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B.Sc., Third year

Part - III
ANCILLARY

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PAPER - XI
GRAPH THEORY

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Graph Theory**

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- Unit - X** Directed Graphs-Directed graphs-Connectivity in digraphs-Strong orientation
of graphs-Eulerian digraphs-Tournaments.

Text Book :

A First Course in Graph Theory (Macmillan) Author : S.A. Choudum

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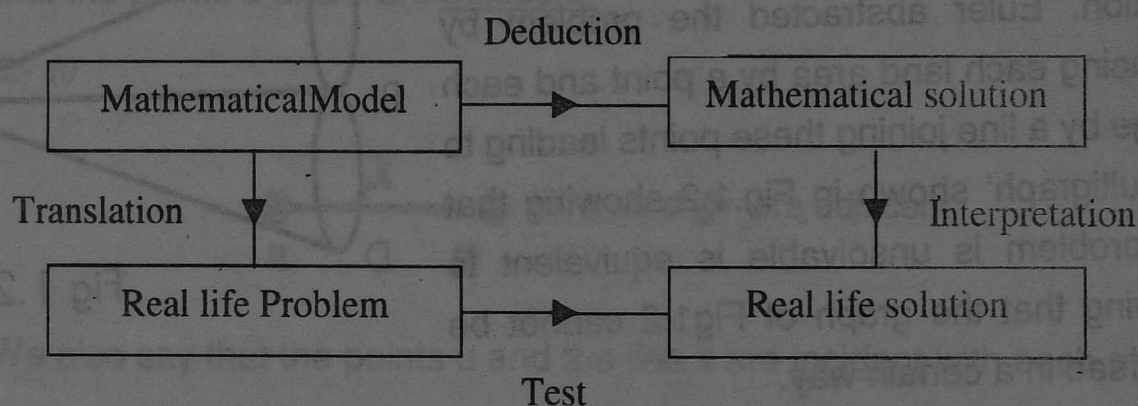
Graph Theory

Unit – I

1. INTRODUCTION: -

In the last three decades graph theory has established itself as a worthwhile mathematical discipline and there are many applications of graph theory to a wide variety of subjects, which include operations research, Physics, Chemistry, Economics, Genetics, Sociology, Linguistics, Engineering, Computer Science etc.

The development of many branches in Mathematics has been necessitated while considering certain real life problems arising in practical life or problems arising in other sciences. Such a development may be roughly described as follows.



Graph theory also has been independently discovered many times through some puzzles that arose from the physical world, consideration of chemical isomers, electrical networks etc. In this chapter we describe two concrete problems that led to the discovery of graphs.

2. THE KONIGSBERG BRIDGE PROBLEM: -

The city of konigsberg was located on the pregel river in Prussia. Euler (1707 – 1782) became the father of graph theory as well as topology. When in 1736 he settled a famous unsolved problem of his day called the Konigsberg Bridge problem.

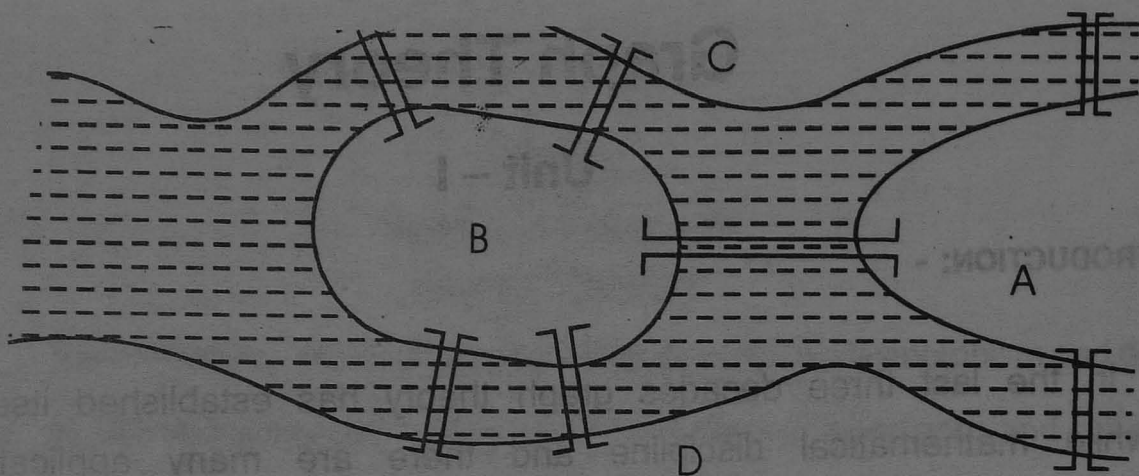


Fig 1.1

The two islands A and B and the two banks C and D of the pregel river are connected by seven bridges as shown in Fig. 1.1

The problem was to start from any one of the land areas, walk across each bridge exactly once and return to the starting point. Euler proved that this problem has no solution. Euler abstracted the problem by replacing each land area by a point and each bridge by a line joining these points leading to a 'multigraph' shown in Fig.1.2 showing that the problem is unsolvable is equivalent to showing that the graph of Fig1.2 cannot be traversed in a certain way.

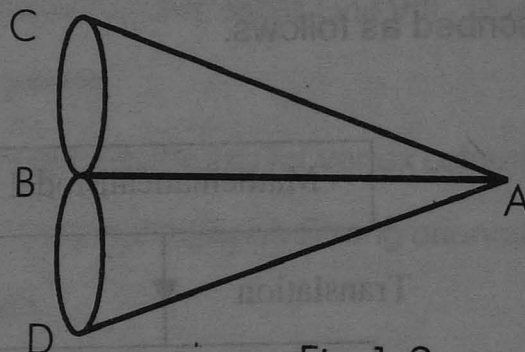


Fig 1.2

Rather than treating this specific situation, Euler generalized the problem and developed a criterion for a given graph to be so traversable, namely, that it is connected and every point is incident with an even number of lines while the graph in Fig.1.2. is connected, not every point is incident with an even number of lines.

3. FOUR COLOUR PROBLEM: -

One of the most famous problems in Graphtheory is the four colour problem. The Problem states that any map on a plane or on the surface of a sphere can be coloured with four colours in such a way that no two adjacent countries have the same colour. This problem can be translated as a problem in Graph theory. We represent each country by a point and join two points by a line if the countries are adjacent. The problem is to colour the points in such way that adjacent points have different colours. This problem was firstposed in

1852 by Francis Guthrie, a post – graduate student at the university college, London.

This problem was finally proved by Appel and Haken in 1976 and they have used 400 pages of arguments and about 1200 hours of computer time on some of the best computers in the world to arrive at the solution.

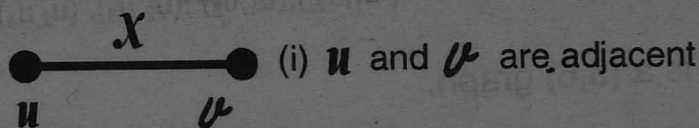
4. GRAPHS: -

DEFINITION:-

A graph G consists of a pair $(V(G), x(G))$ where $V(G)$ is a non – empty finite set whose elements are called points or vertices and $X(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The elements of $X(G)$ are called lines or edges of the graph G .

If $x = \{ u, v \} \in x(G)$, the line x is said to join u and v . We write $x = uv$ and we say that the points u and v are adjacent.

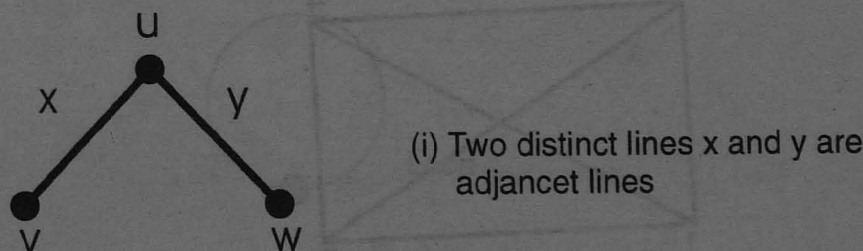
EXAMPLE:- (i)



We also say that the points u and the line x are incident with each other.

If two distinct lines x and y are incident with a common point then they are called adjacent lines.

EXAMPLE:- (ii)

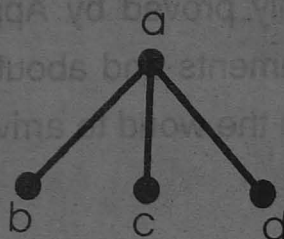


A graph with P points and q lines is called a (p, q) graph.

When there is no possibility of confusion we write $V(G) = V$ and $X(G) = X$.

EXAMPLE:-

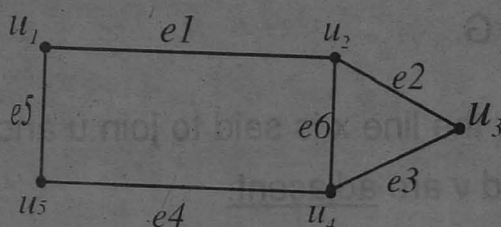
(i)



Let $V = \{ a,b,c,d \}$ and $X = \{ (a,b), (a,c), (a,d) \}$ $G = (V,X)$ is a $(4,3)$ graph.

The points a and b are adjacent whereas b and c are non-adjacent

ii)

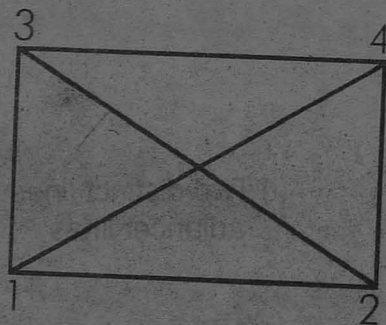


Let $V = \{ u_1, u_2, u_3, u_4, u_5 \}$ and $X = \{ (u_1, u_2), (u_1, u_5), (u_2, u_3), (u_2, u_4), (u_3, u_4), (u_4, u_5) \}$

$G = (v,x)$ is a $(5,6)$ graph.

- 1) u_1 and u_2 are adjacent.
- 2) e_1 , is incident from u_1 to u_2 .
- 3) u_1, u_3, u_4 are adjacent to u_2 .
- 4) The point u_1 and the line e , are incident with each other.

iii)

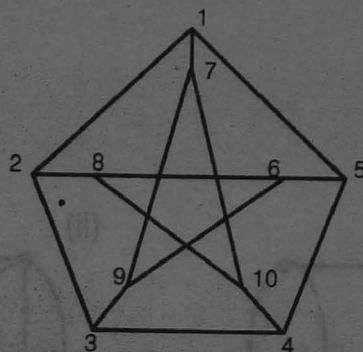


Let $V = \{ 1,2,3,4 \}$ and $X = \{ (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) \}$

$G = (v,x)$ is a $(4,6)$ graph.

The lines $\{1,3\}$ and $\{2,4\}$ intersect in the diagram, Their intersection is not a point of the graph.

iv)



The graph (10,15) is called the Petersen graph.

REMARK:-

It is customary to represent a graph by a diagram and refer to the diagram itself as the graph. Each point is represented by a small dot and each line is represented by a line segment joining the two points with which the line is incident. Thus a diagram of the graph depicts the incidence relation holding between its points and lines. In drawing a graph it is immaterial whether the lines are drawn straight or curved, long or short and what is important is the incidence relation between its points and lines.

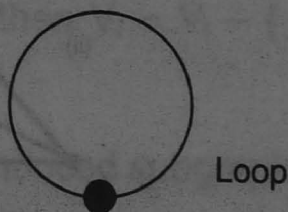
REMARK:-

The definition of a graph does not allow more than one line joining two points. It also does not allow any line joining a point to itself.

DEFINITION:-

In a line joining a point to itself is called a loop.

EXAMPLE:-



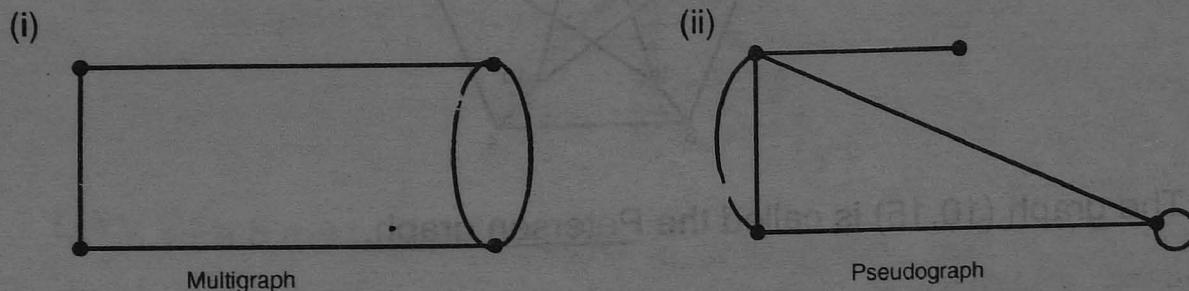
DEFINITION:-

If more than one line joining two vertices are allowed, the resulting object is called a multigraph.

Lines joining the same points are called multiple lines.

If further loops are also allowed, the resulting object is called a pseudograph.

EXAMPLE:-



The konigsberg bridge problem is a multigraph.

NOTE:-

Let G be a (P, q) graph.

Then $q \leq \binom{p}{2}$ and $q = \binom{p}{2}$ if any two distinct points are adjacent.

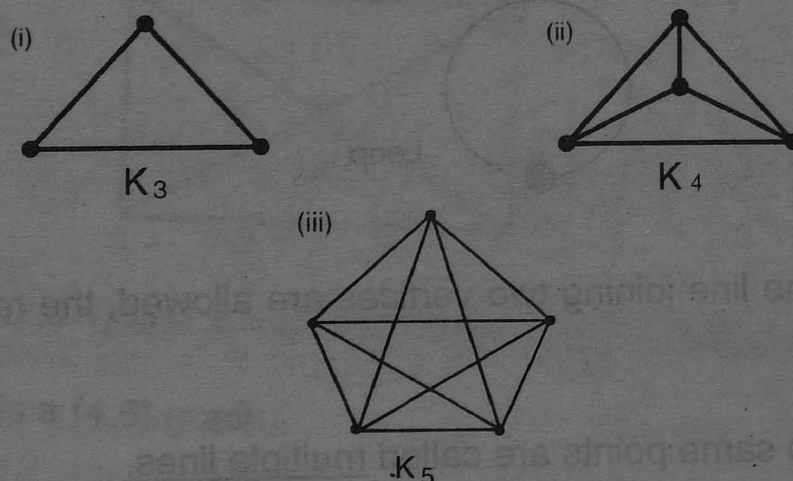
DEFINITION:-

A graph in which any two distinct points are adjacent is called a complete graph.

The complete graph with P points is denoted by K_p .

EXAMPLE:-

K_3 is called a triangle.



DEFINITION:-

A graph whose edge set is empty is called a null graph or a totally disconnected graph.

EXAMPLE:-

(i)

null graph

5. Subgraphs:-

DEFINITION:-

A graph $H = (V_1, X_1)$ is called a subgraph of $G = (V, X)$ if $V_1 \subseteq V$ and $X_1 \subseteq X$

NOTE:-

If H is called a subgraph of G , and G is called a supergraph of H .

DEFINITION:-

A graph $H = (v_1, x_1)$ is called a spanning subgraph of $G = (v, x)$ if $V_1 = V$

DEFINITION:-

A graph $H = (v_1, x_1)$ is called an induced subgraph of G if H is the maximal subgraph of G with the point set V_1 .

This, if H is an induced subgraph of G , Two points are adjacent in H iff they are adjacent in G .

DEFINITION:-

Let $G = (v, x)$ be a graph let $v_i \in v$. The subgraph of G obtained by removing the point v_i and all the lines incident with v_i is called the subgraph obtained by the removal of the point v_i and is denoted by $G - v_i$.

NOTE:-

If $G - v_i = (V_i, X_i)$ then $V_i = V - \{v_i\}$ and $X_i = \{x / x \in X \text{ and } x \text{ is not incident with } v_i\}$

Clearly $G - v_i$ is an induced subgraph of G .

Let $x_j \in X$.

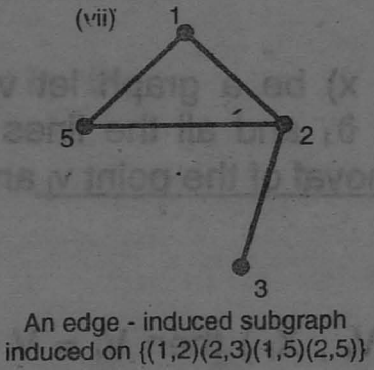
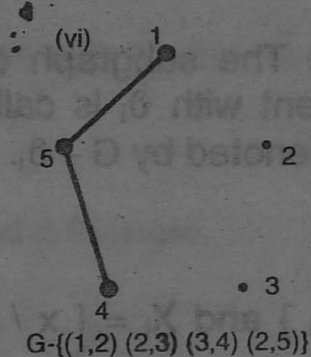
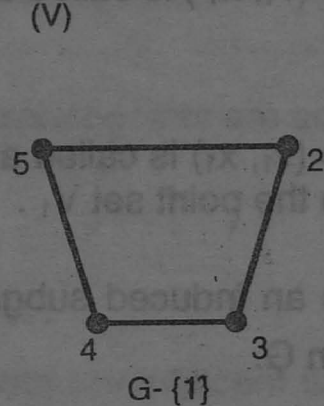
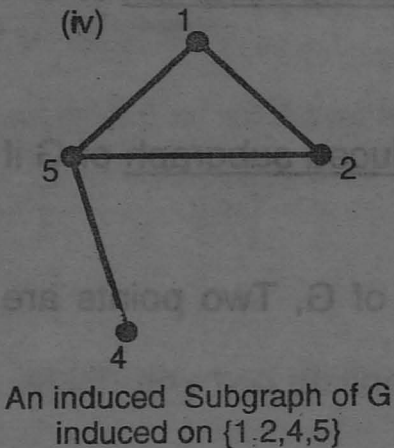
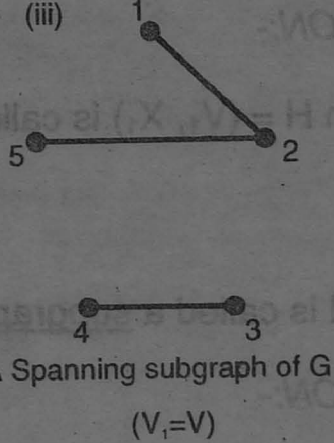
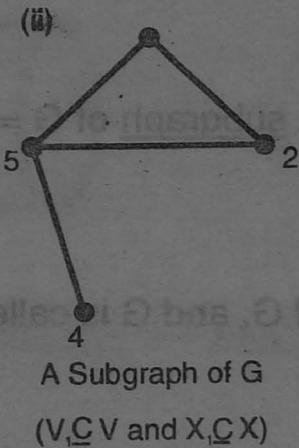
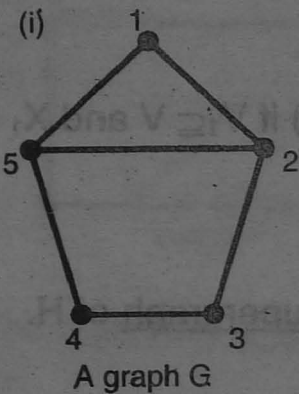
Then $G - X_j = (V, X - \{x_j\})$ is called the subgraph of G obtained by the removal of the line x_j clearly $G - x_j$ is a spanning subgraph of G which contains all the lines of G except x_j .

DEFINITION:-

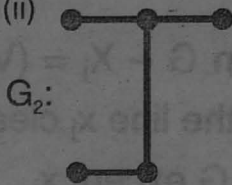
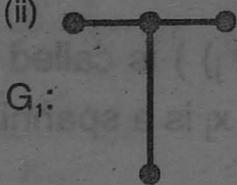
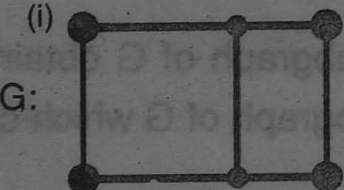
Let $G = (V, X)$ be a graph let v_i, v_j be two points, which are not adjacent in

G . Then $G + v_i, v_j = \{V, X \cup \{v_i, v_j\}\}$ is called the graph obtained by the addition of the line v_i, v_j to G .

EXAMPLES:- (1)

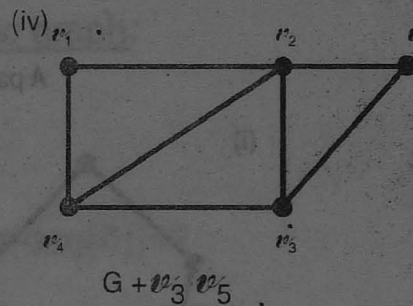
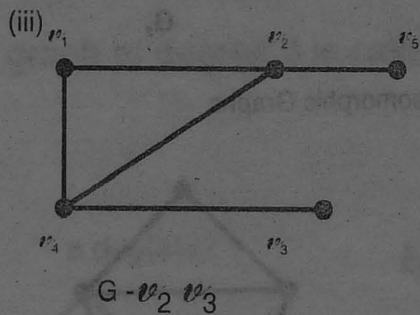
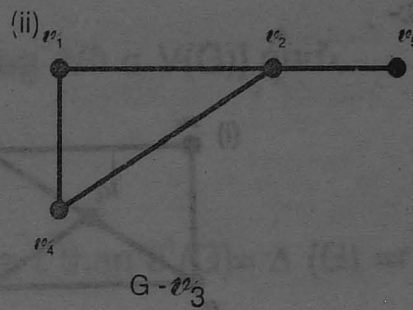
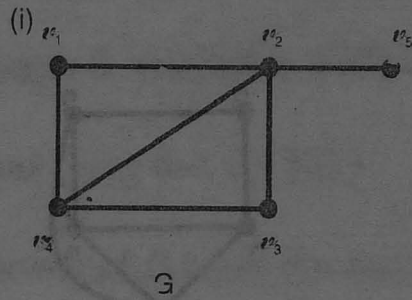


2)

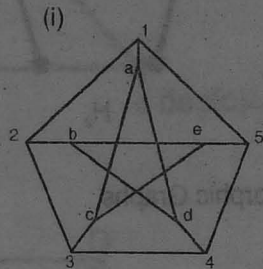


A graph and two subgraphs

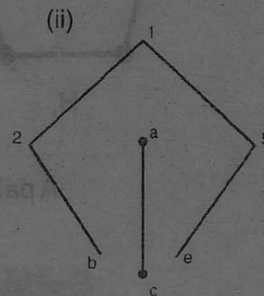
3) A graph plus (or) minus a point or Line



4)



A graph of G



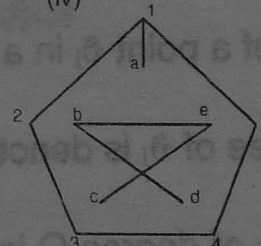
A subgraph of G

(iii)



An included subgraph of G

(iv)



A spanning subgraph of G

6. ISOMORPHISM:

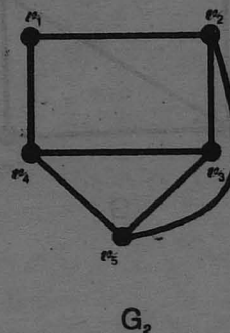
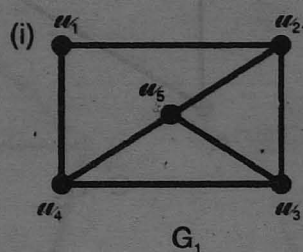
DEFINITION:-

Two graphs $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ are said to be isomorphic if there exists a bijection $f: V_1 \rightarrow V_2$ such that u, v are adjacent in G_1 iff $f(u), f(v)$ are adjacent in G_2 .

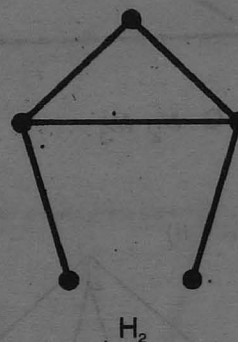
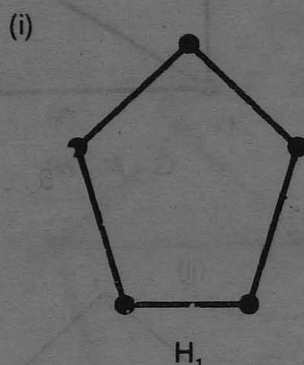
If G_1 is isomorphic to G_2 , we write $G_1 \cong G_2$. The map f is called an

isomorphism from G_1 to G_2 .

EXAMPLES:-



A pair G_1, G_2 of Isomorphic Graphs



A pair H_1, H_2 of Non-Isomorphic Graphs

7. DEGREES:

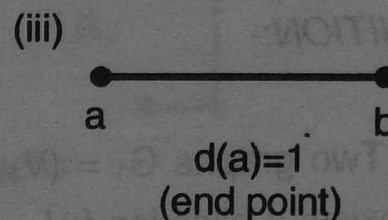
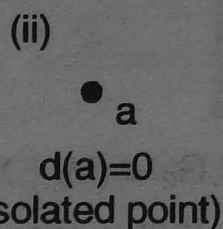
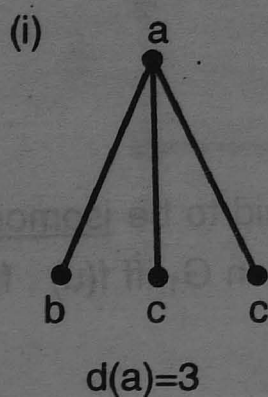
DEFINITION:

The degree of a point v_i in a graph G is the number of lines incident with V_i .

The degree of v_i is denoted by $d_G(v_i)$ or $\deg v_i$ or simply $d(v_i)$

A point v of degree 0 is called an isolated point. A point v of degree 1 is called an end point.

EXAMPLES:-



DEFINITION:-

For any graph G , we define $\delta(G) = \min \{ \deg v / v \in V(G) \}$ and

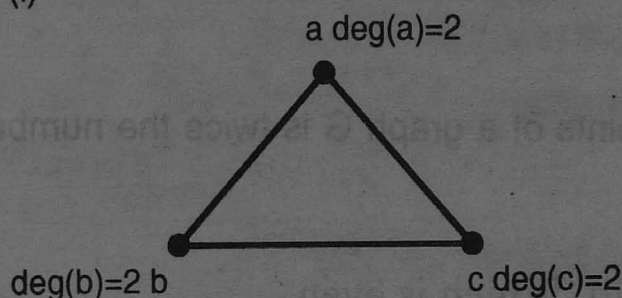
$$\Delta(G) = \max \{ \deg v / v \in V(G) \}$$

If all the points of G have the same degree r then $\delta(G) = \Delta(G) = r$ and in this case G is called a regular graph of degree r .

A regular graph of degree 3 is called a cubic graph:

EXAMPLE:-

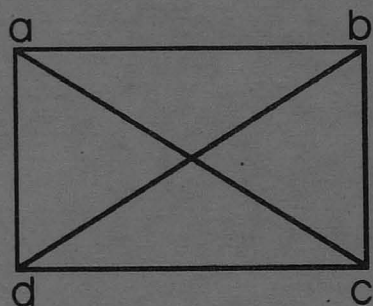
(i)



$$\delta(G) = \min \{ \deg(v) \} = 2$$

$$\Delta(G) = \max \{ \deg(v) \} = 2$$

(ii)



Cubic Graph

$$\deg(a)=3$$

$$\deg(b)=3$$

$$\deg(c)=3$$

$$\deg(d)=3$$

degree 3 is called a cubic graph.

NOTE:-

The complete graph K_p is regular of degree $P - 1$.

THEM:-

The sum of the degrees of the points of a graph G is twice the number of lines, i.e. $\sum_i \deg v_i = 2q$.

PROOF:-

Every line of G is incident with two points. Hence every line contributes

2 to the sum of the degrees of the points.

Hence $\sum \deg v_i = 2q$.

Hence the theorem.

COR:-

If any graph G the number of points of odd degree is even.

PROOF:-

Let $v_1, v_2, \dots, v_k$ denote the points of odd degree in G.

And $\omega_1, \omega_2, \dots, \omega_m$ denote the points of even degree in G.

By THM,

The sum of the degrees of the points of a graph G is twice the number of lines.

$$\sum_{i=1}^k \deg v_i + \sum_{i=1}^m \deg \omega_i = 2q. \text{ Which is even}$$

Further,

$$\sum_{i=1}^m \deg \omega_i \text{ is even}$$

Hence

$$\sum_{i=1}^k \deg v_i \text{ is also even.}$$

But $\deg v_i$ is odd for each i.

Hence K must be even.

Hence the corollary.

EXERCISE:-

1) Every cubic graph has an even number of points.

2) Draw all graphs with 1, 2, 3 and 4 points.

3) Give three examples for a regular graph of degree 2.

UNIT – II

1. WALKS AND CONNECTED GRAPHS.

DEFINITION:-

A walk of a graph G is an alternating sequence of points and lines $v_0, x_1, v_1, x_2, v_2, \dots, v_{n-1}, x_n, v_n$ beginning and ending with points such that each line x_i is incident with v_{i-1} and v_i .

NOTE:-

- The walk joins v_0 and v_n and it is called a $v_0 - v_n$ walk.
 v_0 is called the initial point and v is called the terminal point of the walk.
- The number of lines in the walk, is called the length of this walk.
- A single point is considered as a walk of length 0.

DEFINITION:-

A walk is called a trail if all its lines are distinct.

DEFN:-

A walk is called a path if all its points and lines are distinct.

NOTE:-

- Every path is a trail and a trail need not be a path.
- The graph consisting of a path with n points is denoted by P_n .

EXAMPLE:-

for the graph given in fig.2.1.

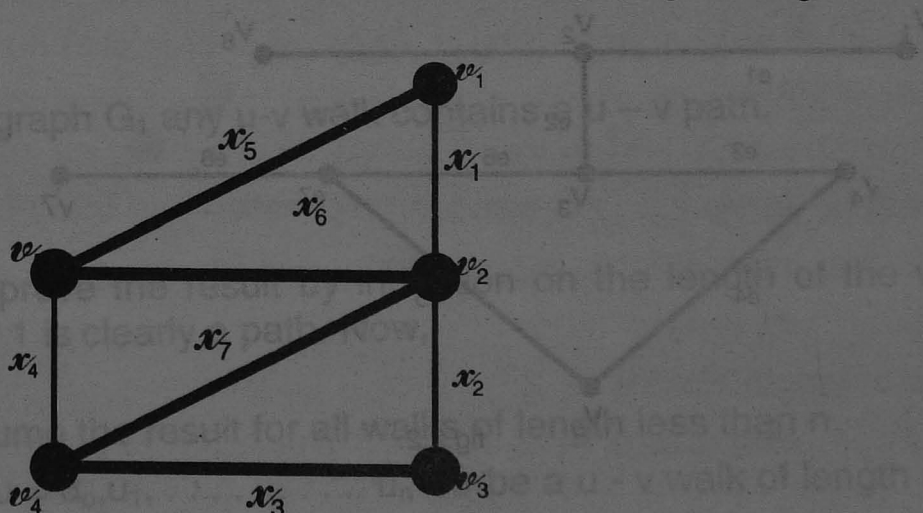


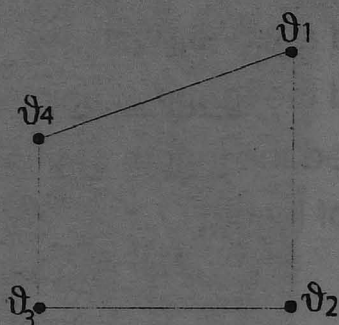
fig.2.1.

- (i) $v_1, x_2, v_2, x_3, v_3, x_4, v_3, x_3, v_4, x_7, v_2, x_1, v_1, x_1, v_2, x_6, v_5$ is a Walk
- (ii) $v_1, x_1, v_2, x_7, v_4, x_3, v_3, x_2, v_2, x_6, v_6$ is a trail but not a path
- (iii) $v_1, x_1, v_2, x_7, v_4, x_4, v_5$ is a path

DEFINITION:-

- i) A $v_0 - v_n$ walk is closed if $v_0 = v_n$
- ii) A closed walk $v_0, v_1, \dots, v_n = v_0$ in which $n \geq 3$ and $v_0, v_1, \dots, v_{n-1}$ are distinct is called a circuit or a cycle of length n .

EXAMPLE :-



v_1, v_2, v_3, v_4, v_1 is a cycle

NOTE:-

- i) The graph consisting of a cycle of length n is denoted by C_n
- ii) C_3 is called a triangle.
- iii) The number of occurrence of the edges in the sequence of a walk W (trail or path) is called length of the walk (trail or path) and is denoted by $L(w)$.

A path on n vertices (of length $n-1$) is denoted by P_n , and that of circuit on n Vertices (of length n) is denoted C_n .

EXAMPLES:-

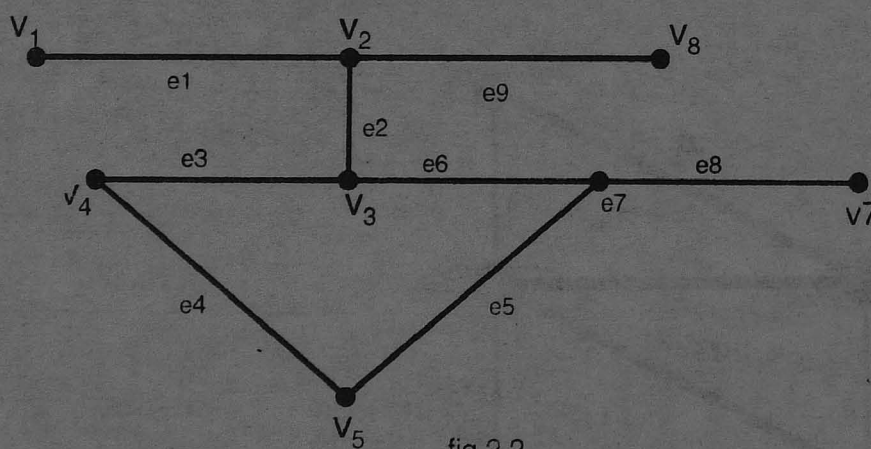


fig.2.2.

Let :

$W_1: v_1, e_1, v_2, e_9, v_8, e_9, v_2, e_2, v_3, e_6, v_6, e_6, v_3, e_3, v_4$

$W_2: v_5, e_5, v_6, e_7, v_6, e_8, v_7$

$W_3: v_1, e_1, v_2, e_2, v_3, e_6, v_6, e_8, v_7$

$W_4: v_4, e_3, v_3, e_6, v_6, e_7, v_6, e_5, v_5, e_4, v_4$

$W_5: v_3, e_3, v_4, e_4, v_5, e_5, v_6, e_6, v_3$

$W_6: v_1, e_1, v_2, e_8, v_7$

$W_7: v_1, e_1, v_2, e_9, v_5, e_9$

Then

i) W_1 is an open walk from v_1 to v_4 of length seven that is not a trail (since e_6 & e_9 is repeated), hence it is also not a path.

ii) W_2 is an open walk from v_5 to v_7 of length three, which is also a trail but not a path (as v_6 is repeated)

iii) W_3 is an open walk from v_1 to v_7 of length four, which is a trail and also a path

iv) W_4 is a close walk of length five that is also a closed trail but not closed path. (Since v_6 is repeated twice)

v) W_5 is a closed walk, which is a closed trail as well as closed path a circuit) of length four.

vi) W_6 is not a walk, since v_2 and v_7 are not the end vertices of the edge e_8 that is cited between them. Hence W_6 is not a trail or a path. Length cannot be defined.

vii) W_7 is not a walk as the sequence of W_7 is ended at an edge e_9 of G .

viii) W_8 is not a walk as no edge occur between the vertices v_2 and v_8 (i.e two vertices v_2 and v_8 are consecutive)

THM:-

In a graph G_1 any u - v walk contains a $u - v$ path.

PROOF:-

We prove the result by induction on the length of the walk. Any walk of length 0 or 1 is clearly a path. Now,

Assume the result for all walks of length less than n .

Let $u = u_0, u_1, \dots, u_n = v$ be a $u - v$ walk of length n .

If all the points of the walk are distinct it is already a path.

If not,

an (Y) i and j such that $0 \leq i < j \leq n$ and $u_i = u_j$.

Now,

$u = u_0, \dots, u_i, u_{j+1}, \dots, u_n = v$ is a $u - v$ walk of length less than n . which by induction hypothesis contains a $u - v$ path.

Hence the theorem.

2. CYCLES IN GRAPHS:-

THM:-

If G is a graph with $\delta(G) \geq 2$, then G contains a cycle of length atleast $\delta + 1$

PROOF:-

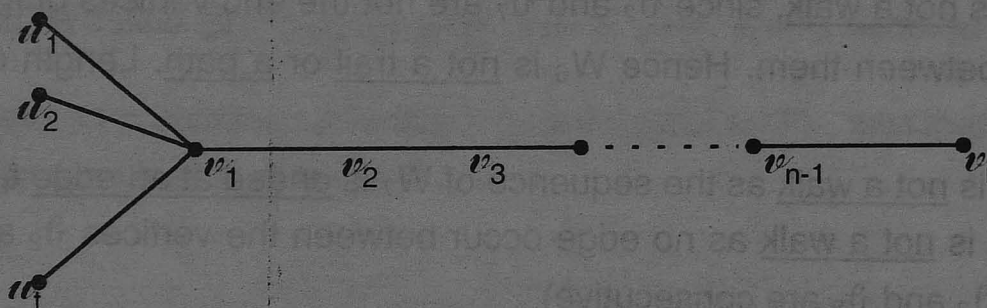
Let P be a path of maximum length in G . Let v_1 and v_n be its initial and terminal vertices.

Let $P : v_1, v_2, v_3, \dots, v_n$.

Let $\deg(v_1) = t + 1$

As $\delta(G) \geq 2$, $t \geq 1$, and so, there are t distinct vertices.

Say $u_1, u_2, \dots, u_t$ adjacent with v_1 besides v_2 .



But then $\{u_1, u_2, \dots, u_t\} \subseteq \{v_3, v_4, \dots, v_n\}$

Because, if, on the contrary,

u_k is a vertex in $\{u_1, u_2, \dots, u_t\} - \{v_3, v_4, \dots, v_n\}$

Then the path $u_k, v_1, v_2, \dots, v_n$ has length greater than that of P .

Now, let j be the greatest integer such that $v_j = u_i$ for some i .

Then $v_1, v_2, \dots, v_j (=u_i)$

v_j is a cycle of length atleast $t + 2$

The proof is complete. Since $t + 2 = \deg(v_1) + 1 \geq \delta + 1$

Hence the thm.

THM:- EVERY (P, q) - GRAPH WITH $q \geq P$ CONTAINS A CYCLE.

PROOF:-

We shall prove the theorem by induction on P .

There is only one (p, q) – graph, namely C_3 , with $q \geq p$, $p \leq 3$ and it contains a cycle.

We now assume that every graph on $P-1$ vertices with $q \geq p-1$ contains a cycle.

Let H be a graph on P vertices with $q(H) \geq p$.

If $\delta(H) \geq 2$.

Then H contains a cycle. (by thm If G is a graph with $\delta(G) \geq 2$, then G contains a cycle of length at least $\delta + 1$)

We take up the case $\delta(H) \leq 1$

Let v be a vertex of degree ≤ 1

Consider the graph $H - v$

We have $P(H-v) = P - 1$ and $q(H-v) \geq q(H) - 1 \geq P - 1$ ($\therefore q(H-v) = P(H-v)$)

So, by induction hypothesis $H - v$ contains a cycle.

Hence H contains a cycle.

Hence the theorem.

DEFINITION:-

Two points u and v of a graph G are said to be connected if there exists a $u-v$ path in G .

DEFINITION:-

A graph G is said to be connected if every pair of its points are connected.

A graph which is not connected is said to be disconnected.

EXAMPLE:-

For $n > 1$ the graph $\overline{K}_n$ consisting of n points and no lines is disconnected.

DEFINITION:-

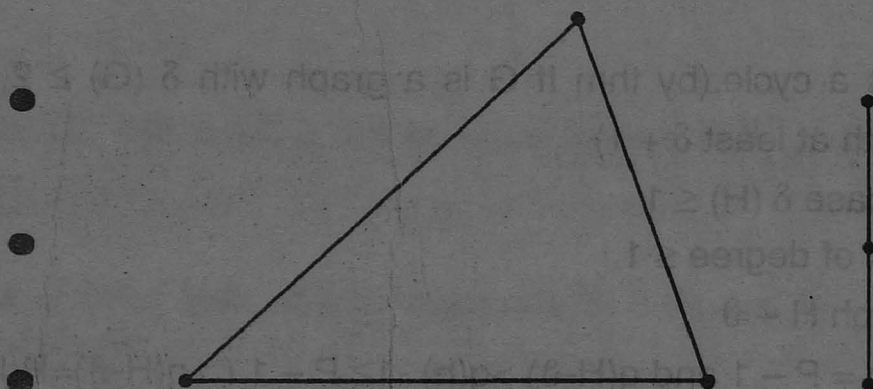
Let G_i denote the induced subgraph of G with vertex set V_i . clearly the

subgraphs $G_1, G_2, \dots, G_n$ are connected and are called the components of G.

