval not containing x=0 is $\varphi_1(x)=|x|^{r_1}$, $\varphi_2(x)=|x|^{r_2}$ in case r_1, r_2 are distinct roots of q, and by $\varphi_1(x)=|x|^{r_1}$, $\varphi_2(x)=|x|^{r_1} \log|x|$ if r_1 is a root of q of multiplicity two.

b) Fin all solutions $x^2y''+2xy'-6y = 0$ for x>0.

14) a) Find a complete general of $p^2x+q^2y=z$

b) Find the general solution of the differential equation

$$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=(x+y)z$$

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