NOTE:-

A graph G is connected iff it has exactly one component.

EXAMPLE:-



A disconnected graph with 5 components

THM:-

If $q \geq p - 1$, then every (p, q) - graph is either connected or contains a cycle.

PROOF:-

Assume the contrary,

Let G be a (p, q) - graph with $q \geq p - 1$ which is neither connected nor contains a cycle.

Let $G_1, G_2, \dots, G_t$ ($t \geq 2$) be the t component of G.

Let G_i have P_i vertices and q_i edges, for each i , ($1 \leq i \leq t$)

Since each of the components is without a cycle.

By THM,

Every (p, q) - graph with $q \geq p$ contains a cycle.

$$\therefore q_i \leq p_i - 1, \text{ for each } i$$

$$\therefore q = q_1 + q_2 + \dots + q_t$$

$$\leq p_1 + p_2 + \dots + p_t - t$$

$$= p - t$$

$$\therefore q \leq p - 2$$

This contradicts our assumption, $\therefore$ The result is proved.

THM:- A CONNECTED (P, q) – GRAPH CONTAINS A CYCLE IFF $q \geq P$

PROOF:- BY THM,

Every (p, q) -graph with $q \geq p$ contains a cycle.

We need to prove only one implication.

If a graph G contains a cycle, then $q(G) \geq p(G)$, This we do by induction on p . When $p \leq 3$, there is only one (p, q) – graph with a cycle namely C_3 and for C_3

$$q(C_3) = 3 = P(C_3)$$

Now,

Assume that every graph on $p-1$, vertices which contains a cycle, has atleast $p-1$ edges, Let G be a graph on P vertices, which contains a cycle C .

If $\delta(G) = 2$.

Then using by thm,

If G is a (P, q) – graph, then $\sum_{v \in V(G)} \deg(v) = 2q$

We get,

$$q(G) = \frac{1}{2} \sum_{v \in V(G)} \deg(v)$$

$$\geq \frac{1}{2} \sum_{v \in V(G)} 2 = p$$

So, next assume that $\delta(G)=1$.

Let v be a vertex of degree 1 consider the graph $G-v$, obviously,

$G-v$ contains the cycle C . So by induction hypothes;

$$q(G-v) \geq p(G-v)$$

we have

$$q(G) = q(G-v) + 1$$

$$\geq p(G-v) + 1$$

$$= p(G)$$

Hence the theorem.

THM:-

The maximum number of lines among all p point graphs with no triangles is $[p^2 / 4]$.

($[x]$ denotes the greatest integer not exceeding the real number x)

PROOF:-

The result can be easily verified for $p \leq 4$. for $p > 4$, we will prove by induction separately for odd p and for even p .

CASE (I):-

For odd p .

Suppose the result is true for all odd $p \leq 2n + 1$

Now,

Let G be a (p, q) graph with $p = 2n + 3$ and no triangles.

If $q=0$, then $q \leq [p^2 / 4]$ Hence let $q > 0$

Let u and v be a pair of adjacent points in G . The subgraph $G^1 = G - \{u, v\}$ has $2n + 1$ points and no triangles.

Hence by induction hypothesis, .

$$q(G^1) \leq \left[\frac{(2n+1)^2}{4} \right] = \left[\frac{4n^2 + 4n + 1}{4} \right]$$

$$= \left[n^2 + n + \frac{1}{4} \right] = n^2 + n \text{ --- } ①$$

Since G has no triangles, no point of G^1 can be adjacent to both u and v in G ---- ②

Now, lines in G are of three types.

i) lines of G^1 ($\leq n^2 + n$ in number by (1))

ii) lines between G^1 and $\{u, v\}$

($\leq 2n + 1$ in number by (2))

iii) line uv .

Hence

$$\begin{aligned}
 q &\leq (n^2 + n) + (2n + 1) + 1 \\
 &= n^2 + 3n + 2 \\
 &= \frac{1}{4} (4n^2 + 12n + 8) \\
 &= \left(\frac{4n^2 + 12n + 9}{4} - \frac{1}{4} \right) \\
 &= \left[\frac{(2n + 3)^2}{4} \right] = \left[\frac{p^2}{4} \right]
 \end{aligned}$$

Also for $p = 2n + 3$, the graph $k_{n+1,n+2}$ has no triangles and has

$$(n+1)(n+2) = n^2 + 3n + 2 = \left[\frac{p^2}{4} \right] \text{ lines.}$$

Hence this maximum q is attained.

CASE (II):-

For even p .

Suppose the result is true for all even $p \leq 2n$

Now, Let G be a (P, q) graph with $P=2n+2$ and no triangles.

As before,

Let u and v be a pair of adjacent points in G .

Let $G^1 = G - \{u, v\}$

Now,

G^1 has $2n$ points and no triangles. Hence by hypothesis.

$$q(G^1) \leq \left[\frac{(2n)^2}{4} \right] = n^2 \text{--- (3)}$$

Lines in G are of three types.

- i) lines of G^1 ($\leq n^2$ in number by (3))
- ii) lines between G^1 and $\{u, v\}$
($\leq 2n$ in number by an argument similar to (2))
- iii) line uv .

Hence

$$q \leq n^2 + 2n + 1 = (n+1)^2 = \frac{(2n+2)^2}{4} = \left\lceil \frac{p^2}{4} \right\rceil$$

Hence the result holds for even p also

we see that for $p=2n+2$.

$K_{n+1,n+1}$ is a $(p, \left\lceil \frac{p^2}{4} \right\rceil)$

Graph without triangles.

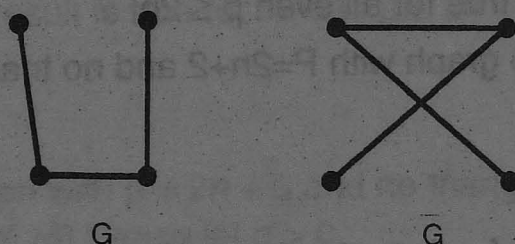
Hence the theorem.

DEFINITION:-

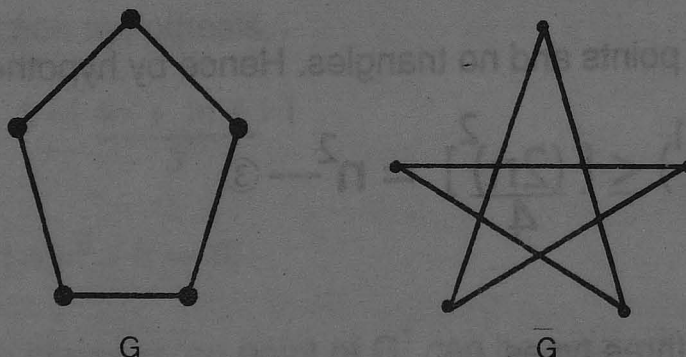
Let $G = (v, x)$ be graph The complement $\bar{G}$ of G is defined to be the graph which has v as its set of points and two points are adjacent in $\bar{G}$ iff they are not adjacent in G . G is said to be a self complementary graph if G is isomorphic to $\bar{G}$.

EXAMPLE:-

(i)



(ii)



A graph and its complement

THM:-

Let G be a graph on at least 6 vertices. Then G or $\bar{G}$ contains a triangle.

PROOF:-

Let v be a vertex in G we have,

$$\deg_G(v) + \deg_{\overline{G}}(v) = P - 1 \geq 5$$

So either $\deg_G(v) \geq 3$ or $\deg_{\overline{G}}(v) \geq 3$

Since G contains 5 points other than v , v must be either adjacent to three points in G or non-adjacent to three points in G . Hence v must be adjacent to three points either in G or in $\overline{G}$. without loss of generality,

Let us assume that v is adjacent to three points u_1, u_2, u_3 , in G .

If two of these three points are adjacent. $\therefore G$ contains a triangle.

Otherwise these three points form a triangle in $\overline{G}$

Hence G or $\overline{G}$ contains a triangle

Hence the theorem.

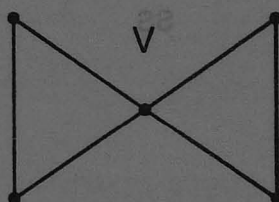
3. CUT – VERTICES AND CUT EDGES:-

DEFINITION: -

A cutpoint of a graph G is a point whose removal increases the number of components.

EXAMPLE:-

(i)



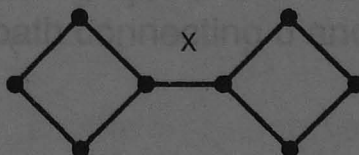
V is a cut-point

DEFINITION:-

A bridge (cut – edge) of a graph G is a line whose removal increases the number of components.

EXAMPLE:-

(i)



X is a bridge

NOTE:-

If v is a cutpoint of a connected graph iff $G - v$ is disconnected.

DEFINITION:-

A graph is said to be trivial if $p = 1$; otherwise it is said to be nontrivial.

DEFINITION:-

For any two points u, v of a graph we define the distance between u and v by

$$(u, v) = \begin{cases} \text{the length of a shortest } u-v \text{ path if such a path exists} \\ \infty \text{ Otherwise} \end{cases}$$

EXAMPLE:-

(i)

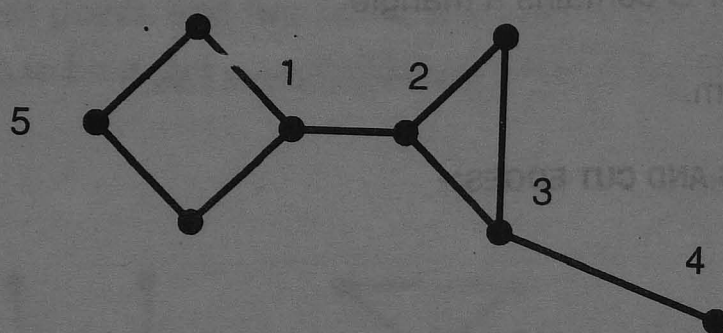


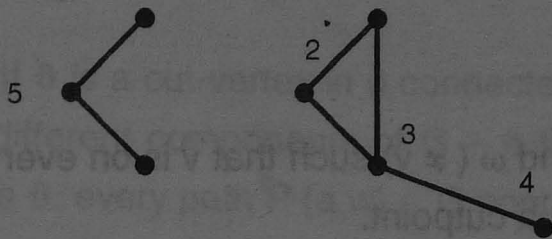
Fig 2.5

For the graph given in Fig 2.5, i) 1, 2 & 3 are cutpoints ii) The lines {1,2} and {3,4} are bridges iii) 5 is a non – cut point.

ss



Cut Point -1



2-Components

Cut Point -2

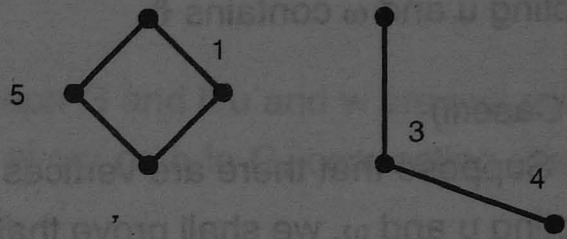
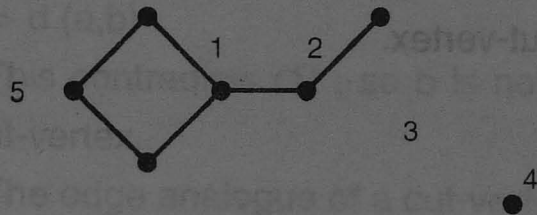


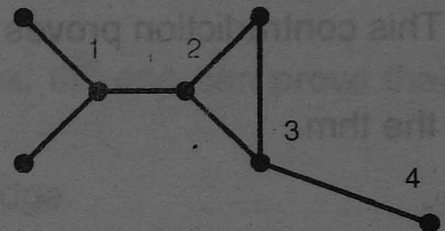
Fig 2.5

Cut Point -3



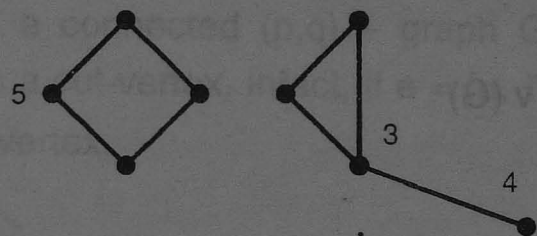
2-Components

Cut Point -5



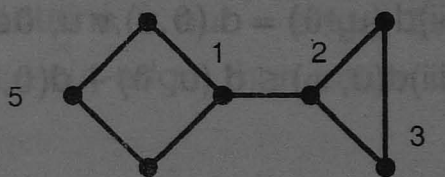
2-Components

Bridge {1,2}



2-Components

Bridge {3,4}



2-Components

NOTE:-

A component of a graph G is a maximal connected subgraph of G and the number of components in a graph G is denoted by $w(G)$

THM:-

A vertex v in a connected graph G is a cut-vertex iff there exist vertices u and w ($u \neq w$) such that every path connecting u and w contains v .

PROOF:-

case (i)

Let v be a cut-vertex.

$G - v$ is disconnected and so it contains at least two components, say C and D .

Let u be a vertex of C .

& w be a vertex of D .

There is no path in $G - v$ connecting u and w . Hence every path in G connecting u and w contains v .

Case:ii)

Suppose that there are vertices u and w ($\neq v$) such that v is on every path connecting u and w . we shall prove that v is a cutpoint.

Let, if possible, v be a non-cutpoint,

So, $G - v$ is connected Since $u, w \in G - v$, there is a path $P(u, w) \in G - v$.

Then $P(u, w)$ is a path in G , which does not contain v .

This contradiction proves that v is a cut-vertex.

Hence the thm.

REMARK:-

Let G be a connected graph and d be the distance function defined above, then $(V(G), d)$ is a metric space. That is, d satisfies:

i) $d(u, v) \geq 0$ and $d(u, v) = 0$ iff $u = v, \forall u, v \in V(G)$

ii) $d(u, v) = d(v, u), \forall u, v \in V(G)$

iii) $d(u, w) \leq d(u, v) + d(v, w), \forall u, v, w \in V(G)$

THM:-

Every non trivial graph contains atleast two vertices which are not cut-vertices.

PROOF:-

It is enough to prove that every connected non-trivial graph contains two non cut-vertices.

Because,

This will imply that, if a graph G contains t components, then it will contains atleast $2t$ ($t \geq 1$) non cut- vertices.

So, Let G be a connected graph with $P(G) \geq 2$ Let a, b be two vertices in G which are at diametrical distance, say d . Then.

i) $d(u, v) \leq d(a, b) \forall u, v \in V(G)$

We claim that a and b are non cut – vertices.

On the contrary, Assume that b is a cut-vertex then $G - b$ is disconnected

Let $c_1, c_2, \dots, c_t$ be its components ($t \geq 2$)

Let,

Without loss of generality, C_1 contain a and Let w be some vertex in C_2
By corollary,

If v is a cut-vertex in a connected graph G and if u and w are two vertices in two different components of $G - v$ then every path in G , connecting u and w contains v . every path $P(a, w) \in G$ contain b .

In particular, a shortest path connecting a and w also contains b . So,

$$d(a, w) \geq d(a, b) + d(b, w)$$

$$\geq d(a, b) + 1 \text{ (since } b \neq w)$$

$$> d(a, b)$$

This contradicts (1) ; so b is not a cut-vertex, Ill^{ly} one can prove that a is not a cut-vertex.

The edge analogue of a cut-vertex is a cut-edge.

Hence the theorem.

THM:-

If a connected (p, q) - graph G ($p \geq 3$) contains a cut - edge, then it contains a cut-vertex, Infact, if $e = (u, v) \in E(G)$ is a cut-edge, then either u or v is a cut-vertex.

PROOF:-

Let $e = (u, v)$ be a cut-edge in G

Then $G - e$ is disconnected Let C and D be the two components.

Without loss of generality,

Assume that $u \in C$ and $v \in D$

Since $p \geq 3$, one of these components, say C , has atleast two vertices.

Clearly $G - u$ is a subgraph of $G - e$.

Hence $G - u$ contains atleast two components namely D and a component of $C - u$.

This proves that $G - u$ is disconnected.

Thus u is a cut-vertex.

Hence the theorem.

THM:-

For any graph G , $\eta(G) \geq p(G) - w(G)$

We prove the theorem by induction on p .

If $P = 1$, then $q(G) = 0$, $W(G) = 1$ and hence the inequality follows.

Assume the inequality if a graph contains less than P vertices and let G be a graph

On P vertices we make 3 cases and prove the inequality in each case.

Case(i):-

G has a vertex v of degree 0

Consider the graph $H = G - v$ We have,

$$P(H) = P(G) - 1, q(H) = q(G) \text{ and } W(H) = W(G) - 1$$

By induction hypothesis.

$$q(H) \geq P(H) - W(H)$$

So,

$$q(G) = q(H) \geq P(H) - W(H) = P(G) - W(G)$$

Case(ii):-

G has a vertex v of degree 1.

Again consider the graph $H = G - v$

v is not a cut-vertex of G .

By cor, A vertex v in a graph G (not necessarily connected) is a cut-vertex iff there are vertices u and w ($\neq v$) such that every path connecting u and w contains v .

So, we have

$$P(H) = P(G) - 1,$$

$$q(H) = q(G) - 1$$

$$\text{And } W(H) = W(G)$$

combining these equations and the induction hypothesis we get the required inequality.

Case(iii):-

$$\delta(G) \geq 2$$

by using them If G is a (P, q) - graph, then $\sum_{v \in G} \deg(v) = 2q$

$$\sum_{v \in G} \deg(v)$$

$$\therefore 2q(G) = \sum_{v \in G} \deg(v) \geq \sum_{v \in G} 2 = 2P(G)$$

$$v \in G \forall v \in G$$

So, $q(G) \geq P(G) > P(G) - W(G)$.

Hence the thm.

EXERCISE:-

III - UNIT

1. Let G be a graph in which every vertex has even degree. Show that G has no cut-edges.
2. Show that a connected graph G is a path iff it contains exactly two non-cut vertices.
3. Show that in a 3-regular graph there is a cut-vertex iff there is a cut-edge.
4. Give an example of a 3-regular graph on 10 vertices which contains a cut-vertex.

Show that there is no 3-regular graph with a cut-edge on less than 10 vertices.

1. Show that if $\delta(G) \geq P - \frac{1}{2}$, then G is connected. Give an example of a disconnected graph G on 8 vertices with $\delta(G) = 3$.



UNIT – III

I. EULERIAN GRAPHS:-

As we have seen in unit – I, Euler's negative solution of the Königsberg Bridge problem constituted the first publicized discovery of graph theory the perambulatory problem of crossing bridges can be abstracted to a graphical one given a graph G , is it possible to find a walk that traverses each line exactly once, goes through all points, and ends at the starting point? A graph for which this is possible is called eulerian. Thus, an eulerian graph has an eulerian trail, a closed trail containing all points and lines. Clearly, an eulerian graph must be connected.

DEFINITION:-

A closed trail containing all points and lines is called an eulerian trail. A graph having an eulerian trail is called an eulerian graph.

NOTE:-

Obviously in an eulerian graph, for every pair of points u and v there exist at least two edge-disjoint $u - v$ trails and consequently there are at least two edge-disjoint $u - v$ paths.

The graph given in Fig 3.1 is an eulerian graph.

EXAMPLE:-

(i)

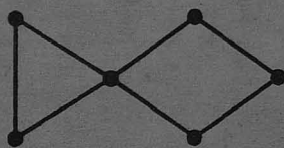
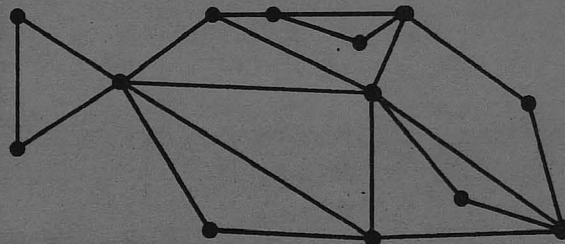


Fig 3.1 (An Eulerian Graph)

(ii)



(An Eulerian Graph)

THEOREM:-

A non-trivial connected graph is eulerian iff it has no vertex of odd-degree.

PROOF :-

Necessity:-

Let G be an eulerian graph.

Let $C(u,u)$ be a closed eulerian trail in G

Let $v \neq u$ be an arbitrary (internal) vertex of C

Let m_v denote the number of occurrences of v .

Since u is the terminal vertex, if it occurs n times internally, it occurs totally $n + 2$ times.

Each occurrence of an internal vertex accounts for two distinct edges whereas initial and terminal vertices account for one edge each.

Moreover,

$C(u,u)$ contains all the edges.

Hence, $\deg_G(v) = 2m_v$

& $\deg_G(u) = 2n + 2$

SUFFICIENCY:

To avoid tedious writing, we shall temporarily, call a graph an 'evengraph' if does not contain a vertex of odd degree.

Let G be a connected even (P,q) – graph.

We shall prove, by induction on q $\therefore G$ is eulerian,

Since G is even and connected $\therefore \deg(v) \geq 2$ Hence $q \geq p$.

So, as the first step in induction we have to prove that if G is a connected even (P,P) Graph.

$\therefore G$ is eulerian

For this,

We note that there is no vertex of degree ≥ 4 in G .

If there is a vertex of degree ≥ 4 in G

We have $q = \frac{1}{2} \sum \deg(v) \geq \frac{1}{2} [2(p-1) + 4]$

$\geq \frac{1}{2} [2p - 2 + 4] \geq \frac{1}{2} \cdot 2[p + 1]$

$= p+1$ which is contradiction Hence every vertex in G is of degree 2.

This implies that G is a cycle.

$\therefore$ Hence it is eulerian.

Hence the theorem.

THM:-

In a connected graph G there is an eulerian trail iff the number of vertices of odd-degree is either zero or two.

PROOF:-

Necessity:

Let $L(u,v)$ be an eulerian trail

If $u = v$, then G is eulerian

Hence, by thm,

A nontrivial connected graph is eulerian iff it has no vertex of odd degree.

The number of vertices of odd – degree in G is zero.

So the necessity is proved in this case next

Let $u \neq v$

Consider the graph H obtained from G , by adding a new vertex x and joining it with u and v . that is H is a graph with vertex set $V(G) \cup \{x\}$ and edge set $E(G)$

$\cup \{ (u,x), (v,x) \}$

Then $x, (x,u), (u,v), (v,x), x$ is a closed eulerian trail in H .

Hence, every vertex in H is of even degree, and so,

u and v are the only two vertices of odd degree in G .

SUFFICIENCY:-

Let the number n of vertices of odd degree in G be either zero or two. We shall prove that G has an eulerian trail. If $n=0$, then by above them, it has a closed eulerian trail afortiori, an eulerian trail.

Hence our assertion stands proved in this case.

Next, let $n = 2$.

Let u and v be the vertices of odd degree. Consider the graph H , defined in the necessary part. H is a connected even graph.

Hence it contains a closed eulerian trail. We can choose this eulerian trail so that x is the initial and terminal vertex.

Let it be $x(x,u), u(u,u), u_1, \dots, u_m(u_m,v), v, (v,x), x$.

But then, $u, (u, u_j), u_1, \dots, u_m, (u_m,v), v$ is an eulerian trail in G connecting u and v .

Hence the theorem.

THM:-

If G is a connected graph with $2n$ ($n \geq 1$) number of odd vertices, then $E(G)$ can be partitioned into n sets $E_1, E_2, \dots, E_n$ such that $\langle E_i \rangle$ is a trail, for each $i, 1 \leq i \leq n$.

PROOF:-

Let $u_1, u_2, \dots, u_{2n}$ be the $2n$ vertices of odd degree in G .

Construct a graph H by adding a new vertex ℓ to G and joining it to the vertices $u_1, u_2, \dots, u_{2n}$. Then every vertex in H has even degree

by them. A nontrivial connected graph is eulerian iff it has no vertex of odd degree. H contains a closed eulerian trail say C . We can choose C so that x is its initial and terminal vertex. So, without loss of generality,

Let

$$C = x, (x, u_1), u_1, \dots, u_2, (u_2, x), x, (x, u_3), u_3, \dots, u_4, (u_4, x), \dots, x, (x, u_{2n-1}), u_{2n-1}, \dots, u_{2n}, (u_{2n}, x), x.$$

But then the subtrails $L(u_1, u_2), L(u_3, u_4), \dots, L(u_{2n-1}, u_{2n})$ of C are edge disjoint and contain all the edges.

Hence the theorem.

FLEURY'S ALGORITHM TO CONSTRUCT A CLOSED EULERIAN TRAIL IN A CONNECTED EVEN GRAPH G .

2. FLEURY'S ALGORITHM:-

STEP 1:- Choose a vertex $v_0 \in V(G)$ and set $W_0 = v_0$

STEP 2:- Having chosen the trail,

$$W = v_0, e_1, v_1, e_2, v_2, \dots, v_{i-1}, e_i, v_i$$

delete the edge e_i ; form the graph $G_i = G_{i-1} - e_i$ ($i \geq 1$)

Where $G_0 = G$; thus $G_i = G - \{e_1, e_2, \dots, e_i\}$

Go to step 3.

STEP 3:-

i) If there is no edge incident at v_i in G_i (that is $\deg G_i(v_i) = 0$)

then stop; W is a closed eulerian trail.

ii) If there is an edge incident at v_i in G_i then choose one of the edges incident at v_i .

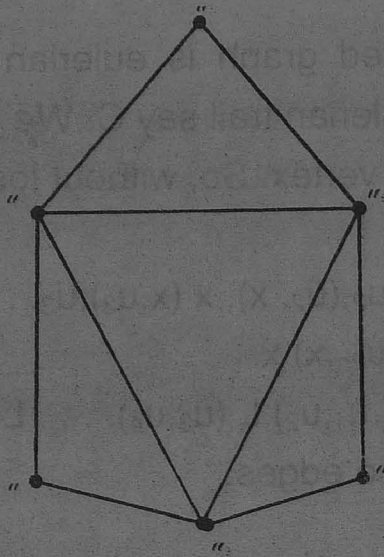
Say $e_{i+1} = (v_i, v_{i+1})$. If there is more than one edge incident at v_i in G_i , then give preference to a non-cut edge over cut-edge.

Go to step 2, with the trail

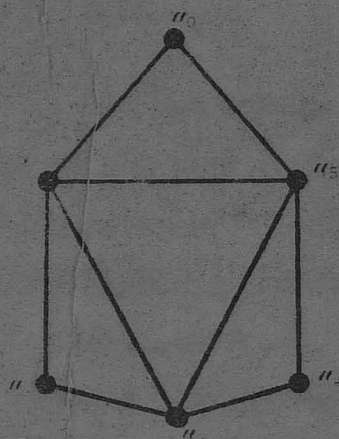
$$W_{i+1} = v_0, e_1, e_2, v_2, \dots, v_{i-1}, e_i, v_i, e_{i+1}, v_{i+1}$$

EXAMPLE:-

We apply the algorithm to find a closed eulerian trail in the following graph G .

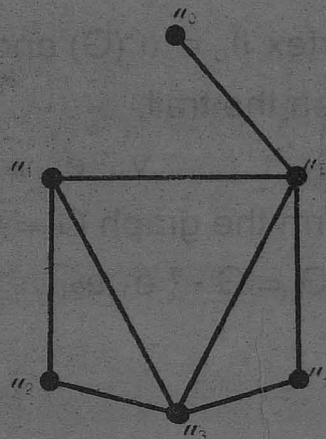


The successive generation of trails W_i and the sequence of 10 graphs G_i are shown below. For brevity we have denoted the long trails as sequence of edges alone.



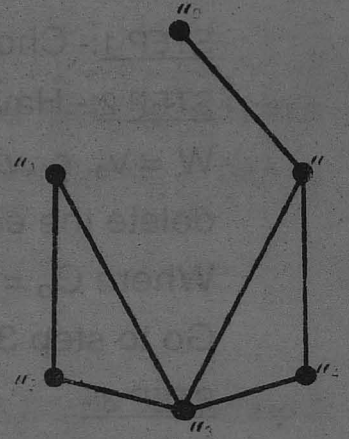
$$W = I$$

$$G = G_0$$



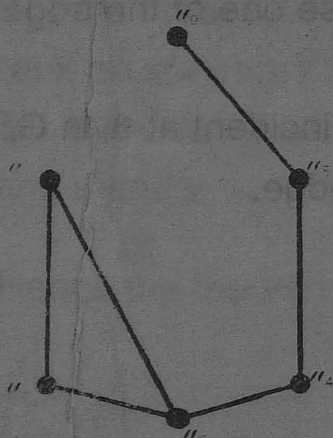
$$W_1 = (u_0, u_1), (u_1, u_2)$$

$$G_1 = G_0 - (u_0, u_1)$$



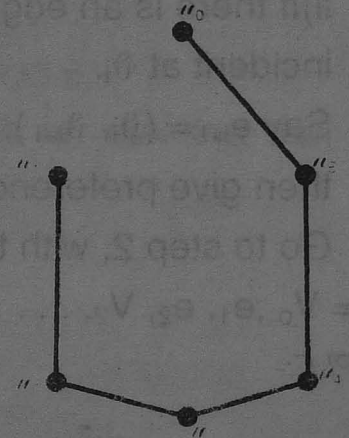
$$W_2 = (u_0, u_1), (u_1, u_2)$$

$$G_2 = G_1 - (u_1, u_2)$$



$$W_3 = (u_0, u_1), (u_1, u_2), (u_2, u_3)$$

$$G_3 = G_2 - (u_2, u_3)$$



$$W_4 = (u_0, u_1), (u_1, u_2), (u_2, u_3)$$

$$G_4 = G_3 - (u_3, u_4)$$

Let $W_n = v_0, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_n, v_n$ be a trail constructed by Fleury's algorithm.

Since (step 3(i)) $\deg_{G_n}(v_n) = 0$, it follows that $v_0 = v_n$.

Otherwise $\deg_G(v_n) = 2m - 1 (>0)$ where m is the number of occurrences of v_n in W_n – a contradiction.

So, W_n is a closed trail.

We assert that W_n contains all the edges of G . On the contrary suppose that W_n does not contain all the edges.

So,

$E(G_n)$ is non – empty Let $H = \langle \{ e_1, e_2, \dots, e_n \} \rangle$.

Since,

$$\deg_G(v_n) = \deg_n(v) - \deg_H(v) \quad n$$

It follows that $\deg_{G_n}(v)$ is even, for every v in G_n . So, none of the edges in G_n is a cut-edge. Let $K = \langle E(G_n) \rangle$.

So, $\deg_K(v) > 0$, for every $v \in K$.

Let i be the greatest integer such that $v_i \in K$. This means that after W_i is constructed we never visit a vertex of K again. Such a vertex v_i does exist because G is connected.

We have $E(G_i) = E(G_n) \cup \{ e_{i+1}, e_{i+2}, \dots, e_n \}$

Because of the maximality of i , e_{i+1} is the only edge in G_i that joins a vertex of K with a vertex in $\langle \{ e_{i+2}, e_{i+3}, \dots, e_n \} \rangle$ and hence it is a cut – edge of G .

NEXT

Let e be an edge incident at v_i in G_n such an edge exists because $v_i \in V(K)$ and $\deg_K(v) > 0$, for every $v \in V(K)$.

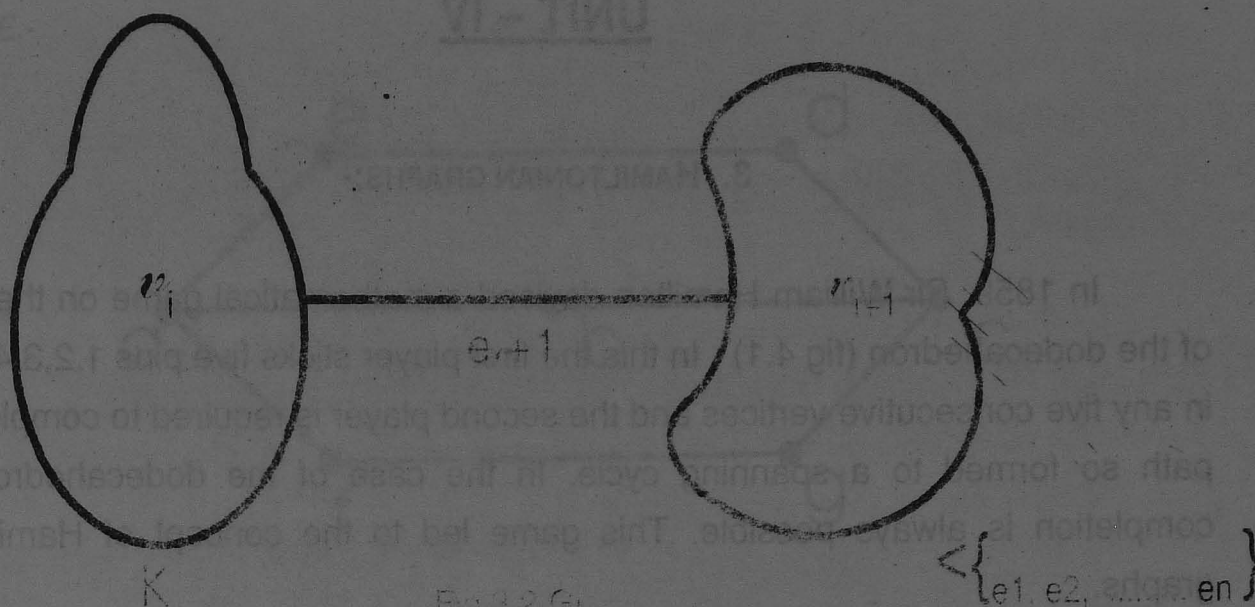


Fig 3.2 G_i

Since G_n has no cut-edges, e is not a cut-edge of G_n .

But G_n is a subgraph of G_i .

Hence e is not a cut-edge of G_i too. This implies that e_{i+1} (which is an edge incident at v_i in G_i) too is not a cut-edge of G_i . Since, we have constructed W_{i+1} by following step 3 (ii) of the algorithm.

This is a contradiction to our earlier assertion that e_{i+1} is a cut-edge of G_i .

Thus our supposition that $E(G_n)$ is nonempty is wrong.

Hence, W_n contains all the edges in G .

Hence the theorem.

EXERCISE:

1. For what values of n , is K_n eulerian?
2. For what values of m and n is $K_{m,n}$ eulerian.
3. If G is a graph in which the degree of every vertex is atleast two then G contains a cycle.
4. The following statements are equivalent for a connected graph G .
 - I. G is eulerian.
 - II. Every point of G has even degree.
 - III. The set of edges of G can be partitioned into cycles.
5. Show that if every block of a connected graph G is eulerian then G is eulerian

UNIT – IV

3. HAMILTONIAN GRAPHS:-

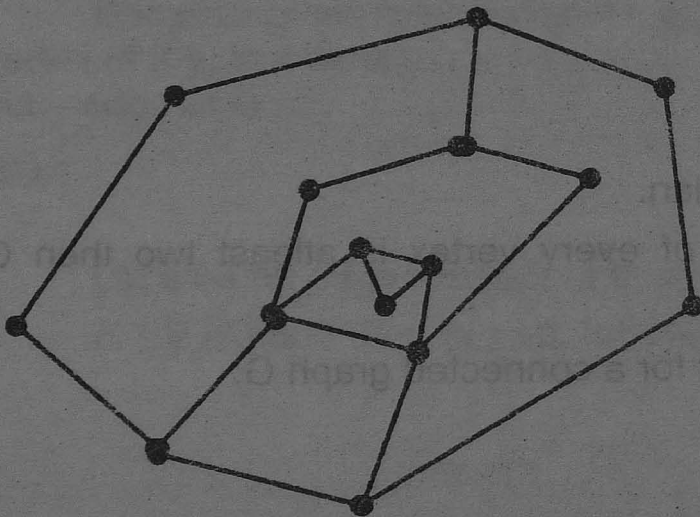
In 1859, Sir William Hamilton devised a mathematical game on the graph of the dodecahedron (fig 4.1) . In this the first player sticks five pins 1,2,3,4 and 5 in any five consecutive vertices and the second player is required to complete the path so formed to a spanning cycle. In the case of the dodecahedron, the completion is always possible. This game led to the concept of Hamiltonian graphs.

DEFINITION:-

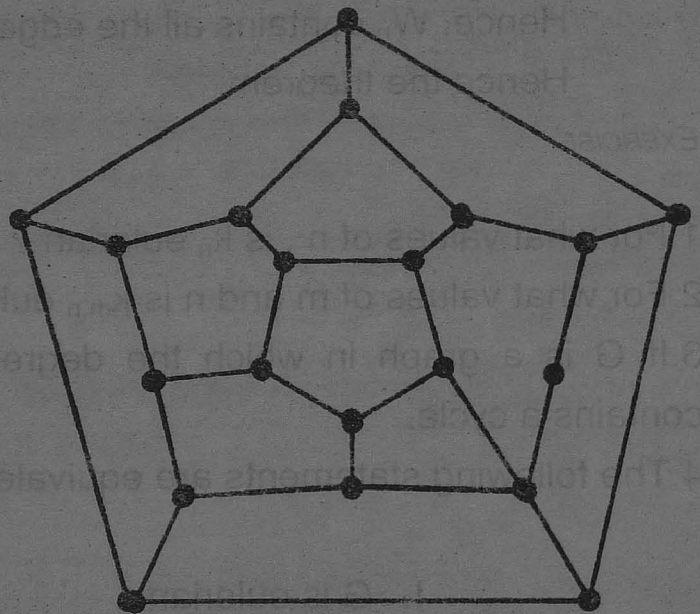
A spanning cycle in a graph is called a Hamiltonian cycle.

A graph having a Hamiltonian cycle is called a Hamiltonian graph.

Several necessary or sufficient conditions for Hamiltonian graph exist, but no elegant characterization of Hamiltonian graphs is known.



Non - hamiltonian Graph



Hamiltonian Graph

DEFINITION:-

A block with two non- adjacent vertices of degree 3 and all other vertices of degree 2 is called a thetagraph.

EXAMPLE:-

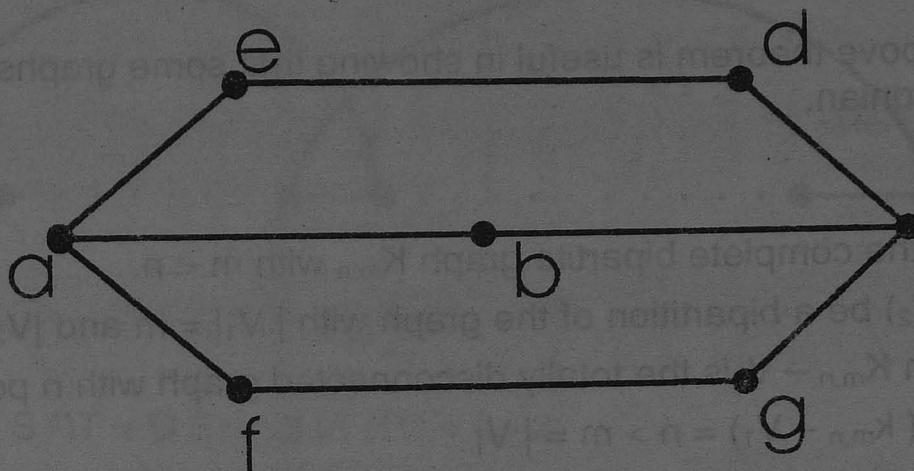


Fig 4.2

Thus a theta graph consists of two vertices of degree 3 and three disjoint paths joining them, each of length at least 2. The graph given in Fig 4.2 is a thetagraph.

A theta graph is obviously nonhamiltonian and every nonhamiltonian 2 – connected graph has a theta subgraph.

THM:- EVERY HAMILTONIAN GRAPH IS 2 – CONNECTED.

PROOF:-

Let G be a Hamiltonian graph. & let Z be a Hamiltonian cycle in G

For any vertex v of G ,

$Z - v$ is connected and hence $G - v$ is also connected.

Hence G has no cut points and thus G is 2-connected.

THM:-

If G is Hamiltonian, then for every non-empty proper subset S of $V(G)$.

$W(G - S) \leq |S|$ where $w(H)$ denotes the numbers of components in any graph H .

PROOF:-

Let Z be a hamultonian cycle of G let s be any non empty propes subject of $v(G)$ now, $w(Z - S) \leq |S|$.

Also $Z - S$ is a spanning subgraph of $G - S$.

Hence $W(G - S) \leq W(Z - S)$.

Hence $W(G - S) \leq |S|$

NOTE:-

- (i) The above theorem is useful in showing that some graphs are non Hamiltonian.

FOR EXAMPLE:-

Consider the complete bipartite graph $K_{m,n}$ with $m < n$.

Let (V_1, V_2) be a bipartition of the graph with $|V_1| = m$ and $|V_2| = n$.

The Graph $K_{m,n} - V_1$ is the totally disconnected graph with n points.

Hence $W(K_{m,n} - V_1) = n > m = |V_1|$

$\therefore K_{m,n}$ is non Hamiltonian.

- ii) The converse of the above theorem is not true.

FOR EXAMPLE:-

The Petersen graph satisfies the condition of the theorem but is nonhamiltonian.

THM:-

If G is a graph with $P \geq 3$ vertices and $\delta \geq P/2$, then G is Hamiltonian.

PROOF:-

Suppose the theorem is false. Let G be a maximal nonhamiltonian graph with P vertices and $\delta \geq P/2$.

Since $P \geq 3$, G cannot be complete.

Let u and v be nonadjacent vertices in G . By the choice of G ,

$G + uv$ is Hamiltonian. Moreover, since G is nonhamiltonian, each Hamiltonian cycle of $G + uv$ must contain the line uv .

Thus G has a spanning path $v_1, v_2, \dots, v_p$ with origin $u = v_1$ and terminus $v = v_p$.

Let $S = \{v_i / uv_{i+1} \in E\}$ & $T = \{v_i / i < p \text{ and } v_i v \in E\}$

Where E is the edge set of G . Clearly $v_p \notin S \cup T$ and hence $|S \cup T| < P$. — ①

Again if

$v_i \in S \cap T$

Then $v_i, v_2, \dots, v_i, v_p, v_{p-1}, \dots, v_{i+1}$

v_i is a hamiltonian cycle in G .

Contrary to the assumption (Refer Fig 4.3)

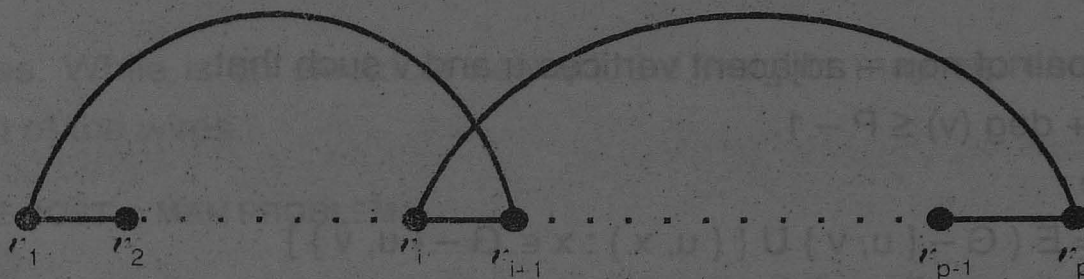


Fig 4.3

Hence $S \cap T = \emptyset$ So that $|S \cap T| = 0 - 2$

Also by the definition of S and T , $d(u) = |S|$ and $d(v) = |T|$

Hence by (1) and (2), $d(u) + d(v) = |S| + |T| = |S \cup T| < P$.

Thus $d(u) + d(v) < P$.

But since $\delta \geq P/2$.

We have $d(u) + d(v) \geq P$. which gives a contradiction

Hence the theorem.

THM:-

Let G be a graph with P points and let u and v be nonadjacent points in G such that $d(u) + d(v) \geq P$. Then G is Hamiltonian iff $G + uv$ is Hamiltonian.

PROOF:-

If G is Hamiltonian, then obviously $G + uv$ is also Hamiltonian. Conversely,

Suppose that $G + uv$ is Hamiltonian but G is not.

Then, as in the above thm, we obtain $d(u) + d(v) < P$

This contradicts the hypothesis that $d(u) + d(v) \geq P$.

Thus $G + uv$ is hamiltonian

$\Rightarrow G$ is hamiltonian.

Hence the theorem.

THM:-

If G is a (p, q) - graph with $P \geq 3$ and $q \geq \underline{P^2 - 3P + 6}$, then G is Hamiltonian.

PROOF:-

If possible,

Let G be non Hamiltonian,

by thm,

If G is hamiltongraph, then $w(G - S) \leq |S|$, for every

non - empty subset S of $V(G)$.

Then a pair of non – adjacent vertices u and v such that
 $\deg(u) + \deg(v) \leq P - 1$.

We have

$$i) \in (G) = E(G - \{u, v\}) \cup \{(u, x) : x \in G - \{u, v\}\}$$

$$\cup \{(v, x) : x \in G - \{u, v\}\}$$

$$ii) |E(G) - \{u, v\}| \leq \binom{p-2}{2}$$

Hence,

$$q \leq \binom{p-2}{2} + \deg(u) + \deg(v)$$

$$\leq \binom{p-2}{2} + p - 1 = \frac{p^2 - 3p + 4}{2}$$

- a contradiction.

Hence the theorem.

2. WEIGHTED GRAPHS:-

There are two practical problems associated with eulerian trails and Hamiltonian cycles. To describe these we need the following concepts.

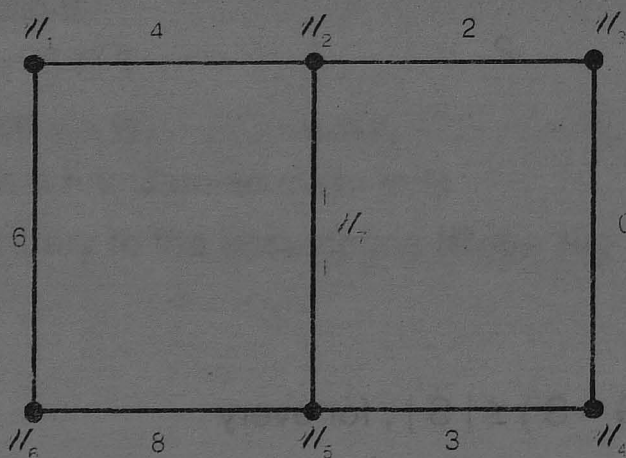
DEFINITION:-

A triple (V, E, w) is called a weighted (general) graph if
 $G = (V, E)$ is a (general) graph and W is a function from E to R .

The number $W(e)$ is called the weight of the edge e . The weight of a subgraph H of G , is the sum of the weights of the edges of H .

EXAMPLE:-

A weighted graph is shown in the following figure. Its total weight is 25



$$w(l) = 4 + 2 + 0 + 3 + 8 + 6 + 1$$

$$w(l) = 25$$

In the road map, for example, the weights may represent the distance between

two cities. While is an electrical network the weights may represent the voltage strength of the wires.

CHINESE POST – MAN PROBLEM:-

A post-man starts from his post-office to deliver the mail. After visiting a number of streets he comes back what route should be take so that he visits every street at least once and the total distance covered is least ?

If the graph of the roadmap is eulerian, then a route along an eulerian trail is of least mileage, since each street is visited exactly once. And we have fleury's algorithm to trace an eulerian trail.

If the graph is not eulerian, we cannot appeal to fleury's algorithm. However, there do exist good algorithm to find a trail of least weight. There are too involved and hence we prefer not to describe them.

TRAVELLING SALES-MAN PROBLEM:-

A traveling sales-man visits a number of cities and comes back to his head-office. What route should he travel so that he comes back after visiting each of the cities exactly once and covers the minimum total distance ? (or covers the distance with minimum expenses ?)

In graph theoretic terminology the problem is to find a Hamiltonian cycle of minimum weight.

We do not have (other than heuristic ones) any algorithm to find a Hamiltonian cycle with weight or without weight.

EXERCISE:-

1. Show that the Petersen graph is non Hamiltonian.
2. Find 4 edge disjoint Hamiltonian cycles in K_{10} .
3. If G is a Hamiltonian connected (P, q) – graph, then show that $q \geq 3P/2$.

UNIT – V

1. BIPARTITE GRAPHS:-

DEFN:-

A graph G is called a bigraph or bipartite graph if V can be partitioned into two disjoint subsets V_1 and V_2 such that every line of G joins a point of V_1 to a point of V_2 (V_1, V_2) is called a bipartition of G .

If further G contains every line joining the points of V_1 to the points of V_2 then G is called a complete bigraph.

If V_1 contains m points and V_2 contains n points then the complete bigraph G is denoted by $K_{m,n}$.

The graph given in Fig 5.1 is $K_{1,3}$. The graph given in Fig 5.2 is $K_{3,3}$. $K_{1,m}$ is called a star for $m \geq 1$.

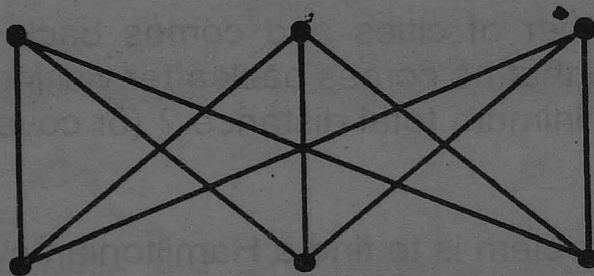


Fig 5.2 (Complete Bigraph) ($K_{3,3}$)

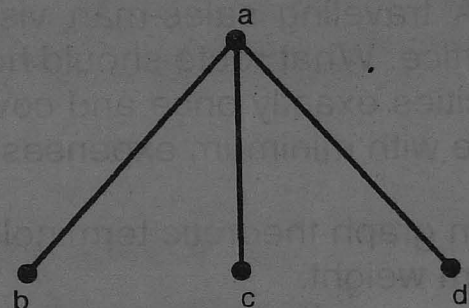
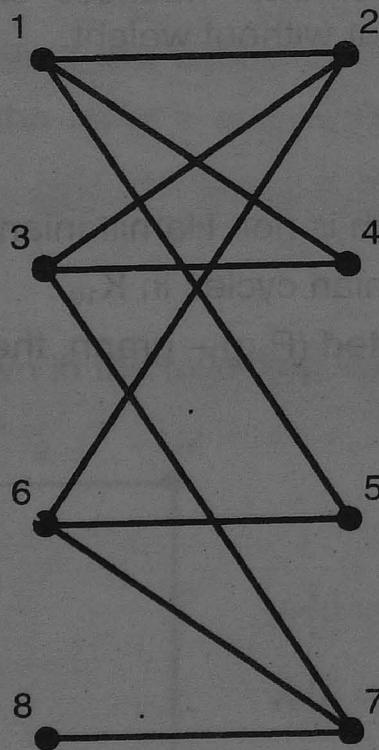


Fig 5.1 ($K_{1,3}$)



Bigraph (or) bipartite graph

The following theorem characterizes bipartite graphs.

THM:-

A (P, q) – graph G is a bipartite graph iff it contains no odd cycles.

PROOF:-

Clearly, a graph is bipartite iff each of its components is bipartite.

So, without loss of generality, we assume that G is connected.

NECESSITY:-

Let G be a bipartite graph with bipartition $[A, B]$ Let $C : v_1, v_2, \dots, v_n, v_{n+1} (=v_1)$, be a cycle of length n in G .

The vertices v_i belong alternately to A and B . Let, without loss of generality,

$v_i \in A$. Then, Since, A and B are disjoint $v_j \in A$ iff j is odd and $v_k \in B$ iff k is even.

Now,

Since $v_1 \in A$ and $(v_1, v_n) \in E$.

We have $v_n \in B$. So n is even.

SUFFICIENCY:-

We give two proofs for sufficiency.

FIRST PROOF:-

Let G be a graph with no odd cycles. We shall prove, by induction of q .

$\therefore G$ is bipartite.

If $q = 0$ or 1 , then obviously G is bipartite. So, assume that if $q = m - 1$, then G is bipartite and let $q = m (> 1)$.

Let (u, v) be an edge in G .

Consider the graph $H = G - (u, v)$

Since G has no odd cycles, H too had no odd cycles.

$\therefore$ by induction hypothesis H is bipartite. We now make two cases and in each case obtain a bipartition of G .

CASE (I):-

u and v are connected in H .

Let $[A, B]$ be a bipartition of H . We claim that u and v are in different sets A and B .

On the contrary suppose that u and v belong to the same set, say A .

Since u and v are connected, there is a path

$P(u, v) : u = u_1, (u_1, u_2), u_2, \dots, u_{n-1}, (u_{n-1}, u_n), u_n (=v)$

The vertices u_i belong alternately to A and B and moreover the terminal vertices u and v belong to A .

Hence the number of vertices u_i is odd $\therefore P(u, v)$ is a path of even length. But then the sequence $P(u, v), (v, u) u$ is a cycle of odd length in G — a contradiction so, our claim holds;

Let $u \in A$ and $v \in B$. But then $[A, B]$ is a bipartition of G too.

CASE (II):-

u and v are not connected in H . Let C be the component in H which contains u . Since H is bipartite, both C and $H - V(C)$ are bipartite.

Let $[A_1, B_1]$ be a bipartition of C and $[A_2, B_2]$ be a bipartition of $H - V(C)$.

We can assume that $u \in A_1$ and $v \in B_2$

(otherwise suitably relabel A_i and B_i)

Then $[A_1 \cup A_2, B_1 \cup B_2]$ is a bipartition of G .

SECOND PROOF:-

Recall that $d(u, v)$ denotes the distance between the vertices u and v .

Let x be an arbitrary vertex in G .

Define the following sets.

$A = \{ v \in V : d(x, v) \text{ is even} \}$, and

$B = \{ v \in V : d(x, v) \text{ is odd} \}$

Clearly, $A \cap B = \emptyset$ and $A \cup B = V(G)$

We claim that no two vertices in A are adjacent. On the contrary suppose that $v_1, v_2 \in A$ are adjacent.

Let $P_1(x, v_1)$ and $P_2(x, v_2)$ be the paths of minimum length.

Since, $v_1, v_2 \in A$, $P_1(x, v_1)$ and $P_2(x, v_2)$

Have even length, say $2m$ and $2n$ respectively. Let u be the last vertex common between $P_1(x, v_1)$ and $P_2(x, v_2)$ (see fig 5.3)

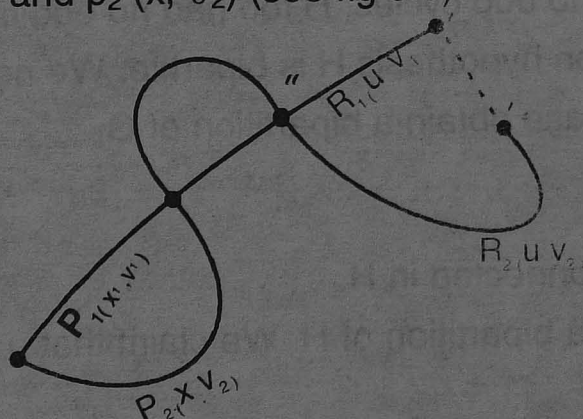


Fig 5.3

Since the paths P_1 and P_2 are of minimum length it follows that the

subpaths $Q_1(x, u)$ and $Q_2(x, u)$ of paths P_1 and P_2 respectively, have the same length, say K .

Now, if $R_1(u, v_1)$ and $R_2(u, v_2)$ denote the subpaths of P_1 and P_2 respectively then the sequence $R_1(u, v_1), (v_1, v_2), R_2(v_2, u)$ is a cycle in G . And its length is $(2m - k) + 1 + (2n - k) - \text{an odd integer}$. This is a contradiction.

Since G has no odd cycles. Hence, no two vertices in A are adjacent. Similarly, no two vertices in B are adjacent. Hence, (A, B) is a bipartition of G .

Hence the theorem.

2. MARRIAGE PROBLEM:-

The problem that we would like to consider is the following.

If there are n boys each knowing several girls then under what conditions can we marry off each of the boys to a girl he knows?

Obviously, we can marry off each of the boys only if

(*) K boys together know atleast K girls, for every $K, 1 \leq K \leq n$.

That (*) is also sufficient to marry off n boys are required, is P. Hall's theorem. Under a different setting this problem becomes a job – assignment problem.

If there are n employees, each capable of handling several jobs, under what conditions can we employ each of the n employees to a job he is capable of handling?

DEFINITIONS:-

Two edges in a graph are said to be independent edges if they are non-adjacent. A subset S of edges is said to be an independent set of edges if no two edges in S are adjacent.

In the marriage problem suppose that there are n boys and m girls. Represent the n boys $B_1, B_2, \dots, B_n$ by n vertices $b_1, b_2, \dots, b_n$ and the m girls $G_1, G_2, \dots, G_m$ by m vertices $g_1, g_2, \dots, g_m$; and join b_i to g_j if the boy B_i knows G_j .

This construction gives rise to a bipartite graph G^* with bipartition $[A, B]$, Where $A = \{b_1, b_2, \dots, b_n\}$ and $B = \{g_1, g_2, \dots, g_m\}$. The following example

illustrates this representation.

Boys	Girls B_i knows
B_1	G_1, G_3
B_2	G_1, G_2, G_3
B_3	G_2, G_1, G_4
B_4	G_3

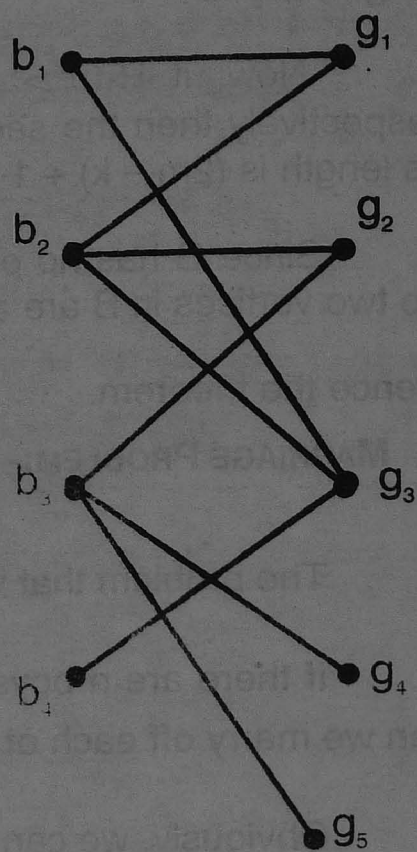


Fig 5.5

A marriage as required is possible in this example one possible marriage is to marry off B_1 to G_1 , B_2 to G_2 , B_3 to G_5 and B_4 to G_3 .

In general, if $G^* [A,B]$ contains n independent edges, say (b_i,g_i) $i = 1,2,..$ Then we can marry off B_i to G_i , $i=1,2.....n$ Hence, in graph theoretic terminology the marriage problem can be stated as follows.

Given a bipartite graph $G[A,B]$ what are the necessary and sufficient conditions it should satisfy so that it contains a set of n ($= | A |$) independent edges.

Note that the boys satisfy the property (*) iff $G^* [A,B]$ satisfies:

$$| N_{G^*} (S) | \geq | S |, \forall S \subseteq A$$

THEOREM :- (HALL'S THEOREM)

Let $G [A,B]$ be a bipartite graph with $| A | = n$ and $| B | = m$. Then there exists a set of n independent edges in G iff

- (H) $| N_G(S) | \geq | S |, \forall S \subseteq A$
- ((H) is called the Hall's condition)

PROOF:-

Let $A = \{x_1, x_2, \dots, x_n\}$, $B = \{y_1, y_2, \dots, y_m\}$

NECESSITY:-

Let $G[A, B]$ contain n independent edges, $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$. Let $S = \{x_{i_1}, x_{i_2}, \dots, x_{i_s}\}$ be a subset of A .

Then $(x_{i_1}, y_{i_1}), \dots, (x_{i_s}, y_{i_s})$ are independent edges and so, $y_{i_1}, y_{i_2}, \dots, y_{i_s}$ are distinct vertices.

Hence, $N(S) \supseteq \{y_{i_1}, y_{i_2}, \dots, y_{i_s}\}$

and we have $|N(S)| \geq |S|$

There are several proofs of sufficiency; we give two proofs.

FIRST PROOF:-

Let G satisfy (H)

We shall prove that G contains n independent edges, by induction on n .

If $n = 1$, then obviously the result holds. So, Let $n \geq 2$ and proceed to the next step in the induction.

We distinguish two cases and prove the assertion separately in each case.

CASE (I) :-

For every proper subset S of A , $|N(S)| \geq |S| + 1$

Since G satisfies (H), $|N\{\ell\}| \geq 1$ and so there is a vertex, say $y \in B$ such that (ℓ, y) is an edge. Consider the subgraph $L = G - \{\ell, y\}$

L is a bipartite graph with bipartition $[A_1, B_1]$ where $A_1 = (x_1, x_2, \dots, x_n)$ and $B_1 = \{y_2, y_3, \dots, y_m\}$

We claim that $L(A_1, B_1)$ satisfies (H)

So, let $T \subseteq A_1$. Then clearly, $N_G(T) \subseteq N_L(T) \cup \{y_1\}$

Hence

$$|N_L(T)| \geq |N_G(T)| - 1$$

$$\geq |T| + 1 - 1, \text{ by our assumption} = |T| \geq 1 + 1$$

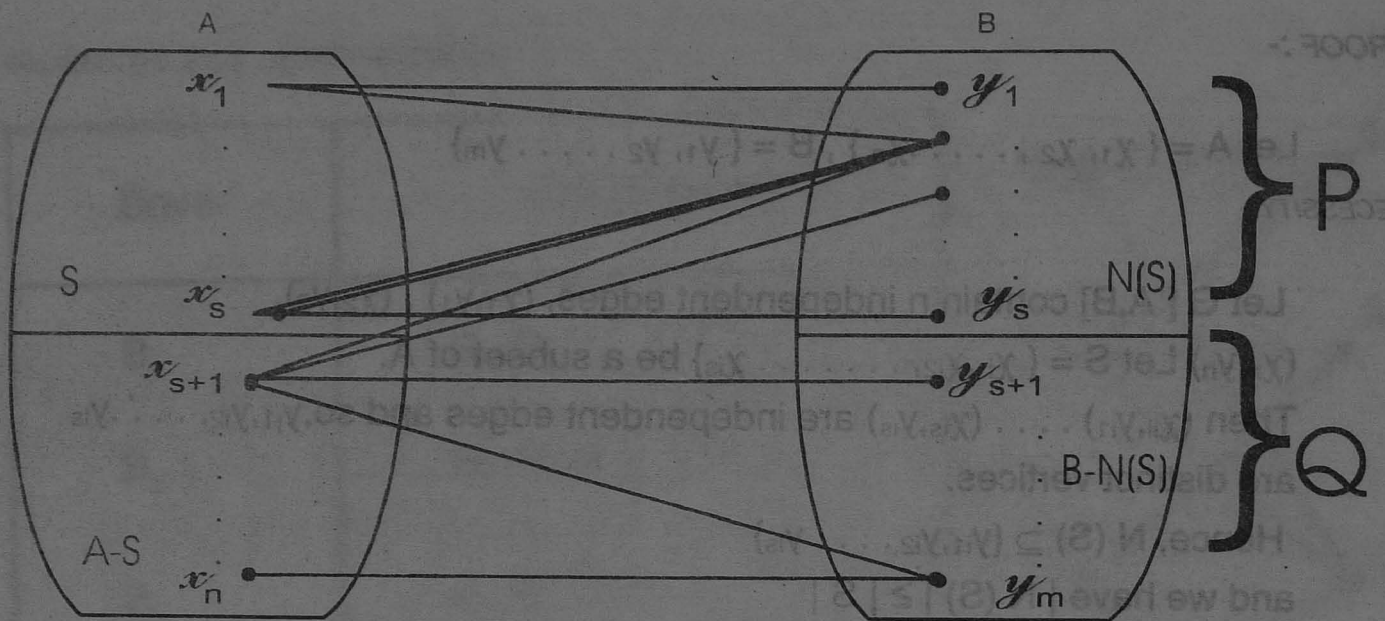
by induction hypothesis $L[A_1, B_1]$ contains $(n - 1)$ independent edges, say $(x_1, y_1), \dots, (x_n, y_n)$. Then $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ are n independent edges in G .

CASE (II) :-

For some proper subset S of A , $|N(S)| = |S|$

For notational convenience,

Let $S = \{x_1, x_2, \dots, x_s\}$ and $N(S) = \{y_1, y_2, \dots, y_s\}$



consider the bipartite graphs $P[S, N(S)]$ and $Q[A - S, B - N(S)]$

We shall prove that both these graphs satisfy (H) and then complete the proof by appealing to the induction hypothesis.

We consider first $P[S, N(S)]$

Let $T \subseteq S$.

Clearly, by our assumption of S , $N_P(T) = N_G(T)$

So $|N_P(T)| = |N_G(T)| \geq |T|$

Then P satisfies (H). Hence, by induction hypothesis P has S independent edges, say

$(x_1, y_1), (x_2, y_2), \dots, (x_s, y_s)$ Next, consider

$Q[A - S, B - N(S)]$

Let $T \subseteq A - S$ Clearly,

$N_G(S \cup T) = N_G(S) \cup N_Q(T)$

$\therefore |N_Q(T)| \geq |N_G(S \cup T)| - |N_G(S)|$

$\geq |S \cup T| - |S|$, (Since G satisfies (H))

$= |S| + |T| - |S|$, (Since S and T are disjoint.)

$= |T|$

Thus $Q[A - S, B - N(S)]$ satisfies (H)

So, by induction hypothesis, Q has $n - s (= |A - S|)$ independent edges, say $(x_{s+1}, y_{s+1}), \dots, (x_n, y_n)$

Then $(x_1, y_1), \dots, (x_s, y_s), (x_{s+1}, y_{s+1}), \dots, (x_n, y_n)$ are n independent edges in G .

SECOND PROOF:-

As before, Let $A = \{x_1, x_2, \dots, x_n\}$ and $B = \{y_1, y_2, \dots, y_m\}$

We first prove the following assertion.

LEMMA:-

If $|N_G(x_1)| \geq 2$

Then there is a $y \in N_G(x_1)$ such that the graph $G - (x_1, y_1)$ satisfies Hall's condition.

PROOF OF THE LEMMA:-

Assume the contrary,

Then there are $y_1, y_2 \in N_G(x_1)$ such that neither $H = G - (x_1, y_1)$ nor $K = G - (x_1, y_2)$ satisfies Hall's condition. So, there exist sets $x_1, x_2 \subseteq X$ such

that

$$(1) |N_H(x_1 \cup \{x_1\})| \leq |X_1| \text{ and } |N_K(x_2 \cup \{x_1\})| \leq |X_2|.$$

where $x_1 \notin x_1, x_2$

we then find that

$$|N_G(X_1 \cup X_2 \cup \{x_1\})| = |N_H(x_1 \cup \{x_1\}) \cup N_K(x_2 \cup \{x_1\})|$$

Since the sets on either side are equal

$$= |N_H(X_1 \cup \{x_1\})| + |N_K(x_2 \cup \{x_1\})|$$

$$- |N_H(X_1 \cup \{x_1\}) \cap N_K(x_2 \cup \{x_1\})|$$

$$\leq |X_1| + |X_2| - |N_G(X_1 \cap X_2)|$$

because of (1) and the fact that $N_G(X_1 \cap X_2) \subseteq N_H(x_1 \cup \{x_1\}) \cap N_K(x_2 \cup \{x_1\})$

$$\leq |x_1| + |x_2| - |x_1 \cap x_2|$$

(Since G satisfies Hall's condition $= |X_1 \cup X_2|$)

This shows that G violates Hall's condition. Hence our lemma follows.

PROOF OF THE THEOREM:-

By repeatedly applying the Lemma to x_1 it follows that there exists a subgraph H of G in which $|N_H(x_1)| = 1$ and which satisfies Hall's condition.

So, applying the lemma to each x_i we eventually obtain a subgraph H of G in which each $x_j \in A$ is adjacent with exactly one $y_j \in B$.

Let $(x_1, y_1), (x_2, y_2) \dots (x_n, y_n) \in E(H)$

But then these edges are n independent edges in G .

Hence the Hall's theorem.

COR: (I):-

Let $G (G > 0)$ be a bipartite graph with bipartition $[A, B]$, where $|A| = n$.

If $\min \deg(x) \geq \max \deg(y)$ then G contains

$x \in A, y \in B$

n independent edges.

PROOF:-

It is enough if we prove that G satisfies Hall's condition.

So, Let $S \subseteq A$.

Let $\delta(X) = \min \deg(x)$ and $\Delta(Y) = \max \deg(y)$.

There are atleast $|S|$

$\delta(X)$ edges incident with the vertices of $N(S)$. Hence, there is a vertex $y \in N(S)$

which is incident with atleast $\frac{|S| \cdot \delta(X)}{|N(S)|}$ edges

$$\text{So, } \Delta(Y) \geq \deg(y) \geq \frac{|S| \cdot \delta(X)}{|N(S)|}$$

Since $\delta(X) \geq \Delta(Y)$, we obtain $|N(S)| \geq |S|$

And thus, we have verified Hall's condition.

COR (II):-

Let $G[A, B]$ be a k -regular ($k \geq 1$)

bipartite graph. Then $E(G)$ can be partitioned into K sets $E_1, E_2, \dots, E_k$ where each E_j is a set of $n (=|A|)$ independent edges.

PROOF:-

We shall prove the theorem by induction on K . At the outset, observe that since $K|A| = |E(G)| = K|B|$ We have $|A| = |B| = n$.

If $K = 1$, then $E(G)$ is set of n independent edges and hence the assertion follows:

So proceed to the next step in the induction. Since, $G[A, B]$ is a k -regular bipartite graph it satisfies the hypothesis of above cor (i), Hence it contains a set, say E_1 of n independent edges.

Since $|A| = |B| = n$, $G - E_1$ is a $(K - 1)$ -regular bipartite graph.

$\therefore$ by induction hypothesis,

$$E(G - E_1) = E_2 \cup \dots \cup E_k \text{ (say)}$$

Where E_i 's are pairwise disjoint sets and each E_j is a set of n independent edges.

Then $E(G) = E_1 \cup E_2 \cup \dots \cup E_k$ is a required partition of $E(G)$.

Hence the corollary.

3. TREES:-

Trees are very important for the sake of their applications to many different fields. Further a tree is the simplest non-trivial type of a graph and in trying to prove a general result or test a conjecture in Graph theory, it is sometimes convenient to first study the situation for trees.

DEFN:-

A graph with no cycle is a cyclic.

A forest is an acyclic graph.

A tree is a connected acyclic graph.

A leaf (or pendant vertex) is a vertex of degree 1.

A spanning subgraph of G is a subgraph with vertex set $V(G)$.

A spanning tree is a spanning subgraph that is a tree.

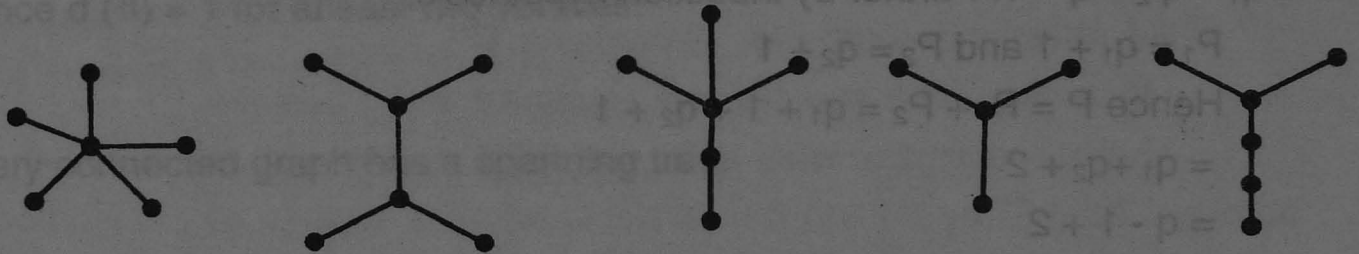


Fig .5.1 gives all trees with 6 vertices.

The following theorem gives several equivalent ways of describing a tree.

THM:-

Let G be a (P, q) graph. The following statement are equivalent.

- G is a tree.
- Every two points of G are joined by a unique path.
- G is connected and $P = q + 1$.
- G is acyclic and $P = q + 1$.

PROOF:-

$1 \Rightarrow 2$

Let u, v be any two points of G . Since G is connected there exists a $u - v$ path in G . Now suppose there exist two distinct $u - v$ paths.

$P_1: u = v_0, v_1, v_2, \dots, v_n = v$ and

$P_2: u = w_0, w_1, w_2, \dots, w_m = v$

Let i be the least positive integer such that $1 \leq i \leq m$ and $w_i \notin P_1$ (such an i exists P_1 & P_2 are distinct.)

Hence $\omega_{j-1} \in P_1 \cap P_2$

Let j be the least positive integer such that $i < j \leq m$ and $W_j \in P_1$

Then the $\omega_{i-1} - \omega_j$ path along P_2 followed by the $W_j W_{i-1}$ path along P_1 form a cycle which is a contradiction.

Hence there exists a unique $u - v$ path in G .

2 $\Rightarrow$ 3:

Clearly G is connected.

We prove $P = q + 1$ by induction on P .

This is trivial for a connected graph with 1 or 2 points. Assume the result for graphs with fewer than P points.

Let G be a graph with P points. Let $x = uv$ be any line G . Since there exists a unique $u-v$ path in G ,

$G - x$ is a disconnected graph with exactly two components G_1 and G_2

Let G_1 be a (P_1, q_1) graph and G_2 be a (P_2, q_2) graph. Then $P_1 + P_2 = P$ and $q_1 + q_2 = q - 1$. Further by induction hypothesis,

$$P_1 = q_1 + 1 \text{ and } P_2 = q_2 + 1$$

$$\text{Hence } P = P_1 + P_2 = q_1 + 1 + q_2 + 1$$

$$= q_1 + q_2 + 2$$

$$= q - 1 + 2$$

$$P = q + 1$$

3 $\Rightarrow$ 4:

We must prove that G is acyclic. Suppose G contains a cycle of length n . There are n points and n lines on this cycle. Fix a point u on the cycle. Consider any one of the remaining $P - n$ points not on the cycle, say v . Since G is connected.

We can find a shortest $u - v$ path in G . Consider the line on this shortest path incident with v the $p - n$ lines thus obtained are all distinct.

$$\text{Hence } q \geq (p - n) + n = p.$$

Which is a contradiction.

$$\text{Since } q + 1 = p.$$

Thus G is acyclic.

4 $\Rightarrow$ 1 :-

Since G is acyclic. To prove that G is a tree.

We need only to prove that G is connected suppose G is not connected.

Let $G_1, G_2, \dots, G_k$ ($k \geq 2$) be the components of G .

Since G is acyclic each of these components is a tree. Hence $q_i + 1 = P_i$,

Where G_i is a (P_i, q_i) graph

$$\therefore \sum_{i=1}^k (q_i + 1) = \sum_{i=1}^k P_i$$

ie) $q + k = p$ and $k \geq 2$, which is a contradiction.

Hence G is connected.

This completes the proof.

COROLLARY:-

Every non-trivial tree G has atleast two vertices of degree 1.

PROOF:-

Since G is non – trivial,
 $d(v) \geq 1$ for all points v .

Also $\sum d(v) = 2q = 2(p-1)$ [Since $q + 1 = p$. $q = p-1$]

$$\sum d(v) = 2p - 2$$

Hence $d(v) = 1$ for atleast two vertices.

THM:-

Every connected graph has a spanning tree.

PROOF:-

Let G be a connected graph.

Let T be a minimal connected spanning subgraph of G .

Then for any line x of T .

$T - x$ is disconnected.

Hence x is a bridge of T .

Hence T is acyclic.

Further T is connected.

Hence is a tree.

COROLLARY:-

Let G be a (p, q) connected graph. Then $q \geq p - 1$.

PROOF:-

Let T be an spanning tree of G . Then the number of lines in T is $P - 1$.

Hence $q \geq p - 1$.

THM:-

Let T be an arbitrary tree on $K + 1$ vertices. If G is a graph with $\delta(G) \geq k$, then G contains (a tree isomorphic with) T .

PROOF:-

We shall prove the theorem by induction on K .

If $k = 1$, then $T = K_2$ and since $\delta(G) \geq 1$, G contains a K_2 .

So, Assume the result for $k = m$.

$\therefore$ Assume that, if T is a tree on $m + 1$ vertices and if G is a graph with $\delta(G) \geq m$.

Then G contains T .

Let $K = m + 1$.

Let T be an arbitrary tree on $m + 2$ vertices and let G be a graph with $\delta(G) \geq m + 1$.

By thm,

Every (p, q) tree T ($p \geq 2$) contains at least two vertices of degree.

T contains a vertex, say v of degree 1. Let u be the unique vertex in T adjacent with v consider the tree $T' = T - v$.

By induction hypothesis, G contains T' .

Since $\deg_G(u) \geq m + 1$ and T' contains $m + 1$ vertices,

There is a vertex $w \in V(G) - V(T')$ incident with $u \in G$. Then the tree, obtained from T' by adding the vertex w and the edge (u, w) , is isomorphic with the given tree T .

Hence the theorem.

4. CONNECTOR PROBLEM:-

We have a certain number of towns and we are given the cost of road construction between any two given towns. The problem is to plan a road-network connecting the towns in such a way that (i) given any two towns one is reachable from the other, and (ii) the total cost of the network is minimum possible.

It is clear that the connector problem can be easily translated into a graph theoretic problem; represent the towns by vertices and join two vertices by an Edge e of weight $w(e)$, if the cost of the road construction between the two corresponding towns is $w(e)$.

The problem is then to find a connected (because of condition (i)) subgraph H of a weighted graph G such that $\sum w(e)$ is minimum among all connected subgraphs H of G .

We may assume that the weights are positive as they represent costs. It is immediate that such a H must be tree; if H contains a cycle, then by deleting an edge of this cycle we get a connected subgraph H' of H whose weight $\in W(e)$ is $\ell \in H$ obviously less than that of H .

A spanning tree of minimum weight in a connected weighted graph G is called an optimal tree (or minimal tree or economy tree) of G . There are several good algorithms to find an optimal tree; in the weighted graph G . In all these algorithms if a tie results, then make an arbitrary choice subject to the stated conditions.

1. J.B. KRUSKAL'S ALGORITHM:-

Step 1:- Select an edge (say) e_1 of minimum weight in G .

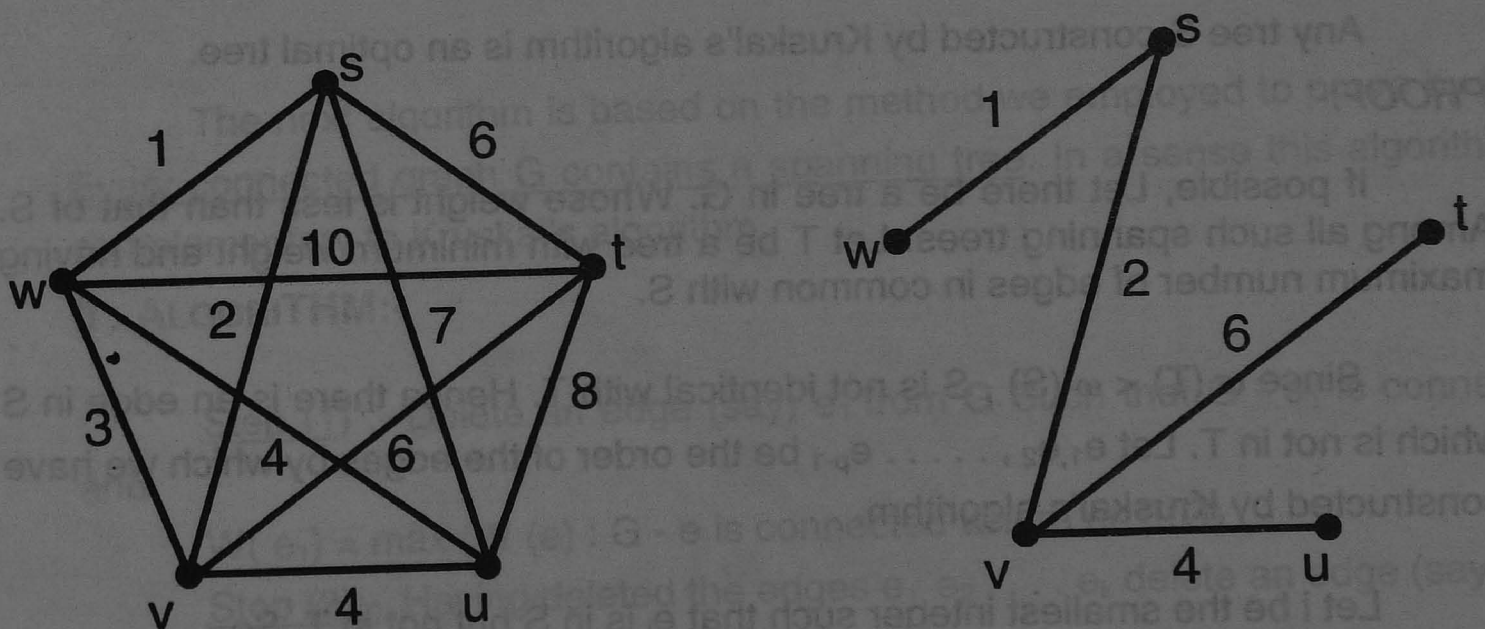
Step 2:- Having selected the edges $e_1, e_2, \dots, e_i$ select an edge $e_{i+1} \in E(G) - \{e_1, e_2, \dots, e_i\}$ subject to the following conditions. (i) The edge induced subgraph $\langle \{e_1, e_2, \dots, e_i, e_{i+1}\} \rangle$ is a cyclic.

(ii) Among all the edges $e \in E(G) - \{e_1, e_2, \dots, e_i\}$, Such that $\langle \{e_1, e_2, \dots, e_i, e\} \rangle$ is a cyclic, e_{i+1} has the minimum weight.

Step 3:- Stop when no more edges can be selected.

EXAMPLE:-

Consider the weighted K_5 shown in below Figure 5-6.



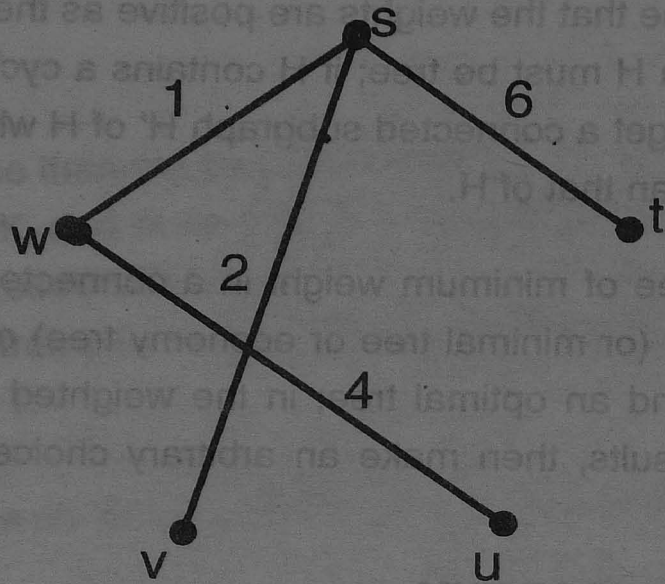


FIGURE 5. 6:-

Two optimal trees of G constructed by Kruskal's algorithm. Notice that both the trees have the same weight 13.

As described in Kruskal's algorithm we successively select the edges (w,s) , (s,v) , (v,u) and (v,t) and obtain the first optimal tree shown in Figure 5-6. Note that at the thirdstage, although the edge (v, w) is of minimum weight it cannot be selected because it forms a cycle with the two edges already selected. Moreover, after ruling out the candidature of (v, w) we have a tie between the edges (w,u) and (v,u) ; we have arbitrarily decided to select (v,u) .

THM:-

Any tree S constructed by Kruskal's algorithm is an optimal tree.

PROOF:-

If possible, Let there be a tree in G . Whose weight is less than that of S . Among all such spanning trees. Let T be a tree with minimum weight and having maximum number of edges in common with S .

Since $\omega(T) < \omega(S)$, S is not identical with T . Hence there is an edge in S which is not in T . Let $e_1, e_2, \dots, e_{p-1}$ be the order of the edges by which we have constructed by Kruskal's algorithm.

Let i be the smallest integer such that e_i is in S but not in T . So,

$$E(S) = \{e_1, e_2, \dots, e_{i-1}, e_i, e_{i+1}, \dots, e_{p-1}\} \&$$

$$E(T) = \{e_1, e_2, \dots, e_{i-1}, e_i, e_{i+1}, \dots, e_{p-1}\}$$

Clearly, $T + e_i$ contains a unique cycle, say C .

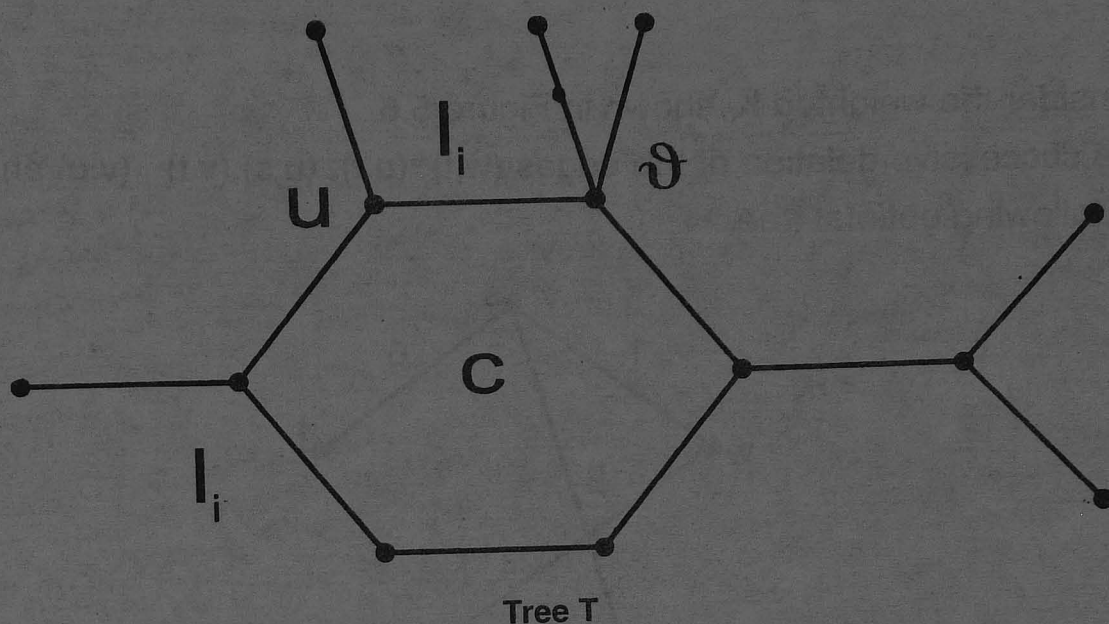


Fig 5.7

Since C is not contained in S , C contains an edge say $e_j (\in E(T))$ which is not in S . Let $T^1 = T + e_i - e_j$.

Then T^1 is a tree with $W(T^1) = W(T) + W(e_i) - W(e_j)$

But $W(T) \leq W(T^1)$; So, $W(e_j) \geq W(e_i)$

Since $\langle \{ (e_1, e_2, \dots, e_{i-1}, e_j) \} \rangle$ is a cyclic, it follows, by our choice (step 2 (ii)) Of e_i , that $W(e_i) \leq W(e_j)$. So $W(e_i) = W(e_j)$ and more over T^1 has one more edge in common with S – a contradiction to our choice of T . This proves that S is indeed a tree of minimum weight in G .

The next algorithm is based on the method we employed to prove by thm, Every connected graph G contains a spanning tree. In a sense this algorithm is complementary to Kruskal's algorithm.

II . ALGORITHM:-

Step (1) :- Delete an edge (say) e_1 from G Such that $G - e_1$ is connected and

$W(e_1) = \max \{ W(e) : G - e \text{ is connected where } e \in E(G) \}$

Step (2):- Having deleted the edges $e_1, e_2, \dots, e_k$ delete an edge (say) e_{k+1} from

$G - \{ e_1, e_2, \dots, e_k \}$ subject to the following conditions.

$G - \{ e_1, e_2, \dots, e_{k+1} \}$ is connected.

$W(e_{k+1}) = \max \{ W(e) : G - \{ e_1, e_2, \dots, e_k, e \} \text{ is connected where } e \in E(G) - \{ e_1, e_2, \dots, e_k \} \}$.

Step (3):- Stop when step 2 cannot be implemented further.

EXAMPLE:-

Consider the weighted K_5 shown in Figure 5.6

The successive deletion of the edges (w,t) , (u,t) , (u,s) , (v,t) , (v,u) and (w,v) yields the following optimal tree.

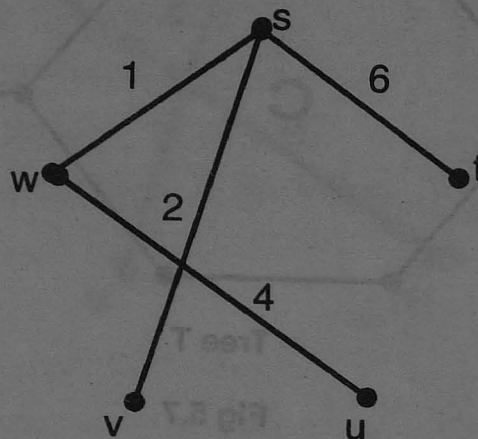


Fig 5.8

THM:-

Any tree S constructed by the above algorithm is an optimal tree.

PROOF:-

At the outset we observe that S is indeed a tree because at every stage we have ensured that the resulting graph is connected and we have stopped the deletion of the edges when every edge in S is a cut-edge.

Hence, by thm, A connected (p,q) – graph is a tree iff every edge is a cut-edge. $\therefore S$ is a tree.

Since S is a tree, we should have deleted $q-p+1$ edges from G to obtain S . Let $e_1, e_2, \dots, e_{q-p+1}$ be the order of the edges of deletion in the algorithm. We shall prove the following assertion. There is an optimal tree T of G which does not contain any of the edges $e_1, e_2, \dots, e_{q-p+1}$; so that T is precisely S .

Since T is optimal, it then follows that S is optimal. To prove the above assertion it is sufficient if we prove the following.

(*) : If T is an optimal tree with $e_k \in T$ and $e_1, e_2, \dots, e_{k-1} \notin T$, then there exists an optimal tree T^1 with $e_1, e_2, \dots, e_{k-1} \in T^1$, $e_k \notin T^1$.

So, let T be an optimal tree with $e_k \in T$ and $e_1, e_2, \dots, e_{k-1} \notin T$.

Since, T is a tree, $T - e_k$ contains exactly two components say T_1 and T_2 .

Since, $G - \{e_1, e_2, \dots, e_k\}$ is connected, there exists an edge $f \in G - \{e_1, e_2, \dots, e_k\}$ which joins a vertex of T_1 with a vertex of T_2 .

So, $T - e_k + f$ is a spanning tree of G .

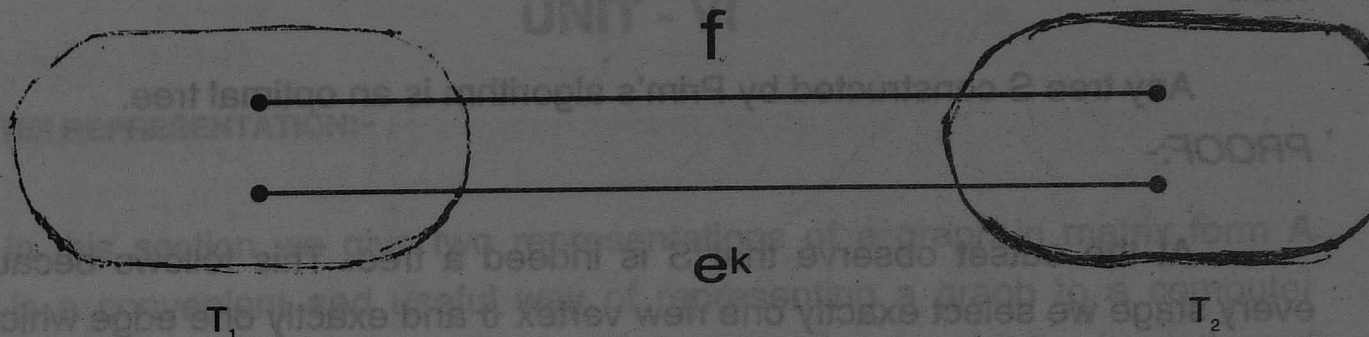


Fig 5.9

Since, $G - \{e_1, e_2, \dots, e_{k-1}, f\}$ is a connected graph and since at the k^{th} stage we have preferred to delete e_k from $G - \{e_1, e_2, \dots, e_{k-1}\}$ it follows that $W(e_k) \geq w(f)$.

$$\text{So, } W(T - e_k + f) = W(T) - W(e_k) + W(f) \leq W(T).$$

Hence, $W(T - e_k + f) = W(T)$, Because T is an optimal tree.

So, $T^1 = T - e_k + f$ is an optimal tree which does not contain the edges $e_1, e_2, \dots, e_{k-1}, e_k$. Thus (*), and hence the theorem, is proved.

II R.C. PRIM'S ALGORITHM:-

STEP (1):- Select a vertex v , (arbitrarily) and an edge (say) (v_1, v_2) which has the least weight among all the edges incident with V_1 select the vertex v_2 .

STEP (2) :- Having selected the vertices $v_1, v_2, \dots, v_k$ and $k-1$ edges (v_i, v_j) ($i < j; 2 \leq j \leq k$), Select an edge (say) (v_i, v_{k+1}) ($1 \leq i \leq k$) and the vertex v_{k+1} (where $v_{k+1} \in G - \{v_1, v_2, \dots, v_k\}$ Such that $W(v_i, v_{k+1}) = \min \{W(v_i, v) : 1 \leq i \leq k \text{ and } v \in G - \{v_1, v_2, \dots, v_k\}\}$

STEP (3) :- Stop when $P-1$ edges have been selected.

EXAMPLE:-

Again consider the weighted K_5 shown in Figure 5-6.

The successive selection of the edges (v,s) , (s,w) , (u,w) and (t,u) generated the following optimal tree.

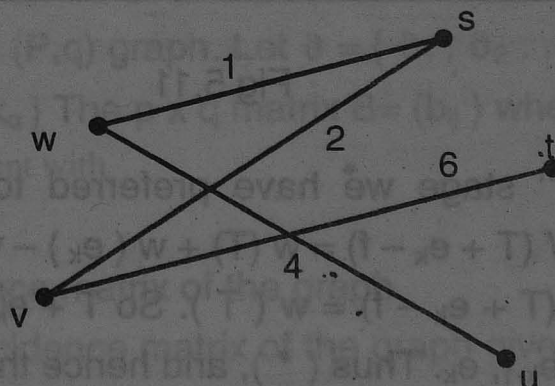


Fig 5.10

THM:-

Any tree S constructed by Prim's algorithm is an optimal tree.

PROOF:-

At the outset observe that S is indeed a tree. This follows because, at every stage we select exactly one new vertex v and exactly one edge which joins v to a vertex in the component already generated; hence at each stage the subgraph generated, is connected and acyclic and therefore is a tree.

Let $e_1, e_2, \dots, e_{p-1}$ (where $e_k = (v_i, v_{k+1})$, $i < k + 1$ and $1 \leq k \leq p - 1$) be the order of edges in S by which they were selected to construct S . To prove the theorem it suffices to prove the following.

(*): If there is an optimal tree T with $e_1, e_2, \dots, e_{k-1} \in T$ and $e_k \notin T$, then there is an optimal tree T^1 with $e_1, e_2, \dots, e_{k-1}, e_k \in T^1$.

So, Let T be an optimal tree with $e_1, e_2, \dots, e_k \in T$ and $e_{k+1} \notin T$. Since $e_k = \{v_i, v_{k+1}\} \notin T$, $T + e_k$ contains a unique cycle. So find an edge $f = (u, u)$ where $u \in \{v_1, v_2, \dots, v_k\}$ and $v \in \{v_{k+1}, \dots, v_p\}$ consider the tree $T^1 = T + e_k - f$.

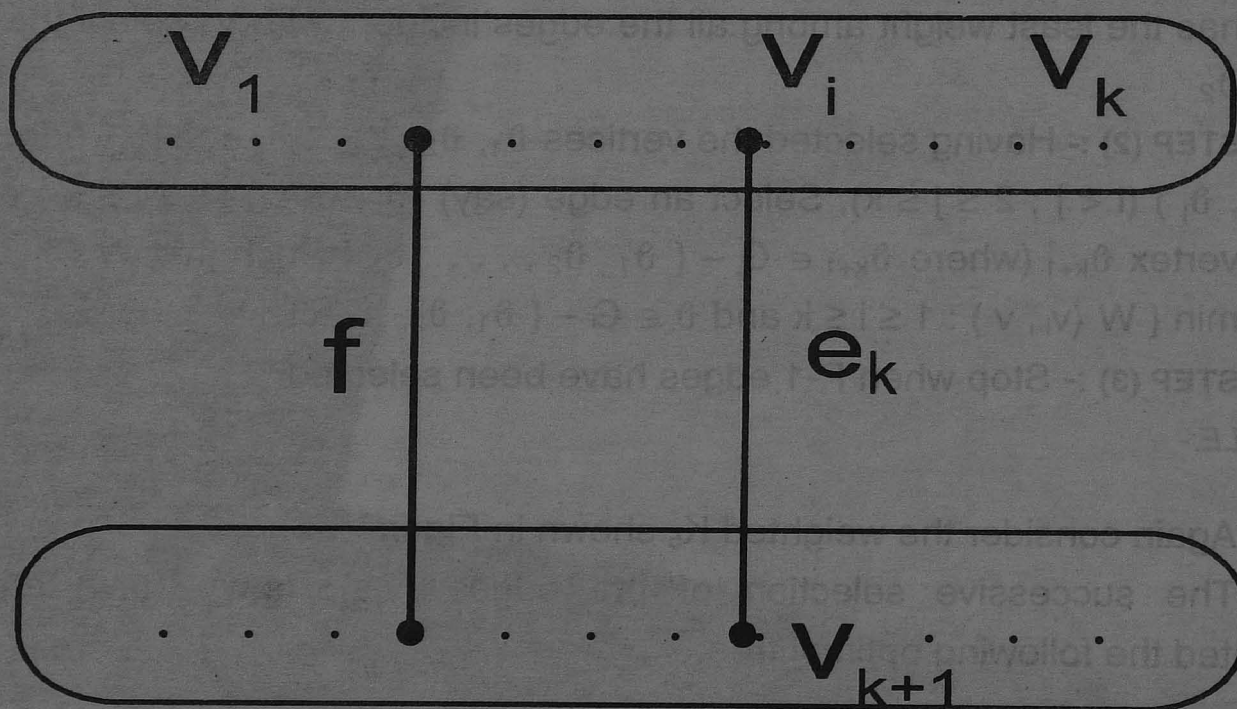


Fig 5.11

Since at the K^{th} stage we have preferred to select e_k it follows that $W(e_k) \leq W(f)$. Hence $W(T + e_k - f) = w(T) + w(e_k) - w(f) \leq w(T)$. Since T is an optimal tree, we get $w(T + e_k - f) = w(T)$. So $T + e_k - f$ is an optimal tree of G containing $e_1, e_2, \dots, e_{k-1}, e_k$. Thus (*), and hence the theorem is proved.

UNIT - VI

1. MATRIX REPRESENTATION:-

In this section we give two representations of a graph in matrix form. A matrix is a convenient and useful way of representing a graph to a computer. Further the algebra of matrices can be used to identify certain properties of graphs.

DEFINITION:-

Let $G = (V, X)$ be a (p, q) graph. Let $V = \{v_1, v_2, \dots, v_p\}$. The $p \times p$ matrix $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i, v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

is called the adjacency matrix of the graph G .

For example the adjacency matrix of the graph given in Fig 6 – 1 is given by

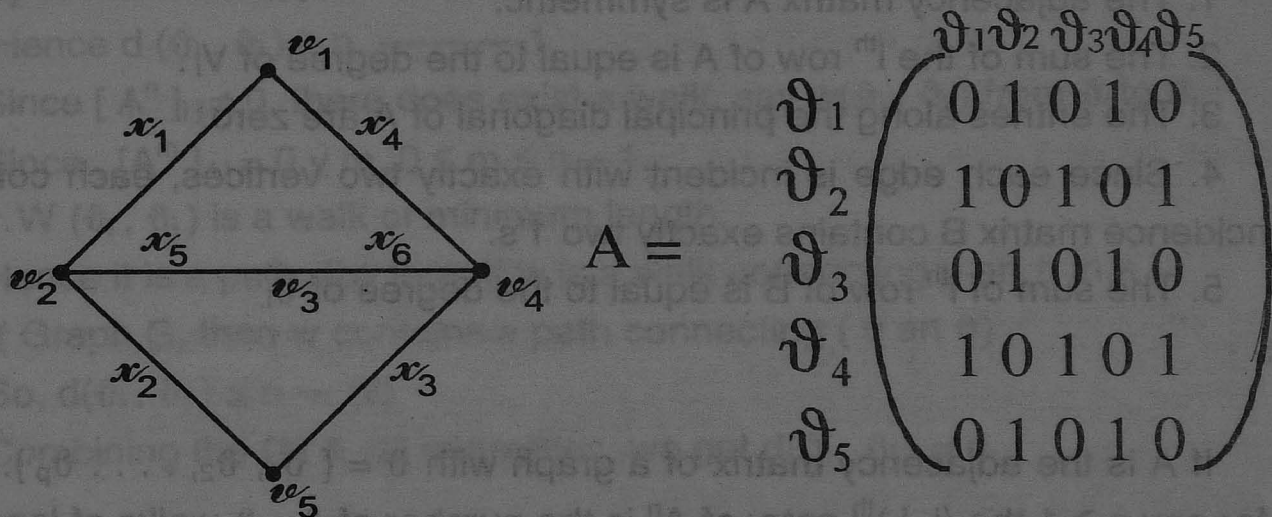


Fig 6.1

DEFINITION:-

Let $G = (V, X)$ be a (p, q) graph. Let $V = \{v_1, v_2, \dots, v_p\}$ and $X = \{x_1, x_2, \dots, x_q\}$. The $p \times q$ matrix $B = (b_{ij})$ where

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } x_j \\ 0 & \text{otherwise} \end{cases}$$

is called the incidence matrix of the graph.

For example the incidence matrix of the graph given in Fig 6-1 is given by

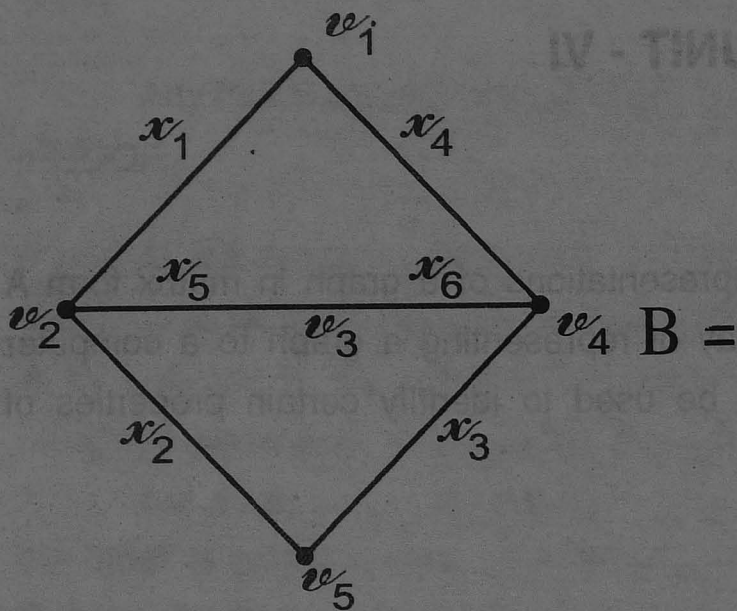


Fig 6.1

$$B = \begin{matrix} & \begin{matrix} e & e & e & e & e & e \end{matrix} \\ & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

We make the following observations regarding the adjacency and incidence matrices of a graph.

1. The adjacency matrix A is symmetric.
2. The sum of the i^{th} row of A is equal to the degree of V_i .
3. The entries along the principal diagonal of A are zero.
4. Since each edge is incident with exactly two vertices, each column of the incidence matrix B contains exactly two 1's.
5. The sum of i^{th} row of B is equal to the degree of v_i .

THM:-

If A is the adjacency matrix of a graph with $v = \{v_1, v_2, \dots, v_p\}$. Prove that for any $n \geq 1$ the $(i, j)^{\text{th}}$ entry of A^n is the number of $v_i - v_j$ walks of length n in G .

PROOF:-

We prove the result by induction on n . The number of $v_i - v_j$ walks of length 1.

$$= \begin{cases} 1 & \text{if } V_i \text{ and } V_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

Hence the result is true for $n = 1$.

We now assume that the result is true for $n - 1$.

Let $A^{n-1} = (a_{ij}^{(n-1)})$ so that $a_{ij}^{(n-1)}$ is number of $v_i - v_j$ walks of length $n-1$ in G .

Now $A^{n-1} A = (a_{ij}^{(n-1)}) (a_{ij})$

Hence $(i, j)^{th}$ entry of $A^n = \sum_{k=1}^p a_{ik}^{(n-1)} a_{kj}$ ----- (1)

Also every $V_i - V_j$ walk of length n in G consists of a $V_i - V_k$ walk of length $n - 1$ followed by a vertex V_j which is adjacent to V_k .

Hence if V_j is adjacent to V_k then $a_{kj} = 1$ and $a_{ik}^{(n-1)} a_{kj}$ represents the number of $V_i - V_j$ walks of length n whose last edge is $V_k V_j$.

Hence the right side of (1) gives the number of $V_i - V_j$ walks of length n in G . This completes the induction and the proof.

THM:-

If G is a connected graph, then the distance between V_i and V_j is the smallest integer (≥ 0) such that $[A^n]_{ij} \neq 0$.

PROOF:-

Let n be the smallest integer such that $[A^n]_{ij} \neq 0$.

By the minimality of n , $[A^m]_{ij} = 0 \forall m, 0 \leq m \leq n - 1$.

By the above thm, there is no walk of length $\leq n - 1$ connecting v_i and v_j .

Hence $d(v_i, v_j) \geq n$ ----- (1)

Since $[A^n]_{ij} \neq 0$, there does exist a walk, say $w(v_i, v_j)$ from v_i to v_j .

Since $[A^m]_{ij} = 0 \forall m, 0 \leq m \leq n - 1$.

$\therefore w(v_i, v_j)$ is a walk of minimum length.

Hence it is a path. (by thm, If w is a walk connecting u and v in a

(Graph G , then w contains a path connecting $(u$ and $v)$)

So, $d(v_i, v_j) \leq n$ --- (2)

Combining the (1) & (2) inequalities, we get $d(v_i, v_j) = n$.

THM:-

A graph G and p vertices is connected iff $(A + I)^{p-1}$ has no zero entries.

PROOF:-

By BINOMIAL THM, (NOTE:- THE MATRICES A AND I COMMUTE)

We have

$$(i) (A + I)^{p-1} = A^{p-1} + \binom{p-1}{1} A^{p-2} + \dots + \binom{p-1}{p-2} A + \binom{p-1}{p-1} A^0$$

At the outset, we observe that every diagonal entry in $(A + I)^{p-1}$ is at least one because of the last term.

1. SUPPOSE THAT G IS CONNECTED:-

Let i, j ($1 \leq i, j \leq p$) be two distinct arbitrary integers. Since G is connected, there is a path of length, say m ($\leq p - 1$).

Connecting V_i and V_j

By thm, Let A is the adjacency matrix of a graph with $\mathcal{V} = \{v_1, v_2, \dots, v_p\}$
P.T for any $n \geq 1$ the $(i, j)^{\text{th}}$ entry of A^n is the number of $u_i - v_j$ walks of length n in G .

$$\therefore [A^m]_{ij} \neq 0.$$

Hence, by (1) $(A + I)_{ij}^{p-1} \neq 0$.

2) Suppose that $[A + I]_{ij}^{p-1} \neq 0 \forall i, j, 1 \leq i, j \leq p$.

Let v_i and v_j be two arbitrary vertices in G .

Since $(A + I)^{p-1}$ has no zero entries it follows, by (1), that there exists m ,

$0 \leq m \leq p - 1$ such that $[A^m]_{ij} \neq 0$

Hence by thm, If A is the adjacency matrix of a graph with $\mathcal{V} = \{v_1, v_2, \dots, v_p\}$

P.T for any $n \geq 1$ the $(i, j)^{\text{th}}$ entry of A^n is the number of $U_i - V_j$ walks of length n in G . There is a walk of length m joining V_i and V_j . So, V_i and V_j are connected. Since V_i and V_j are arbitrary it follows that G is connected.

2. VECTOR SPACES ASSOCIATED WITH GRAPHS:-

We recall the following definitions from linear algebra.

1. Let X be a set. Define.

$\otimes : P(X) \times P(X) \rightarrow P(X)$ by

$\otimes : (S, T) = S \otimes T = S \cup T - S \cap T \forall S, T \in P(X)$ The binary operation $\otimes$ is known as the ring sum or the symmetric difference

$\otimes X(1, S) = 1 \times S = S, \forall S \in P(X)$

The following theorem is an easy exercise in linear algebra.

THM:-

If X is a set, then $(P(X), \otimes, Z_2 \otimes)$ is a vector space over Z_2 of dimension 1×1 . A basis for this vector space is $\{\{x\} : x \in X\}$.

If $G = (V, E)$ is a graph. Then we associate with it the vector space $V(G) = (P(E), Z_2, \otimes)$

Clearly, $V(G)$ is isomorphic with the vector space Z_2^q of q -tuples $(a_1, a_2, \dots, a_q)$ over Z_2

The natural isomorphism $f : V(G) \rightarrow Z_2^q$ is of course given by,

$f(S) = (a_1, a_2, \dots, a_q)$ where $a_i = 0$ if $e_i \notin S = 1$ if $e_i \in S, \forall S \subseteq E(G)$.

UNIT - VII

1. CYCLE SPACE:-

DEFINITION:-

Let (p) denote the set of all cycles in a graph G .

Then the subspace $L(p)$ or $V(G)$ spanned by p is called the cycle space of G .

So, each element of $L(p)$ is a linear combination. $a_1 c_1 \oplus a_2 c_2 \oplus \dots \oplus a_k c_k$ of some k cycles $C_1, C_2, \dots, C_k \in p$, where $a_i \in \mathbb{Z}_2$.

Since, each $a_i \in \mathbb{Z}_2$ is either zero or one, we omit a_i and write a typical linear combination as $C_1 \oplus C_2 \oplus \dots \oplus C_k$.

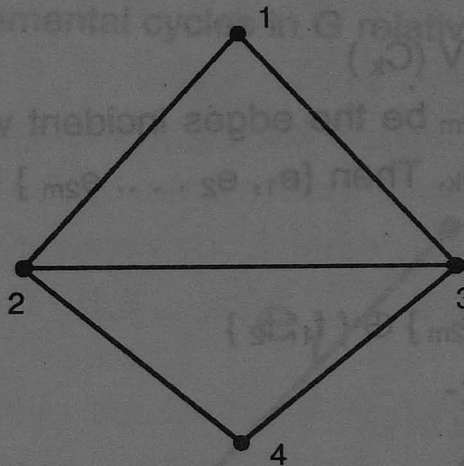


Fig 7.1

The graph of Figure 7-1 contains 3 cycles namely, $C_1 : 1,2,3,1$, $C_2 : 2,3,4,2$ and $C_3 = 1,2,4,3,1$.

We find that $C_1 \oplus C_2 = C_3$; $C_1 \oplus C_3 = C_2$; $C_2 \oplus C_3 = C_1$.

THEOREM:-

Every element of $L(p)$ is a (edge) disjoint union of cycles.

PROOF:-

Let $D = C_1 \oplus C_2 \oplus \dots \oplus C_k$ be an element of $L(p)$.

We shall prove the theorem by induction on k .

If $k = 1$, then D itself is a cycle. Hence the assertion is true. Assume that any linear combination of $n (\geq 1)$ cycles, Where $n < k$, is an edge disjoint union of cycles. Then by induction hypothesis,

$D = C_1 \oplus C_2 \oplus \dots \oplus C_{k-1}$ is an edge disjoint union of cycles.

This prove the assertion for D it suffices, By thm, A connected graph G is eulerian iff $E(G)$ can be expressed as the edge disjoint union of cycles.

To prove that $\deg_D(v)$ is even, $\forall v \in V(G)$ we have $v(D) = v(D^1) \cup v$

(C_k) . Let $V \in (D)$. we distinguish 3 cases and prove in each case $\deg_D(V)$ is even.

Case 1:-

$v \in v(D^1)$ and $v \notin v(C_k)$

We have $\deg_D(v) = \deg_{D^1}(v)$

Since D^1 is an edge disjoint union of cycles. $\therefore \deg_{D^1}(u)$ is even, $\forall u \in v(D^1)$

By thm,

A connected graph G is eulerian iff $E(G)$ can be expressed as the edge disjoint union of cycles.

In particular $\deg_{D^1}(v)$ is even.

Case 2:-

$v \in V(D^1)$ and $v \in V(C_k)$ we have $\deg_D(v) = \deg_{C_k}(v) = 2$.

Case 3:-

$v \in V(D^1) \cap V(C_k)$

Let $e_1, e_2, \dots, e_{2m}$ be the edges incident with V in D^1 . Let f_1, f_2 be the edges incident with V in C_k . Then $\{e_1, e_2, \dots, e_{2m}\} \oplus \{f_1, f_2\}$ is the set of edges incident with V in D .

Hence,

$$\deg_D(V) = |\{e_1, e_2, \dots, e_{2m}\} \oplus \{f_1, f_2\}|$$

WE HAVE 3 POSSIBILITIES:-

$$(i) \{e_1, e_2, \dots, e_{2m}\} \cap \{f_1, f_2\} = \emptyset$$

$$(ii) \{e_1, e_2, \dots, e_{2m}\} \cap \{f_1, f_2\} = \{f_1\} \text{ or } \{f_2\}$$

$$(iii) \{e_1, e_2, \dots, e_{2m}\} \cap \{f_1, f_2\} = \{f_1, f_2\}$$

By the definition of ring sum,

$$|\{e_1, e_2, \dots, e_{2m}\} \oplus \{f_1, f_2\}| = 2m + 2, \text{ when (i) holds}$$

$$= 2m, \text{ when (ii) holds}$$

$$= 2m - 2, \text{ when (iii) holds}$$

So, in each case $\deg_D(v)$ is even.

Hence the theorem

DEFN:-

Let $G = (V, E)$ be a connected graph and T be a spanning tree of G . (Such a T exists by Thm, Every connected graph G contains a spanning tree). The edges of T are called the branches of T and the edges of $E(G) - E(T)$ are called the chords of T .

If e is a chord of T , then $T + e$ contains a unique cycle. Such a cycle is

called a fundamental cycle of G , relative to T .

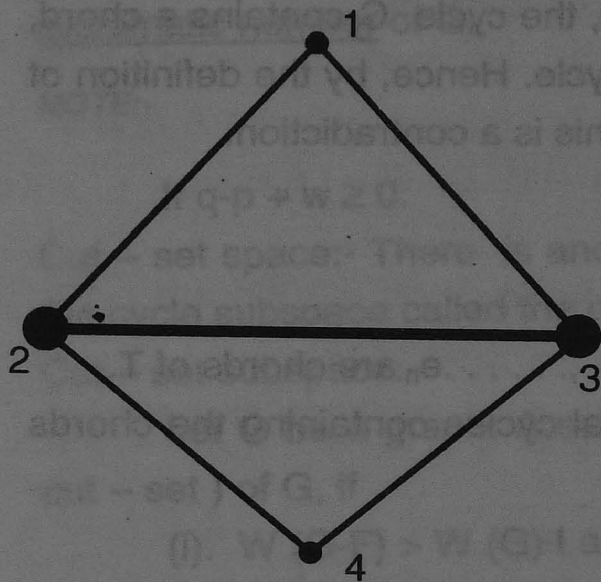
Thus, if C is a fundamental cycle (relative to a spanning tree T) in a graph G , then exactly one of its edges is a chord of T and all the other edges of C are branches of T .

EXAMPLE:-

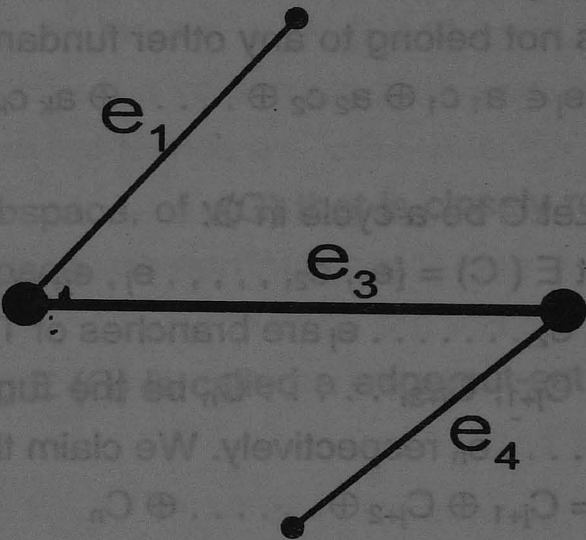
In the following graph e_1, e_3, e_4 are branches of T ; and e_2, e_5 are chords of T .

REMARK:-

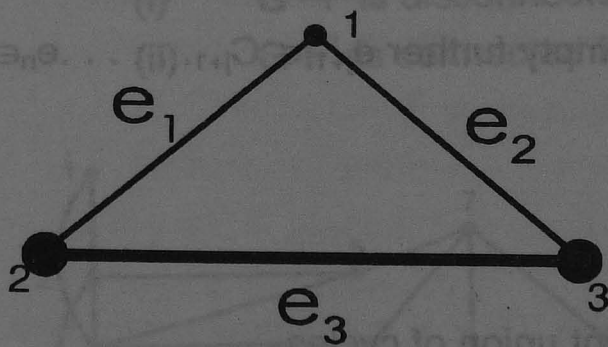
Let G be a connected (p,q) graph and let T be an arbitrary spanning tree of G . The number of branches of T is $p-1$. Hence, the number of chords of T is $q - p + 1$. So, the number of fundamental cycles in G relative to any spanning tree T of G is $q-p+1$.



A graph G



A spanning tree T of G



Two Fundamentals Cycles relative to T

THM:-

Let $G = (V, E)$ be a connected graph let T be a spanning tree of G . Then the set of fundamental cycles (relative to T) in G is a basis for the cycle subspace L (e) Hence, the dimension of L (e) is $q-p+1$.

PROOF:-

We shall prove that,

1). Fundamental cycles are independent in $V(G)$

2). every cycle in G is a finite linear combination of fundamental cycles.

1. Let $C_1, C_2, \dots, C_k$ be K fundamental cycles and let

$$a_1 C_1 \oplus a_2 C_2 \oplus \dots \oplus a_k C_k = \emptyset$$

where $a_i \in \mathbb{Z}_2$ --- (1)

We assert that $a_i = 0, \forall i, 1 \leq i \leq k$.

So that $C_1, C_2, \dots, C_k$ are linearly independent on the contrary, Let $a_j \neq 0$, for some j By the definition of fundamental cycles, the cycle C_j contains a chord, e_j . e_j does not belong to any other fundamental cycle. Hence, by the definition of ring sum, $e_j \in a_1 C_1 \oplus a_2 C_2 \oplus \dots \oplus a_k C_k = \emptyset$. This is a contradiction.

2. Let C be a cycle in G .

$$E(C) = \{e_1, e_2, \dots, e_j, e_{j+1}, \dots, e_n\}$$

Where $e_1, e_2, \dots, e_j$ are branches of T and $e_{j+1}, \dots, e_n$ are chords of T .

Let $C_{j+1}, C_{j+2}, \dots, C_n$ be the fundamental cycles containing the chords $e_{j+1}, e_{j+2}, \dots, e_n$ respectively. We claim that,

$$C = C_{j+1} \oplus C_{j+2} \oplus \dots \oplus C_n$$

On the contrary,

$$C \neq C_{j+1} \oplus C_{j+2} \oplus \dots \oplus C_n$$

So, by the definition of ring sum,

$$C^* = C \oplus C_{j+1} \oplus C_{j+2} \oplus \dots \oplus C_n \text{ is non-empty further } e_{j+1} \in C_{j+1} \oplus \dots \oplus C_n$$

C_n cancel $e_{j+1}, \dots, e_n \in C$ in C^*

$$\text{So, } e \in (C^*) \subseteq e \in (T)$$

By THM,

Every element of $L(e)$ is a (edge) disjoint union of cycles.

$\therefore C^*$ is a disjoint union of cycles and hence in particular contains atleast one cycle.. This is a contradiction.

$\therefore$ The fact that T is acyclic.

Thus (2), and hence our theorem is proved.

THM:-

If G is a (p,q) graph with ω components, then the dimension of the cycle subspace of $V(G)$ is $q-p+\omega$.

PROOF:-

Let $G_1, G_2, \dots, G_w$ be the components of G . Let $T_1, T_2, \dots, T_w$ be spanning trees of $G_1, G_2, \dots, G_w$ respectively. Let T_i be a (p_i, q_i) - tree ($1 \leq i \leq w$)

Let e_i be the set of all fundamental cycles of G_i , relative to T_i ($1 \leq i \leq w$)

Then clearly $p_1 u p_2 u \dots u p_w e_w$ is a basis for the cycle subspace $L(p)$

Hence, the dimension of the cycle subspace is $/ p_1 u p_2 u \dots u p_w / = (q_1 - p_{1+1}) + (q_2 - p_{2+1}) + \dots + (q_w - p_{w+1}) = q - p + w$

Hence the theorem.

DEFINITION:-

If G is a (p,q) - graph with w components, then $p - q + w$ is called the cyclomatic number of G .

NOTE:-

If $q-p + w \geq 0$.

Cut - set space:- There is another subspace, of $v(G)$ that is closely related with the cycle subspace called the cut set space.

Cut - set subspace:-

Let G be a graph. A subset $F \subseteq E(G)$ is called a edge cut-set (or simply cut - set) of G , if

- (i). $W(G-F) > W(G)$ and
- (ii). $W(G-F') = W(G)$ for every proper subset F' of F .

Thus if G is a connected graph then $F \subseteq E(G)$ is a Cut - Set of G , if

- (i) $G - F$ is disconnected and
- (ii) $G - F^1$ is connected for every proper subset F^1 of F

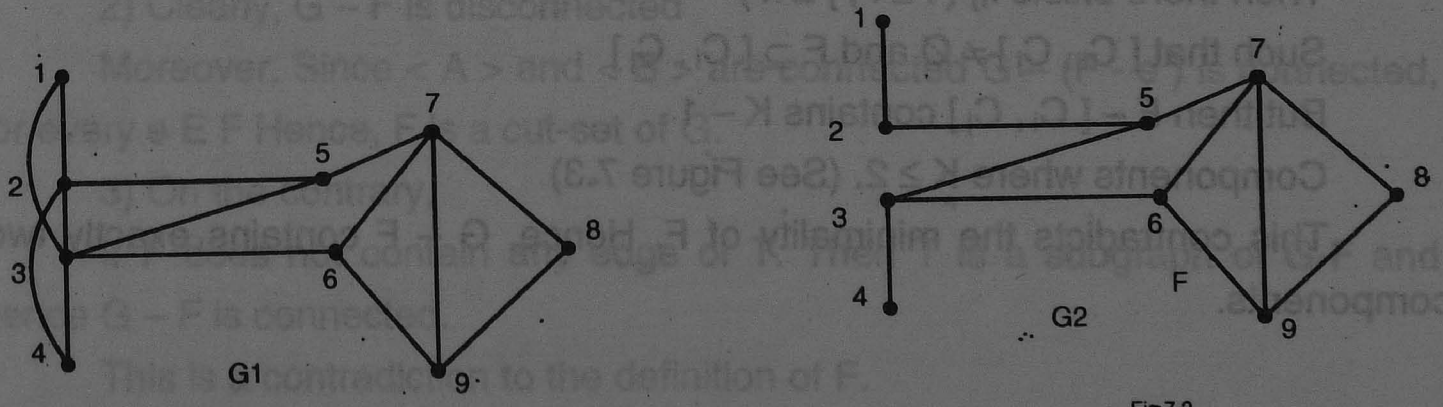


Fig 7.2

In figure 7.2, $\{ (2,5), (3,5), (3,6) \}$ is an edge cut-set of G_1 .

Where as

$F = \{ (2,5), (3,5), (3,6) \}$ is not an edge cut-set of G_2 .

because $\{ (2,5) \} \subseteq F$ and $G_2 - \{ (2,5) \}$ is disconnected.

$\{ (1,2) \}, \{ (3,5), (3,6) \}, \{ (5,7), (6,7), (6,9) \}$ are some of the cut-sets of G_2 .

NOTATION:-

If $A, B \subseteq V(G)$ are two disjoint subsets, then $[A, B] = \{ (u, v) \in E(G); u \in A \text{ and } v \in B \}$

The following result though an easy consequence of definitions, is very often used. Hence, we state it as a theorem for future references.

THM:-

Let G be a connected graph. Then the following are true.

1. If $F \subseteq E(G)$ is a cut-set in G , then $G - F$ contains exactly two components; and F gives rise to a partition of $V(G)$ into two subsets A and B such that $\langle A \rangle$ and $\langle B \rangle$ are connected subgraphs of G ; So, $F = [A, B]$.

2. If A and B are two non-empty subsets whose union is $V(G)$ are such that $\langle A \rangle$ and $\langle B \rangle$ are connected, then $F = [A, B]$ is a cut-set of G .

3. If T is a spanning tree of G , then every cut-set F of G contains atleast one edge of T .

PROOF:-

1). By definition, $G - F$ is disconnected and Hence contains atleast two components suppose that $C_1, C_2, \dots, C_k$ are its components where $k \geq 3$.

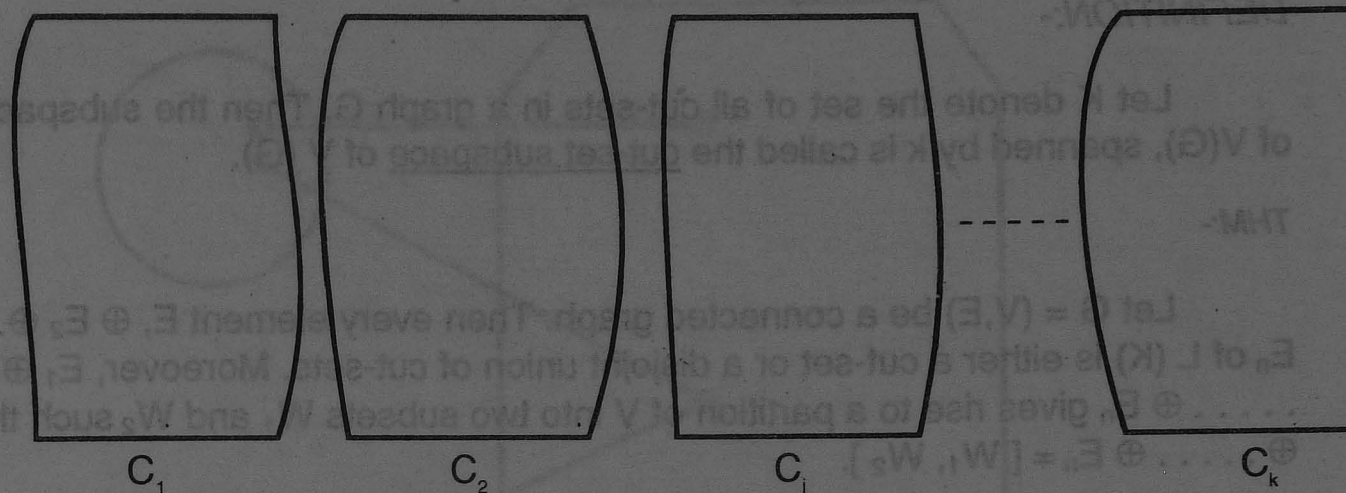
Then there exists i, j ($1 \leq i, j \leq k$)

Such that $[C_i, C_j] \neq \emptyset$ and $F \supseteq [C_i, C_j]$

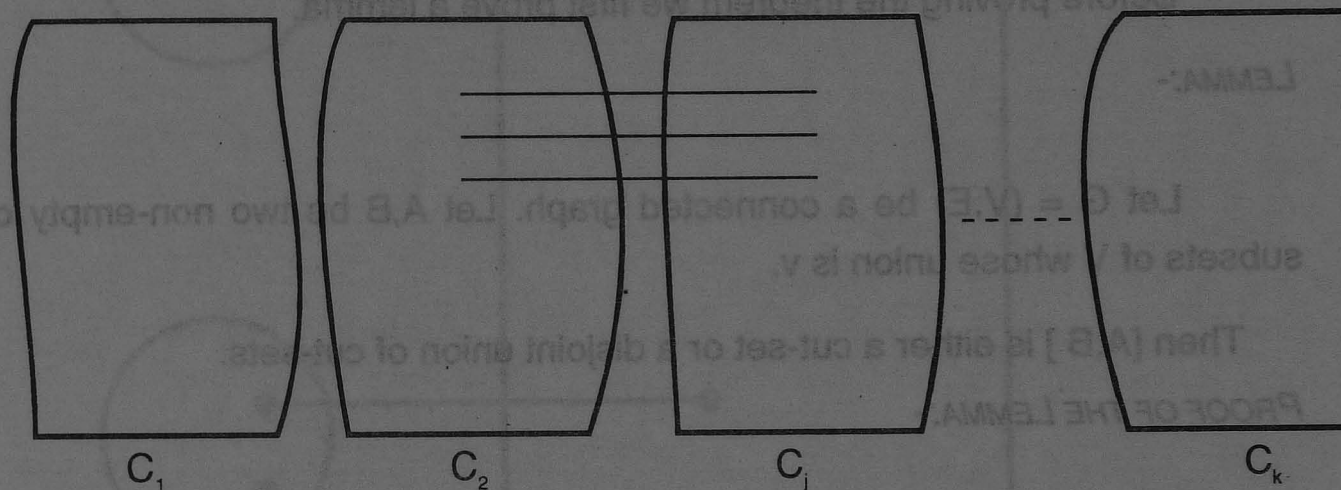
But then $F - [C_i, C_j]$ contains $k - 1$.

Components where $k \geq 2$. (See Figure 7.3)

This contradicts the minimality of F . Hence, $G - F$ contains exactly two components.



$G-F$



$G-(F-[C_i, C_i])$

Let C_1 and C_2 be the two components of $G-F$

If $e = (u, v) \in F$ is an edge with $u, v \in c_i$ for some i ($i = 1, 2$)

Then $G - (F - (u, v))$ is again a disconnected graph.

A contradiction to the minimality of F . Hence, every edge of F has its vertices in different components C_1 and C_2 .

So, if $A = V(C_1)$ and $B = V(C_2)$, then $F = [A, B]$

2) Clearly, $G - F$ is disconnected

Moreover, Since $\langle A \rangle$ and $\langle B \rangle$ are connected $G - (F - e)$ is connected, for every $e \in F$ Hence, F is a cut-set of G .

3) On the contrary,

If F does not contain any edge of T . Then T is a subgraph of $G-F$ and hence $G - F$ is connected.

This is a contradiction to the definition of F .

DEFINITION:-

Let K denote the set of all cut-sets in a graph G . Then the subspace $L(K)$ of $V(G)$, spanned by k is called the cut-set subspace of $V(G)$.

THM:-

Let $G = (V, E)$ be a connected graph. Then every element $E_1 \oplus E_2 \oplus \dots \oplus E_n$ of $L(K)$ is either a cut-set or a disjoint union of cut-sets. Moreover, $E_1 \oplus E_2 \oplus \dots \oplus E_n$ gives rise to a partition of V into two subsets W_1 and W_2 such that $E_1 \oplus \dots \oplus E_n = [W_1, W_2]$.

PROOF:-

Before proving the theorem we first prove a lemma.

LEMMA:-

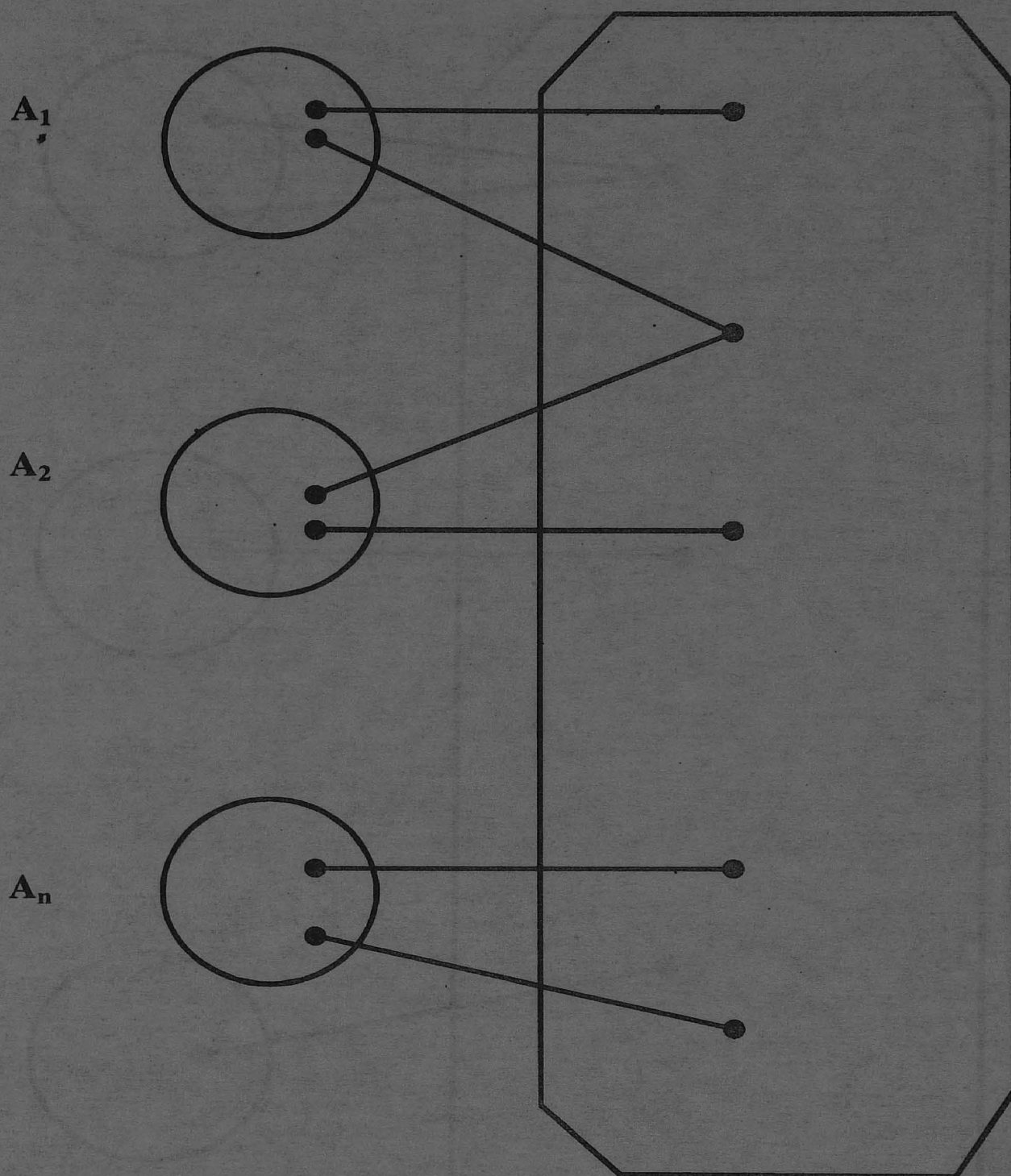
Let $G = (V, E)$ be a connected graph. Let A, B be two non-empty disjoint subsets of V whose union is V .

Then $[A, B]$ is either a cut-set or a disjoint union of cut-sets.

PROOF OF THE LEMMA:-

Let $A_1, A_2, \dots, A_n$ be the $n (\geq 1)$.

Components of $\langle A \rangle$.



$\langle A \rangle$ $\langle B \rangle$: connected or disconnected

So,

$$[A, B] = [A_1, B] \cup [A_2, B] \cup \dots \cup [A_n, B] - 1$$

Since

$$[A_i, A_j] = \emptyset, \text{ for every } i, j \ 1 \leq i, (\neq) j \leq n$$

consider $[A_1, B]$

If $\langle V - A_1 \rangle$ is connected.

Then $[A_1, B] = ([A_1, V - A_1])$ is a cutset.

By the above thm,

So, Let $\langle V - A_1 \rangle$ contain t components, say $C_1, C_2, \dots, C_t$.

A_1

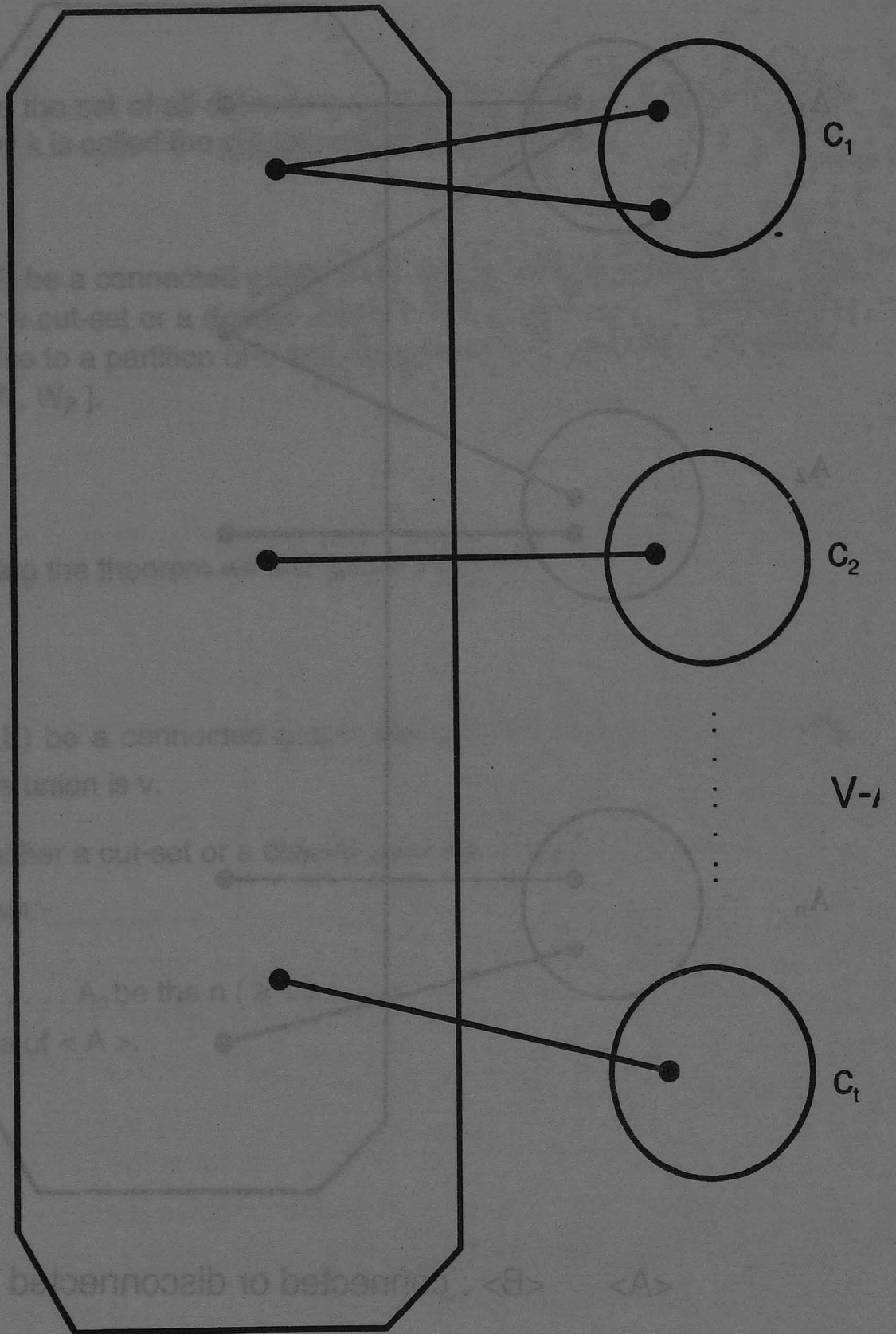


Fig. 7.3

Then $[A_1, B] = [A_1, V - A_1] = [A_1, C_1] \cup \dots \cup [A_t, C_t]$ and $[A_1, C_1], \dots, [A_t, C_t]$ are cut sets in G . by the above thm, (see figure 7.3)

Hence, $[A_1, B]$ is a cut-set or a disjoint union of cut-sets.

III^{ly} we can prove that each of the sets $[A_2, B] \dots [A_n, B]$ is either a cut-set or disjoint union of cut-sets.

Hence, by (i), $[A, B]$ is either a cut-set or disjoint union of cut-sets.

This completes the proof of the lemma.

PROOF OF THE THEOREM:-

We shall prove the theorem by induction on n .

If $n = 1$, then since E_1 is a cut-set, there is nothing to prove.

Assume that the theorem holds whenever $n = k - 1$.

Let $n = k$,

Consider $E_1 \oplus E_2 \oplus \dots \oplus E_k$.

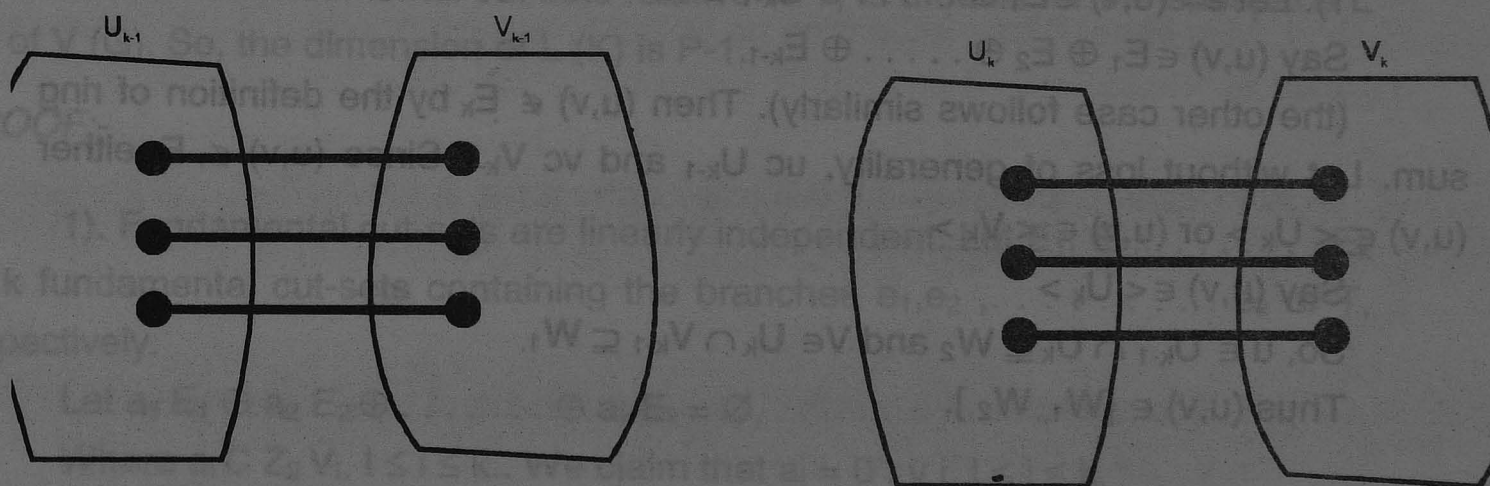
By induction hypothesis,

$E_1 \oplus E_2 \oplus \dots \oplus E_{k-1}$ gives rise to a partition of V into two subsets say U_{k-1} and V_{k-1} such that $E_1 \oplus \dots \oplus E_{k-1} = [U_{k-1}, V_{k-1}]$. Since E_k is a cut-set, it gives rise, by above thm, to a partition of V into two subsets say U_k and V_k , such that $E_k = [U_k, V_k]$.

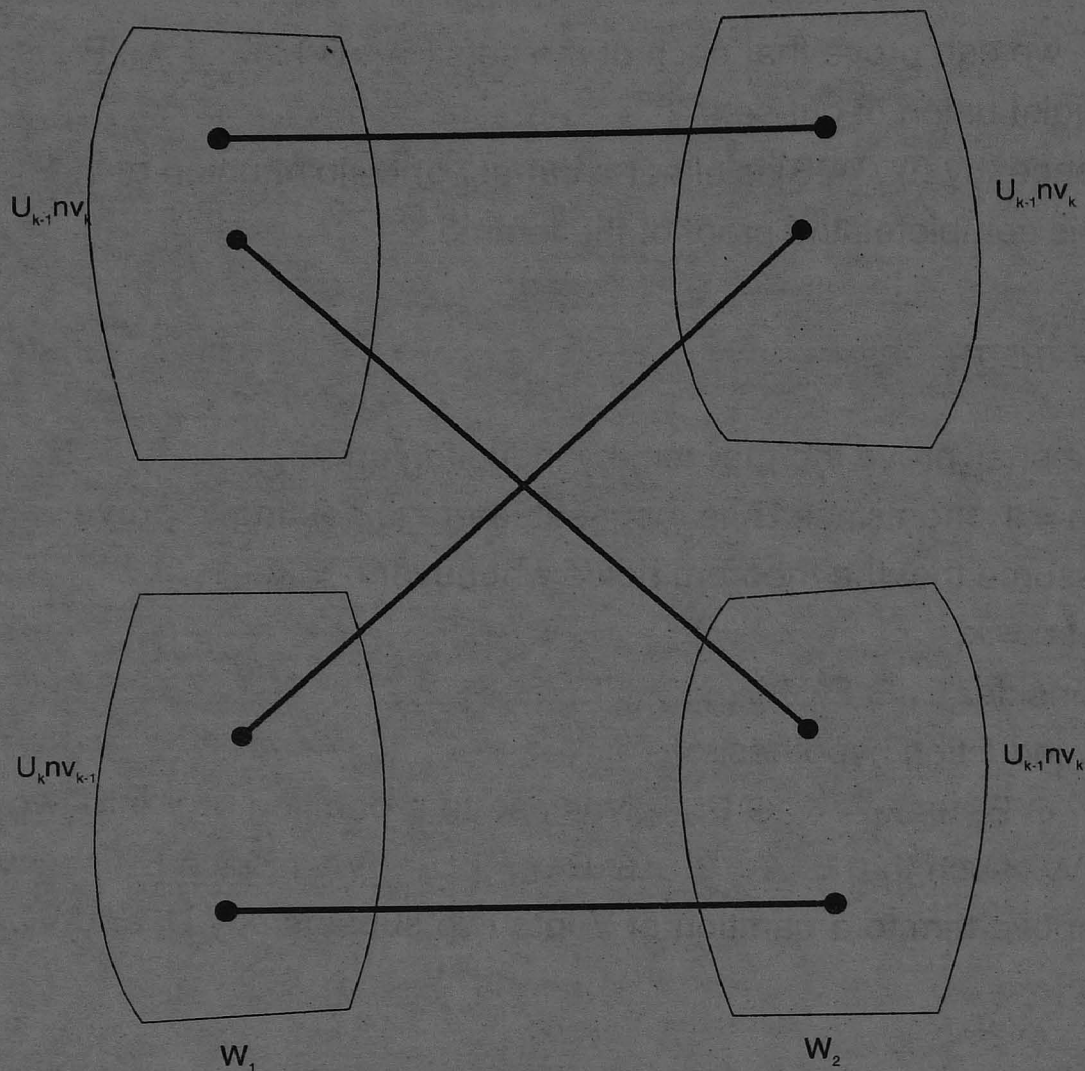
CONSIDER THESE SETS:-

$$W_1 = (U_{k-1} \cap V_k) \cup (U_k \cap V_{k-1}) \text{ and}$$

$$W_2 = (U_{k-1} \cap U_k) \cup (V_{k-1} \cap V_k)$$



$$E_1 \oplus E_2 \oplus \dots \oplus E_{k-1}$$



Clearly, W_1 and W_2 partition V .

We claim that

$$E_1 \oplus \dots \oplus E_k = [W_1, W_2]$$

1). Let $e = (u, v) \in E_1 \oplus \dots \oplus E_{k-1} \oplus E_k$

Say $(u, v) \in E_1 \oplus E_2 \oplus \dots \oplus E_{k-1}$

(the other case follows similarly). Then $(u, v) \notin E_k$ by the definition of ring sum. Let without loss of generality, $u \in U_{k-1}$ and $v \in V_{k-1}$. Since $(u, v) \notin E_k$, either $(u, v) \in \langle U_k \rangle$ or $(u, v) \in \langle V_k \rangle$

Say $(u, v) \in \langle U_k \rangle$

So, $u \in U_{k-1} \cap U_k \subseteq W_2$ and $v \in U_k \cap V_{k-1} \subseteq W_1$.

Thus $(u, v) \in [W_1, W_2]$.

2). Let $(u, v) \in [W_1, W_2]$

Say $u \in U_{k-1} \cap V_k$ and $v \in V_{k-1} \cap V_k$

(Other cases follow similarly). So, $u \in U_{k-1}$ and $u \in V_{k-1}$ and hence $(u, v) \in (U_{k-1}, V_{k-1}] = E_1 \oplus \dots \oplus E_{k-1}$

Since, $u, v \in V_k$

$(u, v) \in \langle V_k \rangle$

Hence, $(u, v) \notin E_k$. Thus $(u, v) \in E_1 \oplus \dots \oplus E_{k-1} \oplus E_k$

Now, By the Lemma it follows that

$[W_1, W_2] = E_1 \oplus E_2 \oplus \dots \oplus E_{k-1} \oplus E_k$ is either a cut-set or a disjoint union of cut-sets.
Hence the theorem.

DEFINITION:-

Let T be a spanning tree of a connected graph G . If e is a branch of T , Then $T - e$ contains exactly two components, say $\langle v_1 \rangle$ and $\langle v_2 \rangle$

Then the set of edges $F = [V_1, V_2]$ is a cut-set of G .

Thus every branch e of T gives rise to an unique edge cut-set, called the fundamental cut-set of G , relative to T .

Since, there are $P-1$ branches of T , there are exactly $P-1$ fundamental cut-sets in a connected (p, q) – graph G .

In general, if G is a graph with ω components, then each of the fundamental cut-sets of a component is called a fundamental cut-set of G . Thus there are $p-\omega$ fundamental cut-sets in a graph G with ω components.

THM:-

Let G be a connected graph on P vertices and let T be a spanning tree of G . The set of all fundamental cut-sets relative to T , is a basis for the subspace $L(K)$ of $V(G)$. So, the dimension of $L(K)$ is $P-1$.

PROOF:-

1). Fundamental cut-sets are linearly independent. Let $E_1, E_2, \dots, E_k$ be k fundamental cut-sets containing the branches $e_1, e_2, \dots, e_k$ of T , respectively.

Let $a_1 E_1 \oplus a_2 E_2 \oplus \dots \oplus a_k E_k = \emptyset$,

Where $a_i \in \mathbb{Z}_2$, $1 \leq i \leq k$. We claim that $a_i = 0$, $\forall i, 1 \leq i \leq k$.

So, that $E_1, E_2, \dots, E_k$ are linearly independent.

If possible, Let $a_j \neq 0$. for some $j, 1 \leq j \leq k$.

But,

Then $e_j \in a_1 E_1 \oplus \dots \oplus a_k E_k = \emptyset$.

Since E_j is the unique fundamental cut-set containing e_j . This contradiction shows that $a_j = 0, \forall j, 1 \leq j \leq k$.

2) Every Cut-set in G is a linear combination of fundamental cut-sets in G .

Let $E^* = \{e_1, e_2, \dots, e_j, e_{j+1}, \dots, e_n\}$ be a cut-set in G , where $e_1, e_2, \dots, e_j$ are branches of T and $e_{j+1}, \dots, e_n$ are chords of T .

Let $E_1, E_2, \dots, E_j$ be the fundamental cut-sets containing $e_1, e_2, \dots, e_j$ respectively.

We claim that

$$E^* = E_1 \oplus E_2 \oplus \dots \oplus E_j.$$

On the contrary,

$$\text{Let } E^* \neq E_1 \oplus E_2 \oplus \dots \oplus E_j$$

So, by the above thm,

$E^* \oplus E_1 \oplus E_2 \oplus E_3 \oplus \dots \oplus E_j$ is a cut-set or a disjoint union of cut-

Sets, Moreover, by the definition of ring sum,

$$E^* \oplus E_1 \oplus E_2 \oplus \dots \oplus E_j \subseteq E(G) - E(T)$$

$\therefore$ Every edge in $E^* \oplus E_1 \oplus E_2 \oplus \dots \oplus E_j$ is a chord.

This contradicts the fact that every cut-set contains a branch of T . Hence our claim holds and the theorem is proved.

DEFINITION:-

Define $\phi : V(G) \times V(G) \rightarrow Z_2$ by $\phi(L, M) = L \cdot M \equiv |L \cap M| \pmod{2}$,

$\forall L, M \in V(G)$ ϕ is said to be an innerproduct over $V(G)$ and $(V(G), \phi)$

Is said to be an inner product . Space: "

$L \cdot M$ is either 0 or 1 according as $|L \cap M|$ is even or odd.

DEFN:-

We define L and $M \in U V(G)$ to be orthogonal iff $L \cdot M = 0$

(i.e. $|L \cap M|$ is even)

THM:-

Let G be a connected graph. The cycle subspace $L(e)$ and the

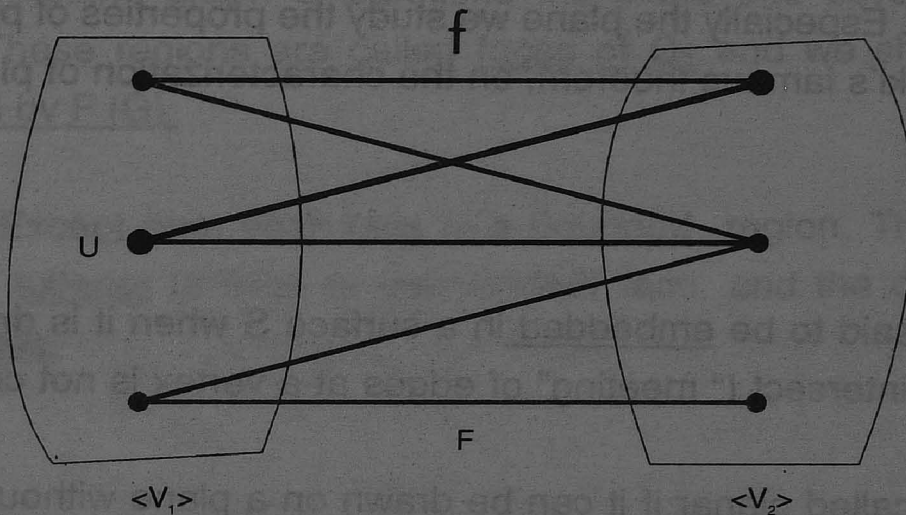
Cut-set subspace $L(K)$ are orthogonal.

PROOF:-

Let T be a spanning tree of G .

It is enough (by standard theorems in linear Algebra). If we prove that every basic vector. (is, every fundamental cycle relative to T) in $L(l)$ is orthogonal

to every basic vector. (ie, every fundamental cut-set relative to T) in L (K). So, Let $C(u,u)$ be a fundamental cycle and Let F be a fundamental cut-Set containing a branch f of T. T – f contains exactly two components, say $\langle V_1 \rangle$ and $\langle V_2 \rangle$. Every edge in F joins a vertex of V_1 with a vertex of V_2 . Let, without loss of generality $u \in V_1$.



Clearly, $E(C) \cap F = \emptyset$, if $E(C) \subseteq \langle V_1 \rangle$ or $\langle V_2 \rangle$
 = positive even number, if $E(C) \cap [V_1, V_2] \neq \emptyset$. So, C and F are orthogonal.
 Hence the theorem.

EXERCISE:-

- 1). Show that a graph G is bipartite iff in a cycle basis of G all the cycles are even.
- 2). What is the cycle basis of a tree ? Show that a connected graph G is a tree iff the cyclomatic number of G is 0.

Unit – VIII

4. PLANAR GRAPHS:-

In this chapter we consider the embedding (drawing without crossings) of graphs on surfaces. Especially the plane we study the properties of planar graphs and state Kuratowski's famous theorem, on the characterization of planar graphs.

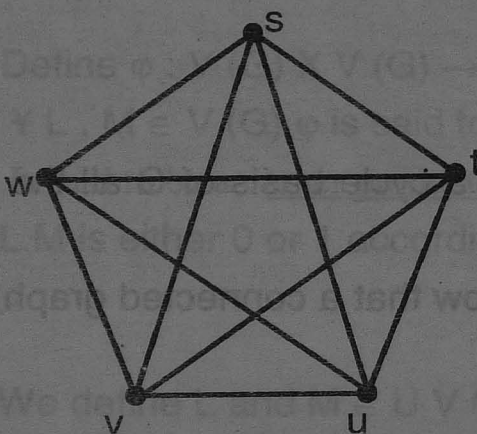
DEFINITION:-

A graph is said to be embedded in a surface S when it is drawn on S so that no two edges intersect ("meeting" of edges at a vertex is not considered an intersection).

A graph is called planar if it can be drawn on a plane without intersecting edges.

A graph is called non-planar if it is not planar.

A graph that is drawn on the plane without intersecting edges is called a plane graph.



<a>

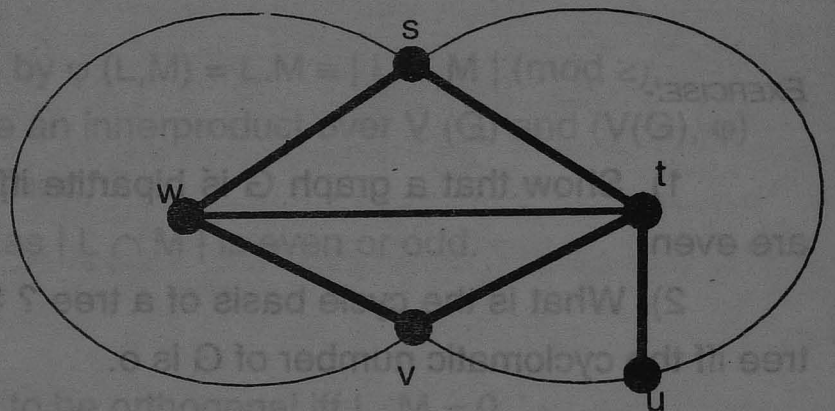


Fig. 8.1

The graph in Fig 8.1 (a) is planar even though it is not plane.

The graph in Fig 8.1 (b) (which is isomorphic to that in Fig 8.1(a)) is plane as it is drawn without intersecting edges.

It is also planar. Thus plane graph is a concept associated with embedding of the graph.

It is obvious that if two graphs are isomorphic and one is planar, then the other is also planar. If two graphs are isomorphic and one is plane, the other need not be plane.

DEFINITIONS:-

1. A plane graph G partitions the plane into several (open connected) regions. These regions are called faces of G and we shall denote the set of faces of G by $F(G)$.

2. Except one, each face is a bounded region. The exceptional face is called the exterior (infinite or unbounded) face, and the other faces are called interior faces.

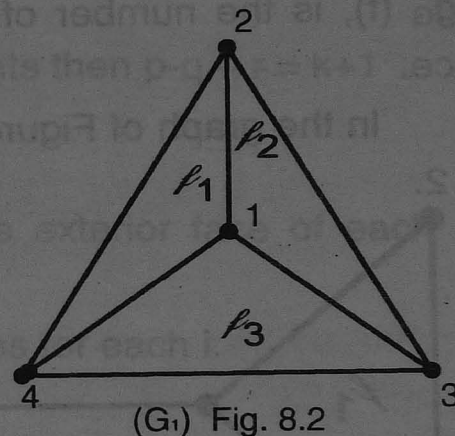
EXAMPLE:-

In figure 8.2, G , has four faces f_1, f_2, f_3 and f_4 .

f_4 is the exterior face.

NOTE:-

The first impression is that the boundary of a face is a cycle; this need not be so.



EXAMPLE:-

In the plane graph of Fig 8.3, the boundary of the face f_1 is $\{ (1,2), (2,3), (3,4), (4,5), (5,6), (6,7), (7,4), (3,1) \}$ which is not a cycle.

The boundary of the face f_2 is $\{ (1,2), (2,3), (3,4) \}$ which is a cycle.

The boundary of the face f_3 is $\{ (4,5), (5,6), (6,7), (7,4), (4,8), (8,9) \}$ which again is not a cycle.

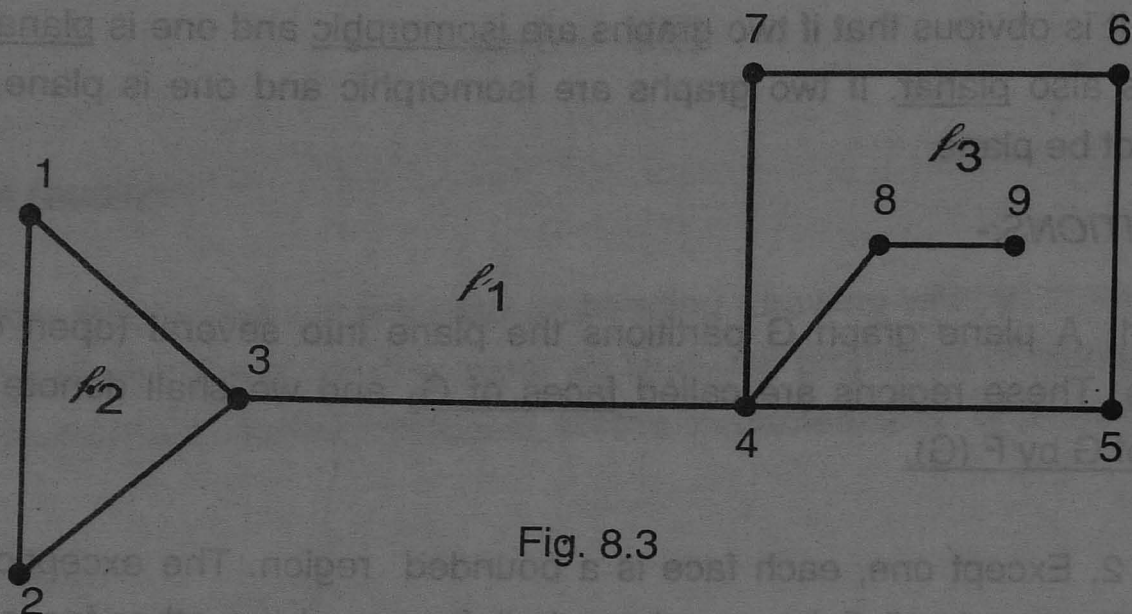


Fig. 8.3

DEFINITION:-

Let G be a plane graph and f be a face of G . The degree of f_1 denoted by $\deg_G(f)$, is the number of edges in the boundary of f with cut-edges counted twice.

In the graph of Figure 8.4 $\deg(f_1) = 3$, $\deg(f_2) = 3$, $\deg(f_3) = 8$, and $\deg(f_4) = 12$.

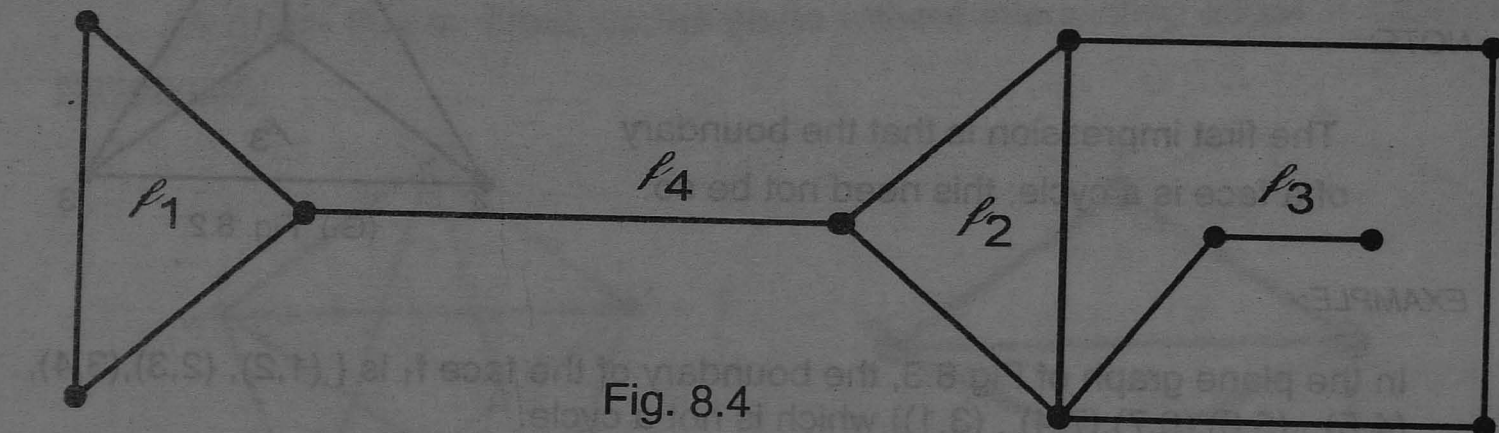


Fig. 8.4

2. EULER FORMULA:-

THM:- (EULER)

If G is a connected plane graph having V , E and F as the set of vertices, edges and faces respectively, then $|V| - |E| + |F| = 2$.

PROOF:-

The proof is by induction on the number of edges of G .

Let $|E| = 0$.

Since G is connected, it is K_1 so that $|V| = 1$, $|F| = 1$ (the infinite face) and hence $|V| - |E| + |F| = 2$.

Now, Let G be a graph as in the theorem and suppose that the theorem is true for all connected plane graphs with at most $|E| - 1$ edges.

Case: (i) If G is a tree, then $|E| = |V| - 1$ and $|F| = 1$

$$\therefore \text{Hence } |V| - |E| + |F| = 2$$

Case: (ii) If G is not a tree,

Let x be an edge contained in some cycle of G . Then $G' = G - x$ is a connected plane graph such that $|V(G')| = |V|$,

$$|E(G)| - |E| = 1 \text{ and } |F(G')| = |F| - 1.$$

Hence by the induction hypothesis.

$$|V(G')| - |E(G')| + |F(G')| = 2$$

$$\text{So that } |V| - (|E| - 1) + (|F| - 1) = 2$$

Hence $|V| - |E| + |F| = 2$, as required. This completes the induction and the proof

THM:-

If G is a plane (p, q) graph with r faces k components then $p - q + r = k + 1$

PROOF:-

Consider a plane embedding of G , such that the exterior face of each component contains all other components.

Let the i^{th} component be a (p_i, q_i) graph with r_i faces for each i .

By Euler formula,

$$p_i - q_i + r_i = 2$$

$$\text{Hence } \sum p_i - \sum q_i + \sum r_i = 2k \text{ ---- } 1$$

$$\text{But } \sum p_i = P, \sum q_i = q \text{ and } \sum r_i - k + 1 = r$$

$$\text{ie. } \sum r_i = r + (k - 1), [\therefore \text{the infinite face is counted } k \text{ times in } \sum r_i]$$

$$\text{Hence } \sum p_i - \sum q_i + \sum r_i = 2k$$

$$\Rightarrow P - q + r + k - 1 = 2k$$

$$\Rightarrow P - q + r = 2k - k + 1$$

$$\Rightarrow P - q + r = k + 1$$

COROLLARY:-

If G is a (p, q) plane graph in which every face is an n cycle then

$$q = \frac{n(p-2)}{n-2}$$

$$n - 2$$

PROOF:-

Every face is an n - cycle hence each edge lies on the boundary of exactly two faces.

Let $f_1, f_2, \dots, f_r$ be the faces of G .

$$2q = \sum_{i=1}^r (\text{number of edges in the boundary of face } f_i) = nr$$

$$i = 1$$

$$\therefore 2q = nr$$

$$\Rightarrow r = 2q/n$$

By Euler's formula, $P - q + r = 2$

$$\therefore P - q + 2q/n = 2$$

$$\Rightarrow np - nq + 2q = 2n$$

$$\Rightarrow q(2 - n) = 2n - np$$

$$\Rightarrow q = \frac{n(2 - p)}{2 - n}$$

$$\Rightarrow q = \frac{n(p - 2)}{(n - 2)}$$

$$\Rightarrow q = \frac{n(p - 2)}{(n - 2)}$$

COROLLARY:-

In any connected plane (p, q) graph $(p \geq 3)$ with r faces $q \geq 3r/2$ and $q \leq 3p - 6$.

PROOF:-

Case (i):- Let G be a tree

Then $r = 1$, $p = q - 1$ and $P \geq 3$

Hence $q \geq 3r/2$ and $q \leq 3p - 6$

Since $P - 1 \leq 3p - 6$ (as $P \geq 3$)

Case (ii):- Let G be not a tree.

Then G have a cycle.

Let $f_1, f_2, \dots, f_r$ be the faces of G .

Since each edge lies on the boundary of the almost 2 faces.

$$2q \geq \sum_{i=1}^r (\text{number of edges in the boundary of face } f_i)$$

$$i = 1$$

ie) $2q \geq 3r$ ($\therefore$ each face is bounded by atleast 3 edges)

ie) $q \geq 3r/2$

By Euler's formula, $P - q + r = 2$

$$\Rightarrow r = 2 + q - p$$

substituting for r in $q \geq 3r/2$ we get ,

$$q \geq 3/2 (2 - p + q)$$

$$q \geq \frac{6-3p+3q}{2} =$$

$$\Rightarrow 2q \geq 6 - 3p + 3q$$

$$\Rightarrow -3q + 2q \geq 6 - 3p$$

$$\Rightarrow -q \geq 6 - 3p$$

ie) $q \leq 3p - 6$. Hence the corollary

DEFINITION:-

A graph is called maximal planar if no line can be added to it without losing planarity.

In a maximal planar graph, each face is a triangle and such a graph is sometimes called a triangulated graph.

COROLLARY:-

Every planar graph G with $P \geq 3$ points has atleast three points of degree less than 6.

PROOF:-

By the corollary,

In any connected plane (p,q) graph $(p \geq 3)$ with r faces $q \geq 3r/2$ and $q \leq 3p - 6$

$$\text{ie) } 2q \leq 6p - 12$$

$$\text{ie) } \sum d_i \leq 6p - 12$$

Where d_i are the degrees of the vertices of G . Since G is connected, $d_i \geq 1$ for every i . If atmost two d_i are less than 6,

$$\text{Then } \sum d_i \geq 1 + 1 + 6 + \dots (p - 2) \text{ times} = 6P - 10$$

$$\text{Whis is } \Rightarrow \Leftarrow$$

Hence $d_i < 6$ for atleast three values of i .

COROLLARY:-

The graph K_5 and $K_{3,3}$ are not planar.

PROOF:-

K_5 is a $(5,10)$ graph

For any planar graph, $q \leq 3p - 6$ (by cor, In any connected plane

(p,q) graph ($P \geq 3$) with r faces

$q \geq 3r/2$ and $q \leq 3p - 6$

But $q = 10$ and $p = 5$ do not satisfy

This inequality.

Hence K_5 is not – planar.

$K_{3,3}$ is a $(6,9)$ bipartite graph and hence has no triangles. If such a graph is planar,

Then by cor, If G is a plane connected (p,q) graph without triangles and $P \geq 3$, then $q \leq 2p - 4$

But $p = 6$ and $q=9$ do not satisfy this inequality. Hence $K_{3,3}$ is not planar.

THM:-

K_5 is non- planar.

PROOF:-

If possible, Let k_5 be planar.

Then k_5 contains a cycle of length 5 Say s,t,u,v,w,s hence without loss of .Generality, any plane embedding of K_5 can be assumed to contain this

Cycle drawn in the form of regular pentagon. Hence the edge wt must lie either wholly inside the pentagon or

Wholly outside the pentagon.

Case (i):-

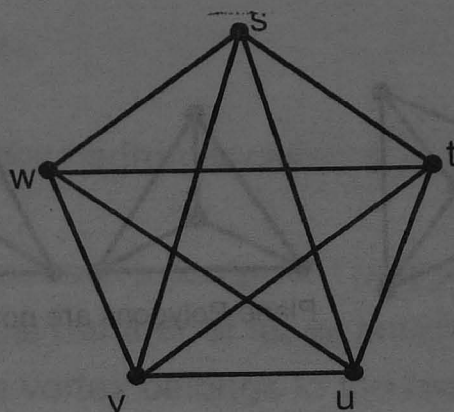
Suppose that wt is wholly inside the pentagon.

Since the edges SV and SU donot cross the edge wt .

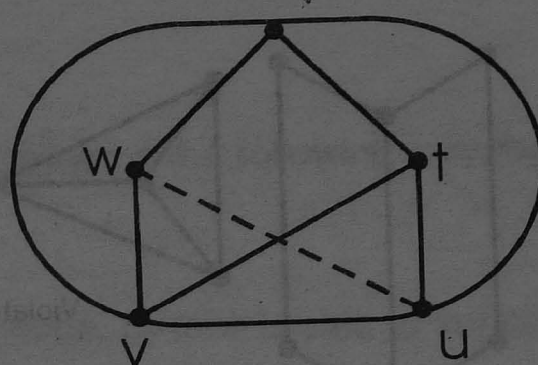
They must both lie outside the pentagon the edge vt cannot cross the edge SU .

Hence Vt must be inside the pentagon But now, the edge Vt and uw Cross each other giving a $\Rightarrow \Leftarrow$

Hence K_5 is non – planar.



<K5>

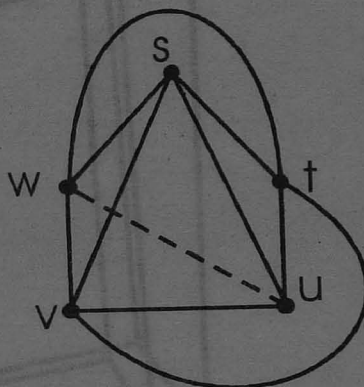


Case (ii):-

Suppose that wt is wholly outside the pentagon.

The argument is quite similar to that of case (i)

Hence K_5 is non-planar.



Hence the theorem.

1. PLATONIC SOLIDS:-

Our next theorem, probably known to ancient mathematicians is concerned with polyhedron.

DEFINITIONS:-

1). A Polyhedron is a finite connected set of plane polygons (called faces) arranged in R^3 in such a way that,

(i). each side of each of the polygons coincides with one and only one other polygon.

(ii). The intersection of any two polygons is either empty or is a side.

The nodes and the sides of the polygons are respectively called the nodes and the sides of the polyhedron.

2) A Polyhedron is said to be convex, if the line segment joining any two points belonging to two distinct polygons, has only its end points common with the polyhedron.

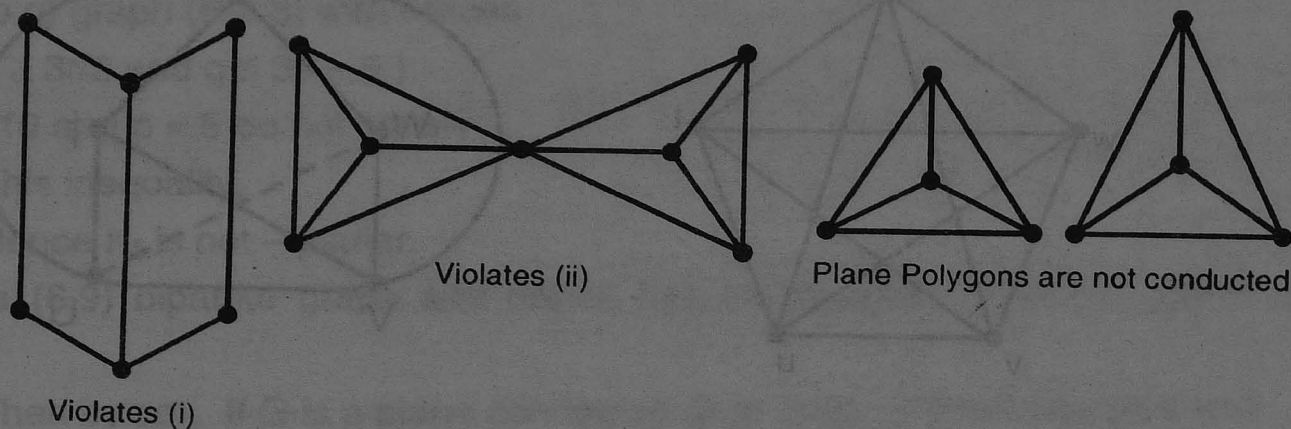


Fig 8.5 Arrangements of Polygons which do not yield Polyhedrons

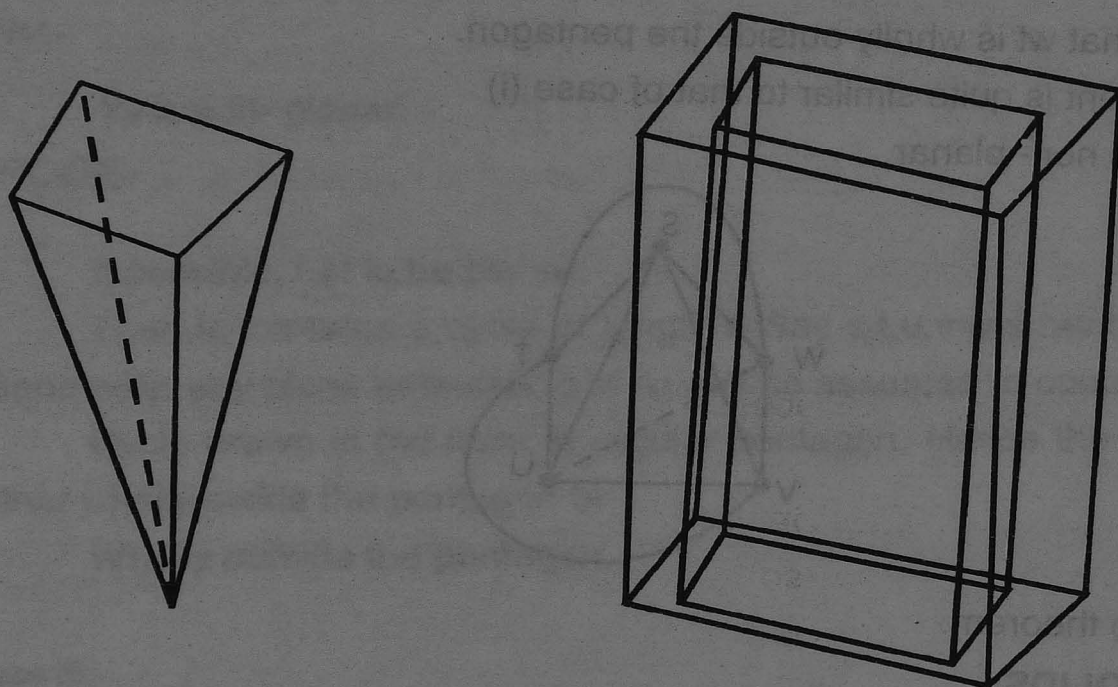


Fig 8.6 Two Polyhedrons which are not complex

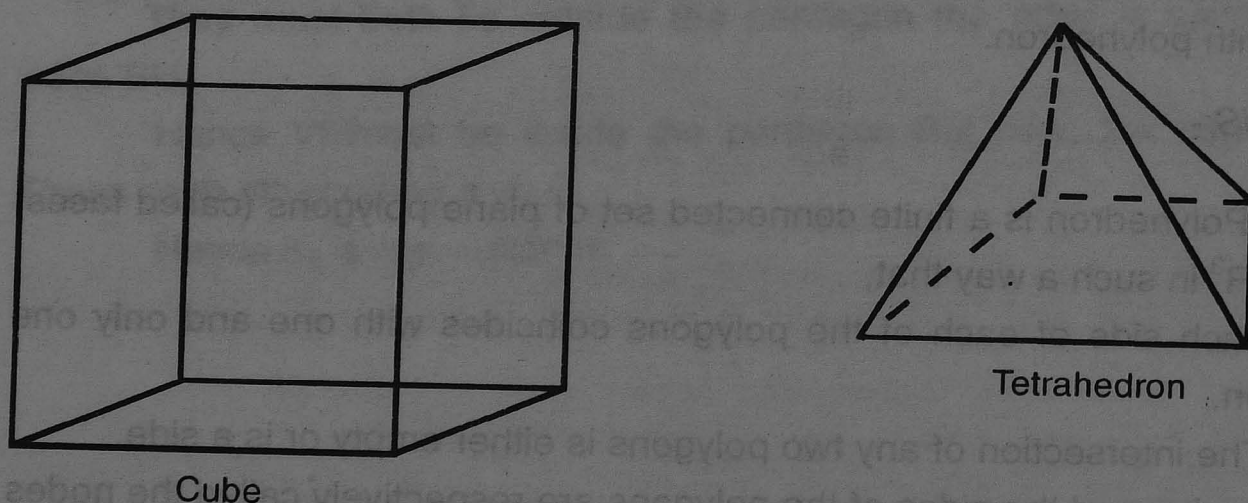


Fig 8.7 Two Convex Polyhedrons

Hereafter, all our polyhedrons are convex.

DEFINITIONS:-

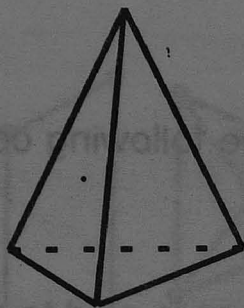
(1). A polyhedron is called a regular polyhedron if the following conditions are satisfied:

(i) its faces are congruent polygons.

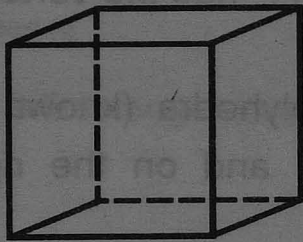
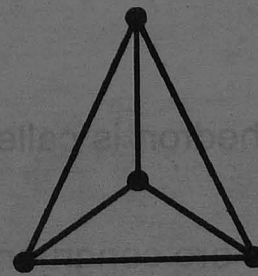
(ii) equal number of faces surround each node. (A face is said to surround a vertex, if the vertex belongs to the face)

(2) The (Polyhedral) graph $G(P)$ of a polyhedron P has the vertex set the set of nodes of P and the edge set the set of sides of P .

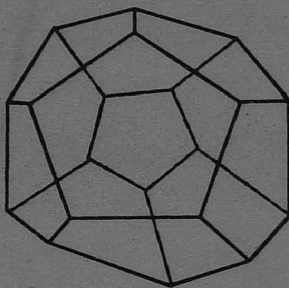
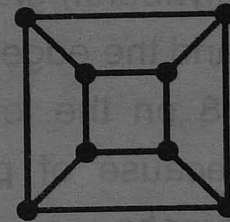
In figure 8.8 on the left we have 5 regular polyhedra (known also as platonic solids, because of plato's interest in them), and on the right their associated plane graphs.



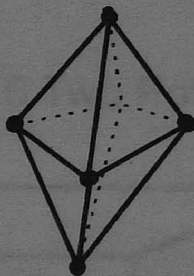
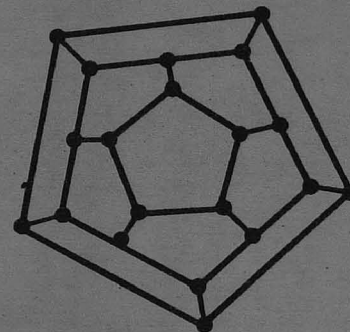
Tetrahedron



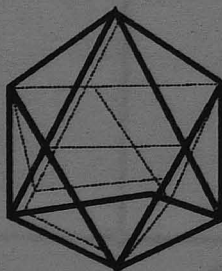
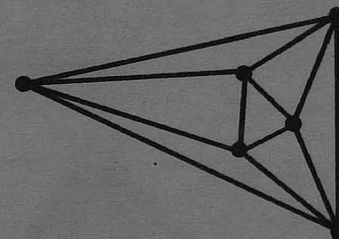
Cube



Dodecahedron



Octahedron



Icosahedron

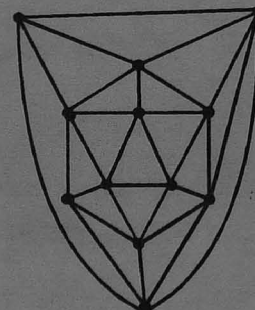


Fig 8.8

That the graph of a polyhedron is planar can be practically demonstrated as follows:-

Take a Polyhedron (P) made of elastic sheet, paint the nodes and the

sides of P by some colour. There upon make a hole in one of the faces of P . Stretch P so that P becomes a plane sheet with the circle of the hole as the boundary. Get a colour print of the nodes and the sides of P by pressing it on a paper. The colour print is precisely the graph of the polyhedron and it is plane.

THM:-

There are exactly five regular polyhedra.

PROOF:-

Let P be a regular polyhedron with v nodes, E sides and F faces. Since, the nodes, sides and faces of P are precisely the vertices, edges and faces of $G(P)$, we have

$$(i). V - E + F = 2$$

$$(ii). \sum_{\text{face } f \in P} \deg(f) = 2E$$

$$(iii). \sum_{\text{vertex } v \in P} \deg(v) = 2E$$

Since P is regular, every node has degree K (say) and every face has degree L (say) Then (ii) and (iii) reduce to $\ell f = 2E$ and $kv = 2E$ respectively

$$iv) KV = LF$$

Substituting $KV = LF$ in (i) we get

$$(v). (K-4) V \neq (L-4) F = -8.$$

Clearly, every face of a polyhedral graph is bounded by atleast 3 edges and so the minimum degree of faces is 3.

Hence, by corollary,

If G is a planar (p,q) - graph, Then $\delta(G) \leq 5$ and

If G is a plane (p,q) graph with $\delta(G) \geq 3$ then there is a face in G of degree ≤ 5 .

$$\therefore 3 \leq k \leq 5 \text{ and } 3 \leq \ell \leq 5.$$

Thus there are 9 possible solutions (k,L) of (V)

Case (1):-

$$(k,L) = (3,3)$$

By (iv) and (v), we get $v = F = 4$

Hence P is a regular polyhedron with 4 nodes, 4 faces with every node and every face of degree 3. So, P is the tetrahedron.

Case (2):-

$$(K,L) = (3,4)$$

From (iv) and (v), we deduce that $V= 8$ and $F = 6$. Thus P is a regular polyhedron, with 8 vertices and 6 faces, in which every node has degree 3 and every face has degree 4. Hence P is the cube.

Case (3):-

$$(K,L) = (3,5)$$

In this case (iv) and (v) yield $v = 20$ and $F = 12$. Thus P is the dodecahedron.

Case (4):-

$$(K,L) = (4,3)$$

By (iv) and (v), we have $V=6$ and $F = 8$ Hence P is the octahedron.

Case (5):-

$$(K,L) = (4,4)$$

with $K=4$ and $L = 4$, (v) reduces to $O = -8$ which is absurd. Hence $(4,4)$ is not a solution of (v)

Case (6):-

$$(K,L) = (4,5)$$

In this case, (v) reduces to $F = - 8$, and obviously there is no polydron with $F=-8$.

Case (7):-

$$(k,L) = (5,3)$$

using (iv) and (v) we get $V = 12$ and $F = 20$. Hence , P is the icosahedran.

Case (8):-

$$(K,L) = (5,4)$$

From (v), We get $V= - 8$; and clearly there is no polyhedron with $V = - 8$.

Case (9):-

$$(K,L) = (5,5)$$

In this case (v) reduces to $V + F = -8$; and clearly there is no polyhedron with $V+F=-8$.

4. DUAL OF A PLANE GRAPH:-

DEFINITION:-

Let G be a plane graph with $V(G) = \{ v_1, v_2, \dots, v_p \}$ $E(G) = \{ e_1, e_2, \dots, e_p \}$ and $F(G) = \{ f_1, f_2, \dots, f_r \}$. The dual (general) graph G^* of G has vertices $f_1^*, f_2^*, \dots, f_r^*$ ($f_i \leftrightarrow f_i^*$) and edges $e_1^*, e_2^*, \dots, e_q^*$ ($e_j \leftrightarrow e_j^*$) where the edge e_j^* , joins the vertices f_m^* and f_n^* iff the edge e_j is common to the faces f_m and f_n in G .

The following figure illustrates a drawing of G^* . In this drawing a point is

chosen in the interior of every region and two such points f_1^* , f_2^* are joined by a line e^* crossing the edge e only, where e is an edge common between the faces f_1 , f_2 .

Though the following statements are intuitively obvious from the definition of the dual and its above drawing, the proofs are tedious and hence they are omitted

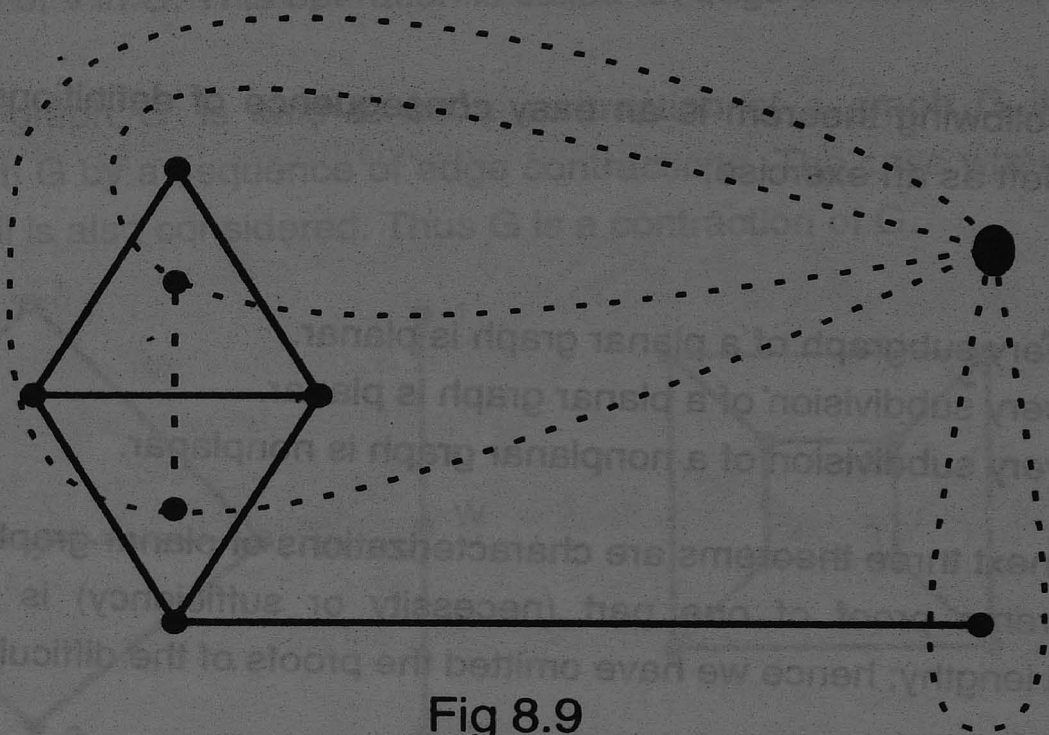


Fig 8.9

- (i) G^* is plane; infact the above drawing gives a plane embedding of G .
- (ii) $(G^*)^* = G$
- (iii) e is a cut-edge in G iff e^* is a loop in G^* .

NOTE:-

We have defined the dual of a plane graph and not a planar graph.

5. CHARACTERIZATION OF PLANAR GRAPHS:-

DEFINITIONS:-

1). The operation of replacing an edge $e = (u,v)$ in a graph G , by a path u,w,v ($w \notin V(G)$) of length z , is called a subdivision of e .

2). A graph H is said to be a subdivision of G if H can be obtained from G by a sequence of edge subdivisions. The case where we do not subdivide at all (ie, the sequence is empty) is also considered. Thus G is a subdivision of G .

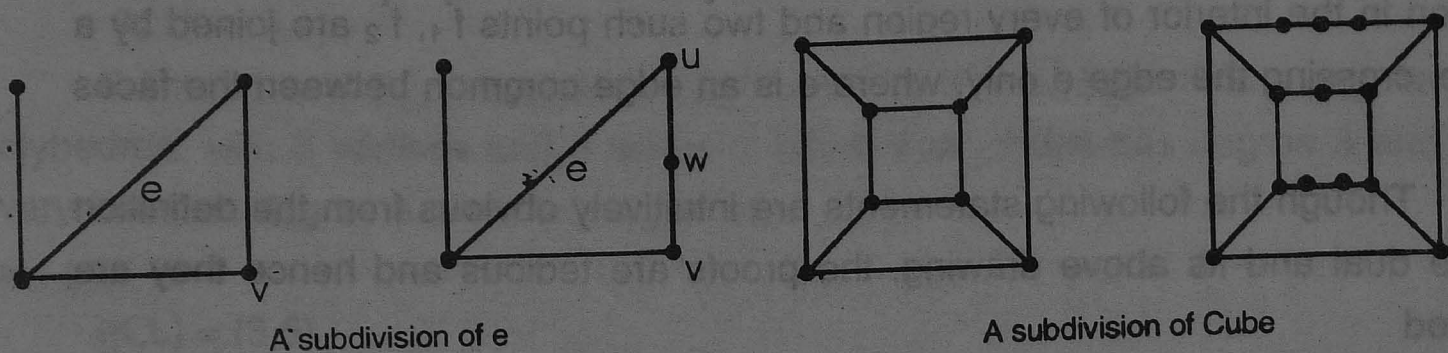


Figure 8.10

The following theorem is an easy consequence of definitions and hence the proof is left as an exercise.

THM:-

- Every subgraph of a planar graph is planar.
- Every subdivision of a planar graph is planar.
- Every subdivision of a nonplanar graph is nonplanar.

The next three theorems are characterizations of planar graphs. In each of these theorems proof of one part (necessity or sufficiency) is unfortunately difficult and lengthy; hence we have omitted the proofs of the difficult parts.

THM: (KURATOWASKI):-

A graph G is a planar iff G contains no subdivision of K_5 or $K_{3,3}$.

PROOF (NECESSITY):-

Suppose G is planar Let G contains a subdivision L of K_5 or $K_{3,3}$ Since K_5 and $K_{3,3}$ are nonplanar L is non planar, by above thm (c).

But then this contradicts, the above thm (a)

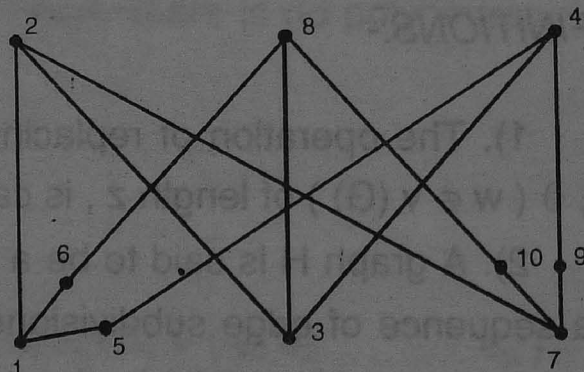
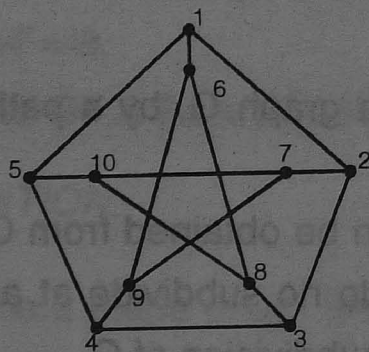


Fig 8.11 Peterson graph contains a subdivision $K_{3,3}$

Our next characterization is, in a sense, dual to Kuratowski thm.

To state it we need the following definitions.

DEFINITIONS:-

1). Let $e = (u,v)$ be an edge in graph G . Delete the vertices u,v from G . Add a new vertex w to $G - (u,v)$ and join w to all those vertices in $V - (u,v)$ which are adjacent to u or v in G . This operation is called an edge contraction of G .

2). A graph G' is said to be a contraction of a graph G if G' can be obtained from G by a sequence of edge contractions. The case where we do not contract at all is also considered. Thus G is a contraction of G .

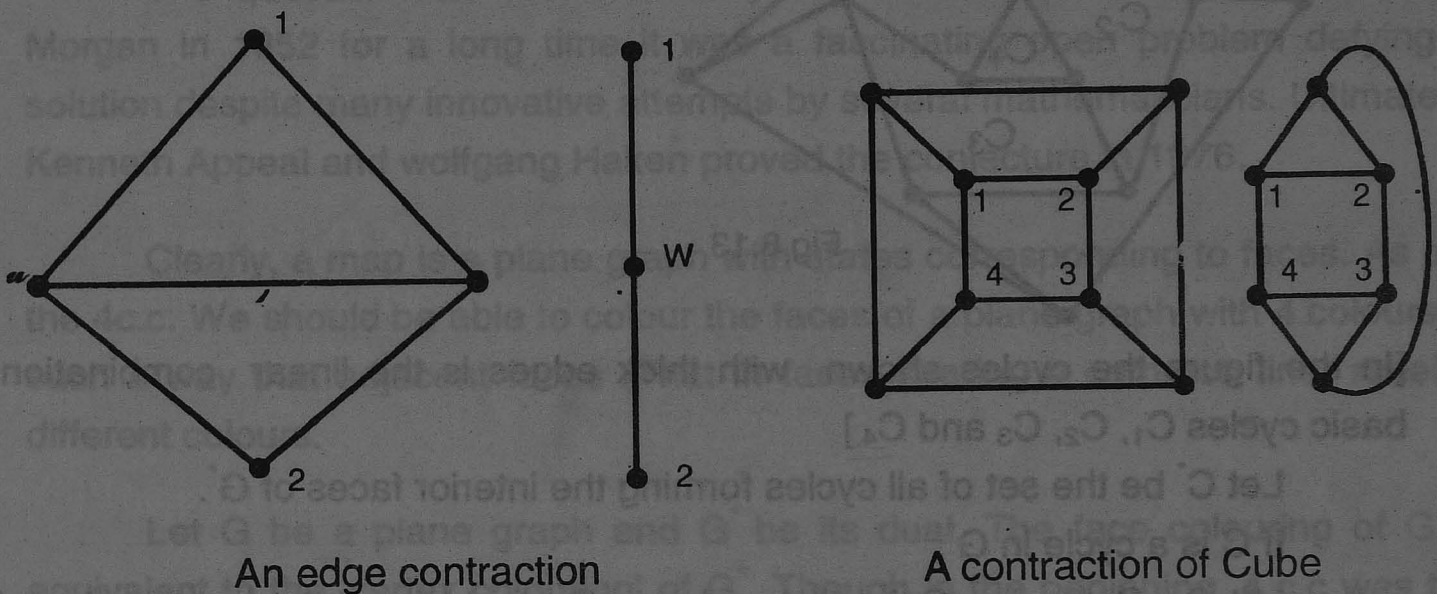


Fig 8.12

THM:- (WAGNER):-

A graph is planar iff it contains no contraction of K_5 or $K_{3,3}$

PROOF:- (SUFFICIENCY):-

Suppose G contains no contraction of K_5 or $K_{3,3}$. Let G be nonplanar.

By Kuratowski's theorem, G contains a subdivision L of K_5 or $K_{3,3}$. By contracting one of the two edges incident to vertices of degree 2 (if any) in L . We obtain a contraction of K_5 or $K_{3,3}$. This contradicts our hypothesis.

Hence G must be planar.

THEOREM:- (S.MACLANE):-

A connected graph G is planar iff there is a (cycle) basis of the cycle space $L(e)$ such that no cyclic edge of G appears in more than two fundamental cycles.

PROOF:-

Let G be a planar connected graph and Let G^* be a plane embedding of G .

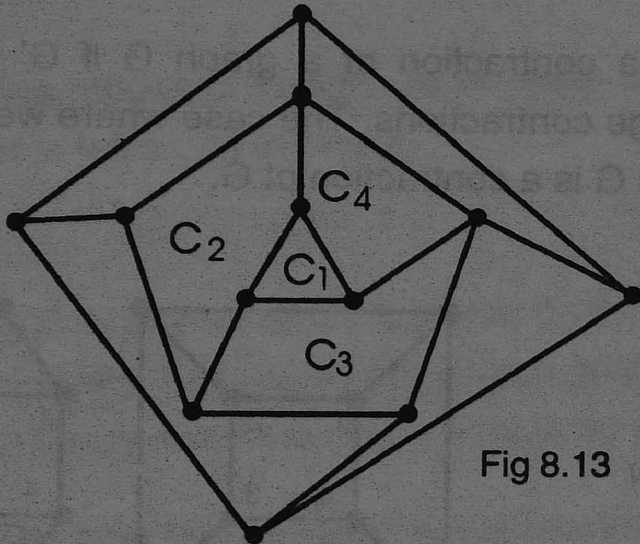


Fig 8.13

[In the figure the cycles shown with thick edges is the linear combination of basic cycles C_1, C_2, C_3 and C_4 .]

Let C^* be the set of all cycles forming the interior faces of G^* .

If C is a cycle in G^* ,

Let $C_1, C_2, \dots, C_t$ be the cycles in G^* that bound the inner region of C .

Then obviously $C = C_1 \oplus C_2 \oplus \dots \oplus C_t$ (see fig 8.13) so C^* spans $L(e)$.

By the Euler formula, $|C^*| = q - p + 1$.

Since the dimension of $L(e)$ is $q - p + 1$, it follows that C^* is a basis of $L(e)$.

Moreover, no cyclic edge of G^* belongs to more than two cycles of C^* .

Hence C^* is the required basis of $L(e)$.

UNIT - IX

1. COLOURINGS

When we colour the map of a country we see to it that its adjacent states are coloured differently, for easy identification of the states. One might ask: what is the minimum number of colours required to colour any map. So that adjacent states receive different colours ?

The famous four colour conjecture states that four colours are sufficient.

The question was first raised by Francis G. Guthrie, in a letter to De Morgan in 1852 for a long time it was a fascinating open problem defying a solution despite many innovative attempts by several mathematicians. Ultimately, Kenneth Appel and Wolfgang Haken proved the conjecture in 1976.

Clearly, a map is a plane graph with states corresponding to faces. As per the 4c.c. We should be able to colour the faces of a plane graph with 4 colours in such a way that adjacent faces (That is faces sharing a common line) receive different colours.

Let G be a plane graph and G^* be its dual. The face colouring of G is equivalent to the 'vertex colouring' of G^* . Though in the beginning, 4.c.c was the main motivation for the study of colourings, subsequently however, the concept of colourings has found applications in such diverse fields as the theory of automata and storage problems in operations research.

2. VERTEX COLOURING:

DEFN:-

1). A K - VERTEX COLOURING (OR BRIEFLY, A K - COLOURING) OF A GRAPH G IS AN ASSIGNMENT OF K - COLOURS TO THE VERTICES OF G SUCH THAT NO TWO ADJACENT VERTICES RECEIVE THE SAME COLOUR.

2). Colours are usually, though not always, represented by positive integers. Thus a K -colouring of G is a function $f: V(G) \rightarrow \{1, 2, \dots, k\}$. Such that $f(u) \neq f(v)$, whenever $(u, v) \in E(G)$.

A 5 - (4 - and 3 -) vertex colouring of the Peterson graph is shown in figure.

(1) EVERY GRAPH WITH P VERTICES IS $P -$ COLOURABLE.

(2) If a graph is k - colourable, then it is t - colourable for every integer $t, t \geq k$.

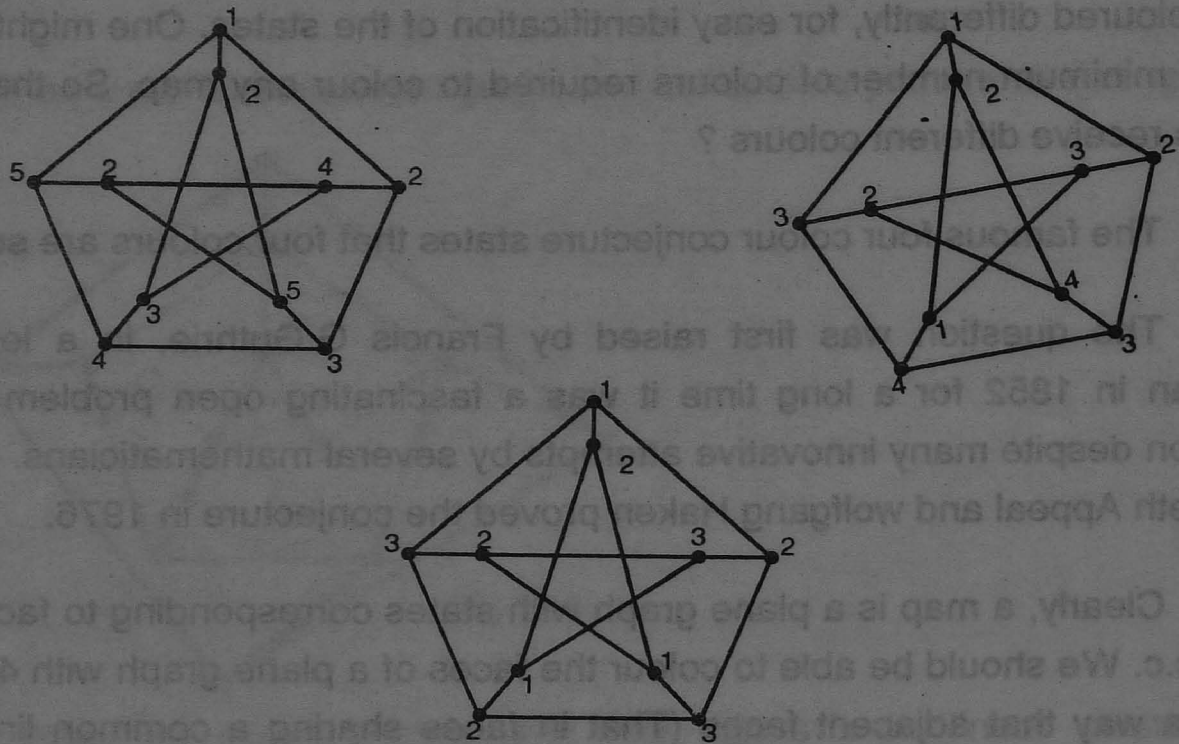


Fig 9.1

DEFINITIONS:-

The chromatic number of a graph G , denoted by $\psi (G)$, is the minimum number of colours required to colour. The vertices of G such that no two adjacent vertices receive the same colour. In other words, $\psi (G)$ is the minimum integer k , such that G has a k - colouring.

EXAMPLE:-

Graph	K_p	K_{p-x}	K_p	$K_{m,n}$	C_{2n}	C_{2n+1}
Chromatic Number	P	$P-1$	1	2	2	3

An alternative definition of a $K -$ colouring can be given by the following theorem.

THM:-

There exists a k - colouring of a graph G iff $V(G)$ can be partitioned into k subsets $V_1, V_2, \dots, V_k$ such that no two vertices in $V_i, i = 1, 2, \dots, K$ are adjacent.

PROOF:- (NECESSITY)

Let $f : V(G) \rightarrow (1, 2, \dots, k)$ be a K -colouring of G .

Let $V_i = \{ u \in V(G) : f(u) = i \}, (1 \leq i \leq k)$

$\therefore V_i$ is the set of vertices coloured i .

Then $V = V_1 \cup \dots \cup V_k$ is a partition and moreover no two vertices in V_i ($1 \leq i \leq k$) are adjacent.

SUFFICIENCY:-

Let $V = V_1 \cup V_2 \cup \dots \cup V_k$ be a partition of $V(G)$ such that no two vertices in V_i ($1 \leq i \leq k$) are adjacent.

Then the function $f: V(G) \rightarrow (1, 2, \dots, k)$, defined by $f(u) = i$, if $u \in V_i$ is a k -colouring of G .

Hence the theorem.

THM:-

If H is a subgraph of a graph G , then $\psi(H) \leq \psi(G)$ proof let $\psi(G) = k$ and let f be a k -colouring of G . Then f restricted to $V(H)$ is a k -colouring of H .

Hence,

$$\psi(H) \leq k = \psi(G)$$

$$\therefore \psi(G) \geq \psi(H).$$

THM:-

If G is a (p, q) -graph, then

$$\psi(G) \geq \frac{p^2}{p^2 - 2q}$$

PROOF:-

Let $\psi(G) = k$.

By thm,

There exists a k -colouring of a graph G iff $V(G)$ can be partitioned into k -subsets $V_1, V_2, \dots, V_k$ such that no two vertices in V_i , $i = 1, 2, \dots, k$ are adjacent.

There is a partition $V = V_1 \cup V_2 \cup \dots \cup V_k$ such that no two vertices in V_i ($1 \leq i \leq k$) are adjacent.

Let $|V_i| = P_i, 1 \leq i \leq k$.

So,

$$q(G) \geq \binom{P}{2} - \sum_{i=1}^K \binom{P_i}{2}$$

Hence

$$2q(G) \leq P^2 - P - \sum^k P_i^2 + \sum^v P_i$$

$$= P^2 - \sum^v P_i^2 \text{ (Since } \sum^k P_i = P \text{)}$$

$$\leq P^2 - (\sum P_i)^2$$

$$\text{(Since, by Cauchy - Schwart inequality } \sum^k P_i^2 \geq (\sum^k P_i)^2 \text{)}$$

$$= P^2 - \frac{P^2}{K}$$

$$\text{Hence } K \geq \frac{P^2}{P^2 - 2q}$$

$$\therefore \Psi(G) \geq \frac{P^2}{P^2 - 2q}$$

The above two results provide lower bounds for the chromatic number.

Our next result gives an upper bound.

THM:-

For any graph G , $\psi(G) \leq \Delta(G) + 1$

PROOF:-

We shall prove the theorem by induction on $P (= |V(G)|)$

If $P = 1$, then the result is obvious. Assume the result for all graphs on $P-1$ vertices and let G be a graph on $P (\geq 2)$ vertices. Let v be a vertex in G of maximum degree and. Let $v_1, v_2, \dots, v_{\Delta}$ be the vertices adjacent to v .

Consider $G - v$.

Using induction hypothesis,

$$(G - v) \leq \Delta(G - v) + 1$$

$$\leq \Delta(G) + 1$$

Hence, there exists a $(\Delta(G) + 1)$ - colouring of $G - v$.

We can now easily extend this colouring to a $(\Delta(G) + 1)$ - colouring of G . Since $\deg(v) = \Delta$, among $\Delta(G) + 1$ colours atleast one colour, say C . Does not appear among the vertices $v_1, v_2, \dots, v_{\Delta}$

Hence, colouring the vertex V with the colour C , we obtain a $(\Delta(G) + 1)$ - colouring of G . So,

$$\psi(G) \leq \Delta(G) + 1$$

THM:-

For any given integer $K (\geq 1)$ there exists a triangle free (that is without triangles) graph with chromatic number K .

PROOF:-

We shall construct the required graphs by induction on k .

For $K = 1$ and 2 ,

K_1 and K_2 are triangle – free graphs with chromatic numbers 1 and 2 respectively.

Now, suppose that we have a triangle free graph G_k with chromatic number K on n vertices $v_1, v_2, \dots, v_n$.

Construct a new graph G_{k+1} from G_k follows.

- 1) add $n + 1$ new vertices $u_1, u_2, \dots, u_n$ and v to G_k ;
- 2) join u_i ($1 \leq i \leq n$) to all those vertices to which v_i is adjacent in G_k .
- 3) join v to every u_i .

Figure 9.2 shows this construction of G_3 from G_2 ($= K_2$) and G_4 from G_3 ($= C_5$)

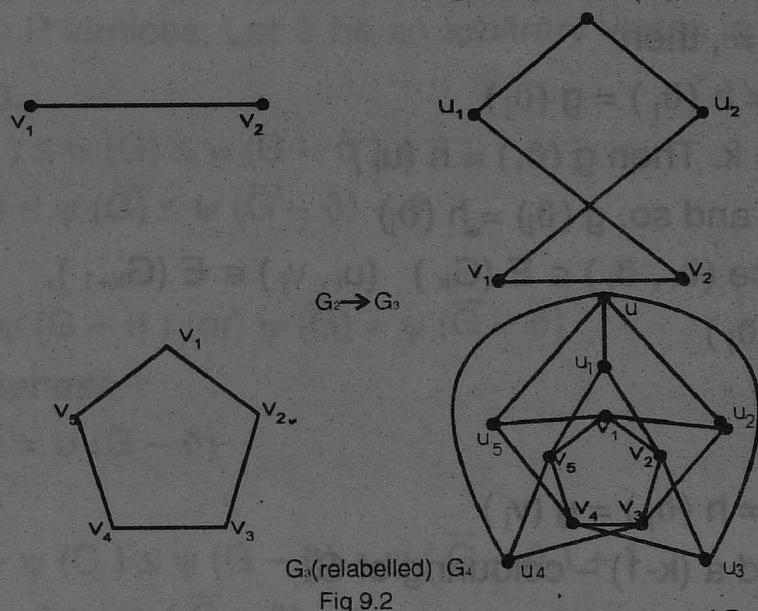


Fig 9.2

We shall prove that G_{k+1} is triangle – free and that $\psi(G_{k+1}) = K + 1$

- 1). G_{k+1} is triangle free.

Let, if possible T be a triangle in G_{k+1} . Since G_k is triangle – free, one of the vertices of T is v or u_i (for some i)

If $v \in T$, then the other two vertices of T are u_g and u_h (for some g and h)

This is a contradiction.

Since no two vertices in $\{u_1, u_2, \dots, u_n\}$ are adjacent.

If $u_i \in T$ and $v \in T$, then the other two vertices of T are v_g and v_h (for some g and h).

But then $\langle v_i, v_g, v_h \rangle$ is a triangle in G_k . Which again is a contradiction.

Hence G_{k+1} does not contain triangles.

- 2). $\psi(G_{k+1}) = K + 1$

Since G_k is a subgraph of G_{k+1} we have $\psi(G_{k+1}) \geq \psi(G_k) = k$.

If $f : V(G_k) \rightarrow \{1, 2, \dots, K\}$ is a k – colouring of G_k , Then $g : V(G_{k+1})$

$\rightarrow \{1, 2, \dots, K + 1\}$ defined by $g(v_i) = f(v_i)$, $1 \leq i \leq n$;

$$g(u_i) = f(v_i), 1 \leq i \leq n;$$

$g(v) = K + 1$, is a $(k+1)$ – colouring of G_{k+1}

Hence $\psi(G_{k+1}) \leq k + 1$

Let, if possible, $\psi(G_{k+1}) = k$; And Let $h : V(G_{k+1}) \rightarrow \{1, 2, \dots, k\}$ be a k colouring of G_{k+1} without loss of generality, Let $h(v) = k$. So, $h(u_i) \neq k$, for $i = 1, 2, \dots, n$.

Now, we define,

$$g : \{v_1, v_2, \dots, v_n\} \rightarrow \{1, 2, \dots, k-1\} \text{ by}$$

$$g(v_i) = h(v_i), \text{ if } h(v_i) \neq k,$$

$$= h(u_i), \text{ if } h(v_i) = k.$$

We shall prove that g is a $(k-1)$ – colouring of G_k which will be a contradiction to the fact that $\psi(G_k) = k$.

Let $(v_i, v_j) \in E(G_k)$

We shall show that $g(v_i) \neq g(v_j)$

If $h(v_i), h(v_j) \neq k$, then

$$g(v_i) = h(v_i) \neq h(v_j) = g(v_j)$$

So, Let $h(v_i) = k$. Then $g(v_i) = h(u_i)$

Also, $h(v_j) \neq k$ and so, $g(v_j) = h(v_j)$

Moreover, Since $(v_i, v_j) \in E(G_k)$ $(u_i, v_j) \in E(G_{k+1})$,

So $h(u_i) \neq h(v_j)$

This shows that

$$g(v_i) = h(u_i) \neq h(v_j) = g(v_j)$$

Thus g is indeed a $(k-1)$ – colouring of G_k .

This $\Rightarrow \Leftarrow$ shows that $\psi(G_{k+1}) \geq k + 1$

Hence, $\psi(G_{k+1}) = k + 1$

Hence the theorem.

THM:-

If G is a graph on p vertices, then

$$1). 2 \sqrt{p} \leq \psi(G) + \psi(\bar{G}) \leq p + 1;$$

$$2). P \leq \psi(G) \psi(\bar{G}) \leq \frac{(P+1)^2}{4}$$

4

PROOF:-

$$1). \psi(G) + \psi(\bar{G}) \geq p.$$

Let $\psi(G) = K$.

By thm, There exists a k – colouring of a graph G iff $V(G)$ can be partitioned into k – subsets $V_1, V_2, \dots, V_k$ such that no two vertices in $V_i, i = 1, 2, \dots$

...k are adjacent.. There exists a partition $V(G) = V_1 \cup V_2 \cup \dots \cup V_k$ such that no two vertices in V_i ($1 \leq i \leq k$) are adjacent

Hence, $\bar{G}$ contains a K_t , where $t = \max_{1 \leq i \leq k} |V_i|$

So, $\psi(\bar{G}) \geq t \geq P/k$

1e) $\psi(G) \cdot \psi(\bar{G}) \geq P$

2). $\psi(G) + \psi(\bar{G}) \geq 2\sqrt{P}$.

This follows from (1) and the inequality $\underline{a+b} \geq \sqrt{ab}$, for any two (+ve) integers a and b. 2

3). $\psi(G) + \psi(\bar{G}) \leq P + 1$

We shall prove this inequality by induction on P. If $P=1$, then the inequality clearly holds. So, assume the inequality for all graphs on $P-1$ vertices. And Let G be a graph on P vertices. Let v be an arbitrary vertex in G and consider $G - v$. We always have,

$$\psi(G - v) \leq \psi(G) \leq \psi(G - v) + 1;$$

$$\psi(\bar{G} - v) \leq \psi(\bar{G}) \leq \psi(\bar{G} - v) + 1.$$

Case (i) :-

$$\psi(G) = \psi(G - v) \text{ (or) } \psi(\bar{G}) = \psi(\bar{G} - v)$$

for definiteness,

$$\text{Let } \psi(\bar{G}) = \psi(\bar{G} - v)$$

We have

$$\psi(G) + \psi(\bar{G}) \leq \psi(G - v) + \psi(\bar{G} - v) + 1$$

$$= \psi(G - v) + \psi(\bar{G} - v) + 1$$

$$[\text{Since } (\bar{G} - v) = (\bar{G} - v)]$$

$$\leq (P-1) + 1 + 1 \text{ (by induction)}$$

$$= P + 1$$

$$\text{Hence } \psi(G) + \psi(\bar{G}) \leq P + 1.$$

Case(2):-

$$\psi(G) = \psi(G - v) + 1 \text{ and } \psi(\bar{G}) = \psi(\bar{G} - v) + 1.$$

By thm,

If v is a vertex in G such that

$$\psi(\bar{G}) = \psi(\bar{G} - v) + 1, \text{ then } \deg_G(v) \geq \psi(\bar{G} - v)$$

$$\psi(\bar{G}) = \psi(G - v) + 1$$

$$\Rightarrow \deg_G(v) \geq \psi(G - v)$$

$$\text{So, } \psi(G) + \psi(\bar{G}) = \psi(G - v) + \psi(G - v) + 2$$

$$\leq \deg_G(v) + \deg_G(v) + 2$$

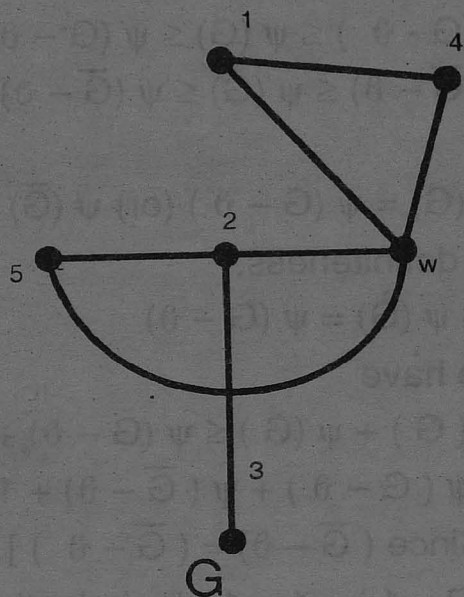
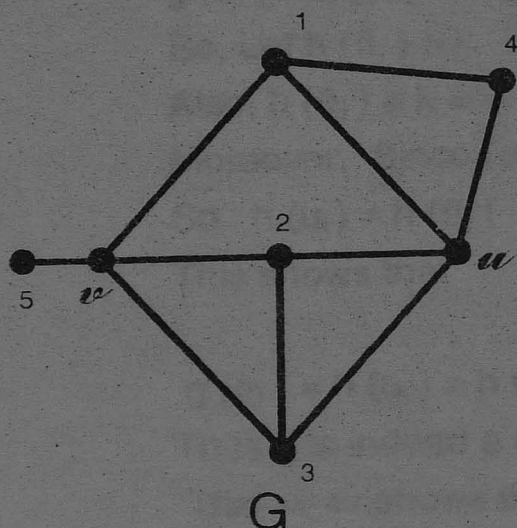
$$\leq (P-1) + 2 = P + 1.$$

$$4) \psi(G) \cdot \psi(\bar{G}) \leq \frac{(P+1)^2}{4}$$

This follows by (3) and the inequality, $(a+b)^2 \geq 4ab$, for any two integers a and b . Hence the theorem.

DEFINITION:-

Let G be a graph and let u and v be two non-adjacent vertices in G . Let H be the graph obtained from $G - \{u, v\}$ by adding a new vertex w and joining it with every vertex which is adjacent with u or v . (If multiple edges are formed, then replace them by single edges). H is called a graph obtained from G by identifying the vertices u and v ; and it is denoted by $G.uv$.



THM:-

Let G be a graph and let u and v be non – adjacent vertices in G . Then

$$\psi(G) = \min \{ \psi((G) + (u, v)), \psi(G.uv) \}$$

PROOF:-

$$1). \psi(G) \leq \min \{ \psi((G) + (u, v)), \psi(G.uv) \}$$

Let $\psi(G.uv) = k$ and Let f be a k – colouring of $G.uv$.

Then $g : V(G) \rightarrow \{1, 2, \dots, k\}$ Defined by

$$g(x) = f(x), \text{ if } x \in V(G) - \{u, v\}$$

$$= f(w), \text{ if } x = u \text{ or } v.$$

is a k – colouring of G .

Hence,

$$\psi(G) \leq k = \psi(G.uv)$$

We always have

$$\psi(G) \leq \psi(G + (u, v))$$

Since G is a subgraph of $G + (u, v)$

$$\text{Hence } \psi(G) \leq \min \{ \psi(G + (u, v)), \psi(G - uv) \}$$

$$2). \psi(G) \geq \min \{ \psi(G + uv), \psi(G - uv) \}$$

Let $\psi(G) = k$ and let f be a k -colouring of G

Then either $f(u) = f(v)$ or $f(u) \neq f(v)$.

In the former case,

$$g : V(G - uv) \rightarrow \{1, 2, \dots, k\}$$

defined by

$$g(x) = f(x), \text{ if } x \in V(G - uv) - \{u, v\}$$

$$= f(u), \text{ if } x = v \text{ is a } k\text{-colouring of } G.$$

$$\text{Hence } \psi(G) = \min \{ \psi(G + uv), \psi(G - uv) \} \leq \psi(G - uv) \leq k$$

In the latter case

$$h : V(G + (u, v)) \rightarrow \{1, 2, \dots, k\}$$

defined by

$$h(x) = f(x), x \in V(G + (u, v)) = V(G) \text{ is a } k\text{-colouring of } G + (u, v)$$

Hence,

$$\min \{ \psi(G + (u, v)), \psi(G - uv) \} \leq \psi(G + (u, v)) \leq k = \psi(G)$$

Now the theorem follows by (1) and (2)

Based on this theorem, we now describe an algorithm to determine the chromatic number of a graph G .

However, this algorithm is not a 'good algorithm' in the terminology of the theory of algorithms.

Step (1) :-

Set $G = G_0$; and let to be the sequence: G_0 .

Step (2) :-

Having constructed a sequence ; $G_0, G_1, \dots, G_m$ of graphs, go to step 3 if some G_L ($0 \leq L \leq m$) is incomplete; go to step 4, if every G_L is complete.

Step (3) :-

Replace every incomplete G_L ($0 \leq L \leq m$) in ; by $G_L + (u, v)$ and $G_L - uv$ where u and v are arbitrarily chosen non-adjacent vertices in G_L ; call the resulting sequence and go to step 2 with

Step (4):-

Stop; $\psi(G) = \min \{ |V(G)|, |V(G_2)|, \dots, |V(G_m)| \}$ (by this thm)

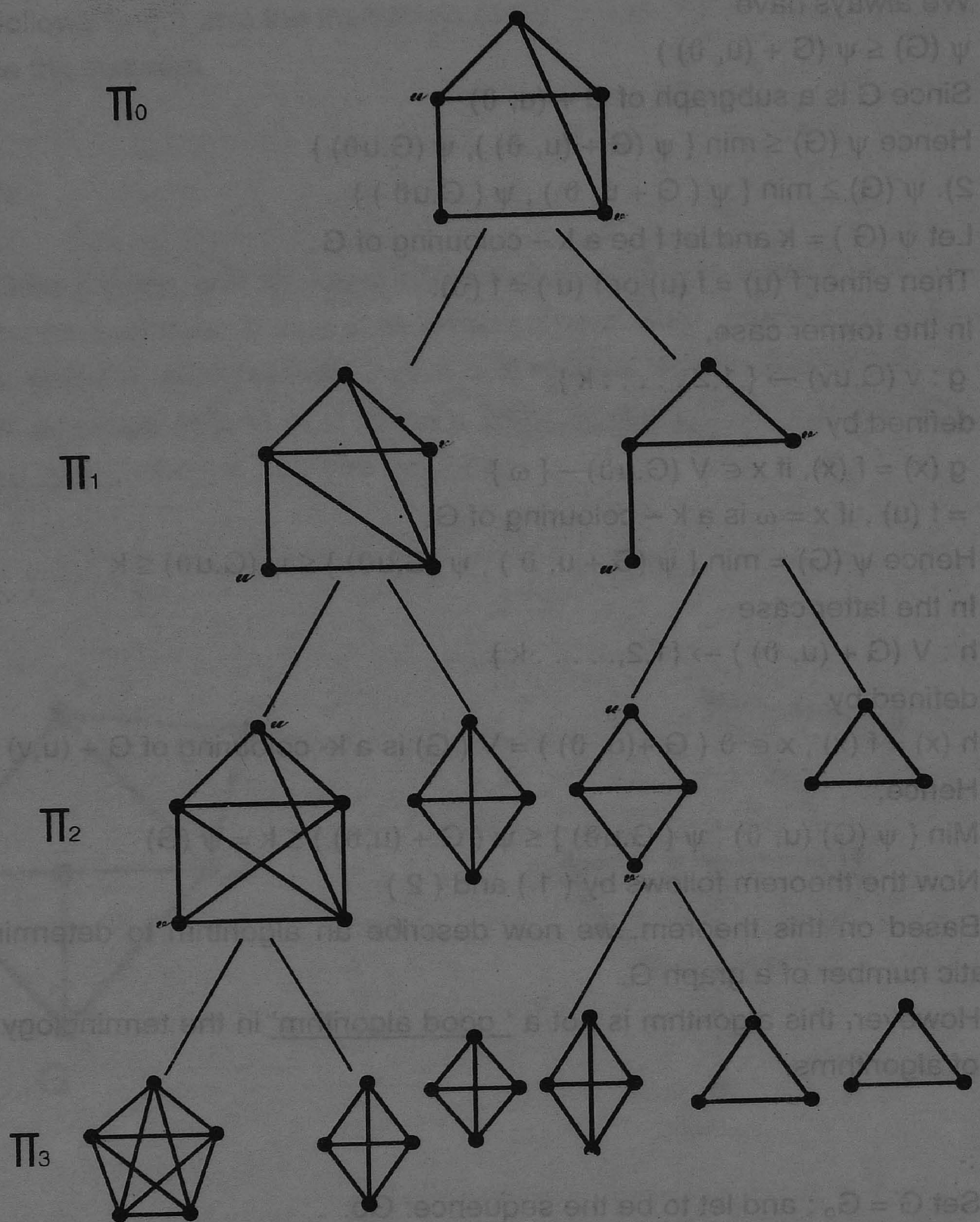


Fig 9.4 The Minimum order of complete graphs in π_3 is 3

Hence, $\psi(G) = 3$ (by this above theorem)

3. EDGE COLOURING:-

DEFINITIONS:-

A K – edge colouring of a graph G is an assignment of K colours to the edges of G such that no two adjacent edges receive the same colour.

Thus a k – edge colouring is a function $f : E(G) \rightarrow \{1, 2, \dots, K\}$ such that $f(x) \neq f(y)$, whenever x and y are adjacent in G .

A graph G is said to be K – edge colourable if there exists a k - edge colouring of G .

The minimum k such that G has a k – edge colouring is said to be the edge – chromatic number of G and is denoted by $\psi_1(G)$.

Thus ψ_1 is the minimum number of colours required to colour the edges of G such that no two adjacent edges receive the same colour.

EXAMPLE:-

A 4 – edge colouring of the Peterson graph is shown in Figure 9.5

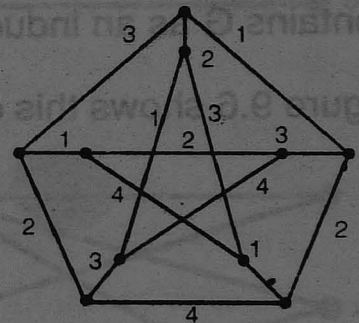


Fig 9.5

THM:-

If G is a bipartite graph (with $q(G) \geq 1$), then $\psi_1(G) = \Delta(G)$

PROOF:-

First we shall construct a $\Delta(G)$ – regular bipartite graph H , containing G as an induced subgraph.

Let $[X, Y]$ be a bipartition of $V(G)$

Let $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$

Let $G' [X' Y']$ be an isomorphic $(x_i : \leftrightarrow x'_i \in X', y_j \leftrightarrow y'_j \in Y')$ of $G (x, y)$ construct the bipartite graph H_1 with bipartition (XUY', YUX') from G and G' by adding the following edges to $E(G) \cup E(G')$.

(i). (x, x') where $x \in X, x' \in X^1$, and $\deg_G(x) = (\deg_{G'}(x')) < \Delta(G)$,

(ii) (y, y') where $y \in Y, y' \in Y'$ and $\deg_G(y) = (\deg_{G'}(y')) < \Delta(G)$

H_1 is a bipartite graph with $\Delta(H_1) = \Delta(G)$ and $\delta(H_1) = \delta(G) + 1$

If $\delta(G) + 1 = \Delta(G)$.

Then H_1 is a $\Delta(G)$ – regular bipartite graph with G as an induced subgraph so we have constructed the required graph.

If $\delta(G) + 1 < \Delta(G)$,

Then take an isomorphic copy of H_1 and construct as above, a bipartite graph H_2 which contains H_1 (and hence G) as an induced subgraph.

H_2 satisfies $\Delta(H_2) = \Delta(G)$ and

$\delta(H_2) = \delta(H_1) + 1 = \delta(G) + 2$

If $\delta(G) + 2 = \Delta(G)$

Then H_2 is the required graph

If $\delta(G) + 2 < \Delta(G)$, we construct as above a bipartite graph H_3 containing H_2 (and hence G) as an induced subgraph.

We repeat these constructions enough number of times to obtain ultimately a $\Delta(G)$ – regular bipartite graph H with $\Delta(H) = \delta(H) = \Delta(G)$, which contains G as an induced subgraph.

Figure 9.6 shows this construction.



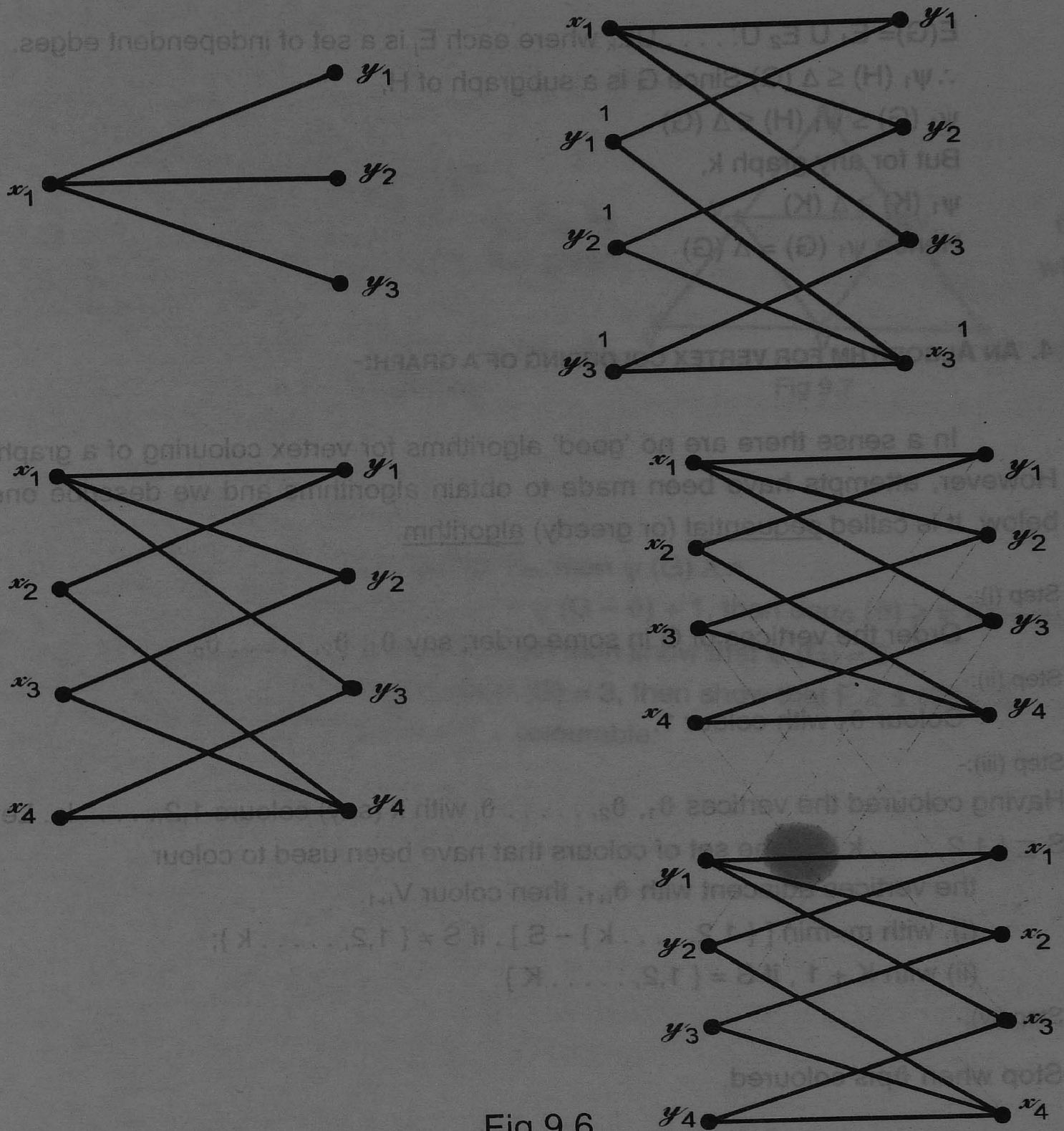


Fig 9.6

We now appeal to corollary,

Let $G[A, B]$ be a K -regular ($K \geq 1$) bipartite graph. Then $E(G)$ can be partitioned into K sets $E_1, E_2, \dots, E_K$ where each E_i is a set of n ($= |A|$) independent edges.

According to it, there is a partition of $E(H)$ into $\Delta(G)$ independent subsets of edges.

Hence by thm,

There exists a K -edge colouring of a graph G , iff there exists a partition $E(G)$

$E(G) = E_1 \cup E_2 \cup \dots \cup E_k$ where each E_j is a set of independent edges.

$\therefore \psi_1(H) \leq \Delta(G)$ Since G is a subgraph of H ,

$\psi_1(G) \leq \psi_1(H) \leq \Delta(G)$

But for any graph k ,

$\psi_1(K) \geq \Delta(K)$

Hence $\psi_1(G) = \Delta(G)$

4. AN ALGORITHM FOR VERTEX COLOURING OF A GRAPH:-

In a sense there are no 'good' algorithms for vertex colouring of a graph. However, attempts have been made to obtain algorithms and we describe one below. It is called sequential (or greedy) algorithm.

Step (i):-

Order the vertices of G in some order; say $v_1, v_2, \dots, v_p$.

Step (ii):-

Colour v_1 with colour 1.

Step (iii):-

Having coloured the vertices $v_1, v_2, \dots, v_j$ with k (say) colours $1, 2, \dots, k$. Let $S \subseteq \{1, 2, \dots, k\}$ be the set of colours that have been used to colour the vertices adjacent with v_{i+1} ; then colour v_{i+1} .

(i). with $m = \min [\{1, 2, \dots, k\} - S]$, if $S \neq \{1, 2, \dots, k\}$;

(ii) with $K + 1$, if $S = \{1, 2, \dots, K\}$

Step (iv):-

Stop when v_p is coloured.

NOTE:-

(1). Step 3 ensures that no two adjacent vertices receive the same colour; so the algorithm is a proper colouring of G .

(2). The algorithm may not colour with the minimum number of colours. The number depends on the order we choose in step.1.

EXAMPLE:-

Consider the following graph

If we choose the ordering : v_1, v_2, v_3, v_5, v_6 then the algorithm colours the graph 4 colours;

If we choose the ordering; v_6, v_2, v_4, v_1, v_3 then it colours with 3 colours.

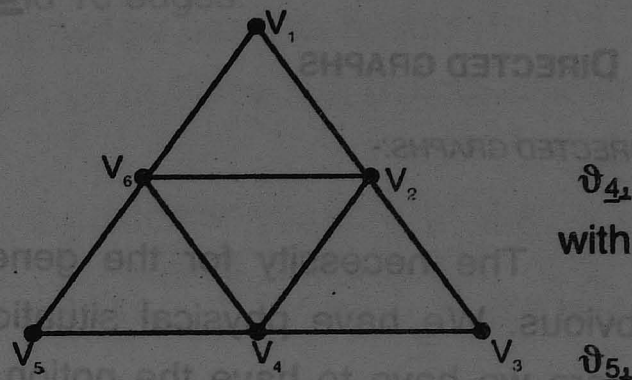


Fig 9.7

EXERCISES:-

1. If G contains a complete subgraph K_n , then $\psi(G) \geq n$.
2. If v is a vertex in G such that $\psi(G) = \psi(G - v) + 1$, then $\deg_G(v) \geq \psi(G - v)$.
3. If G has exactly one cycle of odd length then show that $\psi(G) = 3$.
4. If G is a triangle (P, q) – graph with $\psi(G) = 3$, then show that $P \geq \Delta(G) + 3$.
5. Prove that every planar graph is 6 – colourable.

UNIT - X

1. DIRECTED GRAPHS

DIRECTED GRAPHS:-

The necessity for the generalization of graphs to directed graphs is obvious. We have physical situations, like one-way road maps, tournaments, where we have to have the notion of directedness. Moreover, a graph can be thought of as a special kind of directed graph. The meaning of this statements will be clear soon.

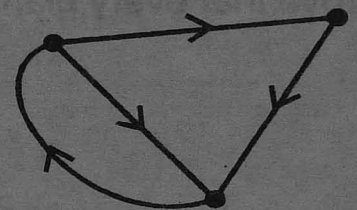
DEFINITIONS:-

A directed graph (diagraph, for short) is an ordered pair $(\mathfrak{V}, A)$ where $\mathfrak{V}$ is a non-empty (finite) set and A is a subset of $\{ (u, v) \in V \times V ; u \neq v \}$

The elements of V are called vertices (or) points and those of A are called arcs (or) directed edges.

EXAMPLE:-

$D = \{ (1,2,3,4) , (1,2) , (2,3) , (1,3) , (3,1) \}$ is a digraph. The diagram representing D is given in Fig 10.1



DEFINITION:-

The indegree $d^-(\mathfrak{v})$ of a vertex $\mathfrak{v}$ in a digraph D is the number of arcs having $\mathfrak{v}$ as its terminal vertex. Fig 10.1

The out degree $d^+(\mathfrak{v})$ of a vertex $\mathfrak{v}$ is the number of arcs having $\mathfrak{v}$ as its vertex. The ordered pair $(d^+(v) , d^-(v))$ is called the degree pair of $\mathfrak{v}$.

EXAMPLE:-

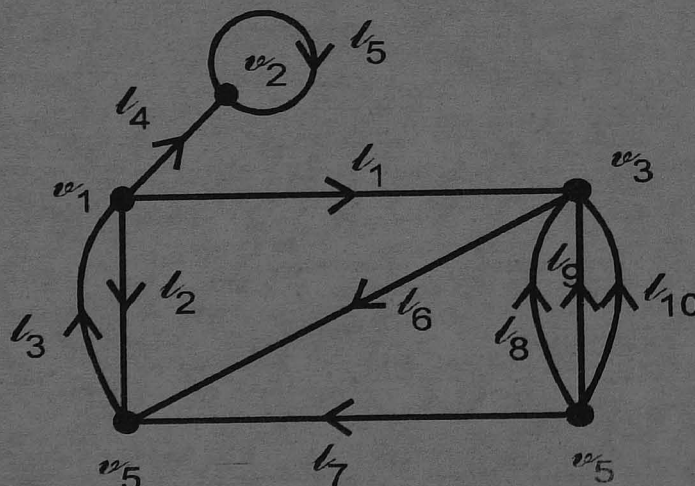


Fig 10.2

Figure 10.2 Directed graph with 5 vertices and 10 edges.

$$d^+(\vartheta_1) = 3, d^-(\vartheta_1) = 1.$$

$$d^+(\vartheta_2) = 1, d^-(\vartheta_2) = 1$$

$$d^+(\vartheta_5) = 4, d^-(\vartheta_5) = 0$$

$$d^+(\vartheta_3) = 1, d^-(\vartheta_3) = 4$$

$$d^+(\vartheta_4) = 1, d^-(\vartheta_4) = 3$$

NOTE:-

- I. An isolated vertex is a vertex in which the in-degree and the out-degree are both equal to zero.
- II. A vertex ϑ in a digraph is called pendant if it is of degree zero.
i.e)if $d^+(v) + d^-(v) = 1$.

NOTE:-

Our definition of a digraph precludes multiple directed arcs and loops.

We now define various kinds of subdigraphs of a directed graph $D = (V, A)$

DEFINITIONS:-

1. A digraph $D' = (W, B)$ is a subdigraph of D , if $W \subseteq V$ and $B \subseteq A$.
2. A Subdigraph $D' = (W, B)$ of D is an induced subdigraph of D if $W \subseteq V$ and $B = A \cap (W \times W)$. Thus two vertices $u, \vartheta \in \omega$, form an Arc in D . lff they form an as c in D .
3. If $S \subseteq V$, then $D - S$ is that sub digraph of D whose vertex set is $v-s$ and arc set is $A \cap (V-S \times V-S)$; thus $D - S$ is obtained from D by deleting all the vertices in S an all the arcs which have at least one vertex in S .
 $D - (\vartheta)$ is denoted by $D - \vartheta$
4. If $B \subseteq A$, then $D - B$ is the sub digraph of D whose vertex set is V and arc set is $A - B$ $D - (e)$ is denoted by $D - e$.

2. CONNECTIVITY IN DIGRAPHS:-

DEFINITION:-

Let $D = (V, A)$ be a digraph. An alternating sequence

$W(u, \vartheta) : (u) = u_0, a_0, u, a, u_2 \dots \dots u_{k-1}, a_{k-1}, u_k (= \vartheta)$ of vertices and arcs where each arc a_t ($0 \leq t \leq k-1$) is (u_t, u_{t+1}) is called a directed walk from u to ϑ ; if $u = \vartheta$, then w is called a closed walk.

The length of W is the number of occurrences of arcs in W .

The directed walk W is called a directed trail if all the arc in w are distinct.

A directed walk w is called a directed path if all the vertices (and hence arcs) in w are distinct.

A closed directed walk from u to v is a directed cycle if all its vertices are distinct except that $u = v$.

DEFINITION:-

The underlying graph G of a directed graph $D = (V, A)$ has the vertex set V and the edge set $E = \{ (u, v) ; (u, v) \in A \}$

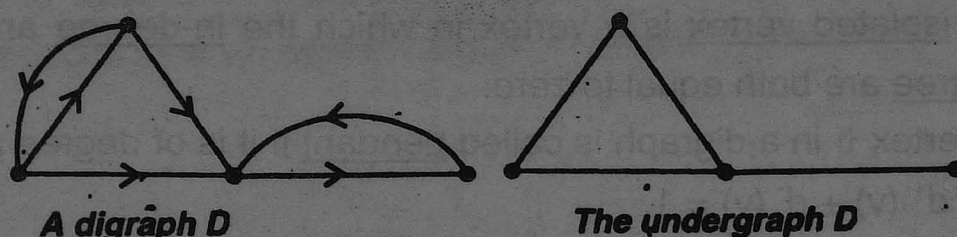


Fig 10.3

DEFINITIONS:-

1. A digraph is said to be weakly connected if its underlying graph is connected, otherwise it is said to be disconnected.
2. A digraph is said to be unilaterally connected if given any two vertices u and v either there exists a directed path from u to v or there exists a directed path from v to u .
3. A digraph is said to be strongly connected (or briefly, strong) if given any two vertices u and v there exists a directed path from u to v and there exists a directed path v to u .

It is obvious that the following implications hold.

Strong connectedness $\Rightarrow$ Unilateral connectedness

$\Rightarrow$ Weakly connectedness

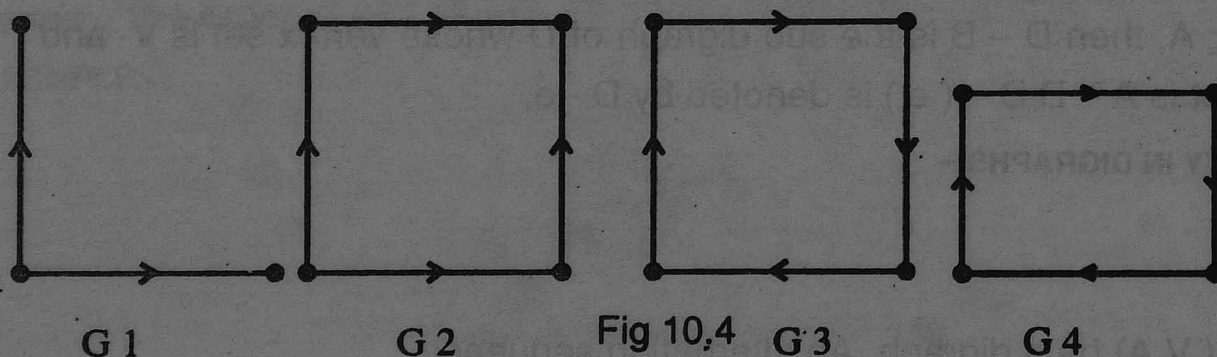


Fig 10.4

In Figure 10.4

G_1 is disconnected ; G_2 is weakly connected but not unilaterally connected.

G_3 is unilaterally connected but not strongly connected and

G_4 is strongly connected

THM:-

A digraph D is strongly connected iff D contains a directed closed walk containing all its vertices.

PROOF:-

1). Let D be a strongly connected digraph. Let $u, v \in V(D)$

There exist directed paths $p(u, v)$ and $p(v, u)$. Hence there is a closed walk in D , namely $p(u, v), p(v, u)$.

Let $W(u, u)$ be a closed walk containing maximum number of vertices. Because of its maximality, We claim that, $W(u, u)$ contains all the vertices of D .

On the contrary, Suppose that there is a vertex w which is not in $W(u, u)$.

Since D is strongly connected, we can show, as above, the existence of a closed directed walk $\omega'(u, u) : P'(u, w), P'(w, u)$. But then $W(u, u), \omega'(u, u)$ is a closed walk containing at least one more vertex namely w .

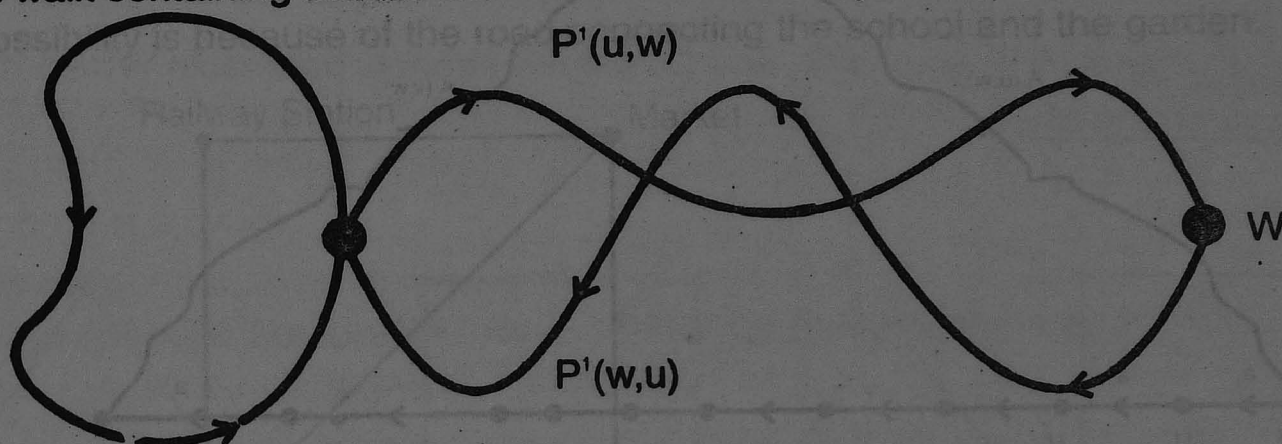


Fig 10.5

This contradiction to the maximality of $W(u, u)$

To prove that $W(u, u)$ indeed contains all the vertices in D .

2) Let D have a directed closed walk z containing all the vertices of D . and,

Let u, v be two arbitrary vertices in D . z contains a directed walk, say $W(u, v)$ from u to v . Among all such walks from u to v .

Let $P(u, v)$ be a directed walk of minimum length. Because of the minimality, We claim that no vertex is repeated in $P(u, v)$. On the contrary suppose that there is a vertex w which occurs more than once in $P(u, v)$.

Let $P(u, v)$ be $(u =) u_1, a_1, u_2, \dots, a_{r-1}, w, a_{t+1}, \dots, u_k (= v)$

is directed path of length less than that of $p(u, v)$. This contradiction shows that $p(u, v)$ is indeed a directed path from u to v . Similarly, we can prove the existence of a directed path from v to u .

Since, u and v are arbitrary. We have proved that D is strong.

Hence the thm.

THM:-

A digraph D is unilaterally connected iff there is a directed walk (not necessarily closed) containing all its vertices.

PROOF:-

(i) Let D be a unilaterally connected digraph.

Let $W(u, v)$ be a directed walk containing maximum number of vertices among all walks in G .

We claim that $W(u, v)$ contains all the vertices in D . On the contrary, Suppose that there is a vertex w not in $w(u, v)$

Let $w(u, v)$ be $(u =) u_1, a_1, u_2, a_2, \dots, u_{n-1}, a_{n-1}, u_n (= v)$

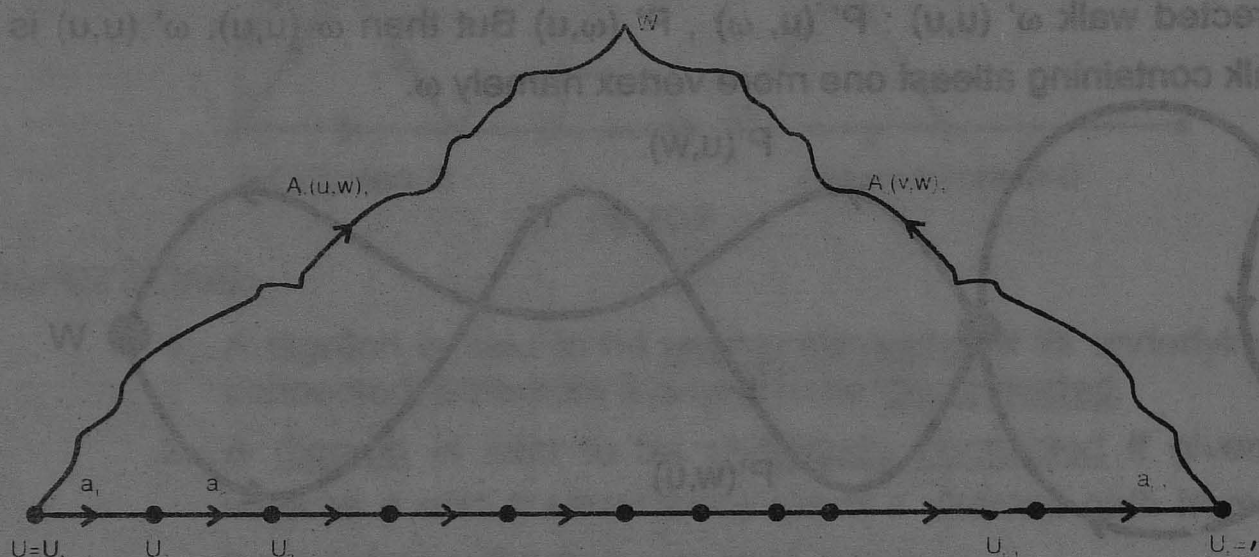


Fig 10.6

ASSERTION – 1:-

If there is a directed walk $A_j(u_j, w)$ from u_j to w , ($1 \leq j \leq n - 1$), Then there is no directed walk $B_j(w, u_{j+1})$ from w to u_{j+1} . On the contrary,

If both the walks exists then the directed walk.

$W'(u, v) : (u =) u_1, a_1, u_2, \dots, u_j, A_j(u_j, w), w, B_j(w, u_{j+1}),$

$u_{j+1}, a_{j+1}, u_{j+2}, \dots, u_{n-1}, u_n (= v)$ has more number of vertices than $w(u, v)$ – a contradiction.

ASSERTION 2:-

There is a directed walk $A_1(u, w)$

On the contrary, If there is no such walk, then there is a directed walk $B_1(w, u_1)$

Since D is unilaterally connected. But then the directed walk, $W''(w, v) : B_1(w, u_1), W(u_1, v)$ has more number of vertices than $w(u, v)$ – a contradictions.

Since D is unilaterally connected, Assertions (1) and (2) imply that D contains directed paths $A_2(u_2, w), \dots, A_{n-1}(u_{n-1}, w)$ and $A_n(u_n, w)$

But then the directed walk

$W^{11}(u, w) = u_1, a_1, u_2, \dots, u_n, A_n(u_n, w)$, w has more number of vertices.

Then $w(u, v)$ again a contradiction. Hence, $W(u, v)$ indeed contains all the vertices in D .

2). Proof for the converse part is similar to that of above theorem.

Hence the theorem.

3. **STRONG – ORIENTATION OF GRAPHS:-**

In this section we are concerned with the following one-way traffic problem. When is it possible to make the roads of a city one-way in such way that every corner of the city is reachable from every other corner ?

Consider the road map of figure 10.7 (a)

It is impossible to make all the roads one-way as required . The impossibility is because of the road connecting the school and the garden.

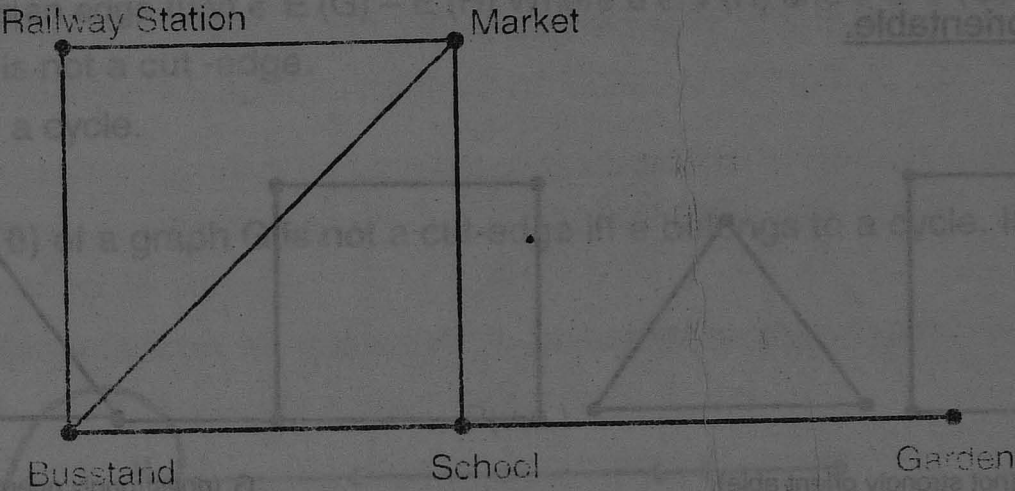


Fig 10.7 (a)

However, if for example, we have an additional road connecting the market and the garden, it is possible to make all the roads one-way as required. (See Figure 10.7 (b)).

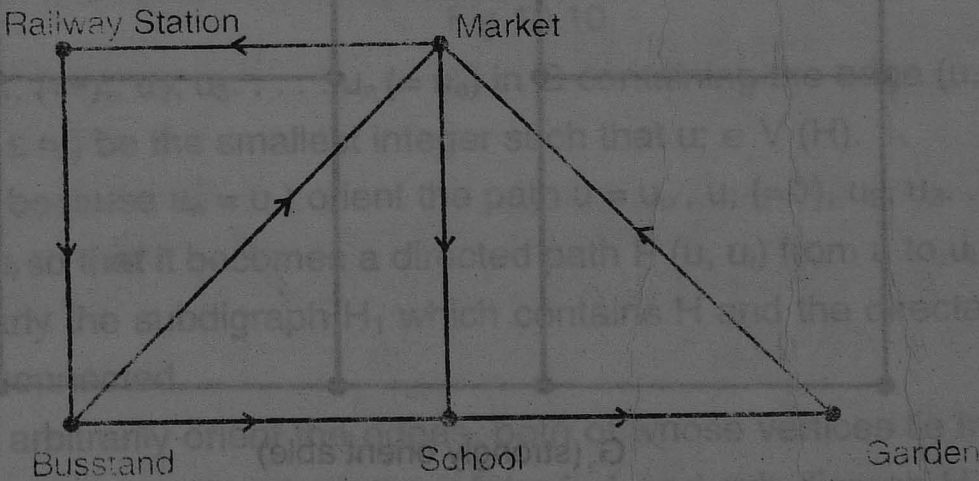


Fig 10.7 (b)

DEFN:-

A graph is said to be strongly orientable if it is possible to given an orientation to each edge of G (in exactly one of the two possible ways) So that the resulting digraph is strongly connected.

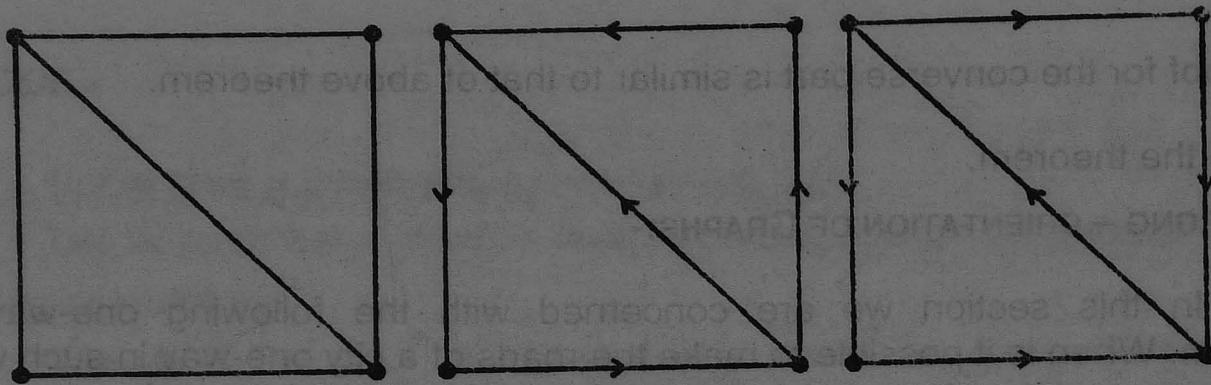
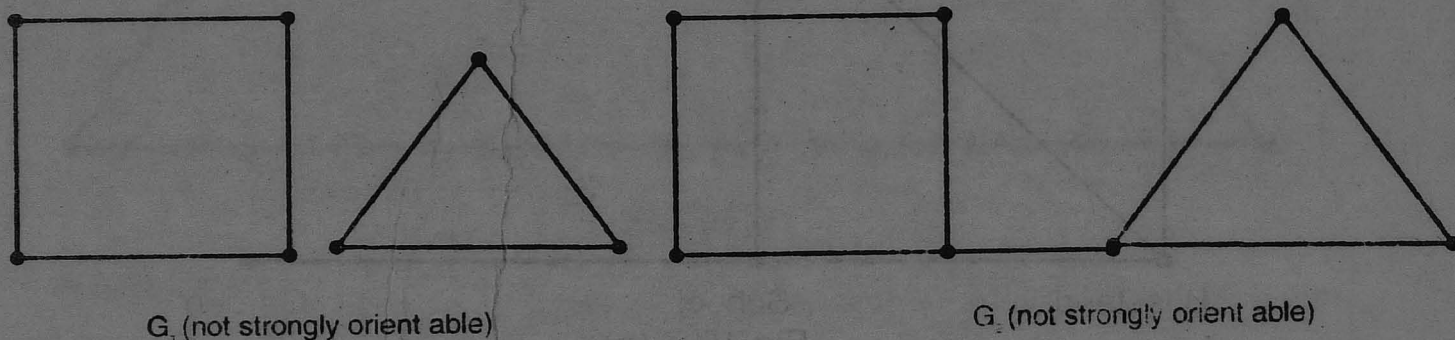


Fig 10.8 A graph G and its two strong Orientations

Thus in graph theoretic terminology the one-way traffic problem is to characterize those graphs which are strongly orientable.

In figure 10.9, graphs G_1 and G_2 are not strongly orientable. Where as G_3 is strongly orientable.



G_3 (strongly orient able)

Fig 10.9

THM:-

A connected graph G is strongly orientable iff G has no cut-edges.

PROOF:-

(1) Suppose that G is strongly orientable to a strongly connected digraph D .

Let (u, v) be an arbitrary edge of G . So, $(u, v) \in A(D)$ or $(v, u) \in A(D)$;

Say $(u, v) \in A(D)$. Since D is strongly connected there exists a directed path $p(v, u)$ from v to u . But then $u, (u, v), p(v, u)$ is a cycle in G containing the edge (u, v)

By thm, An edge $e = (u, v)$ of a graph G is not a cut-edge iff e belongs to a cycle in G . Hence (u, v) is not a cut-edge.

(2). Suppose that G has no cut-edges. Assume for the moment that a proper subgraph H

i.e. $V(G) \neq V(H)$ has been strongly oriented. Since $V(G) \neq V(H)$ and G is connected there is an edge $(u, v) \in E(G) - E(H)$ Where $u \in V(H)$ and $v \in V(G) - V(H)$. Since $e = (u, v)$ is not a cut-edge.

$\therefore$ there is a cycle.

By thm,

An edge $e = (u, v)$ of a graph G is not a cut-edge iff e belongs to a cycle. In G .

Hence a cycle.

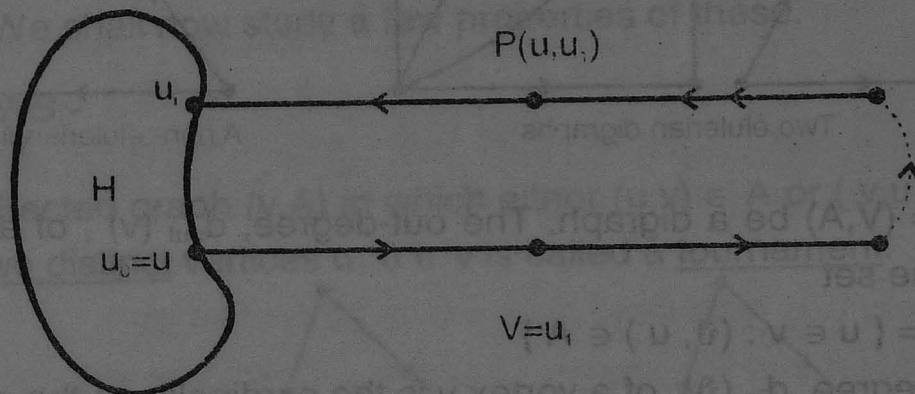


Fig 10.10

C: $(u) u_0, u_1, (=v), u_2, u_3, \dots, u_n (=u_0)$ in G containing the edge (u, v)

Let i ($0 < i \leq n$) be the smallest integer such that $u_i \in V(H)$.

(Note the i exists because $u_n = u$) orient the path $u = u_0, u_1, (=v), u_2, u_3, \dots, u_i$

from u to u_i so that it becomes a directed path $P(u, u_i)$ from u to u_i .

Then clearly the subdigraph H_1 which contains H and the directed path $p(u, u_i)$ is strongly connected.

If we now arbitrarily orient the edges, both of whose vertices lie in $V(H) \cup \{u_0, u_1, \dots, u_i\}$ we get a strong orientation of the induced sub digraph H_2 induced on $V(H) \cup \{u_0, u_1, \dots, u_i\}$ Moreover H_2 contains atleast one more edges.

This gives us a clue to orient the whole graph G .

Since G has no cut-edges, it contains a cycle, say c .

Orient c (in one of the two possible ways), so that it becomes a directed

cycle C^* ; obviously C is strongly connected. If C contains all the vertices of G , then we are through (after arbitrarily orienting the edges of $E(G) - E(C)$).

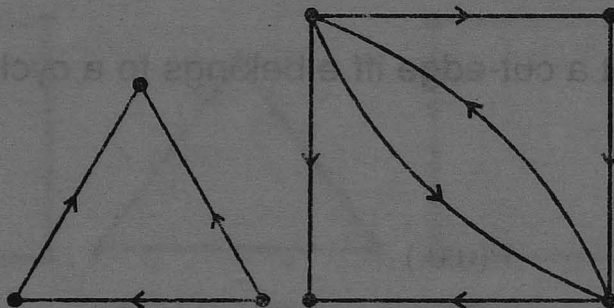
If C^* does not contain all the vertices of G , extend the orientation of C^* to a larger digraph G_1 as explained earlier, with the help of a suitable edge (u, v) . This kind of extension can be continued until we orient all the edges of G .

4. EULERIAN DIGRAPHS:-

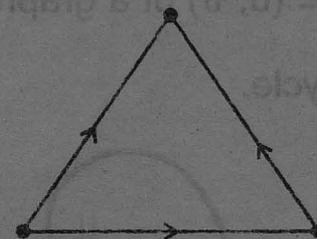
We now define and characterize eulerian directed graphs.

DEFINITIONS:-

1. A closed directed trail in a digraph D is called an eulerian trail if it contains all the edges in D .
2. A digraph is called an eulerian digraph if it contains an eulerian trail.



Two eulerian digraphs



A non-eulerian digraph

DEFINITION:-

Let $D = (V, A)$ be a digraph. The out-degree, $d_{\text{out}}(v)$, of a vertex v is the cardinality of the set

$$N_{\text{out}}(v) = \{ u \in V : (v, u) \in A \}$$

The in-degree, $d_{\text{in}}(v)$, of a vertex v is the cardinality of the set

$$N_{\text{in}}(v) = \{ u \in V : (u, v) \in A \}$$

THM:-

A weak digraph D is Eulerian iff every point of D has equal indegree and outdegree.

PROOF:-

Let D be eulerian and T be an eulerian trail in D .

Each occurrence (occurrence at origin and terminus of T together is to be considered as a single occurrence) of a given point v in T contributes one to $d^-(v)$ and one to $d^+(v)$. Since each arc of D occurs exactly once in T , the contribution of each arc of D to $d^-(v)$ and $d^+(v)$ can be accounted in this way.

Hence $d^-(v) = d^+(v)$ for every point v of D conversely, Let $d^-(v) = d^+(v)$ for every point v of D . Since the trivial digraph is vacuously eulerian, Let D ,

have atleast two points. Hence every point of D has (+ve) indegree and outdegree. Hence D contains a cycle z

The removal of the lines of z results in a spanning subdigraph D_1 in which again $d^-(v) = d^+(v)$ for every point v .

If D_1 has no arcs, then z is an eulerian trail in D . Otherwise, D_1 has a cycle z_1

Continuing the above process, when a digraph D_n with no arc is obtained, we have a partition of the arcs of D into n cycles, $n \geq 2$. Among these n cycles, takes two cycles z_i and z_j having a point v in common. The walk beginning at v and consisting of the cycles z_i and z_j in succession is a closed trail containing the lines of these two cycles.

Continuing this process, we can construct a closed trail containing all the arcs of D . Hence D is eulerian.

5. TOURNAMENTS:-

Tournaments form an interesting subclass of digraphs that have very rich structure . We shall now study a few properties of these.

DEFINITIONS:-

A directed graph (v,A) in which either $(u,v) \in A$ or $(v,u) \in A$ (but not both) for every two distinct vertices $u, v \in v$ is called a tournament.

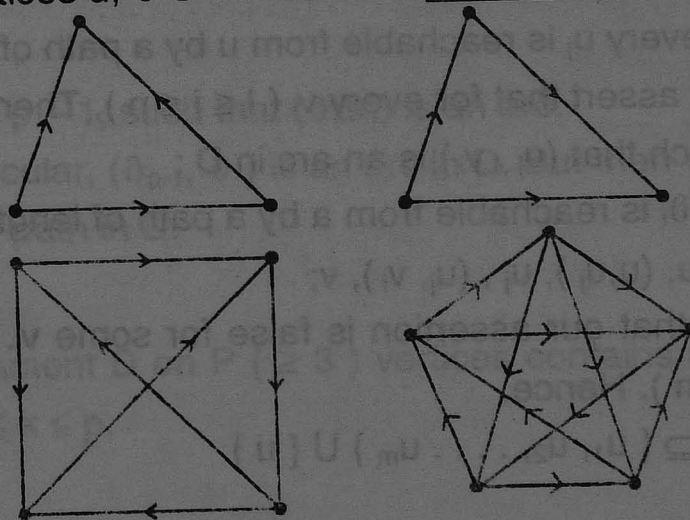


Figure 10.12 : Four tournaments

DEFINITIONS:-

1. A directed path (cycle) in a digraph D is called a directed Hamiltonian path (cycle) if it contains all the vertices.
2. A digraph is called a Hamiltonian digraph if it contains a directed Hamiltonian cycle.

THM:-

Every tournament D contains a vertex from which every other vertex is reachable by a path of length at most 2.

PROOF:-

Let u be a vertex of maximum out-degree in D .

Consider the sets $N_{\text{out}}(u) = \{u_1, u_2, \dots, u_m\}$ and $N_{\text{in}}(u) = \{v_1, v_2, \dots, v_n\}$

Where $m = d_{\text{out}}(u)$ and $n = d_{\text{in}}(u)$

Since D is a tournament, $V(D) - u = N_{\text{out}}(u) \cup N_{\text{in}}(u)$ is a partition.

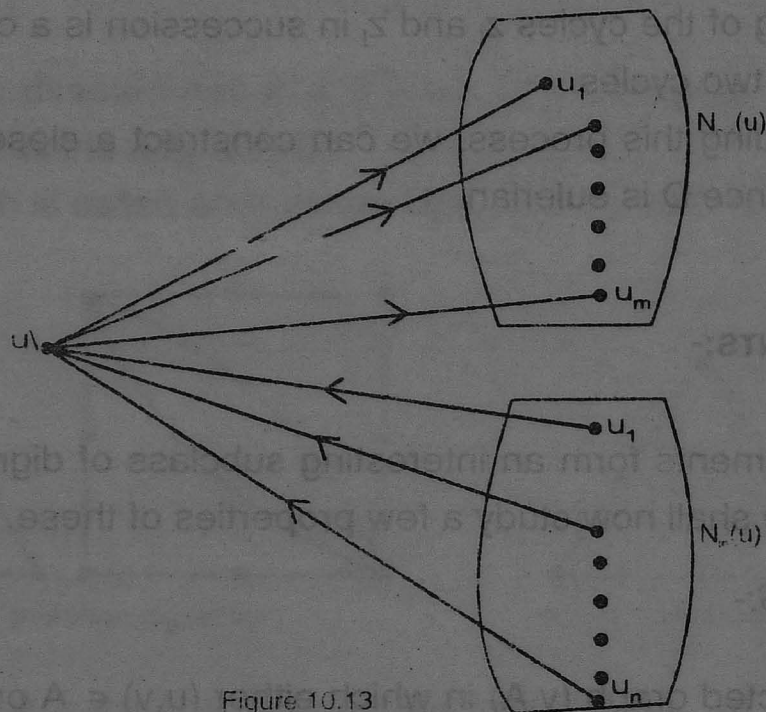


Figure 10.13

Clearly every u_j is reachable from u by a path of length 1.

We next assert that for every v_i ($1 \leq i \leq n$), There is some u_j ($1 \leq j \leq m$) Such that (u_j, v_i) is an arc in D ;

So, that v_i is reachable from u by a path of length 2,

Namely $u, (u, u_j), u_j, (u_j, v_i), v_i$;

Assume that our assertion is false for some v_i . Then (v_i, u_j) is an arc for every j ($1 \leq j \leq m$). Hence,

$$N_{\text{out}}(v_i) \supseteq \{u_1, u_2, \dots, u_m\} \cup \{u\}$$

Thus,

$$\text{Deg}_{\text{out}}(v_i) \geq \text{deg}_{\text{out}}(u) + 1 - \text{contradiction}$$

Hence the thm.

THM:-

Every tournament D contains a directed Hamiltonian path.

PROOF:-

We prove the theorem by induction of $P = |V(D)|$.

If $P = 1$ or 2 . Then the theorem is obvious . Assume that a tournament contains a directed Hamiltonian path if it has $p - 1$ vertices and let D contain P vertices. Let $v \in V(D)$ and consider the tournament $D - v$.

By induction hypothesis, $D - v$ contains a directed Hamiltonian path, say $v_1, v_2, \dots, v_{p-1}$.



Figure 10.14

we make 3 cases and prove that D contains a directed Hamiltonian path in each case.

Case (1):-

(v, v_1) is an arc in D clearly $v, v_1, \dots, v_{p-1}$ is a directed Hamiltonian path in D .

Case (2) :-

(v, v_1) is not arc (and hence (v_1, v) is an arc) but there is a some i ($2 \leq i \leq p - 1$) such that (v, v_i) is an arc.

Let j ($2 \leq j \leq p - 1$) be the smallest integer such that (v, v_j) is an arc in D . This mean that (v, v_{j-1}) is not an arc.

Hence (v_{j-1}, v) is an arc.

But then

$v_1, v_2, \dots, v_{j-1}, v, v_j, v_{j+1}, \dots, v_{p-1}$ is a directed Hamiltonian path in D .

(see figure 10,14)

Case (3):-

There is no i , $1 \leq i \leq p - 1$, such that (v, v_i) is an arc.

This means, in particular, (v_{p-1}, v) is an arc in D . But then $v_1, v_2, \dots, v_{p-1}, v$ is a directed hamiltonian path in D .

THM:-

Every strong tournament D on P (≥ 3) vertices contains a directed cycle of length k , for every k , $3 \leq k \leq p$.

PROOF:-

We first prove that D contains a directed 3 - cycle and then show that, if D contains a directed k - cycle ($3 \leq k \leq p$). then if contains a directed $(k+1)$ cycle so that D contained K - cycle

1). D contains a directed 3 - cycle:

Let $v \in v(D)$ and consider $N_{out}(u)$ and $N_{in}(u)$

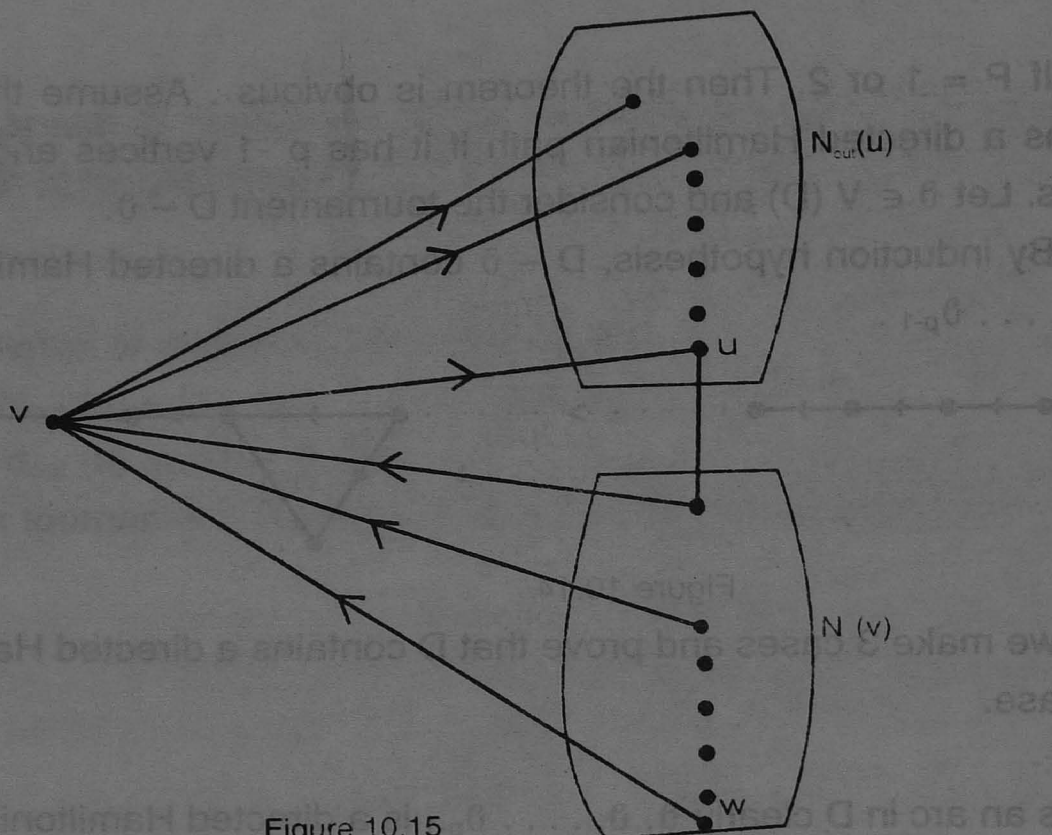


Figure 10.15

Since D is strongly connected, $N_{out}(\vartheta)$ and $N_{in}(\vartheta)$ are non empty.

Moreover, there is an arc of the form (u, w) in D where $u \in N_{out}(\vartheta)$ and $w \in N_{in}(\vartheta)$.

Because every directed path from u to a vertex $w \in N_{in}(\vartheta)$, must pass through a vertex in $N_{out}(\vartheta)$.

But then u, u, w, v is a directed cycles of length 3.

2) Let $C: \vartheta_1, \vartheta_2, \dots, \vartheta_k, \vartheta_1$ be a directed cycle of length $K \in D$ (where $3 \leq k \leq p - 1$) We make two cases and in each case produce a direct $(k+1)$ cycle.

Case(1):-

There is a vertex $u \in D - v(C)$ such that (ϑ_i, ϑ) and (ϑ, ϑ_j) for some i and j ; are arcs in D .

Without loss of generality, Let $(\vartheta, \vartheta) \in A(D)$

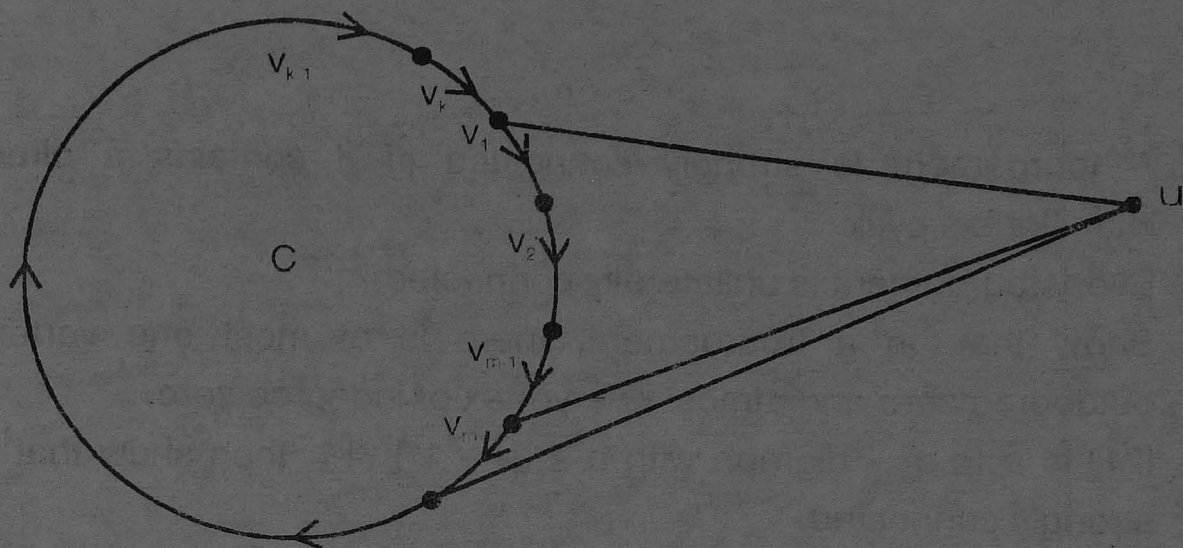


Figure 10.16

Let m ($2 \leq m \leq k$) be the smallest integer, such that $(v, v_m) \in A(D)$.

So, $(v_{m-1}, v) \in A(D)$

Then, $v_1, v_2, \dots, v_{m-1}, v, v_m, v_{m+1}, \dots, v_k, v_1$ is a directed cycle of length $(k+1)$ in D (See figure 10.16)

Case (2):-

For every vertex $v \in v(D) - V(C)$ either (a) $(u, v_i) \in A(D)$, for every i , $1 \leq i \leq k$. or (b) $(v_i, u) \in A(D)$, for every i , $1 \leq i \leq k$.

Let $X = \{u \in v(D) - v(C) ; (u, v_i) \in A(D), \text{ for every } i, 1 \leq i \leq k\}$; and $\vartheta = \{w \in v(D) - v(C) ; (v_i, w) \in A(D), \text{ for every } i, 1 \leq i \leq k\}$

Since D is a strongly connected tournament and since for every $u \in v(D) - V(C)$ either (a) or (b) holds, it follows that

$X \neq \emptyset$, $\vartheta \neq \emptyset$, $X \cap \vartheta \neq \emptyset$ and $X \cup \vartheta = V(D) - V(C)$.

There are vertices $U \in X$ and $w \in \vartheta$ such that (U, w) is an arc in D , because every directed path from a vertex in $V(C)$ to a vertex in ϑ must pass through a vertex in X .

Now, Since $U \in X$ and $w \in \vartheta$, the arcs (U, w) and (w, v_1) are in D .

But then $v_1, U, w, v_3, v_4, \dots, v_k, v_1$ is a directed cycle of length $k+1$ in D .

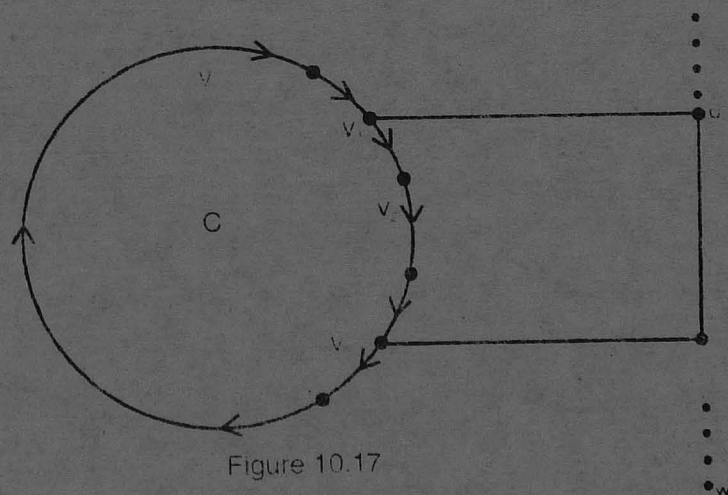


Figure 10.17

Hence the theorem.

EXERCISE:-

1. A tournament is strongly connected iff it contains a directed Hamiltonian cycle.
2. Every tournament is unilaterally connected.
3. Show that, in a tournament, there is at most one vertex of outdegree zero and at most one vertex of indegree zero.
4. If D is a (p, q) – digraph with $q \geq (p - 1)^2 + 1$, then show that D is strongly connected.
5. If D is a weakly connected digraph such that $D - v$ is strong for some vertex $u \in v(D)$, then show that D is unilaterally connected.

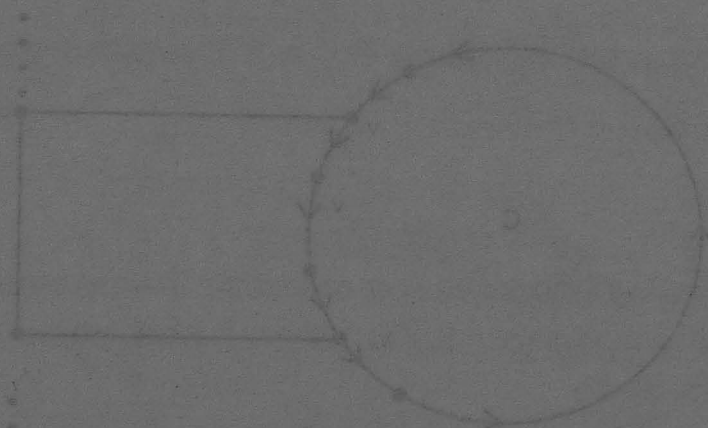


Figure 10.17

Hence the theorem.